Advances in Differential Equations Volume 12, Number 12 (2007), 1415–1436

POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EIGENVALUE PROBLEMS WITH CONCAVE NONLINEARITIES

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(Submitted by: Herbert Amann)

Abstract. In this paper, we prove the existence and nonexistence results for positive solutions to semilinear elliptic boundary value problems, with concave nonlinearities inside a smooth bounded domain and on the boundary. Our approach relies on sub and supersolutions, as well as the Nehari manifold that may contain the critical points for the energy functional associated with the boundary value problem. The fibering method helps us to study the properties of the Nehari manifold.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. We consider the following semilinear elliptic boundary value problem with concave nonlinearities.

$$\begin{cases} -\Delta u = \lambda(m(x) - |u|^{p-1})u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda(\sigma(x) - c|u|^{q-1})u & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $m \in L^\infty(\Omega)$, $\sigma \in W^{1-\frac{1}{r},r}(\partial\Omega)$ for any $r > 1$ and $p, q > 1$, $c \geq 0$, are *arbitrary* constants. Further, $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$ and $\lambda > 0$ is a parameter. By $L^r(\Omega)$, $1 \leq r \leq \infty$, we denote the usual Lebesgue space with the norm $\|\cdot\|_r$, by $L^q(\partial\Omega)$, $1 \leq q \leq \infty$, the set of all functions u defined on $\partial\Omega$ whose usual norm $\|u\|_{q,\partial\Omega}$ is finite. By $W^{m,r}(\Omega)$, $m = 1, 2, 3, \ldots, r > 1$, we denote the usual Sobolev space with the norm $\|\cdot\|_{m,r}$, by $W^{1-\frac{1}{r},r}(\partial\Omega)$, $r > 1$, the set of traces on $\partial\Omega$ of functions in $W^{1,r}(\Omega)$, equipped with the norm $\|\cdot\|_{1-\frac{1}{r},r,\partial\Omega}$. It is well known (see Adams [1]) that the trace operator

Accepted for publication: December 2007.
AMS Subject Classifications: 35J20, 35J25.
Partly supported by the Grant-in-Aid for Scientific Research (C) (2), No. 19540192, Japan Society for the Promotion of Science.

given as $Tu = u|_{\partial\Omega}$, maps $W^{1,r}(\Omega)$ isomorphically and homeomorphically onto $W^{1-\frac{1}{r},r}(\partial\Omega)$ for each $r > 1$. It should be remarked that the functions m and σ may change sign and we have $\sigma \in C^{\theta}(\partial\Omega)$ for any $0 < \theta < 1$ by Sobolev's imbedding theorem.

A function $u \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega) \cap L^{\infty}(\partial\Omega)$ is said to be a *weak solution* of (1.1) if we have

$$\int_{\Omega} \nabla u \nabla \psi dx - \lambda\Big(\int_{\Omega} mu\psi dx + \int_{\partial\Omega} \sigma u\psi da\Big) + \lambda\Big(\int_{\Omega} |u|^{p-1} u\psi dx + c\int_{\partial\Omega} |u|^{q-1} u\psi da\Big) = 0 \quad \text{for all } \psi \in W^{1,2}(\Omega).$$

Here, da represents the surface element of $\partial\Omega$. In fact, a weak solution of (1.1) is of the class $W^{2,r}(\Omega)$ for any $r > 1$ by bootstrap arguments on elliptic regularity, which implies that it is a strong solution, simply called *solution*. If a solution of (1.1) is nonnegative and nonzero in Ω, then it is strictly positive in $\overline{\Omega}$ by the strong maximum principle ([10, Theorem 8.19]) and boundary point lemma ([10, Lemma 3.4]), and it is called *positive solution*.

In this paper, we treat concave homogeneous nonlinearities with coefficients in Sobolev spaces, of the types appearing in (1.1), to discuss the existence and nonexistence of positive solutions. For the nonlinear case when $c > 0$, we use the Nehari manifold for (1.1) with the aid of the related fibering map. The fibering method has been used to study the existence of multiple positive solutions for quasilinear elliptic problems by Drábek and Pohozaev [9] and for semilinear elliptic problems with convex, concave, and convex-concave nonlinearities inside a domain by Brown and Zhang [6], Brown [5], and Brown and Wu [7], respectively. When $c = 0$, the existence and nonexistence results for the case in which the functions m and σ belong to Hölder spaces have been already proved in [14] by using the sub-supersolution method. For the functions in Sobolev spaces, we approach our problem by using some trick based on the linearized eigenvalue problem of (1.1). The trick allows us to reduce (1.1) to a semilinear Neumann boundary value problem with coefficients in Hölder spaces.

Problems similar to (1.1), with concave nonlinearities, have been studied by Amann [2], Hess [11], and Pao [12] by using the sub-supersolution method and bifurcation theory and by Brezis and Oswald [4] by using variational methods.

Throughout this paper, we assume either that

$$m > 0 \text{ on a set of positive measure} \tag{1.2}$$

or that

$$\sigma(x_0) > 0 \text{ for some } x_0 \in \partial\Omega. \tag{1.3}$$

Under this assumption, we introduce a positive principal eigenvalue of the linearized eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda m(x)\varphi & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial \boldsymbol{n}} = \lambda\sigma(x)\varphi & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

An eigenvalue of (1.4) is called *principal* if it has an eigenfunction that does not change sign in Ω. We can prove (see Section 2) that if

$$\int_\Omega m\,dx + \int_{\partial\Omega} \sigma\,da < 0, \tag{1.5}$$

then, there exists a unique positive principal eigenvalue $\lambda_1(m,\sigma)$ of (1.4), characterized by the variational formula

$$\lambda_1(m,\sigma) = \inf\left\{ \frac{\int_\Omega |\nabla\varphi|^2 dx}{\int_\Omega m\varphi^2 dx + \int_{\partial\Omega} \sigma\varphi^2 da} : \varphi \in W^{1,2}(\Omega), \right. \tag{1.6}$$
$$\left. \int_\Omega m\varphi^2 dx + \int_{\partial\Omega} \sigma\varphi^2 da > 0 \right\},$$

and that problem (1.4) has no positive principal eigenvalue if

$$\int_\Omega m\,dx + \int_{\partial\Omega} \sigma\,da \geq 0. \tag{1.7}$$

In the case of (1.7), it is understood that $\lambda_1(m,\sigma) = 0$.

Now, we can state our main results. First, we prove the following nonexistence result of nontrivial solutions for (1.1).

Theorem 1.1. *Assume that either condition* (1.2) *or* (1.3) *holds. If condition* (1.5) *is satisfied, any solution of* (1.1) *identically equals zero in* $\overline{\Omega}$ *when* $0 < \lambda \leq \lambda_1(m,\sigma)$.

Next, we prove the following existence result of positive solutions for (1.1).

Theorem 1.2. *Assume that either condition* (1.2) *or* (1.3) *holds. Then, whenever* $p, q > 1$ *and* $c \geq 0$*, there exists a unique positive solution of* (1.1) *for each* $\lambda > \lambda_1(m,\sigma)$.

Remark 1.3. The uniqueness result for positive solutions of (1.1) is a direct consequence of [12, Theorem 4.6.3].

The rest of this paper is organized as follows: In Section 2, we provide a proof for the existence and uniqueness of a positive principal eigenvalue $\lambda_1(m, \sigma)$ of (1.4) which is a slight modification of the proofs of [14, Theorems 2.1 and 2.2], where the case when the functions m and σ belong to suitable Hölder classes has been established.

