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ELLIPTIC EQUATIONS WITH COMPETING RAPIDLY VARYING NONLINEARITIES AND BOUNDARY BLOW-UP

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Abstract. We prove that $\Delta u + au = b(x)f(u)$ possesses a unique positive solution such that $\lim_{\mathrm{dist}(x,\partial\Omega)\to 0} u(x) = \infty$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $a \in \mathbb{R}$. Here b is a smooth function on $\overline{\Omega}$ which is positive in Ω and may vanish on $\partial\Omega$ (possibly at a very degenerate rate such as $\exp(-[\mathrm{dist}(x,\partial\Omega)]^q)$ with $q < 0$). We assume that f is locally Lipschitz continuous on $[0,\infty)$ with $f(u)/u$ increasing for $u > 0$ and $f(u)$ grows at ∞ faster than any power u^p ($p > 1$). As a distinct feature of this study appears the asymptotic behaviour of the boundary blow-up solution, which breaks up depending on how $b(x)$ vanishes on $\partial\Omega$ and how fast f grows at ∞.

1. Introduction and main results

In this paper we consider semilinear elliptic equations of the form

$$\Delta u + au = b(x)f(u) \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 2$, a is a real parameter and $b \in C^{0,\mu}(\overline{\Omega})$, for some $\mu \in (0,1)$, satisfies $b > 0$ in Ω and $b \equiv 0$ on $\partial\Omega$. With respect to the nonlinearity f, we assume throughout that

$$f \geq 0,\ f \in \mathrm{Lip}_{\mathrm{loc}}[0,\infty) \text{ and } f(u)/u \text{ is increasing on } (0,\infty), \tag{A}$$

where $f \in \mathrm{Lip}_{\mathrm{loc}}[0,\infty)$ means that f is locally Lipschitz continuous on $[0,\infty)$.

A non-negative $C^2(\Omega)$-solution of (1.1) is called a *blow-up* (or *large*) solution if it holds that

$$\lim_{d(x)\to 0} u(x) = +\infty, \quad \text{where } d(x) := \mathrm{dist}(x,\partial\Omega).$$

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Blow-up solutions to equations of the form (1.1) have been extensively studied (especially with $f(u) = u^p$ with $p > 1$) in connection with problems arising in Riemannian geometry, mathematical physics or population dynamics. We refer, for instance, to [3], [8], [9, 12], [13], [15], [16, 17], [20], [21], and [30, 31, 32, 33, 36, 39].

Our aim is to investigate the uniqueness and asymptotic behaviour of the boundary blow-up solutions of (1.1) when $f(u)$ grows faster than any u^p $(p > 1)$ at infinity.

The study of blow-up solutions has been initiated by Bieberbach [5] for the equation $\Delta u = e^u$ in a smooth bounded domain Ω of $\mathbb{R}^2$. He showed that this equation admits a unique positive solution $u \in C^2(\Omega)$ such that $u(x) - \ln(d(x)^{-2})$ is bounded as $d(x) \to 0$. Motivated by a problem from mathematical physics, Rademacher [35] continued this study on smooth bounded domains in $\mathbb{R}^3$. Later, Lazer and McKenna [27] generalized the results of Bieberbach [5] and Rademacher [35] to the case of bounded domains in $\mathbb{R}^N$ satisfying a uniform external sphere condition and for nonlinearities $b(x)e^u$, where b is continuous and positive on $\overline{\Omega}$. Moreover, it is proved in [28] that the blow-up solution of $\Delta u = e^u$ in Ω satisfies

$$u(x) = \ln(2[d(x)]^{-2}) + o(1) \quad \text{as } d(x) \to 0.$$

If $b > 0$ on $\overline{\Omega}$ and f satisfies the following Keller–Osserman condition (see [26], [34]),

$$\Xi(c) := \int_c^\infty \frac{dt}{\sqrt{2F(t)}} < \infty, \quad \text{for every } c > 0, \quad \text{where } F(t) = \int_0^t f(s)\, ds, \tag{1.2}$$

then, for every $a \in \mathbb{R}$, the equation (1.1) has at least one large solution u_a. Furthermore, the blow-up rate of $u_a(x)$ near $\partial\Omega$ can be described by (see, e.g., Theorem 6.8 in [18])

$$\frac{\Xi(u_a(x))}{\sqrt{b(x)}d(x)} \to 1 \quad \text{as } d(x) \to 0.$$

Under the above assumptions on b and f, if one assumes further that

$$\liminf_{u\to\infty} \frac{\Xi(\lambda u)}{\Xi(u)} > 1, \quad \text{for every } \lambda \in (0,1), \tag{1.3}$$

then (1.1) admits only one blow-up solution, since any such solution u_a satisfies

$$\frac{u_a(x)}{\Upsilon(\sqrt{b(x)}d(x))} \to 1 \quad \text{as } d(x) \to 0, \tag{1.4}$$

where Υ is the inverse of Ξ defined by (1.2) (see [1], [2], [4], and [29]).

The main purpose of this paper is to describe the asymptotic behaviour of the blow-up solutions of (1.1) when $b \equiv 0$ on $\partial\Omega$ and f grows faster than any power function u^p ($p > 1$). More precisely, we are concerned with nonlinearities f which are Γ-varying at ∞. Following [37], we say that a non-decreasing function f defined on an interval (A, ∞) is Γ-*varying at* ∞ if $\lim_{u\to\infty} f(u) = \infty$ and there exists $\chi : (A, \infty) \to (0, \infty)$ such that

$$\lim_{u\to\infty} \frac{f(u + \lambda\chi(u))}{f(u)} = e^{\lambda}, \quad \text{for every } \lambda \in \mathbb{R}.$$

The function χ is called an *auxiliary function* and is unique up to asymptotic equivalence.

The Γ-class was introduced by de Haan [14] in connection with extreme value theory.

We see that e^u is Γ-varying at ∞ with the auxiliary function $\chi = 1$. See Example 2.1 in Section 2.2 for other functions that are Γ-varying at ∞.

Note that if f is Γ-varying at ∞ and (A) holds, then (1.2) and (1.3) are automatically verified. Hence, if $b > 0$ on $\overline{\Omega}$, then (1.1) possesses a unique large solution u_a that satisfies (1.4). In sharp contrast, our results show that the hypothesis $b = 0$ on $\partial\Omega$ plays a dividing role for the description of the blow-up rate of the large solutions of (1.1).

We next explain our assumptions on $b(x)$. Let $\mathcal{K}_\ell$ denote (as in [10]) the set of all positive, non-decreasing C^1-functions k defined on $(0, \tau)$, for some $\tau > 0$, which satisfy

$$\lim_{t\to 0^+} \Big(\frac{K(t)}{k(t)}\Big)' =: \ell, \quad \text{where } K(t) := \int_0^t k(s)\, ds. \tag{1.5}$$

It is easy to see that $\ell \in [0, 1]$ and $\lim_{t\to 0^+} K(t)/k(t) = 0$ for every $k \in \mathcal{K}_\ell$. For further details on $\mathcal{K}_\ell$ see Corollary 3.1.

The following functions k belong to $\mathcal{K}_\ell$ with the specified ℓ in each case:

(i) $k(t) = t^q$ for $q \geq 0$ with $\ell = 1/(1 + q)$;
(ii) $k(t) = (-\ln t)^q$ for $q < 0$ with $\ell = 1$;
(iii) $k(t) = \exp(-t^q)$ or $k(t) = \exp(-\exp(t^q))$ for $q < 0$ with $\ell = 0$.

We assume that there exist $k \in \mathcal{K}_\ell$ and some positive constants γ^-, γ^+, and δ such that

$$\gamma^- k^2(d(x)) \leq b(x) \leq \gamma^+ k^2(d(x)) \quad \text{for every } x \in \Omega \text{ with } d(x) \leq 2\delta. \tag{1.6}$$

We will also need the following technical condition:

$$\text{Either} \quad \text{(a)} \lim_{t\to 0^+} \mathcal{N}'(t)\ln K(t) = j \in \mathbb{R} \quad \text{or} \quad \text{(b)} \lim_{t\to 0^+} t\mathcal{N}'(t)/\mathcal{N}(t) = 1, \tag{1.7}$$

where we define $\mathcal{N}(t) := K(t)/k(t)$ for small $t > 0$.

If (1.7) (a) holds, then $j \leq -1$. There are two different model cases for (1.7) (a) corresponding to $j < -1$ and $j = -1$, respectively:

(1) $k(t) = \exp(-t^q)$ with $q < 0$ when $j = -1 + 1/q$;

(2) $k(t) = \exp(-\exp(t^q))$ for $q < 0$ when $j = -1$.

Lemma 3.3 describes precisely all the functions $k \in \mathcal{K}_\ell$ for which (1.7) (a) is satisfied.

A typical example of $k \in \mathcal{K}_0$ satisfying (1.7) (b) corresponds to $K(t) = e^{-(-\ln t)^\alpha}$, $\alpha > 1$. Fortunately, (1.7) (b) is always satisfied when $k \in \mathcal{K}_\ell$ with $\ell \neq 0$, since $\mathcal{N}(t)/t \sim \mathcal{N}'(t) \sim \ell$ as $t \to 0^+$. By $f_1(t) \sim f_2(t)$ as $t \to t_0 \in \overline{\mathbb{R}}$ we mean that $\lim_{t\to t_0} f_1(t)/f_2(t) = 1$.

If (A) holds, then by Theorem 1.28 in [22] we have f is Γ-varying at ∞ if and only if

$$\begin{cases} f(u) \sim \widehat{f}(u) = e^{\int_B^u \frac{dy}{S(y)}} \text{ as } u \to \infty, \text{ where } B > 0 \text{ is a constant and} \\ S \text{ is a positive } C^1\text{-function on } [B, \infty) \text{ such that } \lim\limits_{u\to\infty} S'(u) = 0. \end{cases} \tag{B}$$

Note that $S(u)$ is the auxiliary function of the Γ-varying function f; see Lemma 3.4.

We shall assume (B) and that S satisfies the following additional condition:

$$\text{Either} \quad \text{(a)} \lim_{u\to\infty} S'(u)\ln \widehat{f}(u) = \vartheta \in \mathbb{R} \quad \text{or} \quad \text{(b)} \lim_{u\to\infty} uS'(u)/S(u) = 1. \tag{1.8}$$

If (1.8) (a) is satisfied, then $\vartheta \geq -1$. Examples of functions satisfying (B) and (1.8) are provided by Corollary 1.1 (see Remark 1.4). Note that the cases (a) and (b) in (1.8) (respectively (a) and (b) in (1.7)) are really complementary (cf. Remark 1.1).

Our main results will prove that when f grows faster than any u^p $(p > 1)$, then the vanishing rate of b at $\partial\Omega$ enters into competition with the growth of f at ∞. This produces a split in the asymptotic behaviour of the large solution near $\partial\Omega$, which can be determined when a suitable combination between (1.7) and (1.8) is available.

We first analyse the situation that (1.7) (a) and (1.8) hold.

Theorem 1.1. *Let* (A), (B), (1.6), (1.7) (a) *and* (1.8) *hold. If* (1.8) (a) *is satisfied, then we further assume that* j *and* ϑ *are not both* -1.

Then, for every $a \in \mathbb{R}$, (1.1) *has a unique blow-up solution* u_a, *which satisfies*

$$u_a(x)/\mathcal{E}_0(d(x)) \to 1 \quad \text{as } d(x) \to 0, \tag{1.9}$$

where $\mathcal{E}_0$ *is uniquely defined on some small interval* $(0, \tau)$ *as follows:*

$$\int_{\mathcal{E}_0(t)}^{\infty} \frac{dy}{\widehat{f}(y)\mathcal{Q}(y)} = K^2(t) \quad \text{and} \quad \mathcal{Q}(y) := \begin{cases} \ln \widehat{f}(y) & \text{if (1.8) (a) holds,} \\ y/S(y) & \text{if (1.8) (b) holds.} \end{cases} \tag{1.10}$$

Remark 1.1. Lemma 3.3 shows that (1.7) (a) is valid with $j \neq -1$ if and only if $\lim_{t\to 0^+} t\mathcal{N}'(t)/\mathcal{N}(t) = j/(1+j) \in (1, \infty)$. Similarly, we infer that (1.8) (a) holds for $\vartheta \in (-1, \infty)$ if and only if there exists $\lim_{u\to\infty} uS'(u)/S(u) = \xi \in (-\infty, 1)$, where $\xi = \vartheta/(1 + \vartheta)$. Moreover, if (1.7) (a) is satisfied with $j = -1$ (respectively $\vartheta = -1$ in (1.8) (a)), then $\lim_{t\to 0^+} t\mathcal{N}'(t)/\mathcal{N}(t) = \infty$ (respectively $\lim_{u\to\infty} uS'(u)/S(u) = -\infty$).

Next we want to see what happens to Theorem 1.1 when (1.7) (b) is valid instead of (1.7) (a). A partial answer to this is given by Theorem 1.2 in [7], where it is assumed that (1.6) holds for some $k \in \mathcal{K}_\ell$ with $\ell \neq 0$ (hence, (1.7) (b) is automatically satisfied). However, the case that $\ell = 0$ was left open in [7]. Our next result addresses this gap.

