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A UNIFIED APPROACH FOR MULTIPLE CONSTANT SIGN AND NODAL SOLUTIONS

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Abstract. We consider a nonlinear elliptic equation driven by the p-Laplacian with Dirichlet boundary condition. Using variational techniques, combined with the method of upper-lower solutions and suitable truncation arguments, we establish the existence of at least six nontrivial solutions: two positive, two negative and two nodal (sign-changing) solutions. Our framework of analysis incorporates both coercive and $p-1$-superlinear problems. Also, the result on multiple constant sign solution incorporates the case of concave-convex nonlinearities.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < p < +\infty$. We study the following nonlinear elliptic problem (P):

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The goal of this work is to prove multiplicity results for problem (P) without assuming any symmetry conditions on the right-hand side nonlinearity $s \mapsto f(x,s)$. Also, to determine the sign of the solutions. Our results cover both problems with coercive and indefinite Euler functional and improve several works existing in the literature.

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Recently, multiplicity results for the equations with p-Laplacian without any symmetry conditions on the right-hand side nonlinearity were proved by Jiu and Su [18], Liu and Liu [21], Liu [22]. Their approach uses variational methods combined with Morse theory (critical groups). However, the multiplicity results they prove do not provide information about the sign of all solutions. We also mention the very recent work of Motreanu, Motreanu and Papageorgiou [24], who consider a class of nonlinear eigenvalue problems with the nonlinearity of general polynomial growth, not necessarily subcritical, and using variational and truncation techniques prove the existence of three nontrivial solutions: one positive, the second negative, but they do not determine the sign of the third solution.

The existence of multiple positive solutions was investigated by Ambrosetti, Garcia Azorero and Peral [1], Anello and Cordaro [3], Garcia Azorero, Peral and Manfredi [14]. In [1] and [14], the right-hand side nonlinearity has the form $\lambda|s|^{q-2}s + |s|^{r-2}s$, with $1 < q < p < r < p^*$ and $\lambda > 0$ is a parameter (problems with concave and convex nonlinearities if $p = 2$), where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{otherwise.} \end{cases}$$

The authors prove the existence of a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, the equation has at least two positive solutions. In [1], they use the radial p-Laplacian and the main tool is the Leray–Schauder degree theory. In [14], $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded domain with a smooth boundary and their approach relies on the critical point theory. Anello and Cordaro [3] impose different hypotheses which distinguish their nonlinearity from that in [1] and [14] and they prove the existence of a whole sequence of positive solutions which converges uniformly to zero. Their method of proof is based on an abstract variational principle due to Ricceri [27].

The question of existence of nodal (sign-changing) solutions was investigated for the p-Laplacian only very recently. We have the works of Bartsch and Liu [4], Carl and Motreanu [8], [9], Carl and Perera [10], Papageorgiou and Papageorgiou [26], Zhang, Chen and Li [29], Zhang and Li [30]. Bartsch and Liu [4] use critical point theory for C^1 functionals on ordered Banach spaces. Carl and Motreanu [8] obtain a version of [24] ensuring the existence of two nontrivial opposite constant sign solutions and a third nontrivial solution which is nodal. Carl and Perera [10] extend to the p-Laplacian the method of Dancer and Du [12] for semilinear problems (i.e., $p = 2$), which is based on upper-lower solutions and variational arguments. Carl and Motreanu [9] and Papageorgiou and Papageorgiou [26] prove results ensuring the existence of at least three nontrivial solutions, with sign information, relating the conclusion to the second eigenvalue λ_2 of the p-Laplacian on $W_0^{1,p}(\Omega)$. Finally, Zhang, Chen and Li [29] and Zhang and Li [30] derive a nodal solution by constructing a pseudogradient vector field whose descent flow has appropriate invariance properties.

In this paper, first we develop an abstract study of multiple solutions for problem (P), with sign information, in terms of given upper lower-solutions. Afterwards, we choose suitable upper-lower solutions in order to obtain additional properties. Our study of multiple constant sign solutions covers the problems with concave-convex nonlinearities for $p = 2$. By assuming conditions on the behavior of the nonlinearity towards $\pm\infty$, we present two broad classes of problems for which we are able to go beyond solutions of constant sign and look for nodal solutions. Specifically, we deal with multiple solutions with sign information for coercive problems and for $(p-1)$-superlinear problems, realizing an unified approach for these classes of problems. Inspired by the approach of Dancer and Du [12], where $p = 2$, in order to produce a nodal solution for problem (P), we first generate a smallest positive solution y_+ and a biggest negative solution y_- of (P). Then, using variational techniques on appropriate truncations of the original Euler functional, we find a solution y_0 of (P) belonging to the order interval $[y_-, y_+]$, different from y_- and y_+, so that if $y_0 \neq 0$, then y_0 is nodal. The nontriviality of y_0 is shown by means of a variational characterization of λ_2 given by Cuesta, de Figueiredo and Gossez [11] and the second deformation lemma. Our final result provides six nontrivial solutions: two positive, two negative and two nodal. To the best of our knowledge, no previous work produced two nodal solutions for problems with p-Laplacian and with no symmetry conditions on the nonlinearity.

The rest of the paper is organized as follows: Section 2 contains some mathematical prerequisites. Section 3 presents preliminary results related to the method of upper and lower solutions. Section 4 is devoted to the constant sign solutions, while Section 5 to our abstract results on nodal solutions. Section 6 discusses multiple solutions to coercive problems and Section 7 focuses on multiple solutions to $(p-1)$-superlinear problems.

2. Mathematical background

In the analysis of problem (P), we use some basic facts about the spectrum of the negative p-Laplacian with Dirichlet boundary condition $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ $(\frac{1}{p}+\frac{1}{p'}=1)$. Let $m \in L^\infty(\Omega)_+$, $m \neq 0$, and consider the nonlinear weighted (with weight m) eigenvalue problem:

$$\begin{cases} -\Delta_p u = \hat{\lambda} m(x)|u|^{p-2}u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

The least number $\hat{\lambda} \in \mathbb{R}$, denoted $\hat{\lambda}_1(m)$, for which (2.1) has a nontrivial solution is the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega), m)$. It is positive, isolated, its associated eigenspace is one-dimensional and one has

$$\hat{\lambda}_1(m) = \min\left\{ \frac{\|\nabla u\|_p^p}{\int_\Omega m|u|^p\,dx} \;:\; u \in W_0^{1,p}(\Omega),\ u \neq 0 \right\}. \tag{2.2}$$

Let $\phi_1 \in W_0^{1,p}(\Omega)$ denote the positive normalized (i.e., $\int_\Omega m\phi_1^p\,dx = 1$) eigenfunction corresponding to $\hat{\lambda}_1(m)$. We know that $\phi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, where

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \{u \in C_0^1(\overline{\Omega}) : u(x) > 0,\ \forall x \in \Omega;\ \frac{\partial u}{\partial \nu}(x) < 0,\ \forall x \in \partial\Omega\}.$$

Here $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u(x) = 0 \text{ for all } x \in \partial\Omega\}$, $C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \Omega\}$, and $\nu(x)$ stands for the unit outward normal at $x \in \partial\Omega$.

Because $\hat{\lambda}_1(m)$ is isolated, it follows that

$$\hat{\lambda}_2(m) = \inf\{\hat{\lambda} : \hat{\lambda} \text{ is an eigenvalue of (2.1) and } \hat{\lambda} > \hat{\lambda}_1(m)\} > \hat{\lambda}_1(m),$$

is the second eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega), m)$. The eigenvalues $\hat{\lambda}_1(m)$ and $\hat{\lambda}_2(m)$ exhibit certain monotonicity properties with respect to $m \in L^\infty(\Omega)_+$:

(a) If $m_1(x) \leq m_2(x)$ a.e. on Ω, $m_1 \neq m_2$, then $\hat{\lambda}_1(m_2) < \hat{\lambda}_1(m_1)$ (see (2.2)).

(b) If $m_1(x) < m_2(x)$ a.e. on Ω, then $\hat{\lambda}_2(m_2) < \hat{\lambda}_2(m_1)$ (see Anane-Tsouli [2]).

If $m \equiv 1$, we set $\lambda_1 = \hat{\lambda}_1(m)$ and $\lambda_2 = \hat{\lambda}_2(m)$.

For later use, we now point out that the antimaximum principle (see [16, Theorem 5.1, Remark 5.5]) holds L^∞-locally uniformly with respect to the weight.

Lemma 1. *Given $m, h \in L^\infty(\Omega)_+ \setminus \{0\}$, there is a number $\delta > 0$ such that if $\zeta \in L^\infty(\Omega)_+$ with $\|\zeta - m\|_\infty < \delta$, and if $\hat\lambda_1(\zeta) < \hat\lambda < \hat\lambda_1(\zeta) + \delta$, then any solution u of the Dirichlet problem*

$$\begin{cases} -\Delta_p u = \hat\lambda \zeta(x)|u|^{p-2}u + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $u \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$.

Proof. Arguing by contradiction, there exist sequences $\{\zeta_k\}_{k\ge1} \subset L^\infty(\Omega)_+$ with $\zeta_k \to m$ uniformly on Ω, $\{\hat\lambda_k\}_{k\ge1} \subset \mathbb{R}$ with $\hat\lambda_k > \hat\lambda_1(\zeta_k)$ and $\hat\lambda_k \to \hat\lambda_1(m)$, and $\{u_k\}_{k\ge1} \subset W_0^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta_p u_k = \hat\lambda_k \zeta_k(x)|u_k|^{p-2}u_k + h(x) & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

and $u_k \notin -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$. If $\{u_k\}_{k\ge1}$ were bounded in $L^\infty(\Omega)$, then due to the a priori elliptic estimates (see [20]), $\{u_k\}_{k\ge1}$ would be bounded in $C^{1,\gamma}(\overline{\Omega})$, for some $\gamma \in (0,1)$, so along a subsequence, $u_k \to u$ in $C^1(\overline{\Omega})$, with $u \in C^1(\overline{\Omega})$ solving

$$\begin{cases} -\Delta_p u = \hat\lambda_1(m) m(x)|u|^{p-2}u + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

This contradicts [16, Proposition 4.3, Remark 5.5]. Thus, along a relabeled subsequence, we have that $\|u_k\|_\infty \to \infty$ and $v_k := \frac{u_k}{\|u_k\|_\infty} \to v$ in $C^1(\overline{\Omega})$, for some $v \in C^1(\overline{\Omega})$. By (2.3), we have that

$$\begin{cases} -\Delta_p v_k = \hat\lambda_k \zeta_k(x)|v_k|^{p-2}v_k + \frac{h(x)}{\|u_k\|^{p-1}} & \text{in } \Omega\,, \\ v_k = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.4}$$

so in the limit results in

$$\begin{cases} -\Delta_p v = \hat\lambda_1(m) m(x)|v|^{p-2}v & \text{in } \Omega\,, \\ v = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.5}$$

It follows that either $v > 0$ or $v < 0$ on Ω. The case where $v > 0$ on Ω cannot occur because [16, Proposition 4.1, Remark 5.5] prevents (2.4) to have a positive solution v_k thanks to $\hat\lambda_k > \hat\lambda_1(\zeta_k)$. The case $v < 0$ is also impossible because, in view of the strong maximum principle of Vázquez [28] applied to (2.5), there holds $\frac{\partial v}{\partial \nu} > 0$ on $\partial\Omega$, so $v_k \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ for k sufficiently large, which contradicts the assumption $u_k \notin -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$. $\square$

Next, we recall some basic things in the critical point theory needed later on. Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the sets: $\varphi^c = \{u \in X : \varphi(u) \le c\}$ and $K_c = \{u \in X : \varphi'(u) = 0, \ \varphi(u) = c\}$. The function φ is said to satisfy the Palais–Smale condition at level $c \in \mathbb{R}$ (the $(\mathrm{PS})_c$ condition, for short) if every sequence $\{u_n\}_{n\ge 1} \subset X$ verifying $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$ has a strongly convergent subsequence. We say that φ satisfies the (PS) condition if it satisfies the $(\mathrm{PS})_c$ condition for every $c \in \mathbb{R}$. An essential tool in our variational approach is the second deformation theorem (see, e.g., [15, p. 628]) that we now recall.

