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ON THE SUPERLINEAR LAZER-MCKENNA CONJECTURE: THE NON-HOMOGENEOUS CASE

E.N. Dancer and Sanjiban Santra
School of Mathematics and Statistics
The University of Sydney, NSW 2006 Australia

(Submitted by: Reza Aftabizadeh)

Abstract. We prove the Lazer-McKenna conjecture for the superlinear elliptic problem of the Ambrosetti-Prodi type with a non-homogeneous non-linearity by constructing solutions with sharp peaks. We also compute the critical groups provided the critical points are isolated.

1. Introduction

In this paper, we continue our research on the multiplicity of solutions for the elliptic problem of Ambrosetti-Prodi type. There has been considerable interest in understanding the number of solutions of the elliptic problem

$$\begin{cases} -\Delta u = g(u) - s\varphi_1(y) + \xi(y) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where s is a positive parameter, Ω is a smooth bounded domain in $\mathbb{R}^N$, $\varphi_1(y) > 0$ is an eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition corresponding to the first eigenvalue λ_1, $\xi(y)$ is a given function, $\lim_{t\to\infty}\frac{g(t)}{t} = \mu > \lim_{t\to-\infty}\frac{g(t)}{t} = \nu$, and (ν,μ) contains some eigenvalues of $-\Delta$ in Ω subject to Dirichlet boundary condition. Here, $\mu = +\infty$ and $\nu = -\infty$ are allowed.

A problem of this kind was first studied by Ambrosetti and Prodi [1] and by many authors, especially in the 1980's. We refer the readers to [3] and the references therein for a detailed bibliography on the topic. The main result is that if $g(t)$ grows subcritically at $+\infty$, then (1.1) has at least two solutions: one is a local minimum of the corresponding functional, the other being a solution of the mountain-pass type. Later on, in search of more solutions, Hofer [12] showed (1.1) admits at least four solutions provided g

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has linear growth and $\lambda_n < g'(+\infty) < \lambda_{n+1}$. The central idea of the proof was to take advantage of two asymptotically linear solutions, which can be approximately given by $\frac{t\varphi_1}{g'(\infty)-\lambda_1}$ and $\frac{t\varphi_1}{g'(-\infty)-\lambda_1}$. Since the second of these was almost a linear solution, it is possible to prove that it is local minimum of the associated functional

$$\Phi(u) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - G(u) + t\varphi_1 u + \xi(x)u\Big)dx$$

on the Sobolev space $H_0^1(\Omega)$, where G denotes the primitive of g. He was able to show that the positive solution was a critical point of Morse index n. In fact, $\Phi(c\varphi_1) \to -\infty$ as $c \to \infty$. Hence he applied the mountain-pass theorem to obtain another critical point. Thus we have three solutions, with the local degree of the minimum $+1$, the degree near the mountain pass solution -1, and the degree of the positive solution $(-1)^n$. Now recall that the degree on a big ball is 0. Hence they proved the existence of at least four solutions.

The Lazer-McKenna conjecture [13] states that (1.1) has an unbounded number of solutions as $s \to +\infty$, with $\mu = +\infty$, $\nu < \lambda_1$, and $g(t)$ not growing too rapidly. There was no result on this problem in the partial differential equation setting until recently Breuer et al. [3] showed (1.1) if $g(t) = t^2$, $\xi = 0$ and Ω is a rectangle in $\mathbb{R}^2$, by a partially numerical method. Motivated from the result in [9], the first author and Yan proved in [7] that the Lazer-McKenna conjecture is true if $g(t) = |t|^p$ and $\nu = -\infty$, where $p \in (1, \frac{N+2}{N-2})$ for $N \geq 3$ and $p \in (1, \infty)$ for $N = 2$. In the case that ν is finite, it is shown in [8] that the Lazer-McKenna conjecture is also true if $g(t) = t_+^p + \lambda t$, $\lambda \in (-\infty, \lambda_1)$, for $N \geq 3$ and $p \in (1, \frac{N+2}{N-2})$. For the critical case see [14], [15] and [17] and for the exponential case see [10].

In this paper we consider the problem

$$\begin{cases} -\Delta u = (u^+)^p + (u^-)^q - s\varphi_1(y) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and $1 < q < p < \frac{N+2}{N-2}$ and $N \geq 4$.

The main result of this paper is the following:

Theorem 1.1. *For any positive integer k, there exists an $s_0 > 0$, such that for $s \geq s_0$,* (1.2) *has at least k different solutions.*

Theorem 1.2. *Suppose that the solutions x_s obtained in Theorem* 1.1 *are isolated. Then critical groups of these solutions are given by $C_q(I_s, x_s) =$*

$\delta^{nk}_{q-k}\mathbb{Z}$, *where* $q = 0, 1, \dots$ *and* I_s *is the functional associated to the problem* (1.2).

The method used in this paper to prove Theorem 1.1 is constructive. The advantage of this method is that we can not only prove the existence of many solutions for (1.2) but also obtain the critical groups of these solutions. The disadvantage of this method is that the estimates depend heavily on the non-linear term $g(t)$. Although the results in [7] and [8] are very similar, the detailed estimates of the paper are quite different. In particular, we need a different norm, a different approximate solution and much more careful estimates.

Also, by a minor modification of our proof, we can prove that the results of the paper continue to hold if $|u^+|^p + |u^-|^q$ is replaced by

$$|u^+|^p + |u^-|^q + \sum_{i=1}^{m} a_i (u^+)^{p_i},$$

where $1 < p_1 < p_2 \cdots < p$, $q < p$, $a_i \in \mathbb{R}$.

2. Reduction to the singular elliptic problem

It is very easy to see that by the sub- and supersolution method there exists a unique negative solution $\bar{u}_s$ to the problem (1.2), such that $0 > \bar{u}_s > -\frac{s}{\lambda_1}\varphi_1$, $\forall x \in \Omega$, which follows by maximum principle.

Let us write the negative solution of (1.2) as $u_s = s^{\frac{1}{q}}\bar{u}_s$. Then $u := s^{\frac{1}{q}}(v + \bar{u}_s)$. Then (1.2) transforms into

$$-s^{-\frac{p-1}{q}}\Delta v = (v + \bar{u}_s)^p_+ + s^{-\frac{p-q}{q}}(v + \bar{u}_s)^q_- - s^{-\frac{p-q}{q}}|\bar{u}_s|^q.$$

Then the corresponding singular elliptic problem is

$$\begin{cases} -\varepsilon^2\Delta v = (v + \bar{u}_\varepsilon)^p_+ + \varepsilon^\alpha[(v + \bar{u}_\varepsilon)^q_- - |\bar{u}_\varepsilon|^q] & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\varepsilon^2 = s^{-\frac{p-1}{q}}$ and $0 < \alpha = \frac{2(p-q)}{p-1} < 2$ and $\bar{u}_\varepsilon$ satisfies the problem

$$\begin{cases} -\varepsilon^2\Delta \bar{u}_\varepsilon = \varepsilon^\alpha(|\bar{u}_\varepsilon|^q - \varphi_1) & \text{in } \Omega \\ \bar{u}_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by techniques similar to those in [7], it follows that $0 > \bar{u}_\varepsilon > -\varphi_1^{\frac{1}{q}}$ $\forall x \in \Omega$, and $\bar{u}_\varepsilon$ converges to $-\varphi_1^{\frac{1}{q}}$ on any compact subset of Ω as $\varepsilon \to 0$.

In order to prove Theorem 1.1 we only need to prove that the number of the solutions of (2.1) is unbounded as $\varepsilon \to 0$. Without loss of generality, we assume that $\max_{z\in\Omega} \varphi_1(z) = 1$.

The limiting problem for equation (2.1) is

$$\begin{cases} -\Delta U = (U-1)_+^p \,, U > 0 \ \text{ in } \mathbb{R}^N \\ U(0) = \max\limits_{y\in\mathbb{R}^N} U(y), \quad U(y) \to 0 \ \text{ as } |y| \to \infty, \end{cases} \tag{2.2}$$

which admits a unique solution $U(y) = U(|y|)$ if $N \geq 3$ in $D^{1,2}(\mathbb{R}^N)$. Also by Flucher and Wei [11] it follows that the kernel of the operator $-\Delta v - p(U-1)_+^{p-1} v$, $v \in D^{1,2}(\mathbb{R}^N)$ is spanned by $\{\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_N}\}$, which implies that U is non-degenerate.

For any $z \in \mathbb{R}^N$, let $U_{\varepsilon,z}(y) = U(\frac{y-z}{\varepsilon})$. Let $P_{\varepsilon,\Omega} U_{\varepsilon,z}(y) = V_{\varepsilon,z}(y)$ be the solution of

$$\begin{cases} -\varepsilon^2 \Delta v = (U_{\varepsilon,z} - 1)_+^p & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

and let $\hat{V}_{\varepsilon,z}$ be the solution of

$$\begin{cases} -\varepsilon^2 \Delta v + q\varepsilon^\alpha |\bar{u_\varepsilon}|^{q-1} v = (U_{\varepsilon,z} - 1)_+^p & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Then we have

Theorem 2.1. *Suppose $N \geq 4$. Let $k > 0$ be an integer. Then there is an ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$, (2.1) has a solution of the form*

$$\tilde{v}_\varepsilon = \sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega_\varepsilon,$$

where $\omega_\varepsilon \in H_0^1(\Omega)$ satisfies that as $\varepsilon \to 0$,

$$\varepsilon^2 \int_\Omega |D\omega_\varepsilon|^2 + q\varepsilon^\alpha \int_\Omega |\bar{u_\varepsilon}|^{q-1} \omega_\varepsilon^2 = o(\varepsilon^N)$$

$\frac{|x_{\varepsilon,i} - x_{\varepsilon,j}|}{\varepsilon} \to +\infty$, *for $i \neq j$, $x_{\varepsilon,j} \to \tilde{x}_j \in \Omega$ with $\varphi_1(\tilde{x}_j) = \max_{z\in\Omega} \varphi_1(z)$.*

A direct consequence of Theorem 2.1 implies that (2.1) admits an unbounded number of solutions as $s \to \infty$ and $N \geq 4$.

The result for the cases $N = 2, 3$ is still open. In the case of $N = 2$ the problem (2.2) does not admit a solution. In the case $N = 3$, there are some technical difficulties. But in case in which the domain Ω is a ball, we can show that problem (1.1) admits at least k solutions except for modulo rotational invariance, which follows similarly as in [16].

3. The reduction

Denote

$$\langle u, v\rangle_\varepsilon = \int_\Omega (\varepsilon^2 DuDv + q\varepsilon^\alpha |\bar{u}_\varepsilon|^{q-1} uv),$$

and $\|v\|_\varepsilon^2 = \langle v, v\rangle_\varepsilon$. Let $S = \{x : x \in \Omega, \varphi_1(x) = 1\}$, and let

$$D_{k,\varepsilon} = \Big\{ x : x = (x_1, \dots, x_k) \in \Omega^k, |1 - \varphi_1^{\frac{1}{q}}(x_j)| \le \varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}}, \\ j = 1, 2, \dots, k, \quad U\Big(\frac{|x_i - x_j|}{\varepsilon}\Big) \le \varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}} \Big\},$$

where τ is a small positive constant, $\gamma = \min\{1, \beta\}$ and $\beta > 0$ will be specified later.

Let H be the usual Sobolev space $H_0^1(\Omega)$ and with the norm $\|v\|_\varepsilon$. For any $x \in D_{k,\varepsilon}$, let

$$E_{\varepsilon,x,k} = \Big\{ \omega \in H : \Big\langle \omega, \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}} \Big\rangle_\varepsilon = 0, l = 1, \dots, N, j = 1, \dots, k \Big\},$$

where $x_j = (x_{j1}, \dots, x_{jN}) \in \mathbb{R}^N$. Let

$$I_\varepsilon(v) = \frac{\varepsilon^2}{2}\int_\Omega |Dv|^2 + \frac{q\varepsilon^\alpha}{2}\int_\Omega |\bar{u}_\varepsilon|^{q-1} v^2 - \int_\Omega F_\varepsilon(y, v)dy,$$

where

$$F_\varepsilon(y,t) = \frac{1}{p+1}(t + \bar{u}_\varepsilon)_+^{p+1} + \frac{\varepsilon^\alpha}{q+1}(t + \bar{u}_\varepsilon)_-^{q+1} \\ - \frac{\varepsilon^\alpha}{q+1}|\bar{u}_\varepsilon|^{q+1} - \varepsilon^\alpha |\bar{u}_\varepsilon|^q t + \frac{q\varepsilon^\alpha}{2}|\bar{u}_\varepsilon|^{q-1} t^2.$$

In this section, we will reduce the problem of finding a solution of the form $\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \omega_\varepsilon$ for (2.1) to a finite-dimensional problem. We will prove that for each $x \in D_{k,\varepsilon}$, there is a $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k}$, such that

$$\Big\langle I'_\varepsilon\Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \omega_\varepsilon\Big), \eta \Big\rangle_\varepsilon = 0, \quad \forall \eta \in E_{\varepsilon,x,k}. \tag{3.1}$$

In Section 4, we will choose $x \in D_{k,\varepsilon}$ such that $\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega_\varepsilon$ is a solution of (2.1). Let

$$K(x,\omega) = I_\varepsilon\Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega_\varepsilon\Big),\ x \in D_{k,\varepsilon}, \omega \in E_{\varepsilon,x,k}.$$

