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ELLIPTIC EQUATIONS WITH DECAYING CYLINDRICAL POTENTIALS AND POWER-TYPE NONLINEARITIES

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Abstract. We obtain existence, nonexistence and asymptotic results for solutions to cylindrical equations of the form:

$$-\triangle u+\frac{A}{|y|^{\alpha}}u=f\left(u\right)\ \text{ in }\mathbb{R}^{N},\ x=(y,z)\in\mathbb{R}^{k}\times\mathbb{R}^{N-k},\ N>k\geq 2,$$

where $A,\alpha>0$ and f is continuous and satisfies power-type growth conditions.

1. Introduction and main results

In the recent mathematical literature a great deal of work has been devoted to the study of non-linear elliptic equations of the form

$$-\triangle u+V\left(x\right)u=f\left(x,u\right)\quad\text{in }\mathbb{R}^{N} \tag{1.1}$$

with $\liminf_{|x|\to\infty}V\left(x\right)=0$ (see e.g. [1]-[13], [17], [20]-[24], [29], [33], [36], [37]). The main motivation of this study is the fact that such equations arise in the search for solitary waves of nonlinear evolution equations of the Schrödinger or Klein-Gordon type (cf. [6], [11], [32], [35]). Roughly speaking, a solitary wave is a nonsingular solution which travels as a localized packet in such a way that the physical quantities corresponding to the invariances of the equation are finite and conserved in time. Accordingly, a solitary wave preserves intrinsic properties of particles such as the energy, the angular momentum and the charge, whose finiteness is strictly related to the finiteness of the L^2 norm (cf. Remark 6 below). In particular, equations of type (1.1) with $V\left(x\right)=A\left|y\right|^{-2}$, $x=(y,z)\in\mathbb{R}^{2}\times\mathbb{R}$, $A>0$, give rise to solitary waves with nonvanishing angular momentum (see [7], [8]). Owing to their particle-like behavior, solitary waves can be regarded as a model for

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extended particles and they arise in many problems of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics and plasma physics (see, for instance, [31], [40]).

In this paper we consider semilinear elliptic scalar equations in $\mathbb{R}^N$ featuring autonomous power-type nonlinearities and singular cylindrical potentials decaying to zero at infinity.

Letting $N>k\geq 2$, $p\geq 2$ and $A,\alpha>0$, the model problem for our results is

$$\begin{cases} -\triangle u+\dfrac{A}{|y|^{\alpha}}u=|u|^{p-2}\,u \\ u\in X(\mathbb{R}^N,|y|^{-\alpha}\,dx)\cap L^p(\mathbb{R}^N) \end{cases} \tag{1.2}$$

with $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$ and

$$X(\mathbb{R}^N,|y|^{-\alpha}\,dx):=D^{1,2}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,|y|^{-\alpha}\,dx)\,, \tag{1.3}$$

where

$$D^{1,2}(\mathbb{R}^N)=\left\{u\in L^{2^*}(\mathbb{R}^N):\nabla u\in L^2(\mathbb{R}^N)\right\},\quad 2^*:=\frac{2N}{N-2}\,,$$

is the usual completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the L^2 norm of the gradient and $L^2(\mathbb{R}^N,|y|^{-\alpha}\,dx)$ denotes the Lebesgue space of square integrable functions with respect to the measure $|y|^{-\alpha}\,dx$.

Actually, we also consider more general nonlinearities $f\in C(\mathbb{R},\mathbb{R})$ satisfying

($\mathbf{f}_p$) *$\exists M>0$ such that $|f(s)|\leq M\,|s|^{p-1}$ for all $s\in\mathbb{R}$*

and refer to the notion of solution given by the following definition: we say that $u\in X(\mathbb{R}^N,|y|^{-\alpha}\,dx)$ is a *weak solution* to the equation

$$-\triangle u+\frac{A}{|y|^{\alpha}}u=f(u)\quad \text{in } \mathbb{R}^k\times\mathbb{R}^{N-k} \tag{1.4}$$

if and only if for all $h\in X(\mathbb{R}^N,|y|^{-\alpha}\,dx)$ one has

$$\int_{\mathbb{R}^N}\nabla u\cdot\nabla h\,dx+\int_{\mathbb{R}^N}\frac{A}{|y|^{\alpha}}uh\,dx=\int_{\mathbb{R}^N}f(u)\,h\,dx\,. \tag{1.5}$$

Our results mainly concern existence, nonexistence and asymptotic behavior of weak solutions to equations (1.2) and (1.4), and they rely on compatibility conditions between the exponents α and p.

The radial case of problem (1.2), namely

$$-\triangle u+\frac{A}{|x|^{\alpha}}u=|u|^{p-2}\,u,\qquad u\in X(\mathbb{R}^N,|x|^{-\alpha}\,dx)\cap L^p(\mathbb{R}^N) \tag{1.6}$$

(with $N \geq 3$, $p \geq 2$ and $A, \alpha > 0$), has been studied by several authors (see [10], [18], [36], [37], [38]). In particular Terracini proves in [38] that no positive solution is allowed either for $p \neq 2^*$ and $\alpha = 2$, or for $p = 2^*$ and $\alpha \neq 2$. Indeed, as it is shown in [10], problem (1.6) turns out to have no solution at all whenever the pair (α, p) belongs to the light gray region of the picture below (including boundaries, except for the pair $(\alpha, p) = (2, 2^*)$ and the line $\alpha = 0$), defined as $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where

$$\mathcal{A}_1 := \left\{(\alpha, p) \in \mathbb{R}^2 : \alpha \in (0, 2]\,,\, p \notin (2_\alpha, 2^*)\,,\, p \geq 2\right\} \setminus \{(2, 2^*)\} \tag{1.7}$$

$$\mathcal{A}_2 := \left\{(\alpha, p) \in \mathbb{R}^2 : \alpha \in (2, N)\,,\, p \notin (2^*, 2_\alpha)\,,\, p \geq 2\right\} \tag{1.8}$$

$$\mathcal{A}_3 := \left\{(\alpha, p) \in \mathbb{R}^2 : \alpha \in [N, +\infty)\,,\, p \in [2, 2^*]\right\} \tag{1.9}$$