In Section 3, we introduce the Nehari manifold for the energy functional associated with (1.1) in the space $W^{1,2}(\Omega) \cap L^{p+1}(\Omega) \cap L^{q+1}(\partial\Omega)$ and investigate its properties by using a fibering map defined by the functional. We can understand that the Nehari manifold contains the critical points for the energy functional, meaning that it contains the weak solutions of (1.1). Thus, by studying the properties of the Nehari manifold, we can discuss the nonexistence and the existence of positive solutions by following the same line of arguments. In fact, we can see how drastically the Nehari manifold changes at the positive principal eigenvalue $\lambda_1(m, \sigma)$ (see Proposition 3.2). Section 3 contains the proof of Theorem 1.1, which is a direct consequence of this property of the Nehari manifold.

Section 4 is devoted to the proof of the existence result for the case when $c > 0$. First, we show the existence of a nonnegative, nonzero minimizer for the functional on the Nehari manifold introduced in Section 3. The standard compactness argument based on Sobolev's imbedding theorem for constructing minimizers for functionals does not work for (1.1), because powers p and q are arbitrarily greater than one. We overcome this difficulty by using Fatou's lemma and arguments based on the fibering map (see Lemmas 4.2 and 4.3). Next, we construct a minimizer in the space $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ for the functional by considering a truncated function of the first minimizer. Thus, the second minimizer corresponds to a positive solution of (1.1). Brezis and Oswald [4] have used a similar bootstrap argument to prove the existence of positive solutions by considering a truncated problem for the original problem. However, this technique appears insufficient for application to nonlinear boundary conditions.

In Section 5, we prove the existence result for the linear case when $c = 0$ and end the proof of Theorem 1.2. In this case, the bootstrap technique used for the nonlinear case in Section 4 appears inapplicable. By adopting a trick based on positive eigenfunctions with the principal eigenvalue of (2.1), it is possible to reduce (1.1) with $c = 0$ to a semilinear Neumann boundary value problem with coefficients of Hölder regularity. This implies that we can apply the sub-supersolution method ([2, Theorem 9.4]) to the reduced problem to prove the existence of a positive solution.

2. Positive principal eigenvalues

In this section, we prove the following existence and uniqueness result for positive principal eigenvalues of (1.4), which is an extension of [14, Theorem 2.2] to the lower regularity case of m and σ.

Theorem 2.1. *Assume that either condition* (1.2) *or* (1.3) *holds. Then, there exists a positive principal eigenvalue of* (1.4) *if, and only if, condition* (1.5) *holds. Moreover, it is unique and characterized by formula* (1.6).

Proof. To prove Theorem 2.1, it suffices to verify the following proposition for the associated auxiliary eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda m(x)\phi + \mu(\lambda)\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial \boldsymbol{n}} = \lambda\sigma(x)\phi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Proposition 2.2. *Assume that either condition* (1.2) *or* (1.3) *holds. The following three assertions hold:*

(1) *For any $\lambda \in \mathbb{R}$, there exists a unique principal eigenvalue $\mu_1(\lambda)$ of* (2.1), *given by the formula*

$$\mu_1(\lambda) = \inf\Big\{ \int_\Omega |\nabla v|^2\,dx - \lambda\Big(\int_\Omega mv^2\,dx + \int_{\partial\Omega}\sigma v^2\,da\Big) : \\ v \in W^{1,2}(\Omega), \int_\Omega v^2\,dx = 1\Big\}. \tag{2.2}$$

(2) *Mapping $\lambda \mapsto \mu_1(\lambda)$ is concave and satisfies*

$$\mu_1(\lambda) \to -\infty, \quad \lambda \to \infty. \tag{2.3}$$

(3) *The principal eigenvalue $\mu_1(\lambda)$ has a unique local maximum* (*i.e., global maximum*). *Moreover, the sign of the global maximum point coincides with that of $-(\int_\Omega m\,dx + \int_{\partial\Omega}\sigma\,da)$.*

Indeed, we can easily see that λ is a principal eigenvalue of (1.4) if, and only if, $\mu_1(\lambda) = 0$. Since $\mu_1(0) = 0$, we see from assertions (2) and (3) of Proposition 2.2 that, in order to have the existence of a positive principal eigenvalue of (1.4), it is necessary and sufficient that $\int_\Omega m\,dx + \int_{\partial\Omega}\sigma\,da < 0$. Moreover, the uniqueness is straightforward from assertion (2). Finally, formula (1.6) can be derived from (2.2), just as in the proof of [14, Theorem 2.2]. Now, it remains to verify Proposition 2.2.

Proof of Proposition 2.2. Let S_λ be the energy functional associated with (2.1), defined as

$$S_\lambda(v) = \int_\Omega |\nabla v|^2 dx - \lambda \left(\int_\Omega m v^2 dx + \int_{\partial\Omega} \sigma v^2 da \right), \quad v \in M,$$

where $M = \{v \in W^{1,2}(\Omega) : \|v\|_2 = 1\}$. As shown in [14, Lemma 3.1], we can show that S_λ is bounded below. By using the standard argument by Smoller [13, Chapter 11], we have the existence of a minimizer $\phi_1 \in M$ for S_λ on M; ϕ_1 is nontrivial and nonnegative in Ω:

$$S_\lambda(\phi_1) = \inf_{v \in M} S_\lambda(v).$$

By the Lagrange multiplier rule, $S_\lambda(\phi_1)$ is a principal eigenvalue of (2.1) with eigenfunction ϕ_1. Moreover, ϕ_1 belongs to $W^{2,r}(\Omega)$ for any $r > 1$ by elliptic regularity and is strictly positive in $\overline{\Omega}$ by the strong maximum principle and boundary point lemma. Hence, we see that for all $\psi \in W^{1,2}(\Omega)$,

$$\int_\Omega \nabla\phi_1 \nabla\psi dx - \lambda \left(\int_\Omega m\phi_1\psi dx + \int_{\partial\Omega} \sigma\phi_1\psi da \right) = S_\lambda(\phi_1) \int_\Omega \phi_1\psi dx. \quad (2.4)$$

If μ is any principal eigenvalue of (2.1) with a positive eigenfunction ϕ_2, from (2.4) we have $(\mu - S_\lambda(\phi_1)) \int_\Omega \phi_2\phi_1 dx = 0$. Therefore, the uniqueness of principal eigenvalues for (2.1) follows, thus, assertion (1) has been verified.

Next, we verify assertion (2). The concavity of the mapping $\lambda \mapsto \mu_1(\lambda)$ is from the fact that the mapping $\lambda \mapsto S_\lambda(\lambda)$ is affine. To verify (2.3), we first consider case (1.2). Let $m > 0$ in a measurable set $A \subset \Omega$ with $|A| > 0$. Since $|A| < \infty$, we can choose a compact subset $E \subset \Omega$ such that $E \subset A$ and $|E| > 0$. For $\varepsilon > 0$, we let $G_\varepsilon = \{x \in \Omega : \mathrm{dist}\,(x, E) < \varepsilon\}$. Then, we can choose $\varepsilon_0 > 0$ and $u_0 \in C^1(\Omega)$ with compact support in Ω such that

$$\begin{cases} |G_{\varepsilon_0} \setminus E| < \dfrac{(1/2)\int_E m dx}{\|m^+\|_\infty}, \\ 0 \le u_0 \le 1 \quad \text{in } \Omega, \\ u_0 = 1 \quad \text{in } G_{\varepsilon_0/2}, \\ \operatorname{supp} u_0 \subset G_{\varepsilon_0}. \end{cases}$$

Here, f^+ denotes the positive part of function f. It follows that

$$\begin{aligned} \int_\Omega m u_0^2 dx &= \int_E m u_0^2 dx + \int_{G_{\varepsilon_0} \setminus E} m u_0^2 dx \\ &\ge \int_E m dx - \|m^+\|_\infty |G_{\varepsilon_0} \setminus E| > \frac{1}{2} \int_E m dx > 0. \end{aligned}$$

By letting $v_0 = u_0/\|u_0\|_2$, formula (2.2) gives us the assertion

$$\mu_1(\lambda) \le \int_\Omega |\nabla v_0|^2\, dx - \lambda \int_\Omega m v_0^2\, dx - \lambda \int_{\partial\Omega} \sigma v_0^2\, da$$
$$\le \|u_0\|_2^{-2} \left(\int_\Omega |\nabla u_0|^2 dx - \frac{\lambda}{2} \int_E m dx \right) \longrightarrow -\infty, \quad \lambda \to \infty.$$

Secondly, we consider case (1.3). However, the verification of (2.3) is parallel as in the proof of [14, Theorem 2.2], because $\sigma \in C(\partial\Omega)$. This verifies assertion (2).