Lemma 2. *If $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \le +\infty$, φ satisfies the* $(\mathrm{PS})_c$ *condition for every $c \in [a,b)$, φ has no critical values in (a,b) and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ, then there exists a continuous mapping $h : [0,1] \times (\varphi^b \setminus K_b) \to \varphi^b$ such that for all $u \in \varphi^b \setminus K_b$ we have*

$$h(0,u) = u\,, \quad h(1,u) \in \varphi^a, \quad \varphi(h(t,u)) \le \varphi(h(s,u)) \ \text{ for } 0 \le s \le t \le 1,$$

and for all $u \in \varphi^a$, $h(t,u) = u$ for all $t \in [0,1]$.

Finally, we set some standard notation that will be used in the sequel. Given $u, v \in W_0^{1,p}(\Omega)$, we denote $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$, and if $u \le v$ a.e. on Ω, $[u,v] := \{w \in W_0^{1,p}(\Omega) : u(x) \le w(x) \le v(x) \text{ a.e. on } \Omega\}$.

3. Preliminary results

We state the following hypotheses on the nonlinearity $f(x,s)$ in (P):

$\mathrm{H}(f)_1$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) f is Carathéodory (i.e., for all $s \in \mathbb{R}$, $x \mapsto f(x,s)$ is measurable, and for a.a. $x \in \Omega$, $s \mapsto f(x,s)$ is continuous), and $f(x,0) = 0$ a.e. on Ω;

(ii) there are constants $c > 0$, $1 \le r < +\infty$ and a function $a \in L^\infty(\Omega)_+$ such that

$$|f(x,s)| \le a(x) + c|s|^{r-1} \ \text{ for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}\,;$$

(iii) there exists $\eta \in L^\infty(\Omega)_+$ such that $\eta(x) \ge \lambda_1$ a.e. on Ω, $\eta \ne \lambda_1$ and

$$\liminf_{s\to 0^+} \frac{f(x,s)}{s^{p-1}} \ge \eta(x) \ \text{ uniformly for a.a. } x \in \Omega\,.$$

In a symmetric way, we have:

$\mathrm{H}(f)_2$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying $\mathrm{H}(f)_1$(i)–(ii), and

(iii) there exists $\eta \in L^\infty(\Omega)_+$ such that $\eta(x) \geq \lambda_1$ a.e. on Ω, $\eta \neq \lambda_1$ and

$$\liminf_{s\to 0^-} \frac{f(x,s)}{|s|^{p-2}s} \geq \eta(x) \quad \text{uniformly for a.a. } x \in \Omega\,.$$

We recall the notions of (bounded) upper and lower solutions for problem (P). For more details we refer to the recent monograph [7].

Definition 1. (a) A function $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\overline{u}|_{\partial\Omega} \geq 0$ is an upper solution for problem (P) if

$$\int_\Omega \|\nabla \overline{u}\|^{p-2} \nabla \overline{u} \cdot \nabla v\, dx \geq \int_\Omega f(x, \overline{u}(x)) v(x)\, dx,$$

for all $v \in W_0^{1,p}(\Omega)$ with $v(x) \geq 0$ for a.a. $x \in \Omega$.

(b) A function $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\underline{u}|_{\partial\Omega} \leq 0$ is a lower solution for (P) if

$$\int_\Omega \|\nabla \underline{u}\|^{p-2} \nabla \underline{u} \cdot \nabla v\, dx \leq \int_\Omega f(x, \underline{u}(x)) v(x)\, dx,$$

for all $v \in W_0^{1,p}(\Omega)$ with $v(x) \geq 0$ for a.a. $x \in \Omega$.

Proposition 1. (a) *Under* $\mathrm{H}(f)_1$, *for an upper solution* $\overline{u} \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *and a lower solution* $\underline{u} \in W_0^{1,p}(\Omega)$ *of* (P) *with* $\overline{u} \geq \underline{u} \geq 0$ *a.e. on* Ω, *there exists a solution* $u_0 \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *of problem* (P) *satisfying* $u_0 \in [\underline{u}, \overline{u}]$ *and, if* $\underline{u} = 0$, *with negative energy, that is*

$$\frac{1}{p}\, \|Du_0\|_p^p - \int_\Omega \int_0^{u_0(x)} f(x,s)\, ds\, dx < 0.$$

(b) *Under* $\mathrm{H}(f)_2$, *for a lower solution* $\underline{v} \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *and an upper solution* $\overline{v} \in W_0^{1,p}(\Omega)$ *of* (P) *with* $\underline{v} \leq \overline{v} \leq 0$ *a.e. on* Ω, *there exists a solution* $v_0 \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *of* (P) *satisfying* $v_0 \in [\underline{v}, \overline{v}]$ *and, if* $\overline{v} = 0$, *with negative energy.*

Proof. We only show assertion (a) because (b) holds by the same argument. Given $\overline{u}$ and $\underline{u}$ as required, we introduce the truncation of $f(x,s)$:

$$\tilde{f}_+(x,s) = \begin{cases} f(x, \underline{u}(x)) & \text{if } s < \underline{u}(x) \\ f(x,s) & \text{if } \underline{u}(x) \leq s \leq \overline{u}(x), \\ f(x, \overline{u}(x)) & \text{if } s > \overline{u}(x)\,. \end{cases} \tag{3.1}$$

Setting $\tilde{F}_+(x,s) = \int_0^s \tilde{f}_+(x,\tau)\, d\tau$, we define the functional $\tilde{\varphi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\tilde{\varphi}_+(u) = \frac{1}{p}\, \|\nabla u\|_p^p - \int_\Omega \tilde{F}_+(x, u(x))\, dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have that $\tilde{\varphi}_+ \in C^1(W_0^{1,p}(\Omega))$ and due to the compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, $\tilde{\varphi}_+$ is weakly lower semicontinuous. By (3.1) and $\mathrm{H}(f)_1$(ii), we see that $\tilde{\varphi}_+$ is coercive. Therefore, there exists $u_0 \in [\underline{u}, \overline{u}]$ such that

$$\tilde{\varphi}_+(u_0) = \inf_{[\underline{u},\overline{u}]} \tilde{\varphi}_+ . \tag{3.2}$$

For any $y \in [\underline{u}, \overline{u}]$, let $\xi(t) = \tilde{\varphi}_+(ty + (1-t)u_0)$, $t \in [0,1]$. Then (3.2) ensures that $\xi'(0) \geq 0$, which yields

$$0 \leq \langle -\Delta_p u_0, y - u_0 \rangle - \int_\Omega \tilde{f}_+(x, u_0(x))(y - u_0)(x)\, dx . \tag{3.3}$$

Corresponding to $h \in W_0^{1,p}(\Omega)$ and $\varepsilon > 0$, we set

$$y(x) = \begin{cases} \underline{u}(x) & \text{if } x \in \{u_0 + \varepsilon h \leq \underline{u}\} \\ u_0(x) + \varepsilon h(x) & \text{if } x \in \{\underline{u} < u_0 + \varepsilon h < \overline{u}\}, \\ \overline{u}(x) & \text{if } x \in \{\overline{u} \leq u_0 + \varepsilon h\} \end{cases}$$

obtaining an element $y \in [\underline{u}, \overline{u}]$. Plugging it in (3.3), we get

$$\begin{aligned} 0 \leq{}& \varepsilon \int_\Omega \|\nabla u_0\|^{p-2} (\nabla u_0, \nabla h)_{\mathbb{R}^N}\, dx - \varepsilon \int_\Omega \tilde{f}_+(x, u_0) h\, dx \qquad (3.4)\\ &+ \int_{\{u_0+\varepsilon h \leq \underline{u}\}} (\|\nabla \underline{u}\|^{p-2} (\nabla \underline{u}, \nabla(\underline{u} - u_0 - \varepsilon h))_{\mathbb{R}^N} - \tilde{f}_+(x, \underline{u})(\underline{u} - u_0 - \varepsilon h))\, dx \\ &- \int_{\{\overline{u} \leq u_0+\varepsilon h\}} (\|\nabla \overline{u}\|^{p-2} (\nabla \overline{u}, \nabla(u_0 + \varepsilon h - \overline{u}))_{\mathbb{R}^N} - \tilde{f}_+(x, \overline{u})(u_0 + \varepsilon h - \overline{u}))\, dx \\ &+ \int_{\{u_0+\varepsilon h \leq \underline{u}\}} (\tilde{f}_+(x, \underline{u}) - \tilde{f}_+(x, u_0))(\underline{u} - u_0 - \varepsilon h)\, dx \\ &+ \int_{\{\overline{u} \leq u_0+\varepsilon h\}} (\tilde{f}_+(x, \overline{u}) - \tilde{f}_+(x, u_0))(\overline{u} - u_0 - \varepsilon h)\, dx \\ &- \int_{\{u_0+\varepsilon h \leq \underline{u}\}} (\|\nabla u_0\|^{p-2} \nabla u_0 - \|\nabla \underline{u}\|^{p-2} \nabla \underline{u}, \nabla(u_0 - \underline{u}))_{\mathbb{R}^N}\, dx \\ &- \varepsilon \int_{\{u_0+\varepsilon h \leq \underline{u}\}} (\|\nabla u_0\|^{p-2} \nabla u_0 - \|\nabla \underline{u}\|^{p-2} \nabla \underline{u}, \nabla h)_{\mathbb{R}^N}\, dx \\ &+ \int_{\{\overline{u} \leq u_0+\varepsilon h\}} (\|\nabla \overline{u}\|^{p-2} \nabla \overline{u} - \|\nabla u_0\|^{p-2} \nabla u_0, \nabla(u_0 - \overline{u}))_{\mathbb{R}^N}\, dx \\ &+ \varepsilon \int_{\{\overline{u} \leq u_0+\varepsilon h\}} (\|\nabla \overline{u}\|^{p-2} \nabla \overline{u} - \|\nabla u_0\|^{p-2} \nabla u_0, \nabla h)_{\mathbb{R}^N}\, dx. \end{aligned}$$

Definitions 1 (b) and (a) with $v = (\underline{u} - u_0 - \varepsilon h)^+$ and $v = (u_0 + \varepsilon h - \overline{u})^+$, respectively, give

$$\int_{\{u_0+\varepsilon h\leq \underline{u}\}} (\|\nabla \underline{u}\|^{p-2}(\nabla \underline{u}, \nabla(\underline{u} - u_0 - \varepsilon h))_{\mathbb{R}^N} - \tilde{f}_+(x, \underline{u})(\underline{u} - u_0 - \varepsilon h))\, dx \leq 0, \tag{3.5}$$

$$\int_{\{\overline{u}\leq u_0+\varepsilon h\}} (\|\nabla \overline{u}\|^{p-2}(\nabla \overline{u}, \nabla(u_0 + \varepsilon h - \overline{u}))_{\mathbb{R}^N} - \tilde{f}_+(x, \overline{u})(u_0 + \varepsilon h - \overline{u}))\, dx \geq 0, \tag{3.6}$$

whereas the monotonicity of the mapping $z \in \mathbb{R}^N \mapsto \|z\|^{p-2}z \in \mathbb{R}^N$ implies

$$-\int_{\{u_0+\varepsilon h\leq \underline{u}\}} (\|\nabla u_0\|^{p-2}\nabla u_0 - \|\nabla \underline{u}\|^{p-2}\nabla \underline{u}, \nabla(u_0 - \underline{u}))_{\mathbb{R}^N}\, dx \leq 0, \tag{3.7}$$

$$\int_{\{\overline{u}\leq u_0+\varepsilon h\}} (\|\nabla \overline{u}\|^{p-2}\nabla \overline{u} - \|\nabla u_0\|^{p-2}\nabla u_0, \nabla(u_0 - \overline{u}))_{\mathbb{R}^N}\, dx \leq 0. \tag{3.8}$$