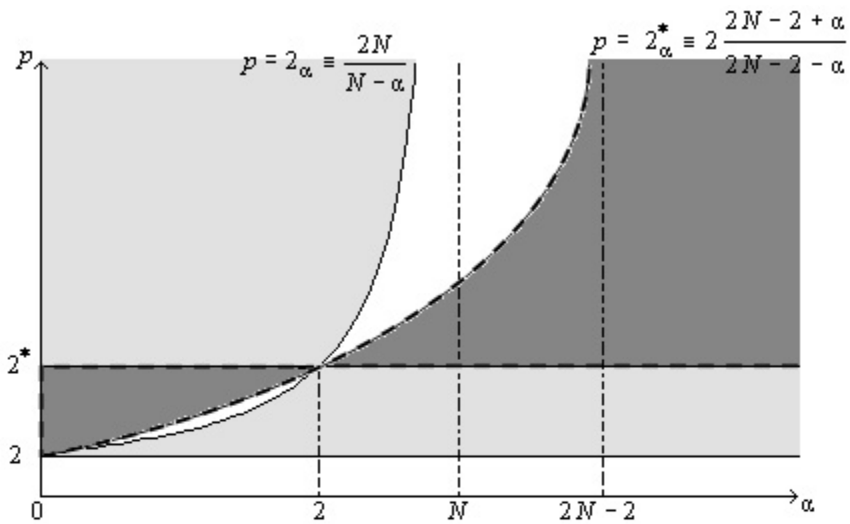
and

$$2_\alpha := \frac{2N}{N - \alpha} \quad \text{for every } \alpha \in (0, N)\ . \tag{1.10}$$

On the other hand, all the positive radial solutions of (1.6) are explicitly known for $(\alpha, p) = (2, 2^*)$ (see [38] again), while existence and multiplicity of radial solutions follow from the results of [36] under the compatibility condition corresponding to the dark gray regions between the lines $p = 2^*$ and

$$p = 2^*_\alpha := 2\frac{2N - 2 + \alpha}{2N - 2 - \alpha} = 2 + \frac{4\alpha}{2N - 2 - \alpha} \tag{1.11}$$

(boundaries excluded), that is, $\alpha \in (0, 2)$ and $p \in (2^*_\alpha, 2^*)$, or $\alpha \in (2, 2N - 2)$ and $p \in (2^*, 2^*_\alpha)$, or $\alpha \geq 2N - 2$ and $p > 2^*$.



Existence for (α, p) in the white region of the picture above is still an open problem.

The strictly cylindrical problem (1.2) has been very much less studied and no existence result is available in the literature, at least at our knowledge. The only result we know is due to Musina [29], who shows that, for $\alpha = 2$ and $p = 2^*$, problem (1.2) does not admit ground state solutions, namely, the infimum

$$S(N,k,A) := \inf_{u \in X(\mathbb{R}^N, |y|^{-2}dx)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + A \int_{\mathbb{R}^N} \frac{u^2}{|y|^2} dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \, dx\right)^{2/2^*}}$$

is not achieved (for $A > 0$ and $N > k \geq 2$). In the following theorem we complete such a result by establishing that an higher level solution actually exists (see also Remark 15).

Theorem 1. *Let $N > k \geq 2$ and $A > 0$. Then the equation*

$$-\triangle u + \frac{A}{|y|^2} u = |u|^{2^*-2} u \quad \text{in } \mathbb{R}^k \times \mathbb{R}^{N-k} \tag{1.12}$$

has at least a nonzero nonnegative weak solution, satisfying

$$u(y,z) = u(|y|, z).$$

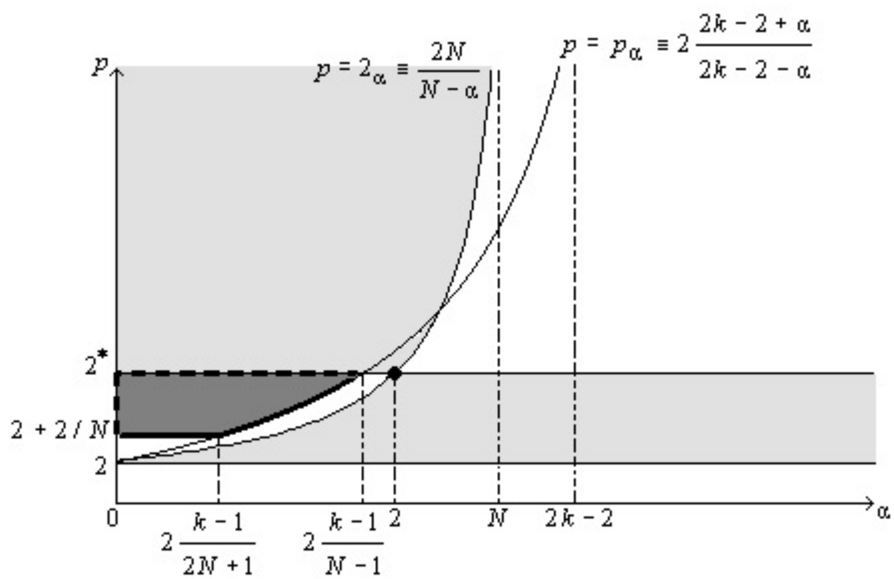
Writing $u(y,z) = u(|y|, z)$ we naturally mean $u(y,z) = u(gy, z)$ for all $g \in O(k)$ and almost every $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