Finally, we verify assertion (3). Let $\phi_1(\lambda)$ be a positive eigenfunction for $\mu_1(\lambda)$, normalized as $\|\phi_1(\lambda)\|_2 = 1$, which implies that for all $\psi \in W^{1,2}(\Omega)$

$$\int_\Omega \nabla\phi_1(\lambda)\nabla\psi dx - \lambda \left(\int_\Omega m\phi_1(\lambda)\psi dx + \int_{\partial\Omega} \sigma\phi_1(\lambda)\psi da \right) \tag{2.5}$$
$$= \mu_1(\lambda) \int_\Omega \phi_1(\lambda)\psi dx,$$

and $\phi_1(\lambda) \in W^{2,r}(\Omega)$ for all $r > 1$, as stated above. Then, we assert that the mappings $\lambda \mapsto \mu_1(\lambda)$ and $\lambda \mapsto \phi_1(\lambda) \in W^{2,r}(\Omega)$ for $r > N$ are both continuously differentiable. In order to verify this assertion, we introduce the following mapping associated with (2.1).

$$\mathcal{F} : \mathbb{R} \times \mathbb{R} \times W^{2,r}(\Omega) \longrightarrow L^r(\Omega) \times W^{1-\frac{1}{r},r}(\partial\Omega) \times \mathbb{R}$$
$$(\lambda, \mu, \phi) \longmapsto \left(-\Delta\phi - \lambda m\phi - \mu\phi,\ \frac{\partial\phi}{\partial \boldsymbol{n}} - \lambda\sigma\phi,\ -1 + \int_\Omega \phi^2 dx \right).$$

Since $W^{2,r}(\Omega) \subset C^1(\overline{\Omega})$ from Sobolev's imbedding theorem, mapping $\mathcal{F}$ is well defined. It is clear that $\mathcal{F}(\lambda, \mu_1(\lambda), \phi_1(\lambda)) = 0$. We can easily check that the Fréchet derivative $\mathcal{F}_{(\mu,\phi)}(\lambda, \mu_1(\lambda), \phi_1(\lambda))$ of $\mathcal{F}$ at $(\lambda, \mu_1(\lambda), \phi_1(\lambda))$ with respect to (μ, ϕ) is continuous and bijective. The open mapping theorem provides that it is homeomorphic. Hence, considering that $\phi_1(\lambda) > 0$ in $\overline{\Omega}$, we apply the implicit function theorem to obtain the desired assertion. Hereby, we differentiate (2.5) with respect to λ and obtain

$$\int_\Omega \nabla\phi_1'(\lambda)\nabla\psi dx - \left(\int_\Omega m\phi_1(\lambda)\psi dx + \int_{\partial\Omega} \sigma\phi_1(\lambda)\psi da \right)$$
$$- \lambda \left(\int_\Omega m\phi_1'(\lambda)\psi dx + \int_{\partial\Omega} \sigma\phi_1'(\lambda)\psi da \right)$$
$$= \mu_1'(\lambda) \int_\Omega \phi_1(\lambda)\psi dx + \mu_1(\lambda) \int_\Omega \phi_1'(\lambda)\psi dx. \tag{2.6}$$

Assertions (2.5) with $\psi = \phi_1'(\lambda)$ and (2.6) with $\psi = \phi_1(\lambda)$ give

$$\mu_1'(\lambda) = -\left(\int_\Omega m\phi_1(\lambda)^2 dx + \int_{\partial\Omega} \sigma\phi_1(\lambda)^2 da\right). \tag{2.7}$$

This implies that λ is a critical point for μ_1, that is, $\mu_1'(\lambda) = 0$ if and only if

$$\int_\Omega m\phi_1(\lambda)^2 dx + \int_{\partial\Omega} \sigma\phi_1(\lambda)^2 da = 0.$$

Moreover, we claim that a critical point for μ_1 is unique if it exists, so that it is the global maximum point. Namely, we can show that $\mu_1(\lambda) < \mu_1(\lambda_0)$ for $\lambda \neq \lambda_0$ provided λ_0 is a critical point for μ_1. Indeed, by the definition of $\mu_1(\lambda)$, it follows that $\mu_1(\lambda) \leq S_\lambda(\phi_1(\lambda_0))$. Since

$$\int_\Omega m\phi_1(\lambda_0)^2 dx + \int_{\partial\Omega} \sigma\phi_1(\lambda_0)^2 da = 0,$$

it follows that $\mu_1(\lambda) \leq \mu_1(\lambda_0)$. Assume to the contrary that $\mu_1(\lambda) = \mu_1(\lambda_0)$ for some $\lambda \neq \lambda_0$. Then, $\phi_1(\lambda)$ satisfies (2.5) and also attains the infimum $\mu_1(\lambda_0)$, which implies that for all $\psi \in W^{1,2}(\Omega)$,

$$\begin{aligned}&\int_\Omega \nabla\phi_1(\lambda)\nabla\psi dx - \lambda_0 \left(\int_\Omega m\phi_1(\lambda)\psi dx + \int_{\partial\Omega} \sigma\phi_1(\lambda)\psi da\right)\\ &= \mu_1(\lambda_0)\int_\Omega \phi_1(\lambda)\psi dx.\end{aligned}$$

It follows that

$$(\lambda - \lambda_0)\left(\int_\Omega m\phi_1(\lambda)\psi dx + \int_{\partial\Omega} \sigma\phi_1(\lambda)\psi da\right) = 0 \quad \text{for all } \psi \in W^{1,2}(\Omega).$$

Since $\phi_1(\lambda)$ is strictly positive in $\overline{\Omega}$, we have $m = 0$ almost everywhere (a.e.) in Ω and $\sigma = 0$ everywhere on $\partial\Omega$, by the same argument as in the proof of assertion (2). This is a contradiction. To prove the existence of a critical point for μ_1, it is essential to note from (2.7) that

$$\mu_1'(0) = -\frac{\int_\Omega m dx + \int_{\partial\Omega} \sigma da}{|\Omega|}, \tag{2.8}$$

where $\phi_1(0) = |\Omega|^{-1/2}$. In fact, assertion (3) follows from assertion (2), (2.8), and the uniqueness of critical points for μ_1, since $\mu_1(0) = 0$. The proof of Proposition 2.2 is complete. □

The proof of Theorem 2.1 is now complete. □

3. NEHARI MANIFOLDS

Let $X = W^{1,2}(\Omega) \cap L^{p+1}(\Omega) \cap L^{q+1}(\partial\Omega)$ be a Banach space with the norm

$$\|\cdot\|_{1,2} + \|\cdot\|_{p+1} + \|\cdot\|_{q+1,\partial\Omega}.$$

We define the associated energy functional $J_\lambda : X \to \mathbb{R}$ as

$$J_\lambda(u) := \frac{1}{2} I_\lambda(u) + \lambda \left(\frac{1}{p+1} \int_\Omega |u|^{p+1} dx + \frac{c}{q+1} \int_{\partial\Omega} |u|^{q+1} da \right),$$

where

$$I_\lambda(u) = \int_\Omega |\nabla u|^2 dx - \lambda \left(\int_\Omega m u^2 dx + \int_{\partial\Omega} \sigma u^2 da \right),$$

and we see that J_λ is of class C^1. Indeed, the Fréchet derivative $J'_\lambda(u)$ at $u \in X$ is given as

$$J'_\lambda(u)\psi = \int_\Omega \nabla u \nabla \psi dx - \lambda \left(\int_\Omega m u \psi dx + \int_{\partial\Omega} \sigma u \psi da \right)$$
$$+ \lambda \left(\int_\Omega |u|^{p-1} u \psi dx + c \int_{\partial\Omega} |u|^{q-1} u \psi da \right), \quad \psi \in X.$$

In this section, we introduce the Nehari manifold associated with J_λ and investigate its properties, directly from which we prove here Theorem 1.1.

