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SEMILINEAR PARABOLIC EQUATIONS IN $L^1(\Omega)$

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Abstract. This paper studies existence, regularity and continuous dependence upon the data of solutions to parabolic semilinear problems of the form: $u'(t) = Au(t) + g[u(t)]$, $u(0) = u_0$. Here, $A : D(A) \to X$ generates an analytic semigroup on a Banach space X and $g : D(g) \to X$. It is assumed that $D(g)$ contains a certain interpolation space of X and $D(A)$; this will allow to treat parabolic partial semilinear problems in the cases where the nonlinear term depends also on the gradient of u.

1. Introduction

Let Ω be a bounded subset of $\mathbb{R}^N$ with regular boundary $\partial\Omega$, let $\mathcal{A}$ be a second order elliptic operator on Ω and let $\mathrm{g} : D(\mathrm{g}) \subset \mathbb{R}^{N+1} \to \mathbb{R}$. In this paper, we study semilinear parabolic partial differential equations such as:

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + \mathrm{g}[x, u(t,x), u_{x_1}(t,x), \cdots, u_{x_n}(t,x)], \quad t > 0,\ x \in \Omega, \\ u(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

Problem of this kind can be rewritten in the following (more general) form. Given a Banach space X of functions defined on Ω, a linear operator $A : D(A) \to X$ and a nonlinear function $g : D(g) \to X$, find $u : [0,T] \to X$ satisfying the problem

$$\begin{cases} u'(t) = Au(t) + g[u(t)],\ t > 0, \\ u(0) = u_0. \end{cases} \tag{1.2}$$

As $\mathcal{A}$ is an elliptic operator, we assume that A generates an analytic semigroup on X. To consider nonlinear terms arising from (1.1), we assume that $D(g)$ contains a certain interpolation space, denoted by X_θ, of X and $D(A)$. We do not require that $u_0 \in D(g)$.

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The main object of this paper is the existence of solutions (1.2) which are differentiable in X in the sense of L^1. To allow the initial datum to be as general as possible, we only require that u is differentiable for $t > 0$. The main assumption is the local Lipschitz continuity of g from X_θ into X. This will correspond when dealing with problem (1.1), to the assumption that g satisfies certain growth conditions, but no sign conditions.

The plan of the paper is as follows: Section 2 contains a number of results concerning linear parabolic equations in a Banach space. In Section 3, we prove existence (see Theorems 3.1 and 3.5), regularity (see Theorems 3.2 and 3.6) and continuous dependence upon the data for solutions of problem (1.2) (see Theorems 3.4 and 3.8). We also show that these solutions can be continued as long as they remain bounded (see Theorems 3.3 and 3.7). This will be achieved by using regularity results for linear parabolic equations and a perturbation argument.

Finally, in Section 4, we apply these results to a class of examples of the form (1.1), where g satisfies certain growth conditions. See Lemmas 4.1, 4.4 and 4.7. Initial values are in some cases allowed to be arbitrary functions in $L^1(\Omega)$, depending on the growth of g. See Theorems 4.2 and 4.3.

Problem (1.1) has been widely studied in $L^1(\Omega)$ with various methods and under various assumptions on g. Such as monotonicity, sign conditions and growth conditions. If $\mathrm{g} = \mathrm{g}(x, u)$ and $\mathrm{g}(x, \cdot)$ is a maximal monotone graph, we refer to [10], where g does not depend on x and to [7] where $\mathrm{g}(\cdot, u)$ is in $L^1(\Omega)$. If $\mathcal{A} = \Delta$, we refer to [13] where (1.1) is studied under the assumption that $\mathrm{g}(x, \cdot)$ is locally Lipschitz continuous from $L^q(\Omega)$ to $L^1(\Omega)$ and to [4], where there is a detailed treatment of the case $\mathrm{g} = |u|^{p-1}u$. Finally, if $\mathrm{g} = \mathrm{g}(u, |\nabla u|)$ satisfies certain sign and growth conditions, existence of generalized solutions have been established by various authors for a number of class of semilinear problems, see e.g. [2], [3] and [6].

2. Linear parabolic equations in L^1-spaces

In this section we collect some known results, as well as establishing new ones, and concerning solutions of parabolic equations. For clarity of exposition, we introduce some notation. Given a Banach space E, we will be concerned with the following spaces of E-valued Bochner measurable functions defined on a bounded interval $[a, b]$:

- $L^1(a, b; E)$ denotes the space of all u such that $\|u(\cdot)\|_E$ is integrable on (a, b);

- $L^\infty(a,b;E)$ denotes the space of all u such that $\|u(\cdot)\|_E$ is essentially bounded on (a,b);
- $L^\infty_\theta(a,b;E)$, for $\theta \in \mathbb{R}$, denotes the space of u such that $t \to t^\theta u(t)$ belongs to $L^\infty(a,b;E)$;
- $C(a,b;E)$ is the space of all continuous functions on $[a,b]$;
- $W^{\alpha,1}(a,b;E)$, $0<\alpha<1$, is the Sobolev space of all $u \in L^1(a,b;E)$ for which
$$\int_a^b dt \int_a^t \|u(t)-u(s)\|_E \,(t-s)^{-1-\alpha} ds < +\infty \;;$$
- $W^{1,1}(a,b;E)$ is the Sobolev space of all functions $u \in L^1(a,b;E)$ having distributional derivatives $u' \in L^1(a,b;E)$.

Finally, we denote by $L^1_+(a,b;E)$ the following space:
$$L^1_+(a,b;E) \;\; := \;\; \{u \in L^1(a',b;E), \forall\; a<a'<b\},$$
and similarly for $C_+(a,b;E)$ and $W^{1,1}_+(a,b;E)$.

Now, let X be a Banach space, whose norm will be denoted by $\|\cdot\|$, and let $A: D(A) \to X$ be the infinitesimal generator of a bounded analytic semigroup $\{S(t)\}_{t\geq 0}$ on X.

We assume without loss of generality for our purposes, that $0 \in \rho(A)$. If this is not the case, we can replace u by $e^{-\beta t}u$, for suitable $\beta>0$, so that A is replaced by $A-\beta I$.

Hence, $D(A)$ is a Banach space under the norm $|x|_{D(A)} := \|Ax\|$, and there exist $M \geq 1$ and $\omega > 0$ verifying, for each $x \in X$ and $t>0$,
$$\|S(t)x\| \leq Me^{-\omega t}\|x\|, \quad t\|AS(t)x\| \leq Me^{-\omega t}\|x\|. \tag{2.1}$$
Now, fix $T>0$. For given $f \in L^1(0,T;X)$ and $x \in X$, we consider the following problem
$$\begin{cases} u'(t) = Au(t) + f(t), & \text{a.e. } t \in]0,T[, \\ u(0) = x. \end{cases} \tag{2.2}$$
A function $u \in C(0,T;X) \cap W^{1,1}_+(0,T;X) \cap L^1_+(0,T;D(A))$, which verifies (2.2), is called a *solution* in the sense of L^1. It is known that such a solution satisfies, for each $t \in [0,T]$,
$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds. \tag{2.3}$$

Conversely, it can be proved that, if the right–hand side of (2.3) is regular, then the function u given by (2.3) satisfies (2.2). As usual, we define *mild solution* of (2.2) the function u given by (2.3).

We now collect some regularity properties of mild solutions that will be used in the sequel. To this end, we introduce the following functions for given $x \in X$ and $f \in L^1(0,T;X)$,

$$U(t) := S(t)x, \qquad V(t) := \int_0^t S(t-s)f(s)\,ds, \qquad t \in [0,T]. \tag{2.4}$$

It follows easily from (2.1) that $U, V \in C(0,T;X)$. To study further regularity properties in the sense of L^1, we introduce the spaces X_α, defined as the real interpolation spaces of X and $D(A)$ (see Butzer and Berens [5, Chapter 3.5]):

$$X_\alpha := \Big\{x \in E : |x|_\alpha := \Big(\int_0^{+\infty} \tau^{-\alpha}\|AS(\tau)x\|\,d\tau\Big) < +\infty\Big\}, \quad 0 < \alpha < 1. \tag{2.5}$$

We recall the following interpolation estimates:

$$t^{1-\alpha}\|AS(t)x\| \leq M' e^{-\omega t}|x|_\alpha. \tag{2.6}$$

The following lemmas collect some properties of the function U.