Then, we prove (3.1) is equivalent to finding a critical point $\omega_{\varepsilon,x}$ for $K(x,\omega)$ in $E_{\varepsilon,x,k}$. We expand $K(x,\omega)$ near $\omega = 0$ as

$$K(x,\omega) = K(x,0) + l_{\varepsilon,x}(\omega) + \tfrac{1}{2} Q_{\varepsilon,x}(\omega,\omega) + R_\varepsilon(\omega),$$

where

$$\begin{aligned} l_{\varepsilon,x}(\omega) \ &= \ \sum_{j=1}^{k} \varepsilon^2 \int_\Omega D\hat{V}_{\varepsilon,x_j} D\omega - \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^p \omega \\ &+ \ \varepsilon^\alpha \Big\{ \int_\Omega |\bar{u}_\varepsilon|^q \omega - \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^q \omega \Big\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} Q_{\varepsilon,x}(\omega,\eta) \ &= \ \varepsilon^2 \int_\Omega D\eta D\omega - p \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^{p-1} \eta\omega \\ &+ \ q\varepsilon^\alpha \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q-1} \eta\omega \quad \forall \omega,\ \eta \in H, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} R_\varepsilon(\omega) = \ & \frac{1}{p+1} \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon + \omega\Big)_+^{p+1} - \frac{1}{p+1} \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^{p+1} \\ & - \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^{p} \omega - \frac{p}{2} \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^{p-1} \omega^2 \\ & + \frac{\varepsilon^\alpha}{q+1} \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon + \omega\Big)_-^{q+1} - \frac{\varepsilon^\alpha}{q+1} \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q+1} \\ & - \varepsilon^\alpha \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q} \omega + \frac{q}{2} \varepsilon^\alpha \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q-1} \omega^2. \end{aligned} \tag{3.4}$$

We will prove in Lemma 3.7 that $l_{\varepsilon,x} : E_{\varepsilon,x,k} \to \mathbb{R}$ is a bounded linear functional. In particular, by the Riesz representation theorem there is an $l_{\varepsilon,x} \in E_{\varepsilon,x,k}$ such that $\langle l_{\varepsilon,x}, \omega\rangle_\varepsilon = l_{\varepsilon,x}(\omega)$. In Lemma 3.8, we will prove

$$|Q_{\varepsilon,x}(\omega,\eta)| \le C\|\omega\|_\varepsilon\|\eta\|_\varepsilon,$$

where C is a constant independent of ε. In particular, there is a bounded linear operator $Q_{\varepsilon,x} : E_{\varepsilon,x,k} \to E_{\varepsilon,x,k}$ such that $\langle Q_{\varepsilon,x}\omega, \eta\rangle_\varepsilon = Q_{\varepsilon,x}(\omega,\eta)$. Thus, finding a critical point for $K(x,\omega)$ in $E_{\varepsilon,x,k}$ is equivalent to solving the following problem in $E_{\varepsilon,x,k}$:

$$l_{\varepsilon,x} + Q_{\varepsilon,x}\omega + R'_\varepsilon(\omega) = 0. \tag{3.5}$$

We will prove in Lemma 3.9 that the operator $Q_{\varepsilon,x}$ is invertible in $E_{\varepsilon,x,k}$. So we see that $l_{\varepsilon,x} + Q_{\varepsilon,x}\omega = 0$ has a unique solution in $E_{\varepsilon,x,k}$ for each fixed $x \in D_{k,\varepsilon}$. In Lemma 3.10, we will prove that if ω belongs to a suitable set, the term $R'_\varepsilon(\omega)$ is a small perturbation term in (3.5). Thus, we can use the contraction mapping theorem to prove that (3.5) has a unique solution for each fixed $x \in D_{k,\varepsilon}$.

In order to prove that $l_{\varepsilon,x}(\omega)$ is a bounded linear functional on H and $Q(\omega,\eta)$ is a bounded quadratic form in H, we need the following lemmas.

Remark 3.1. If we choose $R > 0$, such that $U(y) < 1$ for all $|y| \ge R$, then we see that (2.4) and (2.3) reduce to

$$-\varepsilon^2\Delta\hat{V}_{\varepsilon,z} + q\varepsilon^\alpha|\bar{u_\varepsilon}|^{q-1}\hat{V}_{\varepsilon,z} = 0$$

$$-\Delta V_{\varepsilon,z} = \varepsilon^{-2}(U_{\varepsilon,z} - 1)^p_+ = 0,$$

for all $x \in \Omega \setminus B_{\varepsilon R}(z)$, and also note that as

$$\hat{V}_{\varepsilon,z} \le U_{\varepsilon,z} \le C\frac{\varepsilon^{N-2}}{|x-z|^{N-2}} \quad \forall\, x \in \Omega,$$

we have $\hat{V}_{\varepsilon,z}(\varepsilon R) \sim 1$ and for $N \ge 4$

$$\int_\Omega \hat{V}^3_{\varepsilon,z}dx = O(\varepsilon^N)$$

as

$$\begin{aligned}\int_\Omega \hat{V}^3_{\varepsilon,z}dx &= \int_{B_{\varepsilon R}(z)} \hat{V}^3_{\varepsilon,z}dx + \int_{\Omega\setminus B_{\varepsilon R}(z)} \hat{V}^3_{\varepsilon,z}dx \\ &= O(\varepsilon^N) + O\Big(\varepsilon^N\int_R^\infty \frac{1}{r^{2N-5}}dr\Big) = O(\varepsilon^N).\end{aligned}$$

Lemma 3.2. *There is a constant $R > 0$ large, such that*

$$\hat{V}_{\varepsilon,z} \leq C_1 e^{-\frac{\sqrt{Cq}}{\varepsilon^{1-\alpha/2}}(\min\{|x-z|,R'\})}, \quad \forall x \in \Omega \setminus B_{\varepsilon^{1-\alpha/2}R}(z),$$

where C_1 is a positive constant independent of ε and $R' > R\varepsilon^{1-\alpha/2}$ and is independent of ε.

Proof. First note that for all $x \in \Omega$, it follows by the maximum principle that $w_{\varepsilon,z} \leq V_{\varepsilon,z}$. Let $R_0 = R\varepsilon$ as in Remark 3.1. Choose $0 < R_1 < R_2$ such that $B_{R_0}(z) \subset B_{R_1}(z) \subset \Omega \subset B_{R_2}(z)$, where R_2 is independent of ε. Define

$$m_\varepsilon(x) = \begin{cases} qC\varepsilon^{\alpha-2} & \text{if} \quad |x-z| < R_1 \\ 0 & \text{if} \quad |x-z| \geq R_1, \end{cases}$$

where $|\bar{u}_\varepsilon|^{q-1} \geq C$ in $B_{R_1}(z)$. Hence we have $m_\varepsilon(x) \leq q|\bar{u}_\varepsilon|^{q-1}\varepsilon^{\alpha-2}$ in Ω. Let $w_{\varepsilon,z}$ be a solution of the equation

$$\begin{cases} -\Delta w_{\varepsilon,z} + m_\varepsilon(x) w_{\varepsilon,z} &= \varepsilon^{-2}(U_{\varepsilon,z} - 1)_+^p \quad \text{in} \quad B_{R_2}(z) \\ w_{\varepsilon,z} &= 0 \quad \text{on} \quad \partial B_{R_2}(z). \end{cases}$$

We claim that $\hat{V}_{\varepsilon,z} \leq w_{\varepsilon,z}$ on Ω. Subtracting the above equation from (2.4) we have

$$\begin{cases} -\Delta(\hat{V}_{\varepsilon,z}(x) - w_{\varepsilon,z}) + q\varepsilon^{\alpha-2}|\bar{u}_\varepsilon|^{q-1}(\hat{V}_{\varepsilon,z}(x) - w_{\varepsilon,z}) &\leq 0 \quad \text{in} \quad \Omega \\ (\hat{V}_{\varepsilon,z}(x) - w_{\varepsilon,z}) &\leq 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Hence by the maximum principle we have $\hat{V}_{\varepsilon,z}(x) \leq w_{\varepsilon,z}$ in Ω. Also note that $w_{\varepsilon,z}$ satisfies the problem

$$-\Delta w_{\varepsilon,z} + qC\varepsilon^{\alpha-2} w_{\varepsilon,z} = 0,$$

on $B_{R_1} \setminus B_{R_0}$. Hence, we can write

$$w_{\varepsilon,z}(r) = C_1 u_0(\varepsilon^{-(1-\alpha/2)}r) + C_2 u_1(\varepsilon^{-(1-\alpha/2)}r),$$

where $u_0(r) \sim r^{-\frac{N-1}{2}}e^{-\tau r}$ and $u_1(r) \sim r^{-\frac{N-1}{2}}e^{\tau r}$ as $r \to \infty$ with $u_1(0) = 1$, $\tau = \sqrt{Cq}$ and $C_1 > 0, C_2 < 0$. Moreover, there are corresponding estimates for the derivatives and u_0 and $u_1 > 0$. Since $w_{\varepsilon,z}(r)$is radially decreasing we have $w'_{\varepsilon,z}(r) \leq 0$, and hence by a simple calculation we obtain that

$$\frac{|C_2|}{C_1} \leq e^{-2\tau r}\left|\frac{\tau + \frac{N-1}{2r^2}}{\tau - \frac{N-1}{2r^2}}\right|.$$

Hence, there exists $C > 0$ such that $\frac{|C_2|}{C_1} \leq Ce^{-2\tau R_1}$, which implies that for $r \leq \frac{1}{2}R_1$ we have $w_{\varepsilon,z}(r) \leq C_1 u_0(\varepsilon^{1-\alpha/2}r)$. Also note that there exists a $C_3 > 0$ such that $u_0 \sim C_3 r^{2-N}$ as $r \to 0$. As $w_{\varepsilon,z}(\varepsilon R) \leq K$ we have

$K \geq C_1C_3(\varepsilon^{-1+\alpha/2}\varepsilon R)^{2-N}$, which implies that $C_1 \leq KC_3\varepsilon^{\frac{\alpha}{2}(N-2)}$. This implies that

$$w_{\varepsilon,z} \leq C_0\varepsilon^{\frac{\alpha}{2}(N-2)}u_0(\varepsilon^{-(1-\alpha/2)}r),$$

for all $r \leq \frac{R_1}{2}$. But $w_{\varepsilon,z}(r) \leq w_{\varepsilon,z}(\frac{R_1}{2})$ for all $r \geq \frac{R_1}{2}$. Hence, we have

$$\hat{V}_{\varepsilon,z} \leq C\varepsilon^{\frac{\alpha}{2}(N-2)}u_0(\varepsilon^{-(1-\alpha/2)}r),$$

for all $r \in B_{R_1}(z)\backslash B_{R_0}(z)$. Choose $R_1 = R\varepsilon^{\gamma}$ where $0 < \gamma < 1-\alpha/2$. Choose $R' > R_1$ and independent of ε and, noting the fact that $w_{\varepsilon,z}(r) \leq w_{\varepsilon,z}(R')$ for all $r \leq R_1$, we have for all $z \in \Omega \setminus B_{R\varepsilon^{1-\alpha/2}}(z)$

$$\hat{V}_{\varepsilon,z} \leq C_1 e^{-\frac{\sqrt{Cq}}{\varepsilon^{1-\alpha/2}}(\min\{|x-z|,R_1\})}. \qquad \square$$

Lemma 3.3. *If $\beta < \alpha$, then for any $z \in \Omega$,*

$$\|\hat{V}_{\varepsilon,z} - V_{\varepsilon,z}\|_{L^\infty(\Omega)} \leq C\varepsilon^{\beta},$$

where $C > 0$ is a constant independent of ε.