Theorem 1.2. *Let* (A), (B) *and* (1.6) *hold. If* $\ell = 0$, *then we further assume that* (1.7) (b) *and* (1.8) (a) *are satisfied. Then, for each* $a \in \mathbb{R}$, (1.1) *has a unique blow-up solution* u_a, *which satisfies*

$$u_a(x)/\mathcal{E}(d(x)) \to 1 \quad \text{as } d(x) \to 0, \tag{1.11}$$

where $\mathcal{E}$ *is uniquely defined on some small interval* $(0, \tau)$ *as follows:*

$$\int_{\mathcal{E}(t)}^{\infty} \frac{dy}{\widehat{f}(y)} = tk(t)K(t). \tag{1.12}$$

Remark 1.2. The main novelty of Theorem 1.2 corresponds to the case $\ell = 0$, and an example that could be kept in mind is $k(t) = e^{-(-\ln t)^\alpha}$ with $\alpha > 1$ (see also Remark 1.7). However, even when $\ell \neq 0$, we proceed in a different way than in [7] by providing the asymptotic formula of u_a near $\partial\Omega$ in the form of (1.11) that applies to *every* function k satisfying (1.7) (b) (possibly restricting f if $\ell = 0$).

Remark 1.3. In the framework of Theorem 1.2, the blow-up solution u_a of (1.1) satisfies

$$\frac{u_a(x)}{\Upsilon(\sqrt{d(x)k(d(x))K(d(x))})} \to 1 \quad \text{as } d(x) \to 0, \tag{1.13}$$

where Υ is the inverse of Ξ defined by (1.2). This follows by applying Theorem 1.2 with $k = 1$ and by comparing (1.11) with (1.4) when $b = 1$ on $\partial\Omega$.

In many cases, the asymptotic behaviour of u_a in Theorem 1.1 can be obtained without computing precisely $\mathcal{E}_0$ in (1.10), based upon Remark 5.2. We illustrate below this point, as well as the applicability of Theorem 1.1 via Corollaries 1.1 and 1.2.

Corollary 1.1 (Case $j \neq -1$)**.** *Assume that* (A) *holds and there exist some positive constants* c_1*,* c_2 *and* δ *such that, for every* $x \in \Omega$ *with* $d(x) \leq \delta$*,*

$$c_1[d(x)]^m e^{-\beta[d(x)]^q} \leq b(x) \leq c_2[d(x)]^m e^{-\beta[d(x)]^q}, \tag{1.14}$$

where $m \in \mathbb{R}$*,* $\beta > 0$ *and* $q < 0$*.*

In the following cases, (1.1) *admits a unique blow-up solution* u_a*, for every* $a \in \mathbb{R}$*:*

(1a) $f(u) \sim (\underbrace{\exp \circ \ldots \circ \exp}_{n \text{ times}})(u^p)$ *as* $u \to \infty$*, where* $p > 0$ *and* $n \geq 2$ *is an integer; furthermore, as* $d(x) \to 0$ *we have*

$$u_a(x) \sim \begin{cases} [q \ln d(x)]^{1/p} & \text{if } n = 2, \\ \{(\underbrace{\ln \circ \ldots \circ \ln}_{(n-1) \text{ times}})[1/d(x)]\}^{1/p} & \text{if } n \geq 3. \end{cases}$$

(1b) (i) $f(u) \sim e^{\gamma u^p}$ *as* $u \to \infty$*, where* γ*,* $p > 0$*; moreover, we have*

$$u_a(x) \sim (\beta/\gamma)^{1/p} [d(x)]^{q/p} \quad \text{as } d(x) \to 0.$$

(ii) $f(u) \sim e^{u \ln u}$ *as* $u \to \infty$*; in this case,*

$$u_a(x) \sim \frac{\beta}{q} \frac{[d(x)]^q}{\ln d(x)} \quad \text{as } d(x) \to 0.$$

(1c) (i) $f(u) \sim e^{(\ln u)^\gamma}$ *as* $u \to \infty$*, for some* $\gamma > 1$*; furthermore,*

$$\ln u_a(x) \sim \beta^{1/\gamma} [d(x)]^{q/\gamma} \quad \text{as } d(x) \to 0.$$

(ii) $f(e^{\frac{\ln u}{\ln\ln u}}) \sim Cu$ *as* $u \to \infty$, *for some* $C > 0$*; we have*

$$\ln u_a(x) \sim \frac{\beta}{q} \frac{[d(x)]^q}{\ln d(x)} \quad \text{as } d(x) \to 0.$$

Remark 1.4. Note that (1c) above illustrates (B) and (1.8) (b), while for the rest of the Corollary 1.1 the assumptions (B) and (1.8) (a) hold (with $\vartheta = -1$ in (1a), $\vartheta = (1-p)/p$ in (1b) (i) and $\vartheta = 0$ in (1b) (ii)).

Corollary 1.2 (Case $j = -1$)**.** *Assume that* (A) *holds and there exist some positive constants* c_1*,* c_2 *and* δ *such that, for every* $x \in \Omega$ *with* $d(x) \le \delta$*,*

$$c_1[d(x)]^m e^{\tau[d(x)]^{\widetilde{m}}} e^{-\beta e^{\sigma[d(x)]^q}} \le b(x) \le c_2[d(x)]^m e^{\tau[d(x)]^{\widetilde{m}}} e^{-\beta e^{\sigma[d(x)]^q}}, \quad (1.15)$$

where m*,* $\widetilde{m}$*,* τ*,* β*,* σ *and* q *are real constants such that* $\beta, \sigma > 0$ *and* $q < 0$*.*

In the following cases, (1.1) *admits a unique blow-up solution* u_a*, for every* $a \in \mathbb{R}$*:*

(i) $f(u) \sim e^{\gamma u^p}$ *as* $u \to \infty$*, where* γ*,* $p > 0$*; moreover, it holds that*

$$u_a(x) \sim \left(\frac{\beta}{\gamma}\right)^{1/p} e^{\frac{\sigma}{p}[d(x)]^q} \quad \text{as } d(x) \to 0.$$

(ii) $f(u) \sim e^{u \ln u}$ *as* $u \to \infty$*; in this case, we have*

$$u_a(x) \sim \frac{\beta}{\sigma}[d(x)]^{-q} e^{\sigma[d(x)]^q} \quad \text{as } d(x) \to 0.$$

(iii) $f(u) \sim e^{(\ln u)^\gamma}$ *as* $u \to \infty$*, for some* $\gamma > 1$*; in addition,*

$$\ln u_a(x) \sim \beta^{\frac{1}{\gamma}} e^{\frac{\sigma}{\gamma}[d(x)]^q} \quad \text{as } d(x) \to 0.$$

(iv) $f(e^{\frac{\ln u}{\ln\ln u}}) \sim Cu$ *as* $u \to \infty$*, for some* $C > 0$*; furthermore,*

$$\ln u_a(x) \sim \frac{\beta}{\sigma}[d(x)]^{-q} e^{\sigma[d(x)]^q} \quad \text{as } d(x) \to 0.$$

Remark 1.5. The hypotheses (1.6) and (1.7) (a) are illustrated by (1.14) with $j = -1 + 1/q$ (take k as in (7.1)) and (1.15) with $j = -1$ (choose k as in (7.2)).

Corollaries 1.1 and 1.2 underline a difference in the variation speed of the blow-up solution. To make this more precise, we need some concepts from the theory of regular variation (initiated by Karamata [24, 25]) and its extensions (due to de Haan [14]); see Section 2 for more details.

Let R be a positive measurable function on $[A,\infty)$. We say that R is *regularly varying at ∞ of index $q \in \mathbb{R}$* (and write $R \in RV_q$), if

$$\lim_{u\to\infty} \frac{R(\lambda u)}{R(u)} = \lambda^q \quad \text{for every } \lambda > 0. \tag{1.16}$$

When the index q is zero, then R is called a *slowly varying* function (see [6], [38], and [37]).

Clearly, if $R \in RV_q$, then $L(u) := R(u)/u^q$ is a slowly varying function. Some examples of slowly varying functions are

(1) Any measurable function on $[A,\infty)$ which has a positive limit at infinity.
(2) The logarithm $\ln u$, its iterates $\ln_m u$ (given by $\ln \ln_{m-1} u$) and powers of $\ln_m u$.
(3) $\exp\{\frac{\ln u}{\ln\ln u}\}$ and $\exp\{(\ln u)^\alpha\}$ with $\alpha \in (0,1)$.

If (1.16) holds with $q = \infty$ (respectively, $q = -\infty$), then we refer to R as a *rapidly varying function at ∞ of index ∞* (respectively, $-\infty$) and write $R \in RV_\infty$ (respectively, $R \in RV_{-\infty}$). By λ^∞ for $\lambda > 0$ we mean 0 if $\lambda \in (0,1)$ or ∞ if $\lambda \in (1,\infty)$; when $\lambda = 1$, then $\lambda^\infty = 1$.

We say that R is a *regularly varying function (on the right) at the origin with index $q \in \mathbb{R}$* (and write, $R \in RV_q(0+)$) if $u \longmapsto R(1/u)$ belongs to RV_{-q}. Similarly, R is said to be *rapidly varying at zero of index ∞* (respectively, $-\infty$) if $u \longmapsto R(1/u)$ is rapidly varying at ∞ with index $-\infty$ (respectively, ∞).

Corollaries 1.1 and 1.2 show that, under the assumptions of Theorem 1.1, the function $\mathcal{E}_0$ in (1.10) can manifest any of the possible variation types at zero. More precisely, $\mathcal{E}_0$ can be slowly varying or regularly varying at zero of *any* negative index (cf. (1a) and (1b) in Corollary 1.1) or rapidly varying at zero (of index $-\infty$) according to Corollary 1.2.

We now reveal the conditions that determine the variation of $\mathcal{E}_0$ at zero.

Theorem 1.3 (Variation speed of $\mathcal{E}_0$)**.** *In the framework of Theorem* 1.1, *the variation of $\mathcal{E}_0$ at zero depends upon* (1.7) (a) *and* (1.8) *as follows:*

(1) *When $j \neq -1$, we have the following:*
 (i) *If* (1.8) (a) *holds, then $\mathcal{E}_0$ is regularly varying at zero of index $(1+\vartheta)/(1+j)$;*
 (ii) *If* (1.8) (b) *holds, then $\mathcal{E}_0$ is rapidly varying at zero with index $-\infty$.*
(2) *When $j = -1$, then $\mathcal{E}_0$ is rapidly varying at zero of index $-\infty$, provided that $\vartheta \neq -1$ if* (1.8) (a) *holds.*

Remark 1.6. In contrast to Theorem 1.3, the function $\mathcal{E}$ defined by (1.12) in the context of Theorem 1.2 is always slowly varying at zero (see Remark 8.1).

The next result follows from Theorem 1.2 and Remark 8.1.

Corollary 1.3. *Let* (A), (1.6) *and* (1.7) (b) *hold.*

In each of the following cases, (1.1) *has a unique blow-up solution* u_a, *for every* $a \in \mathbb{R}$*:*

(1a) $f(u) \sim (\underbrace{\exp \circ \ldots \circ \exp}_{n \text{ times}})(u^p)$ *as* $u \to \infty$, *where* $p > 0$ *and* $n \geq 2$ *is an integer; as* $d(x) \to 0$ *we have*

$$u_a(x) \sim \begin{cases} \{(\underbrace{\ln \circ \ldots \circ \ln}_{n \text{ times}})[1/d(x)]\}^{1/p} & \text{if } \ell \neq 0, \\ \{(\underbrace{\ln \circ \ldots \circ \ln}_{n \text{ times}})[1/k(d(x))]\}^{1/p} & \text{if } \ell = 0. \end{cases}$$

(1b) (i) $f(u) \sim e^{\gamma u^p}$ *as* $u \to \infty$, *where* γ, $p > 0$; *moreover, as* $d(x) \to 0$ *we have*

$$u_a(x) \sim \begin{cases} \Big[-\frac{2}{\ell\gamma} \ln d(x) \Big]^{1/p} & \text{if } \ell \neq 0, \\ \Big[-\frac{2}{\gamma} \ln k(d(x)) \Big]^{1/p} & \text{if } \ell = 0. \end{cases}$$

(ii) $f(u) \sim e^{u \ln u}$ *as* $u \to \infty$; *in this case, as* $d(x) \to 0$

$$u_a(x) \sim \begin{cases} \frac{2}{\ell} \frac{\ln(1/d(x))}{\ln \ln(1/d(x))} & \text{if } \ell \neq 0, \\ \frac{2 \ln[1/k(d(x)]}{\ln \ln[1/k(d(x))]} & \text{if } \ell = 0. \end{cases}$$

(1c) (i) $\ell \neq 0$ *and* $f(u) \sim e^{(\ln u)^\gamma}$ *as* $u \to \infty$, *for some* $\gamma > 1$; *furthermore,*

$$\ln u_a(x) \sim \Big[-\frac{2}{\ell} \ln d(x) \Big]^{1/\gamma} \quad \text{as } d(x) \to 0.$$

(ii) $\ell \neq 0$ *and* $f(e^{\frac{\ln u}{\ln \ln u}}) \sim Cu$ *as* $u \to \infty$, *for some* $C > 0$; *we have*

$$\ln u_a(x) \sim \frac{2}{\ell} \frac{\ln(1/d(x))}{\ln \ln(1/d(x))} \quad \text{as } d(x) \to 0.$$

Remark 1.7. (a) Theorem 1.2 and Corollary 1.3 apply if the inequality in (1.6) holds for some $k \in RV_m(0+)$ with $m > 0$ (without necessarily asking k to be in $\mathcal{K}_\ell$). Indeed, by Remark 2.2 (applied at 0), there exists a C^1-function near 0, say $\widetilde{k}$, such that $k(t) \sim \widetilde{k}(t)$ as $t \to 0^+$ and $\lim_{t\to 0^+} t\widetilde{k}'(t)/\widetilde{k}(t) = m > 0$. Thus by Corollary 3.1 and Remark 3.2 we have $\widetilde{k} \in \mathcal{K}_\ell$ with $\ell = 1/(m+1)$. Hence we can apply Theorem 1.2 with $\widetilde{k}$ instead of k; then (1.12) follows since the inverse of $u \longmapsto \int_u^\infty 1/\widehat{f}(y)\,dy$ is slowly varying at 0.