Lemma 2.1. *For each $x \in X$, we have $U, AU \in C_+(0,T;X)$ and $U'(t) = AU(t)$, for each $t > 0$.*

Proof. The result is an easy consequence of (2.1). □

Lemma 2.2. *Let $x \in X$. Then for each $\theta \in (0,1)$ there exists $C_0 = C_0(M,\theta)$ verifying*

(i): $t^\theta |U(t)|_\theta \leq C_0\|x\|$, $t > 0$.

Proof. Using (2.1) and the transformation of variables $\tau = t\sigma$, we obtain, for $t > 0$,

$$\begin{aligned} |U(t)|_\theta &= \int_0^{+\infty} \|\tau^{-\theta} AS(\tau+t)x\,\|d\tau \\ &\leq M\int_0^{+\infty} \tau^{-\theta}(\tau+t)^{-1}\,d\tau\|x\| = Mc' t^{-\theta}\|x\|, \end{aligned} \tag{2.7}$$

where

$$c' := \int_0^{+\infty} \sigma^{-\theta}(\sigma+1)^{-1}\,d\sigma, \tag{2.8}$$

and the result follows with $C_0 := Mc'$. □

Lemma 2.3. *For each $x \in X$, we have $U \in L^1(0,T;X_\alpha)$ and $U \in W^{\alpha,1}(0,T;X)$ for each $\alpha \in (0,1)$. Moreover, $U \in W^{\alpha,1}(0,T;X_{\alpha'-\alpha})$, for each $0 < \alpha < \alpha' < 1$.*

Proof. See Theorems 2,7, and 12 of [8]. □

Lemma 2.4. *For each $x \in X_\alpha$ and $\alpha \in (0,1)$, we have $U', AU \in L^1(0,T;X_\alpha)$. Moreover, $U' = AU$.*

Proof. See Theorems 5 and 15 of [8]. □

Lemma 2.5. *Let $\beta \in (0,1)$ and let $x \in X_\beta$. Then for each $\theta \in [\beta,1)$ there exists $C_0' = C_0'(M,\beta,\theta)$ satisfying*

(*i*) $t^{\theta-\beta}|U(t)|_\theta \le C_0'|x|_\beta$, $t > 0$.

Proof. From (2.6) and an argument similar to the one used in the proof of Lemma 2.2 we have, for $t > 0$,

$$\begin{aligned} |U(t)|_\theta &= \int_0^{+\infty} \|\tau^{-\theta} AS(t+\tau)x\, \| d\tau \\ &\le M' \int_0^{+\infty} \tau^{-\theta}(t+\tau)^{-1+\beta}\, d\tau\, |x|_\beta = M'c''t^{-\theta+\beta}|x|_\beta, \quad (2.9) \end{aligned}$$

where

$$c'' := \int_0^{+\infty} \sigma^{-\theta}(\sigma+1)^{-1+\beta}\, d\sigma, \tag{2.10}$$

therefore, the result is proved with $C_0' := M'c''$. □

The following lemmas collect some properties of the function V.

Lemma 2.6. *For each $f \in L^1(0,T;X)$, we have $V \in L^1(0,T;X_\alpha)$ and $V \in W^{\alpha,1}(0,T;X)$ for each $\alpha \in (0,1)$. Moreover, $V \in W^{\alpha,1}(0,T;X_{\alpha'-\alpha})$, for each $0 < \alpha < \alpha' < 1$.*

Proof. The results are proved in Theorems 17, 18 and 19 of [8]. □

Lemma 2.7. *Let $f \in L^\infty_\alpha(0,T;X)$, for some $\alpha \in [0,1)$. Then for each $\theta \in (0,1)$ we have $V \in C_+(0,T;X_\theta)$. Moreover, there exists $C_1 = C_1(\theta,\alpha,M)$ such that we have*

(*i*) $t^\theta\|V(t)\|_\theta \le C_1\, t^{1-\alpha}\|f\|_{L^\infty_\alpha}$.

Proof. If $f \in C(0,T;D(A))$, then it is known that $V \in C(0,T;D(A))$. Furthermore, using (2.1), we obtain for each $t \in (0,T]$,

$$|V(t)|_\theta = \int_0^{+\infty} \|\tau^{-\theta} \int_0^t AS(\tau+s)f(t-s)\, ds\, \|\, d\tau$$

$$\leq M \int_0^{+\infty} \tau^{-\theta} \int_0^t (\tau + s)^{-1} \|f(t-s)\| \, ds \, d\tau$$
$$= Mc' \int_0^t s^{-\theta} \|f(t-s)\| \, ds, \tag{2.11}$$

where we used the transformation of variable $\tau = s\sigma$ and (2.8). Hence, (i) is proved with

$$C_1 = M \int_0^{+\infty} \sigma^{-\theta} (\sigma + 1)^{-1} \, d\sigma \int_0^1 s^{-\theta} (1-s)^{-\alpha} ds. \tag{2.12}$$

Finally, the conclusion follows from a density argument. □

Lemma 2.8. *Let $f \in L^1(0,T;X)$ satisfy the property*

$$\int_0^T dt \int_0^t (t-s)^{-1} \|f(t) - f(s)\| \, ds < +\infty.$$

Then $V \in L^1(0,T;D(A)) \cap W^{1,1}(0,T;X)$ and satisfies $V'(t) = AV(t)$, for a.e. $t \in (0,T)$.

Proof. Take $f \in C(0,T;D(A))$; then it is known that $V', AV \in C(0,T;X)$ and that $V'(t) = AV(t)$, for each $t \in [0,T]$. Furthermore,

$$\begin{aligned} AV(t) &= \int_0^t AS(t-s)[f(s) - f(t)] \, ds + \int_0^t AS(t-s) f(t) \, ds \\ &= \int_0^t AS(t-s)[f(s) - f(t)] \, ds + S(t)f(t) - f(t). \end{aligned}$$

Therefore, using (2.1),

$$\int_0^T \|AV(t)\| \, dt \tag{2.13}$$
$$\leq M \int_0^T dt \int_0^t (t-s)^{-1} \|f(s) - f(t)\| \, ds + (M+1) \int_0^T \|f(t)\| \, dt,$$

and hence, as $V'(t) = AV(t)$,

$$\int_0^T \|V'(t)\| \, dt \tag{2.14}$$
$$\leq M \int_0^T dt \int_0^t (t-s)^{-1} \|f(s) - f(t)\| \, ds + (M+1) \int_0^T \|f(t)\| \, dt.$$

Therefore, the result follows from (2.13), (2.14) and a density argument. □

3. Semilinear equations

Given $x \in X$, we consider the following problem

$$\begin{cases} u'(t) = Au(t) + g[u(t)], & t > 0, \\ u(0) = x, \end{cases} \tag{3.1}$$

where $g : D(g) \to X$ satisfies the assumptions

(g1) there exist $\theta \in (0,1)$ such that $X_\theta \subset D(g)$,
(g2) there exist $q \geq 1$ and $C > 0$ such that for each $R > 0$ we have $\|g(u) - g(v)\| \leq CR^{q-1}|u - v|_\theta$, for each $|u|_\theta, |v|_\theta \leq R$.

According to the notation of section 1, we introduce the following definitions.

A *solution* on $(0,T)$ of problem (3.1), in the sense of L^1, is a function $u \in C(0,T;X)$ satisfying $u \in W^{1,1}_+(0,T;X) \cap L^1_+(0,T;D(A))$, $g[u(\cdot)] \in L^1(0,T;X)$ and verifying (3.1) on $(0,T)$. It can be proved that if u is such a solution then u satisfies

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} g[u(s)]\, ds. \tag{3.2}$$

Equation (3.2) is usually referred to as the integrated (or mild) version of problem (3.1). Accordingly, a function u satisfying $u(t) \in D(g)$, for a.e. $t \in]0,T[$, and $g[u(\cdot)] \in L^1(0,T;X)$ which verifies (3.2) on $(0,T)$ will be called a *mild solution* of (3.1) on $(0,T)$. It follows from the definition that a mild solution is continuous.

