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MAXIMAL REGULARITY FOR NONSMOOTH PARABOLIC PROBLEMS IN SOBOLEV–MORREY SPACES

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Abstract. This text is devoted to maximal regularity results for second-order parabolic systems on Lipschitz domains of space dimension $n \geq 3$ with diagonal principal part, nonsmooth coefficients, and nonhomogeneous mixed boundary conditions. We show that the corresponding class of initial-value problems generates isomorphisms between two scales of Sobolev–Morrey spaces for solutions and right-hand sides introduced in the first part [12] of our presentation. The solutions depend smoothly on the data of the problem. Moreover, they are Hölder continuous in time and space up to the boundary for a certain range of Morrey exponents. Due to the complete continuity of embedding and trace maps these results remain true for a broad class of unbounded lower-order coefficients.

1. Formulation of the regularity problem

Many instationary drift-diffusion problems are formulated in terms of second-order parabolic boundary-value problems with nonsmooth data. To prove existence and uniqueness results or further qualitative properties like regularity or asymptotic behaviour of solutions it is useful to get a priori estimates for solutions of the original or at least of some auxiliary linear parabolic problem in spaces of bounded or Hölder-continuous functions.

In the first part [12] of our presentation we introduced and discussed in detail new classes of Sobolev–Morrey spaces allowing a satisfactory treatment of the regularity problem for second-order linear parabolic boundary-value problems

$$(\mathcal{E}u)' + \mathcal{A}u + \mathcal{B}u = f \in L^2(S; Y^*), \quad u(t_0) = 0, \tag{1.1}$$

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of drift-diffusion-type on regular sets $G \subset \mathbb{R}^n$ with Lipschitz boundary. The natural choice for the Hilbert space Y in the functional analytic formulation of elliptic and parabolic problems with mixed boundary conditions is the Sobolev space $Y = H_0^1(G)$ and its dual $Y^* = H^{-1}(G)$, see also Gröger and Rehberg [16, 17, 18], and Griepentrog and Recke [10, 14].

In (1.1) the operator $\mathcal{E} \in L^2(S;Y) \to L^2(S;Y^*)$ is associated with the bounded open time interval $S = (t_0, t_1)$ and the map $E \in \mathcal{L}(Y;Y^*)$ via $(\mathcal{E}u)(s) = Eu(s)$ for $s \in S$, $u \in L^2(S;Y)$. Here, $E \in \mathcal{L}(Y;Y^*)$ is defined by

$$\langle Ev, w\rangle_Y = \int_G avw \, d\lambda^n \quad \text{for } v,\, w \in Y.$$

The nonsmooth capacity coefficient $a \in L^\infty(G^\circ)$ satisfies

$$\varepsilon \le \operatorname*{ess\,inf}_{x\in G^\circ} a(x), \quad \operatorname*{ess\,sup}_{x\in G^\circ} a(x) \le \frac{1}{\varepsilon}$$

for some constant $\varepsilon \in (0,1]$. Moreover, we consider nonsmooth diffusivity coefficients $A \in L^\infty(S; L^\infty(G^\circ;\mathbb{S}^n))$ with values in the set $\mathbb{S}^n$ of symmetric $(n\times n)$-matrices, and we assume that for all $\xi \in \mathbb{R}^n$ we have

$$\varepsilon\,\|\xi\|^2 \le \operatorname*{ess\,inf}_{s\in S}\, \operatorname*{ess\,inf}_{x\in G^\circ} A(s)(x)\xi\cdot\xi, \quad \operatorname*{ess\,sup}_{s\in S}\, \operatorname*{ess\,sup}_{x\in G^\circ} A(s)(x)\xi\cdot\xi \le \frac{1}{\varepsilon}\,\|\xi\|^2.$$

With regard to problem (1.1) the principal part $\mathcal{A} : L^2(S;Y) \to L^2(S;Y^*)$ is of the form

$$\langle \mathcal{A}u, w\rangle_{L^2(S;Y)} = \int_S \int_G A(s)\nabla u(s)\cdot\nabla w(s)\, d\lambda^n\, ds \quad \text{for } u,\, w \in L^2(S;Y).$$

Given lower-order coefficients

$$b \in L^\infty(S; L^\infty(G^\circ;\mathbb{R}^n)), \quad b_0 \in L^\infty(S; L^\infty(G^\circ)), \quad b_\Gamma \in L^\infty(S; L^\infty(\Gamma)),$$

which describe drift and damping phenomena, we define $\mathcal{B} : L^2(S;Y) \to L^2(S;Y^*)$ by

$$\begin{aligned}\langle \mathcal{B}u, w\rangle_{L^2(S;Y)} &= \int_S\int_G \big(u(s)b(s)\cdot\nabla w(s) + b_0(s)u(s)w(s)\big)\, d\lambda^n\, ds \\ &\quad + \int_S\int_\Gamma b_\Gamma(s) K_\Gamma u(s) K_\Gamma w(s)\, d\lambda_\Gamma\, ds\end{aligned}$$

for $u, w \in L^2(S;Y)$. Here, $\Gamma = \partial G$ is the Lipschitz boundary of the regular set $G \subset \mathbb{R}^n$, and $K_\Gamma \in \mathcal{L}(H_0^1(G); L^2(\Gamma))$ denotes the trace map.

Using Gröger's functional analytic framework for evolution equations, discussed in detail in [15] and the first part [12] of our presentation, we get unique solvability and well-posedness of problem (1.1) in the Hilbert space

$$W_E(S;Y) = \big\{ u \in L^2(S;Y) : (\mathcal{E}u)' \in L^2(S;Y^*) \big\}.$$

Theorem 1.1 (Unique solvability)**.** *The solution operator associated with problem* (1.1) *is a linear isomorphism between the spaces* $L^2(S;H^{-1}(G))$ *and* $\big\{u \in W_E(S;H_0^1(G)) : u(t_0) = 0\big\}$.

Proof. As we will see it suffices to show that the bounded linear Volterra operator $\mathcal{M} = \mathcal{A} + \mathcal{B} + \alpha\mathcal{E} : L^2(S;Y) \to L^2(S;Y^*)$ is positive definite whenever $\alpha > 1$ is large enough. Due to our assumptions for all $u \in L^2(S;Y)$ we obtain

$$\langle (\mathcal{A} + \alpha\mathcal{E})u, u\rangle_{L^2(S;Y)} \geq \varepsilon \,\|u\|^2_{L^2(S;Y)} + \varepsilon(\alpha - 1)\|u\|^2_{L^2(S;L^2(G^\circ))}.$$

For the trace map $K_\Gamma \in \mathcal{L}(H_0^1(G);L^2(\Gamma))$ the multiplicative inequality [12, Equation (3.1)] holds true: We find some constant $c_G > 0$ such that

$$\|K_\Gamma v\|^2_{L^2(\Gamma)} \leq c_G \|v\|_{H_0^1(G)} \|v\|_{L^2(G^\circ)} \quad \text{for all } v \in H_0^1(G).$$

Applying Young's inequality for all $u \in L^2(S;Y)$ and $\delta > 0$ this yields

$$\big|\langle \mathcal{B}u, u\rangle_{L^2(S;Y)}\big| \leq \frac{\delta L(c_G + 1)}{2} \,\|u\|^2_{L^2(S;Y)} + L\Big(\frac{c_G + 1}{2\delta} + 1\Big)\|u\|^2_{L^2(S;L^2(G^\circ))}.$$

Here, $L = \max\big\{\|b\|_{L^\infty(S;L^\infty(G^\circ;\mathbb{R}^n))}, \|b_0\|_{L^\infty(S;L^\infty(G^\circ))}, \|b_\Gamma\|_{L^\infty(S;L^\infty(\Gamma))}\big\} > 0$ is the common bound of the lower-order coefficients. If we choose $\delta > 0$ small enough and $\alpha > 1$ large enough such that

$$\frac{\delta L(c_G + 1)}{2} < \varepsilon, \quad L\Big(\frac{c_G + 1}{2\delta} + 1\Big) \leq \varepsilon(\alpha - 1),$$

then $\mathcal{M} = \mathcal{A} + \mathcal{B} + \alpha\mathcal{E} : L^2(S;Y) \to L^2(S;Y^*)$ is positive definite. Using [12, Theorem 2.4] the solution operator associated with problem (1.1) maps $L^2(S;H^{-1}(G))$ isomorphically onto $\big\{u \in W_E(S;H_0^1(G)) : u(t_0) = 0\big\}$. □

Following the theory of Ladyzhenskaya, Solonnikov, and Uraltseva [21] it is true that the solution u of problem (1.1) is Hölder continuous in time and space up to the boundary provided that $f \in L^q(S;W^{-1,p}(G))$ and $q > 2$ and $p > n$ with $2/q + n/p < 1$. But in contrast to the case $n = 2$ it has turned out that for $n \geq 3$ it is *not* possible to find $q > 2$ and $p > n$ satisfying $2/q + n/p < 1$ such that maximal regularity

$$u \in L^q(S;W^{1,p}(G^\circ)), \quad (\mathcal{E}u)' \in L^q(S;W^{-1,p}(G)),$$

holds true for every $f \in L^q(S; W^{-1,p}(G))$ without further assumptions on the smoothness of the data; see also Gröger and Rehberg [16, 17, 18].

Fortunately, we have found alternative function spaces for solutions and right-hand sides meeting both the requirements of Hölder continuity *and* maximal regularity in the case $n \geq 3$. The main goal of this text is to prove the following maximal regularity result: For a certain range of parameters $0 \leq \omega < \bar{\omega}_\varepsilon(G)$ with $\bar{\omega}_\varepsilon(G) > n$ the class of problems (1.1) generates linear isomorphisms between two scales of Sobolev–Morrey spaces $\big\{u \in W^\omega_E(S;Y) : u(t_0) = 0\big\}$ and $L^\omega_2(S;Y^*)$ of solutions and functionals, respectively. Here, the function space

$$W^\omega_E(S;Y) = \big\{u \in L^\omega_2(S;Y) : (\mathcal{E}u)' \in L^\omega_2(S;Y^*)\big\} \subset W_E(S;Y)$$

is embedded into a space of Hölder-continuous functions for $\omega > n$, where

$$L^\omega_2(S;Y) \subset L^2(S;Y), \quad L^\omega_2(S;Y^*) \subset L^2(S;Y^*),$$

are suitably chosen Sobolev–Morrey spaces. We refer to the first part [12] for the theory of the above function spaces.

As the starting point for our regularity theory we consider the case $\mathcal{B} = 0$. In the first step we are interested in local estimates for solutions of (1.1) restricted to families of time intervals $I_r(t) = (t - r^2, t) \subset S$, and cubes $Q_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\} \subset G$, regardless of initial or boundary conditions; see Section 2. Here, $t \in S$ and $x \in G$ are fixed, whereas the radius $0 < r \leq 1$ varies in a certain range. One advantage of considering solutions in the function space $W_E(S;Y)$ is that we can completely avoid the technique of Steklov averages. Instead of this method we use integration-by-parts formulae which can be found in Section 1 and Appendix B of the first part [12] of our presentation.

We carry over results well-known for the case of constant capacity coefficients; see Moser [23, 24], Ladyzhenskaya, Solonnikov, and Uraltseva [21], Aronson and Serrin [3], Trudinger [29], and Lieberman [22]. Note that in the case of nonsmooth capacity coefficients a comprehensive regularity theory for (fundamental) solutions of Cauchy's problem can be found in the work of Porper and Eidelman [25, 26] generalizing classical results of Aronson [1, 2].

Based on energy estimates for solutions, in Section 3 we obtain local boundedness results using the Moser iteration technique. As a byproduct, we fill a gap in the proof of Porper and Eidelman [26, Theorem 2] arising from an illegal extension of local solutions to solutions of Cauchy's problem.

Combined with Harnack-type inequalities, see Section 4, this paves the way to estimate the oscillation of solutions, which leads to the Campanato inequality for the spatial gradients of solutions on concentric cubes; see Section 5. To do so, we generalize methods introduced by Kruzhkov [19, 20] and used by Hong-Ming Yin [30] to the case of nonsmooth capacity coefficients. In addition to that, we apply a special variant of the Poincaré inequality contained in Appendix A of the first part [12] of this presentation; see also Struwe [27].

To prove the global regularity result, in Section 6 we define a suitable class of admissible sets consisting of all regular sets $G \subset \mathbb{R}^n$ for which the desired regularity in Sobolev–Morrey spaces holds true for the case $\mathcal{B} = 0$. The invariance of this concept with respect to the principles of localization, Lipschitz transformation, and reflection has already turned out to be successful in the elliptic regularity theory; see Griepentrog and Recke [10, 14]. To show that every regular set is admissible, therefore, it remains to prove the admissibility of some standard cuboids. For that purpose, we use the Campanato inequality for the spatial gradients of solutions on concentric cubes; see Section 5.

Finally, in Section 7 we end up our considerations with isomorphism properties for parabolic operators. For bounded lower-order coefficients the solution operator associated with problem (1.1) maps the Sobolev–Morrey space $L_2^\omega(S;Y^*)$ of linear functionals isomorphically to the Sobolev–Morrey space $\{u \in W_E^\omega(S;Y) : u(t_0) = 0\}$ of solutions for all Morrey exponents $0 \le \omega < \bar{\omega}_\varepsilon(G)$, where $\bar{\omega}_\varepsilon(G) > n$ depends on n, ε, S, and G, only. The solution depends smoothly on the coefficients A, b, b_0, and b_Γ.

Note, that for $\omega \in (n, n+2]$ the embedding and trace operators from $W_E^\omega(S;Y)$ into spaces of Hölder-continuous functions are completely continuous. As a consequence, for $n < \omega < \bar{\omega}_\varepsilon(G)$ all the results remain true if the operator $\mathcal{B}$ contains unbounded lower-order coefficients

$$b \in L_2^\omega(S; L^2(G^\circ;\mathbb{R}^n)), \quad b_0 \in L_2^{\omega-2}(S; L^2(G^\circ)), \quad b_\Gamma \in L_2^{\omega-1}(S; L^2(\Gamma)),$$

belonging to well-known Morrey spaces. Moreover, all the assertions can be generalized to weakly coupled systems; that means to problems with principal parts $\mathcal{E}$ and $\mathcal{A}$ of diagonal structure and operators $\mathcal{B}$ containing strongly coupled lower-order terms.

This allows us to prove the unique solvability and regularity of second-order drift-diffusion problems with linear diffusion terms and nonlinear drift terms which describe, for instance, transport processes of charged particles in semiconductor heterostructures, chemotactical aggregation of biological organisms in heterogeneous environments, or phase separation processes of nonlocally interacting particles; see Gajewski and Skrypnik [4, 5, 6] and Griepentrog [11].

In these applications the drift coefficients b are proportional to the spatial gradients ∇v of interaction potentials v which are solutions to similar quasistationary elliptic or parabolic subproblems having exactly the required regularity $\nabla v \in L_2^\omega(S; L^2(G^\circ; \mathbb{R}^n))$. Hence, in the case $n \geq 3$ our approach avoids artificial assumptions on the smoothness of the data that are in general necessary to prove that, for instance, $\nabla v \in L^q(S; L^p(G^\circ; \mathbb{R}^n))$ holds true for some $q > 2$ and $p > n$ satisfying $2/q + n/p < 1$.

2. Local model problem

Assuming that $\mathcal{B} = 0$, we are looking for local estimates for solutions of problem (1.1) restricted to families of time intervals $I_r(t) = (t - r^2, t) \subset S$, and concentric cubes $Q_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\} \subset G$, regardless of initial or boundary conditions. Here, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ are fixed, and the radius $0 < r \leq 1$ varies in a certain range. Hence, if there is no fear of misunderstanding we shortly write I_r and Q_r, respectively.

Our local model problem describes, for instance, a heat conduction process during the time interval I_r inside a cube Q_r which contains an inhomogeneous material. Its thermal properties are described by a nonsmooth heat capacity coefficient $a \in L^\infty(Q_r)$ which satisfies

$$\varepsilon \leq \operatorname*{ess\,inf}_{y \in Q_r} a(y), \quad \operatorname*{ess\,sup}_{y \in Q_r} a(y) \leq \frac{1}{\varepsilon},$$

and a nonsmooth heat conduction coefficient $A \in L^\infty(I_r; L^\infty(Q_r; \mathbb{S}^n))$ with values in the set $\mathbb{S}^n$ of symmetric $(n \times n)$-matrices satisfying

$$\varepsilon \, \|\xi\|^2 \leq \operatorname*{ess\,inf}_{s \in I_r} \operatorname*{ess\,inf}_{y \in Q_r} A(s)(y)\xi \cdot \xi, \quad \operatorname*{ess\,sup}_{s \in I_r} \operatorname*{ess\,sup}_{y \in Q_r} A(s)(y)\xi \cdot \xi \leq \frac{1}{\varepsilon} \, \|\xi\|^2$$

for all $\xi \in \mathbb{R}^n$ and some ellipticity constant $0 < \varepsilon \leq 1$.

For the functional analytic formulation we choose Hilbert spaces $Y_r = H_0^1(Q_r)$ and $X_r = H^1(Q_r)$. The space $H_r = L^2(Q_r)$ is equipped with the weighted scalar product defined by

$$(v|w)_{H_r} = \int_{Q_r} vw \, d\lambda_a^n \quad \text{for } v, w \in H_r,$$

where λ_a^n is the weighted Lebesgue measure defined as

$$\lambda_a^n(\Omega) = \int_\Omega a \, d\lambda^n \quad \text{for Lebesgue-measurable subsets } \Omega \subset Q_r.$$

We consider the completely continuous embedding $K_r \in \mathcal{L}(X_r; H_r)$ of X_r in H_r. Note that the restriction $K_r|Y_r \in \mathcal{L}(Y_r; H_r)$ has a dense range $K_r[Y_r]$ in H_r. In addition to that, we introduce $\mathcal{E}_r : L^2(I_r; X_r) \to L^2(I_r; Y_r^*)$ as the linear operator associated with I_r and $E_r = (K_r|Y_r)^* J_{H_r} K_r \in \mathcal{L}(X_r; Y_r^*)$.

The next three sections are dedicated to the local regularity properties of functions $v \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ satisfying the homogeneous variational problem

$$\int_{I_r} \langle (\mathcal{E}_r v)'(s), w(s) \rangle_{Y_r} \, ds + \int_{I_r} \int_{Q_r} A(s) \nabla v(s) \cdot \nabla w(s) \, d\lambda^n \, ds = 0, \qquad (2.1)$$

or the inhomogeneous variational problem

$$\int_{I_r} \langle (\mathcal{E}_r v)'(s), w(s) \rangle_{Y_r} \, ds + \int_{I_r} \int_{Q_r} A(s) \nabla v(s) \cdot \nabla w(s) \, d\lambda^n \, ds = \int_{I_r} \langle f(s), w(s) \rangle_{Y_r} \, ds \qquad (2.2)$$

for all test functions $w \in L^2(I_r; Y_r)$ and exterior heat sources $f \in L^2(I_r; Y_r^*)$.

3. Caccioppoli inequalities and local boundedness

Energy estimates. We start our regularity theory with the proof of the local boundedness of solutions to the homogeneous problem (2.1). To do so, we use the following energy estimates:

Lemma 3.1 (Caccioppoli inequalities)**.** *Let $\iota \in C^2(\mathbb{R})$ satisfy $\iota', \iota'' \in BC(\mathbb{R})$ and assume that $\iota''\iota \in BC(\mathbb{R})$ is nonnegative. For all $0 < \delta < r \leq 1$ and every solution $v \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ of (2.1) the estimates*

$$\sup_{s \in \overline{I_\delta}} \int_{Q_\delta} |u(s)|^2 \, d\lambda^n \leq \frac{20}{\varepsilon^2 (r-\delta)^2} \int_{I_r} \int_{Q_r} |u(s)|^2 \, d\lambda^n \, ds, \qquad (3.1)$$

$$\int_{I_\delta} \int_{Q_\delta} \|\nabla u(s)\|^2 \, d\lambda^n \, ds \leq \frac{20}{\varepsilon^2 (r-\delta)^2} \int_{I_r} \int_{Q_r} |u(s)|^2 \, d\lambda^n \, ds, \qquad (3.2)$$

hold true for the composition $u = \iota \circ v \in L^2(I_r; X_r) \cap C(\overline{I_r}; H_r)$.