Proof. It is easy to check that $0 < \hat{V}_{\varepsilon,z} \leq V_{\varepsilon,z} \leq U_{\varepsilon,z}, \ \forall z \in \Omega$. Subtracting (2.4) from (2.3) we have

$$\begin{cases} -\varepsilon^2\Delta(V_{\varepsilon,z} - \hat{V}_{\varepsilon,z}) = q\varepsilon^{\alpha}|\bar{u}_\varepsilon|^{q-1}\hat{V}_{\varepsilon,z} & \text{in } \Omega \\ (V_{\varepsilon,z} - \hat{V}_{\varepsilon,z}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Hence, by global $L^{\tilde{q}}(\Omega)$ estimates we have

$$\|V_{\varepsilon,z} - \hat{V}_{\varepsilon,z}\|_{W^{2,\tilde{q}}(\Omega)} \leq C\varepsilon^{\alpha-2}\|\hat{V}_{\varepsilon,z}\|_{L^{\tilde{q}}(\Omega)}$$
$$= C\varepsilon^{\alpha-2}\Big(\int_{B_{\varepsilon R}(z)}|\hat{V}_{\varepsilon,z}|^{\tilde{q}} + \int_{B_{\varepsilon^{1-\alpha/2}R}(z)\backslash B_{\varepsilon R}(z)}|\hat{V}_{\varepsilon,z}|^{\tilde{q}} + \int_{\Omega\backslash B_{\varepsilon^{1-\alpha/2}R}(z)}|\hat{V}_{\varepsilon,z}|^{\tilde{q}}\Big)^{1/\tilde{q}}.$$

Choose $\tilde{q} > \frac{N}{2}$ and $\beta = \frac{N}{\tilde{q}} + (\alpha - 2) > 0$; we use Lemma 3.2 and the fact that $\hat{V}_{\varepsilon,z}$ is bounded in the interior of the $B_{\varepsilon R}(z)$ and note the fact that for $N \geq 4$,

$$\int_{B_{\varepsilon^{1-\alpha/2}R}(z)\backslash B_{\varepsilon R}(z)}|\hat{V}_{\varepsilon,z}|^{\tilde{q}} \leq \int_{B_{\varepsilon^{1-\alpha/2}R}(z)\backslash B_{\varepsilon R}(z)}|U_{\varepsilon,z}|^{\tilde{q}}$$
$$= O\Big(\varepsilon^N\int_R^{R/\varepsilon^{\alpha/2}}\frac{1}{r^{(N-2)\tilde{q}-N+1}}dr\Big) = O(\varepsilon^N),$$

provided $(N-2)\tilde{q} > N$. Also note that $\frac{N}{2} \geq \frac{N}{N-2}$ if $N \geq 4$. Hence, we get

$$\|V_{\varepsilon,z} - \hat{V}_{\varepsilon,z}\|_{L^\infty(\Omega)} \leq C\varepsilon^{\beta},$$

where $\beta < \alpha$. □

Lemma 3.4. *For any $z \in \Omega$, let $\psi_{\varepsilon,z} = U_{\varepsilon,z} - V_{\varepsilon,z}$. Then*

$$\psi_{\varepsilon,z} = (c_0 + o(1))\varepsilon^{N-2} H(y,z),$$

where c_0 is a positive constant, $H(y,z)$ is the regular part of the Green's function, and $o(1) \to 0$ as $\varepsilon \to 0$.

Proof. For the proof see [8]. □

Lemma 3.5. *There is a constant $R > 0$ large, such that*

$$\sum_{j=1}^{k} V_{\varepsilon,x_j}(y) + \bar{u}_\varepsilon(y) < 0 \ \ \forall y \in \Omega \setminus \cup_{j=1}^{k} B_{\varepsilon R}(x_j).$$

Proof. Note by Remark 3.1 we can choose $R > 0$ such that

$$-\Delta V_{\varepsilon,x_j} = \varepsilon^{-2}(U_{\varepsilon,x_j} - 1)_+^p = 0 \quad \forall y \in \Omega \setminus B_{\varepsilon R}(x_j),$$

and

$$0 \le V_{\varepsilon,x_j} \le U_{\varepsilon,x_j} = o_R(1), \quad \text{on } \partial B_{\varepsilon R}(x_j),$$

where $o_R(1) \to 0$ as $R \to +\infty$. Hence, we obtain

$$-\Delta\Big(\sum_{j=1}^{k} V_{\varepsilon,x_j}\Big) = 0, \ \forall y \in \Omega \setminus \cup_{j=1}^{k} B_{\varepsilon R}(x_j)$$

$$\sum_{j=1}^{k} V_{\varepsilon,x_j}(y) = o_R(1) \ \forall y \in \cup_{j=1}^{k} \partial B_{\varepsilon R}(x_j)$$

and

$$\sum_{j=1}^{k} V_{\varepsilon,x_j}(y) = 0 \ \forall y \in \partial\Omega.$$

Hence, we obtain by the maximum principle

$$\sum_{j=1}^{k} V_{\varepsilon,x_j}(y) = o_R(1) \quad \forall y \in \Omega \setminus \cup_{j=1}^{k} B_{\varepsilon R}(x_j).$$

As $\bar{u}_\varepsilon$ is a negative solution to (2.1) we obtain

$$\sum_{j=1}^{k} V_{\varepsilon,x_j}(y) = o_R(1)|\bar{u}_\varepsilon| \ \forall y \in \Omega \setminus \cup_{j=1}^{k} B_{\varepsilon R}(x_j),$$

which implies that

$$\sum_{j=1}^{k} V_{\varepsilon,x_j}(y) + \bar{u}_\varepsilon < 0 \ \forall y \in \Omega \setminus \cup_{j=1}^{k} B_{\varepsilon R}(x_j). \qquad \square$$

Corollary 3.6. *There is a constant $R > 0$ large, such that*

$$\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j}(y) + \bar{u}_\varepsilon(y) < 0 \ \ \forall y \in \Omega \setminus \cup_{j=1}^{k} B_{\varepsilon R}(x_j).$$

Proof. The proof follows trivially from Lemma 3.5 and the fact that $\hat{V}_{\varepsilon,x_j} \leq V_{\varepsilon,x_j}$. Also note that

$$\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j}(y) = o_R(1)|\bar{u}_\varepsilon| \ \forall y \in \Omega \setminus \cup_{j=1}^{k} B_{\varepsilon R}(x_j). \qquad \square$$

Lemma 3.7. *The functional defined in* (3.2) *is a bounded linear functional from H to $\mathbb{R}$. Moreover,*

$$\|l_{\varepsilon,x}\|_\varepsilon = \varepsilon^{\frac{N}{2}} O\Big(\varepsilon^\gamma + \sum_{i=1}^{k} |1 - \varphi_1^{\frac{1}{q}}(x_j)| + \sum_{i \neq j} U\Big(\frac{|x_i - x_j|}{\varepsilon}\Big)\Big), \tag{3.7}$$

where $\gamma := \min\{\beta, 1\}$.

Proof. Using (2.3) we have

$$\begin{aligned} l_{\varepsilon,x}(\omega) = &\sum_{j=1}^{k} \int_\Omega (U_{\varepsilon,x_j} - 1)_+^p \omega - \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^p \omega \\ &+ \varepsilon^\alpha \Big\{ \int_\Omega |\bar{u}_\varepsilon|^q \omega - \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^q \omega - q \sum_{j=1}^{k} \int_\Omega \hat{V}_{\varepsilon,x_j} |\bar{u}_\varepsilon|^{q-1} \omega \Big\}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} l_{\varepsilon,x}(\omega) = &\sum_{j=1}^{k} \int_\Omega (U_{\varepsilon,x_j} - 1)_+^p \omega - \int_\Omega \Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^p \omega \\ &+ \varepsilon^\alpha O\Big(\sum_{i=1}^{k} \sum_{j=1}^{k} \int_{B_{\varepsilon R}(x_i)} \hat{V}_{\varepsilon,x_j}^2 |\bar{u}_\varepsilon|^{q-2} |\omega| + \sum_{j=1}^{k} \int_{\Omega \setminus \cup B_{\varepsilon R}(x_i)} \hat{V}_{\varepsilon,x_j}^2 |\bar{u}_\varepsilon|^{q-2} |\omega| \Big). \end{aligned}$$

Now, by applying Holder's inequality repeatedly, we have

$$\varepsilon^\alpha O\Big(\sum_{i=1}^k\sum_{j=1}^k\int_{B_{\varepsilon R}(x_i)}\hat V^2_{\varepsilon,x_j}|\bar u_\varepsilon|^{q-2}|\omega|\Big)\le C\varepsilon^\alpha\int_{B_{\varepsilon R}(x_i)}|\omega|$$
$$\le C\varepsilon^\alpha|B_{\varepsilon R}(x_i)|^{\frac12}\Big(\int_{B_{\varepsilon R}(x_i)}|\omega|^2\Big)^{\frac12}\le C\varepsilon^\alpha\varepsilon^{\frac N2}|B_{\varepsilon R}(x_i)|^{\frac12-\frac1{2^*}}\Big(\int_{B_{\varepsilon R}(x_i)}|\omega|^2\Big)^{\frac1{2^*}}$$
$$\le C\varepsilon^\alpha\varepsilon^{\frac N2}\Big(\varepsilon^2\int_\Omega|D\omega|^2\Big)^{\frac12}\le C\varepsilon^{\frac N2+\alpha}\|\omega\|_\varepsilon.$$

Choose $\gamma>0$ such that $0<2-\gamma<2-\alpha$. Then by Hardy's inequality,

$$\Big(\int_{B_{\varepsilon^{1-\frac\gamma2}R}(x_i)\setminus B_{\varepsilon R}(x_i)}|\omega|^2\Big)\le C\varepsilon^{2-\gamma}\int_\Omega|D\omega|^2dx,$$

and since $N\ge 4$ and $\alpha<2$, we have

$$\varepsilon^\alpha O\Big(\sum_{i=1}^k\sum_{j=1}^k\int_{B_{\varepsilon^{1-\frac\gamma2}R}(x_i)\setminus B_{\varepsilon R}(x_i)}\hat V^2_{\varepsilon,x_j}|\bar u_\varepsilon|^{q-2}|\omega|\Big)$$
$$\le C\varepsilon^{2N-4+\alpha}\int_{B_{\varepsilon^{1-\frac\gamma2}R}(x_i)\setminus B_{\varepsilon R}(x_i)}|\omega|$$
$$\le \varepsilon^{2N-4+\alpha}\Big(\int_{B_{\varepsilon^{1-\frac\gamma2}R}(x_i)\setminus B_{\varepsilon R}(x_i)}|\omega|^2\Big)^{\frac12}$$
$$\le \varepsilon^{2N-4+\alpha}\Big(\varepsilon^{2-\gamma}\int_\Omega|D\omega|^2\Big)^{\frac12}\le\varepsilon^{2N-4+\alpha-\frac\gamma2}\|\omega\|_\varepsilon\le\varepsilon^{\frac N2+\alpha}\|\omega\|_\varepsilon.$$

On the other hand, we see that as $\hat V_{\varepsilon,x_j}$ is exponentially small outside the balls $B_{\varepsilon^{1-\frac\gamma2}R}(x_i)$ and noting the fact that $\frac{\hat V_{\varepsilon,x_j}}{|\bar u_\varepsilon|}$ is bounded in $\Omega\setminus\cup B_{\varepsilon^{1-\frac\gamma2}R}(x_i)$,

$$\varepsilon^\alpha O\Big(\sum_{j=1}^k\int_{\Omega\setminus\cup B_{\varepsilon^{1-\frac\gamma2}R}(x_i)}\hat V^2_{\varepsilon,x_j}|\bar u_\varepsilon|^{q-2}|\omega|\Big)$$
$$\le C\varepsilon^\alpha\Big(\sum_{j=1}^k\int_{\Omega\setminus\cup B_{\varepsilon^{1-\gamma/2}R}(x_i)}\hat V_{\varepsilon,x_j}|\bar u_\varepsilon|^{q-1}|\omega|\Big)$$
$$\le C\varepsilon^{\alpha+\frac N2+\alpha}\Big(\int_{\Omega\setminus\cup B_{\varepsilon^{1-\gamma/2}R}(x_i)}|\bar u_\varepsilon|^{\frac{q-1}2}|\omega|\Big)\le C\varepsilon^{\alpha+\frac N2}\|\omega\|_\varepsilon.$$

Also note that

$$
\begin{aligned}
&\int_\Omega \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon\Big)_+^p \omega = \int_{\cup B_{\varepsilon R}(x_i)} \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon\Big)_+^p \omega \\
&= \int_{\cup B_{\varepsilon R}(x_i)} \Big(\sum_{j=1}^k V_{\varepsilon,x_j} + \bar u_\varepsilon\Big)_+^p \omega + \varepsilon^\beta O\Big(\sum_{i=1}^k \sum_{j=1}^k \int_{B_{\varepsilon R}(x_i)} U_{\varepsilon,x_j}^{p-1}|\omega|\Big) \\
&= \int_{\cup B_{\varepsilon R}(x_i)} \Big(\sum_{j=1}^k V_{\varepsilon,x_j} + \bar u_\varepsilon\Big)_+^p \omega + O(\varepsilon^{\beta+\frac{N}{2}})\|\omega\|_\varepsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\cup B_{\varepsilon R}(x_i)} \Big(\sum_{j=1}^k V_{\varepsilon,x_j} + \bar u_\varepsilon\Big)_+^p \omega \\
&= \int_{\cup B_{\varepsilon R}(x_i)} \Big(\sum_{j=1}^k U_{\varepsilon,x_j} + \bar u_\varepsilon\Big)_+^p \omega + O\Big(\sum_{i=1}^k \sum_{j=1}^k \int_{B_{\varepsilon R}(x_i)} U_{\varepsilon,x_j}^{p-1}|\psi_{\varepsilon,x_j}||\omega|\Big) \\
&= \sum_{j=1}^k \int_{B_{\varepsilon R}(x_j)} (U_{\varepsilon,x_j}-1)_+^p \omega dx + O(\varepsilon^{N-2+\frac{N}{2}})\|\omega\|_\varepsilon \\
&+ O\Big(\int_{B_{\varepsilon R}(x_i)} \Big(\sum_{j=1}^k U_{\varepsilon,x_j}\Big)^{p-1} |\bar u_\varepsilon + 1||\omega|\Big) \\
&= \sum_{j=1}^k \int_{B_{\varepsilon_R}(x_j)} (U_{\varepsilon,x_j}-1)_+^p \omega\, dx \\
&+ \varepsilon^{\frac{N}{2}} O\Big(\varepsilon + \sum_{i=1}^k |1-\varphi_1^{\frac{1}{q}}(x_j)| + \sum_{i\neq j} U\Big(\frac{|x_i-x_j|}{\varepsilon}\Big)\Big)\|\omega\|_\varepsilon.
\end{aligned}
$$

This completes the proof. □

Lemma 3.8. *The quadratic form $Q(\omega,\eta)$ defined by* (3.3) *satisfies*

$$
|Q_{\varepsilon,x}(\omega,\eta)| \le C\|\omega\|_\varepsilon \|\eta\|_\varepsilon,
$$

where C is a constant independent of ε.