The next theorem concerns the (strictly cylindrical) case of equation (1.4) with $\alpha < 2$. For $N > k \geq 2$, define

$$p_\alpha := 2\frac{2k-2+\alpha}{2k-2-\alpha} = 2 + \frac{4\alpha}{2k-2-\alpha} \quad \text{for every } \alpha \in (0, 2k-2) \tag{1.13}$$

and

$$\bar{p}_\alpha := \begin{cases} 2 + \frac{2}{N} & \text{if } \ 0 < \alpha \leq 2\frac{k-1}{2N+1} \\ p_\alpha & \text{if } \ 2\frac{k-1}{2N+1} \leq \alpha < 2\frac{k-1}{N-1} \, . \end{cases} \tag{1.14}$$



Observe that $\bar{p}_\alpha < 2^*$ and $2(k-1)/(N-1) < 2$. Let us also point out that the picture is drawn for $N < 2k-2$ but the case $N \geq 2k-2$ can also occur.

Theorem 2. *Let $N \geq k+3 \geq 5$, $A > 0$ and $0 < \alpha < 2(k-1)/(N-1)$. Assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$ together with the following conditions:*

($\mathbf{f}_p'$) *$\exists M' > 0$ and $\exists p \in [\bar{p}_\alpha, 2^*)$ such that $|f'(s)| \leq M' |s|^{p-2}$ for all $s \in \mathbb{R}$*

($\mathbf{f}$) *$\exists \vartheta > 2$ such that $\vartheta F(s) \leq f(s)s < f'(s)s^2$ for all $s \in \mathbb{R} \setminus \{0\}$*

($\mathbf{F}$) *$F \geq 0$ on $(0, +\infty)$ and $\exists \bar{s} > 0$ such that $F(s) > 0$ for all $s \geq \bar{s}$,*

where $F(s) := \int_0^s f(t)\,dt$. Then equation (1.4) has at least a nonzero nonnegative weak solution, which belongs to $L^q(\mathbb{R}^N)$ for all $q \in [\bar{p}_\alpha, 2^*]$, satisfies $u(y,z) = u(|y|, |z|)$ and is nonincreasing with respect to z.

Notice that assumption ($\mathbf{f}_p'$) together with $f(0) = 0$ implies condition ($\mathbf{f}_p$) (with the same exponent p). On the other hand, Theorem 2 applies to problem (1.2) and provides solutions for the pairs (α, p) belonging to the dark gray region of the picture above. As before, $u(y,z) = u(|y|, |z|)$ means $u(y,z) = u(g_1 y, g_2 z)$ for all $(g_1, g_2) \in O(k) \times O(N-k)$ and almost every $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Then the request of nonincreasing with respect to z asks for the following condition:

$$|z_1| \leq |z_2| \Rightarrow u(y, z_1) \geq u(y, z_2) \geq 0 \quad \text{for a.e. } (y, (z_1, z_2)) \in \mathbb{R}^k \times \mathbb{R}^{2(N-k)}.$$

Note that this definition requires nonnegativity.

Remark 3. All the hypotheses of Theorem 2 still hold true if one replaces $f(s)$ with $f(|s|)\,s/|s|$. Thus, as we are interested in nonnegative solutions, Theorem 2 also works if assumptions $(\mathbf{f}_p')$ and $(\mathbf{f})$ only hold for $s>0$.

The existence of solutions to problem (1.2) for pairs (α,p) in the white region of the picture above is an open issue, but the following result shows that, at least under an additional assumption on the summability of the gradient, no solution is allowed if (α,p) is in the same region of nonexistence for the radial problem (1.6), namely $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3$ defined by (1.7)-(1.10). For $N>k\geq 2$ and $\alpha>0$ we set

$$q_k:=\frac{2k}{k-1}\,,\quad q_k^\alpha:=\frac{2k}{k-2+\alpha}\,,\quad q_{k,N}:=\frac{2kN}{(k-2)\,N+2k}\,.$$

Theorem 4. *Let $N>k\geq 2$ and $A>0$. If $(\alpha,p)\in\mathcal{A}$, then the equation*

$$-\triangle u+\frac{A}{|y|^\alpha}u=|u|^{p-2}\,u\quad in\ (\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k}$$

has no nonzero classical solution $u\in C^2((\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k})\cap X(\mathbb{R}^N,|y|^{-\alpha}dx)$ $\cap L^p(\mathbb{R}^N)$ satisfying

$$\nabla u\in L^q_{loc}(\mathbb{R}^N)\quad for\ q=\max\{q_k,q_k^\alpha\}\ or\ q=\max\{q_k,q_{k,N}\}.$$

We observe that $q_k=\max\{q_k,q_k^\alpha\}$ if $\alpha\geq 1$ and $q_k=\max\{q_k,q_{k,N}\}$ if $2k/N\geq 1$, while $q_k^\alpha\geq q_{k,N}>q_k$ if $\alpha\leq 2k/N<1$ and $q_{k,N}\geq q_k^\alpha>q_k$ if $2k/N\leq\alpha<1$. In Theorem 35 of Section 5 we also give a nonexistence result for the case of equation (1.4) with nonlinearities different from a pure power.