Hypothesis $\mathrm{H}(f)_1$(ii) combined with $u_0 \in [\underline{u}, \overline{u}]$, yields

$$\int_{\{u_0+\varepsilon h\leq \underline{u}\}} (\tilde{f}_+(x, \underline{u}) - \tilde{f}_+(x, u_0))(\underline{u} - u_0 - \varepsilon h)\, dx \leq -\varepsilon c_1 \int_{\{u_0+\varepsilon h\leq \underline{u}<u_0\}} h\, dx \tag{3.9}$$

and

$$\int_{\{\overline{u}\leq u_0+\varepsilon h\}} (\tilde{f}_+(x, \overline{u}) - \tilde{f}_+(x, u_0))(\overline{u} - u_0 - \varepsilon h)\, dx \leq \varepsilon c_2 \int_{\{u_0<\overline{u}\leq u_0+\varepsilon h\}} h\, dx, \tag{3.10}$$

for some constants $c_1, c_2 > 0$. Plugging (3.5)–(3.10) in (3.4) leads to

$$\begin{aligned} 0 \leq &\int_\Omega \|\nabla u_0\|^{p-2}(\nabla u_0, \nabla h)_{\mathbb{R}^N}\, dx - \int_\Omega \tilde{f}_+(x, u_0)h\, dx \\ &-c_1 \int_{\{u_0+\varepsilon h\leq \underline{u}<u_0\}} h\, dx + c_2 \int_{\{u_0<\overline{u}\leq u_0+\varepsilon h\}} h\, dx \\ &-\int_{\{u_0+\varepsilon h\leq \underline{u}\}} (\|\nabla u_0\|^{p-2}\nabla u_0 - \|\nabla \underline{u}\|^{p-2}\nabla \underline{u}, \nabla h)_{\mathbb{R}^N}\, dx \\ &+\int_{\{\overline{u}\leq u_0+\varepsilon h\}} (\|\nabla \overline{u}\|^{p-2}\nabla \overline{u} - \|\nabla u_0\|^{p-2}\nabla u_0, \nabla h)_{\mathbb{R}^N}\, dx. \end{aligned} \tag{3.11}$$

Pass now to the limit in (3.11) as $\varepsilon \downarrow 0$. By Stampacchia's theorem (see, e.g., [15, pp. 195–196]), it turns out that

$$0 \leq \int_\Omega \|\nabla u_0\|^{p-2}(\nabla u_0, \nabla h)_{\mathbb{R}^N}\, dx - \int_\Omega \tilde{f}_+(x, u_0)h\, dx\, .$$

Since $h \in W_0^{1,p}(\Omega)$ is arbitrary and $u_0 \in [\underline{u}, \overline{u}]$, we find that u_0 is a solution of (P).

Let us justify that $u_0 \neq 0$. It suffices to check this when $\underline{u} = 0$. From hypothesis $\mathrm{H}(f)_1$(iii) we know that

$$\alpha := \int_\Omega (\lambda_1 - \eta(x))\phi_1(x)^p < 0,$$

and for each $\varepsilon \in (0, -\alpha)$ there is $\delta > 0$ such that

$$\frac{1}{p}\,(\eta(x) - \varepsilon)s^p \leq \int_0^s f(x,\tau)\,d\tau \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0,\delta).$$

Bearing in mind that $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, there is $t \in (0, \frac{\delta}{\|\phi_1\|_\infty})$ with $t\phi_1 \in (0, \overline{u}]$. Then relation (3.2) entails

$$\tilde{\varphi}_+(u_0) \leq \tilde{\varphi}_+(t\phi_1) \leq \frac{t^p}{p}\,(\alpha + \varepsilon) < 0 = \tilde{\varphi}_+(0),$$

hence, $u_0 \neq 0$. In view of $\mathrm{H}(f)_1$(ii)–(iii), there is a constant $c_0 > 0$ such that $\Delta_p u_0 = -f(\cdot, u_0) \leq c_0 u_0^{p-1}$ a.e. on Ω. Then, by the strong maximum principle in [28], we conclude that $u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. $\square$

A useful property of upper-lower solutions is the following one.

Lemma 3. *We assume* $\mathrm{H}(f)_1$(i)–(ii).

(a) *If* $y_1, y_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *are upper solutions for problem* (P), *then* $y := \min\{y_1, y_2\} \in W^{1,p}(\Omega)$ *is also an upper solution for problem* (P).

(b) *If* $w_1, w_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *are lower solutions for problem* (P), *then* $w := \max\{w_1, w_2\} \in W^{1,p}(\Omega)$ *is also a lower solution for problem* (P).

Proof. We provide the argument only for part (a) because (b) can be similarly established. Let y_1, y_2 be as in the statement of (a) and fix an $\varepsilon > 0$. Define the truncation function $\xi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\xi_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if } s \leq -\varepsilon \\ s & \text{if } -\varepsilon < s < \varepsilon \\ \varepsilon & \text{if } s \geq \varepsilon\,. \end{cases}$$

We have that $\xi_\varepsilon((y_1 - y_2)^-) \in W^{1,p}(\Omega)$ with $\nabla \xi_\varepsilon((y_1 - y_2)^-) = \xi_\varepsilon'((y_1 - y_2)^-)\nabla((y_1 - y_2)^-)$ (see Marcus and Mizel [23]). For any test function $\psi \in C_c^1(\Omega)$ with $\psi \geq 0$, the following inequalities hold:

$$\langle -\Delta_p y_1, \xi_\varepsilon((y_1 - y_2)^-)\psi \rangle \geq \int_\Omega f(x, y_1)\xi_\varepsilon((y_1 - y_2)^-)\psi\,dx, \tag{3.12}$$

$$\langle -\Delta_p y_2, (\varepsilon - \xi_\varepsilon((y_1-y_2)^-))\psi\rangle \geq \int_\Omega f(x,y_2)(\varepsilon - \xi_\varepsilon((y_1-y_2)^-))\psi\, dx, \quad (3.13)$$

$$\begin{aligned} &\langle -\Delta_p y_1, \xi_\varepsilon((y_1-y_2)^-)\psi\rangle + \langle -\Delta_p y_2, (\varepsilon - \xi_\varepsilon((y_1-y_2)^-))\psi\rangle \quad (3.14)\\ &\leq \int_\Omega \|\nabla y_1\|^{p-2}(\nabla y_1, \nabla\psi)_{\mathbb{R}^N}\, \xi_\varepsilon((y_1-y_2)^-)\, dx \\ &+ \int_\Omega \|\nabla y_2\|^{p-2}(\nabla y_2, \nabla\psi)_{\mathbb{R}^N}\, (\varepsilon - \xi_\varepsilon((y_1-y_2)^-))\, dx\,. \end{aligned}$$

Adding (3.12) and (3.13), we get by means of (3.14) that

$$\begin{aligned} &\int_\Omega \|\nabla y_1\|^{p-2}(\nabla y_1, \nabla\psi)_{\mathbb{R}^N}\, \frac{1}{\varepsilon}\, \xi_\varepsilon((y_1-y_2)^-)\, dx \quad (3.15)\\ &+ \int_\Omega \|\nabla y_2\|^{p-2}(\nabla y_2, \nabla\psi)_{\mathbb{R}^N} \Big(1 - \frac{1}{\varepsilon}\, \xi_\varepsilon((y_1-y_2)^-)\Big) dx \\ &\geq \int_\Omega f(x,y_1)\frac{1}{\varepsilon}\, \xi_\varepsilon((y_1-y_2)^-)\psi\, dx + \int_\Omega f(x,y_2)\Big(1 - \frac{1}{\varepsilon}\, \xi_\varepsilon((y_1-y_2)^-)\Big)\psi dx. \end{aligned}$$

Passing to the limit as $\varepsilon \downarrow 0$ in (3.15) and noticing that

$$\frac{1}{\varepsilon}\, \xi_\varepsilon((y_1-y_2)^-(x)) \to \chi_{\{y_1<y_2\}}(x) \text{ a.e. on } \Omega \text{ as } \varepsilon \downarrow 0,$$

enable us to derive the inequality

$$\int_\Omega \|\nabla y\|^{p-2}(\nabla y, \nabla\psi)_{\mathbb{R}^N}\, dx \geq \int_\Omega f(x,y)\psi\, dx\,.$$

Since $C^1_{\mathrm{c}}(\Omega)$ is dense in $W^{1,p}_0(\Omega)$, we achieve the desired conclusion. □

4. Constant sign solutions

In this section we focus on the existence of extremal constant sign solutions.

Proposition 2. (a) *If hypotheses* $\mathrm{H}(f)_1$ *hold, then for each upper solution* $\overline{u} \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$ *and each lower solution* $\underline{u} \in W^{1,p}_0(\Omega)$ *of* (P) *with* $\overline{u} \geq \underline{u} \geq 0$ *a.e. on* Ω, $\underline{u} \neq 0$, *problem* (P) *admits a smallest solution* u_* *in* $[\underline{u}, \overline{u}]$. *In addition,* $u_* \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$.

(b) *If hypotheses* $\mathrm{H}(f)_2$ *hold, then for each lower solution* $\underline{v} \in -\operatorname{int}(C^1_0(\overline{\Omega})_+)$ *and each upper solution* $\overline{v} \in W^{1,p}_0(\Omega)$ *of* (P) *with* $\underline{v} \leq \overline{v} \leq 0$ *a.e. on* Ω, $\overline{v} \neq 0$, *problem* (P) *admits a biggest solution* v^* *in* $[\underline{v}, \overline{v}]$. *In addition,* $v^* \in -\operatorname{int}(C^1_0(\overline{\Omega})_+)$.

Proof. We only prove assertion (a) because we can similarly argue in the case of (b). Let $\overline{u}$ and $\underline{u}$ be as in the statement. Let $\mathcal{S}$ be the set of solutions of (P) belonging to the order interval $[\underline{u}, \overline{u}]$ (which is nonempty by Proposition 1 (a)). First, we show that

$$\mathcal{S} \text{ is downward directed.} \tag{4.1}$$

To this end, let $u_1, u_2 \in \mathcal{S}$. By virtue of Lemma 3 (a), $\hat{u} := \min\{u_1, u_2\} \in W_0^{1,p}(\Omega)$ is an upper solution for (P). As in the proof of Proposition 1 (a), truncating $f(x, \cdot)$ with respect to the ordered pair of lower-upper solutions $\{\underline{u}, \hat{u}\}$ (see (3.1)), we find a solution of (P) in $[\underline{u}, \hat{u}] \cap C_0^1(\overline{\Omega})$, thus (4.1) follows.

Now, we make use of Zorn's lemma. Let $C \subset \mathcal{S}$ be a chain. From [13, Corollary 7, p. 336], we can find $\{u_n\}_{n\geq 1} \subset C$ such that $\inf_{n\geq 1} u_n = \inf C$. By $\mathrm{H}(f)_1$(ii) and since $-\Delta_p u_n = f(\cdot, u_n(\cdot))$, we infer that $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. Hence we may assume that $u_n \xrightarrow{\mathrm{w}} \hat{y}$ in $W_0^{1,p}(\Omega)$, $u_n \to \hat{y}$ in $L^p(\Omega)$ and a.e. as $n \to \infty$, for some $\hat{y} \in W_0^{1,p}(\Omega)$. Noting that

$$\langle -\Delta_p u_n, u_n - \hat{y}\rangle = \int_\Omega f(x, u_n(x))(u_n - \hat{y})(x)\, dx \;\to\; 0 \quad \text{as } n \to \infty\,,$$

we deduce, through the (S_+)-property of $-\Delta_p$ on $W_0^{1,p}(\Omega)$, that $u_n \to \hat{y}$ in $W_0^{1,p}(\Omega)$. In the limit, we have $-\Delta_p \hat{y} = f(\cdot, \hat{y}(\cdot))$, which ensures that $\hat{y} = \inf C \in \mathcal{S}$. Then Zorn's lemma provides a minimal element $u_* \in \mathcal{S}$. In order to prove that u_* is the smallest solution of (P) in $[\underline{u}, \overline{u}]$, let $u \in \mathcal{S}$. Due to (4.1), we find $v \in \mathcal{S}$ with $v \leq u_*$ and $v \leq u$. Since u_* is a minimal element of $\mathcal{S}$, it follows that $u_* = v \leq u$. As in the final part of Proposition 1 (a) we conclude that $u_* \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. □

The next result produces positive lower solutions and negative upper solutions.