Proof. Note that

$$\varepsilon^2\int_\Omega D\eta D\omega - p\int_\Omega\Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j}+\bar u_\varepsilon\Big)_+^{p-1}\eta\omega \le C\|\omega\|_\varepsilon\|\eta\|_\varepsilon + C\int_{\cup B_{\varepsilon R}(x_i)}|\omega||\eta|$$

$$\le C\|\omega\|_\varepsilon\|\eta\|_\varepsilon + C\Big(\int_{B_{\varepsilon R}(x_i)}|\omega|^2\Big)^{\frac12}\Big(\int_{B_{\varepsilon R}(x_i)}|\eta|^2\Big)^{\frac12}\le C\|\omega\|_\varepsilon\|\eta\|_\varepsilon.$$

Note also that as

$$\Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j}+\bar u_\varepsilon\Big)_- \le |\bar u_\varepsilon| \ \text{ in } \ \Omega\setminus\cup B_{\varepsilon R}(x_i),$$

we have

$$\begin{aligned}
&q\varepsilon^\alpha\int_\Omega\Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j}+\bar u_\varepsilon\Big)_-^{q-1}\eta\omega\\
&\le C\int_{B_{\varepsilon R}(x_i)}|\omega||\eta| + Cq\varepsilon^\alpha\int_{\Omega\setminus\cup B_{\varepsilon R}(x_i)}\Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j}+\bar u_\varepsilon\Big)_-^{q-1}\omega\eta\\
&\le C\int_{B_{\varepsilon R}(x_i)}|\omega||\eta| + Cq\varepsilon^\alpha\int_{\Omega\setminus\cup B_{\varepsilon R}(x_i)}|\bar u_\varepsilon|^{q-1}\omega\eta\le C\|\omega\|_\varepsilon\|\eta\|_\varepsilon.
\end{aligned}$$

Lemma 3.9. *There is a constant $\rho>0$, independent of ε and $x\in D_{k,\varepsilon}$ such that*

$$\|Q_{\varepsilon,x}\omega\|_\varepsilon\ge\rho\|\omega\|_\varepsilon,\ \forall\omega\in E_{\varepsilon,x,k},\quad x\in D_{k,\varepsilon}.$$

Proof. Let us proceed by contradiction. Suppose that there are $\varepsilon_n\to 0$, $x_n\in D_{k,\varepsilon_n}$ with $x_{j,n}\to x_j\in S$, and $\omega_n\in E_{\varepsilon_n,x_n,k}$ such that $\|\omega_n\|_{\varepsilon_n}=\varepsilon_n^{\frac N2}$ and $\|Q_{\varepsilon_n,x_n}\omega_n\|_{\varepsilon_n}=o(\varepsilon_n^{\frac N2})$. We claim that for any fixed $R>0$ and $j=1,\dots,k$,

$$\int_{B_{\varepsilon_n R}(x_{j,n})}|\omega_n|^2=o(\varepsilon_n^N).$$

For any i define $\tilde\omega_{i,n}(y)=\omega_n(\varepsilon_n y+x_{i,n})$ and $\Omega_n=\{y:\varepsilon_n y+x_{i,n}\in\Omega\}$. Then

$$\int_{\Omega_n}|D\tilde\omega_{i,n}|^2=\varepsilon_n^{-N}\int_\Omega\varepsilon_n^2|D\omega_n|^2$$

and

$$\varepsilon_n^\alpha\int_{\Omega_n}|\tilde u_n|^{q-1}|\tilde\omega_{i,n}|^2=\varepsilon_n^\alpha\varepsilon_n^{-N}\int_\Omega|\bar u_{\varepsilon_n}|^{q-1}|\omega_n|^2,$$

where $\tilde{u}_n(y) = \bar{u}_{\varepsilon_n}(\varepsilon_n y + x_{i,n})$, and hence we have

$$\|\tilde{\omega}_{i,n}\|_{\varepsilon_n} = \varepsilon_n^{-\frac{N}{2}} \|\omega_n\|_{\varepsilon_n} = 1;$$

thereby there exist $\omega_i \in D^{1,2}(\mathbb{R}^N)$ such that

$$\tilde{\omega}_{i,n} \rightharpoonup \omega_i \quad \text{in} \quad D^{1,2}(\mathbb{R}^N)$$

$$\tilde{\omega}_{i,n} \to \omega_i \quad \text{in} \quad L^2_{loc}(\mathbb{R}^N).$$

We claim that

$$-\Delta\omega_i = p(U-1)_+^{p-1}\omega_i \text{ in } \mathbb{R}^N. \tag{3.8}$$

This is equivalent to proving that

$$\int_{\mathbb{R}^N} D\omega_i D\eta - p\int_{\mathbb{R}^N} (U-1)_+^{p-1}\omega_i\eta = 0 \qquad \forall \eta \in C_0^\infty(\mathbb{R}^N).$$

We have $\langle Q_{\varepsilon_n,x_n}\omega_n, \eta\rangle_{\varepsilon_n} = Q_{\varepsilon_n,x_n}(\omega_n, \eta)$, which implies that

$$o(\varepsilon_n^{\frac{N}{2}})\|\eta\|_{\varepsilon_n} = \varepsilon_n^2 \int_\Omega D\omega_n D\eta - p\int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon_n,x_{j,n}} + \bar{u}_{\varepsilon_n}\Big)_+^{p-1}\omega_n\eta$$
$$+ q\varepsilon_n^\alpha \int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon_n,x_{j,n}} + \bar{u}_{\varepsilon_n}\Big)_-^{q-1}\omega_n\eta \quad \forall \eta \in E_{\varepsilon_n,x_n,k}. \tag{3.9}$$

Hence, we obtain

$$o(1)\|\tilde{\eta}\| = \int_{\Omega_n} D\tilde{\omega}_{i,n}D\tilde{\eta} - p\int_{\Omega_n} \Big(\sum_{j=1}^k \tilde{V}_{\varepsilon_n,x_{j,n}} + \bar{u}_{\varepsilon_n}(\varepsilon_n y + x_{i,n})\Big)_+^{p-1}\tilde{\omega}_{i,n}\tilde{\eta}$$
$$+ q\varepsilon_n^\alpha \int_{\Omega_n} \Big(\sum_{j=1}^k \tilde{V}_{\varepsilon_n,x_{j,n}} + \bar{u}_{\varepsilon_n}(\varepsilon_n y + x_{i,n})\Big)_-^{q-1}\tilde{\omega}_{i,n}\tilde{\eta} \ \forall \tilde{\eta} \in \tilde{E}_{\varepsilon_n,x_n,k}, \tag{3.10}$$

where

$$\|\tilde{\eta}\|^2 = \int_{\Omega_n} |D\tilde{\eta}|^2 + q\varepsilon_n^\alpha \int_{\Omega_n} |\tilde{u}_n|^{q-1}|\tilde{\eta}|^2, \qquad \tilde{V}_{\varepsilon_n,x_{j,n}}(y) = \hat{V}_{\varepsilon_n,x_{j,n}}(\varepsilon_n y + x_{i,n}),$$

and

$$\tilde{E}_{\varepsilon_n,x_n,k} = \Big\{\tilde{\eta} : \int_{\Omega_n} D\tilde{W}_{n,j,l}D\tilde{\eta} + q\varepsilon_n^\alpha \int_{\Omega_n} |\tilde{u}_n|^{q-1}\tilde{W}_{n,j,l}\tilde{\eta} = 0\Big\},$$

where $\tilde{u}_n(y) = \bar{u}_{\varepsilon_n}(\varepsilon_n y + x_{i,n})$ and $\tilde{W}_{n,j,l} = \varepsilon_n \frac{\partial \hat{V}_{\varepsilon_n,x_{j,n}}(\varepsilon_n y + x_{i,n})}{\partial x_{jl}}$.

For any $\eta \in C_0^\infty(\mathbb{R}^N)$, we can choose $a_{jln} \in \mathbb{R}$, such that

$$\tilde{\eta}_n = \eta - \sum_{j=1}^{k} \sum_{l=1}^{N} a_{jln} \tilde{W}_{n,j,l} \in \tilde{E}_{\varepsilon_n, x_n, k}.$$

Note that $\tilde{W}_{n,j,l}$ satisfies

$$\begin{cases} -\Delta \tilde{W}_{n,j,l} + q\varepsilon_n^\alpha |\tilde{u}_n|^{q-1} \tilde{W}_{n,j,l} = p\Big(U\Big(y - \frac{x_{i,n}-x_{j,n}}{\varepsilon_n}\Big) - 1\Big)_+^{p-1} \frac{\partial U}{\partial x_l} & \text{in } \Omega_n \\ \tilde{W}_{n,j,l} = 0 & \text{on } \partial\Omega_n. \end{cases}$$

For $i \neq j$, as the support of η is compact we have

$$\begin{aligned} \langle \eta, \tilde{W}_{n,j,l} \rangle &= \int_{\Omega_n \cap \text{ supp } \eta} D\tilde{W}_{n,j,l} D\eta + q\varepsilon_n^\alpha \int_{\Omega_n \cap \text{ supp } \eta} |\tilde{u}_n|^{q-1} \tilde{W}_{n,j,l}\eta \\ &= p \int_{\Omega_n \cap \text{ supp } \eta} \Big(U\Big(y - \frac{x_{i,n} - x_{j,n}}{\varepsilon_n}\Big) - 1\Big)_+^{p-1} \frac{\partial U}{\partial x_l}\Big(y - \frac{x_{i,n} - x_{j,n}}{\varepsilon_n}\Big)\eta = o(1), \end{aligned}$$

while for $i = j$, $|\langle \eta, \tilde{W}_{n,j,l} \rangle| \leq C$ it easily follows from coordinate transformation that $a_{jln} = (I + o(1))^{-1} \langle \tilde{W}_{n,j,l}, \eta \rangle$, where I is the identity matrix. Hence, $a_{jln} \to 0$ as $n \to \infty$ for $i \neq j$.

We claim that

$$\tilde{W}_{n,j,l} \to \frac{\partial U}{\partial x_l} \text{ in } D^{1,2} \text{ as } n \to \infty.$$

It is easy to see that $\tilde{W}_{n,j,l}$ are uniformly bounded in $D^{1,2}(\Omega_n)$ and hence in $L^p(\Omega_n)$.

We will prove $\tilde{W}_{n,j,l}$ converges strongly to V in $D^{1,2}$, where

$$\begin{cases} -\Delta V = p\Big(U - 1\Big)_+^{p-1} \frac{\partial U}{\partial x_l} & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |DV|^2 = p \int_{R^N} \Big(U - 1\Big)_+^{p-1} \frac{\partial U}{\partial x_l} V. \end{cases}$$

Also note that if $\phi \in C_0^\infty(\mathbb{R}^N)$

$$q\varepsilon_n^\alpha |\tilde{u}_n|^{q-1} \tilde{W}_{n,j,l} \to 0 \text{ as } n \to \infty,$$

and hence, if $\tilde{W}_{n,j,l} \rightharpoonup W \in D^{1,2}(\mathbb{R}^n)$ and hence strongly to $\tilde{W}_{n,j,l} \to W$ in $L^2_{loc}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^N} DW D\phi = p \int_{R^N} \Big(U - 1\Big)_+^{p-1} \frac{\partial U}{\partial x_l} \phi,$$

and so $W = V$. Now we have

$$\int_{\Omega_n} |D\tilde{W}_{n,j,l}|^2 + q\varepsilon_n^\alpha \int_{\Omega_n} |\tilde{u}_n|^{q-1}\tilde{W}_{n,j,l}^2 = p\int_{\Omega_n} \Big(U-1\Big)_+^{p-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l}$$
$$\to p\int_{\mathbb{R}^N} \Big(U-1\Big)_+^{p-1}\frac{\partial U}{\partial x_l}W = \int_{\mathbb{R}^N} |DV|^2.$$

As both the terms on the left-hand side are positive, it follows that

$$\limsup_{n\to\infty} \int_{\Omega_n} |D\tilde{W}_{n,j,l}|^2 \leq \int_{\mathbb{R}^N} |DV|^2,$$

but by weak convergence we have

$$\liminf_{n\to\infty} \int_{\Omega_n} |D\tilde{W}_{n,j,l}|^2 \geq \int_{\mathbb{R}^N} |DV|^2.$$

Hence, $\tilde{W}_{n,j,l} \to V$ strongly in $D^{1,2}(\mathbb{R}^N)$ as $D^{1,2}(\mathbb{R}^N)$ is a Hilbert space, and

$$q\varepsilon_n^\alpha \int_{\Omega_n} |\tilde{u}_n|^{q-1}\tilde{W}_{n,j,l}^2 \to 0,$$

as $n \to \infty$. In fact, putting η_n in (3.10) and letting $n \to \infty$ we have

$$\int_{\mathbb{R}^N} D\omega_i D\eta - p\int_{\mathbb{R}^N} (U-1)_+^p\omega_i\eta$$
$$= \sum_{l=1}^k a_{il}\Big(\int_{\mathbb{R}^N} D\omega_i D\frac{\partial U}{\partial x_l} - p\int_{\mathbb{R}^N} (U-1)_+^p\omega_i\frac{\partial U}{\partial x_l}\Big),$$

where $a_{i,l} = \lim_{n\to\infty} a_{i,l,n}$. Using the non-degeneracy result we have

$$\int_{\mathbb{R}^N} D\omega_i D\eta - p\int_{\mathbb{R}^N} (U-1)_+^p\omega_i\eta = 0,$$

which proves our claim.