From the results of [7, Propositions 5 and 6] we know that all nonnegative solutions of equation (1.12), $N>k\geq 2$, $A>0$, are bounded on $\mathbb{R}^N$ (which is not obvious owing to the singularity of the potential) and satisfy

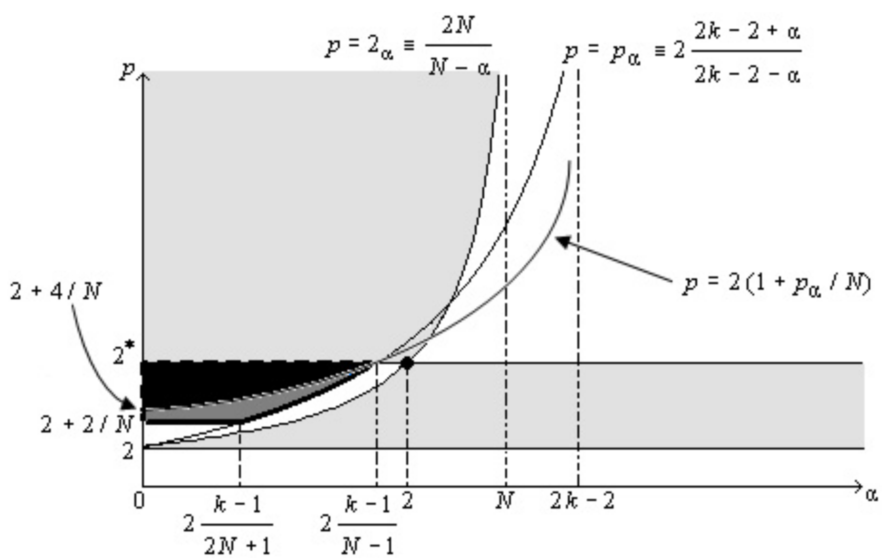
$$\limsup_{|x|\to\infty}|x|^\nu\,u(x)<\infty\quad\text{for every }\nu<\frac{N-2}{2}+\sqrt{\left(\frac{N-2}{2}\right)^2+A}\qquad(1.15)$$

(to be precise, this is proved in [7] for $A=1$, but the argument works for any positive A). In the following theorem we show that the presence of the slower decaying potential $|y|^{-\alpha}$ with $\alpha<2$ forces the solutions found in Theorem 2 to decay exponentially fast at infinity. This kind of inverse relation between the decaying rate of potentials and solutions is a general phenomenon and it is studied in a forthcoming paper [25], where larger classes of potentials and nonlinearities are considered.

Theorem 5. *Let $N > k \geq 2$, $A > 0$ and $0 < \alpha < 2(k-1)/(N-1)$. Assume that $f \in C(\mathbb{R},\mathbb{R})$ satisfies $(\mathbf{f}_p)$ for some $p \in (2 + 2p_\alpha/N, 2^*)$. Then any z-nonincreasing weak solution $u(y,z) = u(|y|,|z|)$ of equation (1.4) is bounded on $\mathbb{R}^N$ and satisfies*

$$\limsup_{|x|\to\infty} e^{\beta|x|^{1-\alpha/2}} u(x) < \infty \qquad \text{for every } \beta < \frac{2\sqrt{A}}{2-\alpha}. \tag{1.16}$$

Observe that, with respect to Theorem 2, Theorem 5 also works in lower dimensions (for which existence is still an open question) but shrinks the range of the admissible exponents p (see picture below).



Remark 6. Thanks to Theorem 5 and the already mentioned results of [7] leading to (1.15), the solutions we find in Theorem 1 (provided that $A > 3/4$ if $N = 3$) and Theorem 2 belong to $H^1(\mathbb{R}^N)$.

2. Preliminaries and notations

Let $N, k \in \mathbb{N}$ be such that $N > k \geq 2$ and let $A, \alpha > 0$. As in (1.3), we define the weighted Sobolev space

$$X := X(\mathbb{R}^N, |y|^{-\alpha}\, dx) := \left\{ u \in D^{1,2}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} \frac{u^2}{|y|^\alpha} dx < +\infty \right\},$$

which is a Hilbert space with respect to the norm defined by

$$\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + A \int_{\mathbb{R}^N} \frac{u^2}{|y|^\alpha} dx \qquad \text{for all } u \in X \,, \tag{2.1}$$

whose inner product we denote by

$$(u \mid v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + A \int_{\mathbb{R}^N} \frac{uv}{|y|^\alpha} dx \qquad \text{for all } u, v \in X \,.$$

Clearly, $X \hookrightarrow D^{1,2}(\mathbb{R}^N)$, whence, by well known embeddings of $D^{1,2}(\mathbb{R}^N)$, one derives $X \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $X \hookrightarrow L^p_{loc}(\mathbb{R}^N)$ for $1 \leq p \leq 2^*$. In particular, the latter embedding is compact if $p < 2^*$ and thus it assures that weak convergence in X implies (up to a subsequence) almost everywhere convergence in $\mathbb{R}^N$.

Of main interest in the following are the subspaces of symmetric functions

$$X_{\mathrm{r}} := X_{\mathrm{r}}(\mathbb{R}^N, |y|^{-\alpha} \, dx) := \{u \in X : u\,(y,z) = u\,(|y|\,, z)\} \tag{2.2}$$

$$X_{\mathrm{s}} := X_{\mathrm{s}}(\mathbb{R}^N, |y|^{-\alpha} \, dx) := \{u \in X : u\,(y,z) = u\,(|y|\,, |z|)\} \,. \tag{2.3}$$

By almost everywhere convergence (up to a subsequence) of X-converging sequences, X_{r} and X_{s} are closed in X and thus they are Hilbert spaces with respect to the same norm (2.1) of X.