Proof. 1. Let $0 < \delta < r \leq 1$ and $\tau \in \overline{I_\delta}$ be fixed. Now, we choose a cut-off function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that for all $y \in \mathbb{R}^n$

$$0 \leq \zeta(y) \leq 1, \quad \|\nabla \zeta(y)\| \leq \frac{2}{r-\delta}, \quad \zeta(y) = \begin{cases} 0 & \text{if } y \in \mathbb{R}^n \setminus Q_r, \\ 1 & \text{if } y \in Q_\delta, \end{cases}$$

and some cut-off function $\vartheta \in C^\infty(\mathbb{R})$ such that for all $s \in \mathbb{R}$ we have

$$0 \le \vartheta(s) \le 1, \quad |\vartheta'(s)| \le \frac{2}{(r-\delta)^2}, \quad \vartheta(s) = \begin{cases} 0 & \text{if } s \le t - r^2, \\ 1 & \text{if } s \ge t - \delta^2. \end{cases}$$

2. Suppose that $v \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ solves the variational equation (2.1). Because of $(\iota^2)'' = 2\big(\iota''\iota + |\iota'|^2\big) \in BC(\mathbb{R})$, the function

$$w = \zeta^2 \cdot \chi_{[t-r^2,\tau]} \cdot \vartheta^2 \cdot (\iota^2)' \circ v \in L^2(I_r; Y_r)$$

is an admissible test function for (2.1). The chain rule [12, Lemma B.1, Equation (B.1)] yields

$$\begin{aligned}
&\int_{I_r} \langle (\mathcal{E}_r v)'(s), w(s) \rangle_{Y_r} \, ds \\
&\qquad = \int_{Q_r} \zeta^2 |u(\tau)|^2 \, a \, d\lambda^n - 2 \int_{t-r^2}^{\tau} \int_{Q_r} \zeta^2 \vartheta(s) \vartheta'(s) |u(s)|^2 \, a \, d\lambda^n \, ds \\
&\qquad\qquad \ge \varepsilon \int_{Q_\delta} |u(\tau)|^2 \, d\lambda^n - \frac{4}{\varepsilon (r-\delta)^2} \int_{I_r} \int_{Q_r} |u(s)|^2 \, d\lambda^n \, ds.
\end{aligned}$$

3. In addition to that, a straightforward calculation leads to

$$\begin{aligned}
&\int_{I_r} \int_{Q_r} A(s) \nabla v(s) \cdot \nabla w(s) \, d\lambda^n \, ds \\
&\qquad = 2 \int_{t-r^2}^{\tau} \int_{Q_r} \zeta^2 \vartheta^2(s) \, A(s) \nabla u(s) \cdot \nabla u(s) \, d\lambda^n \, ds \\
&\qquad + 2 \int_{t-r^2}^{\tau} \int_{Q_r} \zeta^2 \vartheta^2(s) \, \iota''(v(s)) \iota(v(s)) \, A(s) \nabla v(s) \cdot \nabla v(s) \, d\lambda^n \, ds \\
&\qquad\qquad + 4 \int_{t-r^2}^{\tau} \int_{Q_r} \vartheta^2(s) \, \zeta u(s) \, A(s) \nabla \zeta \cdot \nabla u(s) \, d\lambda^n \, ds.
\end{aligned}$$

Due to the nonnegativity of $\iota''\iota \in BC(\mathbb{R})$, Young's inequality, and the positive definiteness of A this yields

$$\begin{aligned}
&\int_{I_r} \int_{Q_r} A(s) \nabla v(s) \cdot \nabla w(s) \, d\lambda^n \, ds \\
&\qquad \ge \int_{t-r^2}^{\tau} \int_{Q_r} \zeta^2 \vartheta^2(s) \, A(s) \nabla u(s) \cdot \nabla u(s) \, d\lambda^n \, ds \\
&\qquad\qquad - 4 \int_{t-r^2}^{\tau} \int_{Q_r} \vartheta^2(s) |u(s)|^2 A(s) \nabla \zeta \cdot \nabla \zeta \, d\lambda^n \, ds,
\end{aligned}$$

and hence,

$$\int_{I_r}\int_{Q_r} A(s)\nabla v(s)\cdot\nabla w(s)\,d\lambda^n\,ds \\ \geq \varepsilon\int_{t-\delta^2}^{\tau}\int_{Q_\delta}\|\nabla u(s)\|^2\,d\lambda^n\,ds - \frac{16}{\varepsilon(r-\delta)^2}\int_{I_r}\int_{Q_r}|u(s)|^2\,d\lambda^n\,ds.$$

4. Summing up the results of the preceeding steps we arrive at

$$\int_{Q_\delta}|u(\tau)|^2\,d\lambda^n + \int_{t-\delta^2}^{\tau}\int_{Q_\delta}\|\nabla u(s)\|^2\,d\lambda^n\,ds \\ \leq \frac{20}{\varepsilon^2(r-\delta)^2}\int_{I_r}\int_{Q_r}|u(s)|^2\,d\lambda^n\,ds.$$

Because $\tau\in\overline{I_\delta}$ was arbitrarily fixed at the beginning, we end up with the inequalities (3.1) and (3.2). □

Remark 3.1. The function $\iota\in C^2(\mathbb{R})$, defined as $\iota(z)=z$ for $z\in\mathbb{R}$, is an admissible composition function in Lemma 3.1. Hence, the solution v itself satisfies the Caccioppoli inequalities (3.1) and (3.2).

Local boundedness. To prove the local boundedness of solutions to the homogeneous problem (2.1) we use the Moser iteration technique; that means a recursive application of Caccioppoli inequalities to suitable powers of the solution; see Moser [23, 24].

Theorem 3.2 (Local boundedness)**.** *Let the convex function $\iota\in C^2(\mathbb{R})$ be nonnegative on* $\operatorname{supp}(\iota'')$*, which is assumed to be compact in $\mathbb{R}$. Then there exists some constant $c=c(n,\varepsilon)>0$ such that for all $0<r\leq 1$ and every solution $v\in W_{E_r}(I_r;X_r)\cap C(\overline{I_r};H_r)$ of* (2.1) *the estimate*

$$\operatorname*{ess\,sup}_{s\in I_{r/2}}\operatorname*{ess\,sup}_{y\in Q_{r/2}}|u(s)(y)|^2\leq c\,⨍_{I_r}⨍_{Q_r}|u(s)|^2\,d\lambda^n\,ds \tag{3.3}$$

holds true for the composition $u=\iota\circ v\in L^2(I_r;X_r)\cap C(\overline{I_r};H_r)$.

Proof. 1. Let $\hat{u}\in L^2(I_r;X_r)\cap C(\overline{I_r};H_r)$ be given and set $\varkappa=1+2/n$. Then for all $0<\delta\leq r\leq 1$ Hölder's inequality yields

$$\int_{I_\delta}\int_{Q_\delta}|\hat{u}(s)|^{2\varkappa}\,d\lambda^n\,ds \\ \leq \int_{I_\delta}\Big(\int_{Q_\delta}|\hat{u}(s)|^{2n/(n-2)}\,d\lambda^n\Big)^{(n-2)/n}\Big(\int_{Q_\delta}|\hat{u}(s)|^2\,d\lambda^n\Big)^{\varkappa-1}ds.$$

Due to the Sobolev inequality we find a constant $c_1 = c_1(n) > 0$ such that

$$\Big(\int_{Q_\delta} |w|^{2n/(n-2)}\,d\lambda^n\Big)^{(n-2)/n} \le c_1 \int_{Q_\delta}\Big(\frac{|w|^2}{\delta^2} + \|\nabla w\|^2\Big)d\lambda^n$$

for all $w \in X_\delta = H^1(Q_\delta)$, which yields

$$\begin{aligned}\int_{I_\delta}\int_{Q_\delta} |\hat u(s)|^{2\varkappa}\,d\lambda^n\,ds \\ &\le \frac{c_1}{\delta^2}\Big(\operatorname*{ess\,sup}_{s\in I_\delta}\int_{Q_\delta}|\hat u(s)|^2\,d\lambda^n\Big)^{\varkappa-1}\int_{I_\delta}\int_{Q_\delta}|\hat u(s)|^2\,d\lambda^n\,ds \\ &\quad + c_1\Big(\operatorname*{ess\,sup}_{s\in I_\delta}\int_{Q_\delta}|\hat u(s)|^2\,d\lambda^n\Big)^{\varkappa-1}\int_{I_\delta}\int_{Q_\delta}\|\nabla\hat u(s)\|^2\,d\lambda^n\,ds.\end{aligned}$$

If $\hat u \in L^2(I_r; X_r) \cap C(\overline{I_r}; H_r)$ satisfies the Caccioppoli inequalities

$$\sup_{s\in\overline{I_\delta}}\int_{Q_\delta}|\hat u(s)|^2\,d\lambda^n \le \frac{20}{\varepsilon^2(\varrho-\delta)^2}\int_{I_\varrho}\int_{Q_\varrho}|\hat u(s)|^2\,d\lambda^n\,ds,$$

$$\int_{I_\delta}\int_{Q_\delta}\|\nabla\hat u(s)\|^2\,d\lambda^n\,ds \le \frac{20}{\varepsilon^2(\varrho-\delta)^2}\int_{I_\varrho}\int_{Q_\varrho}|\hat u(s)|^2\,d\lambda^n\,ds$$

for all $\delta, \varrho > 0$ with $\frac{r}{2} \le \delta < \varrho \le r \le 1$, then we obtain

$$\begin{aligned}\int_{I_\delta}\int_{Q_\delta} |\hat u(s)|^{2\varkappa}\,d\lambda^n\,ds \\ &\le \Big(\frac{c_1\varepsilon^2(\varrho-\delta)^2}{20\delta^2} + c_1\Big)\Big(\frac{20}{\varepsilon^2(\varrho-\delta)^2}\int_{I_\varrho}\int_{Q_\varrho}|\hat u(s)|^2\,d\lambda^n\,ds\Big)^{\varkappa}.\end{aligned}$$

Due to $0 < \frac{r}{2} \le \delta < \varrho \le r \le 1$ and $n\varkappa = n+2$ we have

$$4(\varrho-\delta)^2 \le r^2 \le 4\delta^2, \quad \varrho^{(n+2)\varkappa} \le r^{2\varkappa+n+2} \le (2\delta)^{2\varkappa+n+2},$$

and we find some constant $c_2 = c_2(n,\varepsilon) > 0$ such that

$$\frac{1}{\delta^{n+2}}\int_{I_\delta}\int_{Q_\delta}|\hat u(s)|^{2\varkappa}\,d\lambda^n\,ds \le \frac{c_2\delta^{2\varkappa}}{(\varrho-\delta)^{2\varkappa}}\Big(\frac{1}{\varrho^{n+2}}\int_{I_\varrho}\int_{Q_\varrho}|\hat u(s)|^2\,d\lambda^n\,ds\Big)^{\varkappa}.$$

2. In the following we make use of this estimate for shrinking radii:

$$r_k = \frac{r}{2} + \frac{r}{2^{k+1}} \quad \text{for } k \in \mathbb{N}.$$

Obviously, for all $k \in \mathbb{N}$ we have

$$\frac{r}{2} < r_{k+1} < r_k \le r, \quad r_k - r_{k+1} = \frac{r}{2^{k+2}},$$

and hence,

$$\frac{c_2 r_{k+1}^{2\varkappa}}{(r_k - r_{k+1})^{2\varkappa}} \le 4^{(k+2)\varkappa} c_2 \le c_3^{k+1} \quad \text{for all } k \in \mathbb{N},$$

where $c_3 = c_3(n, \varepsilon) > 0$ is some constant. Setting $\delta = r_{k+1}$ and $\varrho = r_k$ for all $k \in \mathbb{N}$ this yields

$$\frac{1}{r_{k+1}^{n+2}} \int_{I_{r_{k+1}}} \int_{Q_{r_{k+1}}} |\hat{u}(s)|^{2\varkappa} \, d\lambda^n \, ds \le c_3^{k+1} \Big(\frac{1}{r_k^{n+2}} \int_{I_{r_k}} \int_{Q_{r_k}} |\hat{u}(s)|^2 \, d\lambda^n \, ds \Big)^{\varkappa}. \tag{3.4}$$

3. We construct a sequence of smooth functions approximating the convex function $\iota_k \in C(\mathbb{R})$ defined by $\iota_k(z) = |z|^{\varkappa^k}$ for $z \in \mathbb{R}$ and $k \in \mathbb{N}$. To do so, for $k, \ell \in \mathbb{N}$ we define nonnegative convex functions $\iota_k^{\oplus}, \iota_k^{\ominus}, \iota_{k\ell}^{\oplus}, \iota_{k\ell}^{\ominus} \in C(\mathbb{R})$ by

$$\iota_k^{\oplus}(z) = \begin{cases} 0 & \text{if } z \le 0, \\ z^{\varkappa^k} & \text{if } 0 \le z, \end{cases} \qquad \iota_{k\ell}^{\oplus}(z) = \begin{cases} \iota_k^{\oplus}(z) & \text{if } z \le \ell, \\ \varkappa^k \ell^{\varkappa^k - 1}(z - \ell) + \ell^{\varkappa^k} & \text{if } \ell \le z, \end{cases}$$

and

$$\iota_k^{\ominus}(z) = \begin{cases} |z|^{\varkappa^k} & \text{if } z \le 0, \\ 0 & \text{if } 0 \le z, \end{cases} \qquad \iota_{k\ell}^{\ominus}(z) = \begin{cases} \varkappa^k \ell^{\varkappa^k - 1}|z + \ell| + \ell^{\varkappa^k} & \text{if } z \le -\ell, \\ \iota_k^{\oplus}(z) & \text{if } -\ell \le z. \end{cases}$$

Let $\varphi \in C_0^{\infty}(\mathbb{R})$ be some nonnegative function which satisfies

$$\operatorname{supp}(\varphi) \subset (-1, 1), \quad \int_{\mathbb{R}} \varphi(z) \, dz = 1, \quad \varphi(-z) = \varphi(z) \quad \text{for all } z \in \mathbb{R}.$$

Moreover, for $\ell \in \mathbb{N}$ we define $\varphi_\ell^{\oplus}, \varphi_\ell^{\ominus} \in C_0^{\infty}(\mathbb{R})$ by

$$\varphi_\ell^{\oplus}(z) = \ell\varphi(\ell z - 1), \quad \varphi_\ell^{\ominus}(z) = \ell\varphi(\ell z + 1) \quad \text{for } z \in \mathbb{R}.$$

For $k, \ell \in \mathbb{N}$ we consider convolutions $\sigma_{k\ell}^{\oplus}, \sigma_{k\ell}^{\ominus} \in C^{\infty}(\mathbb{R})$ given by

$$\sigma_{k\ell}^{\oplus}(z) = \int_{\mathbb{R}} \iota_{k\ell}^{\oplus}(z - s)\varphi_\ell^{\oplus}(s) \, ds, \quad \sigma_{k\ell}^{\ominus}(z) = \int_{\mathbb{R}} \iota_{k\ell}^{\ominus}(z - s)\varphi_\ell^{\ominus}(s) \, ds \quad \text{for } z \in \mathbb{R}.$$

By construction, for $\ell \to \infty$ and fixed $k \in \mathbb{N}$ the sequences $\big(\sigma_{k\ell}^{\oplus}\big)$ and $\big(\sigma_{k\ell}^{\ominus}\big)$ converge monotonically to $\iota_k^{\oplus}$ and $\iota_k^{\ominus}$: For all $k, \ell \in \mathbb{N}$, and $z \in \mathbb{R}$ we have

$$\sigma_{k\ell}^{\oplus}(z) \le \iota_{k\ell}^{\oplus}(z) \le \iota_k^{\oplus}(z), \quad \lim_{\ell \to \infty} \sigma_{k\ell}^{\oplus}(z) = \iota_k^{\oplus}(z),$$
$$\sigma_{k\ell}^{\ominus}(z) \le \iota_{k\ell}^{\ominus}(z) \le \iota_k^{\ominus}(z), \quad \lim_{\ell \to \infty} \sigma_{k\ell}^{\ominus}(z) = \iota_k^{\ominus}(z).$$

Both the nonnegative and convex functions $\iota_{k\ell} = \iota_{k\ell}^{\oplus} + \iota_{k\ell}^{\ominus} \in C(\mathbb{R})$ and $\sigma_{k\ell} = \sigma_{k\ell}^{\oplus} + \sigma_{k\ell}^{\ominus} \in C^\infty(\mathbb{R})$ approximate $\iota_k = \iota_k^{\oplus} + \iota_k^{\ominus} \in C(\mathbb{R})$ for fixed $k \in \mathbb{N}$: For all $k, \ell \in \mathbb{N}$ and $z \in \mathbb{R}$ we have

$$\iota_k^{\varkappa}(z) = \iota_{k+1}(z), \quad \sigma_{k\ell}(z) \le \iota_{k\ell}(z) \le \iota_k(z), \quad \lim_{\ell\to\infty} \sigma_{k\ell}(z) = \iota_k(z),$$

and $\sigma_{k\ell}'' \in C_0^\infty(\mathbb{R})$. Because of $\iota \in C^2(\mathbb{R})$ and the compactness of $\operatorname{supp}(\iota'')$ in $\mathbb{R}$ we get $\sigma_{k\ell} \circ \iota \in C^2(\mathbb{R})$, $(\sigma_{k\ell} \circ \iota)' = (\sigma_{k\ell}' \circ \iota)\, \iota' \in BC(\mathbb{R})$, and

$$(\sigma_{k\ell} \circ \iota)'' = (\sigma_{k\ell}' \circ \iota)\, \iota'' + (\sigma_{k\ell}'' \circ \iota)|\iota'|^2 \in BC(\mathbb{R}) \quad \text{for all } k, \ell \in \mathbb{N}.$$

Due to our assumption ι is nonnegative on $\operatorname{supp}(\iota'')$. Together with the monotonicity of $\sigma_{k\ell}$ on $[0,\infty)$ and the nonnegativity of ι'' and $\sigma_{k\ell}''$ we obtain that $(\sigma_{k\ell} \circ \iota)''$ is nonnegative, too. Hence, for every $k, \ell \in \mathbb{N}$ the nonnegative function $\sigma_{k\ell} \circ \iota \in C^2(\mathbb{R})$ is an admissible composition function in Lemma 3.1; that means the compositions

$$u_{k\ell} = \sigma_{k\ell} \circ \iota \circ v \in L^2(I_r; X_r) \cap C(\overline{I_r}; H_r)$$

satisfy the Caccioppoli inequalities (3.1) and (3.2). Consequently, from (3.4) it follows that for all $k, \ell \in \mathbb{N}$ we have

$$\frac{1}{r_{k+1}^{n+2}} \int_{I_{r_{k+1}}} \int_{Q_{r_{k+1}}} |u_{k\ell}(s)|^{2\varkappa}\, d\lambda^n\, ds \le c_3^{k+1} \Big(\frac{1}{r_k^{n+2}} \int_{I_{r_k}} \int_{Q_{r_k}} |u_{k\ell}(s)|^2\, d\lambda^n\, ds \Big)^{\varkappa}. \quad (3.5)$$

4. To prove that for all $i \in \mathbb{N}$ we obtain higher integrability $|u|^{\varkappa^{i+1}} \in L^2(I_{r_{i+1}}; H_{r_{i+1}})$ together with the estimate

$$\frac{1}{r_{i+1}^{n+2}} \int_{I_{r_{i+1}}} \int_{Q_{r_{i+1}}} |u(s)|^{2\varkappa^{i+1}}\, d\lambda^n\, ds \le c_3^{i+1} \Big(\frac{1}{r_i^{n+2}} \int_{I_{r_i}} \int_{Q_{r_i}} |u(s)|^{2\varkappa^i}\, d\lambda^n\, ds \Big)^{\varkappa}, \quad (3.6)$$

we proceed by induction: Due to the assumptions on $\iota \in C^2(\mathbb{R})$ the composition $u = \iota \circ v \in L^2(I_r; X_r) \cap C(\overline{I_r}; H_r)$ satisfies the Caccioppoli inequalities. Hence, for $i = 0$ the result follows directly from (3.4). Next, we suppose that (3.6) holds true for $i = k-1$. Because of (3.5) and $u_{k\ell} = \sigma_{k\ell} \circ u \le \iota_k \circ u = |u|^{\varkappa^k}$ this yields the estimate

$$\frac{1}{r_{k+1}^{n+2}}\int_{I_{r_{k+1}}}\int_{Q_{r_{k+1}}}|u_{k\ell}(s)|^{2\varkappa}\,d\lambda^n\,ds \\ \leq c_3^{k+1}\Big(\frac{1}{r_k^{n+2}}\int_{I_{r_k}}\int_{Q_{r_k}}|u(s)|^{2\varkappa^k}\,d\lambda^n\,ds\Big)^{\varkappa}.$$

Due to the monotone convergence of $(\sigma_{k\ell})$ to ι_k and $\iota_k^{\varkappa} = \iota_{k+1}$ we apply Fatou's lemma to the left-hand side and pass to the limit $\ell \to \infty$. This proves (3.6) for the case $i = k$.