(b) Theorem 1.2 and Corollary 1.3 apply if (1.6) holds with $k(t) = e^{-(-\ln t)^\alpha}$ for small $t > 0$, where $\alpha > 1$. In this case, $k \in \mathcal{K}_0$ and $k(t) \sim k^*(t)$ as $t \to 0^+$, where $K^*(t) = \int_0^t k^*(s)\,ds = e^{-(-\ln t)^\alpha} t(-\ln t)^{1-\alpha}/\alpha$. It is easy to check that $\mathcal{N}^*(t) = K^*(t)/k^*(t)$ satisfies $\lim_{t\to 0^+} t(1/\mathcal{N}^*(t))'\mathcal{N}^*(t) = -1$; that is, (1.7) (b) holds with $\mathcal{N}^*$ in place of $\mathcal{N}$. Thus Theorem 1.2 can be applied for k^* instead of k and (1.12) follows as in (a) above.

The basic assumptions of this paper are (A), (B) and (1.6). Note that (B) implies that $f(u)$ grows faster than any power function u^p ($p > 1$) (cf. Proposition 3.10.3 in [6]). Our main results require more information on k at zero and on the growth rate of f at ∞ in the form of (1.7) and (1.8). These assumptions are not merely technical, they play instead a critical role in the asymptotic behaviour of the large solution near $\partial\Omega$ (see Theorems 1.1 and 1.2). We expect a further change in the asymptotic behaviour of the blow-up solution when $\ell = 0$, (1.7) (b) and (1.8) (b) hold, as well as in the case that $j = -1$ in (1.7) (a) is coupled with $\vartheta = -1$ in (1.8) (a), but these cases remain open.

Our analysis and findings here are very different from those in [8] and [12], where $f(u)$ is assumed to vary at ∞ like a power function u^p with $p > 1$. More precisely, in [8] the authors prove that (1.1) has a unique large solution u_a by assuming (A), (1.6) and that if f is regularly varying at ∞ with index p greater than 1 (hence (B) fails). Moreover, the uniqueness is proved in [8] by using an iterative method of Safonov which does not require a precise asymptotic behaviour of u_a near $\partial\Omega$, although it relies on some rough estimates of the blow-up rate of u_a. These estimates lead to a precise asymptotics for u_a near $\partial\Omega$ when the assumption (1.6) is a bit more restrictive, that is, when the limit $\lim_{d(x)\to 0} b(x)/k^2(d(x))$ exists and it is a positive number, say c, for some $k \in \mathcal{K}_\ell$. Then the asymptotic behaviour of u_a can be expressed in a unitary formula that applies equally to the case

$k \in \mathcal{K}_\ell$ with $\ell = 0$ and $\ell \neq 0$ when $f \in RV_p$ with $p > 1$, namely

$$\frac{u_a(x)}{\Upsilon(K(d(x)))} \to \Big[\frac{2+(p-1)\ell}{(p+1)c}\Big]^{1/(p-1)} \quad \text{as } d(x) \to 0, \tag{1.17}$$

where Υ denotes the inverse of Ξ defined by (1.2).

This paper is organized as follows. Section 2 contains some notions and results from the regular variation theory and its extensions, which will be used throughout. The aim of Section 3 is to discuss our framework via Corollary 3.1 and Lemmas 3.3 and 3.4. Section 4 contains two auxiliary results, which are frequently invoked in later sections. We give the proof of Theorem 1.3 in Section 5. Theorem 1.1 and its applications (Corollaries 1.1 and 1.2) are respectively proved in Sections 6 and 7. We conclude the paper with the proof of Theorem 1.2 in Section 8.

2. Regular variation theory and its extensions

2.1. **Karamata's theory.** We present here the properties of regularly varying functions involved in this paper (see [6] or [38]). When the regular variation occurs at infinity and there is no possibility of confusion, we do not mention "at infinity."

Proposition 2.1 (Uniform Convergence Theorem)**.** *Assume that L is slowly varying. Then, $L(\lambda u)/L(u) \to 1$ as $u \to \infty$ holds uniformly on each compact λ-set in $(0,\infty)$.*

Proposition 2.2. *Assume that L is slowly varying. The following hold:*

(i) $\ln L(u)/\ln u \to 0$ *as* $u \to \infty$;
(ii) *For any* $\alpha > 0$, $u^\alpha L(u) \to \infty$, $u^{-\alpha} L(u) \to 0$ *as* $u \to \infty$;
(iii) $(L(u))^\alpha$ *varies slowly for every* $\alpha \in \mathbb{R}$;
(iv) *If* L_1 *varies slowly, so do* $L(u)L_1(u)$ *and* $L(u) + L_1(u)$.

Remark 2.1. Assume that $R \in RV_q$. If $q > 0$ (respectively, $q < 0$), then $\lim_{u\to\infty} R(u) = \infty$ (respectively, 0). However, if $q = 0$ then the behavior of R at infinity cannot be completely described. For instance,

$$L(u) = \exp\Big\{(\ln u)^{1/3}\cos((\ln u)^{1/3})\Big\}$$

is slowly varying with

$$\liminf_{u\to\infty} L(u) = 0, \quad \limsup_{u\to\infty} L(u) = \infty.$$

Proposition 2.3 (Karamata's Theorem; direct half)**.** *Let $R \in RV_q$ be locally bounded in $[A,\infty)$. Then*

(i) *for any* $i \geq -(q+1)$,

$$\lim_{u\to\infty} \frac{u^{i+1}R(u)}{\int_A^u x^i R(x)\,dx} = i+q+1; \tag{2.1}$$

(ii) *for any* $i < -(q+1)$ *(and for* $i = -(q+1)$ *if* $\int^\infty x^{-(q+1)}R(x)\,dx < \infty$*)*

$$\lim_{u\to\infty} \frac{u^{i+1}R(u)}{\int_u^\infty x^i R(x)\,dx} = -(i+q+1). \tag{2.2}$$

Proposition 2.4 (Karamata's Theorem; converse half)**.** *Let R be positive and locally integrable in* $[A,\infty)$.

(i) *If* (2.1) *holds for some* $i > -(q+1)$, *then* $R \in RV_q$.
(ii) *If* (2.2) *is satisfied for some* $i < -(q+1)$, *then* $R \in RV_q$.

Proposition 2.5 (Representation Theorem)**.** *A function $L(u)$ is slowly varying if and only if it can be written in the form*

$$L(u) = M(u)\exp\left\{\int_B^u \frac{\varepsilon(t)}{t}\,dt\right\} \quad (u \geq B) \tag{2.3}$$

for some $B > 0$, *where* $\varepsilon \in C[B,\infty)$ *satisfies* $\lim_{u\to\infty}\varepsilon(u) = 0$ *and* $M(u)$ *is measurable on* $[B,\infty)$ *such that* $\lim_{u\to\infty} M(u) := \widehat{M} \in (0,\infty)$.

By (2.3), we see that $L(u)$ is asymptotically equivalent to $\widehat{L}(u)$ at infinity (that is, $\lim_{u\to\infty} L(u)/\widehat{L}(u) = 1$), where

$$\widehat{L}(u) = \widehat{M}\exp\left\{\int_B^u \frac{\varepsilon(t)}{t}\,dt\right\} \quad (u \geq B). \tag{2.4}$$

Of course, $\widehat{L}(u)$ is a slowly varying function, which is C^1 and

$$\varepsilon(u) = u\widehat{L}'(u)/\widehat{L}(u), \quad \text{for each } u \geq B.$$

A function $\widehat{L}(u)$ of the form (2.4) will be called a *normalised* slowly varying function. Moreover, any function $\widehat{L} \in C^1[B,\infty)$ which is positive and satisfies

$$\lim_{u\to\infty} u\widehat{L}'(u)/\widehat{L}(u) = 0 \tag{2.5}$$

is a normalised slowly varying function.

In general, if $\widehat{R}(u)/u^q$ ($q \in \mathbb{R}$) is a normalised slowly varying function, then we call $\widehat{R}(u)$ a *normalised regularly varying function of index* q and write $\widehat{R} \in NRV_q$.

Notice that $NRV_q \subset RV_q$, since the function $f(u) = u^q + \sin(u^{q+1})$ (defined for large u) is an example that belongs to RV_q but not to NRV_q.

A function $\widehat{R} \in RV_q$ belongs to NRV_q if and only if

$$\widehat{R} \in C^1[B,\infty), \text{ for some } B > 0, \text{ and } \lim_{u\to\infty} u\widehat{R}'(u)/\widehat{R}(u) = q.$$

Remark 2.2. For any $R \in RV_q$, there exists $\widehat{R} \in NRV_q$ such that $\widehat{R}(u)/R(u) \to 1$ as $u \to \infty$. Indeed, let $L(u) := R(u)/u^q$ and use Proposition 2.5 to find $\widehat{L}(u)$ as above. Set $\widehat{R}(u) = u^q \widehat{L}(u)$. Then, we have

$$\widehat{R} \in C^1, \lim_{u\to\infty} \frac{\widehat{R}(u)}{R(u)} = 1, \lim_{u\to\infty} \frac{u\widehat{R}'(u)}{\widehat{R}(u)} = q + \lim_{u\to\infty} \frac{u\widehat{L}'(u)}{\widehat{L}(u)} = q.$$

If H is a non-decreasing function on $\mathbb{R}$, then we define (as in Resnick [37]) the (left-continuous) inverse of H by

$$H^{\leftarrow}(y) = \inf\{s : \; H(s) \geq y\}.$$

Proposition 2.6 (Proposition 0.8 in [37])**.** *We have*

(i) *If $R \in RV_q$ with $-\infty \leq q \leq \infty$, then $\lim_{u\to\infty} \ln R(u)/\ln u = q$.*
(ii) *If $R_1 \in RV_{q_1}$ and $R_2 \in RV_{q_2}$ with $\lim_{u\to\infty} R_2(u) = \infty$, then*

$$R_1 \circ R_2 \in RV_{q_1 q_2}.$$

(iii) *Suppose R is non-decreasing and $R \in RV_q$, $0 \leq q \leq \infty$. Then*

$$R^{\leftarrow} \in RV_{q^{-1}}.$$

(iv) *Suppose R_1 and R_2 are non-decreasing and q-varying with $q \in (0,\infty)$. Then, for $c \in (0,\infty)$, we have*

$$\lim_{u\to\infty} \frac{R_1(u)}{R_2(u)} = c \quad \textit{if and only if} \quad \lim_{u\to\infty} \frac{R_1^{\leftarrow}(u)}{R_2^{\leftarrow}(u)} = c^{-1/q}.$$

Proposition 2.7. *Assume that R is a positive C^1-function on an interval (A,∞) such that there exists $\lim_{u\to\infty} uR'(u)/R(u) = \gamma$. Then R is either regularly varying of index γ or rapidly varying depending on whether $\gamma \in \mathbb{R}$ or $\gamma = \pm\infty$.*

2.2. **Extensions of regular variation theory.** We recall below some concepts and results due to de Haan [14] (see also [37] or [6]).

Definition 2.1 ([37])**.** *A non-decreasing function U defined on (A,∞) is* Γ-varying at ∞ *(written $U \in \Gamma$) if $\lim_{x\to\infty} U(x) = \infty$ and there exists $\chi : (A,\infty) \to (0,\infty)$ such that*

$$\lim_{x\to\infty} \frac{U(x+\lambda\chi(x))}{U(x)} = e^{\lambda}, \quad \forall \lambda \in \mathbb{R}. \tag{2.6}$$

The function χ is called an *auxiliary function* and is unique up to asymptotic equivalence. If (2.6) is satisfied for χ_1 and χ_2, then

$$\lim_{x\to\infty} \chi_1(x)/\chi_2(x) = 1.$$

Conversely, if (2.6) is fulfilled for χ and $\lim_{x\to\infty} \chi_1(x)/\chi(x) = 1$, then (2.6) also holds with χ_1.

Example 2.1 ([14])**.** *The following functions U satisfy* (2.6) *with the specified auxiliary functions χ:*

(1) $U(x) = \exp(x^p)$ *for* $p > 0$ *with* $\chi(x) = \begin{cases} 1 & \text{for } x \le 0, \\ p^{-1}x^{1-p} & \text{for } x > 0. \end{cases}$

(2) $U(x) = \exp(x \ln_+ x)$ *with* $\chi(x) = \begin{cases} 1 & \text{for } x \le 1, \\ (\ln x)^{-1} & \text{for } x > 1. \end{cases}$

(3) $U(x) = \exp(e^x)$ *with* $\chi(x) = e^{-x}$*.*

The next result provides us with many examples of Γ-varying functions.

Proposition 2.8 (Theorem 1.5.6 in [14])**.** *The following hold:*

a) *If U_1 is monotone and regularly varying of index $\rho > 0$ and $U_2 \in \Gamma$ with auxiliary function $\chi(u)$, then $U_1 \circ U_2 \in \Gamma$ with auxiliary function χ/ρ.*
b) *If $U_1 \in \Gamma$ and U_2 has a derivative that varies regularly of index $\rho \in (-1, \infty)$, then $U_1 \circ U_2 \in \Gamma$.*
c) *If $U_1 \in \Gamma$ and U_2 has a derivative belonging to Γ, then $U_1 \circ U_2 \in \Gamma$.*

Example 2.2. Since $U(x) = e^x$ is Γ-varying with auxiliary function $\chi(x) = 1$, by Proposition 2.8 we infer that

$$\exp(x^q) \quad \text{and} \quad (\underbrace{\exp \circ \ldots \circ \exp}_{n \text{ times}})(x^q), \text{ where } n \ge 2 \text{ is an integer},$$

belong to Γ, for every $q > 0$.