Lemma 4. (a) *Under* $\mathrm{H}(f)_1$, *for each function* $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *there exists a lower solution* $\underline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *of* (P) *satisfying* $\overline{u} - \underline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. *Moreover, for every* $\varepsilon \in (0,1)$, $\varepsilon\underline{u}$ *is a lower solution of* (P).

(b) *Under* $\mathrm{H}(f)_2$, *for each function* $\underline{v} \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *there exists an upper solution* $\overline{v} \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *of* (P) *satisfying* $\overline{v} - \underline{v} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. *Moreover, for every* $\varepsilon \in (0,1)$, $\varepsilon\overline{v}$ *is an upper solution of* (P).

Proof. Let $V := \{u \in W_0^{1,p}(\Omega) \;:\; \int_\Omega \phi_1^{p-1} u\, dx = 0\}$. We have the direct sum decomposition $W_0^{1,p}(\Omega) = \mathbb{R}\phi_1 \oplus V$. Defining

$$\lambda_V := \inf\left\{ \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \;:\; u \in V,\; u \neq 0 \right\},$$

there holds $\lambda_V > \lambda_1$. Let $\delta > 0$ be given by Lemma 1 for $h := \phi_1^{p-1}$ and $m := \lambda_1$. Using the continuity of the mapping $m \in L^\infty(\Omega)_+ \setminus \{0\} \mapsto \hat{\lambda}_1(m) \in (0, +\infty)$, we find $\varepsilon \in (0, \min\{\lambda_V - \lambda_1, \lambda_2 - \lambda_1, \delta\})$ such that for all $\zeta \in L^\infty(\Omega)$ with $\lambda_1 \leq \zeta(x) \leq \lambda_1 + \varepsilon$ a.e. on Ω, $\zeta \neq \lambda_1$, we have $\hat{\lambda}_1(\zeta) \in (1-\delta, 1)$. Fix $\zeta \in L^\infty(\Omega)_+$ with

$$\lambda_1 \leq \zeta(x) \leq \min\{\eta(x), \lambda_1 + \varepsilon\} \quad \text{a.e. on } \Omega,\ \zeta \neq \lambda_1 \tag{4.2}$$

and consider the auxiliary boundary value problem:

$$\begin{cases} -\Delta_p u = \zeta(x)|u|^{p-2}u - \phi_1(x)^{p-1} & \text{a.e. on } \Omega\,, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

The functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega \zeta |u|^p\,dx + \int_\Omega \phi_1^{p-1} u\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

is of class C^1 and its critical points are the solutions of (4.3).

Claim 1. φ_0 satisfies the (PS) condition.

Let $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ be a sequence such that $\{\varphi_0(u_n)\}_{n\geq 1}$ is bounded and $\varphi_0'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$. First, we show that $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. Arguing indirectly, we assume that along a subsequence $\|\nabla u_n\|_p \to \infty$ as $n \to \infty$ and set $y_n = \frac{u_n}{\|\nabla u_n\|_p}$, $n \geq 1$. We may suppose that $y_n \xrightarrow{\text{w}} y$ in $W_0^{1,p}(\Omega)$ and $y_n \to y$ in $L^p(\Omega)$, for some $y \in W_0^{1,p}(\Omega)$. We get $\lim_{n\to\infty}\langle -\Delta_p y_n, y_n - y\rangle = 0$. Since $-\Delta_p$ on $W_0^{1,p}(\Omega)$ is an operator of type $(S)_+$, we obtain that $y_n \to y$ in $W_0^{1,p}(\Omega)$, and so $\|\nabla y\|_p = 1$, and

$$-\Delta_p y = \zeta |y|^{p-2} y \quad \text{in } W^{-1,p'}(\Omega)\,. \tag{4.4}$$

By (4.2) and the monotonicity properties of $\hat{\lambda}_1(\cdot)$, $\hat{\lambda}_2(\cdot)$, we see that

$$\hat{\lambda}_1(\zeta) < \hat{\lambda}_1(\lambda_1) = 1 = \hat{\lambda}_2(\lambda_2) < \hat{\lambda}_2(\zeta). \tag{4.5}$$

From (4.4) and (4.5), we infer that $y = 0$, which is a contradiction. So $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ is bounded, and we may assume that $u_n \xrightarrow{\text{w}} u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$. As before, we deduce that $u_n \to u$ in $W_0^{1,p}(\Omega)$.

Claim 2. $\varphi_0|_V \geq 0$.

This follows from the definition of φ_0 since $\zeta(x) \leq \lambda_V$ a.e. on Ω (see (4.2)).

Claim 3. For $t > 0$ large, we have $\varphi_0(\pm t\phi_1) < 0$.

Indeed, using that $\|\phi_1\|_p = 1$, for $t > 0$ it is seen that

$$\varphi_0(\pm t\phi_1) = \frac{t^p}{p}\,\xi \pm t\,, \quad \text{where } \xi := \int_\Omega (\lambda_1 - \zeta(x))\phi_1(x)^p\,dx < 0,$$

which proves the claim.

Claim 4. The auxiliary problem (4.3) has a solution $\underline{\hat{u}} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Claims 1–3 allow to apply the saddle point theorem, and thus we find $\underline{\hat{u}} \in W_0^{1,p}(\Omega)$ such that $\varphi_0'(\underline{\hat{u}}) = 0$. This means that $\underline{\hat{u}}$ is a solution of problem (4.3), hence $\underline{\hat{u}} \neq 0$. From nonlinear regularity theory, we have $\underline{\hat{u}} \in C_0^1(\overline{\Omega})$. Since $\|\zeta - \lambda_1\|_\infty < \delta$ and $\hat{\lambda}_1(\zeta) < 1 < \hat{\lambda}_1(\zeta) + \delta$ (cf. (4.2) and the choice of ε), Lemma 1 ensures that Claim 4 is true.

Let $\tilde{\varepsilon} > 0$ be such that

$$\phi_1^{p-1} - \tilde{\varepsilon}\underline{\hat{u}}^{p-1} \in \operatorname{int}(C_0^1(\overline{\Omega})_+). \tag{4.6}$$

By virtue of hypothesis $\mathrm{H}(f)_1$(iii), we can find $\tilde{\delta} = \tilde{\delta}(\tilde{\varepsilon}) > 0$ such that

$$(\zeta(x) - \tilde{\varepsilon})s^{p-1} \leq f(x,s) \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0,\tilde{\delta}]. \tag{4.7}$$

Take $\beta \in (0,1]$ small enough to fulfill

$$\overline{u} - \beta\underline{\hat{u}} \in \operatorname{int}(C_0^1(\overline{\Omega})_+) \quad \text{and} \quad \beta\underline{\hat{u}}(x) \in [0,\tilde{\delta}] \quad \text{for all } x \in \overline{\Omega}. \tag{4.8}$$

We set $\underline{u} := \beta\underline{\hat{u}}$. By Claim 4, we know that $\underline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, whereas (4.8) yields $\overline{u} - \underline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Using (4.6)–(4.8), we infer that

$$-\Delta_p\underline{u} = \zeta\underline{u}^{p-1} - \beta^{p-1}\phi_1^{p-1} < (\zeta - \tilde{\varepsilon})\underline{u}^{p-1} \leq f(\cdot,\underline{u}(\cdot)) \quad \text{a.e. in } \Omega. \tag{4.9}$$

Taking into account Definition 1 (b), the conclusion of (a) is achieved.

The proof of assertion (b) proceeds in a similar way, so we omit it. □

In order to produce a minimal positive solution and a maximal negative solution for problem (P), we formulate a strengthened version of hypothesis $\mathrm{H}(f)_1$(iii) (resp., $\mathrm{H}(f)_2$(iii)) that dictates a strictly $(p-1)$-linear behavior of the nonlinearity $f(x,\cdot)$ near the origin:

$\mathrm{H}(f)_3$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying $\mathrm{H}(f)_1$ and

(iv) there exists $\hat{\eta} \in L^\infty(\Omega)_+$ such that

$$\limsup_{s\to 0^+} \frac{f(x,s)}{s^{p-1}} \leq \hat{\eta}(x) \quad \text{uniformly for a.a. } x \in \Omega\,.$$

Symmetrically, we consider:

$\mathrm{H}(f)_4$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying $\mathrm{H}(f)_2$ and

(iv) there exist $\hat{\eta} \in L^\infty(\Omega)_+$ such that

$$\limsup_{s \to 0^-} \frac{f(x,s)}{|s|^{p-2}s} \leq \hat{\eta}(x) \text{ uniformly for a.a. } x \in \Omega\,.$$

Proposition 3. (a) *Under* $\mathrm{H}(f)_3$ *, for each upper solution* $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$*, problem* (P) *has a smallest positive solution* y_+ *belonging to* $[0,\overline{u}]$*, which in addition satisfies* $y_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

(b) *Under* $\mathrm{H}(f)_4$ *, for each lower solution* $\underline{v} \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$*, problem* (P) *has a biggest negative solution* y_- *belonging to* $[\underline{v},0]$*, which in addition satisfies* $y_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Proof. We only prove part (a) because for (b) we can argue in a similar fashion. Let $\underline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ be the lower solution for problem (P) obtained in Lemma 4 (a). Let $\varepsilon_n \in (0,1]$, $\varepsilon_n \downarrow 0$, and set $\underline{u}_n = \varepsilon_n \underline{u}$. By Lemma 4 (a), $\underline{u}_n$ is a lower solution of (P). Proposition 2 (a) guarantees that problem (P) admits a smallest solution u_*^n in $[\underline{u}_n, \overline{u}]$. From $-\Delta_p u_*^n = f(\cdot, u_*^n(\cdot))$, we see that the sequence $\{u_*^n\}$ is bounded in $W_0^{1,p}(\Omega)$, so we may assume that $u_*^n \xrightarrow{\mathrm{w}} y_+$ in $W_0^{1,p}(\Omega)$ and $u_*^n \to y_+$ in $L^p(\Omega)$ as $n \to \infty$ for some $y_+ \in W_0^{1,p}(\Omega)$. As in the proof of Proposition 2, we obtain

$$u_*^n \to y_+ \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{4.10}$$

We show that $y_+ \neq 0$. Arguing by contradiction, suppose that $y_+ = 0$. Setting $w_n = \frac{u_*^n}{\|\nabla u_*^n\|_p}$ allows to admit that $w_n \xrightarrow{\mathrm{w}} w$ in $W_0^{1,p}(\Omega)$, $w_n \to w$ in $L^p(\Omega)$ as $n \to \infty$ for some $w \in W_0^{1,p}(\Omega)$. Denoting $h_n := \frac{f(\cdot, u_*^n(\cdot))}{\|\nabla u_*^n\|_p^{p-1}}$, we have

$$-\Delta_p w_n = h_n \text{ for all } n \geq 1. \tag{4.11}$$

Hypothesis $\mathrm{H}(f)_3$ entails that there exists $c_0 > 0$ such that $|f(x,s)| \leq c_0 s^{p-1}$ for a.a. $x \in \Omega$ and all $s \in [0, \|\overline{u}\|_\infty]$. This implies that $\{h_n\}$ is bounded in $L^{p'}(\Omega)$. We may assume that $h_n \xrightarrow{\mathrm{w}} h$ in $L^{p'}(\Omega)$ for some $h \in L^{p'}(\Omega)$. Therefore, by (4.11), we obtain $\lim_{n\to\infty}\langle -\Delta_p w_n, w_n - w\rangle = 0$, thereby $w_n \to w$ in $W_0^{1,p}(\Omega)$ and $\|w\| = 1$.

By (4.10), it turns out at least for a relabeled subsequence that $u_*^n(x) \to 0$ a.e. on Ω. Hypothesis $\mathrm{H}(f)_3$ implies for every $\varepsilon > 0$ that a.e. on Ω,

$$(\eta(x) - \varepsilon) w_n(x)^{p-1} \leq h_n(x) \leq (\hat{\eta}(x) + \varepsilon) w_n(x)^{p-1},$$

for n sufficiently large. Recalling that $\varepsilon > 0$ is arbitrary, it follows that $h(x) = k(x) w(x)^{p-1}$ a.e. on Ω with $k \in L^\infty(\Omega)_+$, $\eta(x) \leq k(x) \leq \hat{\eta}(x)$ a.e. on Ω. Passing to the limit as $n \to \infty$ in (4.11), we obtain $-\Delta_p w = k|w|^{p-2}w$,

$w \neq 0$. Because $k(x) \geq \lambda_1$ a.e. on Ω, $k \neq \lambda_1$, we have that $\hat{\lambda}_1(k) < \hat{\lambda}_1(\lambda_1) = 1$, so w changes sign which contradicts that $w \geq 0$. This proves that $y_+ \neq 0$.