Since $\omega_i \in D^{1,2}(\mathbb{R}^N)$, it follows by [11] that

$$\omega_i = \sum_{l=1}^N a_l \frac{\partial U}{\partial x_l} \quad \text{for some}\, a_l \in \mathbb{R}. \tag{3.11}$$

On the other hand,

$$\int_{\Omega_n} D\tilde{W}_{n,j,l} D\tilde{\omega}_{i,n} + q\varepsilon_n^\alpha \int_{\Omega_n} |\tilde{u}_n|^{q-1}\tilde{W}_{n,j,l}\tilde{\omega}_{i,n} = 0,$$

which implies that

$$\int_{\mathbb{R}^N} D\Big(\frac{\partial U}{\partial x_l}\Big)D\omega_i = 0. \tag{3.12}$$

From (3.11) and (3.12) we get $\omega_i = 0$. Hence for any $R > 0$ we have

$$\int_{B_{\varepsilon_n R}(x_{j,n})} |\omega_n|^2 = o(\varepsilon_n^N);$$

hence,

$$\int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon_n, x_{j,n}} + \bar{u}_{\varepsilon_n} \Big)_+^{p-1} \omega_n^2 = o(\varepsilon_n^N),$$

and

$$\int_{\cup_{j=1}^k B_{\varepsilon_n R}(x_{j,n})} \Big(\sum_{j=1}^k \hat{V}_{\varepsilon_n, x_{j,n}} + \bar{u}_{\varepsilon_n} \Big)_-^{q-1} \omega_n^2 = o(\varepsilon_n^N).$$

Also by Taylor's expansion and the proof of Corollary 3.6 we have

$$\begin{aligned}
&\varepsilon_n^\alpha \int_{\Omega \setminus \cup_{i=1}^k B_{\varepsilon_n R}(x_{i,n})} \Big(\sum_{j=1}^k \hat{V}_{\varepsilon_n, x_{j,n}} + \bar{u}_{\varepsilon_n} \Big)_-^{q-1} \omega_n^2 - \varepsilon_n^\alpha \int_{\Omega \setminus \cup_{i=1}^k B_{\varepsilon_n R}(x_{i,n})} |\bar{u}_{\varepsilon_n}|^{q-1} \omega_n^2 \\
&= O\Big(\varepsilon^\alpha \int_{\Omega \setminus \cup_{i=1}^k B_{\varepsilon_n R}(x_{i,n})} \frac{\hat{V}_{\varepsilon x_j}}{|\bar{u}_\varepsilon|} |\bar{u}_\varepsilon|^{q-1} |\omega_n|^2 \Big) = \varepsilon_n^N o_R(1);
\end{aligned}$$

hence, it follows that

$$\begin{aligned}
o(\varepsilon_n^N) &\geq |\langle Q_{\varepsilon_n, x_n}, \omega_n \rangle_{\varepsilon_n}| \\
&\geq \|\omega_n\|_{\varepsilon_n}^2 - p \int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon_n, x_{j,n}} + \bar{u}_{\varepsilon_n} \Big)_+^{p-1} \omega_n^2 \\
&\quad + q\varepsilon_n^\alpha \int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon_n, x_{j,n}} + \bar{u}_{\varepsilon_n} \Big)_-^{q-1} \omega_n^2 - q\varepsilon_n^\alpha \int_\Omega |\bar{u}_{\varepsilon_n}|^{q-1} \omega_n^2 \\
&\geq \|\omega_n\|_{\varepsilon_n}^2 - o(\varepsilon_n^N) - o(\varepsilon_n^{\alpha+N}) - \varepsilon_n^N o_R(1),
\end{aligned}$$

which implies a contradiction. □

Lemma 3.10. *Let $R_\varepsilon(\omega)$ be the functional defined by* (3.4)*. Let $\omega \in H$ with $|\omega(y)| \leq \frac{1}{2}|\bar{u}_\varepsilon(y)|$. Let $R_\varepsilon(\omega) = J_1(\omega) + J_2(\omega)$, where*

$$\begin{aligned}
J_1(\omega) :&= \frac{1}{p+1} \int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon, x_j} + \bar{u}_\varepsilon + \omega \Big)_+^{p+1} - \frac{1}{p+1} \int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon, x_j} + \bar{u}_\varepsilon \Big)_+^{p+1} \\
&\quad - \int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon, x_j} + \bar{u}_\varepsilon \Big)_+^{p} \omega - \frac{p}{2} \int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon, x_j} + \bar{u}_\varepsilon \Big)_+^{p-1} \omega^2,
\end{aligned}$$

and

$$J_2(\omega):=\frac{\varepsilon^\alpha}{q+1}\int_\Omega\Big(\sum_{j=1}^k\hat V_{\varepsilon,x_j}+\bar u_\varepsilon+\omega\Big)_-^{q+1}-\frac{\varepsilon^\alpha}{q+1}\int_\Omega\Big(\sum_{j=1}^k\hat V_{\varepsilon,x_j}+\bar u_\varepsilon\Big)_-^{q+1}$$
$$-\varepsilon^\alpha\int_\Omega\Big(\sum_{j=1}^k\hat V_{\varepsilon,x_j}+\bar u_\varepsilon\Big)_-^{q}\omega+\frac{q}{2}\varepsilon^\alpha\int_\Omega\Big(\sum_{j=1}^k\hat V_{\varepsilon,x_j}+\bar u_\varepsilon\Big)_-^{q-1}\omega^2.$$

Then we have

$$|J_1(\omega)|\le C\{\varepsilon^{N(1-\frac{\min\{p+1,3\}}{2})}\|\omega\|_\varepsilon^{\min\{p+1,3\}}\}$$
$$\|J_1'(\omega)\|_\varepsilon\le C\{\varepsilon^{N(1-\frac{\min\{p+1,3\}}{2})}\|\omega\|_\varepsilon^{\min\{p,2\}}\}$$
$$\|J_1''(\omega)\|_\varepsilon\le C\{\varepsilon^{N(1-\frac{\min\{p+1,3\}}{2})}\|\omega\|_\varepsilon^{\min\{p-1,1\}}\},$$

and

$$|J_2(\omega)|\le C\{\varepsilon^{N(1-\frac{\min\{q+1,3\}}{2})+\alpha}\|\omega\|_\varepsilon^{\min\{q+1,3\}}+o(1)\|\omega\|_\varepsilon^2\}$$
$$\|J_2'(\omega)\|_\varepsilon\le C\{\varepsilon^{N(1-\frac{\min\{q+1,3\}}{2})+\alpha}\|\omega\|_\varepsilon^{\min\{q,2\}}+o(1)\|\omega\|_\varepsilon\}.$$

Proof. It is enough to prove the estimate for J_1 and J_2. The other estimates follow similarly. Since $|\omega(y)|\le\frac12|\bar u_\varepsilon(y)|$, we can choose an $R>0$ such that for all $y\in\Omega\setminus\cup_{j=1}^k B_{\varepsilon R}(x_j)$

$$\Big(\sum_{j=1}^k\hat V_{\varepsilon,x_j}+\bar u_\varepsilon+\omega\Big)_+=0\quad\text{and}\quad\Big(\sum_{j=1}^k\hat V_{\varepsilon,x_j}+\bar u_\varepsilon\Big)_+=0,$$

$$|J_1(\omega)|\le C\int_{B_{\varepsilon R}(x_i)}|\omega|^{\min\{p+1,3\}}dx\le C\varepsilon^{N(1-\frac{\min\{p+1,3\}}{2^*})}\Big(\int_\Omega|\omega|^{2^*}\Big)^{\frac{\min\{p+1,3\}}{2^*}}$$
$$\le C\varepsilon^{N(1-\frac{\min\{p+1,3\}}{2^*})}\Big(\int_\Omega|D\omega|^2\Big)^{\frac{\min\{p+1,3\}}{2}}\le C\varepsilon^{N(1-\frac{\min\{p+1,3\}}{2})}\|\omega\|_\varepsilon^{\min\{p+1,3\}}.$$

Now, let us calculate the estimates for $J_2(\omega)$. Note that

$$J_2(\omega):=J_2(\omega,\cup_{j=1}^k B_{\varepsilon R}(x_j))+J_2(\omega,\Omega\setminus\cup_{j=1}^k B_{\varepsilon R}(x_j)),$$

where $J_2(\omega,\cup_{j=1}^k B_{\varepsilon R}(x_j))=J_2(\omega)$ restricted to the domain $\cup_{j=1}^k B_{\varepsilon R}(x_j)$. It is very easy to see that

$$|J_2(\omega,\cup_{j=1}^k B_{\varepsilon R}(x_j))|\le C\varepsilon^{N(1-\frac{\min\{q+1,3\}}{2})+\alpha}\|\omega\|_\varepsilon^{\min\{q+1,3\}}$$

$$J_2(\omega, \Omega \setminus \cup_{j=1}^k B_{\varepsilon R}(x_j)) \leq \varepsilon^\alpha o\Big(q \int_{\Omega \setminus \cup_{j=1}^k B_{\varepsilon R}(x_j)} \Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q-1} \omega^2\Big)$$

$$\leq o(1) q \varepsilon^\alpha \int_{\Omega \setminus \cup_{j=1}^k B_{\varepsilon R}(x_j)} |\bar{u}_\varepsilon|^{q-1}\omega^2 \leq o(1)\|\omega\|_\varepsilon^2. \tag{3.13}$$

Similarly, we calculate the other estimate. □

Proposition 3.11. *There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there is a C^1 map $\omega_{\varepsilon,x} : D_{k,\varepsilon} \to H$, such that $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k}$ with $|\omega_{\varepsilon,x}(y)| \leq \frac{1}{2}|\bar{u}_\varepsilon(y)|$,*

$$\Big\langle I'_\varepsilon\Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x}\Big), \eta\Big\rangle_\varepsilon = 0, \ \forall \eta \in E_{\varepsilon,x,k}.$$

Moreover, we have

$$\|\omega_{\varepsilon,x}\|_\varepsilon = \varepsilon^{\frac{N}{2}} O(\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}+\sigma}),$$

where $\sigma > 0$ is a small number.

Proof. We have $l_{\varepsilon,x} + Q_{\varepsilon,x}\omega + R'_\varepsilon(\omega) = 0$. As $Q_{\varepsilon,x}^{-1}$ exists, the above equation is equivalent to solving

$$\omega + Q_{\varepsilon,x}^{-1} l_{\varepsilon,x} + Q_{\varepsilon,x}^{-1} R'_\varepsilon(\omega) = 0.$$

Let

$$G(\omega) := -Q_{\varepsilon,x}^{-1} l_{\varepsilon,x} - Q_{\varepsilon,x}^{-1} R'_\varepsilon(\omega) \quad \forall \omega \in E_{\varepsilon,x,k}.$$

Hence, the problem reduces to finding a fixed point of the map G.