Now, the Nehari manifold $S(\lambda)$ for J_λ is introduced as

$$S(\lambda) = \left\{ u \in X : J'_\lambda(u)u = 0 \right\}.$$

Let the set of nontrivial elements of $S(\lambda)$ be

$$S^+(\lambda) = S(\lambda) \setminus \{0\}.$$

It should be remarked that the nontrivial solutions of (1.1) are contained in $S^+(\lambda)$. In order to understand $S^+(\lambda)$ well, we next introduce the fibering map $t \mapsto \Phi_u(t)$ associated with J_λ in the following manner. For $u \in X \setminus \{0\}$, we set

$$\Phi_u(t) := J_\lambda(tu), \quad t > 0.$$

By a direct calculation,

$$\Phi'_u(t) = t I_\lambda(u) + \lambda \left(t^p \int_\Omega |u|^{p+1} dx + c t^q \int_{\partial\Omega} |u|^{q+1} da \right), \tag{3.1}$$

from which we see that $\Phi'_u(t) = 0$ if and only if

$$I_\lambda(tu) + \lambda \left(\int_\Omega |tu|^{p+1} dx + c \int_{\partial\Omega} |tu|^{q+1} da \right) = 0,$$

that is, $tu \in S^+(\lambda)$. Furthermore, for $t>0$ satisfying $\Phi_u'(t)=0$, we obtain

$$\Phi_u''(t) = I_\lambda(u) + \lambda\left(pt^{p-1}\int_\Omega |u|^{p+1}dx + cqt^{q-1}\int_{\partial\Omega}|u|^{q+1}da\right) \tag{3.2}$$

$$= \lambda\left\{(p-1)t^{p-1}\int_\Omega |u|^{p+1}dx + c(q-1)t^{q-1}\int_{\partial\Omega}|u|^{q+1}da\right\} > 0. \tag{3.3}$$

From (3.1) and (3.2), we observe that $\Phi_u'(0)=0$, $\Phi_u''(0)=I_\lambda(u)$, and Φ_u is strictly positive for any sufficiently small value of $t>0$ when $I_\lambda(u)=0$, from (3.3) that Φ_u is strictly convex for $t>0$, satisfying $\Phi_u'(t)=0$, and from the definition of Φ_u that $\Phi_u(t)\to\infty$ as $t\to\infty$. Thus, we obtain the following result.

Proposition 3.1. *Let $u\in X\setminus\{0\}$. Then, we have the following two assertions.*

(1) *If $I_\lambda(u)<0$, then there exists a unique local minimum point $t(u)>0$ for Φ_u. More precisely, Φ_u is strictly decreasing in $t\in(0,t(u))$ and increasing in $t\in(t(u),\infty)$. In this case, $t(u)u\in S^+(\lambda)$.*

(2) *If $I_\lambda(u)\geq 0$, then there is no local maximum and minimum points for Φ_u. More precisely, Φ_u is strictly increasing in $t\in(0,\infty)$. In this case, $u\notin S^+(\lambda)$.*

Proposition 3.1 provides that the structure of $S^+(\lambda)$ depends on $\lambda_1(m,\sigma)$ as follows.

Proposition 3.2. (1) *$S^+(\lambda)\neq\emptyset$ if $\lambda>\lambda_1(m,\sigma)$. Conversely, when*

$$\int_\Omega m\,dx + \int_{\partial\Omega}\sigma\,da < 0,$$

we have $S^+(\lambda)=\emptyset$ if $0<\lambda\leq\lambda_1(m,\sigma)$.
(2) *$S^+(\lambda)$ is bounded in $W^{1,2}(\Omega)$.*

Proof. First, we use an eigenfunction ϕ_1 with the principal eigenvalue $\lambda_1(m,\sigma)$ of (2.1) to prove assertion (1). By elliptic regularity and using Sobolev's imbedding theorem, we have $\phi_1\in C^1(\overline{\Omega})$, so that $\phi_1\in X\setminus\{0\}$. It is clear that $I_\lambda(\phi_1)=\mu_1(\lambda)\int_\Omega\phi_1^2dx$. Since $\mu_1(\lambda)<0$ for $\lambda>\lambda_1(m,\sigma)$, we have $I_\lambda(\phi_1)<0$. Hence, assertion (1) of Proposition 3.1 is applicable and thus, we obtain $t(\phi_1)\phi_1\in S^+(\lambda)$. Meanwhile, when $0<\lambda\leq\lambda_1(m,\sigma)$, it is easy to see from (1.6) that $I_\lambda(u)\geq 0$ for all $u\in W^{1,2}(\Omega)$. Hence, assertion (2) of Proposition 3.1 is applicable and thus we have $S^+(\lambda)=\emptyset$.

Next, we use a contradiction argument to prove assertion (2). Assume to the contrary that there is a sequence $\{u_j\}\subset S^+(\lambda)$ such that $\|u_j\|_{1,2}\to\infty$.

By substituting $v_j = \frac{u_j}{\|u_j\|_{1,2}}$, we have $\|v_j\|_{1,2} = 1$ and

$$I_\lambda(v_j) + \lambda \left(\|u_j\|_{1,2}^{p-1} \int_\Omega |v_j|^{p+1} dx + c\|u_j\|_{1,2}^{q-1} \int_{\partial\Omega} |v_j|^{q+1} da \right) = 0.$$

Since v_j is bounded in $W^{1,2}(\Omega)$ by the compactness argument, there exist its subsequence still denoted by the same notation v_j, and $v_0 \in W^{1,2}(\Omega)$ such that

$$\begin{aligned} v_j &\longrightarrow v_0 \quad \text{weakly in } W^{1,2}(\Omega), \\ v_j &\longrightarrow v_0 \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \end{aligned} \tag{3.4}$$

By weakly lower semicontinuity,

$$\int_\Omega |\nabla v_0|^2 dx \leq \liminf_{j\to\infty} \int_\Omega |\nabla v_j|^2 dx.$$

It follows that

$$\begin{aligned} I_\lambda(v_0) &\leq \liminf_{j\to\infty} I_\lambda(v_j) \\ &= \liminf_{j\to\infty} (-\lambda) \left(\|u_j\|_{1,2}^{p-1} \int_\Omega |v_j|^{p+1} dx + c\|u_j\|_{1,2}^{q-1} \int_{\partial\Omega} |v_j|^{q+1} da \right) \leq 0. \end{aligned}$$

Since $\|u_j\|_{1,2} \to \infty$, it follows that

$$\int_\Omega |v_j|^{p+1} dx \longrightarrow 0 \quad \text{and} \quad \int_{\partial\Omega} |v_j|^{q+1} da \longrightarrow 0, \quad j \to \infty. \tag{3.5}$$