Definition 2.2 ([37])**.** *A non-negative, non-decreasing function W defined on (A, ∞) is called* Π-varying *(written $W \in \Pi$) if there exists $\beta : (A, \infty) \to (0, \infty)$ such that*

$$\lim_{u\to\infty} \frac{W(\lambda u) - W(u)}{\beta(u)} = \ln \lambda, \quad \forall \lambda > 0. \tag{2.7}$$

Note that β is called an *auxiliary function* and is unique up to asymptotic equivalence.

The next result gives a convenient relationship between Π and Γ.

Proposition 2.9 (Proposition 0.9 in [37]). *The following hold:*

(1) *If $U \in \Gamma$ with auxiliary function χ, then $U^{\leftarrow} \in \Pi$ with auxiliary function $\beta(u) = \chi \circ U^{\leftarrow}(u)$.*
(2) *If $W \in \Pi$ with auxiliary function β, then $W^{\leftarrow} \in \Gamma$ with auxiliary function $\chi(u) = \beta \circ W^{\leftarrow}(u)$.*

Proposition 2.10 (Proposition 0.11 in [37]). *Suppose $W : \mathbb{R}^+ \to \mathbb{R}^+$ is absolutely continuous with density w.*

(a) *If $w \in RV_{-1}$, then $W \in \Pi$ with auxiliary function $\beta(u) = uw(u)$.*
(b) *If w is non-increasing and $W \in \Pi$, then $w \in RV_{-1}$.*

3. On the framework

We recall that h is a normalised regularly varying function at zero with index $m \in \mathbb{R}$ if h is a positive C^1-function on $(0, \tau)$, for some $\tau > 0$, such that $\lim_{t\to 0^+} th'(t)/h(t) = m$. In this case, we write $h \in NRV_m(0+)$ or alternatively $u \longmapsto h(1/u) \in NRV_{-m}$.

Lemma 3.1. *Let h be a decreasing C^2-function on some interval $(0, \tau)$, for some $\tau > 0$, such that $\lim_{t\to 0^+} h(t) = \infty$. If there exists*

$$\lim_{t\to 0^+} \frac{h(t)h''(t)}{[h'(t)]^2} = \alpha \in \mathbb{R}, \tag{3.1}$$

then $\alpha \geq 1$. Furthermore, we have the following:

(a) *Relation (3.1) holds with $\alpha \neq 1$ if and only if $t \longmapsto -h'(t)$ is normalised regularly varying at zero of index $\alpha/(1-\alpha)$.*
(b) *Relation (3.1) holds with $\alpha = 1$ if and only if $\lim_{t\to 0^+} th''(t)/h'(t) = -\infty$ with h being of the form*

$$h(t) = c_0 \exp\Big(\int_t^{\tau_0} \frac{ds}{\zeta(s)}\Big), \quad \forall t \in (0, \tau_0), \tag{3.2}$$

where c_0 and τ_0 are positive constants and ζ is a positive C^1-function on $(0, \tau_0)$ such that $\lim_{t\to 0^+} \zeta'(t) = 0$.

Proof. If (3.1) is satisfied, then there exists $t_0 \in (0, \tau)$ and a continuous function φ on $(0, t_0)$ such that $\lim_{t\to 0^+} \varphi(t) = 0$ and

$$\frac{h''(s)}{h'(s)} = (\alpha + \varphi(s))\frac{h'(s)}{h(s)}, \quad \forall s \in (0, t_0). \tag{3.3}$$

Fix $\varepsilon \in (0, t_0)$ and let $t \in (\varepsilon, t_0)$ be arbitrary. By integrating (3.3) over (t, t_0), we find

$$h'(t) = \frac{h'(t_0)}{[h(t_0)]^\alpha}[h(t)]^\alpha \exp\Big(- \int_t^{t_0} \frac{\varphi(s)h'(s)}{h(s)}\, ds\Big). \tag{3.4}$$

Let h^{-1} denote the inverse of h, which exists on $(0, t_0)$ since h is decreasing on $(0, \tau)$. For $u > 0$ large, we define

$$\eta(u) = -\frac{h'(t_0)}{[h(t_0)]^\alpha} \exp\Big(\int_{h(t_0)}^{u} \frac{\varphi(h^{-1}(z))}{z}\, dz\Big),$$

which is a normalised slowly varying function (see Proposition 2.5 and (2.4)). Hence, we can rewrite (3.4) as follows:

$$-\frac{h'(t)}{\eta(h(t))[h(t)]^\alpha} = 1, \quad \forall t \in (\varepsilon, t_0). \tag{3.5}$$

By integrating (3.5) over (ε, t_0) we obtain

$$\int_{h(t_0)}^{u} \frac{dy}{\eta(y)y^\alpha} = t_0 - h^{-1}(u), \quad \text{where } u = h(\varepsilon). \tag{3.6}$$

If we assume that $\alpha < 1$, then by Proposition 2.2 (ii), $\lim_{y\to\infty} y^{1-\alpha}/\eta(y) = \infty$. As $\varepsilon \to 0$ we have $u = h(\varepsilon) \to \infty$ such that the left-hand side of (3.6) goes to ∞, while the right-hand side of (3.6) is finite. Hence, we conclude that $\alpha \geq 1$.

Proof of (a). Assume that (3.1) holds with $\alpha > 1$. Letting $\varepsilon \to 0^+$ in (3.6) we find

$$\int_{h(t)}^{\infty} \frac{dy}{\eta(y)y^\alpha} = t, \quad \forall t \in (0, t_0). \tag{3.7}$$

By Proposition 2.3 (ii) we have $u \longmapsto 1/\int_u^\infty 1/[y^\alpha \eta(y)]\, dy$ belongs to $NRV_{\alpha-1}$. Thus, by (3.7) and Proposition 2.6, we deduce $h(1/u) \in NRV_{1/(\alpha-1)}$; that is, $h \in NRV_{1/(1-\alpha)}(0+)$. Using (3.5) we infer that $-h' \in NRV_{\alpha/(1-\alpha)}(0+)$.

Conversely, if $t \longmapsto -h'(t) \in NRV_{\alpha/(1-\alpha)}(0+)$ then $\lim_{t\to 0^+} th''(t)/h'(t) = \alpha/(1-\alpha)$. Hence, $\lim_{t\to 0^+} h(t)/[th'(t)] = 1-\alpha$ and (3.1) follows immediately.

Proof of (b). Suppose that (3.1) holds with $\alpha = 1$. By (3.5) and (3.7) we obtain

$$-\frac{th'(t)}{h(t)} = \eta(h(t)) \int_{h(t)}^{\infty} \frac{dy}{y\eta(y)}. \tag{3.8}$$

Since $\lim_{t\to 0^+} h(t) = \infty$, by Proposition 2.3 (ii) we see that the right-hand side of (3.8) tends to ∞ as $t \to 0^+$. This, together with (3.1), implies that

$$\lim_{t\to 0^+} \frac{th'(t)}{h(t)} = \lim_{t\to 0^+} \frac{th''(t)}{h'(t)} = -\infty. \tag{3.9}$$

Hence, by Proposition 2.7 (applied at zero), $-h'$ is rapidly varying at zero of index $-\infty$.

If we set $\zeta(t) = -h(t)/h'(t)$ for $t > 0$ small, then $\lim_{t\to 0^+} \zeta'(t) = 0$; cf. (3.1). This shows that h is of the form (3.2).

Conversely, from h satisfying (3.2) we see that $-h(t)/h'(t) = \zeta(t)$, for every $t \in (0, \tau_0)$. Differentiating this equality and using the properties of ζ, we prove (3.1) with $\alpha = 1$. □

Remark 3.1. If, in the framework of Lemma 3.1, h satisfies (3.1) then

$$\lim_{t\to 0^+} h(t)/h'(t) = 0. \tag{3.10}$$

Indeed, if $\alpha > 1$ in (3.1), then Lemma 3.1 (a) proves that $h \in NRV_{1/(1-\alpha)}(0+)$; that is, $\lim_{t\to 0^+} th'(t)/h(t) = 1/(1-\alpha)$. If $\alpha = 1$, then (3.10) follows from Lemma 3.1 (b).

With an argument similar to that of Lemma 3.1, one can deduce the following:

Lemma 3.2. *Let g be an increasing C^2-function on some interval $(0, \tau)$ with $\tau > 0$ such that $\lim_{t\to 0^+} g(t) = 0$. If there exists*

$$\lim_{t\to 0^+} \frac{g(t)g''(t)}{[g'(t)]^2} = p \in \mathbb{R}, \tag{3.11}$$

then $p \le 1$ and $\lim_{t\to 0^+} g(t)/g'(t) = 0$. Moreover, we have the following:

(a) *Relation (3.11) holds with $p < 1$ if and only if g' is normalised regularly varying at zero of index $p/(1-p)$.*
(b) *Relation (3.11) holds with $p = 1$ if and only if $\lim_{t\to 0^+} tg''(t)/g'(t) = \infty$ with g being of the form*

$$g(t) = c_1 \exp\Big(- \int_t^{\tau_1} \frac{ds}{\zeta_1(s)} \Big), \quad \forall t \in (0, \tau_1), \tag{3.12}$$

for some positive constants c_1, τ_1 and a positive function $\zeta_1 \in C^1(0, \tau_1)$ such that $\lim_{t\to 0^+} \zeta_1'(t) = 0$.

Alternatively, Lemma 3.2 can be recovered from Lemma 3.1 with $h = 1/g$. Lemma 3.2 enables us to give a precise description of a class larger than $\mathcal{K}_\ell$.

Let $\overline{\mathcal{K}}_\ell$ denote the set of all positive C^1-functions k defined on $(0,\nu)$ with $\nu>0$ such that $\lim_{t\to 0^+} K(t)=0$ and $\lim_{t\to 0^+} (K(t)/k(t))'=\ell\in\mathbb{R}$, where $K(t)=\int_0^t k(s)\,ds$.

The characterization of $\mathcal{K}_\ell$ is provided by [12]. However, we give here a different proof, which also applies to understanding the structure of $\overline{\mathcal{K}}_\ell$.

Corollary 3.1 (Characterization of $\overline{\mathcal{K}}_\ell$)**.** *If $k\in\overline{\mathcal{K}}_\ell$ (respectively, $\mathcal{K}_\ell$), then $\ell\in[0,\infty)$ (respectively, $\ell\in[0,1]$) and $\lim_{t\to 0^+} K(t)/k(t)=0$. Moreover, we have*

(i) *$k\in\overline{\mathcal{K}}_\ell$ with $\ell\neq 0$ if and only if $\lim_{t\to 0^+} K(t)=0$ and k is a normalised regularly varying function at zero with index $(1-\ell)/\ell$;*

(ii) *$k\in\overline{\mathcal{K}}_0$ if and only if $\lim_{t\to 0^+} tk'(t)/k(t)=\infty$ with K being of the form*

$$K(t)=c_1\exp\Big(-\int_t^{\tau_1}\frac{ds}{\zeta_1(s)}\Big),\quad \forall t\in(0,\tau_1), \tag{3.13}$$

for some positive constants c_1 and τ_1 and a positive function $\zeta_1\in C^1(0,\tau_1)$ such that $\lim_{t\to 0^+}\zeta_1'(t)=0$.

Proof. Apply Lemma 3.2 with $g=K$. □

Remark 3.2. The relationship between $\mathcal{K}_\ell$ and $\overline{\mathcal{K}}_\ell$ is described below:

(a) $\mathcal{K}_\ell\equiv\overline{\mathcal{K}}_\ell$, for every $\ell\in[0,1)$;

(b) $\mathcal{K}_1\subset\overline{\mathcal{K}}_1$ and the inclusion is strict, since for instance

$$k(t)=\exp\{(-\ln t)^{1/3}\cos((-\ln t)^{1/3})\},\quad \text{for small } t>0,$$

does not belong to $\mathcal{K}_1$, while $k\in\overline{\mathcal{K}}_1$. Indeed, k is not monotone (see Remark 2.1), but $\lim_{t\to 0^+} K(t)=0$ and $\lim_{t\to 0^+} tk'(t)/k(t)=0$, which proves that k is normalised slowly varying at zero.

(c) $\overline{\mathcal{K}}_\ell\neq\emptyset=\mathcal{K}_\ell$, for every $\ell\in(1,\infty)$. For example, $k(t)=t^{-\alpha}$ with $\alpha\in(0,1)$ belongs to $\overline{\mathcal{K}}_\ell$ where $\ell=1/(1-\alpha)$.

Lemma 3.3. *Let k be a positive C^1-function on $(0,\nu)$, for some $\nu>0$, such that $\lim_{t\to 0^+} K(t)=0$. If* (1.7) (a) *holds, then $j\le -1$. Moreover, we have*

(a) *k satisfies* (1.7) (a) *with $j<-1$ if and only if the function $\mathcal{N}:=K/k$ is normalised regularly varying at zero of index $j/(1+j)$, or equivalently*

$$K(t)=c_1\exp\Big(-\int_t^{\tau_1}\frac{ds}{\mathcal{N}(s)}\Big),\quad \forall t\in(0,\tau_1),$$

where c_1 and τ_1 are positive constants and $\mathcal{N}$ is a normalised regularly varying function at zero of index $j/(1+j)$.