Using (4.10), we obtain $-\Delta_p y_+ = f(\cdot, y_+(\cdot))$ and as in the last part of the proof of Proposition 1 (a), $y_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Let $\hat{y}$ be another positive solution in $[0, \overline{u}]$. As before, from nonlinear regularity theory and the nonlinear strong maximum principle in [28], we have $\hat{y} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Take an $\hat{\varepsilon} \in (0,1)$ such that $\hat{\varepsilon}\underline{u} \leq \hat{y}$. Then for n large we have $\underline{u}_n = \varepsilon_n \underline{u} \leq \hat{y} \leq \overline{u}$. Since u_*^n is the smallest solution in $[\underline{u}_n, \overline{u}]$, we derive $u_*^n \leq \hat{y}$. It follows from (4.10) that $y_+ \leq \hat{y}$, which completes the proof. □

5. Nodal solutions

Producing nodal solutions for (P) requires a strengthening of hypotheses:

H$(f)_5$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying H$(f)_1$(i)–(ii), and

(iii) there exists a constant $\mu_0 > \lambda_2$ such that

$$\mu_0 < \liminf_{s \to 0} \frac{f(x,s)}{|s|^{p-2}s} \quad \text{uniformly for a.a. } x \in \Omega\,.$$

Theorem 1. (a) *Under* H$(f)_5$, *for each upper solution* $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and each lower solution* $\underline{v} \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$, *problem* (P) *has at least three (nontrivial) solutions* $u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, $v_0 \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and* $y_0 \in C_0^1(\overline{\Omega})$ *having negative energies, and satisfying*

$$\underline{v} \leq v_0 \leq y_0 \leq u_0 \leq \overline{u},\ \ y_0 \neq 0,\ \ y_0 \neq u_0,\ \ y_0 \neq v_0.$$

(b) *If, in addition, hypotheses* H$(f)_3$(iv) *and* H$(f)_4$(iv) *hold, then the solution* y_0 *can be chosen to be nodal.*

Proof. The existence of opposite constant sign solutions $u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $v_0 \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ of problem (P) follows from Proposition 1 by choosing $\underline{u} = \overline{v} = 0$. To produce the third nontrivial solution, we define the functions $y_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $y_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ as follows: in case (a) set $y_+ := u_0$, $y_- := v_0$, while for (b) let y_+ and y_- be the minimal positive solution and maximal negative solution of (P), respectively, obtained in Proposition 3. We introduce the following truncations of the nonlinearity $f(x,s)$:

$$\hat{f}_+(x,s) = \begin{cases} 0 & \text{if } s \leq 0 \\ f(x,s) & \text{if } 0 < s < y_+(x) \\ f(x, y_+(x)) & \text{if } s \geq y_+(x)\,, \end{cases}$$

$$\hat{f}_-(x,s) = \begin{cases} f(x,y_-(x)) & \text{if } s \le y_-(x) \\ f(x,s) & \text{if } y_-(x) < s < 0 \\ 0 & \text{if } s \ge 0\,, \end{cases}$$

$$\hat{f}(x,s) = \begin{cases} f(x,y_-(x)) & \text{if } s \le y_-(x) \\ f(x,s) & \text{if } y_-(x) < s < y_+(x) \\ f(x,y_+(x)) & \text{if } s \ge y_+(x)\,. \end{cases}$$

Corresponding to these truncations we set

$$\hat{F}_\pm(x,s) = \int_0^s \hat{f}_\pm(x,\tau)\,d\tau\,, \quad \hat{F}(x,s) = \int_0^s \hat{f}(x,\tau)\,d\tau,$$

and then introduce the following C^1-functionals on $W_0^{1,p}(\Omega)$:

$$\hat{\varphi}_\pm(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega \hat{F}_\pm(x,u(x))\,dx,$$

$$\hat{\varphi}(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega \hat{F}(x,u(x))\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

In what follows, we will use the order intervals in $W_0^{1,p}(\Omega)$: $T_+ = [0, y_+]$, $T_- = [y_-, 0]$ and $T = [y_-, y_+]$. The critical points of $\hat{\varphi}_+$ are located in T_+, the critical points of $\hat{\varphi}_-$ in T_- and the critical points of $\hat{\varphi}$ in T. We do the proof for $\hat{\varphi}_+$ because the others are similar. Suppose $u \in W_0^{1,p}(\Omega)$ is a critical point of $\hat{\varphi}_+$. Acting in $-\Delta_p u = \hat{f}_+(\cdot, u(\cdot))$ with $(u - y_+)^+ \in W_0^{1,p}(\Omega)$ and using that y_+ is a solution of (P), we obtain

$$\begin{aligned} &\langle -\Delta_p u + \Delta_p y_+, (u-y_+)^+ \rangle \\ &= \int_\Omega (\hat{f}_+(x,u(x)) - f(x,y_+(x)))(u-y_+)^+(x)\,dx = 0. \end{aligned}$$

Since the mapping $z \in \mathbb{R}^N \mapsto \|z\|^{p-2} z \in \mathbb{R}^N$ is strictly monotone, we infer that $u \le y_+$. Similarly, we show that $u \ge 0$, therefore $u \in T_+$.

By hypothesis $\mathrm{H}(f)_5$(iii) we find $\mu \in (\lambda_2, \mu_0)$ and $\delta > 0$ such that

$$\frac{f(x,s)}{|s|^{p-2}s} > \mu \quad \text{for a.a. } x \in \Omega \text{ and all } s \in (-\delta,\delta),\ s \ne 0. \tag{5.1}$$

For $\varepsilon > 0$ with $\varepsilon\phi_1(x) \le \min\{\delta, y_+(x)\}$ on Ω, by (5.1), we see that

$$\hat{\varphi}_+(\varepsilon\phi_1) < \frac{\varepsilon^p}{p} \int_\Omega (\lambda_1 - \mu)\phi_1(x)^p\,dx < 0\,.$$

Since $\hat{\varphi}_+$ is coercive and weakly lower semicontinuous, there exists $\hat{y}_0 \in W_0^{1,p}(\Omega)$ satisfying $\hat{\varphi}_+(\hat{y}_0) = \inf_{W_0^{1,p}(\Omega)} \hat{\varphi}_+ < 0 = \hat{\varphi}_+(0)$. Therefore, $\hat{y}_0$ is a nonzero critical point of $\hat{\varphi}_+$ and so $\hat{y}_0$ is located in T_+. Hence, in case (a), we may assume that $\hat{y}_0 = y_+$ because otherwise the conclusion holds with $y_0 = \hat{y}_0$). In case (b), we also have $\hat{y}_0 = y_+$ due to the fact that $\{0, y_+\}$ are the only critical points of $\hat{\varphi}_+$ in view of the extremality property of y_+. Thereby, we may assume that y_+ is the unique global minimizer of $\hat{\varphi}_+$. Consequently, y_+ is a local $C_0^1(\overline{\Omega})$-minimizer of $\hat{\varphi}$. Then it turns out that y_+ is a local $W_0^{1,p}(\Omega)$-minimizer of $\hat{\varphi}$ (see [14]). We may assume that y_+ is an isolated local minimizer of $\hat{\varphi}$. Indeed, if this is not the case, we can find a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that $u_n \to y_+$ in $W_0^{1,p}(\Omega)$, $u_n \neq y_+$, $\hat{\varphi}(u_n) = \hat{\varphi}(y_+)$ and $\hat{\varphi}'(u_n) = 0$. This results in $u_n \in T$ with $u_n \neq y_+$, $u_n \neq y_-$, $u_n \neq 0$ for all n. We have produced a sequence of distinct solutions for problem (P) which are nodal in case (b).

Similarly, working with $\hat{\varphi}_-$ on T_-, we deduce that $y_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ is an isolated local minimizer of $\hat{\varphi}$.

We observe that $\hat{\varphi}$ has a global minimizer $z_0 \in W_0^{1,p}(\Omega)$ with $\hat{\varphi}(z_0) < 0$, hence $z_0 \neq 0$. Thus $z_0 \in T$, so if $z_0 \neq y_+$ and $z_0 \neq y_-$, then z_0 is the third desired solution of (P) (nodal in case (b)).

It remains to study the cases $z_0 = y_+$ or $z_0 = y_-$. Let us suppose that $z_0 = y_+$ (the other case can be analogously treated). We can find $\delta > 0$ such that

$$\hat{\varphi}(y_+) \leq \hat{\varphi}(y_-) < \inf\{\hat{\varphi}(u) \,:\, u \in \partial B_\delta(y_-)\},$$

with $\partial B_\delta(y_-) = \{u \in W_0^{1,p}(\Omega) \,:\, \|u - y_-\| = \delta\}$ (see [25]). Since $\hat{\varphi}$ verifies the (PS) condition (because it is coercive), we can apply the mountain pass theorem (see, e.g., [15, p. 646]) and produce a critical point $y_0 \in W_0^{1,p}(\Omega)$ of $\hat{\varphi}$ with

$$\hat{\varphi}(y_-) < \hat{\varphi}(y_0) = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} \hat{\varphi}(\gamma(t)), \tag{5.2}$$

where $\Gamma = \{\gamma \in C([-1,1], W_0^{1,p}(\Omega)) \,:\, \gamma(-1) = y_-,\ \gamma(1) = y_+\}$.

The next step in the proof is to show that $\hat{\varphi}(y_0) < 0$. According to (5.2), this reduces to find a path $\overline{\gamma}_0 \in \Gamma$ such that

$$\hat{\varphi}(\overline{\gamma}_0(t)) < 0 \quad \text{for all } t \in [-1,1]. \tag{5.3}$$

Denote $\partial B_1^{L^p(\Omega)} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$, $S = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p(\Omega)}$ endowed with the $W_0^{1,p}(\Omega)$-topology and $S_C = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p(\Omega)} \cap C_0^1(\overline{\Omega})$ equipped with the $C_0^1(\overline{\Omega})$-topology. Evidently, S_C is dense in S in the $W_0^{1,p}(\Omega)$-topology. Setting $\Gamma_0 = \{\gamma \in C([-1,1], S) \,:\, \gamma(-1) = -\phi_1,\ \gamma(1) = \phi_1\}$ and $\Gamma_{0,C} = \{\gamma \in C([-1,1], S_C) \,:\, \gamma(-1) = -\phi_1,\ \gamma(1) = \phi_1\}$, we have that $\Gamma_{0,C}$ is dense Γ_0. Recall from Cuesta, de Figueiredo and Gossez [11] the following variational characterization of $\lambda_2 > 0$:

$$\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \|\nabla u\|_p^p .$$

Thus, we can find $\hat{\gamma}_0 \in \Gamma_{0,C}$ such that

$$\max\{\|\nabla u\|_p^p :\ u \in \hat{\gamma}_0([-1,1])\} < \mu . \tag{5.4}$$

Since $\hat{\gamma}_0([-1,1])$ is compact and $-y_-, y_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, there is $\varepsilon > 0$ such that $|\varepsilon u(x)| \le \delta$ for all $x \in \overline{\Omega}$, all $u \in \hat{\gamma}_0([-1,1])$, and $\varepsilon u \in [y_-, y_+]$ whenever $u \in \hat{\gamma}_0([-1,1])$ (see [8], [24]). Then, by (5.1), (5.4) and since $\hat{\gamma}_0([-1,1]) \subset \partial B_1^{L^p(\Omega)}$, we have

$$\hat{\varphi}(\varepsilon u) \le \frac{\varepsilon^p}{p}\|\nabla u\|_p^p - \frac{\varepsilon^p}{p}\mu\|u\|_p^p < 0 \ \text{ for all } u \in \hat{\gamma}_0([-1,1]).$$