By Hölder's inequality,

$$\begin{aligned} \int_\Omega |v_0|^2 dx &\leq 2 \left(\int_\Omega |v_0 - v_j|^2 dx + \int_\Omega |v_j|^2 dx \right) \\ &\leq 2 \left\{ \int_\Omega |v_0 - v_j|^2 dx + C_p' \left(\int_\Omega |v_j|^{p+1} dx \right)^{\frac{2}{p+1}} \right\}, \end{aligned}$$

with some constant $C_p' > 0$. Passing to the limit, we have $\int_\Omega |v_0|^2 dx = 0$, so that

$$v_0 = 0 \quad \text{a.e. in } \Omega. \tag{3.6}$$

In the same way, conditions (3.4) and (3.5) give

$$v_0 = 0 \quad \text{a.e. on } \partial\Omega.$$

Hence, it follows that

$$0 = I_\lambda(v_0) = \liminf_{j\to\infty} I_\lambda(v_j).$$

In view of (3.4), this ensures the existence of a subsequence of v_j, still denoted by v_j, such that

$$\int_\Omega |\nabla v_j|^2 dx \longrightarrow \int_\Omega |\nabla v_0|^2 dx, \quad j \to \infty,$$

which implies that $\|v_j\|_{1,2} \to \|v_0\|_{1,2}$, so that $\|v_0\|_{1,2} = 1$, a contradiction for (3.6). The proof of Proposition 3.2 is complete. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let u be a solution of (1.1), then, $u \in C^1(\overline{\Omega})$. By definition, we have

$$I_\lambda(u) + \lambda \left(\int_\Omega |u|^{p+1} dx + c \int_{\partial\Omega} |u|^{q+1} da \right) = 0,$$

that is, $u \in S(\lambda)$. If $0 < \lambda \leq \lambda_1(m, \sigma)$, then assertion (1) of Proposition 3.2 implies that $u = 0$. Thus, the proof of Theorem 1.1 is complete. □

4. MINIMIZERS

This section is devoted to the existence of a positive solution of (1.1) for the case when $c > 0$. Without loss of generality, we may consider the case when $c = 1$.

First, we prove the existence of a minimizer for functional J_λ on $S^+(\lambda)$, which is a solution of (1.1) with $c = 1$ in a weak sense.

Proposition 4.1. *Let $\lambda > \lambda_1(m, \sigma)$. Then, we have $u_0 \geq 0$ such that*

$$J_\lambda(u_0) = \min_{u \in S^+(\lambda)} J_\lambda(u) < 0. \tag{4.1}$$

Moreover, we obtain

$$J'_\lambda(u_0)\psi = 0 \quad \textit{for all } \psi \in X. \tag{4.2}$$

Proof. First, we verify assertion (4.1). If $u \in S^+(\lambda)$, we see that

$$J_\lambda(u) = \lambda \left\{ \left(\frac{1}{p+1} - \frac{1}{2} \right) \int_\Omega |u|^{p+1} dx + \left(\frac{1}{q+1} - \frac{1}{2} \right) \int_{\partial\Omega} |u|^{q+1} da \right\} < 0.$$

Meanwhile, it is clear that $J_\lambda(u) \geq \frac{1}{2} I_\lambda(u)$. By using the continuous imbedding $W^{1,2}(\Omega) \subset L^2(\partial\Omega)$, we find from assertion (2) of Proposition 3.2 that I_λ is bounded in $S^+(\lambda)$, so that

$$-\infty < \inf_{u \in S^+(\lambda)} J_\lambda(u) < 0.$$

Let $u_j \in S^+(\lambda)$ be a minimizing sequence, which implies that

$$J_\lambda(u_j) \longrightarrow \inf_{u \in S^+(\lambda)} J_\lambda(u).$$

It follows that

$$\lambda\Big\{\Big(\frac{1}{p+1}-\frac{1}{2}\Big)\int_\Omega |u_j|^{p+1}dx + \Big(\frac{1}{q+1}-\frac{1}{2}\Big)\int_{\partial\Omega} |u_j|^{q+1}da\Big\} \longrightarrow \inf_{u \in S^+(\lambda)} J_\lambda(u). \tag{4.3}$$

From assertion (2) of Proposition 3.2, we see that u_j is bounded in $W^{1,2}(\Omega)$. Then, there exist a subsequence of u_j, denoted by the same notation, and $u_0 \in W^{1,2}(\Omega)$ such that

$$u_j \longrightarrow u_0 \quad \text{weakly in } W^{1,2}(\Omega), \tag{4.4}$$

$$u_j \longrightarrow u_0 \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{4.5}$$

Now, we consider the fact that the standard compactness argument breaks down because of the condition that $p > 1$ and $q > 1$ are arbitrary. The following two lemmas play an essential role in overcoming the difficulty.

Lemma 4.2. *$u_0 \in L^{p+1}(\Omega) \cap L^{q+1}(\partial\Omega)$.*

Proof. From (4.3), we see that $\int_\Omega |u_j|^{p+1}dx$ and $\int_{\partial\Omega} |u_j|^{q+1}da$ are both bounded. Condition (4.5) gives us a subsequence of u_j, denoted by the same notation, satisfying that

$$u_j \longrightarrow u_0 \quad \text{a.e. in } \Omega, \tag{4.6}$$

$$u_j \longrightarrow u_0 \quad \text{a.e. on } \partial\Omega. \tag{4.7}$$

By Fatou's lemma, it follows that

$$\int_\Omega |u_0|^{p+1}dx = \int_\Omega \liminf_{j\to\infty} |u_j|^{p+1}dx \le \liminf_{j\to\infty} \int_\Omega |u_j|^{p+1}dx < \infty, \tag{4.8}$$

$$\int_{\partial\Omega} |u_0|^{q+1}da = \int_{\partial\Omega} \liminf_{j\to\infty} |u_j|^{q+1}da \le \liminf_{j\to\infty} \int_{\partial\Omega} |u_j|^{q+1}da < \infty. \tag{4.9}$$

The proof of Lemma 4.2 is complete. □

Lemma 4.3. *Let u_j be the minimizing sequence such that* (4.4) *through* (4.7) *hold. Then, there exists a subsequence of u_j, denoted by the same notation, such that we have the assertion*

$$\begin{cases} \|\nabla u_j\|_2 \longrightarrow \|\nabla u_0\|_2, \\ \|u_j\|_{p+1} \longrightarrow \|u_0\|_{p+1}, \\ \|u_j\|_{q+1,\partial\Omega} \longrightarrow \|u_0\|_{q+1,\partial\Omega} \quad \text{as } j \to \infty. \end{cases} \tag{4.10}$$

Proof. By weakly lower semicontinuity (see (4.8) and (4.9)), we have

$$I_\lambda(u_0) + \lambda \left(\int_\Omega |u_0|^{p+1} dx + \int_{\partial\Omega} |u_0|^{q+1} da \right) \\ \leq \liminf_{j\to\infty} I_\lambda(u_j) + \lambda \left(\liminf_{j\to\infty} \int_\Omega |u_j|^{p+1} dx + \liminf_{j\to\infty} \int_{\partial\Omega} |u_j|^{q+1} da \right). \quad (4.11)$$

If we assume to the contrary that assertion (4.10) does not hold, it follows from (4.11) that

$$I_\lambda(u_0) + \lambda \left(\int_\Omega |u_0|^{p+1} dx + \int_{\partial\Omega} |u_0|^{q+1} da \right) \\ < \liminf_{j\to\infty} I_\lambda(u_j) + \lambda \left(\liminf_{j\to\infty} \int_\Omega |u_j|^{p+1} dx + \liminf_{j\to\infty} \int_{\partial\Omega} |u_j|^{q+1} da \right) \\ \leq \liminf_{j\to\infty} \left\{ I_\lambda(u_j) + \lambda \left(\int_\Omega |u_j|^{p+1} dx + \int_{\partial\Omega} |u_j|^{q+1} da \right) \right\} = 0. \quad (4.12)$$