(b) *k satisfies* (1.7) (a) *with* $j = -1$ *if and only if* $\lim_{t\to 0^+} t\mathcal{N}'(t)/\mathcal{N}(t) = \infty$ *with K being of the form*

$$K(t) = \exp\Big(-c_0 \exp\Big(\int_t^{\tau_0} \frac{ds}{\zeta(s)} \Big)\Big), \quad t \in (0, \tau_0),$$

where c_0 and τ_0 are positive constants and ζ is a positive C^1-function on $(0, \tau_0)$ such that $\lim_{t\to 0^+} \zeta'(t) = 0$.

Proof. Apply Lemma 3.1 with $h = -\ln K$. □

Lemma 3.4. *Let f be continuous and increasing on (A, ∞) such that*

$$\lim_{u\to\infty} f(u) = \infty.$$

If (B) *holds, then f is Γ-varying at ∞ with auxiliary function S.*

Proof. Since $\lim_{u\to\infty} S'(u) = 0$ from (B), we have

$$\lim_{u\to\infty} \frac{\widehat{f}(u)\widehat{f}''(u)}{[\widehat{f}'(u)]^2} = 1,$$

which implies that

$$1 = \lim_{u\to\infty} \frac{u\widehat{f}''(\widehat{f}^{-1}(u))}{[\widehat{f}'(\widehat{f}^{-1}(u))]^2} = \lim_{u\to\infty} \frac{-u[\widehat{f}^{-1}(u)]''}{[\widehat{f}^{-1}(u)]'}.$$

It follows that $u \longmapsto [\widehat{f}^{-1}(u)]'$ belongs to RV_{-1}. Hence, $\widehat{f}^{-1}$ is Π-varying with auxiliary function $u/\widehat{f}'(\widehat{f}^{-1}(u))$; cf. Proposition 2.10. By Proposition 2.9 we infer that $\widehat{f} \in \Gamma$ with auxiliary function $\widehat{f}(u)/\widehat{f}'(u) = S(u)$. This concludes the proof. □

4. Auxiliary results

Throughout this section, we assume that (A), (B) and (1.8) are satisfied.

Remark 4.1. If $\mathcal{Q}$ is given by (1.10), then $\mathcal{Q} \circ \widehat{f}^{-1}$ is normalised slowly varying at ∞ and

$$\begin{cases} \text{if (1.8) (a) holds, then } \lim_{u\to\infty} \mathcal{Q}(u)S'(u) = \vartheta \text{ and } \lim_{u\to\infty} \mathcal{Q}(u)S(u)/u = 1+\vartheta, \\ \text{if (1.8) (b) holds, then } \lim_{u\to\infty} \mathcal{Q}(u)S'(u) = \lim_{u\to\infty} \mathcal{Q}(u)S(u)/u = 1. \end{cases} \tag{4.1}$$

We only check that if (1.8) (a) holds, then $\lim_{u\to\infty} \mathcal{Q}(u)S(u)/u = 1+\vartheta$. Indeed, by l'Hôpital's rule and Remark 1.1, we find

$$\lim_{u\to\infty} \frac{\mathcal{Q}(u)S(u)}{u} = \lim_{u\to\infty} \frac{\int_B^u \frac{dy}{S(y)}}{u/S(u)} = \lim_{u\to\infty} \frac{1}{1-uS'(u)/S(u)} = 1+\vartheta. \tag{4.2}$$

We define $\mathcal{L}$ as follows:

$$\int_{\mathcal{L}(u)}^{\infty} \frac{dy}{\widehat{f}(y)\mathcal{Q}(y)} = \frac{1}{u} \quad \text{for large } u>0, \tag{4.3}$$

which implies that

$$\int_{\widehat{f}(\mathcal{L}(u))}^{\infty} \frac{[\widehat{f}^{-1}(z)]'}{z\,\mathcal{Q}(\widehat{f}^{-1}(z))}\,dz = \frac{1}{u} \quad \text{for } u>0 \text{ sufficiently large.} \tag{4.4}$$

The next result plays an important role in the proof of Theorem 1.1.

Lemma 4.1 (Properties of $\mathcal{L}$)**.** *The function $\mathcal{L}$ in* (4.3) *satisfies the following:*

(i) $u^2\mathcal{L}'(u) \sim \mathcal{Q}(\mathcal{L}(u))f(\mathcal{L}(u))$ *as* $u\to\infty$;

(ii) $\widehat{f}\circ\mathcal{L} \in NRV_1$;

(iii) $\mathcal{L}$ *is slowly varying at* ∞ *and* $\mathcal{L}' \in NRV_{-1}$;

(iv) $\displaystyle\lim_{u\to\infty} u(\mathcal{Q}\circ\mathcal{L})'(u) = \lim_{u\to\infty}\frac{(\mathcal{Q}\circ\mathcal{L})(u)}{\ln u} = \begin{cases} 1 & \text{if (1.8) (a) holds,}\\ 0 & \text{if (1.8) (b) holds.}\end{cases}$

(v) $\displaystyle\lim_{u\to\infty} \mathcal{Q}(\mathcal{L}(u))\left(1+\frac{u\mathcal{L}''(u)}{\mathcal{L}'(u)}\right) = \begin{cases} \vartheta & \text{if (1.8) (a) holds,}\\ 1 & \text{if (1.8) (b) holds.}\end{cases}$

Proof. (i) By differentiating (4.3), we obtain

$$u^2\mathcal{L}'(u) = \mathcal{Q}(\mathcal{L}(u))\widehat{f}(\mathcal{L}(u)) \quad \text{for large } u>0. \tag{4.5}$$

(ii) Since $[\widehat{f}^{-1}(u)]' \in RV_{-1}$, by Remark 4.1 and Proposition 2.3 we have

$$\Theta(u) = \int_u^{\infty} \frac{[\widehat{f}^{-1}(z)]'}{z\,\mathcal{Q}(\widehat{f}^{-1}(z))}\,dz \in RV_{-1} \ \text{ and } \lim_{u\to\infty}\frac{u\Theta'(u)}{\Theta(u)} = -1. \tag{4.6}$$

By (4.4) and (4.6), we deduce that $\widehat{f}\circ\mathcal{L} \in NRV_1$.

(iii) By (ii) and Proposition 2.6, we see that $\mathcal{L}$ is slowly varying. Using Remark 4.1 and (ii), we have that $\mathcal{Q}\circ\mathcal{L}$ is normalised slowly varying. This and (4.5) imply that $\mathcal{L}' \in NRV_{-1}$.

(iv) By (ii) we have

$$\frac{u\mathcal{L}'(u)}{(S\circ\mathcal{L})(u)} = \frac{u(\widehat{f}\circ\mathcal{L})'(u)}{(\widehat{f}\circ\mathcal{L})(u)} \to 1 \quad \text{as } u\to\infty. \tag{4.7}$$

If (1.8) (a) holds, then the assertion of (iv) follows from (4.7) and (1.10). If (1.8) (b) holds, then by (1.10) we find

$$u(\mathcal{Q}\circ\mathcal{L})'(u) = \frac{u\mathcal{L}'(u)}{(S\circ\mathcal{L})(u)}\Big[1-\frac{\mathcal{L}(u)S'(\mathcal{L}(u))}{S(\mathcal{L}(u))}\Big] \to 0 \quad \text{as } u\to\infty.$$

(v) From (4.5) and a simple computation, we obtain

$$\mathcal{Q}(\mathcal{L}(u))\Big(1+\frac{u\mathcal{L}''(u)}{\mathcal{L}'(u)}\Big) = \widehat{f}'(\mathcal{L}(u))\mathcal{Q}(\mathcal{L}(u))\Big[\frac{\mathcal{Q}(\mathcal{L}(u))}{u}-\frac{1}{\widehat{f}'(\mathcal{L}(u))}\Big]+u(\mathcal{Q}\circ\mathcal{L})'(u). \tag{4.8}$$

In view of (ii) and (4.5), we find

$$\widehat{f}'(\mathcal{L}(u))\,\mathcal{Q}(\mathcal{L}(u)) \sim u \quad \text{as } u\to\infty. \tag{4.9}$$

Using l'Hôpital's rule, we arrive at

$$\lim_{u\to\infty} u\Big[\frac{\mathcal{Q}(\mathcal{L}(u))}{u}-\frac{1}{\widehat{f}'(\mathcal{L}(u))}\Big] = \lim_{u\to\infty}\left[-u(\mathcal{Q}\circ\mathcal{L})'(u)+\mathcal{Q}(\mathcal{L}(u))S'(\mathcal{L}(u))\right]. \tag{4.10}$$

From (4.1) and (4.8)–(4.10), we conclude the claim of (v). □

In the proof of Theorems 1.2, we shall use $\mathcal{G}$, which is defined as

$$\int_{\mathcal{G}(u)}^{\infty}\frac{dy}{\widehat{f}(y)} = \frac{1}{u} \quad \text{for large } u>0. \tag{4.11}$$

A result similar to Lemma 4.1 is available for $\mathcal{G}$, namely

Lemma 4.2 (Properties of $\mathcal{G}$)**.** *The function $\mathcal{G}$ in* (4.11) *satisfies the following:*

(i) $u^2\mathcal{G}'(u)\sim f(\mathcal{G}(u))$ *as* $u\to\infty$*;*
(ii) $\widehat{f}\circ\mathcal{G}\in NRV_1$*;*
(iii) $\mathcal{G}$ *is slowly varying at* ∞ *and* $\mathcal{G}'\in NRV_{-1}$*;*
(iv) $\displaystyle\lim_{u\to\infty} u(\mathcal{Q}\circ\mathcal{G})'(u) = \lim_{u\to\infty}\frac{(\mathcal{Q}\circ\mathcal{G})(u)}{\ln u} = \begin{cases} 1 & \textit{if } (1.8)\ (a)\ \textit{holds},\\ 0 & \textit{if } (1.8)\ (b)\ \textit{holds}.\end{cases}$
(v) $\displaystyle\lim_{u\to\infty}\mathcal{Q}(\mathcal{G}(u))\left(1+\frac{u\mathcal{G}''(u)}{\mathcal{G}'(u)}\right) = \begin{cases} \vartheta & \textit{if } (1.8)\ (a)\ \textit{holds},\\ 1 & \textit{if } (1.8)\ (b)\ \textit{holds}.\end{cases}$

The proof of Lemma 4.2 is omitted, since it differs only slightly from that of Lemma 4.1.

Remark 4.2. The properties (i)–(iii) of Lemma 4.2 do not require the assumption (1.8).

5. Proof of Theorem 1.3

In what follows, for $k \in \mathcal{K}_0$ we define

$$\Psi(t) := 1/K^2(t) \quad \text{for } t > 0 \text{ sufficiently small.} \tag{5.1}$$

Remark 5.1. If (1.7) (a) holds, then

$$-\frac{d}{dt}\left(\frac{\Psi'(t)}{\Psi(t)}\right)\frac{\Psi^2(t)}{[\Psi'(t)]^2}\ln\Psi(t) = \left(\frac{K(t)}{k(t)}\right)'\ln K(t) \to j \quad \text{as } t \to 0^+, \tag{5.2}$$

and, by Remark 3.1 (with $h = -\ln K$), we have

$$\frac{4\Psi(t)}{\Psi'(t)}\ln\Psi(t) = \frac{\Psi'(t)\ln\Psi(t)}{k^2(t)\Psi^2(t)} = \frac{4K(t)}{k(t)}\ln K(t) \to 0 \quad \text{as } t \to 0^+. \tag{5.3}$$

From (1.10), (4.3) and (5.1), we have

$$\mathcal{E}_0(t) = (\mathcal{L} \circ \Psi)(t) \quad \text{for } t > 0 \text{ small.} \tag{5.4}$$

By differentiating (5.4) and using (4.7), we obtain

$$\frac{\mathcal{E}_0'(t)}{S(\mathcal{E}_0(t))} \sim \frac{\Psi'(t)}{\Psi(t)} \quad \text{as } t \to 0^+. \tag{5.5}$$

By Lemma 4.1 (iv) and (5.4), we find

$$\lim_{t\to 0^+}\frac{\mathcal{Q}(\mathcal{E}_0(t))}{\ln\Psi(t)} = \begin{cases} 1 & \text{if (1.8) (a) holds,} \\ 0 & \text{if (1.8) (b) holds.} \end{cases} \tag{5.6}$$

By (5.5), (4.1) and (5.6), it follows that

$$\frac{t\mathcal{E}_0'(t)}{\mathcal{E}_0(t)} \sim \begin{cases} \dfrac{(1+\vartheta)t\Psi'(t)}{\Psi(t)\ln\Psi(t)} & \text{as } t \to 0^+ \quad \text{if (1.8) (a) holds,} \\[2ex] \dfrac{t\Psi'(t)}{\Psi(t)\mathcal{Q}(\mathcal{E}_0(t))} & \text{as } t \to 0^+ \quad \text{if (1.8) (b) holds.} \end{cases} \tag{5.7}$$

We now distinguish two cases according to $j = -1$ or $j \neq -1$ in (1.7) (a).
Case 1: $j \neq -1$. By Remark 1.1 we have

$$\lim_{t\to 0^+}\frac{t\Psi'(t)}{\Psi(t)\ln\Psi(t)} = \lim_{t\to 0^+}\frac{t}{\mathcal{N}(t)\ln K(t)} = \frac{1}{1+j}. \tag{5.8}$$

From (5.6)–(5.8) we infer that

$$\lim_{t\to 0^+} \frac{t\mathcal{E}_0'(t)}{\mathcal{E}_0(t)} = \begin{cases} \dfrac{1+\vartheta}{1+j} & \text{if (1.8) (a) holds,} \\ -\infty & \text{if (1.8) (b) holds.} \end{cases} \tag{5.9}$$

Case 2: $j=-1$ and (1.8) (a) holds with $\vartheta \neq -1$. If we set $h(t) = -\ln K(t)$, then h satisfies (3.1) with $\alpha = 1$. Using (3.9) we find

$$\lim_{t\to 0^+} \frac{t\Psi'(t)}{\Psi(t)\ln \Psi(t)} = \lim_{t\to 0^+} \frac{tk(t)}{K(t)\ln K(t)} = -\infty. \tag{5.10}$$

From (5.7) and (5.10) it follows that

$$\lim_{t\to 0^+} t\mathcal{E}_0'(t)/\mathcal{E}_0(t) = -\infty. \tag{5.11}$$

By (5.9), (5.11) and Proposition 2.7 (applied at zero), we conclude the assertions of Theorem 1.3.