We begin to prove existence of mild solutions of problem (3.1). We will treat the cases $q < 1/\theta$ and $q \geq 1/\theta$, separately.

Theorem 3.1. *Let g satisfy assumptions* (g1) *and* (g2) *with $q \in [1, 1/\theta[$ and let $x \in X$. There exists $T_0 = T_0(\|x\|)$ such that if $T \leq T_0$ then equation* (3.2) *admits a unique solution on $(0,T)$ satisfying $u \in L^\infty_\theta(0,T;X_\theta)$. Moreover, $u \in C_+(0,T;X_\theta)$ and*

(i) *$u \in L^1(0,T;X_\alpha) \cap W^\alpha(0,T;X)$ and $u \in W^{\alpha,1}(0,T;X_{\alpha'-\alpha})$, for each $0 < \alpha < \alpha' < 1$.*

Proof. To solve (3.2), we use a fixed point argument. For given $T > 0$ and $R > 0$ we denote by $B_T(R)$ the subset of $L^\infty_\theta(0,T;X_\theta)$ given by

$$B_T(R) := \{u \in L^\infty_\theta(0,T;X_\theta); \|u\|_{L^\infty_\theta(0,T;X_\theta)} \leq R\}. \tag{3.3}$$

Now, fix $v \in B_T(R)$; it follows from the definition that $|v(t)|_\theta \le t^{-\theta}R$, for almost every $t \in (0,T)$. Thus, from assumption (g2), we have

$$\|g[v(t)]\| \le \|g[v(t)] - g(0)\| + \|g(0)\| \le CR^q\, t^{-q\theta} + \|g(0)\|. \tag{3.4}$$

Hence, if $v \in B_T(R)$, then $g(v)$ belongs to $L^1(0,T;X)$ and we can consider the function

$$u(v) := e^{tA}x + \int_0^t e^{(t-s)A} g[v(s)]\, ds. \tag{3.5}$$

We want to prove that there exist R and T such that the application $v \to \mathcal{T}(v) := u(v)$ is a contraction in $B_T(R)$.

From (2.7), (2.11) and (3.4), we have for each $t \in (0,T]$,

$$\begin{aligned} |u(v)(t)|_\theta &\le Mc'\big(t^{-\theta}\|x\| + \int_0^t s^{-\theta}\|g[v(t-s)]\|\, ds\,\big) \qquad (3.6)\\ &\le Mc'\big(t^{-\theta}\|x\| + (1-\theta)^{-1}t^{1-\theta}\|g(0)\| + CR^q \int_0^t s^{-\theta}(t-s)^{-q\theta}\, ds\,\big). \end{aligned}$$

Therefore, using the transformation of variable $s = t\sigma$ and setting

$$C' := \int_0^1 \sigma^{-\theta}(1-\sigma)^{-q\theta}\, d\sigma, \tag{3.7}$$

we obtain

$$t^\theta |u(v)(t)|_\theta \le Mc'\big(\|x\| + (1-\theta)^{-1}t\|g(0)\| + CC'R^q t^{1-q\theta}\,\big).$$

Now, fix R_0 satisfying

$$2Mc'\|x\| \le R_0, \tag{3.8}$$

and choose $T_0 > 0$ verifying

$$2Mc'\big((1-\theta)^{-1}T_0\|g(0)\| + C\,C'R_0^q T_0^{1-q\theta}\big) \le R_0. \tag{3.9}$$

Then, if $T \le T_0$, we have that $\mathcal{T}$ maps $B_T(R_0)$ into itself. To accomplish the proof that $\mathcal{T}$ is a contraction, let $v_1, v_2 \in B_T(R_0)$. From (2.11) and assumption (g2), we have

$$\begin{aligned} |\mathcal{T}(v_2)(t) - \mathcal{T}(v_1)(t)|_\theta &\le Mc' \int_0^t s^{-\theta}\|g[v_2(t-s)] - g[v_1(t-s)]\|\, ds\\ &\le Mc'\, CR_0^{q-1} \int_0^t s^{-\theta}(t-s)^{(1-q)\theta}|v_2(t-s) - v_1(t-s)|_\theta\, ds. \end{aligned}$$

Hence, using (3.7)

$$|\mathcal{T}(v_2)(t) - \mathcal{T}(v_1)(t)|_\theta \le Mc'CR_0^{q-1}\|v_2 - v_1\|_{L^\infty_\theta(X_\theta)} \int_0^t s^{-\theta}(t-s)^{-q\theta}\, ds$$

$$= Mc'\,CC'R_0^{q-1}t^{1-\theta-q\theta}\|v_2 - v_1\|_{L^\infty_\theta(X_\theta)}, \tag{3.10}$$

so that if $T \le T_0$, with T_0 satisfying (3.9), we have

$$\|\mathcal{T}(v_2) - \mathcal{T}(v_1)\|_{L^\infty_\theta(0,T;X_\theta)} \le 1/2\|v_2 - v_1\|_{L^\infty_\theta(0,T;X_\theta)}.$$

Therefore, $\mathcal{T}$ is a contraction. Consequently, there exists a unique $u \in B_T(R_0)$ satisfying $u = \mathcal{T}(u)$, that is u is a solution of (3.2). In addition, using Lemmas 2.1 and 2.7 and the fact that u is a solution of (3.2), we find that $u \in C_+(0,T;X_\theta)$. Finally, assertion (i) follows from Lemmas 2.3 and 2.6. □

We now establish existence of differentiable solutions.

Theorem 3.2. *Let g satisfy assumptions* (g1) *and* (g2) *with $q < 1/\theta$. Let $x \in X$ and denote by u the solution of* (3.2) *given by Theorem* 3.1. *Then $u \in L^1_+(0,T;D(A)) \cap W^{1,1}_+(0,T;X)$ and satisfies* (3.1).

Proof. Let R_0 and T_0 be the numbers given by (3.8) and (3.9) and let u be the solution of (3.2), given by Theorem 3.1. We want to prove that the function $t \to f(t) := g[u(t)]$ satisfies the assumption of Lemma 2.8.

For $0 < s \le t \le T \le T_0$, we have

$$|u(s)|_{\theta,1} \le s^{-\theta}R_0, \quad |u(t)|_{\theta,1} \le t^{-\theta}R_0 \le s^{-\theta}R_0, \tag{3.11}$$

so that from assumption (g2),

$$\|f(t) - f(s)\| \le CR_0^{q-1}s^{\theta(1-q)}\,|u(t) - u(s)|_\theta.$$

Therefore,

$$\int_0^T dt\int_0^t (t-s)^{-1}\|f(t) - f(s)\|\,ds \le CR_0^{q-1}I_1, \tag{3.12}$$

where

$$I_1 := \int_0^T dt\int_0^t s^{\theta(1-q)}(t-s)^{-1}|u(t) - u(s)|_\theta\,ds.$$

Now

$$\begin{aligned} I_1 \le{}& \int_0^T dt\int_0^{t/2} s^{\theta(1-q)}(t-s)^{-1}|u(t) - u(s)|_\theta\,ds \\ &+ \int_0^T dt\int_{t/2}^t s^{\theta(1-q)}(t-s)^{-1}|u(t) - u(s)|_\theta\,ds =: J_1 + J_2. \end{aligned} \tag{3.13}$$

Furthermore, from the fact $2s < t$ and (3.11),

$$J_1 \leq 2\int_0^T t^{-1}\,dt \int_0^{t/2} s^{\theta(1-q)}|u(t)-u(s)|_\theta\,ds$$
$$\leq 4R\int_0^T t^{-1}\,dt\int_0^{t/2} s^{-q\theta}\,ds = 2^{1+q\theta}R_0(1-q\theta)^{-2}T^{1-q\theta}, \tag{3.14}$$