So, the path $\gamma_0 = \varepsilon\hat{\gamma}_0$ joining $-\varepsilon\phi_1$ and $\varepsilon\phi_1$ verifies

$$\hat{\varphi}|_{\gamma_0([-1,1])} < 0 . \tag{5.5}$$

Let us apply Lemma 2 for $\hat{\varphi}_+$ and with $a := \hat{\varphi}_+(y_+) = \inf\hat{\varphi}_+ < \hat{\varphi}_+(\varepsilon\phi_1) =: b$. The previous analysis allows us to suppose that $K_b = \emptyset$ and that $\hat{\varphi}_+$ has no critical values in (a, b). Lemma 2 provides a continuous mapping $h : [0,1]\times\hat{\varphi}_+^b \to \hat{\varphi}_+^b$ such that for all $u \in \hat{\varphi}_+^b$ we have $h(0,u) = u$, $h(1,u) = y_+$ and $\hat{\varphi}_+(h(t,u)) \le \hat{\varphi}_+(u)$ whenever $t \in [0,1]$. The path $\gamma_+ : [0,1] \to W_0^{1,p}(\Omega)$ defined by $\gamma_+(t) = h(t, \varepsilon\phi_1)^+$ for all $t \in [0,1]$ joins $\varepsilon\phi_1$ and y_+. By (5.5), it satisfies

$$\hat{\varphi}|_{\gamma_+([0,1])} < 0 . \tag{5.6}$$

Similarly, we construct a path γ_- which joins $-\varepsilon\phi_1$ and y_- such that

$$\hat{\varphi}|_{\gamma_-([0,1])} < 0 . \tag{5.7}$$

If we concatenate paths γ_- , γ_0 , γ_+ , we produce a path $\overline{\gamma}_0 \in \Gamma$ which fulfills (5.3) (see (5.5)–(5.7)). In view of (5.2), y_0 is the desired third solution of (P) which belongs to $C_0^1(\overline{\Omega})$ owing to the nonlinear regularity theory. In case (b), y_0 is a nodal solution of (P) thanks to the extremality properties of y_+ , y_- . □

Remark 1. In the works of Zhang, Chen and Li [29] and Zhang and Li [30] the quotient $\frac{f(s)}{|s|^{p-2}s}$ (f is independent of $x \in \Omega$ and locally Lipschitz) has finite limits as $s \to 0^\pm$ and as $s \to \pm\infty$, which is important in their analysis. In Zhang and Li [30] it is treated the low dimensionality $N < p$ that permits to the authors to exploit the compact embedding of $W_0^{1,p}(\Omega)$ into $C(\overline{\Omega})$. The approach in both papers is completely different being based on the invariance properties of the descent flow of a pseudogradient vector field. In Bartch and Liu [4], the nonlinearity $f(x,s)$ is continuous, they employ the Ambrosetti and Rabinowitz condition (so their problem is $(p-1)$-superlinear), assume that for some $m > 0$, $s \mapsto f(x,s) + m|s|^{p-2}s$ is increasing and, when $N \geq 6$, require a technical condition on the exponent $p > 1$. Their approach uses critical point theory for C^1 functionals for ordered Banach spaces. Theorem 1 also holds if $\mathrm{H}(f)_1$(ii) is replaced with the weaker assumption that f is bounded on bounded sets. With this change, it extends the main results in Carl and Motreanu [9] and Papageorgiou and Papageorgiou [26].

6. Multiple solutions to coercive problems

Here we deal with problems (P) where the corresponding Euler functional is coercive. In this respect, we formulate the conditions:

$\mathrm{H}(f)'_1$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying $\mathrm{H}(f)_1$ and

(iv) there exists $\vartheta \in L^\infty(\Omega)_+$ such that $\vartheta(x) \leq \lambda_1$ a.e. on Ω, $\vartheta \neq \lambda_1$ and

$$\limsup_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} \leq \vartheta(x) \ \text{ uniformly for a.a. } x \in \Omega\,.$$

or, symmetrically,

$\mathrm{H}(f)'_2$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying $\mathrm{H}(f)_1$ and

(iv) there exists $\vartheta \in L^\infty(\Omega)_+$ such that $\vartheta(x) \leq \lambda_1$ a.e. on Ω, $\vartheta \neq \lambda_1$ and

$$\limsup_{s\to-\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \vartheta(x) \ \text{ uniformly for a.a. } x \in \Omega\,.$$

A first existence and multiplicity result focusses on constant sign solutions.

Proposition 4. (a) *Under hypotheses* $\mathrm{H}(f)'_1$, *problem* (P) *has a solution* $u_0 \in \mathrm{int}\,(C^1_0(\overline{\Omega})_+)$ *with negative energy. If, in addition,* $\mathrm{H}(f)_3$ (iv) *holds, then problem* (P) *has a smallest positive solution* $u_0 \in \mathrm{int}\,(C^1_0(\overline{\Omega})_+)$.

(b) *Under hypotheses* $\mathrm{H}(f)'_2$, *problem* (P) *has a solution* $v_0 \in -\mathrm{int}\,(C^1_0(\overline{\Omega})_+)$ *with negative energy. If, in addition,* $\mathrm{H}(f)_4$ (iv) *holds, problem* (P) *has a biggest negative solution* $v_0 \in -\mathrm{int}\,(C^1_0(\overline{\Omega})_+)$.

Proof. We only prove (a) because the proof of (b) follows the same pattern.
Claim: Under hypotheses $\mathrm{H}(f)_1$(i)–(ii) and $\mathrm{H}(f)'_1$(iv), there exists an upper solution $\overline{u} \in \mathrm{int}\,(C^1_0(\overline{\Omega})_+)$ of problem (P).

In view of $\mathrm{H}(f)'_1$(iv), there exists a constant $\xi_0 > 0$ such that

$$\|\nabla u\|_p^p - \int_\Omega \vartheta(x)|u(x)|^p\,dx \geq \xi_0 \|\nabla u\|_p^p \ \text{ for all } u \in W^{1,p}_0(\Omega).$$

By virtue of $\mathrm{H}(f)_1$(ii) and $\mathrm{H}(f)'_1$(iv), given $\varepsilon \in (0, \xi_0\lambda_1)$, we can find $\gamma_\varepsilon \in L^\infty(\Omega)_+$, $\gamma_\varepsilon \neq 0$, such that

$$f(x,s) < (\vartheta(x)+\varepsilon)s^{p-1} + \gamma_\varepsilon(x) \quad \text{for a.a. } x \in \Omega \text{ and all } s \geq 0. \tag{6.1}$$

Let $K_\varepsilon : L^p(\Omega) \to L^{p'}(\Omega)$ be the nonlinear operator defined by $K_\varepsilon(u)(\cdot) = (\vartheta(\cdot)+\varepsilon)|u(\cdot)|^{p-2}u(\cdot)$. Clearly, K_ε is bounded, continuous and $K_\varepsilon|_{W_0^{1,p}(\Omega)}$ is completely continuous. Taking into account that $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is maximal monotone, the operator $-\Delta_p - K_\varepsilon : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is pseudomonotone. Moreover, using (2.2), we have

$$\langle -\Delta_p u - K_\varepsilon(u), u\rangle \geq \left(\xi_0 - \frac{\varepsilon}{\lambda_1}\right)\|\nabla u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $\varepsilon < \xi_0\lambda_1$, we infer that the operator $-\Delta_p - K_\varepsilon$ is coercive, so $-\Delta_p - K_\varepsilon$ is surjective (see, e.g. [15, p. 336]). Thus there exists $\overline{u} \in W_0^{1,p}(\Omega)$ satisfying

$$\begin{cases} -\Delta_p \overline{u} = (\vartheta(x)+\varepsilon)|\overline{u}|^{p-2}\overline{u} + \gamma_\varepsilon(x) & \text{a.e. on } \Omega\,, \\ \overline{u} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{6.2}$$

Acting on (6.2) with the test function $-\overline{u}^- \in W_0^{1,p}(\Omega)$, we obtain

$$\xi_0\|\nabla \overline{u}^-\|_p^p \leq \|\nabla \overline{u}^-\|_p^p - \int_\Omega \vartheta|\overline{u}^-|^p\,dx \leq \varepsilon\|\overline{u}^-\|_p^p \leq \frac{\varepsilon}{\lambda_1}\|\nabla \overline{u}^-\|_p^p\,.$$

The choice of ε ensures that $\overline{u} \geq 0$. From (6.2), we have $\overline{u} \in C_0^1(\Omega)\setminus\{0\}$ and $-\Delta_p\overline{u} \in L^\infty(\Omega)_+$. The nonlinear strong maximum principle of Vázquez [28] yields $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Combining (6.1) and (6.2) establishes the claim.

Now, the conclusion follows from Proposition 1 with $\overline{u}$ in the Claim and $\underline{u} = 0$, while the final assertion is obtained from Proposition 3. □

Remark 2. Proposition 4 improves [25, Proposition 7] by dropping the assumption

$$\liminf_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} \geq -\vartheta_0 \quad \text{uniformly for a.a. } x \in \Omega \quad \text{with some } \vartheta_0 > 0.$$

Let us now state:

$\mathrm{H}(f)'_3$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying assumptions $\mathrm{H}(f)_1$(i)–(ii), $\mathrm{H}(f)_5$(iii), $\mathrm{H}(f)'_1$(iv), and $\mathrm{H}(f)'_2$(iv).

Theorem 2. (a) *Under* $\mathrm{H}(f)'_3$, *problem* (P) *has at least three (nontrivial) solutions* $u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, $v_0 \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and* $y_0 \in C_0^1(\overline{\Omega})$ *satisfying* $v_0 \leq y_0 \leq u_0$, $y_0 \neq 0$, $y_0 \neq u_0$, $y_0 \neq v_0$ *and with negative energy.*

(b) *If, in addition, hypotheses* $\mathrm{H}(f)_3$(iv) *and* $\mathrm{H}(f)_4$(iv) *hold, then the solution* y_0 *can be chosen to be nodal.*

Proof. Let $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ be the upper solution of (P) constructed in the proof of Proposition 4. In the same way, we find a lower solution $\underline{v} \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ of (P). Then we apply Theorem 1 for the functions $\overline{u}$, $\underline{v}$. $\square$

Remark 3. Theorem 2 improves the multiplicity results of Liu and Liu [21] and Liu [22] for coercive problems by providing information about the sign of the third solution. In Motreanu, Motreanu and Papageorgiou [25, Theorem 2, Remark 4], four solutions are obtained, two of constant sign and the other two nontrivial, but with no information about location or sign properties.

7. Multiple solutions to $(p-1)$-superlinear problems

Another important class of problems, which fit into the general framework of Proposition 1 consists of certain parametric problems (P_λ):

$$\begin{cases} -\Delta_p u = f(x, u(x), \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We consider the following hypotheses on the nonlinearity $f(x, s, \lambda)$:

$\mathrm{H}(f)_1''$ $f : \Omega \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, is a function such that

(i) for all $(s, \lambda) \in \mathbb{R} \times (0, \bar{\lambda})$, $f(\cdot, s, \lambda)$ is measurable, and for a.a. $x \in \Omega$, all $\lambda \in (0, \bar{\lambda})$, $f(x, \cdot, \lambda)$ is continuous, $f(x, 0, \lambda) = 0$;

(ii) there are functions $a(\cdot, \lambda) \in L^\infty(\Omega)_+$ $(\lambda \in (0, \bar{\lambda}))$ with $\|a(\cdot, \lambda)\|_\infty \to 0$ as $\lambda \downarrow 0$, and constants $c > 0$, $r > p$ such that

$$|f(x, s, \lambda)| \le a(x, \lambda) + c|s|^{r-1} \text{ for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R}, \text{ all } \lambda \in (0, \bar{\lambda});$$

(iii) for every $\lambda \in (0, \bar{\lambda})$, there exists $\eta = \eta(\lambda) \in L^\infty(\Omega)_+$ such that $\eta(x) \ge \lambda_1$ a.e. on Ω, $\eta \ne \lambda_1$ and

$$\liminf_{s \to 0^+} \frac{f(x, s, \lambda)}{s^{p-1}} \ge \eta(x) \text{ uniformly for a.a. } x \in \Omega.$$

Symmetrically, we formulate the following conditions:

$\mathrm{H}(f)_2''$ $f : \Omega \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, satisfies $\mathrm{H}(f)_1''$(i)–(ii) and

(iii) for every $\lambda \in (0, \bar{\lambda})$, there exists $\eta = \eta(\lambda) \in L^\infty(\Omega)_+$ such that $\eta(x) \ge \lambda_1$ a.e. on Ω, $\eta \ne \lambda_1$ and

$$\liminf_{s \to 0^-} \frac{f(x, s, \lambda)}{|s|^{p-2} s} \ge \eta(x) \text{ uniformly for a.a. } x \in \Omega.$$

First, we are concerned with constant sign solutions.