Remark 5.2. The function $\mathcal{E}_0$ defined by (1.10) in the context of Theorem 1.1 satisfies

$$\ln \widehat{f}(\mathcal{E}_0(t)) \sim -2\ln K(t) \quad \text{as } t \to 0^+. \tag{5.12}$$

Indeed, by Lemma 4.1 (ii) and Proposition 2.6 (i), we have $\ln(\widehat{f}\circ\mathcal{L})(u) \sim \ln u$ as $u \to \infty$. This, jointly with (5.4), proves (5.12).

6. Proof of Theorem 1.1

We shall use a comparison principle that has been proved in [11] (see [19] for a version corresponding to the p-Laplacian case). For the reader's convenience, we state it below.

Lemma 6.1 (Comparison Principle)**.** *Let f be continuous on $(0,\infty)$ such that $f(u)/u$ is increasing for $u>0$. Let $p \not\equiv 0$ be a continuous non-negative function on Ω.*

If $u_1, u_2 \in C^2(\Omega)$ are positive such that

$$\begin{cases} \limsup\limits_{d(x,\partial\Omega)\to 0} (u_2 - u_1)(x) \le 0 \\ -\Delta u_1 - au_1 + p(x)f(u_1) \ge 0 \ge -\Delta u_2 - au_2 + p(x)f(u_2) \quad \text{in } \Omega, \end{cases}$$

then $u_1 \ge u_2$ in Ω.

We are now ready to present the proof of Theorem 1.1. Thus, we assume throughout this section that (A), (B), (1.6), (1.7) (a) and (1.8) are satisfied. If (1.8) (a) is verified, then we further assume that $j \neq -1$ if $\vartheta = -1$.

Since f satisfies the Keller–Osserman condition (1.2) and $b > 0$ in Ω, (1.1) admits blow-up solutions, for every $a \in \mathbb{R}$; see Theorem 1.1 in [9].

We take $\delta > 0$ small such that the following holds:

(a) $d(x)$ is a C^2-function on the set $\{x \in \overline{\Omega} : d(x) < 2\delta\}$ (use Lemma 14.16 in [23]);
(b) k is non-decreasing on $(0, 2\delta)$;
(c) $|\nabla d(x)| = 1$, for every $x \in \Omega_{2\delta} := \{x \in \Omega : d(x) < 2\delta\}$.

Notation. For $\nu > 0$, we set $\Omega_\nu := \{x \in \Omega : d(x) < \nu\}$.

Let $\sigma \in (0, \delta)$ be arbitrary. We define C_1 as follows:

$$C_1 = \begin{cases} \vartheta - j & \text{if (1.8) (a) holds,} \\ 1 & \text{if (1.8) (b) holds.} \end{cases} \tag{6.1}$$

Since $j \neq -1$ if $\vartheta = -1$ in (1.8) (a), by Remark 1.1 we see that C_1 is a positive constant. Let $\lambda^\pm$ be such that $\lambda^+ \in (4C_1/\gamma^-, \infty)$ and $\lambda^- \in (0, 4C_1/\gamma^+)$, where $\gamma^\pm$ appear in (1.6).

With $\mathcal{L}$ and Ψ given by (4.3) and (5.1) respectively, we define

$$\begin{cases} v_\sigma^+(x) := \mathcal{L}(\lambda^+ \Psi(d(x) - \sigma)) & \text{for every } x \in \Omega_{2\delta} \setminus \overline{\Omega}_\sigma, \\ v_\sigma^-(x) := \mathcal{L}(\lambda^- \Psi(d(x) + \sigma)) & \text{for every } x \in \Omega_{2\delta-\sigma}. \end{cases} \tag{6.2}$$

We divide the rest of the argument into three steps:

Step 1: If $\delta > 0$ is small enough, then v_σ^+ and v_σ^- are upper and lower solutions of (1.1) near the boundary:

$$\begin{cases} -\Delta v_\sigma^+(x) - a v_\sigma^+(x) + b(x) f(v_\sigma^+(x)) \geq 0 & \text{for every } x \in \Omega_{2\delta} \setminus \overline{\Omega}_\sigma, \\ -\Delta v_\sigma^-(x) - a v_\sigma^-(x) + b(x) f(v_\sigma^-(x)) \leq 0 & \text{for every } x \in \Omega_{2\delta-\sigma}. \end{cases} \tag{6.3}$$

Step 2: Every blow-up solution u_a of (1.1) satisfies (1.9), where $\mathcal{E}_0$ is given by (1.10).

Step 3: Uniqueness of the blow-up solution of (1.1).

Details of Step 1. Using (1.6) and (b), we have

$$\begin{cases} -\Delta v_\sigma^+(x) - a v_\sigma^+(x) + b(x) f(v_\sigma^+(x)) \geq E^+(x) \Delta v_\sigma^+(x), & \forall x \in \Omega_{2\delta} \setminus \overline{\Omega}_\sigma, \\ -\Delta v_\sigma^-(x) - a v_\sigma^-(x) + b(x) f(v_\sigma^-(x)) \leq E^-(x) \Delta v_\sigma^-(x), & \forall x \in \Omega_{2\delta-\sigma}, \end{cases} \tag{6.4}$$

where we set

$$E^\pm(x) := -1 - \frac{a v_\sigma^\pm(x)}{\Delta v_\sigma^\pm(x)} + \frac{\gamma^\mp k^2(d(x)\mp\sigma) f(v_\sigma^\pm(x))}{\Delta v_\sigma^\pm(x)}. \tag{6.5}$$

We conclude (6.3) from (6.4), our choice of $\lambda^\pm$, and the following result.

Lemma 6.2. *If C_1 and $E^\pm$ are respectively given by* (6.1) *and* (6.5), *then*

$$\lim_{d(x)\mp\sigma\to 0} E^\pm(x) = -1 + \frac{\gamma^\mp\lambda^\pm}{4C_1}. \tag{6.6}$$

Proof. From the definition of $v_\sigma^\pm$ in (6.2), we derive

$$\begin{aligned}\Delta v_\sigma^\pm(x) &= \operatorname{div}\big(\lambda^\pm \mathcal{L}'(\lambda^\pm\Psi(d(x)\mp\sigma))\Psi'(d(x)\mp\sigma)\nabla d(x)\big) \\ &= \lambda^\pm\left(\frac{\Psi'}{\Psi}\right)'(d(x)\mp\sigma)\mathcal{L}'(\lambda^\pm\Psi(d(x)\mp\sigma))\Psi(d(x)\mp\sigma) \\ &+ \lambda^\pm\mathcal{L}'(\lambda^\pm\Psi(d(x)\mp\sigma))\Psi'(d(x)\mp\sigma)\Delta d(x) \\ &+ \lambda^\pm[\Psi'(d(x)\mp\sigma)]^2\Big(\lambda^\pm\mathcal{L}''(\lambda^\pm\Psi(d(x)\mp\sigma)) + \frac{\mathcal{L}'(\lambda^\pm\Psi(d(x)\mp\sigma))}{\Psi(d(x)\mp\sigma)}\Big).\end{aligned} \tag{6.7}$$

We rearrange the above sum in a convenient manner to reveal the dominant term. To this aim, we define $T^\pm$, T_1 and T_2, on some small interval $(0,\tau)$, by

$$\begin{cases} T^\pm(t) = \lambda^\pm \dfrac{[\Psi'(t)]^2}{\Psi(t)}\,\dfrac{\mathcal{L}'(\lambda^\pm\Psi(t))}{\mathcal{Q}(\mathcal{E}_0(t))}, \\ T_1(t) = \dfrac{d}{dt}\left(\dfrac{\Psi'(t)}{\Psi(t)}\right)\dfrac{\Psi^2(t)}{[\Psi'(t)]^2}\mathcal{Q}(\mathcal{E}_0(t)), \quad T_2(t) = \dfrac{\Psi(t)}{\Psi'(t)}\,\mathcal{Q}(\mathcal{E}_0(t)), \end{cases} \tag{6.8}$$

while, for $u>0$ large, we set

$$T_3^\pm(u) = \mathcal{Q}(\mathcal{L}(u/\lambda^\pm))\Big(1 + \frac{u\mathcal{L}''(u)}{\mathcal{L}'(u)}\Big). \tag{6.9}$$

We rewrite (6.7) in the following way:

$$\Delta v_\sigma^\pm = T^\pm(d(x)\mp\sigma)\left[T_1(d(x)\mp\sigma) + T_2(d(x)\mp\sigma)\Delta d + T_3^\pm(\lambda^\pm\Psi(d(x)\mp\sigma))\right]. \tag{6.10}$$

Since $f(u)\sim\widehat{f}(u)$ as $u\to\infty$, from Lemma 4.1 and (5.4), we infer that

$$\lim_{t\to 0^+}\frac{T^\pm(t)}{k^2(t)(f\circ\mathcal{L})(\lambda^\pm\Psi(t))} = \frac{4}{\lambda^\pm}\lim_{t\to 0^+}\frac{\Psi^2(t)\,\mathcal{L}'(\Psi(t))}{f(\mathcal{E}_0(t))\,\mathcal{Q}(\mathcal{E}_0(t))} = \frac{4}{\lambda^\pm}. \tag{6.11}$$

By Remark 5.1 and (5.6), we have

$$\lim_{t\to 0^+} T_2(t) = 0 \quad \text{and} \quad \lim_{t\to 0^+} T_1(t) = \begin{cases} -j & \text{if (1.8) (a) holds,} \\ 0 & \text{if (1.8) (b) holds.} \end{cases} \tag{6.12}$$

Since $\mathcal{Q} \circ \mathcal{L}$ is slowly varying, by Lemma 4.1 (v) we have

$$\lim_{u\to\infty} T_3^{\pm}(u) = \begin{cases} \vartheta & \text{if (1.8) (a) holds,} \\ 1 & \text{if (1.8) (b) holds.} \end{cases}$$

Hence, when $d(x) \mp \sigma \to 0$ the second factor on the right-hand side of (6.10) converges to the constant C_1 defined by (6.1). This, jointly with (6.11), proves that

$$\lim_{d(x)\mp\sigma\to 0} \frac{k^2(d(x)\mp\sigma) f(v_\sigma^{\pm}(x))}{\Delta v_\sigma^{\pm}(x)} = \frac{\lambda^{\pm}}{4C_1}. \tag{6.13}$$

Using (6.8), (6.10) and Lemma 4.1 (iii), we find

$$\lim_{d(x)\mp\sigma\to 0} \frac{v_\sigma^{\pm}(x)}{\Delta v_\sigma^{\pm}(x)} = \lim_{t\to 0^+} \frac{\mathcal{L}(\Psi(t))}{C_1 T^{\pm}(t)} = \frac{1}{C_1} \lim_{t\to 0^+} \frac{\mathcal{E}_0(t)}{\mathcal{E}_0'(t)} \frac{\Psi(t)\, \mathcal{Q}(\mathcal{E}_0(t))}{\Psi'(t)}. \tag{6.14}$$

In view of (6.13) and (6.14), to complete the proof of Lemma 6.2 it suffices to show that

$$G_0(t) := \frac{\mathcal{E}_0(t)}{\mathcal{E}_0'(t)} \frac{\Psi(t)}{\Psi'(t)} \mathcal{Q}(\mathcal{E}_0(t)) \to 0 \quad \text{as } t \to 0^+. \tag{6.15}$$

Proof of (6.15). Using (5.5) and (4.1), we obtain

$$\lim_{t\to 0^+} G_0(t) = \lim_{t\to 0^+} \left[\frac{\Psi(t)}{\Psi'(t)}\right]^2 \mathcal{Q}^2(\mathcal{E}_0(t)) \times \begin{cases} \dfrac{1}{1+\vartheta} & \text{if (1.8) (a) holds with } \vartheta \neq -1, \\ 1 & \text{if (1.8) (b) holds.} \end{cases}$$

Then, (6.15) follows from (5.3) and (5.6).

We now consider the case that (1.8) (a) is satisfied with $\vartheta = -1$. By our assumption, we have $j \neq -1$. Using (5.5), (5.8) and (5.6), we find

$$G_0(t) \sim \left[\frac{\Psi(t)}{\Psi'(t)}\right]^2 \frac{\mathcal{E}_0(t)}{S(\mathcal{E}_0(t))} \mathcal{Q}(\mathcal{E}_0(t)) \sim (1+j)^2 \frac{t^2 \mathcal{E}_0(t)}{S(\mathcal{E}_0(t)) \ln \Psi(t)} \quad \text{as } t \to 0^+. \tag{6.16}$$

By (5.5), (5.8) and l'Hôpital's rule, we derive that

$$\lim_{t\to 0^+} \frac{\ln S(\mathcal{E}_0(t))}{-\ln t} = \lim_{t\to 0^+} \frac{\ln(\ln \Psi(t))}{\ln t} = \frac{1}{1+j}. \tag{6.17}$$

By letting $\vartheta = -1$ in (5.9), we get $\lim_{t\to 0^+} \ln \mathcal{E}_0(t)/\ln t = 0$. This and (6.17) show that

$$\lim_{t\to 0^+} \ln\left[\frac{t^2\mathcal{E}_0(t)}{S(\mathcal{E}_0(t))\ln\Psi(t)}\right] = 2\lim_{t\to 0^+} \ln t = -\infty. \tag{6.18}$$

From (6.16) and (6.18), we conclude (6.15). This finishes the proof of Lemma 6.2. □

Details of Step 2. Let w be a positive function in $C^2(\Omega_\delta)\cap C(\Omega_\delta\cup\partial\Omega)$ such that

$$-\Delta w \geq aw - b(x)f(w) \quad \text{in } \Omega_\delta, \qquad w=\infty \quad \text{on } \partial\Omega_\delta\cap\Omega. \tag{6.19}$$

One can take w as the minimal positive solution of (1.1) in Ω_δ, subject to $u=\infty$ on $\partial\Omega_\delta\cap\Omega$ and $u=1$ on $\partial\Omega$. Lemma 2.3 in [11] ensures the existence of such a solution.