In particular, $u_0 \in X \setminus \{0\}$ and $I_\lambda(u_0) < 0$. Hence, assertion (1) of Proposition 3.1 is applicable and we obtain $t(u_0)u_0 \in S^+(\lambda)$. This gives

$$I_\lambda(u_0) + \lambda \left(t(u_0)^{p-1} \int_\Omega |u_0|^{p+1} dx + t(u_0)^{q-1} \int_{\partial\Omega} |u_0|^{q+1} da \right) = 0. \quad (4.13)$$

Condition (4.12) together with (4.13) provides

$$\int_\Omega |u_0|^{p+1} dx + \int_{\partial\Omega} |u_0|^{q+1} da < t(u_0)^{p-1} \int_\Omega |u_0|^{p+1} dx + t(u_0)^{q-1} \int_{\partial\Omega} |u_0|^{q+1} da,$$

so that $1 < t(u_0)$, since the mapping

$$t \longmapsto t^{p-1} \int_\Omega |u_0|^{p+1} dx + t^{q-1} \int_{\partial\Omega} |u_0|^{q+1} da,$$

is strictly increasing in $t \in [0, \infty)$. Again by using assertion (1) of Proposition 3.1, it follows that $\Phi_{u_0}(t(u_0)) < \Phi_{u_0}(1)$, which implies that

$$J_\lambda(t(u_0)u_0) < J_\lambda(u_0) < \liminf_{j\to\infty} J_\lambda(u_j) = \inf_{u \in S^+(\lambda)} J_\lambda(u).$$

This is contradictory to the assertion that $t(u_0)u_0 \in S^+(\lambda)$. The proof of Lemma 4.3 is complete. □

Now, assertion (4.1) follows immediately from Lemma 4.3. Indeed, we have only to consider $|u_0|$ in place of u_0.

Next, we use the Lagrange multiplier rule to verify assertion (4.2). Since $J_\lambda(u_0) < 0$, we have $J_\lambda(u_0) = \min_{u \in S(\lambda)} J_\lambda(u)$. We put $K_\lambda(u) = J'_\lambda(u)u$, then $K_\lambda : X \to \mathbb{R}$ is of class C^1. Indeed, the Fréchet derivative $K'_\lambda(u)$ at $u \in X$ is given as

$$K'_\lambda(u)\psi = 2\left\{\int_\Omega \nabla u \nabla \psi dx - \lambda\left(\int_\Omega mu\psi dx + \int_{\partial\Omega} \sigma u \psi da\right)\right\}$$
$$+ \lambda\left((p+1)\int_\Omega |u|^{p-1}u\psi dx + (q+1)\int_{\partial\Omega} |u|^{q-1}u\psi da\right), \quad \psi \in X.$$

From the condition that $K_\lambda(u_0) = 0$, it follows that

$$K'_\lambda(u_0)u_0 = \lambda\left((p-1)\int_\Omega |u_0|^{p+1}dx + (q-1)\int_{\partial\Omega} |u_0|^{q+1}da\right) > 0, \quad (4.14)$$

which implies that $K'_\lambda(u_0)$ is surjective. By the Lagrange multiplier rule (see [15, Proposition 43.19]), there exists a constant $\Lambda \in \mathbb{R}$ such that

$$J'_\lambda(u_0)\psi - \Lambda K'_\lambda(u_0)\psi = 0 \quad \text{for all } \psi \in X.$$

Put $\varphi = u_0$ and then

$$J'_\lambda(u_0)u_0 - \Lambda K'_\lambda(u_0)u_0 = 0.$$

Since $J'_\lambda(u_0)u_0 = K_\lambda(u_0) = 0$ and $K'_\lambda(u_0)u_0 \neq 0$ from (4.14), we have $\Lambda = 0$. Assertion (4.2) has been now verified. The proof of Proposition 4.1 is complete. □

Next, we show that it is possible to choose a minimizer for J_λ on $S^+(\lambda)$ belonging to $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$. From the definition of J_λ, we see that

$$J_\lambda(u) = \frac{1}{2}\int (|\nabla u|^2 dx + \lambda\left(\int_\Omega G(x,u)dx + \int_{\partial\Omega} H(x,u)da\right), \quad (4.15)$$

where

$$G(x,u) = \frac{1}{p+1}|u|^{p+1} - \frac{m(x)}{2}u^2, \qquad H(x,u) = \frac{1}{q+1}|u|^{q+1} - \frac{\sigma(x)}{2}u^2.$$

Put

$$G_\infty(u) = \frac{1}{p+1}|u|^{p+1} - \frac{\|m^+\|_\infty}{2}u^2, \quad H_\infty(u) = \frac{1}{q+1}|u|^{q+1} - \frac{\|\sigma^+\|_{\infty,\partial\Omega}}{2}u^2,$$

then, we find that $G_\infty(u)$ and $H_\infty(u)$ are strictly increasing in $u \in [\|m^+\|_\infty^{\frac{1}{p-1}}, \infty)$ and $u \in [\|\sigma^+\|_{\infty,\partial\Omega}^{\frac{1}{q-1}}, \infty)$, respectively. Hence, for any x, $G(x,u)$ and $H(x,u)$ are both strictly increasing in $u \in [t_0, \infty)$ with the constant

$$t_0 = \max\left\{\|m^+\|_\infty^{\frac{1}{p-1}},\ \|\sigma^+\|_{\infty,\partial\Omega}^{\frac{1}{q-1}}\right\}. \quad (4.16)$$

Let $\lambda > \lambda_1(m,\sigma)$, and let $u_0 \in S^+(\lambda)$ the minimizer for J_λ just as in Proposition 4.1. By $v_0 \in W^{1,2}(\Omega)$, we denote the function truncated at t_0

$$v_0 = \min\{u_0,\ t_0\} \quad \text{in } \Omega. \tag{4.17}$$

Then, $v_0 \in L^\infty(\Omega)$. Moreover, since we can verify (see Appendix) that

$$v_0|_{\partial\Omega} = \min\{u_0|_{\partial\Omega}, t_0\} \quad \text{a.e. on } \partial\Omega, \tag{4.18}$$

it follows that

$$v_0|_{\partial\Omega} = \begin{cases} u_0|_{\partial\Omega} & \text{on } \{x \in \partial\Omega : u_0|_{\partial\Omega} \le t_0\}, \\ t_0 & \text{on } \{x \in \partial\Omega : u_0|_{\partial\Omega} > t_0\}, \end{cases}$$

implying that $v_0 \in L^\infty(\partial\Omega)$.

Now, combining Proposition 4.1 and the following proposition allows us to confirm that the truncated function $v_0 \ge 0$ is indeed a weak solution of (1.1) with $c = 1$ and is thus a positive solution of the problem as we desire.

Proposition 4.4. *Let $v_0 \in L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ be the truncated function defined by* (4.17)*. Then, we obtain*

$$J_\lambda(v_0) \le J_\lambda(u_0), \tag{4.19}$$

$$v_0 \in S^+(\lambda). \tag{4.20}$$

This implies that v_0 is a minimizer for J_λ on $S^+(\lambda)$.