For any $\omega_1 \in E_{\varepsilon,x,k}$ and $\omega_2 \in E_{\varepsilon,x,k}$ with $\|\omega_1\|_\varepsilon \leq \varepsilon^{\frac{N}{2}} \varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}$, $\|\omega_2\|_\varepsilon \leq \varepsilon^{\frac{N}{2}} \varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}$ and using Lemma 3.10,

$$\begin{aligned} \|G(\omega_1) - G(\omega_2)\|_\varepsilon &\leq C\|R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2)\|_\varepsilon \\ &\leq C\|J'_1(\omega_1) - J'_1(\omega_2)\|_\varepsilon + C\|J'_2(\omega_1) - J'_2(\omega_2)\|_\varepsilon. \end{aligned}$$

Note that

$$\|J'_1(\omega_1) - J'_1(\omega_2)\|_\varepsilon \leq \max_{t \in [0,1]} \|J''_1(t\omega_1 + (1-t)\omega_2)\|_\varepsilon \|\omega_1 - \omega_2\|_\varepsilon.$$

Let $\Omega' = \cup_{j=1}^k B_{\varepsilon R}(x_j)$,

$$\langle J'_2(\omega_1) - J'_2(\omega_2), \eta\rangle_\varepsilon = \varepsilon^\alpha \int_\Omega \Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon + \omega_1\Big)_-^q \eta$$

$$
\begin{aligned}
&- \varepsilon^\alpha \int_\Omega \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon + \omega_2 \Big)_-^q \eta + q\varepsilon^\alpha \int_\Omega \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon \Big)_-^{q-1} (\omega_1 - \omega_2)\eta \\
&= \varepsilon^\alpha \int_{\Omega'} \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon + \omega_1 \Big)_-^q \eta - \varepsilon^\alpha \int_{\Omega'} \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon + \omega_2 \Big)_-^q \eta \\
&+ q\varepsilon^\alpha \int_{\Omega'} \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon \Big)_-^{q-1} (\omega_1 - \omega_2)\eta + \varepsilon^\alpha \int_{\Omega\setminus\Omega'} \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon + \omega_1 \Big)_-^q \eta \\
&- \varepsilon^\alpha \int_{\Omega\setminus\Omega'} \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon + \omega_2 \Big)_-^q \eta + q\varepsilon^\alpha \int_{\Omega\setminus\Omega'} \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon \Big)_-^{q-1} (\omega_1 - \omega_2)\eta,
\end{aligned}
$$

$$
\begin{aligned}
&\langle J_2'(\omega_1) - J_2'(\omega_2), \eta \rangle_\varepsilon \\
&\le C\varepsilon^\alpha \int_{\Omega'} |\omega_1 - \omega_2||\eta| + Cq\varepsilon^\alpha o(1) \int_{\Omega\setminus\Omega'} \Big(\sum_{j=1}^k \hat V_{\varepsilon,x_j} + \bar u_\varepsilon \Big)_-^{q-1} |\omega_1 - \omega_2||\eta|.
\end{aligned}
$$

Hence, we have by Holder's inequality

$$
\begin{aligned}
&\langle J_2'(\omega_1) - J_2'(\omega_2), \eta \rangle_\varepsilon \\
&\le C\varepsilon^\alpha \int_{B_{\varepsilon R}(x_j)} |\omega_1 - \omega_2||\eta| + Cq\varepsilon^\alpha o(1) \int_{\Omega\setminus\Omega'} |\bar u_\varepsilon|^{q-1} |\omega_1 - \omega_2||\eta| \\
&\le C\varepsilon^\alpha \|\omega_1 - \omega_2\|_\varepsilon \|\eta\|_\varepsilon + Co(1)\|\omega_1 - \omega_2\|_\varepsilon \|\eta\|_\varepsilon \le o(1)\|\omega_1 - \omega_2\|_\varepsilon \|\eta\|_\varepsilon.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|J_1'(\omega_1) - J_1'(\omega_2)\|_\varepsilon &\le C \max_{t\in[0,1]} \|J_1''(t\omega_1 + (1-t)\omega_2)\|_\varepsilon \|\omega_1 - \omega_2\|_\varepsilon \\
&\le \varepsilon^\sigma \|\omega_1 - \omega_2\|_\varepsilon, \qquad (3.14)
\end{aligned}
$$

where $\sigma > 0$ is a small number and

$$
\|J_2'(\omega_1) - J_2'(\omega_2)\|_\varepsilon \le o(1)\|\omega_1 - \omega_2\|_\varepsilon. \qquad (3.15)
$$

Hence, G is a contraction map.

Also, with $\omega \in E_{\varepsilon,x,k}$ with $\|\omega\|_\varepsilon \le \varepsilon^{\frac{N}{2}} \varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}$, we have

$$
\begin{aligned}
\|G(\omega)\|_\varepsilon &\le C\|l_{\varepsilon,x}\|_\varepsilon + C\|R_\varepsilon'\omega\|_\varepsilon \\
&\le C\varepsilon^{\frac{N}{2}} \varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}} + C\varepsilon^{\frac{N}{2}} \varepsilon^{\frac{\gamma\tau}{\min\{q,2\}} + \sigma} \le \varepsilon^{\frac{N}{2}} \varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}, \qquad (3.16)
\end{aligned}
$$

as $\|l_{\varepsilon,x}\|_\varepsilon \leq C\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}}$ if $l_{\varepsilon,x} \in E_{\varepsilon,x,k}$. Thus, $G : S_{\varepsilon,x,k} \to E_{\varepsilon,x,k} \cap B_{\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}}(0)$ is a contraction map, where

$$S_{\varepsilon,x,k} := E_{\varepsilon,x,k} \cap B_{\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}}(0) \cap \{\omega : |\omega| \leq \frac{1}{2}|\bar{u}_\varepsilon|\}.$$

Now we will prove below that for any $\omega \in S_{\varepsilon,x,k}$, we have $|G(\omega)| \leq \frac{1}{2}|\bar{u}_\varepsilon|$. As a result of the claim, G maps $S_{\varepsilon,x,k}$ into itself for each $x \in D_{k,\varepsilon}$, there is a unique C^1 map $\omega_{\varepsilon,x}$ such that

$$\|\omega_{\varepsilon,x}\|_\varepsilon = \|G(\omega_{\varepsilon,x})\|_\varepsilon \leq C\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}+\sigma}.$$

Now we prove our claim. Let $\omega_1 = G(\omega)$. Then ω_1 satisfies

$$Q_{\varepsilon,x}\omega_1 = -l_{\varepsilon,x} - R'_\varepsilon(\omega) \text{ in } E_{\varepsilon,x,k}.$$

So there exist $\beta_{j,i} \in \mathbb{R}$, such that

$$Q_{\varepsilon,x}\omega_1 + l_{\varepsilon,x} + R'_\varepsilon(\omega) = \sum_{j=1}^{k}\sum_{i=1}^{N}\beta_{j,i}\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{ji}} \text{ in } H.$$

Hence, for any $\eta \in H$ we have

$$\langle Q_{\varepsilon,x}\omega_1, \eta\rangle_\varepsilon + \langle l_{\varepsilon,x}, \eta\rangle_\varepsilon + \langle R'_\varepsilon(\omega), \eta\rangle_\varepsilon = \sum_{j=1}^{k}\sum_{i=1}^{N}\beta_{j,i}\Big\langle\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{ji}}, \eta\Big\rangle_\varepsilon$$

$$= \sum_{j=1}^{k}\sum_{i=1}^{N}\beta_{j,i}\int_\Omega p(U_{\varepsilon,x_j} - 1)_+^{p-1}\frac{\partial U_{\varepsilon,x_j}}{\partial x_{ji}}\eta. \tag{3.17}$$

By the definition of $Q_{\varepsilon,x}$, $l_{\varepsilon,x}$ and $R_\varepsilon(\omega)$ it follows from (3.17) that ω_1 satisfies

$$\begin{aligned}
&-\varepsilon^2\Delta\omega_1 - p\Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^{p-1}\omega_1 + q\varepsilon^\alpha\Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q-1}\omega_1 \qquad (3.18)\\
&= -\sum_{j=1}^{k}(U_{\varepsilon,x_j} - 1)_+^{p-1} + p\Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^{p}\\
&- \varepsilon^\alpha\Big\{|\bar{u}_\varepsilon|^q - \Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q} - q\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}|\bar{u}_\varepsilon|^{q-1}\Big\}\\
&- \Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon + \omega\Big)_+^{p} + \Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^{p} + p\Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+^{p-1}\omega
\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon^\alpha \Big\{ \Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon + \omega \Big)_-^q - \Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon \Big)_-^q \\
&+ q \Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon \Big)_-^{q-1} \omega \Big\} + p \sum_{j=1}^k \sum_{i=1}^N \beta_{j,i} (U_{\varepsilon,x_j} - 1)_+^{p-1} \frac{\partial U_{\varepsilon,x_j}}{\partial x_{ji}} := g(y).
\end{aligned}
$$

Letting $\eta = \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{ji}}$ in (3.17), we have

$$
\begin{aligned}
& \Big| \Big\langle Q_{\varepsilon,x} \omega_1, \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{ji}} \Big\rangle_\varepsilon + \Big\langle l_{\varepsilon,x}, \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{ji}} \Big\rangle_\varepsilon + \Big\langle R'_\varepsilon(\omega), \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{ji}} \Big\rangle_\varepsilon \Big| \\
\leq\ & C(\|\omega_1\|_\varepsilon + \|l_{\varepsilon,x}\|_\varepsilon + \|R'_\varepsilon(\omega)\|_\varepsilon) \Big\| \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{ji}} \Big\|_\varepsilon \\
\leq\ & C\varepsilon^{\frac{N}{2} + \frac{\gamma\tau}{\min\{q,2\}}} \varepsilon^{\frac{N}{2}-1} = C\varepsilon^{N-1+\frac{\gamma\tau}{\min\{q,2\}}}.
\end{aligned}
\tag{3.19}
$$

Note that $\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}}$ satisfies the equation

$$
\begin{cases}
-\varepsilon^2 \Delta \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}} + q\varepsilon^\alpha |\bar{u}_\varepsilon|^{q-1} \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}} = p(U_{\varepsilon,z} - 1)_+^{p-1} \frac{\partial U_{\varepsilon,x_j}}{\partial x_{jl}} & \text{in } \Omega \\
\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}} = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{3.20}
$$

and

$$
\Big| \frac{\partial U_{\varepsilon,x_j}}{\partial x_{ji}} \Big| \leq \frac{C}{\varepsilon} U_{\varepsilon,x_j}.
$$

Now we claim that there exists a $0 < \beta' < 1$ such that

$$
\Big| \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}} - \frac{\partial V_{\varepsilon,x_j}}{\partial x_{jl}} \Big| \leq C\varepsilon^{\beta'-1}.
$$

To prove this, consider the problems

$$
\begin{cases}
-\varepsilon^2 \Delta Z_\varepsilon + q\varepsilon^\alpha |\bar{u}_\varepsilon|^{q-1} Z_\varepsilon = p(U_{\varepsilon,x_j} - 1)_+^{p-1} \Big| \frac{\partial U_{\varepsilon,x_j}}{\partial x_{jl}} \Big| & \text{in } \Omega \\
Z_\varepsilon = 0 & \text{on } \partial\Omega
\end{cases}
\tag{3.21}
$$

$$
\begin{cases}
-\varepsilon^2 \Delta W_\varepsilon = p(U_{\varepsilon,x_j} - 1)_+^{p-1} \Big| \frac{\partial U_{\varepsilon,x_j}}{\partial x_{jl}} \Big| & \text{in } \Omega \\
W_\varepsilon = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{3.22}
$$

Then it follows by the maximum principle that $0 \le \left|\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}}\right| \le Z_\varepsilon \le W_\varepsilon$ in Ω. Also, by global L^q estimates it follows that

$$\|W_\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon^{\frac{N}{q'}-3},$$

where $q' > \frac{N}{2}$ is so chosen that $(2 - \frac{N}{q'})$ is very small. Also it easily follows by the maximum principle that

$$W_\varepsilon \le C\varepsilon^{\frac{N}{q'}-3}|V_{\varepsilon,x_j}| \text{ in } \Omega \setminus B_{\varepsilon R}(x_j).$$

Now we have from (3.20) and from the equation satisfied by $\frac{\partial V_{\varepsilon,x_j}}{\partial x_{jl}}$, by using global L^p estimates and the fact that $q' > \frac{N}{2}$,

$$\begin{aligned}
&\Big\|\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}} - \frac{\partial V_{\varepsilon,x_j}}{\partial x_{jl}}\Big\|_{L^\infty(\Omega)} \le C\varepsilon^{\alpha-2}\Big\|\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}}\Big\|_{L^{q'}(\Omega)} \\
&= C\varepsilon^{\alpha-2}\Big(\int_{B_{\varepsilon R}(x_j)}\Big|\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}}\Big|^{q'}\Big)^{\frac{1}{q'}} + C\varepsilon^{\alpha-2}\Big(\int_{\Omega\setminus B_{\varepsilon R}(x_j)}\Big|\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}}\Big|^{q'}\Big)^{\frac{1}{q'}} \\
&\le C\varepsilon^{\alpha-2}\varepsilon^{\frac{2N}{q'}-3} + C\varepsilon^{\alpha-2}\Big(\int_{\Omega\setminus B_{\varepsilon R}(x_j)}|W_\varepsilon|^{q'}\Big)^{\frac{1}{q'}}.
\end{aligned}$$

Also note that if $N \ge 4$, we have

$$\int_{\Omega\setminus B_{\varepsilon R}(x_j)}|W_\varepsilon|^{q'} = \varepsilon^{q'(N/q'-3)}O(\varepsilon^N).$$

Hence, choosing $\beta' = \alpha + 2(\frac{N}{q'} - 2) > 0$, we have

$$\Big\|\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}} - \frac{\partial V_{\varepsilon,x_j}}{\partial x_{jl}}\Big\|_{L^\infty(\Omega)} \le C\varepsilon^{\beta'-1}.$$

Hence, we have using [8],

$$\begin{aligned}
&\Big\langle\frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{ji}}, \frac{\partial \hat{V}_{\varepsilon,x_h}}{\partial x_{hl}}\Big\rangle_\varepsilon = p\int_\Omega (U_{\varepsilon,x_j}-1)_+^{p-1}\frac{\partial U_{\varepsilon,x_j}}{\partial x_{ji}}\frac{\partial \hat{V}_{\varepsilon,x_h}}{\partial x_{hl}} \\
&= p\int_\Omega (U_{\varepsilon,x_j}-1)_+^{p-1}\frac{\partial U_{\varepsilon,x_j}}{\partial x_{ji}}\Big\{\frac{\partial \hat{V}_{\varepsilon,x_h}}{\partial x_{hl}} - \frac{\partial V_{\varepsilon,x_h}}{\partial x_{hl}}\Big\} \\
&\qquad + p\int_\Omega (U_{\varepsilon,x_j}-1)_+^{p-1}\frac{\partial U_{\varepsilon,x_j}}{\partial x_{ji}}\frac{\partial V_{\varepsilon,x_h}}{\partial x_{hl}} = C\varepsilon^{N-2}(\delta_{ijhl} + o(1)),
\end{aligned}$$

where

$$\delta_{ijhl} = \begin{cases} 1 & \text{if } i = j = h = l \\ 0 & \text{otherwise.} \end{cases}$$