Let u_a denote an arbitrary blow-up solution of (1.1). Using (A) and (6.19), we find

$$-\Delta(u_a+w) \geq a(u_a+w) - b(x)f(u_a+w) \quad \text{in } \Omega_\delta.$$

Since $u_a+w=\infty > v_\sigma^-$ on $\partial\Omega_\delta$, by Step 1 and Lemma 6.1 we deduce

$$u_a+w\geq v_\sigma^- \quad \text{in } \Omega_\delta. \tag{6.20}$$

Similarly, one can derive that

$$v_\sigma^+ + w \geq u_a \quad \text{in } \Omega_\delta\setminus\overline{\Omega}_\sigma. \tag{6.21}$$

Letting $\sigma\to 0$ in (6.20) and (6.21), we infer that

$$\mathcal{L}(\lambda^-\Psi(d(x))) - w(x) \leq u_a(x) \leq \mathcal{L}(\lambda^+\Psi(d(x))) + w(x), \quad \forall x\in\Omega_\delta. \tag{6.22}$$

But w is uniformly bounded on $\partial\Omega$ and $\mathcal{L}$ is slowly varying at ∞. Hence, dividing (6.22) by $\mathcal{L}(\Psi(d(x)))$ and passing to the limit $d(x)\to 0$, we obtain

$$\lim_{d(x)\to 0}\frac{u_a(x)}{\mathcal{L}(\Psi(d(x)))} = 1, \tag{6.23}$$

which completes Step 2.

Details of Step 3. Step 2 allows us to use a standard argument (as in [8]) to deduce the uniqueness of the blow-up solution. Let u_1 and u_2 be arbitrary blow-up solutions of (1.1). It suffices to prove that $u_1\leq u_2$ in Ω.

We define $U_\varepsilon = (1+\varepsilon)u_2$, where $\varepsilon>0$ is arbitrary. By (A) and Step 2, we infer that

$$-\Delta U_\varepsilon \geq aU_\varepsilon - b(x)f(U_\varepsilon) \quad \text{in } \Omega \quad \text{and} \quad \lim_{d(x)\to 0}(u_1-U_\varepsilon)(x) = -\infty.$$

From Lemma 6.1 we deduce $u_1 \le U_\varepsilon$ in Ω. Letting $\varepsilon \to 0$ we conclude Step 3 and the proof of Theorem 1.1.

Remark 6.1. If $\ell = 0$ in the framework of Theorem 1.2, then the asymptotic behaviour of the blow-up solution found in Theorem 1.1 is not valid anymore. Indeed, the hypothesis (1.7) (b) implies that

$$j = \lim_{t\to 0^+} \mathcal{N}'(t) \ln K(t) = -\infty, \tag{6.24}$$

so that the dominant term in the expression of $\Delta v_\sigma^\pm(x)$ in (6.10) cannot be $T^\pm(d(x) \pm \sigma)$ as $d(x) \mp \sigma \to 0$ (see (6.12)).

7. Applications of Theorem 1.1

In this section we prove Corollaries 1.1 and 1.2.

7.1. **Proof of Corollary 1.1.** By (1.14) we infer that (1.6) holds for $k \in \mathcal{K}_0$ defined, for small $t > 0$, by

$$k(t) = t^{m/2} e^{-(\beta/2)t^q} \Big(1 - \frac{m-2q+2}{\beta q} t^{-q}\Big) = \frac{d}{dt}\Big[-\frac{2}{\beta q} t^{m/2-q+1} e^{-(\beta/2)t^q}\Big]. \tag{7.1}$$

Moreover, k satisfies (1.7) (a) with $j = (1-q)/q$ and

$$-2\ln(K(t)) \sim \beta t^q \quad \text{as } t \to 0^+.$$

Corollary 1.1 is a consequence of Theorem 1.1 and Remark 5.2. Indeed, the hypotheses (B) and (1.8) are satisfied in each of the cases (1a), (1b) and (1c):

(1a) $\widehat{f}(u) = (\underbrace{\exp \circ \ldots \circ \exp}_{n \text{ times}})(u^p)$ such that

$$\ln \widehat{f}(\mathcal{E}_0(t)) = (\underbrace{\exp \circ \ldots \circ \exp}_{(n-1) \text{ times}})(\mathcal{E}_0^p(t)) \sim \beta t^q \quad \text{as } t \to 0^+.$$

It follows that, as $t \to 0^+$,

$$\mathcal{E}_0(t) \sim \begin{cases} [q \ln t]^{1/p} & \text{if } n = 2, \\ \{(\underbrace{\ln \circ \ldots \circ \ln}_{(n-1) \text{ times}})[1/t]\}^{1/p} & \text{if } n \ge 3. \end{cases}$$

(1b) (i) $\widehat{f}(u) = e^{\gamma u^p}$ such that

$$\ln \widehat{f}(\mathcal{E}_0(t)) = \gamma [\mathcal{E}_0(t)]^p \sim \beta t^q \quad \text{as } t \to 0^+.$$

(ii) $\widehat{f}(u) = e^{u \ln u}$ such that

$$\ln \widehat{f}(\mathcal{E}_0(t)) = \mathcal{E}_0(t) \ln \mathcal{E}_0(t) \sim \beta t^q \quad \text{as } t \to 0^+.$$

Therefore, we have

$$\ln \mathcal{E}_0(t) \sim q \ln t \quad \text{and} \quad \mathcal{E}_0(t) \sim \frac{\beta}{q} \frac{t^q}{\ln t} \quad \text{as } t \to 0^+.$$

(1c) (i) $\widehat{f}(u) = e^{(\ln u)^\gamma}$ such that

$$\ln \widehat{f}(\mathcal{E}_0(t)) = [\ln \mathcal{E}_0(t)]^\gamma \sim \beta t^q \quad \text{as } t \to 0^+.$$

(ii) $\widehat{f}(e^{\frac{\ln u}{\ln \ln u}}) = C$ such that

$$\ln \widehat{f}(\mathcal{E}_0(t)) \sim (\ln \mathcal{E}_0(t)) \cdot \ln \ln \mathcal{E}_0(t) \sim \beta t^q \quad \text{as } t \to 0^+.$$

Hence, we infer that

$$\ln \ln \mathcal{E}_0(t) \sim q \ln t \quad \text{and} \quad \ln \mathcal{E}_0(t) \sim \frac{\beta}{q} \frac{t^q}{\ln t} \quad \text{as } t \to 0^+.$$

Using (1.9) we complete the proof of Corollary 1.1.

7.2. **Proof of Corollary 1.2.** We show that (1.15) implies that (1.6) holds for some function $k \in \mathcal{K}_0$ that satisfies (1.7) (a) with $j = -1$. Let us define k by

$$k(t) = \frac{d}{dt}\Big[-\frac{2}{\beta \sigma q} t^{\frac{m}{2}+1-q} e^{\frac{\tau}{2} t^{\widetilde{m}} - \sigma t^q} e^{-\frac{\beta}{2} e^{\sigma t^q}} \Big] \quad \text{for small } t > 0. \tag{7.2}$$

A simple calculation shows that

$$k(t) \sim t^{\frac{m}{2}} e^{\frac{\tau}{2} t^{\widetilde{m}}} e^{-\frac{\beta}{2} e^{\sigma t^q}} \text{ and } \ln K(t) \sim -\frac{\beta}{2} e^{\sigma t^q} \quad \text{as } t \to 0^+, \tag{7.3}$$

as well as

$$\frac{d}{dt}\Big[\frac{k(t)}{K(t)}\Big] \sim -\frac{\beta \sigma^2 q^2}{2} t^{2q-2} e^{\sigma t^q} \text{ and } \Big(\frac{k(t)}{K(t)}\Big)^2 \sim \frac{\beta^2 \sigma^2 q^2}{4} t^{2q-2} e^{2\sigma t^q} \text{ as } t \to 0^+. \tag{7.4}$$

Using (7.3) and (7.4), we infer that $k \in \mathcal{K}_0$ and (1.7) (a) is satisfied with $j = -1$. In light of Remark 1.4, we see that the hypotheses of Theorem 1.1 are satisfied in each of the cases (i)–(iv). Hence, for every $a \in \mathbb{R}$, (1.1) has a unique blow-up solution u_a and

$$\lim_{d(x) \to 0} u_a(x)/\mathcal{E}_0(d(x)) = 1, \tag{7.5}$$

where $\mathcal{E}_0$, defined by (1.10), satisfies (cf. Remark 5.2 and (7.3))

$$\ln \widehat{f}(\mathcal{E}_0(t)) \sim -2\ln K(t) \sim \beta e^{\sigma t^q} \quad \text{as } t \to 0^+. \tag{7.6}$$

With the choice of $\widehat{f}$ made accordingly in the proof of Corollary 1.1, (7.6) becomes

(i) $\gamma[\mathcal{E}_0(t)]^p \sim \beta e^{\sigma t^q}$ as $t \to 0^+$;
(ii) $\mathcal{E}_0(t)\ln \mathcal{E}_0(t) \sim \beta e^{\sigma t^q}$ as $t \to 0^+$;
(iii) $[\ln \mathcal{E}_0(t)]^\gamma \sim \beta e^{\sigma t^q}$ as $t \to 0^+$;
(iv) $[\ln \mathcal{E}_0(t)] \ln\ln \mathcal{E}_0(t) \sim \beta e^{\sigma t^q}$ as $t \to 0^+$.

Using (7.5) and the above, we conclude the proof of Corollary 1.2.

8. Proof of Theorem 1.2

Let (A), (B) and (1.6) be satisfied. If $k \in \mathcal{K}_\ell$ with $\ell \neq 0$, then there is no further assumption. If (1.6) holds for $k \in \mathcal{K}_0$, then we further impose (1.7) (b) and (1.8) (a).

Let $\sigma \in (0,\delta)$ be fixed and let $\delta > 0$ be small such that (1.6) and (a)–(c) in Section 6 hold.

By Remark 6.1 we need to modify the argument of Theorem 1.1 with regard to the construction of v_σ^+ and v_σ^- satisfying (6.3). To this aim, we need the following result.

Lemma 8.1. *There exists a C^2-function Λ on $(0,\tau)$ such that the following hold:*

(i) $1/\Lambda(t) \sim tk(t)K(t)$ *as* $t \to 0^+$;
(ii) $\ln \Lambda(t) \sim -2\ln K(t)$ *as* $t \to 0^+$;
(iii) $[\ln \Lambda(t)]' \sim -2/\mathcal{N}(t)$ *as* $t \to 0^+$;
(iv) $[\ln \Lambda(t)]'' \sim 2k^2(t)\Lambda(t)$ *as* $t \to 0^+$.
(v) $t \longmapsto -[\ln \Lambda(t)]' \in NRV_{-1}(0+)$ *and* $t \longmapsto \ln \Lambda(t) \in NRV_0(0+)$.

Proof. By (1.7) (b) we have $\mathcal{N} \in NRV_1(0+)$ and $\mathcal{N}'$ is slowly varying at 0. From Remark 2.2 (applied at 0), there exists a C^2-function $\widehat{\mathcal{N}}$ on $(0,\tau)$ with $\tau > 0$ such that

$$\widehat{\mathcal{N}}'(t) \sim \mathcal{N}'(t) \ \text{ as } t \to 0^+ \quad \text{and} \quad \widehat{\mathcal{N}}' \in NRV_0(0+). \tag{8.1}$$

Thus, we infer that

$$\widehat{\mathcal{N}}(t) \sim \mathcal{N}(t) \ \text{ as } t \to 0^+ \quad \text{and} \quad \widehat{\mathcal{N}} \in NRV_1(0+). \tag{8.2}$$

By (8.1) and (8.2), $t \longmapsto [\ln \widehat{\mathcal{N}}(t)]'$ belongs to $NRV_{-1}(0+)$; that is,

$$[\ln \widehat{\mathcal{N}}(t)]'' \sim -\frac{\widehat{\mathcal{N}}'(t)}{t\widehat{\mathcal{N}}(t)} \sim -\frac{1}{t^2} \quad \text{as } t \to 0^+. \tag{8.3}$$

Let us define Λ as follows:

$$\Lambda(t) = \frac{\widehat{\mathcal{N}}(t)}{tK^2(t)}, \quad \text{for small } t > 0. \tag{8.4}$$

From (8.2) and (8.4), we see that Λ is a C^2-function that satisfies (i).