5. Applying the estimates (3.6) for $i \in \{0, \dots, k-1\}$ recursively, we get

$$\frac{1}{r_k^{n+2}}\int_{I_{r_k}}\int_{Q_{r_k}}|u(s)|^{2\varkappa^k}\,d\lambda^n\,ds \leq c_3^{p_k(\varkappa)}\Big(\frac{1}{r_0^{n+2}}\int_{I_{r_0}}\int_{Q_{r_0}}|u(s)|^2\,d\lambda^n\,ds\Big)^{\varkappa^k}$$

for all $k \in \mathbb{N}$, where we have introduced the polynomial

$$p_k(\varkappa) = \sum_{i=0}^{k-1}(k-i)\varkappa^i \quad \text{for } k \in \mathbb{N}.$$

Because of the property

$$\varkappa^{-k}p_k(\varkappa) = \sum_{i=0}^{k-1}(k-i)\varkappa^{i-k} = \sum_{i=1}^{k} i\varkappa^{-i} \leq \frac{\varkappa}{(\varkappa-1)^2} \quad \text{for all } k \in \mathbb{N},$$

we find some constant $c_4 = c_4(n, \varepsilon) > 0$ such that

$$\Big(\frac{1}{r_k^{n+2}}\int_{I_{r_k}}\int_{Q_{r_k}}|u(s)|^{2\varkappa^k}\,d\lambda^n\,ds\Big)^{\varkappa^{-k}} \leq \frac{c_4}{r^{n+2}}\int_{I_r}\int_{Q_r}|u(s)|^2\,d\lambda^n\,ds.$$

Finally, passing to the limit $k \to \infty$ we end up with

$$\operatorname*{ess\,sup}_{s\in I_{r/2}}\operatorname*{ess\,sup}_{y\in Q_{r/2}}|u(s)(y)|^2 \leq c_5 ⨍_{I_r}⨍_{Q_r}|u(s)|^2\,d\lambda^n\,ds,$$

where $c_5 = c_5(n, \varepsilon) > 0$ is some constant. □

Remark 3.2. Note that the function $\iota \in C^2(\mathbb{R})$, given by $\iota(z) = z$ for $z \in \mathbb{R}$, is an admissible composition function in Theorem 3.2. Hence, the solution v itself is locally bounded and satisfies (3.3).

4. Harnack-type inequalities

To estimate the oscillation of solutions we need not only local boundedness but also Harnack-type inequalities concerning level sets of nonnegative solutions to the homogeneous problem (2.1); see Kruzhkov [19, 20] for the case of constant heat capacity coefficients.

Let $\Omega \subset \mathbb{R}^n$ be open and $w : \Omega \to \mathbb{R}$ be some Lebesgue-measurable function. Then for every value $z \in \mathbb{R}$ we introduce the level set

$$N_z(w, \Omega) = \{y \in \Omega : w(y) \geq z\}.$$

Lemma 4.1 (Measure estimate)**.** *There exist constants $0 < \kappa_1, \kappa_2, \theta < 1$ depending on n and ε, only, such that for all $0 < r \leq 1$ and every nonnegative solution $v \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ of (2.1) which satisfies*

$$⨍_{I_r} \lambda_a^n\big(N_1(v(s), Q_r)\big)\, ds \geq \frac{1}{2}\, \lambda_a^n(Q_r), \tag{4.1}$$

the following pointwise estimate holds true:

$$\lambda_a^n\big(N_\theta(v(\tau), Q_{\kappa_2 r})\big) \geq \frac{1}{4}\, \lambda_a^n(Q_{\kappa_2 r}) \quad \textit{for all } \tau \in I_{\kappa_1 r}. \tag{4.2}$$

Proof. 1. Let $0 < \kappa_1 < \frac{1}{2}$ be some constant. Assume that for each $s \in (t - r^2, t - \kappa_1^2 r^2)$ the inequality

$$\lambda_a^n\big(N_1(v(s), Q_r)\big) < \frac{\frac{1}{2} - \kappa_1^2}{1 - \kappa_1^2}\, \lambda_a^n(Q_r)$$

holds true. Then by integration over I_r we get the relation

$$\begin{aligned}\int_{t-r^2}^{t-\kappa_1^2 r^2} \lambda_a^n\big(N_1(v(s), Q_r\big)\, ds + \int_{t-\kappa_1^2 r^2}^{t} \lambda_a^n\big(N_1(v(s), Q_r\big)\, ds \\ < \int_{t-r^2}^{t-\kappa_1^2 r^2} \frac{\frac{1}{2} - \kappa_1^2}{1 - \kappa_1^2}\, \lambda_a^n(Q_r)\, ds + \kappa_1^2 r^2\, \lambda_a^n(Q_r) = \frac{1}{2}\, r^2 \lambda_a^n(Q_r),\end{aligned}$$

which is a contradiction to (4.1).

Therefore, we have proved that for every constant $0 < \kappa_1 < \frac{1}{2}$ there exists some $\tau_1 \in (t - r^2, t - \kappa_1^2 r^2)$ such that

$$\lambda_a^n\big(N_1(v(\tau_1), Q_r)\big) \geq \frac{\frac{1}{2} - \kappa_1^2}{1 - \kappa_1^2}\, \lambda_a^n(Q_r). \tag{4.3}$$

2. Let $0 < \theta < \frac{1}{2}$ be some constant that will be fixed later. We construct a sequence of smooth functions approximating the nonnegative convex function $\iota \in C(\mathbb{R})$ given by

$$\iota(z) = \begin{cases} -\frac{z}{\theta} - \ln \theta & \text{if } z \leq 0, \\ -\ln(z + \theta) & \text{if } 0 \leq z \leq 1 - \theta, \\ 0 & \text{if } 1 - \theta \leq z. \end{cases}$$

To do so, let $\varphi \in C_0^\infty(\mathbb{R})$ be some nonnegative function which satisfies

$$\operatorname{supp}(\varphi) \subset (-1,1), \quad \int_{\mathbb{R}} \varphi(z)\,dz = 1, \quad \varphi(-z) = \varphi(z) \quad \text{for all } z \in \mathbb{R}.$$

For $k \in \mathbb{N}$ we define $\varphi_k \in C_0^\infty(\mathbb{R})$ by

$$\varphi_k(z) = k\varphi(kz+1) \quad \text{for } z \in \mathbb{R},$$

and we introduce nonnegative convex functions $\iota_k \in C^\infty(\mathbb{R})$ by

$$\iota_k(z) = \int_{\mathbb{R}} \iota(z-s)\varphi_k(s)\,ds \quad \text{for } z \in \mathbb{R},\ k \in \mathbb{N}.$$

By construction, for $k \to \infty$ the sequence (ι_k) converges monotonically to ι. Moreover, for all $k \in \mathbb{N}$ we have $\iota_k'' \in C_0^\infty(\mathbb{R})$ and

$$0 \le \iota_k(z) \le \iota(z) \le \ln \tfrac{1}{\theta} \ \text{ for all } z \ge 0, \quad \iota(z) = \iota_k(z) = 0 \ \text{ for all } z \ge 1.$$

Calculating the derivatives

$$\begin{aligned}
\iota_k'(z) &= -\frac{1}{\theta}\int_z^\infty \varphi_k(s)\,ds - \int_{z-(1-\theta)}^{z} \frac{\varphi_k(s)}{z+\theta-s}\,ds,\\
\iota_k''(z) &= \varphi_k(z-(1-\theta)) + \int_{z-(1-\theta)}^{z} \frac{\varphi_k(s)}{(z+\theta-s)^2}\,ds,
\end{aligned}$$

and using Hölder's inequality, for all $k \in \mathbb{N}$ and $z \ge 0$ we obtain

$$\begin{aligned}
|\iota_k'(z)|^2 &= \left| \int_{z-(1-\theta)}^{z} \frac{\varphi_k(s)}{z+\theta-s}\,ds \right|^2\\
&\le \left(\int_{z-(1-\theta)}^{z} \varphi_k(s)\,ds \right)\left(\int_{z-(1-\theta)}^{z} \frac{\varphi_k(s)}{(z+\theta-s)^2}\,ds \right)\\
&\le \int_{z-(1-\theta)}^{z} \frac{\varphi_k(s)}{(z+\theta-s)^2}\,ds \le \iota_k''(z).
\end{aligned}$$

3. Let $0 < \kappa_1 < \frac{1}{2}$ and $0 < \kappa_2 < 1$ be given constants that will be fixed later. We choose a cut-off function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that for all $y \in \mathbb{R}^n$

$$0 \le \zeta(y) \le 1, \quad \|\nabla\zeta(y)\| \le \frac{2}{(1-\kappa_2)r}, \quad \zeta(y) = \begin{cases} 0 & \text{if } y \in \mathbb{R}^n \setminus Q_r,\\ 1 & \text{if } y \in Q_{\kappa_2 r}. \end{cases}$$

Furthermore, let $\tau_1 \in (t-r^2, t-\kappa_1^2 r^2)$ and $\tau_2 \in I_{\kappa_1 r}$ be fixed. Because $\iota_k \in C^\infty(\mathbb{R})$ and $\iota_k'' \in C_0^\infty(\mathbb{R})$ holds true, for all $k \in \mathbb{N}$ the function

$$w_k = \zeta^2 \cdot \chi_{[\tau_1,\tau_2]} \cdot \iota_k' \circ v \in L^2(I_r; Y_r)$$

is an admissible test function for (2.1). Using the chain rule [12, Lemma B.1, Equation (B.1)], for all $k \in \mathbb{N}$ we get

$$\int_{I_r} \langle (\mathcal{E}_r v)'(s), w_k(s)\rangle_{Y_r}\, ds = \int_{Q_r} \zeta^2 \iota_k(v(\tau_2))\, d\lambda_a^n - \int_{Q_r} \zeta^2 \iota_k(v(\tau_1))\, d\lambda_a^n.$$

4. Additionally, by a straightforward calculation, for all $k \in \mathbb{N}$ we obtain

$$\begin{aligned}\int_{I_r}\int_{Q_r} A(s)\nabla v(s)\cdot\nabla w_k(s)\, d\lambda^n\, ds &= \int_{\tau_1}^{\tau_2}\int_{Q_r} \zeta^2 \iota_k''(v(s)) A(s)\nabla v(s)\cdot\nabla v(s)\, d\lambda^n\, ds \\ &\quad + 2\int_{\tau_1}^{\tau_2}\int_{Q_r} \zeta A(s)\nabla\zeta\cdot\nabla(\iota_k\circ v)(s)\, d\lambda^n\, ds.\end{aligned}$$

Applying the relation $\iota_k'' \ge |\iota_k'|^2$ on $[0,\infty)$ and the positive definiteness of A, for all $k \in \mathbb{N}$ we get

$$\begin{aligned}\int_{I_r}\int_{Q_r} A(s)\nabla v(s)\cdot\nabla w_k(s)\, d\lambda^n\, ds &\ge \int_{\tau_1}^{\tau_2}\int_{Q_r} \zeta^2 A(s)\nabla(\iota_k\circ v)(s)\cdot\nabla(\iota_k\circ v)(s)\, d\lambda^n\, ds \\ &\quad + 2\int_{\tau_1}^{\tau_2}\int_{Q_r} \zeta A(s)\nabla\zeta\cdot\nabla(\iota_k\circ v)(s)\, d\lambda^n\, ds.\end{aligned}$$

Hence, Young's inequality yields some constant $c_1 = c_1(\varepsilon, n) > 0$ such that

$$\begin{aligned}\int_{I_r}\int_{Q_r} A(s)\nabla v(s)\cdot\nabla w_k(s)\, d\lambda^n\, ds &\ge \frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\int_{Q_r} \zeta^2\|\nabla(\iota_k\circ v)(s)\|^2\, d\lambda^n\, ds - c_1\int_{\tau_1}^{\tau_2}\int_{Q_r}\|\nabla\zeta\|^2\, d\lambda^n\, ds.\end{aligned}$$

5. Summing up the results of the preceeding steps and using the properties of the cut-off functions we find some constant $c_2 = c_2(\varepsilon, n) > 0$ such that for all $k \in \mathbb{N}$ we have

$$\begin{aligned}\int_{Q_r} \zeta^2 \iota_k(v(\tau_2))\, d\lambda_a^n &+ \frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\int_{Q_r} \zeta^2\|\nabla(\iota_k\circ v)(s)\|^2\, d\lambda^n\, ds \\ &\le \int_{Q_r} \zeta^2 \iota_k(v(\tau_1))\, d\lambda_a^n + \frac{c_2}{\kappa_2^n(1-\kappa_2)^2}\,\lambda_a^n(Q_{\kappa_2 r}).\end{aligned} \tag{4.4}$$

Neglecting the second integral term on the left-hand side, we pass to the limit $k \to \infty$ in the two remaining integrals: The monotone convergence of (ι_k) to ι on $[0,\infty)$ yields

$$\lim_{k\to\infty} \int_{Q_r} \zeta^2 \iota_k(v(\tau_2))\, d\lambda_a^n = \int_{Q_r} \zeta^2 \iota(v(\tau_2))\, d\lambda_a^n \geq \int_{Q_{\kappa_2 r}\setminus N_\theta(v(\tau_2),Q_{\kappa_2 r})} \iota(v(\tau_2))\, d\lambda_a^n.$$

Because $v(\tau_2) + \theta \leq 2\theta < 1$ and, hence, $\iota(v(\tau_2)) \geq \ln \frac{1}{2\theta} > 0$ holds true λ_a^n-almost everywhere on $Q_{\kappa_2 r} \setminus N_\theta(v(\tau_2), Q_{\kappa_2 r})$, it follows that

$$\lim_{k\to\infty} \int_{Q_r} \zeta^2 \iota_k(v(\tau_2))\, d\lambda_a^n \geq \lambda_a^n\big(Q_{\kappa_2 r} \setminus N_\theta(v(\tau_2), Q_{\kappa_2 r})\big) \ln \frac{1}{2\theta}. \tag{4.5}$$

Using the same argument as above, we get

$$\lim_{k\to\infty} \int_{Q_r} \zeta^2 \iota_k(v(\tau_1))\, d\lambda_a^n = \int_{Q_r} \zeta^2 \iota(v(\tau_1))\, d\lambda_a^n \leq \int_{Q_r\setminus N_1(v(\tau_1),Q_r)} \iota(v(\tau_1))\, d\lambda_a^n.$$

Note, that λ_a^n-almost everywhere on Q_r we have $\iota(v(\tau_1)) \leq \ln \frac{1}{\theta}$. This yields

$$\lim_{k\to\infty} \int_{Q_r} \zeta^2 \iota_k(v(\tau_1))\, d\lambda_a^n \leq \big(\lambda_a^n(Q_r) - \lambda_a^n\big(N_1(v(\tau_1), Q_r)\big)\big) \ln \frac{1}{\theta}. \tag{4.6}$$

Passing to the limit $k \to \infty$ in (4.4) we use (4.5) und (4.6) to get

$$\lambda_a^n\big(Q_{\kappa_2 r} \setminus N_\theta(v(\tau_2), Q_{\kappa_2 r})\big) \ln \frac{1}{2\theta} \leq \big(\lambda_a^n(Q_r) - \lambda_a^n\big(N_1(v(\tau_1), Q_r)\big)\big) \ln \frac{1}{\theta} + \frac{c_2}{\kappa_2^n(1-\kappa_2)^2}\, \lambda_a^n(Q_{\kappa_2 r}).$$

In view of (4.3), for every $0 < \kappa_1 < \frac{1}{2}$ there exists some $\tau_1 \in (t - r^2, t - \kappa_1^2 r^2)$ such that

$$\lambda_a^n\big(Q_{\kappa_2 r} \setminus N_\theta(v(\tau_2), Q_{\kappa_2 r})\big) \ln \frac{1}{2\theta} \leq \frac{1}{2(1-\kappa_1^2)}\, \lambda_a^n(Q_r)\, \ln \frac{1}{\theta} + \frac{c_2}{\kappa_2^n(1-\kappa_2)^2}\, \lambda_a^n(Q_{\kappa_2 r}).$$

Due to $\varepsilon \leq \operatorname{ess\,inf}_{y\in Q_r} a(y)$ and $\operatorname{ess\,sup}_{y\in Q_r} a(y) \leq 1/\varepsilon$ we obtain

$$\lambda_a^n(Q_r) \leq \Big(1 + \frac{1-\kappa_2^n}{\varepsilon^2 \kappa_2^n}\Big) \lambda_a^n(Q_{\kappa_2 r}),$$

which yields

$$\lambda_a^n\big(Q_{\kappa_2 r}\setminus N_\theta(v(\tau_2),Q_{\kappa_2 r})\big)\le \frac{c_2}{\kappa_2^n(1-\kappa_2)^2\ln\frac{1}{2\theta}}\,\lambda_a^n(Q_{\kappa_2 r}) + \frac{1}{1-\kappa_1^2}\Big(1+\frac{1-\kappa_2^n}{\varepsilon^2\kappa_2^n}\Big)\Big(\frac{1}{2}+\frac{\ln 2}{2\ln\frac{1}{2\theta}}\Big)\lambda_a^n(Q_{\kappa_2 r}).$$

Here, we fix constants $0<\kappa_1<\frac{1}{2}$ and $0<\kappa_2<1$ such that

$$\frac{1}{1-\kappa_1^2}\Big(1+\frac{1-\kappa_2^n}{\varepsilon^2\kappa_2^n}\Big)\le\frac{9}{8}.$$

After that, we choose $0<\theta<\frac{1}{2}$ such that both

$$\frac{\ln 2}{2\ln\frac{1}{2\theta}}\le\frac{1}{18}\quad\text{and}\quad\frac{c_2}{\kappa_2^n(1-\kappa_2)^2\ln\frac{1}{2\theta}}\le\frac{1}{8}.$$

Indeed, we have found three constants $0<\kappa_1,\kappa_2,\theta<1$ depending on ε and n, only, such that for all $\tau_2\in I_{\kappa_1 r}$ the estimate

$$\lambda_a^n\big(Q_{\kappa_2 r}\setminus N_\theta(v(\tau_2),Q_{\kappa_2 r})\big)\le\frac{3}{4}\,\lambda_a^n(Q_{\kappa_2 r})$$

holds true, which proves the desired result. □

Theorem 4.2 (Harnack-type inequality)**.** *We find constants* $0<\gamma<\frac{1}{2}$ *and* $0<\kappa<\frac{1}{2}$ *depending on* n *and* ε*, only, such that for all* $0<r\le 1$ *and every nonnegative solution* $v\in W_{E_r}(I_r;X_r)\cap C(\overline{I_r};H_r)$ *of* (2.1) *satisfying*

$$\fint_{I_r}\lambda_a^n\big(N_1(v(s),Q_r)\big)\,ds\ge\frac{1}{2}\,\lambda_a^n(Q_r),$$

the following estimate holds true:

$$\operatorname*{ess\,inf}_{s\in I_{\kappa r}}\operatorname*{ess\,inf}_{y\in Q_{\kappa r}} v(s)(y)\ge\gamma. \tag{4.7}$$

Proof. 1. In view Lemma 4.1 and estimate (4.2) we find $0<\kappa_1,\kappa_2,\theta<1$ depending on ε and n, only, such that

$$\lambda^n\big(N_\theta(v(\tau),Q_{\kappa_2 r})\big)\ge\frac{1}{4}\,\varepsilon^2\lambda^n(Q_{\kappa_2 r})\quad\text{for all }\tau\in I_{\kappa_1 r}. \tag{4.8}$$

2. Let $\gamma>0$ be some constant with $\gamma^2<\frac{\theta}{2}$ which will be fixed later. We take a sequence of smooth functions approximating the nonnegative convex

function $\iota \in C(\mathbb{R})$ defined as

$$\iota(z) = \begin{cases} -\frac{z}{\gamma^2} - \ln\frac{\gamma^2}{\theta} & \text{if } z \le 0, \\ -\ln\frac{z+\gamma^2}{\theta} & \text{if } 0 \le z \le \theta - \gamma^2, \\ 0 & \text{if } \theta - \gamma^2 \le z. \end{cases}$$

To that end, let $\varphi \in C_0^\infty(\mathbb{R})$ be some nonnegative function which satisfies

$$\operatorname{supp}(\varphi) \subset (-1,1), \quad \int_{\mathbb{R}} \varphi(z)\,dz = 1, \quad \varphi(-z) = \varphi(z) \quad \text{for all } z \in \mathbb{R}.$$

For $k \in \mathbb{N}$ we define $\varphi_k \in C_0^\infty(\mathbb{R})$ by

$$\varphi_k(z) = k\varphi(kz+1) \quad \text{for } z \in \mathbb{R},$$

and we construct nonnegative convex functions $\iota_k \in C^\infty(\mathbb{R})$ by

$$\iota_k(z) = \int_{\mathbb{R}} \iota(z-s)\varphi_k(s)\,ds \quad \text{for } z \in \mathbb{R},\ k \in \mathbb{N}.$$

By construction, for $k \to \infty$ the sequence (ι_k) converges monotonically to ι. Furthermore, for all $k \in \mathbb{N}$ we have $\iota_k'' \in C_0^\infty(\mathbb{R})$ and

$$0 \le \iota_k(z) \le \iota(z) \le \ln\tfrac{\theta}{\gamma^2} \ \text{ for all } z \ge 0, \quad \iota(z) = \iota_k(z) = 0 \ \text{ for all } z \ge \theta.$$

Using the same arguments as in Step 2 of the proof of Lemma 4.1 we get $|\iota_k'(z)|^2 \le \iota_k''(z)$ for all $k \in \mathbb{N}$ and $z \ge 0$.