For any $f \in C\,(\mathbb{R}, \mathbb{R})$ satisfying $(\mathbf{f}_p)$ for some $p > 2$, we set

$$I\,(u) := \frac{1}{2}\,\|u\|^2 - \int_{\mathbb{R}^N} F\,(u)\, dx, \tag{2.4}$$

where $F\,(s) := \int_0^s f\,(t)\, dt$. Thanks to condition $(\mathbf{f}_p)$, this defines a real functional $I : X \cap L^p(\mathbb{R}^N) \to \mathbb{R}$ on the Banach space $X \cap L^p(\mathbb{R}^N)$ equipped with the norm $\|\cdot\| + \|\cdot\|_{L^p(\mathbb{R}^N)}$. Moreover, by standard computations, I is of class C^1 and has Fréchet derivative $I'\,(u)$ at any $u \in X \cap L^p(\mathbb{R}^N)$ given by

$$I'\,(u)\, h := (u \mid h) - \int_{\mathbb{R}^N} f\,(u)\, h \, dx \qquad \text{for all } h \in X \cap L^p\left(\mathbb{R}^N\right) \,. \tag{2.5}$$

Notice that I is the Euler functional associated to equation (1.4) but its critical points are not weak solutions in general, since they do not necessarily satisfy (1.5) for all $h \in X$.

We conclude this preliminary section by summarizing the notations of most frequent use in the following.

Notations.

- Given $N, k \in \mathbb{N}$, we shall always write $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

- Given $f \in C(\mathbb{R}, \mathbb{R})$, we always denote $F(s) := \int_0^s f(t)\,dt$ for all $s \in \mathbb{R}$.
- $\mathbb{N}$ is the set of natural numbers, including 0.
- For any $r \in \mathbb{R}$ we set $r_+ := (|r| + r)/2$ and $r_- := (|r| - r)/2$, so that $r = r_+ - r_-$ with $r_+, r_- \geq 0$.
- The open ball $B_r(\xi_0) := \left\{\xi \in \mathbb{R}^d : |\xi - \xi_0| < r\right\}$ shall be simply denoted by B_r when $\xi_0 = 0$. $\bar{B}_r$ stands for the closure of B_r and σ_d denotes the $(d-1)$-dimensional measure of the unit sphere ∂B_r of $\mathbb{R}^d$.
- $|A|$ and χ_A respectively denote the d-dimensional Lebesgue measure and the characteristic function of any measurable set $A \subseteq \mathbb{R}^d$, $d \geq 1$. We set $A^c := \mathbb{R}^d \setminus A$.
- $O(d)$ is the orthogonal group of $\mathbb{R}^d$.
- By $\to$ and $\rightharpoonup$ we respectively mean *strong* and *weak* convergence in a Banach space E, whose dual space is denoted by E'.
- $\hookrightarrow$ denotes *continuous* embeddings.
- $C_c^\infty(\Omega)$ is the space of the infinitely differentiable real functions with compact support in the open set $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$.
- For any open set $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, $D_0^{1,2}(\Omega)$ denotes the usual Sobolev space defined as the closure of $C_c^\infty(\Omega)$ in $D^{1,2}(\Omega) = \{u \in L^{2^*}(\Omega) : \nabla u \in L^2(\Omega)\}$ with respect to the norm $\|u\|_{L^{2^*}(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$. Recall that $\|\nabla u\|_{L^2(\Omega)}$ is an equivalent norm on $D_0^{1,2}(\Omega)$.

3. Existence for the critical problem

In this section we focus on equation (1.2) in the critical case considered in Theorem 1, namely

$$-\triangle u + \frac{A}{|y|^2} u = |u|^{2^*-2} u \quad \text{in } \mathbb{R}^k \times \mathbb{R}^{N-k}. \tag{3.1}$$

As we are interested in nonnegative solutions, we set

$$f(t) := \chi_{(0,+\infty)}(t)\,|t|^{2^*-1} \qquad \text{for all } t \in \mathbb{R} \tag{3.2}$$

and look at (3.1) as the Euler-Lagrange equation of the corresponding functional (2.4), that is,

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{A}{2}\int_{\mathbb{R}^N} \frac{u^2}{|y|^2}dx - \int_{\mathbb{R}^N} F(u)\,dx,$$

where

$$F(t) = \frac{1}{2^*}\chi_{(0,+\infty)}(t)\,|t|^{2^*} \qquad \text{for all } t \in \mathbb{R}\,. \tag{3.3}$$

By the continuous embedding $X = X(\mathbb{R}^N, |y|^{-2}\,dx) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, this functional belongs to $C^1(X,\mathbb{R})$ and the variational approach to equation (3.1) is quite standard (see Lemma 10 below). The only difficulty is that we cannot guarantee the fulfilment of the Palais-Smale condition, owing to the lack of compactness due to z-translation invariance. Such a difficulty will be faced in Subsection 3.1 with the aid of a version of the Concentration-Compactness Principle, due to S. Solimini [34].

In order to state and use the result of Solimini, we have to preliminarily introduce a group of rescaling operators, of which we also remark some basic properties.