Concerning J_2 we have, using the fact $t-s<s$,

$$J_2 \leq \int_0^T\,dt\int_{t/2}^t (t-s)^{-1+\theta-q\theta}|u(t)-u(s)|_\theta\,ds \leq \|u\|_{W^{q\theta-\theta,1}(X_\theta)}. \tag{3.15}$$

Finally, combining (3.12), (3.13), (3.14) and (3.15), we obtain

$$\int_0^T dt\int_0^t (t-s)^{-1}\|f(t)-f(s)\|\,ds$$
$$\leq CR_0^{q-1}\big[\,2^{1+q\theta}R_0(1-q\theta)^{-2}T^{1-q\theta}+\|u\|_{W^{q\theta-\theta,1}(X_\theta)}\,\big].$$

Hence, from property (ii) of Lemma 3.1, we find that the function $f(t) := g[u(t)]$ satisfies the assumption of Lemma 2.8. Consequently, using Lemmas 2.1 and 2.8, we have that the solution of (3.2) is actually a solution (in the sense of L^1) of problem (3.1). □

Theorem 3.2 proves existence of local solutions of problem (3.1). The following result ensures that these solutions can be continued as long as they remain bounded.

Theorem 3.3. *Let g satisfy assumptions* (g1) *and* (g2) *with $q \in [1, 1/\theta)$. Let $x \in X$ and denote by $u : [0,\tau) \to X_\theta$ the maximally defined solution of* (3.1) *given by Theorem* 3.2*. Then, either $\tau = +\infty$, or*

(*i*) $\limsup_{t\to\tau}|u(t)|_\theta = +\infty$.

Proof. From Theorem 3.2, there exists a unique local solution of (3.1). Using standard continuation arguments, we obtain that there exists a unique maximally defined solution $u : [0,\tau) \to X_\theta$ and we have $u \in L^\infty_\theta(0,T;X_\theta)$, for each $T < \tau$. Assume, for contradiction, that $\tau < +\infty$ and that there exists R such that

$$|u(t)|_\theta \leq R, \quad \text{for a.e. } t \in (\tau/4, \tau), \tag{3.16}$$

then, from (g2),

$$\|g[u(t)]\| \leq CR^q + \|g(0)\|, \quad \text{for a.e. } t \in (\tau/4,\tau). \tag{3.17}$$

Furthermore, from (3.2) we have, for $\tau/2 \le s \le t < \tau$,

$$\|u(t) - u(s)\| \le \int_0^s \|(e^{(t-\sigma)A} - e^{(s-\sigma)A})g[u(\sigma)]\|d\sigma \qquad (3.18)$$
$$+ \int_s^t \|e^{(t-\sigma)A}g[u(\sigma)]\|d\sigma =: I_1 + I_2,$$

We estimate I_1. Using (2.1) and (3.17) we find

$$I_1 = \int_0^s \| \int_{s-\sigma}^{t-\sigma} Ae^{\sigma' A}g[u(\sigma)]d\sigma'\|d\sigma \le M \int_0^s \|g[u(\sigma)]\| \int_{s-\sigma}^{t-\sigma} \sigma'^{-1}\, d\sigma'\, d\sigma$$
$$= M \int_0^{\tau/4} \|g[u(\sigma)]\| [\log(t-\sigma) - \log(s-\sigma)]\, d\sigma$$
$$+ M\,[CR^q + \|g(0)\|] \int_{\tau/4}^s [\log(t-\sigma) - \log(s-\sigma)]\, d\sigma =: J_1 + J_2.$$

Now $s \ge \tau/2$ and $\sigma \le \tau/4$ imply $s - \sigma \ge \tau/4$, so that

$$\log(t-\sigma) - \log(s-\sigma) = \log[1 + (t-s)(s-\sigma)^{-1}] \le \log[1 + 4(t-s)\tau^{-1}$$

therefore,

$$J_1 \le \log[1 + 4(t-s)\tau^{-1}] \int_0^{\tau/4} \|g[u(\sigma)]\|d\sigma.$$

Furthermore,

$$J_2 = [CR^q + \|g(0)\|]$$
$$\times \left[(t-\tau/4)\log(t-\tau/4) - (s-\tau/4)\log(s-\tau/4) - (t-s)\log(t-s) \right].$$

Finally, from (2.1) and (3.17)

$$I_2 \le M[CR^q + \|g(0)\|](t-s).$$

Summarizing, we find that u is uniformly continuous on $[\tau/2, \tau)$. Hence, there exists $u_1 := \lim_{t\to\tau} u(\tau)$ and we have $u_1 \in X_\theta$, from (3.16). Therefore, using standard continuation arguments, we can prove that u can be extended over an interval $[0, \tau']$ with $\tau' > \tau$. This is a contradiction and proves the theorem. □

The following theorem proves continuous dependence upon the data.

Theorem 3.4. *Let g satisfy* (g1) *and* (g2) *with $q \in [1, 1/\theta)$. Let $x_1, x_2 \in X$ and let $u_1 : (0, \tau_1) \to X_\theta$ and $u_2 : (0, \tau_2) \to X_\theta$ be the corresponding maximally defined solutions of* (3.1). *Choose $T_* < \inf(\tau_1, \tau_2)$ and set $R_* := \max\{\|u_i\|_{L^\infty_\theta(0,T_*;X_\theta)}\}$, $i = 1, 2$. Then there exists $C_2 = C_2(R_*)$ satisfying* (*i*) $t^{\theta}|u_2(t)-u_1(t)|_{\theta} \le C_2\|x_2-x_1\|$, $t\in(0,T_*]$.

Proof. By assumption we have $t^{\theta}|u_i(t)|_{\theta} \le R_*$, for $t\in(0,T_*]$, $i=1,2$. Furthermore, choose $0<\tau\le T_*$ satisfying

$$2Mc'C\,C'R_*^{q-1}\tau^{1-q\theta}\le 1. \tag{3.19}$$

By an argument similar to the one used in the proof of (3.10), we find for $t\in(0,\tau]$,

$$\begin{aligned}&|u_2(t)-u_1(t)|_{\theta}\\ &\le Mc'\big[t^{-\theta}\|x_2-x_1\|+CC'R_*^{q-1}t^{1-\theta-q\theta}\|u_2-u_1\|_{L^{\infty}_{\theta}(0,\tau;X_{\theta})}\big]\\ &\le Mc'\big[t^{-\theta}\|x_2-x_1\|+CC'R_*^{q-1}t^{-\theta}\tau^{1-q\theta}\|u_2-u_1\|_{L^{\infty}_{\theta}(0,\tau;X_{\theta})}\big],\end{aligned}$$

so that, from (3.19),

$$\|u_2-u_1\|_{L^{\infty}_{\theta}(0,\tau;X_{\theta})}\le 2Mc'\,\|x_2-x_1\|,$$

which proves (i) on $(0,\tau]$. As τ, given by (3.19), depends only on R_* we can iterate this argument and prove the desired result. □

We now study problem (3.1) in the case $q\ge 1/\theta$. As before we begin to prove existence of mild solutions.

Theorem 3.5. *Let g satisfy assumptions* (g1) *and* (g2)*, with $q\in[1/\theta,+\infty)$, and fix $\beta\in((q\theta-1)/(q-1),\theta)$. Then, for each $x\in X_{\beta}$, there exists $T_1=T_1(|x|_{\beta})$ such that if $T\le T_1$, then equation* (3.2) *admits a unique solution u on $(0,T)$ satisfying $u\in L^{\infty}_{\theta-\beta}(0,T;X_{\theta})$. Moreover, we have $u\in C_+(0,T;X_{\theta})$ and*

(*i*) *$u\in L^1(0,T;X_{\alpha})\cap W^{\alpha,1}(0,T;X)$ and $u\in W^{\alpha,1}(0,T;X_{\alpha'-\alpha})$, for each $0<\alpha<\alpha'<1$.*

Proof. We proceed as in the proof of Theorem 3.1. First, we note that the assumption $(q\theta-1)/(q-1)<\beta$ implies $q(\theta-\beta)<1$.