3. We choose some cut-off function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that for all $y \in \mathbb{R}^n$

$$0 \le \zeta(y) \le 1, \quad \|\nabla\zeta(y)\| \le \frac{2}{(1-\kappa_2)r}, \quad \zeta(y) = \begin{cases} 0 & \text{if } y \in \mathbb{R}^n \setminus Q_r, \\ 1 & \text{if } y \in Q_{\kappa_2 r}. \end{cases}$$

Moreover, let $\tau_1 = t - \kappa_1^2 r^2$ and $\tau_2 \in I_{\kappa_1 r}$ be fixed. Since $\iota_k \in C^\infty(\mathbb{R})$ and $\iota_k'' \in C_0^\infty(\mathbb{R})$ hold true, for all $k \in \mathbb{N}$ the function

$$w_k = \zeta^2 \cdot \chi_{[\tau_1,\tau_2]} \cdot \iota_k' \circ v \in L^2(I_r; Y_r)$$

is an admissible test function for (2.1). Following exactly the same arguments as in Steps 3 and 4 of the proof of Lemma 4.1, we get an estimate analogous to (4.4): We obtain

$$\begin{aligned} \int_{Q_r} \zeta^2 \iota_k(v(\tau_2))\,d\lambda_a^n + \frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\int_{Q_r} \zeta^2 \|\nabla(\iota_k \circ v)(s)\|^2\,d\lambda^n\,ds \\ \le \int_{Q_r} \zeta^2 \iota_k(v(\tau_1))\,d\lambda_a^n + \frac{c_1}{\kappa_2^n(1-\kappa_2)^2}\,\lambda_a^n(Q_{\kappa_2 r}) \end{aligned} \tag{4.9}$$

for some constant $c_1 = c_1(n,\varepsilon) > 0$.

Due to the fact that $\iota_k(z) \le \ln \frac{\theta}{\gamma^2}$ holds true for all $z \ge 0$ and $k \in \mathbb{N}$, we estimate the first term of the right-hand side by

$$\int_{Q_r} \zeta^2 \iota_k(v(\tau_1))\, d\lambda_a^n \le \lambda_a^n(Q_r)\, \ln \frac{\theta}{\gamma^2}.$$

Neglecting the first term on the left-hand side of (4.9), this yields

$$\int_{I_{\kappa_1 r}} \int_{Q_{\kappa_2 r}} \|\nabla(\iota_k \circ v)(s)\|^2 \, d\lambda^n \, ds \le c_2 r^n \ln \frac{3\theta}{\gamma^2} \tag{4.10}$$

for all $k \in \mathbb{N}$, where $c_2 = c_2(n, \varepsilon) > 0$ is some constant.

In view of (4.8) we apply a weighted version of the Poincaré inequality [12, Lemma A.2, Equation (A.1)] to find a constant $c_3 = c_3(\varepsilon, n) > 0$ such that

$$\begin{aligned} \int_{I_{\kappa_1 r}} \int_{Q_{\kappa_2 r}} \left| \iota_k(v(s)) - \fint_{N_\theta(v(s), Q_{\kappa_2 r})} \iota_k(v(s))\, d\lambda^n \right|^2 d\lambda^n \, ds \\ \le c_3 (\kappa_2 r)^2 \int_{I_{\kappa_1 r}} \int_{Q_{\kappa_2 r}} \|\nabla(\iota_k \circ v)(s)\|^2 \, d\lambda^n \, ds. \end{aligned}$$

Using the fact that for all $s \in I_{\kappa_1 r}$ we have $v(s) \ge \theta$ and, hence, $\iota_k(v(s)) = 0$ λ^n-almost everywhere on $N_\theta(v(s), Q_{\kappa_2 r})$, the mean value in the integrand of the left-hand side vanishes. Remembering (4.10) this yields some constant $c_4 = c_4(n, \varepsilon) > 0$ such that

$$\int_{I_{\kappa_1 r}} \int_{Q_{\kappa_2 r}} |\iota_k(v(s))|^2 \, d\lambda^n \, ds \le c_4 r^{n+2} \ln \frac{3\theta}{\gamma^2}. \tag{4.11}$$

4. For every $k \in \mathbb{N}$ the nonnegative convex function $\iota_k \in C^\infty(\mathbb{R})$ satisfies $\iota_k'' \in C_0^\infty(\mathbb{R})$. Due to Theorem 3.2 we find a constant $c_5 = c_5(n, \varepsilon) > 0$ such that for $\kappa = \frac{1}{2} \min\{\kappa_1, \kappa_2\}$ and all $k \in \mathbb{N}$ we obtain the estimate

$$\operatorname*{ess\,sup}_{s \in I_{\kappa r}} \operatorname*{ess\,sup}_{y \in Q_{\kappa r}} |\iota_k(v(s)(y))|^2 \le c_5 \fint_{I_{2\kappa r}} \fint_{Q_{2\kappa r}} |\iota_k(v(s))|^2 \, d\lambda^n \, ds.$$

Hence, applying (4.11) and using the monotone convergence of (ι_k) to ι on $[0, \infty)$, we arrive at

$$\operatorname*{ess\,sup}_{s \in I_{\kappa r}} \operatorname*{ess\,sup}_{y \in Q_{\kappa r}} |\iota(v(s)(y))|^2 \le c_6 \ln \frac{3\theta}{\gamma^2}, \tag{4.12}$$

where $c_6 = c_6(n, \varepsilon) > 0$ is some constant.

In view of the properties of logarithmic and quadratic functions we fix some constant $\gamma > 0$ depending on n and ε, only, such that

$$\gamma^2 < \min\left\{\tfrac{\theta}{2}, \theta^2\right\}, \quad c_6\left(\ln 3\theta - \ln\gamma^2\right) < (\ln\theta - \ln\gamma)^2.$$

Using (4.12) for all $s \in I_{\kappa r}$ this yields

$$\left(\ln\frac{\theta}{v(s)+\gamma^2}\right)^2 \le c_6 \ln\frac{3\theta}{\gamma^2} \le \left(\ln\frac{\theta}{\gamma}\right)^2$$

λ^n-almost everywhere on $Q_{\kappa r} \setminus N_{\theta-\gamma^2}(v(s), Q_{\kappa r})$. Therefore, for all $s \in I_{\kappa r}$ we obtain $v(s) \ge \gamma - \gamma^2 > 0$ λ^n-almost everywhere on $Q_{\kappa r} \setminus N_{\theta-\gamma^2}(v(s), Q_{\kappa r})$. Note that by definition for all $s \in I_{\kappa r}$ we get $v(s) \ge \theta - \gamma^2 > 0$ λ^n-almost everywhere on $N_{\theta-\gamma^2}(v(s), Q_{\kappa r})$.

Finally, by setting $\gamma^* = \min\left\{\theta - \gamma^2, \gamma - \gamma^2\right\}$ we have obtained two constants $0 < \gamma^*, \kappa < \frac{1}{2}$ depending on n and ε, only, such that the desired estimate

$$\operatorname*{ess\,inf}_{s\in I_{\kappa r}} \operatorname*{ess\,inf}_{y\in Q_{\kappa r}} v(s)(y) \ge \gamma^*$$

holds true. □

5. Campanato inequalities

Using both local boundedness and the Harnack-type inequality we prove the De Giorgi–Moser–Nash inequality to estimate the oscillation of solutions. The proofs use ideas of Troianiello [28] and Hong-Ming Yin [30].

Theorem 5.1 (De Giorgi–Moser–Nash inequality)**.** *There exist constants $0 < \nu < 1$ and $c > 0$ depending on n and ε, only, such that for all $0 < \delta \le r \le 1$ and every solution $v \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ of (2.1) the following estimate holds true:*

$$\operatorname*{ess\,sup}_{s,\hat{s}\in I_{\delta/2}} \operatorname*{ess\,sup}_{y,\hat{y}\in Q_{\delta/2}} |v(s)(y) - v(\hat{s})(\hat{y})|^2 \le c\left(\frac{\delta}{r}\right)^{2\nu} \fint_{I_r} \fint_{Q_r} |v(s)|^2 \, d\lambda^n \, ds. \quad (5.1)$$

Proof. 1. Let $0 < \varrho \le \frac{r}{2}$ be given and consider an essentially bounded function $v \in W_{E_\varrho}(I_\varrho; X_\varrho) \cap C(\overline{I_\varrho}; H_\varrho)$ which satisfies

$$\int_{I_\varrho} \langle (\mathcal{E}_\varrho v)'(s), w(s)\rangle_{Y_\varrho} \, ds + \int_{I_\varrho}\int_{Q_\varrho} A(s)\nabla v(s)\cdot\nabla w(s) \, d\lambda^n \, ds = 0 \quad (5.2)$$

for all $w \in L^2(I_\varrho; Y_\varrho)$. We define the bounds $m_*, m^* \in \mathbb{R}$ by

$$m_* = \operatorname*{ess\,inf}_{s\in I_\varrho} \operatorname*{ess\,inf}_{y\in Q_\varrho} v(s)(y) \le \operatorname*{ess\,sup}_{s\in I_\varrho} \operatorname*{ess\,sup}_{y\in Q_\varrho} v(s)(y) = m^*. \quad (5.3)$$

In the following step we prove that there exist constants $0 < \gamma, \kappa < \frac{1}{2}$ depending on n, ε, only, and $M_*, M^* \in \mathbb{R}$ such that both

$$M_* \leq \operatorname*{ess\,inf}_{s \in I_{\kappa\varrho}} \operatorname*{ess\,inf}_{y \in Q_{\kappa\varrho}} v(s)(y) \leq \operatorname*{ess\,sup}_{s \in I_{\kappa\varrho}} \operatorname*{ess\,sup}_{y \in Q_{\kappa\varrho}} v(s)(y) \leq M^* \tag{5.4}$$

and

$$M^* - M_* \leq (1-\gamma)(m^* - m_*) \tag{5.5}$$

holds true:

2. In the case $m_* = m^*$ the statement is obviously true. Hence, assume that $m_* < m^*$ and let $z_* \in [m_*, m^*]$ be the supremum of all $z \in [m_*, m^*]$ which satisfy

$$\fint_{I_\varrho} \lambda_a^n\big(\{y \in Q_\varrho : v(s)(y) < z\}\big)\, ds \leq \frac{1}{2}\, \lambda_a^n(Q_\varrho).$$

Introducing the level sets

$$F_k(s) = \big\{y \in Q_\varrho : v(s)(y) \leq z_* - \tfrac{1}{k}\big\},$$
$$F(s) = \big\{y \in Q_\varrho : v(s)(y) < z_*\big\},$$

for all $s \in \overline{I_\varrho}$ and $k \in \mathbb{N}$ we get $F_k(s) \subset F_{k+1}(s)$ and $\cup_{k=1}^\infty F_k(s) = F(s)$, which yields

$$\fint_{I_\varrho} \lambda_a^n(F(s))\, ds = \lim_{k\to\infty} \fint_{I_\varrho} \lambda_a^n(F_k(s))\, ds \leq \frac{1}{2}\, \lambda_a^n(Q_\varrho).$$

In other words, we have

$$\fint_{I_\varrho} \lambda_a^n\big(\{y \in Q_\varrho : v(s)(y) < z_*\}\big)\, ds \leq \frac{1}{2}\, \lambda_a^n(Q_\varrho). \tag{5.6}$$

Analogously, introducing the level sets

$$G_k(s) = \big\{y \in Q_\varrho : v(s)(y) < z_* + \tfrac{1}{k}\big\},$$
$$G(s) = \big\{y \in Q_\varrho : v(s)(y) \leq z_*\big\},$$

for all $s \in \overline{I_\varrho}$ and $k \in \mathbb{N}$ we get $G_{k+1}(s) \subset G_k(s)$ and $\cap_{k=1}^\infty G_k(s) = G(s)$, which yields

$$\fint_{I_\varrho} \lambda_a^n(G(s))\, ds = \lim_{k\to\infty} \fint_{I_\varrho} \lambda_a^n(G_k(s))\, ds \geq \frac{1}{2}\, \lambda_a^n(Q_\varrho).$$

Hence, we also get

$$\fint_{I_\varrho} \lambda_a^n\big(\{y \in Q_\varrho : v(s)(y) > z_*\}\big)\, ds \leq \frac{1}{2}\, \lambda_a^n(Q_\varrho). \tag{5.7}$$

2.1. In the case $m_* < z_*$ the nonnegative function

$$v_* = \frac{v - m_*}{z_* - m_*} \in W_{E_\varrho}(I_\varrho; X_\varrho) \cap C(\overline{I_\varrho}; H_\varrho)$$

solves (5.2) as well as v. By construction, from (5.6) we get the estimate

$$\fint_{I_\varrho} \lambda_a^n\big(\{y \in Q_\varrho : v_*(s)(y) \geq 1\}\big)\, ds \geq \frac{1}{2}\, \lambda_a^n(Q_\varrho).$$

Applying Theorem 4.2 there exist two constants $0 < \gamma, \kappa < \frac{1}{2}$ depending on n and ε, only, such that the Harnack-type inequality (4.7)

$$\operatorname*{ess\,inf}_{s \in I_{\kappa\varrho}} \operatorname*{ess\,inf}_{y \in Q_{\kappa\varrho}} v_*(s)(y) \geq \gamma$$

holds true. Hence, setting

$$M_* = m_* + \gamma(z_* - m_*) = z_* - (1-\gamma)(z_* - m_*),$$

we get

$$M_* \leq \operatorname*{ess\,inf}_{s \in I_{\kappa\varrho}} \operatorname*{ess\,inf}_{y \in Q_{\kappa\varrho}} v(s)(y),$$

which remains true in the case $z_* = m_*$ due to (5.3).

2.2. Analogously to Step 2.1, in the case $z_* < m^*$ the nonnegative function

$$v^* = \frac{m^* - v}{m^* - z_*} \in W_{E_\varrho}(I_\varrho; X_\varrho) \cap C(\overline{I_\varrho}; H_\varrho)$$

solves (5.2), too. From (5.7) we obtain

$$\fint_{I_\varrho} \lambda_a^n\big(\{y \in Q_\varrho : v^*(s)(y) \geq 1\}\big)\, ds \geq \frac{1}{2}\, \lambda_a^n(Q_\varrho),$$

and Theorem 4.2 yields

$$\operatorname*{ess\,inf}_{s \in I_{\kappa\varrho}} \operatorname*{ess\,inf}_{y \in Q_{\kappa\varrho}} v^*(s)(y) \geq \gamma,$$

where the constants $0 < \gamma, \kappa < \frac{1}{2}$ are the same as in Step 2.1. Therefore, setting

$$M^* = m^* - \gamma(m^* - z_*) = z_* + (1-\gamma)(m^* - z_*),$$

we get

$$\operatorname*{ess\,sup}_{s \in I_{\kappa\varrho}} \operatorname*{ess\,sup}_{y \in Q_{\kappa\varrho}} v(s)(y) \leq M^*,$$

which remains true in the case $z_* = m^*$ because of (5.3). Summing up the results of Steps 2.1 and 2.2 we have shown both (5.4) and (5.5).

3. For $0 < \varrho \le \frac{r}{2}$ we define the oscillation of v with respect to I_ϱ and Q_ϱ by

$$o(\varrho) = \operatorname*{ess\,sup}_{s,\hat{s}\in I_\varrho} \operatorname*{ess\,sup}_{y,\hat{y}\in Q_\varrho} |v(s)(y) - v(\hat{s})(\hat{y})|.$$

A recursive application of (5.3), (5.4), and (5.5) (see Step 1) to shrinking radii $\varrho = \frac{1}{2}\kappa^i r$ yields

$$o\big(\tfrac{1}{2}\kappa^i r\big) \le (1-\gamma)^i\, o\big(\tfrac{r}{2}\big) \quad \text{for all } i \in \mathbb{N}.$$

For every pair of radii $0 < \delta \le r$ we choose $i \in \mathbb{N}$ such that $\kappa^{i+1} r < \delta \le \kappa^i r$. In the case $o\big(\frac{\delta}{2}\big) > 0$ we obtain

$$\ln o\big(\tfrac{\delta}{2}\big) - \ln o\big(\tfrac{r}{2}\big) \le \ln \frac{1}{1-\gamma} + (i+1)\ln(1-\gamma) \le \ln\frac{1}{1-\gamma} + \frac{\ln(1-\gamma)}{\ln\kappa}\ln\frac{\delta}{r}.$$

Setting $\nu = \frac{\ln(1-\gamma)}{\ln\kappa} \in (0,1)$, we get

$$o\big(\tfrac{\delta}{2}\big) \le \frac{o\big(\frac{r}{2}\big)}{1-\gamma}\Big(\frac{\delta}{r}\Big)^\nu,$$

which holds true also in the trivial case $o\big(\frac{\delta}{2}\big) = 0$. Hence, due to Remark 3.2 concerning the local boundedness of v, for all $0 < \delta \le r \le 1$ we end up with

$$\operatorname*{ess\,sup}_{s,\hat{s}\in I_{\delta/2}} \operatorname*{ess\,sup}_{y,\hat{y}\in Q_{\delta/2}} |v(s)(y) - v(\hat{s})(\hat{y})|^2 \le c\Big(\frac{\delta}{r}\Big)^{2\nu} \fint_{I_r}\fint_{Q_r} |v(s)|^2\, d\lambda^n\, ds,$$

where $c = c(n,\varepsilon) > 0$ is some constant. □

Campanato inequalities. Due to the De Giorgi–Moser–Nash inequality we get the Campanato inequality for the spatial gradients of solutions to the homogeneous problem (2.1).