Proof. First, we verify (4.19). By [8, Theorem 2.8], we have

$$\nabla v_0 = \chi_{\{u_0<t_0\}} \nabla u_0 \quad \text{in } \Omega,$$

where χ_A is the characteristic function of set $A \subset \Omega$. It follows that

$$\int_\Omega |\nabla v_0|^2 dx \le \int_\Omega |\nabla u_0|^2 dx. \tag{4.21}$$

For $x \in \Omega$ satisfying $u_0(x) \le t_0$, we have $v_0(x) = u_0(x)$, so that $G(x, v_0(x)) = G(x, u_0(x))$. For $x \in \Omega$ satisfying $u_0(x) > t_0$, we have $G(x, v_0(x)) = G(x, t_0) \le G(x, u_0(x))$, since the mapping $u \mapsto G(x,u)$ is increasing in $u \ge t_0$. Hence,

$$G(x, v_0(x)) \le G(x, u_0(x)) \quad \text{a.e. } x \in \Omega. \tag{4.22}$$

In the same manner, condition (4.16) provides

$$H(x, v_0(x)) \le H(x, u_0(x)) \quad \text{a.e. } x \in \partial\Omega. \tag{4.23}$$

By noting (4.15), assertion (4.19) follows directly from (4.21), (4.22), and (4.23).

Next, we show how the fibering method works in verifying (4.20). From the definition of K_λ,

$$K_\lambda(u) = \int_\Omega |\nabla u|^2 dx + \lambda \left(\int_\Omega g(x,u)dx + \int_{\partial\Omega} h(x,u)da \right),$$

where

$$g(x,u) = |u|^{p+1} - m(x)u^2, \qquad h(x,u) = |u|^{q+1} - \sigma(x)u^2.$$

We put

$$g_\infty(u) = |u|^{p+1} - \|m^+\|_\infty u^2, \qquad h_\infty(u) = |u|^{q+1} - \|\sigma^+\|_{\infty,\partial\Omega} u^2.$$

We note that $g_\infty(u)$ and $h_\infty(u)$ are both increasing in $u \geq t_1$ for the constant

$$t_1 = \max\left\{ \left(\frac{2}{p+1}\|m^+\|_\infty\right)^{\frac{1}{p-1}}, \left(\frac{2}{q+1}\|\sigma^+\|_{\infty,\partial\Omega}\right)^{\frac{1}{q-1}} \right\}.$$

It follows that $g(x,u)$ and $h(x,u)$ are both increasing in $u \geq t_1$ independently of x. Since $t_1 < t_0$, it follows that $g(x,u)$ and $h(x,u)$ are both increasing in $u \geq t_0$ independently of x. Hence, $K_\lambda(v_0) \leq K_\lambda(u_0) = 0$. However, we can show that

$$K_\lambda(v_0) = 0. \tag{4.24}$$

In order to prove this, we assume on the contrary that $K_\lambda(v_0) < 0$, which reads as

$$I_\lambda(v_0) + \lambda \left(\int_\Omega |v_0|^{p+1} dx + \int_{\partial\Omega} |v_0|^{q+1} da \right) < 0.$$

In particular, $v_0 \in X \setminus \{0\}$ and $I_\lambda(v_0) < 0$. Then, assertion (1) of Proposition 3.1 is applicable and we obtain $t(v_0)v_0 \in S^+(\lambda)$, which implies that

$$I_\lambda(v_0) + \lambda \left(t(v_0)^{p-1} \int_\Omega |v_0|^{p+1} dx + t(v_0)^{q-1} \int_{\partial\Omega} |v_0|^{q+1} da \right) = 0.$$

Consequently,

$$\int_\Omega |v_0|^{p+1} dx + \int_{\partial\Omega} |v_0|^{q+1} da < t(v_0)^{p-1} \int_\Omega |v_0|^{p+1} dx + t(v_0)^{q-1} \int_{\partial\Omega} |v_0|^{q+1} da.$$

Again, by using the property that the mapping

$$t \mapsto t^{p-1} \int_\Omega |v_0|^{p+1} dx + t^{q-1} \int_{\partial\Omega} |v_0|^{q+1} da,$$

is strictly increasing in $t \geq 0$, we have $1 < t(v_0)$. By assertion (1) of Proposition 3.1 and (4.19),

$$J_\lambda(t(v_0)v_0) < J_\lambda(v_0) \leq J_\lambda(u_0) = \inf_{u \in S^+(\lambda)} J_\lambda(u).$$

This contradicts the assertion that $t(v_0)v_0 \in S^+(\lambda)$. Assertion (4.24) (or (4.20)) has been verified. The proof of Proposition 4.4 is now complete. □

Theorem 1.2 has been verified in the case when $c > 0$. □

5. Linear boundary conditions

In this section, we discuss the existence of a positive solution to (1.1) for the case when $c = 0$ and end the proof of Theorem 1.2. We recall from Section 2 that the principal eigenvalue $\mu_1(\lambda)$ of (2.1) is negative for any $\lambda > \lambda_1(m, \sigma)$, having a nonnegative eigenfunction $\phi_1(\lambda) \in W^{1,2}(\Omega)$ that is normalized as $\|\phi_1(\lambda)\|_2 = 1$. We also recall that $\phi_1(\lambda) \in W^{2,r}(\Omega)$ for each $r > 1$, thus that $\phi_1(\lambda) \in C^{1+\theta}(\overline{\Omega})$ for all $0 < \theta < 1$ by Sobolev's imbedding theorem, and that $\phi_1(\lambda) > 0$ in $\overline{\Omega}$.

Now, we consider the existence of a positive solution of the semilinear Neumann boundary value problem with coefficients in Hölder spaces

$$\begin{cases} -\Delta v - \dfrac{2\nabla\phi_1(\lambda) \cdot \nabla v}{\phi_1(\lambda)} = -\mu_1(\lambda)v - \lambda\phi_1(\lambda)^{p-1}v^p & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

It is clear that $\psi \equiv K$, where K is a positive constant, is a supersolution of (5.1) if K is sufficiently large. The following proposition plays an essential role in constructing a smaller subsolution of (5.1) for $\lambda > \lambda_1(m, \sigma)$.

Proposition 5.1. *Let $\mu_1(\lambda)$ and $\phi_1(\lambda)$ be as above. Then, the smallest eigenvalue $\gamma_1(\lambda)$ for the linearized eigenvalue problem*

$$\begin{cases} -\Delta w - \dfrac{2\nabla\phi_1(\lambda) \cdot \nabla w}{\phi_1(\lambda)} = -\mu_1(\lambda)w + \gamma(\lambda)w & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

is given by the formula

$$\gamma_1(\lambda) = \mu_1(\lambda) - \int_\Omega \frac{\phi_1(\lambda)^2}{w_1^2} |\nabla w_1|^2 dx. \tag{5.2}$$

Here, w_1 is an eigenfunction with $\gamma_1(\lambda)$, which is strictly positive in $\overline{\Omega}$. In particular, $\gamma_1(\lambda) < 0$ for any $\lambda > \lambda_1(m, \sigma)$.