Next, we denote by $B'_T(R)$, for given $T>0$ and $R>0$, the subset of $L^{\infty}_{\theta-\beta}(0,T;X_{\theta})$ given by

$$B'_T(R):=\{u\in L^{\infty}_{\theta-\beta}(0,T;X_{\theta});\|u\|_{L^{\infty}_{\theta-\beta}(0,T;X_{\theta})}\le R\}. \tag{3.20}$$

Thus, if $v\in B'_T(R)$ then from assumption (g2) we have, for a.e. $t\in(0,T)$,

$$\|g[v(t)]\|\le\|g[v(t)]-g(0)\|+\|g(0)\|\le CR^q\,t^{q(\beta-\theta)}+\|g(0)\|.$$

Hence, $v\in B'_T(R)$ implies that $g(v)\in L^1(0,T;X)$ and we can consider the function $u(v)$ given by (3.5). We want to prove that there exist R_1 and T_1 such that if $T\le T_1$ then $v\to\mathcal{T}(v):=u(v)$ is a contraction in $B'_T(R_1)$.

Using (2.9) and (2.11) and proceeding as in (3.6) we find

$$
\begin{aligned}
&|u(v)(t)|_\theta \qquad (3.21)\\
&\le M'c''t^{\beta-\theta}|x|_\beta + Mc'\big[(1-\theta)^{-1}t^{1-\theta}\|g(0)\| + CR^q\int_0^t s^{-\theta}(t-s)^{q(\beta-\theta)}\,ds\,\big]\\
&\le M'c''t^{\beta-\theta}|x|_\beta + Mc'\big[(1-\theta)^{-1}t^{1-\theta}\|g(0)| + CC''R^q t^{1-\theta+q(\beta-\theta)}\,\big],
\end{aligned}
$$

where

$$C'' =: \int_0^1 s^{-\theta}(1-s)^{q(\beta-\theta)}\,ds. \qquad (3.22)$$

Now, note that $(q\theta-1)/(q-1) < \beta$ implies $1-\beta+q(\beta-\theta) > 0$. Hence, for fixed R_1 verifying

$$2M'c''|x|_\beta \le R_1, \qquad (3.23)$$

we can choose $T_1 > 0$ satisfying

$$2Mc'\big[(1-\theta)^{-1}T_1^{1-\beta}\|g(0)\| + C\,C''R_1^q T_1^{1-\beta+q(\beta-\theta)}\big] \le R_1. \qquad (3.24)$$

Therefore, we can repeat the arguments used in the proof of Theorem 3.1 and prove that if T_1 satisfies (3.24), with R_1 verifying (3.23), and $T \le T_1$, then $\mathcal{T}$ is a contraction in $B'_T(R_1)$. Consequently, there exists a unique $u \in B'_T(R_1)$ satisfying $u = \mathcal{T}(u)$, that is u is a solution of (3.2) on $(0,T)$. The remainder properties can be proved as in Theorem 3.1 □

Theorem 3.6. *Let θ and q satisfy assumptions* (g1) *and* (g2)*, with $q \ge 1/\theta$, and take $\beta \in ((q\theta-1)/(q-1),\theta)$. Let $x \in X_\beta$ and denote by u the solution of* (3.2) *given by Theorem* 3.5*. Then $u \in L^1(0,T;D(A)) \cap W^{1,1}(0,T;X)$ and satisfies* (3.1)*.*

Proof. The results can be proved using Lemma 2.4 and an argument similar to the one of the proof of Lemma 3.2. □

Theorem 3.7. *Let g satisfy assumptions* (g1) *and* (g2) *with $q \in [1/\theta,+\infty)$ and take $\beta \in ((q\theta-1)/(q-1),\theta)$. Let $x \in X_\beta$ and denote by $u : [0,\tau) \to X_\theta$ the maximally defined solution of* (3.1) *given by Theorem* 3.6*. Then either $\tau = +\infty$ or*

(*i*) $\limsup_{t\to\tau} |u(t)|_\theta = +\infty$.

Proof. The proof is similar to the one of Theorem 3.3. □

Finally, the following theorem proves continuous dependence upon the data.

Theorem 3.8. *Let g satisfy assumptions* (g1) *and* (g2) *with* $q \in [1/\theta, +\infty)$ *and take* $\beta \in ((q\theta-1)/(q-1), \theta)$. *Furthermore, let* $x_1, x_2 \in X_\beta$ *and denote by* $u_1 : (0,\tau_1) \to X_\theta$ *and* $u_2 : (0,\tau_2) \to X_\theta$ *the corresponding maximally defined solutions of* (3.1). *Choose* $T_* < \inf(\tau_1, \tau_2)$ *and set* $R_* := \max\{\|u_i\|_{L^\infty_\theta(0,T_*;X_\theta)}\}$, $i = 1, 2$. *Then there exist* $C_2' = C_2'(R_*)$ *satisfying*

(*i*) $t^{\theta-\beta}|u_2(t) - u_1(t)|_\theta \le C_2'|x_2 - x_1|_\beta$, $t \in (0, T_*]$.

Proof. The proof is similar to the one of Theorem 3.4. □

Remark 3.9. In Theorems *3.5* and *3.6*, we assume $u_0 \in X_\beta$ with $\beta < \theta$. Therefore, we do not require $u_0 \in D(g)$.

4. Semilinear parabolic partial differential equations

In this section, we illustrate how the general results of the preceding sections can be used to study, in a unified way, a class of semilinear parabolic partial differential equations.

Let Ω be a bounded subset of $\mathbf{R}^N$ with C^2 boundary $\partial\Omega$ and let $\mathcal{A}$ be an elliptic operator in Ω of order 2

$$\mathcal{A}u = \sum_{i,j=1}^{N} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{N} b_i(x)u_{x_i} + c(x)u,$$

where $a_{ij} \in C^1(\overline{\Omega})$ and $b_i, c \in C(\overline{\Omega})$ are given functions. We want to study problems of the following kind

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + \mathrm{g}[x, u(t,x), u_{x_1}(t,x), \cdots, u_{x_n}(t,x)]\ ,\ t > 0,\ x \in \Omega. \\ u(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x)\ ,\ x \in \Omega. \end{cases} \tag{4.1}$$

Here g satisfies certain growth bounds, that will be specified later, and $u_0 \in L^1(\Omega)$.

It is well known that the realization of $\mathcal{A}$, with homogeneous Dirichlet boundary conditions, in the space $L^1(\Omega)$ generates an analytic semigroup. See [1], [11], [12]. Hence, we can study (4.1) using the results of the preceding sections, with A defined by,

$$D(A) = \{u \in W_0^{1,1}(\Omega) : \mathcal{A}u \in L^1(\Omega)\}; \qquad Au = \mathcal{A}u, \tag{4.2}$$

where $\mathcal{A}u$ is understood in the sense of distributions.

Moreover, setting $X := L^1(\Omega)$, we have the following characterization of the interpolation spaces X_α, $\alpha \in (0,1)$, defined by (2.5) (see [9])

$$X_\alpha := \begin{cases} W^{2\alpha,1}(\Omega)\ , & \text{if } \ 2\alpha < 1 \\ B^{1,1}(\Omega)\ , & \text{if } \ 2\alpha = 1 \\ W_0^{2\alpha,1}(\Omega)\ , & \text{if } \ 2\alpha > 1\ . \end{cases} \tag{4.3}$$

Here, $W^{\alpha,1}(\Omega)$ are the Sobolev spaces of fractional order and $B^{1,1}(\Omega)$ is the Besov space.

We begin to study (4.1) in the case where the nonlinear term does not depend on spatial derivatives of u. We assume that

(i1) $\mathrm{g} : D(\mathrm{g}) \subset \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable, $(x,0) \in D(\mathrm{g})$, $g(\cdot,0) \in L^1(\Omega)$;
(i2) there exist $r \geq 1$ and $C > 0$ such that
$|\mathrm{g}(x,v) - \mathrm{g}(x,u)| \leq C\,(|u|^{r-1} + |v|^{r-1})|u - v|$.