Lemma 5.2 (Campanato inequality)**.** *There exist constants $c > 0$ and $\bar{\omega} \in (n, n+2)$ depending on n and ε, only, such that for all $0 < \delta \le r \le 1$ and every solution $v \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ of (2.1) we have*

$$\int_{I_\delta}\int_{Q_\delta} \|\nabla v(s)\|^2\, d\lambda^n\, ds \le c\Big(\frac{\delta}{r}\Big)^{\bar{\omega}} \int_{I_r}\int_{Q_r} \|\nabla v(s)\|^2\, d\lambda^n\, ds.$$

Proof. 1. First, we consider the case $0 < \delta \le \frac{r}{4}$. Setting

$$\bar{v} = \fint_{I_{2\delta}}\fint_{Q_{2\delta}} v(s)\, d\lambda^n\, ds,$$

the difference $v - \bar{v} \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ satisfies (2.1) as well as v. In view of the Caccioppoli inequality (3.2) and the local boundedness (see Remarks 3.1 and 3.2) this leads to the estimate

$$\begin{aligned}\int_{I_\delta}\int_{Q_\delta} \|\nabla v(s)\|^2 \, d\lambda^n \, ds &\leq \frac{20}{\varepsilon^2 \delta^2} \int_{I_{2\delta}}\int_{Q_{2\delta}} |v(s) - \bar{v}|^2 \, d\lambda^n \, ds \\ &\leq c_1 \delta^n \operatorname*{ess\,sup}_{s \in I_{2\delta}} \operatorname*{ess\,sup}_{y \in Q_{2\delta}} |v(s)(y) - \bar{v}|^2,\end{aligned}$$

where $c_1 = c_1(n, \varepsilon) > 0$ is some constant. Due to the relation

$$\operatorname*{ess\,inf}_{s \in I_{2\delta}} \operatorname*{ess\,inf}_{y \in Q_{2\delta}} v(s)(y) \leq \overline{v} \leq \operatorname*{ess\,sup}_{s \in I_{2\delta}} \operatorname*{ess\,sup}_{y \in Q_{2\delta}} v(s)(y),$$

this yields

$$\int_{I_\delta}\int_{Q_\delta} \|\nabla v(s)\|^2 \, d\lambda^n \, ds \leq c_1 \delta^n \operatorname*{ess\,sup}_{s, \hat{s} \in I_{2\delta}} \operatorname*{ess\,sup}_{y, \hat{y} \in Q_{2\delta}} |v(s)(y) - v(\hat{s})(\hat{y})|^2. \tag{5.8}$$

2. Introducing the mean value

$$\hat{v} = ⨍_{I_r} ⨍_{Q_r} v(s) \, d\lambda^n \, ds,$$

again we make use of the fact, that $v - \hat{v}$ satisfies (2.1) as well as v. We apply the De Giorgi–Moser–Nash inequality (5.1) to the function

$$v - \hat{v} \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$$

to estimate its oscillation: We find two constants $c_2 > 0$ and $0 < \nu < 1$ depending on n and ε, only, such that for all $0 < \delta \leq \frac{r}{4}$

$$\operatorname*{ess\,sup}_{s, \hat{s} \in I_{2\delta}} \operatorname*{ess\,sup}_{y, \hat{y} \in Q_{2\delta}} |v(s)(y) - v(\hat{s})(\hat{y})|^2 \leq c_2 \Big(\frac{\delta}{r}\Big)^{2\nu} ⨍_{I_r} ⨍_{Q_r} |v(s) - \hat{v}|^2 \, d\lambda^n \, ds.$$

Together with (5.8) for $0 < \delta \leq \frac{r}{4}$ we obtain

$$\int_{I_\delta}\int_{Q_\delta} \|\nabla v(s)\|^2 \, d\lambda^n \, ds \leq \frac{c_3}{r^2} \Big(\frac{\delta}{r}\Big)^{\bar{\omega}} \int_{I_r}\int_{Q_r} |v(s) - \hat{v}|^2 \, d\lambda^n \, ds,$$

where $\bar{\omega} = n + 2\nu \in (n, n+2)$ and $c_3 = c_3(n, \varepsilon) > 0$ are constants. Hence, using the Poincaré inequality, see [12, Theorem A.3], we find some constant $c_4 = c_4(\varepsilon, n) > 0$ such that

$$\begin{aligned}\int_{I_\delta}\int_{Q_\delta} \|\nabla v(s)\|^2 \, d\lambda^n \, ds &\leq c_4 \Big(\frac{\delta}{r}\Big)^{\bar{\omega}} \int_{I_r}\int_{Q_r} \|\nabla v(s)\|^2 \, d\lambda^n \, ds \\ &\quad + c_4 \Big(\frac{\delta}{r}\Big)^{\bar{\omega}} \int_{I_r} \|(\mathcal{E}_r v)'(s)\|^2_{H^{-1}(Q_r)} \, ds.\end{aligned}$$

Since $v \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ satisfies the variational equation (2.1), for all $0 < \delta \le \frac{r}{4}$ we arrive at the sought-for estimate

$$\int_{I_\delta} \int_{Q_\delta} \|\nabla v(s)\|^2 \, d\lambda^n \, ds \le c_5 \Big(\frac{\delta}{r}\Big)^{\bar{\omega}} \int_{I_r} \int_{Q_r} \|\nabla v(s)\|^2 \, d\lambda^n \, ds,$$

where $c_5 = c_5(\varepsilon, n) > 0$ is some constant. Obviously, a relation of this type holds true in the case $\frac{r}{4} \le \delta \le r$, too. □

We conclude our local regularity theory with the Campanato inequality for the spatial gradients of solutions to the inhomogeneous problem (2.2). This estimate serves as the starting point of our global regularity theory for second-order parabolic boundary-value problems in Lipschitz domains with nonsmooth coefficients and mixed boundary conditions in Sobolev–Morrey spaces.

Theorem 5.3 (Campanato inequality)**.** *We find constants $n < \bar{\omega} < n+2$ and $c > 0$ depending on n and ε, only, such that for all $0 < \delta \le r \le 1$, every functional $f \in L^2(I_r; Y_r^*)$, and every solution $u \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ of the variational equation* (2.2) *we have*

$$\int_{I_\delta} \int_{Q_\delta} \|\nabla u(s)\|^2 \, d\lambda^n \, ds \\ \le c \Big(\frac{\delta}{r}\Big)^{\bar{\omega}} \int_{I_r} \int_{Q_r} \|\nabla u(s)\|^2 \, d\lambda^n \, ds + c \int_{I_r} \|f(s)\|^2_{Y_r^*} \, ds. \quad (5.9)$$

Proof. 1. Let $u_0 \in W_{E_r|Y_r}(I_r; Y_r)$ be the function which solves (2.2) and satisfies $u_0(t - r^2) = 0$ (see Theorem 1.1). Using $w = u_0$ as a test function and having in mind the Sobolev–Friedrichs inequality

$$\int_{Q_r} |u_0(s)|^2 \, d\lambda^n \le 4r^2 \int_{Q_r} \|\nabla u_0(s)\|^2 \, d\lambda^n \quad \text{for } s \in I_r,$$

we apply Young's inequality to obtain the following estimate:

$$\varepsilon \int_{I_r} \int_{Q_r} \|\nabla u_0(s)\|^2 \, d\lambda^n \, ds \le \int_{I_r} \langle f(s), u_0(s) \rangle_{Y_r} \, ds \\ \le c_1 \int_{I_r} \|f(s)\|^2_{Y_r^*} \, ds + \frac{\varepsilon}{2} \int_{I_r} \int_{Q_r} \|\nabla u_0(s)\|^2 \, d\lambda^n \, ds,$$

where $c_1 = c_1(\varepsilon, n)$ is some constant. Consequently, we get

$$\int_{I_r} \int_{Q_r} \|\nabla u_0(s)\|^2 \, d\lambda^n \, ds \le \frac{2c_1}{\varepsilon} \int_{I_r} \|f(s)\|^2_{Y_r^*} \, ds. \quad (5.10)$$

2. Let $u \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ be a solution of (2.2). Then the difference $v = u - u_0 \in W_{E_r}(I_r; X_r) \cap C(\overline{I_r}; H_r)$ solves the homogeneous problem (2.1). Due to Lemma 5.2 and $v = u - u_0$, for all $0 < \delta \le r \le 1$ we obtain the estimate

$$\begin{aligned} \int_{I_\delta}\int_{Q_\delta} \|\nabla v(s)\|^2 \, d\lambda^n \, ds &\le c_2 \Big(\frac{\delta}{r}\Big)^{\bar\omega} \int_{I_r}\int_{Q_r} \|\nabla v(s)\|^2 \, d\lambda^n \, ds \\ &\le 2c_2 \Big(\frac{\delta}{r}\Big)^{\bar\omega} \int_{I_r}\int_{Q_r} \big(\|\nabla u(s)\|^2 + \|\nabla u_0(s)\|^2\big) \, d\lambda^n \, ds, \end{aligned}$$

where $\bar\omega \in (n, n+2)$ and $c_2 > 0$ are two constants depending on n and ε.

In view of $u = u_0 + v$ and estimate (5.10) this yields the existence of some constant $c_3 = c_3(n, \varepsilon) > 0$ such that the desired inequality holds true. $\square$

6. Global regularity for a model problem

Let $S = (t_0, t_1)$ be a bounded open interval, $G \subset \mathbb{R}^n$ a regular set, and $0 < \varepsilon \le 1$ some constant. To formulate our model problem we consider the following type of parabolic operators.

Definition 6.1 (Parabolic operator)**.** 1. The pair of leading coefficients (a, A) is called ε-definite with respect to S and G° if $a \in L^\infty(G^\circ)$ fulfills

$$\varepsilon \le \operatorname*{ess\,inf}_{y \in G^\circ} a(y), \quad \operatorname*{ess\,sup}_{y \in G^\circ} a(y) \le \frac{1}{\varepsilon},$$

and $A \in L^\infty(S; L^\infty(G^\circ; \mathbb{S}^n))$ satisfies the ellipticity condition

$$\varepsilon\, \|\xi\|^2 \le \operatorname*{ess\,inf}_{s \in S} \operatorname*{ess\,inf}_{y \in G^\circ} A(s)(y)\xi \cdot \xi, \quad \operatorname*{ess\,sup}_{s \in S} \operatorname*{ess\,sup}_{y \in G^\circ} A(s)(y)\xi \cdot \xi \le \frac{1}{\varepsilon}\, \|\xi\|^2$$

for all $\xi \in \mathbb{R}^n$. Here $\mathbb{S}^n$ is the set of symmetric $(n \times n)$-matrices.

2. Let the pair (a, A) of leading coefficients be ε-definite with respect to S and G°. Consider the operator $E \in \mathcal{L}(H_0^1(G); H^{-1}(G))$ associated with a and introduce its time-dependent counterpart $\mathcal{E} : L^2(S; H_0^1(G)) \to L^2(S; H^{-1}(G))$ as usual by $(\mathcal{E}u)(s) = Eu(s)$ for $u \in L^2(S; H_0^1(G))$ and $s \in S$. Moreover, for $u, w \in L^2(S; H_0^1(G))$ we define the bounded linear operator $\mathcal{A} : L^2(S; H_0^1(G)) \to L^2(S; H^{-1}(G))$ by

$$\langle \mathcal{A}u, w \rangle_{L^2(S; H_0^1(G))} = \int_S \int_G A(s) \nabla u(s) \cdot \nabla w(s) \, d\lambda^n \, ds.$$

3. We define the parabolic operator

$$\mathcal{P} : \big\{ u \in W_E(S; H_0^1(G)) : u(t_0) = 0 \big\} \to L^2(S; H^{-1}(G)),$$

associated with the maps $\mathcal{E}$ and $\mathcal{A}$, by setting

$$\mathcal{P}u = (\mathcal{E}u)' + \mathcal{A}u \quad \text{for } u \in W_E(S; H_0^1(G)) \text{ with } u(t_0) = 0.$$

We formulate the model problem to find a solution $u \in W_E(S; H_0^1(G))$ of

$$\mathcal{P}u = f \in L^2(S; H^{-1}(G)), \quad u(t_0) = 0. \tag{6.1}$$

Applying Theorem 1.1, the operator $\mathcal{P}$ is an isomorphism between the Hilbert spaces $\{u \in W_E(S; H_0^1(G)) : u(t_0) = 0\}$ and $L^2(S; H^{-1}(G))$: For every $f \in L^2(S; H^{-1}(G))$ the initial-value problem (6.1) admits a uniquely determined solution $u \in W_E(S; H_0^1(G))$. This section is dedicated to the maximal regularity properties of the parabolic operator $\mathcal{P}$. To that end we introduce the concept of admissibility for regular sets $G \subset \mathbb{R}^n$:

Definition 6.2 (Admissible sets)**.** 1. Let $\varepsilon \in (0, 1]$ and $F \subset G \subset \mathbb{R}^n$ be two regular sets. We denote by $\bar{\omega}_\varepsilon(F, G) \in [0, n + 2]$ the supremum of all $\bar{\omega} \in [0, n + 2]$ such that for every $\omega \in [0, \bar{\omega})$, all bounded open intervals $S = (t_0, t_1)$, every functional $f \in L_2^\omega(S; H^{-1}(G))$, and all coefficients (a, A) being ε-definite with respect to S and G°, for the solution $u \in W_E(S; H_0^1(G))$ to the model problem (6.1) the estimate

$$\|\mathcal{R}_{S,F} u\|_{L_2^\omega(S; H^1(F^\circ))} \le c_1 \Big(\|f\|_{L_2^\omega(S; H^{-1}(G))} + \|u\|_{W_E(S; H_0^1(G))} \Big),$$

holds true, where $c_1 > 0$ is some constant which depends on n, ε, ω, S, G, and F, only. In the case $F = G$ we set $\bar{\omega}_\varepsilon(G) = \bar{\omega}_\varepsilon(G, G)$.

2. Let $F \subset G \subset \mathbb{R}^n$ be two regular sets. The set F is called admissible with respect to G if and only if $\bar{\omega}_\varepsilon(F, G) > n$ for all $\varepsilon \in (0, 1]$. We call G admissible if and only if $\bar{\omega}_\varepsilon(G) > n$ for all $\varepsilon \in (0, 1]$.

Theorem 6.1. *If $G \subset \mathbb{R}^n$ is admissible, then for every $0 \le \omega < \bar{\omega}_\varepsilon(G)$ the restriction $\mathcal{P}_\omega$ of the parabolic operator $\mathcal{P}$ associated with the coefficients (a, A) being ε-definite with respect to $S = (t_0, t_1)$ and G° is a linear isomorphism from $\{u \in W_E^\omega(S; H_0^1(G)) : u(t_0) = 0\}$ onto $L_2^\omega(S; H^{-1}(G))$.*

Proof. Let $G \subset \mathbb{R}^n$ be admissible and $0 \le \omega < \bar{\omega}_\varepsilon(G)$ be some given parameter. In view of the above definition, for every $f \in L_2^\omega(S; H^{-1}(G))$ the solution $u \in W_E(S; H_0^1(G))$ of problem (6.1) belongs to $L_2^\omega(S; H_0^1(G))$ and satisfies the estimate

$$\|u\|_{L_2^\omega(S; H_0^1(G))} \le c_1 \Big(\|f\|_{L_2^\omega(S; H^{-1}(G))} + \|u\|_{W_E(S; H_0^1(G))} \Big), \tag{6.2}$$

where $c_1 > 0$ is some constant depending on n, ε, ω, S, and G, only. Using [12, Remark 3.2, Theorem 5.6] this yields $\mathcal{A}u \in L_2^\omega(S; H^{-1}(G))$ and, hence, maximal regularity $(\mathcal{E}u)' = f - \mathcal{A}u \in L_2^\omega(S; H^{-1}(G))$ with a norm estimate

$$\|(\mathcal{E}u)'\|_{L_2^\omega(S;H^{-1}(G))} \le c_2\Big(\|f\|_{L_2^\omega(S;H^{-1}(G))} + \|u\|_{W_E(S;H_0^1(G))}\Big), \tag{6.3}$$

where $c_2 > 0$ is some constant depending on n, ε, ω, S, and G, only.

Since the operator $\mathcal{P}^{-1}$ maps $L^2(S; H^{-1}(G))$ continuously into the space $W_E(S; H_0^1(G))$ (see Theorem 1.1) and $L_2^\omega(S; H^{-1}(G))$ is continuously embedded into $L^2(S; H^{-1}(G))$, the above estimates (6.2) and (6.3) lead to

$$\|\mathcal{P}^{-1} f\|_{W_E^\omega(S;H^1(G^\circ))} \le c_3 \|f\|_{L_2^\omega(S;H^{-1}(G))} \quad \text{for all } f \in L_2^\omega(S; H^{-1}(G)),$$

where $c_3 = c_3(n, \varepsilon, \omega, S, G) > 0$ is some constant.

From the theory of function spaces $L_2^\omega(S; H^{-1}(G))$ (see [12, Theorem 5.6]) it follows that the restriction $\mathcal{P}_\omega$ of the parabolic operator $\mathcal{P}$ is a bounded linear operator from $\big\{u \in W_E^\omega(S; H_0^1(G)) : u(t_0) = 0\big\}$ into $L_2^\omega(S; H^{-1}(G))$. Combining both results, we have proved the isomorphism property. □

Remark 6.1. We want to emphasize that for admissible sets $G \subset \mathbb{R}^n$ in the case $n < \omega < \bar{\omega}_\varepsilon(G)$ the solution $u = \mathcal{P}^{-1} f \in \mathfrak{L}_2^{\omega+2}(S; L^2(G^\circ))$ is Hölder continuous in time and space up to the boundary; see [12, Theorem 3.4, Theorem 6.8]. Hence, the aim of this section is to prove the admissibility of all regular sets $G \subset \mathbb{R}^n$.

Invariance principles for admissible sets. In the following we prove that the concept of admissibility is invariant with respect to localization, transformation and reflection.

Lemma 6.2 (Localization). *Let $G \subset \mathbb{R}^n$ be a regular set and $\{U_1, \dots, U_m\}$ and $\{V_1, \dots, V_m\}$ be two open coverings of the set $\overline{G}$ such that the inclusion $V_i \subset U_i$ holds true, and $V_i \cap G$ is admissible with respect to $U_i \cap G$. Then the set G is admissible.*

Proof. 1. Let $\varepsilon \in (0, 1]$ and take a smooth partition $\{\chi_1, \dots, \chi_m\} \subset C_0^\infty(\mathbb{R}^n)$ of unity subordinate to the open covering $\{V_1, \dots, V_m\}$ of $\overline{G}$. We choose some $\delta > 0$ such that $Q_\delta(x) \subset V_i$ holds true for every $x \in \operatorname{supp}(\chi_i)$ and $i \in \{1, \dots, m\}$. Since $V_i \cap G$ is admissible with respect to $U_i \cap G$ we choose $\bar{\omega} \in (n, n+2]$ satisfying

$$\bar{\omega} \le \bar{\omega}_\varepsilon(V_i \cap G, U_i \cap G) \quad \text{for all } i \in \{1, \dots, m\}.$$

2. Let the coefficients (a, A) be ε-definite with respect to S and G°. For every $i \in \{1, \dots, m\}$ we define the restriction $a_i \in L^\infty(U_i \cap G^\circ)$, the associated operator $E_i \in \mathcal{L}(H_0^1(U_i \cap G); H^{-1}(U_i \cap G))$, and the bounded linear

map $\mathcal{A}_i : L^2(S; H_0^1(U_i \cap G)) \to L^2(S; H^{-1}(U_i \cap G))$ by

$$\langle \mathcal{A}_i v, w \rangle_{L^2(S;H_0^1(U_i\cap G))} = \int_S \int_{U_i \cap G} A(s)\nabla v(s) \cdot \nabla w(s)\, d\lambda^n\, ds$$

for $v, w \in L^2(S; H_0^1(U_i \cap G))$.

3. Let $\omega \in (0, \bar{\omega}]$ be fixed. For every functional $f \in L_2^\omega(S; H^{-1}(G))$, the corresponding solution $u \in W_E(S; H_0^1(G))$ of the problem

$$(\mathcal{E}u)' + \mathcal{A}u = f, \quad u(t_0) = 0,$$

and every $i \in \{1, \dots, m\}$ we define the function

$$u_i = \mathcal{R}_{S,U_i\cap G}(\chi_i u) \in W_{E_i}(S; H_0^1(U_i \cap G))$$

and the functional $f_{0i} \in L^2(S; H^{-1}(U_i \cap G))$ by

$$\begin{aligned} \langle f_{0i}, w \rangle_{L^2(S;H_0^1(U_i\cap G))} &= \int_S \int_{U_i\cap G} u(s)A(s)\nabla\chi_i \cdot \nabla w(s)\, d\lambda^n\, ds \\ &\quad - \int_S \int_{U_i\cap G} w(s)A(s)\nabla u(s) \cdot \nabla\chi_i\, d\lambda^n\, ds \end{aligned}$$

for $w \in L^2(S; H_0^1(U_i \cap G))$. Using [12, Lemma 6.2, Lemma 6.3] we obtain

$$\begin{aligned} \langle (\mathcal{E}_i u_i)' + \mathcal{A}_i u_i - f_{0i}, w \rangle_{L^2(S;H_0^1(U_i\cap G))} &= \langle (\mathcal{E}u)' + \mathcal{A}u, \mathcal{Z}_{S,G}(\chi_i w) \rangle_{L^2(S;H_0^1(G))} \\ &= \langle f, \mathcal{Z}_{S,G}(\chi_i w) \rangle_{L^2(S;H_0^1(G))} \end{aligned}$$

for all $w \in L^2(S; H_0^1(U_i \cap G))$. Thus, setting

$$f_i = f_{0i} + f_{1i}, \quad f_{1i} = \mathcal{L}_{S,U_i\cap G}(\chi_i f) \in L^2(S; H^{-1}(U_i \cap G)),$$

for every $i \in \{1, \dots, m\}$ the function $u_i \in W_{E_i}(S; H_0^1(U_i \cap G))$ solves the localized problem

$$(\mathcal{E}_i u_i)' + \mathcal{A}_i u_i = f_i, \quad u_i(t_0) = 0. \tag{6.4}$$

4. Due to the continuous embedding of $W_E(S; H_0^1(G))$ in $L_2^2(S; L^2(G^\circ))$ (see [12, Theorem 3.4, Theorem 6.8]) we get

$$\|uA\nabla\chi_i\| \in L_2^2(S; L^2(G^\circ)), \quad -A\nabla u \cdot \nabla\chi_i \in L^2(S; L^2(G^\circ));$$

see [12, Remark 3.2]. Using [12, Theorem 5.6] for $\mu = \min\{\omega, 2\}$ we obtain $f_{0i} \in L_2^\mu(S; H^{-1}(U_i \cap G))$, and we find some constant $c_1 > 0$ depending on ε, G, and the above partition of unity such that

$$\|f_{0i}\|_{L_2^\mu(S;H^{-1}(U_i\cap G))} \le c_1 \|u\|_{W_E(S;H_0^1(G))} \quad \text{for all } i \in \{1, \dots, m\}.$$

Due to [12, Lemma 5.2, Lemma 5.3] we get $f_{1i} \in L_2^\mu(S; H^{-1}(U_i \cap G))$ and

$$\|f_{1i}\|_{L_2^\mu(S;H^{-1}(U_i\cap G))} \le c_2 \|f\|_{L_2^\mu(S;H^{-1}(G))} \quad \text{for all } i \in \{1,\dots,m\},$$

where the constant $c_2 > 0$ depends on the partition of unity.