Definition 7. *Fix $\lambda > 0$ and $x \in \mathbb{R}^N$. For any $u \in L^p(\mathbb{R}^N)$ with $1 < p < \infty$ we define*

$$T_{\lambda,x}u := \lambda^{-(N-2)/2} u\left(\lambda^{-1}\cdot + x\right).$$

Clearly, $T_{\lambda,x}u \in L^p(\mathbb{R}^N)$ for all $u \in L^p(\mathbb{R}^N)$ and in particular $T_{\lambda,x}u \in D^{1,2}(\mathbb{R}^N)$ if $u \in D^{1,2}(\mathbb{R}^N)$. Moreover, by direct computations, it is easy to see that the linear operators $u \in L^{2^*}(\mathbb{R}^N) \mapsto T_{\lambda,x}u \in L^{2^*}(\mathbb{R}^N)$ and $u \in D^{1,2}(\mathbb{R}^N) \mapsto T_{\lambda,x}u \in D^{1,2}(\mathbb{R}^N)$ are isometric. Notice that

$$T_{\lambda,x}^{-1} = T_{1/\lambda,-\lambda x} \quad \text{and} \quad T_{\lambda_1,x_1}T_{\lambda_2,x_2} = T_{\lambda_1\lambda_2,x_1/\lambda_2+x_2}\,. \tag{3.4}$$

Similarly, for any $\tilde{z} = (0,z) \in \mathbb{R}^N$ and $\lambda > 0$, one plainly deduces that the linear operator $u \mapsto T_{\lambda,\tilde{z}}u$ is an isometry of both X and the symmetric subspace $X_{\mathrm{r}} = X_{\mathrm{r}}(\mathbb{R}^N, |y|^{-2}\,dx)$ introduced in (2.2).

The following proposition is proved in [7].

Proposition 8. *Let $1 < p < \infty$ and assume that $\{\lambda_n\} \subset (0,+\infty)$ and $\{x_n\} \subset \mathbb{R}^N$ are such that $\lambda_n \to \lambda \neq 0$ and $x_n \to x$. Then $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^N)$ implies $T_{\lambda_n,x_n}u_n \rightharpoonup T_{\lambda,x}u$ in $L^p(\mathbb{R}^N)$.*

We now recall the above mentioned result of Solimini [34].

Theorem 9. *If $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ is bounded, then, up to a subsequence, either $u_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$ or there exist $\{\lambda_n\} \subset (0,+\infty)$ and $\{x_n\} \subset \mathbb{R}^N$ such that $T_{\lambda_n,x_n}u_n \rightharpoonup u$ in $L^{2^*}(\mathbb{R}^N)$ and $u \neq 0$.*

3.1. **Proof of Theorem 1 .** Here we assume $N > k \geq 2$ and $A > 0$, and we give the proof of Theorem 1, which will be achieved through several lemmas.

In the first one we collect some standard but useful properties of I on X_{r}, which derive from the Principle of Symmetric Criticality [30], the Sobolev embedding and other well known arguments (a detailed proof of *iv* can be found for instance in [7]).

Lemma 10. *i) Every critical point of $I_{|X_{\mathrm{r}}}$ is nonnegative and weakly solves equation* (3.1).

ii) *$I_{|X_{\mathrm{r}}}$ has a mountain-pass geometry, namely, there exist $\rho > 0$ and $\bar{u} \in X_{\mathrm{r}}$ with $\|\bar{u}\| > \rho$ such that*

$$\min_{u \in X_{\mathrm{r}}, \|u\| \leq \rho} I(u) = I(0) = 0, \qquad \inf_{u \in X_{\mathrm{r}}, \|u\| = \rho} I(u) > 0 \quad \text{and} \quad I(\bar{u}) < 0\,.$$

iii) *Every Palais-Smale sequence for $I_{|X_{\mathrm{r}}}$ is bounded in X_{r}.*

iv) *For any $h \in X_{\mathrm{r}}$ the mapping $I'(\cdot)\, h : X_{\mathrm{r}} \to \mathbb{R}$ is sequentially weakly continuous.*

In order to prove Theorem 1 and according to Lemma 10.*i*, we look for nonzero critical points of $I_{|X_{\mathrm{r}}}$.

The starting point is the bounded Palais-Smale sequence $\{v_n\} \subset X_{\mathrm{r}}$ provided by Lemma 10.*ii-iii* (see [39, Theorem 2.9]), which is such that $I(v_n) \to c > 0$ and $I'(v_n) \to 0$ in X_{r}', where c is the mountain-pass level of $I_{|X_{\mathrm{r}}}$ defined by

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \ \Gamma := \{\gamma \in C([0,1], X_{\mathrm{r}}) : \gamma(0) = 0, I(\gamma(1)) < 0\}\,. \tag{3.5}$$

As $\{v_n\}$ is bounded, it must satisfy one of the alternatives allowed by Theorem 9. The next lemma shows that the first one cannot occur.

Lemma 11. *The sequence $\{v_n\}$ does not tend to zero in $L^{2^*}(\mathbb{R}^N)$.*

Proof. For sake of contradiction, assume that $v_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$, so that (3.2) and (3.3) imply

$$\int_{\mathbb{R}^N} |f(v_n)\, v_n|\, dx \leq \|v_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \to 0$$

and

$$0 \leq \int_{\mathbb{R}^N} F(v_n)\, dx \leq \frac{1}{2^*} \|v_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \to 0\,.$$

Therefore, since $I'(v_n) \to 0$ in X_{r}' and $\{v_n\}$ is bounded in X_{r}, we conclude

$$\|v_n\|^2 = I'(v_n)\, v_n + \int_{\mathbb{R}^N} f(v_n)\, v_n dx \to 0$$

and thus

$$I(v_n) = \frac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} F(v_n)\, dx \to 0\,,$$

which contradicts $I(v_n) \to c > 0$. □

Thanks to Lemma 11, Theorem 9 implies that there exist $\{\lambda_n\} \subset (0, +\infty)$, $\{x_n\} \subset \mathbb{R}^N$ and $v \in L^{2^*}(\mathbb{R}^N)$ such that (up to a subsequence)

$$T_{\lambda_n, x_n} v_n \rightharpoonup v \text{ in } L^{2^*}\left(\mathbb{R}^N\right) \quad \text{and} \quad v \neq 0\,.$$

Letting $x_n =: (y_n, z_n)$, $\tilde{y}_n := (y_n, 0)$ and $\tilde{z}_n := (0, z_n)$, so that $x_n = \tilde{y}_n + \tilde{z}_n$, we now define

$$u_n := T_{\lambda_n, \tilde{z}_n} v_n$$

and exploit the invariances of equation (3.1) to get the following lemma.