Hence, it follows that

$$|\beta_{ji}| \le C\varepsilon^{1+\frac{\gamma\tau}{\min\{q,2\}}}.$$

Let $\tilde{\omega}_1(y) = \omega_1(\varepsilon y)$. Then, from (3.18) we have

$$-\Delta\tilde{\omega}_1 + q\varepsilon^{\alpha}\Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q-1}\tilde{\omega}_1 = g(\varepsilon y). \tag{3.23}$$

Note that for $x \in D_{k,\varepsilon}$

$$\Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+ = 0 \text{ in } \Omega \setminus \cup_{j=1}^{k} B_{\varepsilon R}(x_j).$$

Also, as ω is bounded we obtain that

$$\begin{aligned}|g(y)| &\le C\sum_{j\ne i} U\Big(\frac{|x_i - x_j|}{\varepsilon}\Big) \\ &\qquad + C\sum_{j=1}^{k}|1-\varphi_1(x_j)| + C|\omega(y)|^p + C\varepsilon^{\alpha} + C\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}} \\ &\le C|\omega(y)|^p + C\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}.\end{aligned}$$

So for any $\tilde{q} > 1$ large, and $z \in \mathbb{R}^N$ and noting the fact that $|\omega| \le \frac{1}{2}|\bar{u}_\varepsilon| \le C$ we obtain

$$\begin{aligned}&\Big(\int_{B_2(z)}|g(\varepsilon y)|^{\tilde{q}}\Big)^{\frac{1}{\tilde{q}}} \le C\Big(\int_{B_2(z)}|\omega(\varepsilon y)|^{p\tilde{q}}\Big)^{\frac{1}{\tilde{q}}} + C\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}} \\ &= C\varepsilon^{-\frac{N}{\tilde{q}}}\Big(\int_{B_{2\varepsilon}(\varepsilon z)}|\omega|^{p\tilde{q}}\Big)^{\frac{1}{\tilde{q}}} + C\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}} \\ &\le C\varepsilon^{-\frac{N}{\tilde{q}}}\Big(\int_{B_{2\varepsilon}(\varepsilon z)}|\omega|^2\Big)^{\frac{1}{\tilde{q}}} + C\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}} \le C\varepsilon^{-\frac{N}{\tilde{q}}}\|\omega\|_\varepsilon^{\frac{2}{\tilde{q}}} + C\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}.\end{aligned}$$

As $\|\omega\|_\varepsilon^2 \le C\varepsilon^{N+\frac{2\gamma\tau}{\min\{q,2\}}}$, we have

$$\Big(\int_{B_2(z)}|g(\varepsilon y)|^{\tilde{q}}\Big)^{\frac{1}{\tilde{q}}} \le C\varepsilon^{\frac{\gamma\tau}{\tilde{q}\min\{q,2\}}}.$$

Also note that by Holder's inequality we have

$$\|\tilde{\omega}_1\|^2_{L^2(B_2(z))} = \int_{B_2(z)} |\omega_1(\varepsilon y)|^2 = \varepsilon^{-N}\Big(\int_{B_{2\varepsilon}(\varepsilon z)} |\omega_1|^2\Big)$$
$$\leq \varepsilon^{-N}\Big(\varepsilon^2 \int_{\Omega} |D\omega_1|^2\Big) \leq \varepsilon^{-N}\|\omega_1\|^2_{\varepsilon}.$$

Choosing $\tilde{q} > \frac{N}{2}$ and applying local L^p estimates, we obtain

$$\|\tilde{\omega}_1\|_{L^\infty(B_1(z))} \leq C\|\tilde{\omega}_1\|_{L^2(B_2(z))} + C\|g(\varepsilon y)\|_{L^{\tilde{q}}(B_2(z))}$$
$$\|\tilde{\omega}_1\|_{L^\infty(B_1(z))} \leq C\varepsilon^{-\frac{N}{2}}\|\omega_1\|_{\varepsilon} + C\|g(\varepsilon y)\|_{L^{\tilde{q}}(B_2(z))} \leq C\varepsilon^{\frac{\gamma\tau}{\tilde{q}\min\{q,2\}}}.$$

This implies $|\omega_1| \leq C\varepsilon^{\frac{\gamma\tau}{\tilde{q}\min\{q,2\}}}$, and hence $\omega_1 \to 0$ uniformly in Ω.

Again, since $|\omega| \leq \frac{1}{2}|\bar{u}_\varepsilon|$, we can choose a $R > 0$ large such that for any $y \in \Omega \setminus \cup_{j=1}^k B_{\varepsilon R}(x_j)$

$$\Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon + \omega\Big)_+ = 0,\ \Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_+ = 0,\ \Big(U_{\varepsilon,x_j} - 1\Big)_+ = 0.$$

As a result we obtain from (3.18)

$$\begin{aligned} & -\varepsilon^2\Delta\omega_1 + q\varepsilon^\alpha\Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q-1}\omega_1 \\ = \; & -\varepsilon^\alpha\Big\{|\bar{u}_\varepsilon|^q - \Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^q - q\sum_{j=1}^k \hat{V}_{\varepsilon,x_j}|\bar{u}_\varepsilon|^{q-1}\Big\} \\ - \; & \varepsilon^\alpha\Big\{\Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon + \omega\Big)_-^q - \Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^q \\ + \; & q\Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon\Big)_-^{q-1}\omega\Big\} \text{ in } \Omega \setminus \cup_{j=1}^k B_{\varepsilon R}(x_j). \end{aligned} \tag{3.24}$$

Choose $0 < \gamma < 2$ such that $1 - \gamma/2 < 1 - \alpha/2$. First note that in $\cup_{j=1}^k\Big(B_{\varepsilon^{1-\gamma/2}R}(x_j) \setminus B_{\varepsilon R}(x_j)\Big)$, we have $|\omega_1| \leq \mu|\bar{u}_\varepsilon|$, where $\mu > 0$ can be chosen sufficiently small. Let $T = \Omega \setminus \cup_{j=1}^k B_{\varepsilon^{1-\gamma/2}R}(x_j)$. Then it follows that

$$|\omega_1| \leq \mu|\bar{u}_\varepsilon| \,\text{on}\, \partial T,$$

where $\mu > 0$ can be chosen as small as we can, which follows by the fact that $\bar{u}_\varepsilon$ is bounded below in the interior boundary of T. Let $\omega_1 = \tilde{\eta}_1 + \tilde{\eta}_2$. Then the above equation can be written in the forms

$$\begin{cases} -\varepsilon^2 \Delta \tilde{\eta}_1 + q\varepsilon^\alpha |\bar{u}_\varepsilon|^{q-1} \tilde{\eta}_1 = Kq\varepsilon^\alpha |\bar{u}_\varepsilon|^{q-2}\omega^2 & \text{in } T \\ |\tilde{\eta}_1| \leq \mu |\bar{u}_\varepsilon| & \text{on } \partial T \end{cases} \tag{3.25}$$

$$\begin{cases} -\varepsilon^2 \Delta \tilde{\eta}_2 + q\varepsilon^\alpha |\bar{u}_\varepsilon|^{q-1} \tilde{\eta}_2 = f_\varepsilon & \text{in } T \\ \tilde{\eta}_2 = 0 & \text{on } \partial T, \end{cases} \tag{3.26}$$

where $K > 0$ and f_ε is exponentially small in T. Note that $\frac{1}{4}|\bar{u}_\varepsilon|$ is a supersolution to the problem (3.25). Hence $\tilde{\eta}_1 \leq \mu|\bar{u}_\varepsilon|$ in T. Also $-\mu|\bar{u}_\varepsilon|$ is a subsolution of the problem (3.25). Hence we have

$$-\mu|\bar{u}_\varepsilon| \leq \tilde{\eta}_1 \leq \mu|\bar{u}_\varepsilon| \text{ in } T.$$

For the problem (3.26), by the maximum principle it follows that

$$|\tilde{\eta}_2| \leq Z_\varepsilon(x) \text{ in } T,$$

where $Z_\varepsilon(x)$ satisfies the problem

$$\begin{cases} -\varepsilon^2 \Delta Z_\varepsilon = |f_\varepsilon| & \text{in } T \\ Z_\varepsilon = 0 & \text{on } \partial T. \end{cases} \tag{3.27}$$

Hence, $Z_\varepsilon(x)$ is exponentially small. Hence in T we have

$$|\tilde{\eta}_2| \leq \frac{Z_\varepsilon(x)}{|\bar{u}_\varepsilon|}|\bar{u}_\varepsilon| \leq \tau|\bar{u}_\varepsilon|,$$

where $\tau > 0$ can be chosen to be very small. This together implies that $|\omega| \leq \frac{1}{2}|\bar{u}_\varepsilon|$ in T and hence in Ω. □

4. Existence of interior peak solutions

Lemma 4.1. *For any positive integer k, we have*

$$\begin{aligned} I_\varepsilon\Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j}\Big) &= k\varepsilon^N A + \varepsilon^N B \sum_{j=1}^k (\varphi_1^{\frac{1}{q}}(x_j) - 1) - c'\varepsilon^N \sum_{i<j} U\Big(\frac{|x_i - x_j|}{\varepsilon}\Big) \\ &\quad + \varepsilon^N O\Big(\varepsilon^\gamma + \sum_{j=1}^k |\varphi_1^{\frac{1}{q}}(x_j) - 1| + \sum_{i<j} U^{\sigma+1}\Big(\frac{|x_i - x_j|}{\varepsilon}\Big)\Big), \end{aligned}$$

where σ is a small positive constant and

$$A = \int_{\mathbb{R}^N} (U-1)_+^p U, \qquad B = \int_{\mathbb{R}^N} (U-1)_+^{p+1}.$$

Proof. First note that

$$I_\varepsilon\Big(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j}\Big) = \sum_{j=1}^k I_\varepsilon(\hat{V}_{\varepsilon,x_j}) + \frac{1}{2}\sum_{i\neq j} \varepsilon^2 \int_\Omega D\hat{V}_{\varepsilon,x_j} D\hat{V}_{\varepsilon,x_i}$$
$$+ \frac{q}{2}\varepsilon^\alpha \sum_{i\neq j} \int_\Omega |\bar{u}_\varepsilon|^{q-1} \hat{V}_{\varepsilon,x_j}\hat{V}_{\varepsilon,x_i} - \int_\Omega F\Big(y, \sum_{j=1}^k \hat{V}_{\varepsilon,x_j}\Big) + \sum_{j=1}^k \int_\Omega F\Big(y, \hat{V}_{\varepsilon,x_j}\Big).$$

Using (2.4), Lemma 3.3 and Corollary 3.6 we have

$$\frac{\varepsilon^2}{2}\int_\Omega |D\hat{V}_{\varepsilon,x_j}|^2 + \frac{q\varepsilon^\alpha}{2}\int_\Omega |\bar{u}_\varepsilon|^{q-1}\hat{V}^2_{\varepsilon,x_j} = \frac{1}{2}\int_\Omega (U_{\varepsilon,x_j}-1)_+^p \hat{V}_{\varepsilon,x_j}$$
$$= \frac{1}{2}\int_\Omega (U_{\varepsilon,x_j}-1)_+^p V_{\varepsilon,x_j} + O(\varepsilon^{N+\beta})$$
$$= \frac{1}{2}\int_\Omega (U_{\varepsilon,x_j}-1)_+^p U_{\varepsilon,x_j} + O(\varepsilon^{N+\beta}) + O(\varepsilon^{N+1})$$
$$= \frac{1}{2}\varepsilon^N \int_{\mathbb{R}^N} (U-1)_+^p U + O(\varepsilon^{N+\beta}) + O(\varepsilon^{N+1}).$$

Also using Lemma 3.3 and Corollary 3.6 we have

$$\frac{1}{p+1}\int_\Omega (\hat{V}_{\varepsilon,x_j} + \bar{u}_\varepsilon)_+^{p+1} = \frac{1}{p+1}\int_\Omega (V_{\varepsilon,x_j}+\bar{u}_\varepsilon)_+^{p+1} + O(\varepsilon^{N+\beta})$$
$$= \frac{1}{p+1}\int_\Omega (U_{\varepsilon,x_j}-1)_+^{p+1} + \int_{B_{\varepsilon R}(x_j)} (U_{\varepsilon,x_j}-1)_+^p(\bar{u}_\varepsilon+1)$$
$$+ O\Big(\int_{B_{\varepsilon R}(x_j)} (U_{\varepsilon,x_j}-1)_+^{p-1}(\bar{u}_\varepsilon+1)^2\Big) + O(\varepsilon^{N+\beta}).$$

Hence, by using mean value theorem we have

$$\frac{1}{p+1}\int_\Omega (\hat{V}_{\varepsilon,x_j}+\bar{u}_\varepsilon)_+^{p+1} = \frac{\varepsilon^N}{p+1}\int_{\mathbb{R}^N} (U-1)_+^{p+1}$$
$$- \varepsilon^N(\varphi_1^{\frac{1}{q}}(x_j)-1)\int_{\mathbb{R}^N}(U-1)_+^p + \varepsilon^N O(\varepsilon + |1-\varphi_1^{\frac{1}{q}}(x_j)|^2) + O(\varepsilon^{N+\beta}).$$