In view of the admissibility of $V_i \cap G$ with respect to $U_i \cap G$ there exists some constant $c_3 > 0$ depending on n, ε, μ, S, G, the coverings $\{U_1,\dots,U_m\}$, $\{V_1,\dots,V_m\}$, and the partition of unity, only, such that for every $i \in \{1,\dots,m\}$ the solution $u_i \in W_{E_i}(S; H_0^1(U_i \cap G))$ to the localized problem (6.4) satisfies the estimate

$$\|\mathcal{R}_{S,V_i\cap G} u_i\|_{L_2^\mu(S;H_0^1(V_i\cap G))} \le c_3\Big(\|f\|_{L_2^\mu(S;H^{-1}(G))} + \|u\|_{W_E(S;H_0^1(G))}\Big).$$

In view of [12, Remark 3.3] we arrive at

$$u = \sum_{i=1}^m \chi_i u = \sum_{j=1}^m \mathcal{Z}_{S,G} u_i \in L_2^\mu(S; H_0^1(G))$$

together with the estimate

$$\|u\|_{L_2^\mu(S;H_0^1(G))} \le c_4\Big(\|f\|_{L_2^\mu(S;H^{-1}(G))} + \|u\|_{W_E(S;H_0^1(G))}\Big),$$

where $c_4 > 0$ is some constant depending on n, ε, μ, S, G, δ, the partition of unity, and the coverings $\{U_1,\dots,U_m\}$, $\{V_1,\dots,V_m\}$.

5. We complete the proof using iterative arguments: Since Step 4 and [12, Theorem 5.6] yields

$$(\mathcal{E}u)' = f - \mathcal{A}u \in L_2^\mu(S; H^{-1}(G)),$$

and the embedding of $W_E^\mu(S; H_0^1(G))$ into $\mathfrak{L}_2^{\mu+2}(S; L^2(G^\circ))$ is continuous (see [12, Theorem 6.8]) there exist some constants $c_5, c_6 > 0$ depending on n, ε, μ, S, G, the partition of unity, and $\{U_1,\dots,U_m\}$ and $\{V_1,\dots,V_m\}$ such that

$$\|u\|_{W_E^\mu(S;H_0^1(G))} \le c_5\Big(\|f\|_{L_2^\mu(S;H^{-1}(G))} + \|u\|_{W_E(S;H_0^1(G))}\Big),$$

$$\|u\|_{\mathfrak{L}_2^{\mu+2}(S;L^2(G^\circ))} \le c_6\Big(\|f\|_{L_2^\mu(S;H^{-1}(G))} + \|u\|_{W_E(S;H_0^1(G))}\Big).$$

Using [12, Theorem 3.4] for $\mu = \min\{\omega, 4\}$ and every $i \in \{1,\dots,m\}$ we obtain

$$\|uA\nabla\chi_i\| \in L_2^\mu(S; L^2(G^\circ)), \quad -A\nabla u \cdot \nabla\chi_i \in L_2^{\mu-2}(S; L^2(G^\circ)).$$

Applying [12, Theorem 5.6] we get $f_{0i} \in L_2^\mu(S; H^{-1}(U_i \cap G))$ for every $i \in \{1,\dots,m\}$ together with a constant $c_7 > 0$ depending on n, ε, μ, S, G, the partition of unity, and $\{U_1,\dots,U_m\}$ and $\{V_1,\dots,V_m\}$ such that

$$\|f_{0i}\|_{L_2^\mu(S;H^{-1}(U_i\cap G))}\le c_7\Big(\|f\|_{L_2^\mu(S;H^{-1}(G))}+\|u\|_{W_E(S;H_0^1(G))}\Big).$$

Using [12, Lemma 5.2 and Lemma 5.3] we see that $f_{1i}\in L_2^\mu(S;H^{-1}(U_i\cap G))$ and

$$\|f_{1i}\|_{L_2^\mu(S;H^{-1}(U_i\cap G))}\le c_8\,\|f\|_{L_2^\mu(S;H^{-1}(G))}\quad\text{for all } i\in\{1,\dots,m\},$$

where $c_8>0$ depends on the partition of unity. As in Step 4 the admissibility of $V_i\cap G$ with respect to $U_i\cap G$ yields $u\in L_2^\mu(S;H_0^1(G))$ and

$$\|u\|_{L_2^\mu(S;H_0^1(G))}\le c_9\Big(\|f\|_{L_2^\mu(S;H^{-1}(G))}+\|u\|_{W_E(S;H_0^1(G))}\Big),$$

where $c_9>0$ is some constant depending on n, ε, μ, S, G, δ, the partition of unity, and the coverings $\{U_1,\dots,U_m\}$ and $\{V_1,\dots,V_m\}$. Repeating these arguments, after a finite number of analogous steps we arrive at $\mu=\omega$, which proves the admissibility of G. □

Lemma 6.3 (Transformation)**.** *Let $F\subset G\subset\mathbb{R}^n$ be two regular sets and T some Lipschitz transformation from an open neighborhood of $\overline{G}$ into $\mathbb{R}^n$. Then $F_*=T[F]$ is admissible with respect to $G_*=T[G]$, if and only if F is admissible with respect to G.*

Proof. 1. Let $L\ge 1$ be a Lipschitz constant of T and $\varepsilon_*\in(0,1]$. We consider coefficients (a_*,A_*) being ε_*-definite with respect to S and G_*° and the map $E_*\in\mathcal{L}(H_0^1(G_*);H^{-1}(G_*))$ associated with a_*. Moreover, we define the bounded linear map $\mathcal{A}_*:L^2(S;H_0^1(G_*))\to L^2(S;H^{-1}(G_*))$ by

$$\langle\mathcal{A}_*v_*,w_*\rangle_{L^2(S;H_0^1(G_*))}=\int_S\int_{G_*}A_*(s)\nabla v_*(s)\cdot\nabla w_*(s)\,d\lambda^n\,ds$$

for $v_*,w_*\in L^2(S;H_0^1(G_*))$.

Due to the properties of the Jacobi matrix DT and its determinant JT the pair (a,A) of transformed coefficients

$$a=|JT|\cdot T_*a_*,\quad A=|JT|\cdot((DT)^{-1})^*(T_*A_*)(DT)^{-1},$$

is ε-definite with respect to S and G° with $\varepsilon=\varepsilon_*/L^{n+2}$. We introduce the operator $E\in\mathcal{L}(H_0^1(G);H^{-1}(G))$ associated with a and the bounded linear map $\mathcal{A}:L^2(S;H_0^1(G))\to L^2(S;H^{-1}(G))$ by

$$\langle\mathcal{A}v,w\rangle_{L^2(S;H_0^1(G))}=\int_S\int_G A(s)\nabla v(s)\cdot\nabla w(s)\,d\lambda^n\,ds$$

for $v, w \in L^2(S; H_0^1(G))$. Due to the chain rule and the change-of-variable formula we have both $\mathcal{E}_* = \mathcal{T}^* \mathcal{E} \mathcal{T}_*$ and $\mathcal{A}_* = \mathcal{T}^* \mathcal{A} \mathcal{T}_*$.

2. Suppose that F is admissible with respect to G and fix the parameter $0 \le \omega < \bar{\omega}_\varepsilon(F, G)$. For every functional $f^* \in L_2^\omega(S; H^{-1}(G_*))$ the problem

$$(\mathcal{E}_* u_*)' + \mathcal{A}_* u_* = f^*, \quad u_*(t_0) = 0,$$

admits a uniquely determined solution $u_* \in W_{E_*}(S; H_0^1(G_*))$. Using the invariance of the Morrey spaces with respect to Lipschitz transformations (see [12, Lemma 5.4 and Lemma 6.4]), we have that the functions $u = \mathcal{T}_* u_* \in W_E(S; H_0^1(G))$ and $f \in L_2^\omega(S; H^{-1}(G))$ defined by $\mathcal{T}^* f = f^*$ satisfy

$$\begin{aligned} \langle (\mathcal{E}u)' + \mathcal{A}u, \mathcal{T}_* w_* \rangle_{L^2(S; H_0^1(G))} &= \langle \mathcal{T}^* (\mathcal{E} \mathcal{T}_* u_*)' + \mathcal{T}^* \mathcal{A} \mathcal{T}_* u_*, w_* \rangle_{L^2(S; H_0^1(G_*))} \\ &= \langle (\mathcal{E}_* u_*)' + \mathcal{A}_* u_*, w_* \rangle_{L^2(S; H_0^1(G_*))} \\ &= \langle f, \mathcal{T}_* w_* \rangle_{L^2(S; H_0^1(G))} \end{aligned}$$

for all $w_* \in L^2(S; H_0^1(G_*))$. Applying [12, Lemma 4.4] we obtain that $u = \mathcal{T}_* u_* \in W_E(S; H_0^1(G))$ solves the transformed problem

$$(\mathcal{E}u)' + \mathcal{A}u = f, \quad u(t_0) = 0.$$

3. Due to the admissibility of F with respect to G we find some constant $c_1 > 0$ depending on n, ε, ω, S, F, and G such that

$$\|\mathcal{R}_{S,F} u\|_{L_2^\omega(S; H^1(F^\circ))} \le c_1 \Big(\|f\|_{L_2^\omega(S; H^{-1}(G))} + \|u\|_{W_E(S; H_0^1(G))} \Big).$$

In view of the invariance of the Morrey spaces with respect to Lipschitz transformations (see [12, Lemma 4.4, Lemma 5.4, and Lemma 6.4]), we end up with the estimate

$$\|\mathcal{R}_{S,F_*} u_*\|_{L_2^\omega(S; H^1(F_*^\circ))} \le c_2 \Big(\|f^*\|_{L_2^\omega(S; H^{-1}(G_*))} + \|u_*\|_{W_{E_*}(S; H_0^1(G_*))} \Big),$$

where the constant $c_2 > 0$ depends on n, ε, ω, T, S, F, and G. This proves the admissibility of F_* with respect to G_*. The proof of the inverse statement can be done in the same manner. □

Lemma 6.4 (Reflection)**.** *If Q_ϱ is admissible with respect to Q for some $0 < \varrho \le 1$, then Q_ϱ^+ and Q_ϱ^- are admissible with respect to Q^+ and Q^-, respectively.*

Proof. 1. Let $0 < \varepsilon \le 1$. We consider coefficients (a^-, A^-) being ε-definite with respect to S and Q^- and the map $E^- \in \mathcal{L}(H_0^1(Q^-); H^{-1}(Q^-))$ associated with a^-. Furthermore, we define the bounded linear map $\mathcal{A}^-$:

$L^2(S; H_0^1(Q^-)) \to L^2(S; H^{-1}(Q^-))$ by

$$\langle \mathcal{A}^- u^-, w^- \rangle_{L^2(S;H_0^1(Q^-))} = \int_S \int_{Q^-} A^-(s)\nabla v^-(s) \cdot \nabla w^-(s)\, d\lambda^n\, ds$$

for u^-, $w^- \in L^2(S; H_0^1(Q^-))$.

The pair (a, A) of reflected coefficients

$$a = R^+ a^-, \quad A = \mathcal{R}^+ A^-,$$

is ε-definite with respect to S and Q. Let $E \in \mathcal{L}(H_0^1(Q); H^{-1}(Q))$ be associated with a and the bounded linear operator $\mathcal{A} : L^2(S; H_0^1(Q)) \to L^2(S; H^{-1}(Q))$ be defined as

$$\langle \mathcal{A}v, w \rangle_{L^2(S;H_0^1(Q))} = \int_S \int_Q A(s)\nabla v(s) \cdot \nabla w(s)\, d\lambda^n\, ds$$

for $v, w \in L^2(S; H_0^1(Q))$. Note that the properties of the reflection ensure both the relations $\mathcal{E}\mathcal{R}^- = \mathcal{R}^-\mathcal{E}^-$ and $\mathcal{A}\mathcal{R}^- = \mathcal{R}^-\mathcal{A}^-$.

2. Assume that Q_ϱ is admissible with respect to Q for some $\varrho \in (0, 1]$ and let $0 \le \omega < \bar{\omega}_\varepsilon(Q_\varrho, Q)$ be fixed. For every functional $f^- \in L_2^\omega(S; H^{-1}(Q^-))$ the problem

$$(\mathcal{E}^- u^-)' + \mathcal{A}^- u^- = f^-, \quad u^-(t_0) = 0,$$

has a uniquely determined solution $u^- \in W_{E^-}(S; H_0^1(Q^-))$. In view of the invariance of the Morrey spaces with respect to antireflection (see [12, Lemma 5.5 and Lemma 6.5]), the function $u = \mathcal{R}^- u^- \in W_E(S; H_0^1(Q))$ and the functional $f = \mathcal{R}^- f^- \in L_2^\omega(S; H^{-1}(Q))$ satisfy the identity

$$\begin{aligned}\langle (\mathcal{E}u)' + \mathcal{A}u, w \rangle_{L^2(S;H_0^1(Q))} &= \langle (\mathcal{E}\mathcal{R}^- u^-)' + \mathcal{A}\mathcal{R}^- u^-, w \rangle_{L^2(S;H_0^1(Q))} \\ &= \langle \mathcal{R}^-(\mathcal{E}^- u^-)' + \mathcal{R}^-\mathcal{A}^- u^-, w \rangle_{L^2(S;H_0^1(Q))} \\ &= \langle \mathcal{R}^- f^-, w \rangle_{L^2(S;H_0^1(Q))}\end{aligned}$$

for all $w \in L^2(S; H_0^1(Q))$. Thus, $u = \mathcal{R}^- u^- \in W_E(S; H_0^1(Q))$ solves the reflected problem

$$(\mathcal{E}u)' + \mathcal{A}u = f, \quad u(t_0) = 0.$$

3. The admissibility of Q_ϱ with respect to Q yields some constant $c_1 > 0$ depending on n, ε, ω, ϱ, and S such that

$$\|\mathcal{R}_{S,Q_\varrho} u\|_{L_2^\omega(S;H^1(Q_\varrho))} \le c_1 \Big(\|f\|_{L_2^\omega(S;H^{-1}(Q))} + \|u\|_{W_E(S;H_0^1(Q))} \Big).$$

Consequently, the invariance of the Morrey spaces $L_2^\omega(S; H^{-1}(Q^-))$ under antireflection (see [12, Lemma 4.5, Lemma 5.5, and Lemma 6.5]) leads to the estimate

$$\|\mathcal{R}_{S,Q_\varrho^-} u^-\|_{L_2^\omega(S;H^1(Q_\varrho^-))} \le c_2\Big(\|f^-\|_{L_2^\omega(S;H^{-1}(Q^-))} + \|u^-\|_{W_{E^-}(S;H_0^1(Q^-))}\Big),$$

where the constant $c_2 > 0$ depends on n, ε, ω, ϱ, and S. This yields the admissibility of Q_ϱ^- with respect to Q^-. Analogously, we prove that Q_ϱ^+ is admissible with respect to Q^+. □

Admissibility of regular sets. To prove the admissibility of all regular sets $G \subset \mathbb{R}^n$, we begin with the unit cube Q and the half-cubes Q^+, Q^-, and $Q^\pm$. In a first step we show that the cube Q_ϱ is admissible with respect to the unit cube Q for every $0 < \varrho < 1$. We use the Campanato inequality for the spatial gradient of solutions on concentric cubes; see Theorem 5.3.

Lemma 6.5. *For $0 < \varrho < 1$ the cube Q_ϱ is admissible with respect to Q.*

Proof. 1. Let $\varepsilon \in (0, 1]$. We consider coefficients (a, A) which are ε-definite with respect to S and Q, the operator $E \in \mathcal{L}(H_0^1(Q); H^{-1}(Q))$ associated with a, and the bounded linear map $\mathcal{A} : L^2(S; H_0^1(Q)) \to L^2(S; H^{-1}(Q))$ defined by

$$\langle \mathcal{A}v, w\rangle_{L^2(S;H_0^1(Q))} = \int_S \int_Q A(s)\nabla v(s) \cdot \nabla w(s)\, d\lambda^n\, ds$$

for $v, w \in L^2(S; H_0^1(Q))$. Let $u \in W_E(S; H_0^1(Q))$ be the solution of the problem

$$(\mathcal{E}u)' + \mathcal{A}u = f, \quad u(t_0) = 0,$$

where $f \in L^2(S; H^{-1}(Q))$ is some given functional.

We define ε-definite coefficients (a, A_0) with respect to $S_0 = (t_0 - 1, t_1)$ and Q by setting

$$A_0(s) = \begin{cases} A(s) & \text{if } s \in S, \\ (\delta_{ij}) & \text{otherwise,} \end{cases}$$

and extensions $u_0 \in W_E(S_0; H_0^1(Q))$ and $f_0 \in L^2(S_0; H^{-1}(Q))$ by

$$u_0(s) = \begin{cases} u(s) & \text{if } s \in S, \\ 0 & \text{otherwise,} \end{cases} \qquad f_0(s) = \begin{cases} f(s) & \text{if } s \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_0 \in W_E(S_0; H_0^1(Q))$ solves the extended problem

$$(\mathcal{E}_0 u_0)' + \mathcal{A}_0 u_0 = f_0, \quad u(t_0 - 1) = 0,$$

where the operator $\mathcal{E}_0 : L^2(S_0; H_0^1(Q)) \to L^2(S_0; H^{-1}(Q))$ is associated with S_0 and $E \in \mathcal{L}(H_0^1(Q); H^{-1}(Q))$, and the bounded linear map $\mathcal{A}_0 : L^2(S_0; H_0^1(Q)) \to L^2(S_0; H^{-1}(Q))$ is defined by

$$\langle \mathcal{A}_0 v, w\rangle_{L^2(S_0;H_0^1(Q))} = \int_{S_0}\int_Q A_0(s)\nabla v(s)\cdot\nabla w(s)\, d\lambda^n\, ds$$

for $v, w \in L^2(S_0; H_0^1(Q))$.

2. In the next steps we make use of the local regularity properties of $u_0 \in W_E(S_0; H_0^1(Q))$: Let $0 < \varrho < 1$ be given. We fix $t \in S$ and $x \in Q_\varrho$ arbitrarily, and we consider radii $0 < \delta \le 1-\varrho$. Furthermore, we introduce the operator $\mathcal{E}_\delta : L^2(I_\delta(t); H^1(Q_\delta(x))) \to L^2(I_\delta(t); H^{-1}(Q_\delta(x)))$ associated with $I_\delta(t)$ and $E_\delta \in \mathcal{L}(H^1(Q_\delta(x)); H^{-1}(Q_\delta(x)))$, which is defined by

$$\langle E_\delta v, w\rangle_{H^1(Q_\delta(x))} = \int_{Q_\delta(x)} avw\, d\lambda^n \quad \text{for } v \in H^1(Q_\delta(x)),\ w \in H_0^1(Q_\delta(x)).$$

Then for all $t \in S$, $x \in Q_\varrho$, and $0 < \delta \le 1-\varrho$ the restriction

$$v = \mathcal{R}_{I_\delta(t),Q_\delta(x)} u_0 \in W_{E_\delta}(I_\delta(t); H^1(Q_\delta(x))) \cap C(\overline{I_\delta(t)}; L^2(Q_\delta(x)))$$

of u_0 satisfies the localized variational equation

$$\int_{I_\delta(t)} \langle (\mathcal{E}_\delta v)'(s), w(s)\rangle_{H_0^1(Q_\delta(x))}\, ds + \int_{I_\delta(t)}\int_{Q_\delta(x)} A_0(s)\nabla v(s)\cdot\nabla w(s)\, d\lambda^n\, ds = \int_{I_\delta(t)} \langle L_{Q_\delta(x)} f_0(s), w(s)\rangle_{H_0^1(Q_\delta(x))}\, ds$$

for all $w \in L^2(I_\delta(t); H_0^1(Q_\delta(x)))$.