Proposition 5. (a) *Under hypotheses* $\mathrm{H}(f)''_1$, *for all* $b > 0$, *there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that if* $\lambda \in (0, \lambda^*)$, *problem* (P_λ) *has a solution* $u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *with* $\|u_0\|_\infty < b$ *and negative energy.*

(b) *Under hypotheses* $\mathrm{H}(f)''_2$, *for all* $b > 0$, *there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that if* $\lambda \in (0, \lambda^*)$, *problem* (P_λ) *has a solution* $v_0 \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *with* $\|v_0\|_\infty < b$ *and negative energy.*

Proof. Let $e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ be such that $-\Delta_p e = 1$, and let $b > 0$.

Claim 1. There exists $\lambda^* \in (0, \bar{\lambda})$ such that for all $\lambda \in (0, \lambda^*)$, there is $\xi_1 = \xi_1(\lambda) \in (0, \frac{b}{\|e\|_\infty})$ satisfying

$$\|a(\cdot, \lambda)\|_\infty + c(\xi_1 \|e\|_\infty)^{r-1} < \xi_1^{p-1}. \tag{7.1}$$

For the proof of Claim 1, we refer to Proposition 3.1 in [24].

Claim 2. Under hypotheses $\mathrm{H}(f)''_1$ (i)–(ii), for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has an upper solution $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ with $\|\overline{u}\|_\infty < b$.

We fix $\lambda \in (0, \lambda^*)$, with λ^* corresponding to b in Claim 1. For $\xi_1 \in (0, \frac{b}{\|e\|_\infty})$ given by Claim 1, we set $\overline{u} = \xi_1 e$. Then $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, $\|\overline{u}\|_\infty < b$ and $-\Delta_p \overline{u} = \xi_1^{p-1} \in L^\infty(\Omega)_+$. By (7.1) and hypothesis $\mathrm{H}(f)''_1$ (ii)$''$, we see that

$$-\Delta_p \overline{u} > \|a(\cdot, \lambda)\|_\infty + c\|\overline{u}\|_\infty^{r-1} \geq f(x, s, \lambda) \text{ a.e. on } \Omega, \text{ all } s \in [0, \overline{u}(x)]. \tag{7.2}$$

Thus $\overline{u}$ satisfies the claim. We complete the proof of (a) by applying Proposition 1 (a) with $\overline{u}$ in Claim 2 and $\underline{u} = 0$. Similarly, we derive assertion (b) from Proposition 1 (b). □

Remark 4. Proposition 5 improves [24, Proposition 3.1] by relaxing the assumption $\operatorname{ess\,inf} \eta > \lambda_1$ and dropping the local sign condition.

We strengthen our assumptions on $f(x, s, \lambda)$ to be $(p-1)$-superlinear:

$\mathrm{H}(f)''_3$ $f : \Omega \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, is a function satisfying $\mathrm{H}(f)''_1$ with $r < p^*$ (r in $\mathrm{H}(f)''_1$(ii)) and

(iv) for every $\lambda \in (0, \bar{\lambda})$, there exist $M = M(\lambda) > 0$ and $\mu = \mu(\lambda) > p$ such that

$$0 < \mu F(x, s, \lambda) \leq f(x, s, \lambda)s \text{ for a.a. } x \in \Omega, \text{ all } s \geq M,$$

where $F(x, s, \lambda) = \int_0^s f(x, \tau, \lambda)\, d\tau$;

(v) there exists $b > 0$ such that $f(x, s, \lambda) > 0$ for a.a. $x \in \Omega$, all $s \in (0, b)$ and all $\lambda \in (0, \bar{\lambda})$.

Symmetrically, we state:

H$(f)''_4$ $f : \Omega \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, satisfies H$(f)''_2$ with $r < p^*$ (r in H$(f)''_1$(ii)), and

(iv) for every $\lambda \in (0, \bar{\lambda})$, there exist $M = M(\lambda) > 0$ and $\mu = \mu(\lambda) > p$ such that

$$0 < \mu F(x, s, \lambda) \le f(x, s, \lambda)s \quad \text{for a.a. } x \in \Omega, \text{ all } s \le -M;$$

(v) there exists $b > 0$ such that $f(x, s, \lambda) < 0$ for a.a. $x \in \Omega$, all $s \in (-b, 0)$ and all $\lambda \in (0, \bar{\lambda})$.

Theorem 3. (a) *Under* H$(f)''_3$, *there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that if* $\lambda \in (0, \lambda^*)$, (P_λ) *has at least two (positive) solutions* $u_0, \hat{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, $u_0 \le \hat{u}$, $u_0 \ne \hat{u}$, $\|u_0\|_\infty < b$.

(b) *Under* H$(f)''_4$, *there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that if* $\lambda \in (0, \lambda^*)$, (P_λ) *has at least two (negative) solutions* $v_0, \hat{v} \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$, $\hat{v} \le v_0$, $\hat{v} \ne v_0$, $\|v_0\|_\infty < b$.

Proof. Since (b) is the symmetric counterpart of (a), we only prove (a). Apply Proposition 5 with $b > 0$ given by H$(f)''_3$(v) to get $\lambda^* \in (0, \bar{\lambda})$. Let $\lambda \in (0, \lambda^*)$, and let $u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ be the solution of (P_λ) with $\|u_0\|_\infty < b$ supplied by Proposition 5 (a). We introduce the following truncation of $f(\cdot, \cdot, \lambda)$:

$$\overline{f}_+(x, s) = \begin{cases} f(x, u_0(x), \lambda) & \text{if } s \le u_0(x) \\ f(x, s, \lambda) & \text{if } s > u_0(x), \end{cases} \tag{7.3}$$

for all $(x, s) \in \Omega \times \mathbb{R}$. Let $\overline{F}_+(x, s) = \int_0^s \overline{f}_+(x, \tau)\, d\tau$, and define the C^1-functional $\overline{\varphi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\overline{\varphi}_+(u) = \frac{1}{p} \|\nabla u\|_p^p - \int_\Omega \overline{F}_+(x, u(x))\, dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Consider the boundary value problem:

$$\begin{cases} -\Delta_p u = \overline{f}_+(x, u(x)) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{7.4}$$

Denote by $\overline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ the upper solution of problem (P_λ) obtained in Claim 2 of the proof of Proposition 5. From (7.3), hypothesis H$(f)''_3$(v) and since $0 < u_0 \le \overline{u} < b$, we see that $\overline{f}_+(x, 0) = f(x, u_0(x), \lambda) \ge 0$ a.e. on Ω and

$$-\Delta_p \overline{u} > f(x, \overline{u}(x), \lambda) = \overline{f}_+(x, \overline{u}(x)) \quad \text{a.e. on } \Omega,$$

(see (7.2)). So $\{0, \overline{u}\}$ form an ordered pair of lower-upper solutions for problem (7.4). Since $\overline{\varphi}_+$ is weakly lower semicontinuous and $\overline{\varphi}_+|_{[0,\overline{u}]}$ is coercive, we find $\tilde{u} \in [0, \overline{u}]$ such that

$$\overline{\varphi}_+(\tilde{u}) = \inf_{[0,\overline{u}]} \overline{\varphi}_+. \tag{7.5}$$

As in the proof of Proposition 1, we infer that $\tilde{u}$ is a solution of (7.4) and

$$\int_{\{u_0>\tilde{u}\}} (\|\nabla\tilde{u}\|^{p-2}\nabla\tilde{u}, \nabla(\tilde{u}-u_0))_{\mathbb{R}^N}\,dx = \int_{\{u_0>\tilde{u}\}} f(x,u_0,\lambda)(\tilde{u}-u_0)\,dx$$
$$= \int_{\{u_0>\tilde{u}\}} (\|\nabla u_0\|^{p-2}\nabla u_0, \nabla(\tilde{u}-u_0))_{\mathbb{R}^N}\,dx,$$

hence $u_0 \le \tilde{u}$. So $\tilde{u}$ is a solution of (P_λ). Since $\tilde{u} \ge u_0 > 0$ and $\Delta_p\tilde{u} = -f(\cdot,\tilde{u},\lambda) \le c_0\tilde{u}^{p-1}$ a.e. on Ω, for some $c_0 > 0$ (by $\mathrm{H}(f)''_1$(ii)–(iii)), the strong maximum principle in [28], ensures that $\tilde{u} \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$.

We may assume that $\tilde{u} = u_0$, because otherwise we have the desired second positive solution of (P_λ). By $\mathrm{H}(f)''_3$(v), (7.2) and since $u_0 \le \overline{u} < b$, we have that

$$0 \le -\Delta_p u_0(x) = f(x,u_0(x),\lambda) < -\Delta_p\overline{u}(x) \quad \text{a.e. on } \Omega.$$

Then, by virtue of Guedda and Véron [17, Proposition 2.2], we infer that $\overline{u} - u_0 \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$. This, in conjunction with $u_0 \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$ and (7.5), guarantees that u_0 is a local $C^1_0(\overline{\Omega})$-minimizer of the functional $\overline{\varphi}_+$. Hence, by [14, Theorem 1.1], we obtain that u_0 is a local $W^{1,p}_0(\Omega)$-minimizer of $\overline{\varphi}_+$. We may admit that u_0 is an isolated local minimizer of $\overline{\varphi}_+$, because otherwise, arguing as above, we generate a sequence of distinct positive solutions of (P_λ). So we find $\rho > 0$ such that

$$\overline{\varphi}_+(u_0) < \inf\{\overline{\varphi}_+(u) \,:\, \|\nabla(u-u_0)\|_p = \rho\}\,. \tag{7.6}$$

Note that hypothesis $\mathrm{H}(f)''_3$(iv) implies that $F(x,s,\lambda) \ge c_3 s^\mu - c_4$ for a.a. $x \in \Omega$ and all $s \ge 0$, with $c_3, c_4 > 0$. We see that

$$\overline{\varphi}_+(t\phi_1) \to -\infty \quad \text{as } t \to +\infty. \tag{7.7}$$