Also,

$$\frac{\varepsilon^\alpha}{q+1}\int_\Omega (\hat{V}_{\varepsilon,x_j}+\bar{u}_\varepsilon)_-^{q+1} - \frac{\varepsilon^\alpha}{q+1}\int_\Omega |\bar{u}_\varepsilon|^{q+1} - \varepsilon^\alpha \int_\Omega |\bar{u}_\varepsilon|^q \hat{V}_{\varepsilon,x_j}$$

$$+\frac{q}{2}\varepsilon^{\alpha}\int_{\Omega}|\bar{u}_{\varepsilon}|^{q-1}\hat{V}^{2}_{\varepsilon,x_j}=\varepsilon^{\alpha}O\Big(\int_{\Omega}|\bar{u}_{\varepsilon}|^{q-2}\hat{V}^{3}_{\varepsilon,x_j}\Big)$$
$$=\varepsilon^{\alpha}O\Big(\int_{\Omega\setminus B_{\varepsilon^{1-\frac{\alpha}{2}}R}(x_j)}|\bar{u}_{\varepsilon}|^{q-2}\hat{V}^{3}_{\varepsilon,x_j}+\int_{B_{\varepsilon^{1-\frac{\alpha}{2}}R(x_j)\setminus B_{\varepsilon R}(x_j)}}|\bar{u}_{\varepsilon}|^{q-2}\hat{V}^{3}_{\varepsilon,x_j}$$
$$+\int_{B_{\varepsilon R}(x_j)}|\bar{u}_{\varepsilon}|^{q-2}\hat{V}^{3}_{\varepsilon,x_j}\Big)=O(\varepsilon^{N+\alpha});$$

hence,

$$I_{\varepsilon}(\hat{V}_{\varepsilon,x_j})=\varepsilon^{N}A+\varepsilon^{N}(\varphi_1^{\frac{1}{q}}(x_j)-1)B+\varepsilon^{N}O(\varepsilon+|1-\varphi_1^{\frac{1}{q}}(x_j)|^2)+O(\varepsilon^{N+\beta}),\quad (4.1)$$

where

$$A=\int_{R^N}(U-1)^p_+U\,dx\quad\text{ and }\quad B=\int_{\mathbb{R}^N}(U-1)^p_+\,dx,$$

On the other hand,

$$\frac{1}{2}\sum_{i\neq j}\varepsilon^2\int_{\Omega}D\hat{V}_{\varepsilon,x_j}D\hat{V}_{\varepsilon,x_i}+\frac{q\varepsilon^{\alpha}}{2}\sum_{i\neq j}\int_{\Omega}|\bar{u}_{\varepsilon}|^{q-1}\hat{V}_{\varepsilon,x_j}\hat{V}_{\varepsilon,x_i}\quad (4.2)$$
$$=\frac{1}{2}\sum_{i\neq j}\int_{\Omega}(U_{\varepsilon,x_j}-1)^p_+\hat{V}_{\varepsilon,x_i}=\frac{1}{2}\sum_{i\neq j}\int_{\Omega}(U_{\varepsilon,x_j}-1)^p_+V_{\varepsilon,x_i}+O(\varepsilon^{N+\beta})$$
$$=\frac{1}{2}\sum_{i\neq j}\int_{\Omega}(U_{\varepsilon,x_j}-1)^p_+U_{\varepsilon,x_i}+O(\varepsilon^{N+\beta})=c'\varepsilon^N\sum_{i<j}U\Big(\frac{x_i-x_j}{\varepsilon}\Big)+O(\varepsilon^{N+\beta}),$$

and using Lemma 3.3 and using Taylor's expansion repeatedly we have

$$\frac{1}{p+1}\int_{\Omega}\Big(\Big(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}+\bar{u}_{\varepsilon}\Big)^{p+1}_{+}-\sum_{j=1}^{k}\Big(\hat{V}_{\varepsilon,x_j}+\bar{u}_{\varepsilon}\Big)^{p+1}_{+}\Big)\quad (4.3)$$
$$=\frac{1}{p+1}\int_{\Omega}\Big(\Big(\sum_{j=1}^{k}V_{\varepsilon,x_j}+\bar{u}_{\varepsilon}\Big)^{p+1}_{+}-\sum_{j=1}^{k}\Big(V_{\varepsilon,x_j}+\bar{u}_{\varepsilon}\Big)^{p+1}_{+}\Big)+O(\varepsilon^{N+\beta})$$
$$=\frac{1}{p+1}\sum_{i=1}^{k}\int_{B_{\varepsilon R}(x_i)}\Big(\Big(\sum_{j=1}^{k}U_{\varepsilon,x_j}-\varphi_1^{\frac{1}{q}}(x_i)\Big)^{p+1}_{+}-\Big(U_{\varepsilon,x_j}-\varphi_1^{\frac{1}{q}}(x_i)\Big)^{p+1}_{+}\Big)$$
$$+O(\varepsilon^{N+\beta})+O(\varepsilon^{N+1})$$
$$=\sum_{i\neq j}\int_{B_{\varepsilon R}(x_i)}\Big(U_{\varepsilon,x_j}-\varphi_1^{\frac{1}{q}}(x_i)\Big)^{p}_{+}U_{\varepsilon,x_j}+\varepsilon^N O\Big(\varepsilon+\sum_{i<j}U^{1+\sigma}\Big(\frac{|x_i-x_j|}{\varepsilon}\Big)\Big)$$

$$+ O(\varepsilon^{N+\beta}) = 2c'\varepsilon^N \sum_{i<j} U\Big(\frac{|x_i - x_j|}{\varepsilon}\Big) + O(\varepsilon^{N+\beta})$$

$$+ \varepsilon^N O\Big(\varepsilon + \sum_{j=1}^{k} |1 - \varphi_1^{\frac{1}{q}}(x_j)| + \sum_{i<j} U^{1+\sigma}\Big(\frac{|x_i - x_j|}{\varepsilon}\Big)\Big).$$

Also, note that the other terms are of the order $\varepsilon^{N+\beta}$. Hence, combining (4.1), (4.2) and (4.3) we have the required result. □

Proof of Theorem 2.1. Consider $\max_{x \in D_{k,\varepsilon}} F_\varepsilon(x)$, where

$$F_\varepsilon(x) = I\Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x}\Big).$$

In order to prove that $\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x}$ is a solution of (2.1) we only need to prove that x is a critical point of F_ε. This follows from Lemma 4.1 and that for any $x \in D_{k,\varepsilon}$

$$F_\varepsilon(x) = I_\varepsilon\Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j}\Big) + O(\|l_{\varepsilon,x}\|_\varepsilon \|\omega_{\varepsilon,x}\|_\varepsilon + \|\omega_{\varepsilon,x}\|_\varepsilon^2 + R_\varepsilon(\omega_{\varepsilon,x_\varepsilon})) \tag{4.4}$$

$$= I_\varepsilon\Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j}\Big) + \varepsilon^N O\Big(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}+\sigma}\Big) = I_\varepsilon\Big(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j}\Big) + O\Big(\varepsilon^{N+\frac{2\gamma\tau}{\min\{q,2\}}+\sigma}\Big)$$

$$= k\varepsilon^N A + \varepsilon^N B \sum_{j=1}^{k} (\varphi_1^{\frac{1}{q}}(x_j) - 1) - c'\varepsilon^N \sum_{i<j} U\Big(\frac{|x_i - x_j|}{\varepsilon}\Big)$$

$$+ O\Big(\varepsilon^{N+\frac{2\gamma\tau}{\min\{q,2\}}+\sigma}\Big),$$

if we choose $\sigma > 0$ small. Let $x_\varepsilon \in D_{k,\varepsilon}$ be a maximum point of $F_\varepsilon(x)$ in $D_{k,\varepsilon}$. Choose $\tilde{x} = (\tilde{x}_{\varepsilon,1}, \ldots, \tilde{x}_{\varepsilon,k})$ such that $d(\tilde{x}_{\varepsilon,x_j}, S) = \varepsilon^{\frac{1}{2}}$, $j = 1, \ldots, k$, and

$$|\tilde{x}_{\varepsilon,x_i} - \tilde{x}_{\varepsilon,x_j}| \geq \frac{1}{2k}\varepsilon^{\frac{1}{2}}, \quad i \neq j.$$

Then we have $|\varphi_1(\tilde{x}_{\varepsilon,x_j}) - 1| \leq cd^2(\tilde{x}_{\varepsilon,x_j}, S) \leq c\varepsilon$ and $U\big(\frac{\tilde{x}_{\varepsilon,x_i} - \tilde{x}_{\varepsilon,x_j}}{\varepsilon}\big) \leq c\varepsilon^{\frac{N-2}{2}}$, $i \neq j$, which implies that $\tilde{x} \in D_{k,\varepsilon}$. From (4.4) we have

$$F_\varepsilon(\tilde{x}_\varepsilon) = \varepsilon^N kA + O\Big(\varepsilon^{N+\frac{2\gamma\tau}{\min\{q,2\}}+\sigma}\Big), \tag{4.5}$$

and from (4.5) and the fact that $F_\varepsilon(\tilde{x}_\varepsilon) \le F_\varepsilon(x_\varepsilon)$, we have

$$\varepsilon^N B \sum_{j=1}^{k} (\varphi_1^{\frac{1}{q}}(x_{\varepsilon,j}) - 1) - c' \varepsilon^N \sum_{i<j} U\Big(\frac{|x_{\varepsilon,i} - x_{\varepsilon,j}|}{\varepsilon}\Big) \ge O\Big(\varepsilon^{N + \frac{2\gamma\tau}{\min\{q,2\}} + \sigma}\Big).$$

Hence, we have

$$0 \le 1 - \varphi_1^{\frac{1}{q}}(x_{\varepsilon,j}) \le O\Big(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}} + \sigma}\Big),$$

and

$$U\Big(\frac{|x_{\varepsilon,i} - x_{\varepsilon,j}|}{\varepsilon}\Big) \le O\Big(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}} + \sigma}\Big), \quad i \ne j.$$

That is, x_ε is an interior point of $D_{k,\varepsilon}$. Hence, x_ε is a critical point of $F_\varepsilon(x)$.

5. Computation of critical groups

In order to compute the critical groups of these solutions, provided they are isolated but may be degenerate, we basically use the shifting theorem; see [4] and [5]. Note that we are searching for solutions to the problem (2.1) with k-peaks. Now we prove Theorem 1.2.

Proof. We have

$$I_\varepsilon(v) = \frac{\varepsilon^2}{2} \int_\Omega |Dv|^2 - \int_\Omega F_\varepsilon(y, v) dy,$$

where

$$F_\varepsilon(y,t) = \frac{1}{p+1}(t + \bar{u}_\varepsilon)_+^{p+1} + \frac{\varepsilon^\alpha}{q+1}(t + \bar{u}_\varepsilon)_-^{q+1} - \frac{\varepsilon^\alpha}{q+1}|\bar{u}_\varepsilon|^{q+1} - \varepsilon^\alpha |\bar{u}_\varepsilon|^q t$$

$\forall v \in H_0^1(\Omega)$. Hence, I_ε can be written in the form

$$I_\varepsilon(v) = \frac{1}{2} \int_{\Omega_\varepsilon} |Dv|^2 - \int_{\Omega_\varepsilon} F_\varepsilon(\varepsilon y, v) dy,$$

$\forall v \in H_0^1(\Omega_\varepsilon)$ where $\Omega_\varepsilon = \Omega/\varepsilon$. Note that the mountain-pass solution U of the limiting problem (2.2) has Morse index one. Also note that by [11] the operator $-\Delta v - p(U-1)_+^{p-1} v$ on $D^{1,2}(\mathbb{R}^N)$ has exactly one negative eigenvalue, a zero eigenvalue of multiplicity n, and the rest of the eigenvalues all positive. Since we are looking for solutions with k peaks, the linearized problem has exactly k negative eigenvalues which are not small, Nk small eigenvalues, and other eigenvalues that are all positive and not small. Also note that we are looking for critical points of the form $\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \omega$. That is, basically we are decomposing $H = E_{\varepsilon,x,k} \bigoplus E_{\varepsilon,x,k}^\perp$, where $dim\ E_{\varepsilon,x,k}^\perp = nk$. Thus we can apply implicit function theorem on $E_{\varepsilon,x,k}$ to conclude that for sufficiently small $\varepsilon > 0$, there exists a $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k}$ satisfying $PI'_\varepsilon(z+\omega) = 0$, where P is the orthogonal projection onto $E_{\varepsilon,x,k}$. Thus we have reduced the problem into a maximization problem in nk dimensions.

Let $\tilde{I}_\varepsilon$ be the reduced functional defined on $\mathbb{R}^{nk}$. Let x_ε be an isolated critical point of I_ε. Hence, by the shifting theorem and the formula for the critical groups of an isolated maximum (see [4]), the critical groups of x_ε are given by

$$C_q(I_\varepsilon, x_\varepsilon) = C_{q-k}(\tilde{I}_\varepsilon, x_\varepsilon) = \delta^{nk}_{q-k}\mathbb{Z},$$

where $q = 0, 1, \ldots$. This completes the proof. □

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