3. Using the Campanato inequality (5.9) (see Theorem 5.3), we find constants $\bar\omega \in (n, n+2]$ and $c_1 > 0$ depending on n and ε, only, such that for all $t \in S$, $x \in Q_\varrho$, and $0 < \delta \le r \le 1-\varrho$ we have

$$\int_{I_\delta(t)}\int_{Q_\delta(x)} \|\nabla u_0(s)\|^2\, d\lambda^n\, ds \le c_1\Big(\frac{\delta}{r}\Big)^{\bar\omega} \int_{I_r(t)}\int_{Q_r(x)} \|\nabla u_0(s)\|^2\, d\lambda^n\, ds + c_1 \int_{I_r(t)} \|L_{Q_r(x)} f_0(s)\|^2_{H^{-1}(Q_r(x))}\, ds.$$

Let $\omega \in [0, \bar\omega)$ be fixed and $f \in L_2^\omega(S; H^{-1}(Q))$. For all $t \in S$, $x \in Q_\varrho$, and $0 < \delta \le r \le 1-\varrho$ we obtain

$$\int_{I_\delta(t)}\int_{Q_\delta(x)} \|\nabla u_0(s)\|^2\, d\lambda^n\, ds$$

$$\le c_1\Big(\frac{\delta}{r}\Big)^{\bar\omega}\int_{I_r(t)}\int_{Q_r(x)}\|\nabla u_0(s)\|^2\,d\lambda^n\,ds + c_1 r^{\omega}[f]^2_{L_2^{\omega}(S;H^{-1}(Q))}.$$

Note, that the integral on the left-hand side is a nonnegative and nondecreasing function of the radius $0<\delta\le 1-\varrho$. Hence, for all $0<\delta\le r\le 1-\varrho$ the application of an elementary inequality yields

$$\int_{I_\delta(t)}\int_{Q_\delta(x)}\|\nabla u_0(s)\|^2\,d\lambda^n\,ds \le c_2\Big(\frac{\delta}{r}\Big)^{\omega}\int_{I_r(t)}\int_{Q_r(x)}\|\nabla u_0(s)\|^2\,d\lambda^n\,ds + c_2\delta^{\omega}[f]^2_{L_2^{\omega}(S;H^{-1}(Q))}, \quad (6.5)$$

where the constant $c_2>0$ depends on n, ε, ω, $\bar\omega$, and ϱ; see Giaquinta [7, 8]. After specifying $r=1-\varrho$ and dividing by δ^{ω} we take the supremum over all $0<\delta\le 1-\varrho$, $t\in S$, and $x\in Q_\varrho$ to estimate the Morrey seminorm

$$[\|\nabla\mathcal{R}_{S,Q_\varrho}u\|]^2_{L_2^{\omega}(S;L^2(Q_\varrho))}\le c_3\Big(\|\nabla u(s)\|^2_{L^2(S;L^2(Q;\mathbb{R}^n))}+[f]^2_{L_2^{\omega}(S;H^{-1}(Q))}\Big),$$

where $c_3>0$ depends on n, ε, ω, $\bar\omega$, and ϱ, only.

4. Applying the Poincaré inequality to $v=\mathcal{R}_{I_\delta(t),Q_\delta(x)}u_0$ (see [12, Theorem A.3]), for all $t\in S$, $x\in Q_\varrho$, and $0<\delta\le 1-\varrho$ we get

$$\int_{I_\delta(t)}\int_{Q_\delta(x)}\Big|v(s)-⨍_{I_\delta(t)}⨍_{Q_\delta(x)}v(\tau)\,d\lambda^n\,d\tau\Big|^2\,d\lambda^n\,ds \le c_4\delta^2\int_{I_\delta(t)}\Big(\int_{Q_\delta(x)}\|\nabla v(s)\|^2\,d\lambda^n+\|L_{Q_\delta(x)}(\mathcal{E}_\delta v)'(s)\|^2_{H^{-1}(Q_\delta(x))}\Big)\,ds,$$

where $c_4=c_4(n,\varepsilon)>0$. Since the restriction $v=\mathcal{R}_{I_\delta(t),Q_\delta(x)}u_0$ solves the localized variational equation (see Step 2), we find some constant $c_5=c_5(\varepsilon,n)>0$ such that

$$\int_{I_\delta(t)}\int_{Q_\delta(x)}\Big|u_0(s)-⨍_{I_\delta(t)}⨍_{Q_\delta(x)}u_0(\tau)\,d\lambda^n\,d\tau\Big|^2\,d\lambda^n\,ds \le c_5\delta^2\int_{I_\delta(t)}\Big(\int_{Q_\delta(x)}\|\nabla u_0(s)\|^2\,d\lambda^n+\|L_{Q_\delta(x)}f_0(s)\|^2_{H^{-1}(Q_\delta(x))}\Big)\,ds$$

holds true for all $t\in S$, $x\in Q_\varrho$, and $0<\delta\le 1-\varrho$. Remembering estimate (6.5) for $r=1-\varrho$ this yields

$$\int_{I_\delta(t)}\int_{Q_\delta(x)}\Big|u_0(s)-⨍_{I_\delta(t)}⨍_{Q_\delta(x)}u_0(\tau)\,d\lambda^n\,d\tau\Big|^2\,d\lambda^n\,ds$$

$$\le \frac{c_6\delta^{\omega+2}}{(1-\varrho)^\omega}\int_S\int_Q \|\nabla u(s)\|^2\,d\lambda^n\,ds + c_6\delta^{\omega+2}[f]^2_{L_2^\omega(S;H^{-1}(Q))},$$

where the constant $c_6 > 0$ depends on n, ε, ω, $\bar\omega$, and ϱ, only. After applying the minimal property of the integral mean value to the left-hand side and dividing by $\delta^{\omega+2}$ we take the supremum over all $0 < \delta \le 1-\varrho$, $t \in S$, and $x \in Q_\varrho$ to obtain an estimate of the Campanato seminorm,

$$[\mathcal{R}_{S,Q_\varrho} u]^2_{\mathfrak{L}_2^{\omega+2}(S;L^2(Q_\varrho))} \le c_7\Big(\|\nabla u(s)\|^2_{L^2(S;L^2(Q;\mathbb{R}^n))} + [f]^2_{L_2^\omega(S;H^{-1}(Q))}\Big),$$

where $c_7 > 0$ depends on n, ε, ω, $\bar\omega$, and ϱ, only.

5. Using [12, Theorem 3.4] and the estimates for the seminorms of $\mathcal{R}_{S,Q_\varrho} u$ (see Steps 3 and 4), we find some constant $c_8 > 0$ depending on n, ε, ω, $\bar\omega$, and ϱ, only, such that

$$\|\mathcal{R}_{S,Q_\varrho} u\|_{L_2^\omega(S;H^1(Q_{\varrho'}))} \le c_8\Big(\|f\|_{L_2^\omega(S;H^{-1}(Q))} + \|u\|_{L^2(S;H_0^1(Q))}\Big).$$

Consequently, Q_ϱ is admissible with respect to Q for every $0 < \varrho < 1$. □

Lemma 6.6. *The unit cube Q is admissible.*

Proof. Since Q is a regular set, we find an atlas $\big\{(T_1, U_1), \dots, (T_m, U_m)\big\}$ for Q (see [12, Lemma 4.2]) and radii $0 < \varrho' < \varrho < 1$ such that the systems $\{V_1', \dots, V_m'\}$ and $\{V_1, \dots, V_m\}$ defined by

$$V_i' = T_i^{-1}[Q_{\varrho'}], \quad V_i = T_i^{-1}[Q_\varrho] \quad \text{for } i \in \{1, \dots, m\}$$

are open coverings of $\overline{Q}$. Using Lemma 6.5 the cube $Q_{\varrho'}$ is admissible with respect to Q_ϱ. Hence, applying Lemma 6.4 the half-cube $Q^-_{\varrho'}$ is admissible with respect to Q^-_ϱ. Consequently, Lemma 6.3 yields the admissibility of $V_i' \cap Q$ with respect to $V_i \cap Q$ for every $i \in \{1, \dots, m\}$. Due to Lemma 6.2 the result follows. □

Lemma 6.7. *The half-cubes Q^+, Q^-, and $Q^\pm$ are admissible sets.*

Proof. Because of Lemmas 6.4 and 6.6 both the half-cubes Q^+ and Q^- are admissible. Note that there exists a Lipschitz transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$ which maps Q^+ onto $Q^\pm$; see Griepentrog, Höppner, Kaiser, and Rehberg [9, 13]. Hence, Lemma 6.3 yields the admissibility of $Q^\pm$. □

Theorem 6.8 (Maximal regularity)**.** *For every regular set $G \subset \mathbb{R}^n$ there exists some parameter $\bar\omega_\varepsilon(G) \in (n, n+2]$ such that for every $0 \le \omega < \bar\omega_\varepsilon(G)$ the restriction $\mathcal{P}_\omega$ of the parabolic operator $\mathcal{P}$ associated with the coefficients (a, A) being ε-definite with respect to $S = (t_0, t_1)$ and G° is a linear isomorphism from $\big\{u \in W_E^\omega(S; H_0^1(G)) : u(t_0) = 0\big\}$ onto $L_2^\omega(S; H^{-1}(G))$.*

Proof. Since G is regular, we find an atlas $\{(T_1, U_1), \dots, (T_m, U_m)\}$ for G (see [12, Lemma 4.2]) and $\varrho \in (0, 1)$ such that the system $\{V_1, \dots, V_m\}$ defined by

$$V_i = T_i^{-1}[Q_\varrho] \quad \text{for } i \in \{1, \dots, m\},$$

is an open covering of the closure $\overline{G}$. Applying Lemma 6.7, all the half-cubes Q_ϱ^+, Q_ϱ^-, and $Q_\varrho^\pm$ are admissible sets. Using Lemma 6.6 the cube Q_ϱ is admissible, too. Hence, Lemma 6.3 yields the admissibility of the intersection $V_i \cap G$ for every $i \in \{1, \dots, m\}$. Due to Lemma 6.2 we arrive at the admissibility of the set G. In view of Theorem 6.1 this yields the desired isomorphism property for $\mathcal{P}_\omega$. □

Remark 6.2. Let $S = (t_0, t_\ell)$ be some bounded open interval. Due to the above result, for every $0 \le \omega < \bar{\omega}_\varepsilon(G)$ we find some constant $c_1 > 0$ depending on ε, n, ω, G, and S such that for all coefficients (a, A) being ε-definite with respect to S and G°, and every $f \in L_2^\omega(S; H^{-1}(G))$, the solution $u \in W_E(S; H_0^1(G))$ of problem (6.1) satisfies the estimate

$$\|u\|_{W_E^\omega(S;H_0^1(G))} \le c_1 \|f\|_{L_2^\omega(S;H^{-1}(G))}. \tag{6.6}$$

We fix some $t_1 \in S$ and consider the subinterval $S_1 = (t_0, t_1)$ of S. In the following we show that estimate (6.6) remains true with the same constant $c_1 > 0$ when both the solution u and the functional f are restricted to $u_1 \in W_E^\omega(S_1; H_0^1(G))$ and $f_1 \in L_2^\omega(S_1; H^{-1}(G))$, respectively. To do so, we introduce the interval $S_0 = (t_1 + t_0 - t_\ell, t_1)$ which contains S_1 and has the same length as S. We introduce ε-definite coefficients (a, A_0) with respect to S_0 and G° by setting

$$A_0(s) = \begin{cases} A(s) & \text{if } s \in S_1, \\ (\delta_{ij}) & \text{otherwise,} \end{cases}$$

and define extensions $u_0 \in W_E^\omega(S_0; H_0^1(G))$ and $f_0 \in L_2^\omega(S_0; H^{-1}(G))$ by

$$u_0(s) = \begin{cases} u(s) & \text{if } s \in S_1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_0(s) = \begin{cases} f(s) & \text{if } s \in S_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_0 \in W_E^\omega(S_0; H_0^1(Q))$ solves the extended problem

$$(\mathcal{E}_0 u_0)' + \mathcal{A}_0 u_0 = f_0, \quad u(t_0 + t_1 - t_\ell) = 0,$$

and satisfies estimate (6.6) with the same constant $c_1 > 0$. Because of the construction of the extensions and the definition of the norm in the corresponding Morrey spaces we obtain the desired estimate,

$$\begin{aligned}\|u_1\|_{W_E^\omega(S_1;H_0^1(G))} &= \|u_0\|_{W_E^\omega(S_0;H_0^1(G))}\\ &\le c_1\|f_0\|_{L_2^\omega(S_0;H^{-1}(G))} = c_1\|f_1\|_{L_2^\omega(S_1;H^{-1}(G))}.\end{aligned}$$

7. Maximal regularity for problems with lower-order terms

In this section we conclude with isomorphism properties of second-order linear parabolic operators with lower-order terms. Suppose that $\varepsilon \in (0,1]$, $G \subset \mathbb{R}^n$ is a regular set, and $\Gamma = \partial G$ denotes its Lipschitz boundary. Throughout this section we assume that the parabolic operator $\mathcal{P}$ is associated with the pair of leading coefficients (a, A) being ε-definite with respect to some bounded open interval $S = (t_0, t_\ell)$ and G°.

Bounded lower-order coefficients. In order to generalize the isomorphism result for $\mathcal{P}$ (see Theorem 6.8), we consider bounded linear operators generated by lower-order terms:

Definition 7.1. Given a set of lower-order coefficients

$$b \in L^\infty(S; L^\infty(G^\circ;\mathbb{R}^n)), \quad b_0 \in L^\infty(S; L^\infty(G^\circ)), \quad b_\Gamma \in L^\infty(S; L^\infty(\Gamma)),$$

we define the bounded linear map $\mathcal{B} : L^2(S; H_0^1(G)) \to L^2(S; H^{-1}(G))$ by

$$\begin{aligned}\langle \mathcal{B}u, w\rangle_{L^2(S;H_0^1(G))} &= \int_S\int_G \big(u(s)b(s)\cdot\nabla w(s) + b_0(s)u(s)w(s)\big)\, d\lambda^n\, ds\\ &\quad + \int_S\int_\Gamma b_\Gamma(s)K_\Gamma u(s)K_\Gamma w(s)\, d\lambda_\Gamma\, ds\end{aligned}$$

for $u, w \in L^2(S; H_0^1(G))$.

Using Theorem 1.1 the operator $\mathcal{P}+\mathcal{B}$ is a linear isomorphism from $\big\{u \in W_E(S; H_0^1(G)) : u(t_0) = 0\big\}$ onto $L^2(S; H^{-1}(G))$: For every $f \in L^2(S; H^{-1}(G))$ the initial-value problem

$$\mathcal{P}u + \mathcal{B}u = f, \quad u(t_0) = 0, \tag{7.1}$$

admits a uniquely determined solution $u \in W_E(S; H_0^1(G))$. We show that the isomorphism property between the corresponding Sobolev–Morrey spaces carries over from $\mathcal{P}$ to $\mathcal{P}+\mathcal{B}$:

Lemma 7.1 (Continuity)**.** *For every $\omega \in [0, n+2]$ the restriction $\mathcal{B}_\omega$ of $\mathcal{B}$ to $\big\{u \in W_E^\omega(S; H_0^1(G)) : u(t_0) = 0\big\}$ is a bounded linear map into $L_2^\omega(S; H^{-1}(G))$.*

Proof. The embedding from $W_E^\omega(S;H_0^1(G))$ into $\mathfrak{L}_2^{\omega+2}(S;L^2(G^\circ))$ and the trace map from $W_E^\omega(S;H_0^1(G))$ into $\mathfrak{L}_2^{\omega+1}(S;L^2(\Gamma))$ are continuous (see [12, Theorem 6.8 and Theorem 6.11]). Due to [12, Remark 3.2 and Remark 3.5] and [12, Theorem 3.4, Theorem 3.6, and Theorem 5.6] the continuity of the map $\mathcal{B}_\omega$ from $\{u \in W_E^\omega(S;H_0^1(G)) : u(t_0) = 0\}$ into $L_2^\omega(S;H^{-1}(G))$ follows. □

Theorem 7.2 (Maximal regularity)**.** *Let $0 \le \omega < \bar{\omega}_\varepsilon(G)$ be given. For every pair (a, A) of leading coefficients being ε-definite with respect to S and G° and all lower-order coefficients*

$$b \in L^\infty(S;L^\infty(G^\circ;\mathbb{R}^n)), \quad b_0 \in L^\infty(S;L^\infty(G^\circ)), \quad b_\Gamma \in L^\infty(S;L^\infty(\Gamma)),$$

$\mathcal{P}_\omega + \mathcal{B}_\omega$ is a linear isomorphism from $\{u \in W_E^\omega(S;H_0^1(G)) : u(t_0) = 0\}$ onto $L_2^\omega(S;H^{-1}(G))$.

Proof. 1. First, we prove the surjectivity of $\mathcal{P}_\omega + \mathcal{B}_\omega$: Let $f \in L_2^\omega(S;H^{-1}(G))$ be given and $u \in W_E(S;H_0^1(G))$ be the unique solution of problem (7.1). Consequently, $u \in W_E(S;H_0^1(G))$ solves the model problem

$$\mathcal{P}u = (\mathcal{E}u)' + \mathcal{A}u = f - \mathcal{B}u, \quad u(t_0) = 0.$$

Due to [12, Theorem 6.8 and Theorem 6.11] we know that both the embedding operator from $W_E(S;H_0^1(G))$ into $\mathfrak{L}_2^2(S;L^2(G^\circ))$ and the trace map from $W_E(S;H_0^1(G))$ into $\mathfrak{L}_2^1(S;L^2(\Gamma))$ are bounded. Using [12, Theorem 3.4 and Theorem 3.6] we get that $u \in L_2^2(S;L^2(G^\circ))$ and $\mathcal{K}_{S,\Gamma}u \in L_2^1(S;L^2(\Gamma))$. Applying [12, Theorem 5.6] for $\mu = \min\{\omega, 2\}$ we then obtain $f - \mathcal{B}u \in L_2^\mu(S;H^{-1}(G))$, which leads to $u \in W_E^\mu(S;H_0^1(G))$ (see Theorem 6.8).

We apply a bootstrap argument: The embedding from $W_E^\mu(S;H_0^1(G))$ into $\mathfrak{L}_2^{\mu+2}(S;L^2(G^\circ))$ and the trace map from $W_E^\mu(S;H_0^1(G))$ into the space $\mathfrak{L}_2^{\mu+1}(S;L^2(\Gamma))$ are continuous (see [12, Theorem 6.8 and Theorem 6.11]). Using [12, Theorem 3.4 and Theorem 3.6] for $\mu = \min\{\omega, 4\}$ we obtain $u \in L_2^\mu(S;L^2(G^\circ))$ and $\mathcal{K}_{S,\Gamma}u \in L_2^{\mu-1}(S;L^2(\Gamma))$. Therefore, by [12, Theorem 5.6] and Theorem 6.8 this yields $f - \mathcal{B}u \in L_2^\mu(S;H^{-1}(G))$ and $u \in W_E^\mu(S;H_0^1(G))$. After a finite number of analogous steps we arrive at $\mu = \omega$, which yields the surjectivity of $\mathcal{P}_\omega + \mathcal{B}_\omega$.