Lemma 12. *The sequence $\{u_n\}$ is bounded in X_{r} and satisfies*

$$I(u_n) \to c, \quad I'(u_n) \to 0 \textit{ in } X_{\mathrm{r}}' \quad \textit{and} \quad u_n(\,\cdot + \lambda_n \tilde{y}_n) \rightharpoonup v \textit{ in } L^{2^*}\left(\mathbb{R}^N\right).$$

Proof. Since the operators $T_{\lambda_n, \tilde{z}_n}$ are isometrics of X_{r}, $\{u_n\} \subset X_{\mathrm{r}}$ is bounded because $\{v_n\} \subset X_{\mathrm{r}}$ is bounded. Moreover, recalling (3.4), we have

$$T_{1, \lambda_n \tilde{y}_n} u_n = T_{1, \lambda_n \tilde{y}_n} T_{\lambda_n, \tilde{z}_n} v_n = T_{\lambda_n, x_n} v_n \rightharpoonup v \text{ in } L^{2^*}\left(\mathbb{R}^N\right).$$

Finally, since $f(\lambda t) = \lambda^{2^*-1} f(t)$ for all $t \in \mathbb{R}$ and $\lambda > 0$, by easy computation we obtain $I(u_n) = I(v_n)$ and

$$\begin{aligned}
I'(u_n)\, h &= (u_n \mid h) - \int_{\mathbb{R}^N} f(u_n)\, h\, dx \\
&= \lambda_n^{-\frac{N-2}{2}} \left(v_n\left(\lambda_n^{-1} \cdot + \tilde{z}_n\right) \mid h\right) - \lambda_n^{-\frac{N-2}{2}(2^*-1)} \int_{\mathbb{R}^N} f\left(v_n\left(\lambda_n^{-1} \cdot + \tilde{z}_n\right)\right) h\, dx \\
&= \lambda_n^{-\frac{N-2}{2}} \lambda_n^{N-2} \left(v_n \mid h(\lambda_n \cdot - \lambda_n \tilde{z}_n)\right) \\
&\quad - \lambda_n^{-\frac{N-2}{2}(2^*-1)} \lambda_n^N \int_{\mathbb{R}^N} f(v_n)\, h(\lambda_n \cdot - \lambda_n \tilde{z}_n)\, dx \\
&= \left(v_n \mid \lambda_n^{\frac{N-2}{2}} h(\lambda_n \cdot - \lambda_n \tilde{z}_n)\right) - \int_{\mathbb{R}^N} f(v_n)\, \lambda_n^{\frac{N-2}{2}} h(\lambda_n \cdot - \lambda_n \tilde{z}_n)\, dx \\
&= \left(v_n \mid T_{\lambda_n^{-1}, -\lambda_n \tilde{z}_n} h\right) - \int_{\mathbb{R}^N} f(v_n)\, T_{\lambda_n^{-1}, -\lambda_n \tilde{z}_n} h\, dx
\end{aligned}$$

$= I'(v_n)\, T_{\lambda_n^{-1}, -\lambda_n \tilde{z}_n} h$

for all $h \in X_{\mathrm{r}}$, which implies $\|I'(u_n)\|_{X_{\mathrm{r}}'} = \|I'(v_n)\|_{X_{\mathrm{r}}'}$ again by the fact that $T_{\lambda_n^{-1}, -\lambda_n \tilde{z}_n}$ is an isometry of X_{r}. □

The key step in the proof of Theorem 1 is the removal of translations from the sequence $u_n(\,\cdot\, + \lambda_n \tilde{y}_n)$. This is the topic of Lemma 14, where we will take advantage of the following elementary proposition (see [7] for a proof).

Proposition 13. *Let $\{\eta_n\} \subset \mathbb{R}^k$ be such that $\lim_{n\to\infty} |\eta_n| = +\infty$ and fix $R > 0$. Then for any $m \in \mathbb{N} \setminus \{0, 1\}$ there exists $n_m \in \mathbb{N}$ such that for any $n > n_m$ one can find $g_1, ..., g_m \in O(k)$ satisfying the condition: $i \neq j \Rightarrow B_R(g_i \eta_n) \cap B_R(g_j \eta_n) = \varnothing$.*

Lemma 14. *There exists $u \in X_{\mathrm{r}}$, $u \neq 0$, such that (up to a subsequence) $u_n \rightharpoonup u$ in X_{r}.*

Proof. As $\{u_n\} \subset X_{\mathrm{r}}$ is bounded (Lemma 12), we can assume that (up to a subsequence) $u_n \rightharpoonup u$ in X_{r}. If $u \neq 0$ the proof is complete. We are now going to show by contradiction that $u = 0$ is impossible. So, assume

$$u_n \rightharpoonup 0 \quad \text{in } X_{\mathrm{r}}\,. \tag{3.6}$$

Letting $\tilde{T}_n := T_{1, \lambda_n \tilde{y}_n}$ for brevity, we recall from Lemma 12 that $\tilde{T}_n u_n \rightharpoonup v \neq 0$ in $L^{2^*}(\mathbb{R}^N)$. First, we deduce that