By differentiating (8.4) we conclude (iii) and (ii), since

$$-\frac{\Lambda'(t)}{\Lambda(t)} = \frac{1}{\mathcal{N}(t)}\Big(2 - \widehat{\mathcal{N}}'(t)\,\frac{\mathcal{N}(t)}{\widehat{\mathcal{N}}(t)} + \frac{\mathcal{N}(t)}{t}\Big) \sim \frac{2}{\mathcal{N}(t)} \quad \text{as } t \to 0^+. \tag{8.5}$$

From (8.5), (1.7) (b) and (8.3), we obtain

$$[\ln \Lambda(t)]'' = \frac{2\mathcal{N}'(t)}{\mathcal{N}^2(t)} + [\ln \widehat{\mathcal{N}}(t)]'' + \frac{1}{t^2} \sim \frac{2}{t\mathcal{N}(t)} \sim 2\Lambda(t)k^2(t) \quad \text{as } t \to 0^+,$$

which proves (iv). This and (iii) imply that $t \longmapsto -[\ln \Lambda(t)]' \in NRV_{-1}(0+)$, since

$$t[\ln \Lambda(t)]'' \sim -[\ln \Lambda(t)]' \quad \text{as } t \to 0^+.$$

Using (ii) and (6.24), we get

$$\lim_{t\to 0^+} \frac{t[\ln \Lambda(t)]'}{\ln \Lambda(t)} = \lim_{t\to 0^+} \frac{t}{\mathcal{N}(t)\ln K(t)} = 0, \tag{8.6}$$

which completes the proof of Lemma 8.1. □

Construction of $v_\sigma^\pm$ satisfying (6.3). Using $\mathcal{G}$ and Λ given respectively by (4.11) and (8.4), we define

$$\begin{cases} v_\sigma^+(x) := \mathcal{G}(\lambda^+\Lambda(d(x)-\sigma)), & \text{for every } x \in \Omega_{2\delta} \setminus \overline{\Omega}_\sigma, \\ v_\sigma^-(x) := \mathcal{G}(\lambda^-\Lambda(d(x)+\sigma)), & \text{for every } x \in \Omega_{2\delta-\sigma}, \end{cases} \tag{8.7}$$

where λ^+ and λ^- are positive constants such that $\lambda^+ \in (2/\gamma^-, \infty)$ and $\lambda^- \in (0, 2/\gamma^+)$.

Since (6.4) remains valid, to conclude (6.3) it suffices to prove

Lemma 8.2. *The functions $E^\pm$ defined by* (6.5) *satisfy*

$$\lim_{d(x)\mp\sigma\to 0} E^\pm(x) = -1 + \frac{\gamma^\mp\lambda^\pm}{2}. \tag{8.8}$$

Proof. We first prove that

$$\lim_{d(x)\mp\sigma\to 0}\frac{k^2(d(x)\mp\sigma)f(v_\sigma^\pm(x))}{\Delta v_\sigma^\pm(x)}=\frac{\lambda^\pm}{2}. \tag{8.9}$$

We split the proof of (8.9) into two parts depending on whether (1.6) is satisfied for $k\in\mathcal{K}_\ell$ with $\ell\neq 0$ or $\ell=0$.

Part A: Assume that (1.6) holds for $k\in\mathcal{K}_\ell$ and $\ell\neq 0$.

In this case, Λ given by Lemma 8.1 satisfies

$$\frac{t\Lambda'(t)}{\Lambda(t)}\sim-\frac{2}{\ell},\quad \frac{[\Lambda'(t)]^2}{[\Lambda(t)]^3}\sim\frac{4}{\ell}k^2(t)\quad\text{and}\quad \frac{d}{dt}\left[\frac{\Lambda(t)}{\Lambda'(t)}\right]\sim-\frac{\ell}{2}\ \text{ as } t\to 0^+. \tag{8.10}$$

For $t>0$ small enough, we set

$$\begin{cases} D^\pm(t)=(\lambda^\pm)^2\mathcal{G}''(\lambda^\pm\Lambda(t))[\Lambda'(t)]^2,\\ D_1^\pm(t)=\dfrac{1}{\lambda^\pm}\dfrac{\mathcal{G}'(\lambda^\pm\Lambda(t))}{\mathcal{G}''(\lambda^\pm\Lambda(t))}\dfrac{\Lambda''(t)}{[\Lambda'(t)]^2}\ \text{ and }\ D_2^\pm(t)=D_1^\pm(t)\dfrac{\Lambda'(t)}{\Lambda''(t)}. \end{cases} \tag{8.11}$$

From (8.7) and (8.11), we derive that

$$\begin{aligned}\Delta v_\sigma^\pm(x)&=\operatorname{div}\left(\lambda^\pm\mathcal{G}'(\lambda^\pm\Lambda(d(x)\mp\sigma))\Lambda'(d(x)\mp\sigma)\nabla d(x)\right)\\ &=D^\pm(d(x)\mp\sigma)\left\{1+D_1^\pm(d(x)\mp\sigma)+D_2^\pm(d(x)\mp\sigma)\Delta d(x)\right\}.\end{aligned} \tag{8.12}$$

Using Lemma 4.2 (iii) and (8.10), we deduce that

$$\lim_{t\to 0^+}D_1^\pm(t)=-\lim_{t\to 0^+}\frac{\Lambda(t)\Lambda''(t)}{[\Lambda'(t)]^2}=-1-\frac{\ell}{2}. \tag{8.13}$$

Since $\lim_{t\to 0^+}\Lambda'(t)/\Lambda''(t)=0$ (see (8.10)), by (8.13) we get $\lim_{t\to 0^+}D_2^\pm(t)=0$. It follows that

$$\lim_{d(x)\mp\sigma\to 0}\left[1+D_1^\pm(d(x)\mp\sigma)+D_2^\pm(d(x)\mp\sigma)\Delta d(x)\right]=-\frac{\ell}{2}\neq 0. \tag{8.14}$$

By (8.10) and Lemma 4.2 (i), (iii), we obtain

$$D^\pm(d(x)\mp\sigma)\sim-\frac{4}{\ell\lambda^\pm}k^2(d(x)\mp\sigma)f(v_\sigma^\pm(x))\quad\text{as } d(x)\mp\sigma\to 0. \tag{8.15}$$

From (8.12), (8.14) and (8.15), we achieve (8.9).

Part B: Let (1.6) hold for $k\in\mathcal{K}_0$. Then, we also assume (1.7) (b) and (1.8) (a).

For $t > 0$ sufficiently small, we define

$$\begin{cases} P^{\pm}(t) := \lambda^{\pm}\Lambda(t)\mathcal{G}'(\lambda^{\pm}\Lambda(t))[\ln\Lambda(t)]'', \quad P_1(t) := \dfrac{[\ln\Lambda(t)]'}{[\ln\Lambda(t)]''}, \\ P_2^{\pm}(t) := \left[\dfrac{\Lambda'(t)}{\Lambda(t)}\right]^2 \dfrac{1}{[\ln\Lambda(t)]''}\left[1 + \dfrac{\lambda^{\pm}\Lambda(t)\mathcal{G}''(\lambda^{\pm}\Lambda(t))}{\mathcal{G}'(\lambda^{\pm}\Lambda(t))}\right]. \end{cases} \tag{8.16}$$

From (8.7) and (8.16), we obtain

$$\begin{aligned} \Delta v_\sigma^{\pm}(x) &= \operatorname{div}\left(\lambda^{\pm}\mathcal{G}'(\lambda^{\pm}\Lambda(d(x)\mp\sigma))\Lambda'(d(x)\mp\sigma)\nabla d(x)\right) \\ &= P^{\pm}(d(x)\mp\sigma)\left[1 + P_1(d(x)\mp\sigma)\Delta d(x) + P_2^{\pm}(d(x)\mp\sigma)\right]. \end{aligned} \tag{8.17}$$

Based upon Lemmas 8.1 and 4.2, we see that

$$\begin{cases} \lim_{t\to 0^+} P_1(t) = 0 \quad \text{and} \quad P^{\pm}(t) \sim \dfrac{2}{\lambda^{\pm}} k^2(t) f(\mathcal{G}(\lambda^{\pm}\Lambda(t))) \quad \text{as } t \to 0^+; \\ \lim_{t\to 0^+} P_2(t) = \vartheta \lim_{t\to 0^+} \dfrac{[\Lambda'(t)/\Lambda(t)]^2}{[\ln\Lambda(t)][\ln\Lambda(t)]''} = \vartheta \lim_{t\to 0^+} \dfrac{t}{\mathcal{N}(t)\ln K(t)} = 0 \text{ (cf. (6.24)).} \end{cases} \tag{8.18}$$

In view of (8.17) and (8.18), we conclude (8.9).

We next show that

$$\lim_{d(x)\mp\sigma\to 0} \frac{v_\sigma^{\pm}(x)}{\Delta v_\sigma^{\pm}(x)} = 0. \tag{8.19}$$

By Lemma 4.2 (ii), we have

$$\lim_{t\to 0^+} \frac{\Lambda(t)\widehat{f}'(\mathcal{G}(\Lambda(t)))\mathcal{G}'(\Lambda(t))}{\widehat{f}(\mathcal{G}(\Lambda(t)))} = 1.$$

Thus, using $S(u) = \widehat{f}(u)/\widehat{f}'(u)$ and Lemma 4.2 (i), we find

$$\frac{(\mathcal{G}\circ\Lambda)'(t)}{S(\mathcal{G}(\Lambda(t)))} \sim \frac{\Lambda'(t)}{\Lambda(t)} \quad \text{and} \quad \widehat{f}'(\mathcal{G}(\Lambda(t))) \sim \Lambda(t) \quad \text{as } t \to 0^+. \tag{8.20}$$

From (8.20) and Lemma 8.1 (i), (iii), we infer that

$$2k^2(t) \sim -\frac{[\ln\Lambda(t)]'}{t\Lambda(t)} \sim -\frac{[\ln\Lambda(t)]'}{t\widehat{f}'(\mathcal{G}(\Lambda(t)))} \quad \text{as } t \to 0^+. \tag{8.21}$$

By (8.9), Lemma 4.2 (iii), and (8.21), it follows that

$$\lim_{d(x)\mp\sigma\to 0} \frac{v_\sigma^{\pm}(x)}{\Delta v_\sigma^{\pm}(x)} = \lim_{t\to 0^+} \frac{\mathcal{G}(\Lambda(t))}{2k^2(t) f(\mathcal{G}(\Lambda(t)))} = \lim_{t\to 0^+} \frac{-t\mathcal{G}(\Lambda(t))}{[\ln\Lambda(t)]' S(\mathcal{G}(\Lambda(t)))}. \tag{8.22}$$

We now claim that

$$t \longmapsto \mathcal{G}(\Lambda(t)) \quad \text{and} \quad t \longmapsto S(\mathcal{G}(\Lambda(t))) \quad \text{are slowly varying at 0.} \tag{8.23}$$

Indeed, by (8.20), (1.8) (a) and (8.6), we have

$$\lim_{t\to 0^+} \frac{t\frac{d}{dt}[S(\mathcal{G}(\Lambda(t)))]}{S(\mathcal{G}(\Lambda(t)))} = \lim_{t\to 0^+} tS'(\mathcal{G}(\Lambda(t)))\frac{\Lambda'(t)}{\Lambda(t)} = \vartheta \lim_{t\to 0^+} \frac{t[\ln \Lambda(t)]'}{\ln \Lambda(t)} = 0.$$

Hence, $t \longmapsto S(\mathcal{G}(\Lambda(t)))$ is (normalised) slowly varying at 0. From (1.8) (a) we have $S \in RV_{\vartheta/(1+\vartheta)}$ if $\vartheta \neq -1$ and $S \in RV_{-\infty}$ if $\vartheta = -1$ (see Remark 1.1 and Proposition 2.7). Thus, by Proposition 2.6 we derive that $t \longmapsto \mathcal{G}(\Lambda(t))$ is slowly varying at 0.

By (8.23) and Lemma 8.1 (v), we deduce that

$$t \longmapsto \frac{-t\mathcal{G}(\Lambda(t))}{[\ln \Lambda(t)]' S(\mathcal{G}(\Lambda(t)))} \quad \text{belongs to } RV_2(0+).$$

This, jointly with (8.22) and Proposition 2.2 (ii), leads to (8.19).

Finally, from (8.9), (8.19) and the definition of $E^{\pm}$ in (6.5), we reach (8.8). This completes the proof of Lemma 8.2.

Let u_a be an arbitrary blow-up solution of (1.1). Using (8.7) and the same line of argument as in Step 2 in the proof of Theorem 1.1, we arrive at

$$\lim_{d(x)\to 0} \frac{u_a(x)}{\mathcal{G}(\Lambda(d(x)))} = 1. \tag{8.24}$$

From (1.12) and (4.11) we have

$$\mathcal{G}\Big(\frac{1}{tk(t)K(t)}\Big) = \mathcal{E}(t) \quad \text{for } t > 0 \text{ small.} \tag{8.25}$$

Since $\mathcal{G}$ is slowly varying (cf. Lemma 4.2), by Lemma 8.1 we find $\mathcal{E}(t) \sim \mathcal{G}(\Lambda(t))$ as $t \to 0^+$. Thus, using (8.24) we obtain (1.11). The uniqueness assertion follows now as in Step 3 in the proof of Theorem 1.1. This concludes the proof of Theorem 1.2. □

Remark 8.1. Under the assumptions of Theorem 1.2, the function $\mathcal{E}$ in (1.12) is slowly varying at 0 (use (8.23)). Moreover, as $t \to 0^+$ we have

$$\ln \widehat{f}(\mathcal{E}(t)) \sim -2\ln K(t) \sim \begin{cases} -\dfrac{2}{\ell}\ln t & \text{if } k \in \mathcal{K}_\ell \text{ with } \ell \neq 0, \\ -2\ln k(t) & \text{if } k \in \mathcal{K}_0. \end{cases} \tag{8.26}$$

Indeed, (8.26) follows from (8.25), Lemma 4.2 (ii) and Proposition 2.6 (i).

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