2. In view of Lemma 7.1 the operator $\mathcal{B}_\omega$ is a bounded linear map from the space $\{u \in W_E^\omega(S;H_0^1(G)) : u(t_0) = 0\}$ into $L_2^\omega(S;H^{-1}(G))$. By definition the same holds true for $\mathcal{P}_\omega$ and, therefore, for the sum $\mathcal{P}_\omega + \mathcal{B}_\omega$, too. The unique solvability of the problem (7.1), and the surjectivity (see Step 1), yields that the operator $\mathcal{P}_\omega + \mathcal{B}_\omega$ maps $\{u \in W_E^\omega(S;H_0^1(G)) : u(t_0) = 0\}$

onto $L_2^\omega(S;H^{-1}(G))$. Therefore, by the inverse mapping theorem it is a linear isomorphism between these spaces. □

Theorem 7.3 (Continuous dependence)**.** *Let $\varepsilon \in (0,1]$ and $0 \le \omega < \bar{\omega}_\varepsilon(G)$ be given constants. Then for every pair (a,A) of leading coefficients being ε-definite with respect to S and G° and all lower-order coefficients*

$$b \in L^\infty(S;L^\infty(G^\circ;\mathbb{R}^n)), \quad b_0 \in L^\infty(S;L^\infty(G^\circ)), \quad b_\Gamma \in L^\infty(S;L^\infty(\Gamma)),$$

the assignment $(A,b,b_0,b_\Gamma) \mapsto (\mathcal{P}+\mathcal{B})^{-1}$ is a continuous map from the metric space of admissible coefficients equipped with the metric d defined by

$$\begin{aligned} d\big((A,b,b_0,b_\Gamma),(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)\big) \\ = \|A-\underline{A}\|_{L^\infty(S;L^\infty(G^\circ;\mathbb{S}^n))} + \|b-\underline{b}\|_{L^\infty(S;L^\infty(G^\circ;\mathbb{R}^n))} \\ + \|b_0-\underline{b}_0\|_{L^\infty(S;L^\infty(G^\circ))} + \|b_\Gamma-\underline{b}_\Gamma\|_{L^\infty(S;L^\infty(\Gamma))}, \end{aligned}$$

into the Banach space $\mathcal{L}(L_2^\omega(S;H^{-1}(G));W_E^\omega(S;H_0^1(G)))$ of solution maps corresponding to problem (7.1)*.*

Proof. We consider the operators $\mathcal{P}$, $\mathcal{B}$, $\underline{\mathcal{P}}$, and $\underline{\mathcal{B}}$ that are associated with the sets (a,A,b,b_0,b_Γ) and $(a,\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)$ of admissible coefficients, respectively. Using the same arguments as in the proof of Lemma 7.1, for all $u \in W_E^\omega(S;H_0^1(G))$ we obtain

$$\begin{aligned} \|\mathcal{P}u+\mathcal{B}u-\underline{\mathcal{P}}u-\underline{\mathcal{B}}u\|_{L_2^\omega(S;H^{-1}(G))} \\ \le c_1 d\big((A,b,b_0,b_\Gamma),(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)\big)\,\|u\|_{W_E^\omega(S;H_0^1(G))}, \end{aligned} \tag{7.2}$$

where $c_1 = c_1(n,\varepsilon,\omega,S,G) > 0$ is some constant. Therefore, for every fixed set $(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)$ of admissible coeffcients we find some constant $\delta > 0$ such that for all admissible coefficients (A,b,b_0,b_Γ) which satisfy

$$d\big((A,b,b_0,b_\Gamma),(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)\big) < \delta, \tag{7.3}$$

the following relation holds true:

$$2\,\|(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|_{\mathcal{L}(L_2^\omega;W_E^\omega)}\|\mathcal{P}+\mathcal{B}-\underline{\mathcal{P}}-\underline{\mathcal{B}})\|_{\mathcal{L}(W_E^\omega;L_2^\omega)} < 1.$$

Using the identities

$$\begin{aligned} \mathcal{P}+\mathcal{B} = (\underline{\mathcal{P}}+\underline{\mathcal{B}})(\mathcal{I}-(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}(\underline{\mathcal{P}}+\underline{\mathcal{B}}-\mathcal{P}-\mathcal{B})), \\ (\mathcal{P}+\mathcal{B})^{-1}-(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1} = (\mathcal{P}+\mathcal{B})^{-1}(\underline{\mathcal{P}}+\underline{\mathcal{B}}-\mathcal{P}-\mathcal{B})(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}, \end{aligned}$$

for all admissible coefficients (A,b,b_0,b_Γ) which satisfy (7.3) the above estimates and the von Neumann expansion leads to

$$\|(\mathcal{P}+\mathcal{B})^{-1}\|_{\mathcal{L}(L_2^\omega;W_E^\omega)} \le 2\,\|(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|_{\mathcal{L}(L_2^\omega;W_E^\omega)}$$

and, consequently,

$$\|(\mathcal{P}+\mathcal{B})^{-1}-(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|_{\mathcal{L}(L_2^\omega;W_E^\omega)} \leq 2\,\|(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|^2_{\mathcal{L}(L_2^\omega;W_E^\omega)}\|\underline{\mathcal{P}}+\underline{\mathcal{B}}-\mathcal{P}-\mathcal{B}\|_{\mathcal{L}(W_E^\omega;L_2^\omega)}.$$

Applying (7.2) we end up with the desired estimate

$$\|(\mathcal{P}+\mathcal{B})^{-1}-(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|_{\mathcal{L}(L_2^\omega;W_E^\omega)} \leq 2c_1 d\big((A,b,b_0,b_\Gamma),(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)\big)\,\|(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|^2_{\mathcal{L}(L_2^\omega;W_E^\omega)}$$

for all admissible coefficients (A,b,b_0,b_Γ) which satisfy (7.3). □

Unbounded lower-order coefficients. It turns out that for the most interesting range of parameters $n<\omega<\bar{\omega}_\varepsilon(G)$ the above results for the parabolic operator $\mathcal{P}+\mathcal{B}$ remain true under weaker assumptions on the lower-order coefficients. Corresponding to [12, Theorem 5.6] it is sufficient to suppose that

$$b\in L_2^\omega(S;L^2(G^\circ;\mathbb{R}^n)),\quad b_0\in L_2^{\omega-2}(S;L^2(G^\circ)),\quad b_\Gamma\in L_2^{\omega-1}(S;L^2(\Gamma)).$$

Lemma 7.4 (Complete continuity)**.** *For every $\omega\in(n,n+2]$ the restriction $\mathcal{B}_\omega$ of $\mathcal{B}$ to $\{u\in W_E^\omega(S;H_0^1(G)) : u(t_0)=0\}$ is a completely continuous map into $L_2^\omega(S;H^{-1}(G))$.*

Proof. Let $\omega\in(n,n+2]$ be fixed and take $\sigma\in(n,\omega)$. Then the embedding from $W_E^\omega(S;H_0^1(G))$ into $\mathfrak{L}_2^{\sigma+2}(S;L^2(G^\circ))$ and the trace map $\mathcal{K}_{S,\Gamma}$ from $W_E^\omega(S;H_0^1(G))$ into $\mathfrak{L}_2^{\sigma+1}(S;L^2(\Gamma))$ are completely continuous (see [12, Theorem 6.9 and Theorem 6.12]). Following [12, Remark 3.2 and Remark 3.5] and [12, Theorem 3.4, Theorem 3.6, and Theorem 5.6] this yields that the operator $\mathcal{B}_\omega$ is a completely continuous map from $\{u\in W_E^\omega(S;H_0^1(G)) : u(t_0)=0\}$ into $L_2^\omega(S;H^{-1}(G))$. □

Theorem 7.5 (Maximal regularity)**.** *Let $\varepsilon\in(0,1]$ and $n<\omega<\bar{\omega}_\varepsilon(G)$ be given constants. For every pair (a,A) of leading coefficients being ε-definite with respect to S and G° and all lower-order coefficients*

$$b\in L_2^\omega(S;L^2(G^\circ;\mathbb{R}^n)),\quad b_0\in L_2^{\omega-2}(S;L^2(G^\circ)),\quad b_\Gamma\in L_2^{\omega-1}(S;L^2(\Gamma)),$$

$\mathcal{P}_\omega+\mathcal{B}_\omega$ is a linear isomorphism from $\{u\in W_E^\omega(S;H_0^1(G)) : u(t_0)=0\}$ onto $L_2^\omega(S;H^{-1}(G))$.

Proof. 1. Let $n<\omega<\bar{\omega}_\varepsilon(G)$ be given. Since $\mathcal{P}_\omega$ is an isomorphism between $\{u\in W_E^\omega(S;H_0^1(G)) : u(t_0)=0\}$ and $L_2^\omega(S;H^{-1}(G))$ (see Theorem 6.8) and $\mathcal{B}_\omega$ is completely continuous from $\{u\in W_E^\omega(S;H_0^1(G)) : u(t_0)=0\}$ into

$L_2^\omega(S; H^{-1}(G))$ (see Lemma 7.4), the sum $\mathcal{P}_\omega + \mathcal{B}_\omega$ is a Fredholm operator of index zero between these spaces. Hence, it suffices to prove the injectivity of the linear operator $\mathcal{P}_\omega + \mathcal{B}_\omega$.

2. Suppose that $u \in W_E^\omega(S; H_0^1(G))$ is a solution of the homogeneous initial-value problem

$$\mathcal{P}u + \mathcal{B}u = 0, \quad u(t_0) = 0. \tag{7.4}$$

For fixed $t_1 \in S$ we consider the subinterval $S_1 = (t_0, t_1)$ of S, the restriction $u_1 \in W_E^\omega(S_1; H_0^1(G))$ of u and the restriction $f_1 \in L_2^\omega(S_1; H^{-1}(G))$ of $\mathcal{B}u \in L_2^\omega(S; H^{-1}(G))$. Due to Remark 6.2 we get

$$\|u_1\|_{W_E^\omega(S_1;H_0^1(G))} \le c_1 \|f_1\|_{L_2^\omega(S_1;H^{-1}(G))}, \tag{7.5}$$

where the constant $c_1 > 0$ may depend on S but not on t_1. To estimate the right-hand side of (7.5) we use [12, Theorem 6.8 and Theorem 6.11] and [12, Remark 3.2, Remark 3.5, and Theorem 5.6] to find a constant $c_2 = c_2(n, G) > 0$ such that

$$\|f_1\|_{L_2^\omega(S_1;H^{-1}(G))} \le c_2 c_{\mathcal{B}} \|u_1\|_{C(\overline{S_1};C(\overline{G}))}, \tag{7.6}$$

where $c_{\mathcal{B}} > 0$ is given by

$$c_{\mathcal{B}}^2 = \|b\|^2_{L_2^\omega(S;L^2(G^\circ;\mathbb{R}^n))} + \|b_0\|^2_{L_2^{\omega-2}(S;L^2(G^\circ))} + \|b_\Gamma\|^2_{L_2^{\omega-1}(S;L^2(\Gamma))}.$$

To estimate the left-hand side of (7.5) we consider the interval $S_0 = (t_1 + t_0 - t_\ell, t_1)$, which contains S_1 and has the same length as S, and we define the zero extension $u_0 \in W_E^\omega(S_0; H_0^1(G))$ by

$$u_0(s) = \begin{cases} u(s) & \text{if } s \in S_1, \\ 0 & \text{otherwise.} \end{cases}$$

In view of the continuity of the embedding from $W_E^\omega(S_0; H_0^1(G))$ into the Hölder space $C^{0,\alpha}(\overline{S_0}; C(\overline{G}))$ for $\alpha = (\omega - n)/4$ (see [12, Theorem 3.4 and Theorem 6.8]) and the definition of the norms in the corresponding Morrey and Hölder spaces, the above construction yields

$$\begin{aligned} \|u_1\|_{C^{0,\alpha}(\overline{S_1};C(\overline{G}))} &\le \|u_0\|_{C^{0,\alpha}(\overline{S_0};C(\overline{G}))} \\ &\le c_3 \|u_0\|_{W_E^\omega(S_0;H_0^1(G))} = c_3 \|u_1\|_{W_E^\omega(S_1;H_0^1(G))}, \end{aligned}$$

where the constant $c_3 > 0$ may depend on S but not on t_1. Together with (7.5) and (7.6) this leads to the key estimate

$$\|u_1\|_{C^{0,\alpha}(\overline{S_1};C(\overline{G}))} \le c_4 \|u_1\|_{C(\overline{S_1};C(\overline{G}))}, \tag{7.7}$$

where the constant $c_4 = c_1 c_2 c_3 c_{\mathcal{B}} > 0$ does not depend on t_1.

3. Because $t_1 \in S$ was arbitrarily fixed at the beginning we may choose

$$t_k = t_0 + \tfrac{k}{\ell}(t_\ell - t_0) \quad \text{for } k \in \{1, \dots, \ell\},$$

where $\ell \in \mathbb{N}$, $\ell > 1$ is large enough to satisfy the condition

$$2c_4(t_\ell - t_0)^\alpha < \ell^\alpha. \tag{7.8}$$

Furthermore, for $k \in \{1, \dots, \ell\}$ we introduce the intervals $S_k = (t_{k-1}, t_k)$ and the restrictions $u_k \in W_E^\omega(S_k; H_0^1(G))$ of $u \in W_E^\omega(S; H_0^1(G))$.

We prove that for every $k \in \{1, \dots, \ell - 1\}$ from $u(t_{k-1}) = 0$ it follows that $u(s) = 0$ for all $s \in \overline{S_k}$. To do so, we proceed by induction: Starting from $k = 1$ and using (7.7), condition (7.8) ensures that for all $s \in \overline{S_1}$ we have

$$\|u(s) - u(t_0)\|_{C(\overline{G})} \le (s - t_0)^\alpha \|u_1\|_{C^{0,\alpha}(\overline{S_1}; C(\overline{G}))} \le \tfrac{1}{2}\|u_1\|_{C(\overline{S_1}; C(\overline{G}))}.$$

Since $u(t_0) = 0$ this leads to $u(s) = 0$ for all $s \in \overline{S_1}$.

Assuming that $u(t_{k-1}) = 0$ holds true for some $k \in \{1, \dots, \ell - 1\}$, we apply (7.7) and (7.8) to $u_k \in W_E^\omega(S_k; H_0^1(G))$ to get

$$\|u(s) - u(t_{k-1})\|_{C(\overline{G})} \le (s - t_{k-1})^\alpha \|u_k\|_{C^{0,\alpha}(\overline{S_k}; C(\overline{G}))} \le \tfrac{1}{2}\|u_k\|_{C(\overline{S_k}; C(\overline{G}))}$$

for all $s \in \overline{S_k}$. Therefore, $u(t_{k-1}) = 0$ yields $u(s) = 0$ for all $s \in \overline{S_k}$.

Hence, we have proved that $u = 0$ is the unique solution of the homogeneous problem (7.4) in the space $W_E^\omega(S; H_0^1(G))$. Following Step 1, the linear operator $\mathcal{P}_\omega + \mathcal{B}_\omega$ is an injective Fredholm operator of index zero and, consequently, a linear isomorphism between $\{u \in W_E^\omega(S; H_0^1(G)) : u(t_0) = 0\}$ and $L_2^\omega(S; H^{-1}(G))$. □

Theorem 7.6 (Continuous dependence)**.** *Let $\varepsilon \in (0, 1]$ and $n < \omega < \bar{\omega}_\varepsilon(G)$ be given constants. Then, for every pair (a, A) of leading coefficients being ε-definite with respect to S and G° and all lower-order coefficients*

$$b \in L_2^\omega(S; L^2(G^\circ; \mathbb{R}^n)), \quad b_0 \in L_2^{\omega-2}(S; L^2(G^\circ)), \quad b_\Gamma \in L_2^{\omega-1}(S; L^2(\Gamma)),$$

the assignment $(A, b, b_0, b_\Gamma) \mapsto (\mathcal{P} + \mathcal{B})^{-1}$ is a continuous map from the metric space of admissible coefficients equipped with the metric d, defined by

$$\begin{aligned} d\big((A, b, b_0, b_\Gamma), (\underline{A}, \underline{b}, \underline{b}_0, \underline{b}_\Gamma)\big) &= \|A - \underline{A}\|_{L^\infty(S; L^\infty(G^\circ; \mathbb{S}^n))} + \|b - \underline{b}\|_{L_2^\omega(S; L^2(G^\circ; \mathbb{R}^n))} \\ &\quad + \|b_0 - \underline{b}_0\|_{L_2^{\omega-2}(S; L^2(G^\circ))} + \|b_\Gamma - \underline{b}_\Gamma\|_{L_2^{\omega-1}(S; L^2(\Gamma))}, \end{aligned}$$

into the Banach space $\mathcal{L}(L_2^\omega(S; H^{-1}(G)); W_E^\omega(S; H_0^1(G)))$ of solution maps corresponding to problem (7.1).

Proof. Consider the maps $\mathcal{P}$, $\mathcal{B}$, $\underline{\mathcal{P}}$, and $\underline{\mathcal{B}}$ that are associated with the sets (a,A,b,b_0,b_Γ) and $(a,\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)$ of admissible coefficients, respectively. By the same arguments as in the proof of Lemma 7.4 for all $u \in W_E^\omega(S;H_0^1(G))$ we get

$$\|\mathcal{P}u + \mathcal{B}u - \underline{\mathcal{P}}u - \underline{\mathcal{B}}u\|_{L_2^\omega(S;H^{-1}(G))} \le c_1 d\big((A,b,b_0,b_\Gamma),(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)\big)\,\|u\|_{W_E^\omega(S;H_0^1(G))},$$

where $c_1 = c_1(n,\varepsilon,\omega,S,G) > 0$ is some constant. Hence, for every fixed set $(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)$ of admissible coefficients there exists a constant $\delta > 0$ such that for all admissible coefficients (A,b,b_0,b_Γ) which satisfy

$$d\big((A,b,b_0,b_\Gamma),(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)\big) < \delta, \tag{7.9}$$

the relation

$$2\,\|(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|_{\mathcal{L}(L_2^\omega;W_E^\omega)}\|\mathcal{P}+\mathcal{B}-\underline{\mathcal{P}}-\underline{\mathcal{B}})\|_{\mathcal{L}(W_E^\omega;L_2^\omega)} < 1$$

holds true. Now we repeat exactly the same arguments as in the proof of Theorem 7.3 to get the estimate

$$\|(\mathcal{P}+\mathcal{B})^{-1} - (\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|_{\mathcal{L}(L_2^\omega;W_E^\omega)} \le 2c_1 d\big((A,b,b_0,b_\Gamma),(\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)\big)\,\|(\underline{\mathcal{P}}+\underline{\mathcal{B}})^{-1}\|^2_{\mathcal{L}(L_2^\omega;W_E^\omega)}$$

for all admissible coefficients (A,b,b_0,b_Γ) which satisfy (7.9). ☐

Remark 7.1. All the results can be generalized to weakly coupled systems, which means to problems with principal parts $\mathcal{E}$ and $\mathcal{A}$ of diagonal structure and operators $\mathcal{B}$ containing strongly coupled lower-order terms.

Remark 7.2. One problem left open is the continuous dependence of the solution $u \in W_E^\omega(S;H_0^1(G))$ to problem (7.1) on the ε-definite capacity coefficient a. Since the space $W_E^\omega(S;H_0^1(G))$ depends highly sensitive on that coefficient, a result in the spirit of Theorem 7.6 cannot be true in general. Nevertheless, it would be interesting to know whether the quantity

$$\|(\mathcal{E}u)' - (\underline{\mathcal{E}}\underline{u})'\|_{L_2^\omega(S;H^{-1}(G))} + \|u-\underline{u}\|_{L_2^\omega(S;H_0^1(G))}$$

can be estimated in terms of $\|f-\underline{f}\|_{L_2^\omega(S;H^{-1}(G))}$ and the modified distance

$$\begin{aligned} d\big((a,A,b,b_0,b_\Gamma),&(\underline{a},\underline{A},\underline{b},\underline{b}_0,\underline{b}_\Gamma)\big)\\ &= \|a-\underline{a}\|_{L^\infty(G^\circ)} + \|A-\underline{A}\|_{L^\infty(S;L^\infty(G^\circ;\mathbb{S}^n))} + \|b-\underline{b}\|_{L_2^\omega(S;L^2(G^\circ;\mathbb{R}^n))}\\ &\quad + \|b_0-\underline{b}_0\|_{L_2^{\omega-2}(S;L^2(G^\circ))} + \|b_\Gamma-\underline{b}_\Gamma\|_{L_2^{\omega-1}(S;L^2(\Gamma))}, \end{aligned}$$

defined for admissible coefficients (a, A, b, b_0, b_Γ) and $(\underline{a}, \underline{A}, \underline{b}, \underline{b}_0, \underline{b}_\Gamma)$. Here, $u \in W^\omega_E(S; H^1_0(G))$ and $\underline{u} \in W^\omega_{\underline{E}}(S; H^1_0(G))$ are solutions to the problems

$$(\mathcal{E}u)' + \mathcal{A}u + \mathcal{B}u = f \in L^\omega_2(S; H^{-1}(G)), \quad u(t_0) = 0,$$
$$(\underline{\mathcal{E}u})' + \underline{\mathcal{A}u} + \underline{\mathcal{B}u} = \underline{f} \in L^\omega_2(S; H^{-1}(G)), \quad \underline{u}(t_0) = 0.$$

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