To check that $\overline{\varphi}_+$ satisfies the (PS) condition, let $\{u_n\}_{n\ge1} \subset W^{1,p}_0(\Omega)$ be a sequence such that $\{\overline{\varphi}_+(u_n)\}_{n\ge1}$ is bounded and $\overline{\varphi}'_+(u_n) \to 0$ in $W^{-1,p'}(\Omega)$. Then ,we have

$$\frac{1}{p}\,\|\nabla u_n\|^p_p - \int_\Omega \overline{F}_+(x,u_n)\,dx \le M_1 \quad \text{for all } n \ge 1, \tag{7.8}$$

for some $M_1 > 0$, and

$$\langle -\Delta_p u_n, v\rangle - \int_\Omega \overline{f}_+(x,u_n)v\,dx \le \|\overline{\varphi}'_+(u_n)\|_{W^{-1,p'}}\,\|\nabla v\|_p\,, \tag{7.9}$$

for all $v \in W_0^{1,p}(\Omega)$ and all $n \geq 1$. Setting $v = -u_n^- \in W_0^{1,p}(\Omega)$ in (7.9) and using that $p > 1$ and $f(z, u_0, \lambda)u_n^- \geq 0$ a.e. on Ω, it follows that $\{u_n^-\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. By (7.8) and (7.9) with $v = -u_n^+$, we obtain

$$\left(\frac{\mu}{p} - 1\right) \|\nabla u_n^+\|_p^p + \int_{\{u_n \geq M_0\}} (f(x, u_n, \lambda)u_n - \mu F(x, u_n, \lambda))\, dx$$
$$\leq M_2(1 + \|\nabla u_n^+\|_p),$$

for all $n \geq 1$, with some $M_2 > 0$, where $M_0 := \max\{M, \|u_0\|_\infty\}$. Combining with $\mathrm{H}(f)_3''$ (iv), it turns out that $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, we may assume that $u_n \xrightarrow{\mathrm{w}} u$ in $W_0^{1,p}(\Omega)$, $u_n \to u$ in $L^r(\Omega)$ and a.e. on Ω for some $u \in W_0^{1,p}(\Omega)$, and $|u_n(x)| \leq k(x)$ for a.a. $x \in \Omega$, for all $n \geq 1$, with $k \in L^r(\Omega)_+$. Since $\langle \overline{\varphi}_+'(u_n), u_n - u\rangle \to 0$, it follows that $\lim_{n\to\infty}\langle -\Delta_p u_n, u_n - u\rangle = 0$. Knowing that $-\Delta_p$ on $W_0^{1,p}(\Omega)$ is an operator of type $(S)_+$, we infer that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Therefore, $\overline{\varphi}_+$ satisfies the (PS) condition. This fact, (7.6) and (7.7) allow to apply the mountain pass theorem. We find a critical point $\hat{u} \in W_0^{1,p}(\Omega)$ of the functional $\overline{\varphi}_+$ satisfying $\hat{u} \neq u_0$. As previously done for $\tilde{u}$, we can show that $\hat{u} \geq u_0$, and so $\hat{u}$ is a solution of (P_λ). Furthermore, from Ladyzhenskaya and Uraltseva [19, Theorem 7.1], we have that $\hat{u} \in L^\infty(\Omega)$ and then applying the regularity result of Lieberman [20], we obtain $\hat{u} \in C_0^1(\overline{\Omega})$. By $\mathrm{H}(f)_1''$(ii)–(iii), there is a constant $c_0 > 0$ such that $\Delta_p \hat{u} = -f(\cdot, \hat{u}, \lambda) \leq c_0 \hat{u}^{p-1}$ a.e. on Ω. Since $\hat{u} \geq u_0 > 0$, by the strong maximum principle of Vázquez [28], we conclude that $\hat{u} \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$. □

More insight in our multiplicity study can be achieved under further conditions on $f(x, s, \lambda)$:

$\mathrm{H}(f)_5''$ $f : \Omega \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, is a function satisfying $\mathrm{H}(f)_1''$ (i)$''$–(ii)$''$, and

(iii) for all $\lambda \in (0, \bar{\lambda})$ there exists $\mu > \lambda_2$ for which we have

$$\mu < \liminf_{s\to 0} \frac{f(x, s, \lambda)}{|s|^{p-2}s} \quad \text{uniformly for a.a. } x \in \Omega;$$

$\mathrm{H}(f)_6''$ $f : \Omega \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, is a function satisfying $\mathrm{H}(f)_5''$, and

(iv) for every $\lambda \in (0, \bar{\lambda})$, there exists $\hat{\eta} = \hat{\eta}(\lambda) \in L^\infty(\Omega)_+$ such that

$$\limsup_{s\to 0} \frac{f(x, s, \lambda)}{|s|^{p-2}s} \leq \hat{\eta}(x) \quad \text{uniformly for a.a. } x \in \Omega.$$

Theorem 4. (a) *Under* $\mathrm{H}(f)''_5$, *for all* $b > 0$ *there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that if* $\lambda \in (0, \lambda^*)$, (P_λ) *has at least three (nontrivial) solutions* $u_0 \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$, $v_0 \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *and* $y_0 \in C_0^1(\overline{\Omega})$ *with negative energy and*

$$-b < v_0 \le y_0 \le u_0 < b\,,\ y_0 \ne 0,\ y_0 \ne u_0,\ y_0 \ne v_0.$$

(b) *Under* $\mathrm{H}(f)''_6$, *for all* $b > 0$ *there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that if* $\lambda \in (0, \lambda^*)$, (P_λ) *has at least three (nontrivial) solutions* $u_0 \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$, $v_0 \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *and* $y_0 \in C_0^1(\overline{\Omega})$ *with the properties in part* (a) *and, in addition,* y_0 *nodal.*

(c) *Under* $\mathrm{H}(f)''_5$ *with* $r < p^*$, $\mathrm{H}(f)''_3(\mathrm{iv})$–$(\mathrm{v})$ *and* $\mathrm{H}(f)''_4(\mathrm{iv})$–$(\mathrm{v})$, *there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that if* $\lambda \in (0, \lambda^*)$, (P_λ) *has at least five (nontrivial) solutions* $u_0, \hat{u} \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$, $v_0, \hat{v} \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *and* $y_0 \in C_0^1(\overline{\Omega})$ *with negative energy and*

$$-b < \hat{v} \le v_0 \le y_0 \le u_0 \le \hat{u} < b,\ y_0 \ne 0,\ y_0 \ne u_0,\ y_0 \ne v_0,\ u_0 \ne \hat{u},\ v_0 \ne \hat{v}.$$

(d) *Under* $\mathrm{H}(f)''_6$ *with* $r < p^*$, $\mathrm{H}(f)''_3(\mathrm{iv})$–$(\mathrm{v})$ *and* $\mathrm{H}(f)''_4(\mathrm{iv})$–$(\mathrm{v})$, *there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that if* $\lambda \in (0, \lambda^*)$, (P_λ) *has at least five (nontrivial) solutions* $u_0, \hat{u} \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$, $v_0, \hat{v} \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *and* $y_0 \in C_0^1(\overline{\Omega})$ *with the properties in part* (c) *and, in addition,* y_0 *nodal.*

Proof. (a) For an arbitrary $b > 0$, consider λ^* given by Proposition 5 (a). Fix $\lambda \in (0, \lambda^*)$. It was shown in Claim 2 in the proof of Proposition 5, that problem (P_λ) admits an upper solution $\overline{u} \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ with $\|\overline{u}\|_\infty < b$ and a lower solution $\underline{v} \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ satisfying $\|\underline{v}\|_\infty < b$. Then, we may invoke Theorem 1 (a) that ensures the conclusion.

(b) We proceed as for (a) by applying Theorem 1 (b) instead of Theorem 1 (a).

(c) Let $b > 0$ denote the minimum between the numbers b in the statements of $\mathrm{H}(f)''_3(\mathrm{v})$ and $\mathrm{H}(f)''_4(\mathrm{v})$. Arguing as in part (a), for all $\lambda \in (0, \lambda^*)$ we get three solutions v_0, y_0, u_0 of problem (P_λ) with the required properties. The reasoning in the proof of Theorem 3 provides two more solutions $\hat{v}$, $\hat{u}$ of opposite constant sign satisfying the conclusion.

(d) The argument is the same as for (c) excepting that we make use of part (b) in place of part (a). □

Remark 5. Let $p < r < p^*$ and assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $g(x, 0) = 0$ a.e. on Ω. Defining $G(x, s) = \int_0^s g(x, \tau)\, d\tau$, we suppose that

(a) there exist $\hat{c}_0 > 0$ and $1 \le q < p$ such that

$$|g(x, s)| \le \hat{c}_0(|s|^{q-1} + 1) \ \text{ for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R}\,;$$

(b) $\liminf_{s\to 0} \frac{g(x,s)}{|s|^{p-2}s} = +\infty$ uniformly for a.a. $x \in \Omega$;

(c) there exist $M_0 > 0$, $\mu \in (\max\{p, \frac{r}{p}\}, r)$, $c_1, c_2 > 0$ and $r_0 \in [0, r)$ such that

$$-c_1|s|^r \leq \mu G(x,s) \leq g(x,s)s + c_2|s|^{r_0} \quad \text{for a.a. } x \in \Omega, \text{ all } |s| \geq M_0\,;$$

(d) $g(x,s)s \geq 0$ for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$.

Set $f(x,s,\lambda) = |s|^{r-2}s + \lambda g(x,s)$, $\lambda \in (0, \max\{\frac{1}{pc_1}, \frac{p}{rc_1}\})$. Then $f(x,s,\lambda)$ satisfies the hypotheses of Theorem 4 (c), which yields five nontrivial solutions, two positive, two negative and one between the maximal negative and the minimal positive, for the corresponding problem (P_λ). A particular case is when $g(x,s) = g(x) = |s|^{q-2}s$, with $1 < q < p$, so $f(x,s,\lambda) = |s|^{r-2}s + \lambda|s|^{q-2}s$ (concave-convex nonlinearity when $q < p = 2$). Theorem 4 (a), (c) extends the work of Ambrosetti, Garcia Azorero and Peral [1] and Garcia Azorero, Peral and Manfredi [14]. Theorem 4 (a) also extends [24], while Theorem 4 (b) extends [8]. Theorem 4 partially generalizes the existence result of Boccardo, Escobedo and Peral [6].

Finally, to obtain an additional nodal solution, we strengthen the assumptions $\mathrm{H}(f)''_6$:

$\mathrm{H}(f)''_7$ $f : \overline{\Omega} \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, is a function such that

(i) for all $\lambda \in (0, \bar{\lambda})$, $f(\cdot, \cdot, \lambda) \in C(\overline{\Omega} \times \mathbb{R})$ and for all $(x, \lambda) \in \overline{\Omega} \times (0, \bar{\lambda})$, $f(x, 0, \lambda) = 0$;

(ii) $\mathrm{H}(f)''_1$ (ii) with $r < p^*$ holds true;

(iii) for every $\lambda \in (0, \bar{\lambda})$, there exist $\mu_0 = \mu_0(\lambda) > \lambda_2$ and $\hat{\eta} = \hat{\eta}(\lambda) \in L^\infty(\Omega)_+$ such that

$$\mu_0 < \liminf_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \leq \limsup_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \leq \hat{\eta}(x),$$

uniformly for a.a. $x \in \Omega$;

(iv) for every $\lambda \in (0, \bar{\lambda})$, there exist $M = M(\lambda) > 0$ and $\mu = \mu(\lambda) > p$ such that

$$0 < \mu F(x,s,\lambda) \leq f(x,s,\lambda)s \quad \text{for all } x \in \Omega, \text{ all } |s| \geq M;$$

(v) there exist $b_- < 0 < b_+$ such that for all $\lambda \in (0, \bar{\lambda})$ we have

$$f(x, b_-, \lambda) = 0 = f(x, b_+, \lambda) \quad \text{for all } x \in \Omega,$$

$$f(x,s,\lambda) < 0 \quad \text{for all } x \in \Omega, \text{ all } s \in (b_-, 0),$$

$$f(x,s,\lambda) > 0 \quad \text{for all } x \in \Omega, \text{ all } s \in (0, b_+).$$

Theorem 5. *If hypotheses* $\mathrm{H}(f)''_7$ *hold, then there exists* $\lambda^* \in (0, \bar{\lambda})$ *such that for every* $\lambda \in (0, \lambda^*)$ *the corresponding problem* (P_λ) *has at least six solutions* $u_0, \hat{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, $v_0, \hat{v} \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and* $y_0, w_0 \in C_0^1(\overline{\Omega})$ *both nontrivial, nodal.*

Proof. The positive number λ^* and the five solutions u_0, $\hat{u}$, v_0, $\hat{v}$, y_0 come from Theorem 4 (d). An additional nodal solution $w_0 \in C_0^1(\overline{\Omega})$, $w_0 \neq 0$, is got from Bartsch, Liu and Weth [5, Theorem 1.1] since our hypotheses are stronger than the required ones therein. Let us now justify that $y_0 \neq w_0$. Towards this, we note that Theorem 4 ensures the a priori estimate $\|y_0\|_\infty < \min\{b_+, |b_-|\}$. On the other hand, according to Bartsch, Liu and Weth [5, Theorem 1.1] we know that $\max_{\overline{\Omega}} w_0 \geq b_+$ and $\min_{\overline{\Omega}} w_0 \leq b_-$. Hence, the solutions y_0 and w_0 of problem (P_λ) are distinct, both nodal. □

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