Now, let $g : D(g) \to L^1(\Omega)$ be defined as

$$D(g) = \{u \in L^1(\Omega) : \mathrm{g}(\cdot, u(\cdot)) \in L^1(\Omega)\}, \quad g(u)(x) := \mathrm{g}(x, u(x)), \tag{4.4}$$

then we have:

Lemma 4.1. *Assume that* (i1) *and* (i2) *hold and let* r *be given by* (i2)*. Then if* $N \leq 2$ *the function* g *given by* (4.4) *satisfies assumptions* (g1) *and* (g2) *of section* 3*, for each* $r \geq 1$*. If* $N \geq 3$*, then* g *satisfies* (g1) *and* (g2)*, if* $r < N/(N-2)$*.*

Proof. If $r = 1$ the result is trivial for each N; therefore, we consider only the case $r > 1$.

We begin to study the case $N = 1$. Let r satisfy (i2) and set

$$\theta := (r-1)/(2r), \tag{4.5}$$

then $\theta < 1/2$ and we have, using Sobolev's embedding theorem and (4.3),

$$X_{\theta,1} \subset L^r(\Omega). \tag{4.6}$$

Moreover, from (i2),

$$|\mathrm{g}[x,u(x)]| \leq |\mathrm{g}[x,u(x)] - \mathrm{g}(x,0)| + |\mathrm{g}(x,0)| \leq C|u(x)|^r + |\mathrm{g}(x,0)|.$$

Hence, if θ is given by (4.5), we find $X_{\theta,1} \subset D(g)$, which proves (g1).

To prove (g2), we use Hölder's inequality and (4.6) to obtain for $|u|_{\theta,1}$, $|v|_{\theta,1} < R$,

$$\|g(v) - g(u)\| \leq C \int_\Omega (|u(x)|^{r-1} + |v(x)|^{r-1})|v(x) - u(x)|\,dx$$

$$\leq C\Big[\int_\Omega (|u(x)|^{r-1}+|v(x)|^{r-1})^{r/r-1}dx\Big]^{(r-1)/r}\Big[\int_\Omega |v(x)-u(x)|^r\,dx\Big]^{1/r}$$
$$\leq C\,c^* R^{r-1}|v-u|_{\theta,1},$$

where c^* depends on r and the embedding (4.6). Hence, g satisfies (g2), with $q = r$.

Next, we consider the case $N = 2$. Set

$$\theta := 1 - 1/r, \quad \text{if } r \neq 2; \quad \theta := 1/2 + \varepsilon, \quad \text{if } r = 2, \tag{4.7}$$

where $\varepsilon \in (0, 1/2)$ is arbitrarily fixed.

Then again, using Sobolev's embedding theorems and (4.3), we have

$$X_{\theta,1} \subset L^r(\Omega). \tag{4.8}$$

Hence, we can repeat the above arguments with θ given by (4.7) and $q = r$.

Finally, we consider the case $N > 2$ and assume that $r < N/(N-2)$. Set

$$\theta := N(r-1)/2r, \tag{4.9}$$

then $\theta \in (0, 1)$ and, from Sobolev's embedding theorems,

$$X_{\theta,1}(\Omega) \subset L^r(\Omega). \tag{4.10}$$

Hence, we can repeat the above arguments, with θ given by (4.9) and $q = r$.

Therefore, we proved that g satisfies (g1) and (g2) with $q = r$ and θ given by (4.5), (4.7) and (4.9), for $N = 1$, $N = 2$ and $N \geq 3$, respectively. □

Assuming properties (i1) and (i2), we can apply the results of Section 3, with A and g given by (4.2) and (4.4), to the problem

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + \mathrm{g}[x, u(t,x)], \ t > 0, x \in \Omega \\ u(t,x) = 0\ ,\ t > 0, x \in \partial\Omega \\ u(0,x) = u_0(x)\ ,\ x \in \Omega, \end{cases} \tag{4.11}$$

in the all the cases where r and N satisfy the conditions of Lemma 4.1. As an example, let us study in a detailed way, the cases $N = 2$ and $N = 3$.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^2$, let* g *satisfy* (i1) *and* (i2) *and let r be given by* (i2). *Set $\theta := 1 - 1/r$, if $r \neq 2$, and $\theta := 1/2 + \varepsilon$ for fixed $\varepsilon \in (0, 1/6)$, if $r = 2$.*

If $r < 2$ for each $u_0 \in L^1(\Omega)$, there exists a unique solution u of problem (4.11)*, defined on a maximal interval $(0, \tau(u_0))$, satisfying $u \in L^\infty_\theta(0,T; W^{2\theta,1}(\Omega))$, for each $T < \tau(u_0)$. Moreover, $u \in W^{1,1}_+(0,T; L^1(\Omega)) \cap L^1_+(0,T; W^{1,1}_0(\Omega))$ and $\mathcal{A}u \in L^1_+((0,T) \times \Omega)$.*

Furthermore, if $u_0, v_0 \in L^1(\Omega)$ *and* u, v *are the corresponding maximally defined solutions of* (4.11)*, then for each* $T < \inf\{\tau(u_0), \tau(v_0)\}$*, there exists* $C = C(T, \|u_0\|_{L^1}, \|v_0\|_{L^1})$ *verifying*

$$t^{\theta}|v(t,\cdot) - u(t,\cdot)|_{W^{2\theta,1}} \leq C\|v_0 - u_0\|_{L^1}, \ t \in (0,T].$$

If $r = 2$*, take* $\beta \in (2\varepsilon, 1/2)$*. Then, for each* $u_0 \in W^{2\beta,1}(\Omega)$ *problem* (4.11) *admits a unique solution* u*, defined on a maximal interval* $(0, \tau(u_0))$ *satisfying* $u \in L^{\infty}_{\theta-\beta}(0,T; W^{2\theta,1}(\Omega))$*, for each* $T < \tau(u_0)$*. Moreover,* $u \in W^{1,1}(0,T; L^1(\Omega))$*,* $u \in L^1(0,T; W^{1,1}_0(\Omega))$ *and* $\mathcal{A}u \in L^1((0,T) \times \Omega)$*. Furthermore, if* $u_0, v_0 \in W^{2\beta,1}(\Omega)$ *and* u, v *are the corresponding solutions of* (4.11)*, then for each* $T < \inf\{\tau(u_0), \tau(v_0)\}$ *there exists* $C = C(T, |u_0|_{W^{2\beta,1}}, |v_0|_{W^{2\beta,1}})$ *satisfying*

$$t^{\theta-\beta}|v(t,\cdot) - u(t,\cdot)|_{W^{2\theta,1}} \leq C|v_0 - u_0|_{W^{2\beta,1}}, \ t \in (0,T].$$

If $r \in (2,3)$*, choose* $\beta \in (1 - 1/(r-1), 1/2)$*. Then, for each* $u_0 \in W^{2\beta,1}(\Omega)$ *problem* (4.11)*, admits a unique solution* u*, defined on a maximal interval* $(0, \tau(u_0))$*, satisfying* $u \in L^{\infty}_{\theta-\beta}(0,T; W^{2\theta,1}(\Omega))$ *for each* $T < \tau(u_0)$*. Moreover,* $u \in W^{1,1}(0,T; L^1(\Omega))$*,* $u \in L^1(0,T; W^{1,1}_0(\Omega))$ *and* $\mathcal{A}u \in L^1((0,T) \times \Omega)$*. Furthermore, if* $u_0, v_0 \in W^{2\beta,1}(\Omega)$ *and* u, v *are the corresponding solutions of* (4.11)*, then for each* $T < \inf\{\tau(u_0), \tau(u_0)\}$*, there exists* $C = C(T, |u_0|_{W^{2\beta,1}}, |v_0|_{W^{2\beta,1}})$ *satisfying*

$$t^{\theta-\beta}|v(t,\cdot) - u(t,\cdot)|_{W^{2\theta,1}} \leq C|v_0 - u_0|_{W^{2\beta,1}}, \ t \in (0,T].$$

If $r \geq 3$*, choose* $\beta \in (1 - 1/(r-1), 1 - 1/r)$*. Then, for each* $u_0 \in W^{2\beta,1}_0(\Omega)$ *the conclusions of the statement for* $r \in (2,3)$ *hold.*

Proof. Using (4.7) of Lemma 4.1 the results follow from Theorems 3.2 and 3.4, if $r < 2$, and Theorems 3.6 and 3.8, if $r \geq 2$. Let us note that also in the case $r \geq 2$ we do not require $u_0 \in D(g)$. □

If $N = 3$ we have the following result.