Proof. We use the generalized Picone identity (cf. [3]) to obtain (5.2). By a direct computation,

$$\begin{aligned}&\left(\frac{\phi_1(\lambda)}{w_1}\right)\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left(w_1^2\frac{\partial}{\partial x_j}\left(\frac{\phi_1(\lambda)}{w_1}\right)\right)\\ &= -\lambda m\phi_1(\lambda)^2 - 2\mu_1(\lambda)\phi_1(\lambda)^2 + 2\frac{\phi_1(\lambda)}{w_1}\nabla\phi_1(\lambda)\nabla w_1 + \gamma_1(\lambda)\phi_1(\lambda)^2.\end{aligned}$$

Green's formula gives us

$$\begin{aligned}&\int_\Omega \left(\frac{\phi_1(\lambda)}{w_1}\right)\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left(w_1^2\frac{\partial}{\partial x_j}\left(\frac{\phi_1(\lambda)}{w_1}\right)\right)dx\\ &= -\int_\Omega w_1^2\left|\nabla\left(\frac{\phi_1(\lambda)}{w_1}\right)\right|^2 dx - \int_{\partial\Omega}\lambda\sigma\phi_1(\lambda)^2\,da.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_1(\lambda) &= 2\mu_1(\lambda) - \int_\Omega w_1^2\left|\nabla\left(\frac{\phi_1(\lambda)}{w_1}\right)\right|^2 dx\\ &\quad + \lambda\left(\int_\Omega m\phi_1(\lambda)^2dx + \int_{\partial\Omega}\lambda\sigma\phi_1(\lambda)^2da\right) - 2\int_\Omega \frac{\phi_1(\lambda)}{w_1}\nabla\phi_1(\lambda)\nabla w_1 dx.\end{aligned}$$

We note here that

$$w_1^2\left|\nabla\left(\frac{\phi_1(\lambda)}{w_1}\right)\right|^2 = |\nabla\phi_1(\lambda)|^2 - 2\frac{\phi_1(\lambda)}{w_1}\nabla\phi_1(\lambda)\nabla w_1 + \frac{\phi_1(\lambda)^2}{w_1^2}|\nabla w_1|^2.$$

Hence, we have

$$\begin{aligned}\gamma_1(\lambda) &= \mu_1(\lambda) - \int_\Omega \frac{\phi_1(\lambda)^2}{w_1^2}|\nabla w_1|^2dx\\ &\quad + \left\{\mu_1(\lambda) - \int_\Omega |\nabla\phi_1(\lambda)|^2dx + \lambda\left(\int_\Omega m\phi_1(\lambda)^2dx + \int_{\partial\Omega}\sigma\phi_1(\lambda)^2da\right)\right\}\\ &= \mu_1(\lambda) - \int_\Omega \frac{\phi_1(\lambda)^2}{w_1^2}|\nabla w_1|^2dx.\end{aligned}$$

Proposition 5.1 has been proved. □

Since $\gamma_1(\lambda) < 0$ for $\lambda > \lambda_1(m,\sigma)$, we can show that εw_1 is a subsolution of (5.1) for any $\varepsilon > 0$ small enough. The sub-supersolution method ([2, Theorem 9.4]) can be applied to obtain a positive solution $v(\lambda) \in C^2(\overline{\Omega})$ of (5.1) for each $\lambda > \lambda_1(m,\sigma)$ such that $\varepsilon w_1 \le v(\lambda) \le K$ in $\overline{\Omega}$. Finally, we can easily check that function $u = \phi_1(\lambda)v(\lambda)$ is a positive solution of the original problem (1.1). The proof of Theorem 1.2 is now complete. □

Appendix A. Appendix

For readers' convenience we give a proof for the following result, which has been used for the verification of (4.18).

Proposition A.1. *For any $u \in W^{1,2}(\Omega)$ and $t \in \mathbb{R}$, we have*

$$(u \wedge t)|_{\partial\Omega} = u|_{\partial\Omega} \wedge t \quad \text{a.e. on } \partial\Omega. \tag{A.1}$$

Here, $u \wedge v = \min\{u, v\}$ for functions u and v.

Proof. For $u \in W^{1,2}(\Omega)$, it is possible to choose $\{u_j\} \subset C^1(\overline{\Omega})$ such that $u_j \to u$ in $W^{1,2}(\Omega)$. Then, it follows that $u_j|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^2(\partial\Omega)$ and assertion (A.1) holds for u_j. Without loss of generality, we may assume that $u_j \to u$ a.e. in Ω.

Now, there exists a constant $K_1 > 0$ such that

$$\begin{aligned}
&\int_{\partial\Omega} |(u \wedge t)|_{\partial\Omega} - u|_{\partial\Omega} \wedge t|^2 da \\
&\le K_1 \left\{ \int_{\partial\Omega} |(u \wedge t)|_{\partial\Omega} - (u_j \wedge t)|_{\partial\Omega}|^2 da + \int_{\partial\Omega} |(u_j \wedge t)|_{\partial\Omega} - u_j|_{\partial\Omega} \wedge t|^2 da \right. \\
&\qquad \left. + \int_{\partial\Omega} |u_j|_{\partial\Omega} \wedge t - u|_{\partial\Omega} \wedge t|^2 da \right\} \\
&= K_1 \left\{ \int_{\partial\Omega} |(u \wedge t)|_{\partial\Omega} - (u_j \wedge t)|_{\partial\Omega}|^2 da + \int_{\partial\Omega} |u_j|_{\partial\Omega} \wedge t - u|_{\partial\Omega} \wedge t|^2 da \right\} \\
&=: (I_1)_j + (I_2)_j.
\end{aligned}$$

By a simple consideration, we find that

$$|u_j|_{\partial\Omega} \wedge t - u|_{\partial\Omega} \wedge t| \le |u_j|_{\partial\Omega} - u|_{\partial\Omega}| \quad \text{a.e. on } \partial\Omega. \tag{A.2}$$

This implies that $(I_2)_j \to 0$. For $(I_1)_j$, we note that there exists a constant $K_2 > 0$ such that

$$\begin{aligned}
(I_1)_j &\le K_2 \|u \wedge t - u_j \wedge t\|_{1,2}^2 \\
&= K_2 \left\{ \|u \wedge t - u_j \wedge t\|_2^2 + \|\nabla(u \wedge t) - \nabla(u_j \wedge t)\|_2^2 \right\}.
\end{aligned}$$

In the same manner as in (A.2), we have $\|u \wedge t - u_j \wedge t\|_2^2 \to 0$. Again by using the formula

$$\nabla(v \wedge t) = \chi_{\{v<t\}} \nabla v \quad \text{a.e. in } \Omega \qquad \text{for } v \in W^{1,2}(\Omega),$$

it follows that

$$\begin{aligned}|\nabla(u\wedge t)-\nabla(u_j\wedge t)|^2 &= |\chi_{\{u<t\}}\nabla u-\chi_{\{u_j<t\}}\nabla u_j|^2\\ &\le 2\left(|\chi_{\{u_j<t\}}\nabla(u_j-u)|^2+|\chi_{\{u_j<t\}}-\chi_{\{u<t\}}|^2|\nabla u|^2\right)\\ &\le 2\left(|\nabla(u_j-u)|^2+|\chi_{\{u_j<t\}}-\chi_{\{u<t\}}|^2|\nabla u|^2\right) \quad \text{a.e. in } \Omega. \quad (A.3)\end{aligned}$$

From [8, Corollary 2.1], we have

$$\nabla u = 0 \quad \text{a.e. on } \{u=t\}.$$

Therefore, it follows that

$$\int_\Omega |\chi_{\{u_j<t\}}-\chi_{\{u<t\}}|^2|\nabla u|^2dx = \int_{\{u<t\}\cup\{u>t\}} |\chi_{\{u_j<t\}}-\chi_{\{u<t\}}|^2|\nabla u|^2dx.$$

The condition that $u_j \to u$ a.e. in Ω implies that

$$\chi_{\{u_j<t\}}-\chi_{\{u<t\}} \longrightarrow 0 \quad \text{a.e. in } \{u<t\}\cup\{u>t\}.$$

Hence, the Lebesgue convergence theorem shows that

$$\int_{\{u<t\}\cup\{u>t\}} |\chi_{\{u_j<t\}}-\chi_{\{u<t\}}|^2|\nabla u|^2dx \longrightarrow 0.$$

Back to (A.3), this implies that $(I_1)_j \to 0$. We conclude that

$$\int_{\partial\Omega} |(u\wedge t)|_{\partial\Omega}-u|_{\partial\Omega}\wedge t|^2da = 0,$$

from which assertion (A.1) follows. The proof of Proposition A.1 is now complete. □

Acknowledgements. The author would like to thank the referee for his many valuable comments and suggestions.

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