Theorem 4.3. *Let us consider problem* (4.11) *with* $\Omega \subset \mathbb{R}^3$ *and* g *satisfying* (i1) *and* (i2) *with* $r < 3$*. Set* $\theta := (r-1)3/2r$*. Then,*

If $r < 5/3$ *for each* $u_0 \in L^1(\Omega)$*, there exists a unique solution* u *of problem* (4.11)*, defined on a maximal interval* $(0, \tau(u_0))$*, satisfying* $u \in L^{\infty}_{\theta}(0,T; W^{2\theta,1}(\Omega))$*, for each* $T < \tau(u_0)$*. Moreover,* $u \in W^{1,1}_{+}(0,T; L^1(\Omega))$*,* $u \in L^1_{+}(0,T; W^{1,1}_0(\Omega))$ *and* $\mathcal{A}u \in L^1_{+}((0,T) \times \Omega)$*. Furthermore, if* $u_0, v_0 \in L^1(\Omega)$ *and* u, v *are the corresponding solutions of* (4.11)*, then for each*

$T < \inf\{\tau(u_0), \tau(v_0)\}$ there exists $C = C(T, \|u_0\|_{L^1}, \|v_0\|_{L^1})$ verifying

$$t^\theta |v(t,\cdot) - u(t,\cdot)|_{W^{2\theta,1}} \leq C\|v_0 - u_0\|_{L^1},\ t \in (0,T].$$

If $r \in [5/3, 3)$ fix $\beta \in (3/2 - 1/(r-1), 3/2 - 3/2r))$. Then for each $u_0 \in W^{2\beta,1}(\Omega)$ there exists a unique solution u of problem (4.11), defined on a maximal interval $(0, \tau(u_0))$, satisfying $u \in L^\infty_{\theta-\beta}(0,T; W^{2\theta,1}(\Omega))$, for each $T < \tau(u_0)$. Moreover, $u \in W^{1,1}(0,T; L^1(\Omega))$, $u \in L^1(0,T; W^{1,1}_0(\Omega))$ and $\mathcal{A}u \in L^1((0,T)\times\Omega)$. Furthermore, if $u_0, v_0 \in W^{2\beta,1}(\Omega)$ and u, v are the corresponding solutions of (4.11), then for each $T < \inf\{\tau(u_0), \tau(v_0)\}$ there exists $C = C(T, |u_0|_{W^{2\beta,1}}, |v_0|_{W^{2\beta,1}})$ verifying

$$t^{\theta-\beta} |v(t,\cdot) - u(t,\cdot)|_{W^{2\theta,1}} \leq C|v_0 - u_0|_{W^{2\beta,1}},\ t \in (0,T].$$

Proof. Using (4.9) of Lemma 4.1, the results follow from Theorems 3.2 and 3.4, if $r < 5/3$, and Theorems 3.6 and 3.8, if $r \in [5/3, 3)$. □

We now turn to problem (4.1). For brevity of exposition, we do not treat in details all the possible situations but we limit ourselves to some simple, though significant examples, in the cases $N = 1$ and $N = 2$.

Consider the following problem:

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + \gamma(x)\, u^n(t,x) u_x^m(t,x) + \gamma_0(x),\ t > 0,\ x \in (0,1), \\ u(t,0) = 0, u(t,1) = 0, \quad t > 0, \\ u(0,x) = u_0(x),\ x \in (0,1), \end{cases} \tag{4.12}$$

where

$$\mathcal{A}u = (a(x)u_x(x))_x + b(x)u_x(x) + c(x)u(x),$$

with $a \in C^1[0,1]$ and $b, c \in L^\infty(0,1)$, and $\gamma \in L^\infty(0,1)$, $\gamma_0 \in L^1(0,1)$ and $n \geq 0, m \geq 1$.

As before, we use the results of section 3, with $A : W^{2,1}(0,1) \cap W^{1,1}_0(0,1) \to L^1(0,1)$ and $g : D(g) \to L^1(0,1)$ defined as

$$(Au)(x) = (a(x)u_x(t,x))_x + b(x)u_x(x) + c(x)u(x), \tag{4.13}$$

$$D(g) = \{u \in W^{1,1}(0,1), \gamma\, u^n u_x^m \in L^1(0,1\}; \tag{4.14}$$

$$g(u)(x) = \gamma(x)u^n(x)u_x^m(x) + \gamma_0(x).$$

Lemma 4.4. *The function g given by (4.14) verifies (g1) and (g2) of Section 3.*

Proof. Assume that $m = 1$. For $\theta > 1/2$ we have

$$X_{\theta,1} \subset W^{1,1}(0,1) \subset L^\infty(\Omega), \tag{4.15}$$

so that $X_{\theta,1} \subset D(g)$. Furthermore,

$$\begin{aligned}
&\|g(v) - g(u)\| \\
&\leq \|\gamma\|_{L^\infty} \int_\Omega \big[\, |v(x)|^n |v_x(x) - u_x(x)| + |v^n(x) - u^n(x)|\, |u_x(x)| \,\big]\, dx \\
&\leq \|\gamma\|_{L^\infty} \big[\, \|v\|^n_{L^\infty} \|v_x - u_x\|_{L^1} + k' \|v - u\|_{L^\infty} (\|u\|^{n-1}_{L^\infty} + \|v\|^{n-1}_{L^\infty}) \|u_x\|_{L^1} \,\big],
\end{aligned}$$

where k' depends on n. Hence, from (4.15) there exists C verifying, for $|u|_{\theta,1}, |v|_{\theta,1} \leq R$,

$$\|g(v) - g(u)\| \leq CR^n |v - u|_{\theta,1},$$

which proves that g satisfies (g2) with $q = n + 1$.

If $m > 1$, take $\theta = 1 - 1/2m$. From (4.3) and embedding theorems, we have

$$X_{1-1/2m,1} \subset W^{1,1}(0,1) \subset L^\infty(0,1), \quad X_{1/2-1/2m,1} \subset L^m(0,1). \tag{4.16}$$

Thus, if $u \in X_{1-1/2m,1}$ then $u \in L^\infty(0,1)$ and $u_x \in L^m(0,1)$ so that $X_{1-1/2m,1} \subset D(g)$. Moreover,

$$\begin{aligned}
&\|g(v) - g(u)\| \\
&\leq \|\gamma\|_{L^\infty} \Big[\int_0^1 \big[\, |v(x)|^n |v_x^m(x) - u_x^m(x)| + |v^n(x) - u^n(x)|\, |u_x(x)|^m \,\big]\, dx \\
&\leq \|\gamma\|_{L^\infty} \Big[\|v\|^n_{L^\infty} k^* \int_0^1 |v_x(x) - u_x(x)| (|u_x(x)|^{m-1} + |v_x(x)|^{m-1})\, dx \\
&\qquad + \|v - u\|_{L^\infty} k' (\|v\|^{n-1}_{L^\infty} + \|u\|^{n-1}_{L^\infty}) \|u_x\|^m_{L^m},
\end{aligned}$$

where k^* depends on m. Now Hölder's inequality implies

$$\begin{aligned}
&\int_0^1 |v_x(x) - u_x(x)| (|u_x(x)|^{m-1} + |v_x(x)|^{m-1})\, dx \\
&\leq \|v_x - u_x\|_{L^m} k'' (\|u_x\|^{m-1}_{L^m} + |v_x\|^{m-1}_{L^m}),
\end{aligned}$$

where k'' depends on m. Thus, using (4.16), we find that there exists K depending on n, m and the embedding estimates, such that if $|u|_{1-1/2m,1}$, $|v|_{1-1/2m,1} < R$, then

$$\|g(u) - g(v)\| \leq KR^{n+m-1} |u - v|_{1-1/2m,1},$$

which proves (g2), with $q = n + m$. Therefore, the Lemma is proved with θ and q given by:

$$\theta > 1/2,\ q = n + 1,\ \text{if } m = 1; \quad \theta = 1 - 1/2m,\ q = n + m,\ \text{if } m > 1. \tag{4.17}$$

$\square$

Using Lemma 4.4 and the results of Section 3, we can prove existence results for problem (4.12), for all n and m. As an example, we consider the cases $n = 1, m = 1$ and $n = 0, m = 2$.

Theorem 4.5. *Consider problem* (4.12) *for* $n = m = 1$. *Fix* $\theta > 1/2$ *and choose* $\beta \in (2\theta - 1, 1/2)$. *Then, for each* $u_0 \in W^{2\beta,1}(0,1)$, *there exists a unique solution* u, *defined on a maximal interval* $(0, \tau(u_0))$, *of problem* (4.12) *satisfying* $u \in L^\infty_{\theta-\beta}(0,T; W^{2\theta,1}(0,1))$ *for each* $T < \tau(u_0)$. *Moreover,* $u \in W^{1,1}(0,T; L^1(0,1))$, $u \in L^1(0,T; W^{2,1}(0,1) \cap W^{1,1}_0(0,1))$. *Furthermore, if* $u_0, v_0 \in W^{2\beta,1}(0,1)$ *and* u, v *are the corresponding solutions and* $T < \inf\{\tau(u_0), \tau(v_0)\}$, *then there exists* $C = C(T, |u_0|_{W^{2\beta,1}}, |v_0|_{W^{2\beta,1}})$ *verifying*

$$t^{\theta-\beta}|v(t,\cdot) - u(t,\cdot)|_{W^{2\theta,1}} \le C|v_0 - u_0|_{W^{2\beta,1}}.$$

Proof. The results follow from Lemma 4.4 and Theorems 3.6 and 3.8. □

Theorem 4.6. *Consider problem* (4.12) *for* $n = 0$ *and* $m = 2$. *Take* $\theta = 3/4$ *and choose* $\beta \in (1/2, 3/4)$. *Then, for each* $u_0 \in W^{2\beta,1}_0(0,1)$, *there exists a unique solution* u *defined on a maximal interval* $(0, \tau(u_0))$, *of problem* (4.12) *satisfying* $u \in L^\infty_{3/4-\beta}(0,T; W^{3/2,1}(0,1))$, *for each* $T < \tau(u_0)$. *Moreover,* $u \in W^{1,1}(0,T; L^1(0,1))$ *and* $u \in L^1(0,T; W^{2,1}(0,1) \cap W^{1,1}_0(0,1))$.

Furthermore, if $u_0, v_0 \in W^{2\beta,1}(0,1)$ *and* u, v *are the corresponding solutions and* $T < \inf\{\tau(u_0), \tau(v_0)\}$, *then there exists* $C = C(T, |u_0|_{W^{2\beta,1}}, |v_0|_{W^{2\beta,1}})$ *verifying*

$$t^{3/4-\beta}|v(t,\cdot) - u(t,\cdot)|_{W^{3/2,1}} \le C|v_0 - u_0)|_{W^{2\beta,1}}.$$

Proof. The results follow from Lemma 4.4 and Theorems 3.6 and 3.8. □

We conclude this section with the following example of (4.1) in the case $\Omega \subset \mathbb{R}^2$

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + u(x)[\gamma_1(x)u_{x_1}(x) + \gamma_2(x)u_{x_2}(x)] + \gamma_0(x), \\ \qquad\qquad t > 0, x \in \Omega \\ u(t,x) = 0, \quad t > 0, x \in \partial\Omega \\ u(0,x) = u_0(x), \ x \in \Omega, \end{cases} \tag{4.18}$$

where $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ and $\gamma_0 \in L^1(\Omega)$. As before, let $g : D(g) \to L^1(\Omega)$ be defined as

$$\begin{cases} D(g) = \{u \in W^{1,1}(\Omega); \gamma_1 u u_{x_1} + \gamma_2 u u_{x_2} \in L^1(\Omega)\}; \\ g(u)(x) := \gamma_1(x)u(x)u_{x_1}(x) + \gamma_2(x)u(x)u_{x_2}(x) + \gamma_0(x). \end{cases} \tag{4.19}$$

Lemma 4.7. *The function g verifies* (g1) *and* (g2) *of Section* 3 *with $\theta = 3/4$ and $q = 2$.*

Proof. From (4.3) and Sobolev embedding theorems we have,

$$X_{3/4,1} \subset L^4(\Omega), \quad X_{1/4,1} \subset L^{4/3}(\Omega), \tag{4.20}$$

so that, if $u \in X_{3/4,1}$, then $u \in L^4(\Omega)$ and $u_{x_1}, u_{x_1} \in L^{4/3}(\Omega)$. Furthermore, setting,

$$K' := \|\gamma_1\|_{L^\infty} + \|\gamma_2\|_{L^\infty},$$

we obtain from (4.19)

$$\begin{aligned}\|g(u)\| &\le (\|\gamma_1\|_{L^\infty} + \|\gamma_2\|_{L^\infty}) \int_\Omega |u(x)|(|u_{x_1}(x)| + |u_{x_2}(x)|)\,dx + \|\gamma_0\|_{L^1} \\ &\le 2K'\|u\|_{L^4}(\|u_{x_1}\|_{L^{4/3}} + \|u_{x_2}\|_{L^{4/3}}) + \|\gamma_0\|_{L^1},\end{aligned}$$

which proves that $X_{3/4,1} \subset D(g)$.

In a similar way, we find if $u, v \in X_{3/4,1}$

$$\begin{aligned}\|g(v) - g(u)\| \quad &\le \quad K'\big[\|v-u\|_{L^4}(\|v_{x_1}\|_{L^{4/3}} + \|v_{x_2}\|_{L^{4/3}}) \\ &\qquad + \|u\|_{L^4}(\|v_{x_1} - u_{x_1}\|_{L^{4/3}} + (\|v_{x_2} - u_{x_2}\|_{L^{4/3}}\big].\end{aligned}$$

Therefore, there exists K'' depending on K' and (4.15) such that, for $|u|_{3/4,1}$, $|v|_{3/4,1} \le R$,

$$\|g(v) - g(u)\| \quad \le \quad K''R\,|v-u|_{3/4,1},$$

which completes the proof. □

Theorem 4.8. *Choose $\beta \in (1/2, 3/4)$. Then for each $u_0 \in W_0^{2\beta,1}(\Omega)$ there exists a unique u, defined on a maximal interval $(0, \tau(u_0))$, satisfying $u \in L^\infty_{3/2-\beta}(0,T; W^{3/2,1}(\Omega))$ for each $T < \tau(u_0)$, solution of problem* (4.18)*. Moreover, $u \in W^{1,1}(0,T;L^1(\Omega))$, $u \in L^1(0,T;W_0^{1,1}(\Omega))$ and $\mathcal{A}u \in L^1((0,T)\times\Omega)$. Furthermore, if $u_0, v_0 \in W^{2\beta,1}(\Omega)$ and u, v are the corresponding solutions and $T < \inf\{\tau(u_0), \tau(v_0)\}$, then there exists $C = C(T, |u_0|_{W^{2\beta,1}}, |v_0|_{W^{2\beta,1}})$ verifying*

$$t^{3/4-\beta}|v(t,\cdot) - u(t,\cdot)|_{W^{3/2,1}} \le C|v_0 - u_0)|_{W^{2\beta,1}}.$$

Proof. The results follow from Lemma 4.7 and Theorems 3.6 and 3.8. □

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