$$\lim_{n\to\infty} |\lambda_n \tilde{y}_n| = +\infty\,. \tag{3.7}$$

Otherwise, up to a subsequence, $\lambda_n \tilde{y}_n \to \tilde{y}_0 \in \mathbb{R}^k \times \{0\}$ and $T_{1, -\lambda_n \tilde{y}_n} \tilde{T}_n u_n \rightharpoonup T_{1, -\tilde{y}_0} v$ in $L^{2^*}(\mathbb{R}^N)$ by Proposition 8, but, since $T_{1, -\lambda_n \tilde{y}_n} \tilde{T}_n = T_{1,0}$, this means $u_n \rightharpoonup T_{1, -\tilde{y}_0} v \neq 0$ in $L^{2^*}(\mathbb{R}^N)$ and thus it contradicts (3.6). Since $v \neq 0$, there exist $\delta > 0$ and $D \subseteq \mathbb{R}^N$ with $|D| \neq 0$ such that either $v > \delta$ or $v < -\delta$ almost everywhere in D. Fixing $R > 0$ such that $|B_R \cap D| > 0$, by weak convergence we obtain

$$\Big| \int_{\mathbb{R}^N} \tilde{T}_n u_n \chi_{B_R \cap D} dx \Big| \to \Big| \int_{\mathbb{R}^N} v \chi_{B_R \cap D} dx \Big| \geq \delta\, |B_R \cap D| > 0\,. \tag{3.8}$$

On the other hand

$$\begin{aligned} \Big| \int_{\mathbb{R}^N} \tilde{T}_n u_n \chi_{B_R \cap D} dx \Big| &\leq \int_{B_R} |\tilde{T}_n u_n| dx = \int_{B_R} |u_n(\,\cdot\, + \lambda_n \tilde{y}_n)|\, dx \\ &= \int_{B_R(\lambda_n \tilde{y}_n)} |u_n|\, dx \leq C \Big(\int_{B_R(\lambda_n \tilde{y}_n)} |u_n|^{2^*}\, dx \Big)^{1/2^*}, \end{aligned} \tag{3.9}$$

where $C > 0$ only depends on R and N. From (3.8) and (3.9) we now deduce that

$$\liminf_{n\to\infty} \int_{B_R(\lambda_n \tilde{y}_n)} |u_n|^{2^*} dx > 0$$

and hence, up to a subsequence, we can assume

$$\inf_n \int_{B_R(\lambda_n \tilde{y}_n)} |u_n|^{2^*} dx > \varepsilon \quad \text{for some } \varepsilon > 0\,. \tag{3.10}$$

This will yield a contradiction. Indeed, using (3.7), from Proposition 13 it readily follows that for every $m \in \mathbb{N} \setminus \{0,1\}$ there exists $n_m \in \mathbb{N}$ such that for any $n > n_m$ one can find $g_1, ..., g_m \in O(k)$ satisfying the condition $i \neq j \Rightarrow B_R(\lambda_n (g_i y_n, 0)) \cap B_R(\lambda_n (g_j y_n, 0)) = \varnothing$. As a consequence, using (3.10) and the fact that $u_n \in X_{\mathrm{r}}$, we get

$$\int_{\mathbb{R}^N} |u_n|^{2^*} dx \geq \sum_{i=1}^m \int_{B_R(\lambda_n(g_i y_n,0))} |u_n|^{2^*} dx = \sum_{i=1}^m \int_{B_R(\lambda_n \tilde{y}_n)} |u_n|^{2^*} dx > m\varepsilon$$

for every $m \in \mathbb{N}\setminus\{0,1\}$ and $n > n_m$. This finally implies $\|u_n\|_{L^{2^*}(\mathbb{R}^N)} \to +\infty$ as $n \to \infty$, which is a contradiction, since $\{u_n\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$. □

We are now able to easily conclude the proof of Theorem 1.

Proof of Theorem 1. Since $u_n \rightharpoonup u \neq 0$ in X_{r} (Lemma 14) and $I'(u_n) \to 0$ in X_{r}' (Lemma 12), from Lemma 10.*iv* we deduce that u is a nonzero critical point of $I_{|X_{\mathrm{r}}}$. The proof is then completed by Lemma 10.*i*. □

Remark 15. Since the sequence of approximating solutions $\{u_n\}$ is a Palais-Smale sequence at the mountain-pass level c defined in (3.5) (Lemma 12), by standard arguments exploiting the properties of the projection on the Nehari manifold

$$\mathcal{N}_{\mathrm{r}} := \left\{ w \in X_{\mathrm{r}} \setminus \{0\} : \|w\|^2 = \|w\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \right\}$$

(see [39, Lemma 4.1, Theorem 4.2]) it is not difficult to see that the nonzero nonnegative solution found in Theorem 1 is a ground state solution for equation (3.1) restricted to X_{r}, namely, u achieves the infimum

$$S_{\mathrm{r}}(N,k,A) := \inf_{w \in X_{\mathrm{r}} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla w|^2 dx + A \int_{\mathbb{R}^N} \frac{w^2}{|y|^2} dx}{\left(\int_{\mathbb{R}^N} |w|^{2^*} dx \right)^{2/2^*}}\,.$$

4. Existence for the sub-critical problem

In this section we will deduce Theorem 2 by studying the non-critical case of equation (1.4), namely

$$-\triangle u+\frac{A}{|y|^{\alpha}}u=f(u)\quad \text{in } \mathbb{R}^{k}\times\mathbb{R}^{N-k}, \tag{4.1}$$

where f satisfies $(\mathbf{f}_p)$ for some $p>2$, $p\neq 2^*$. In this case the presence of a potential vanishing at infinity creates some difficulties in the variational approach to the equation. For example, it prevents the use of well known $H^1(\mathbb{R}^N)$ embeddings, so that we cannot assure that the energy space $X=X(\mathbb{R}^N,|y|^{-\alpha}\,dx)$ is contained in any $L^p(\mathbb{R}^N)$ with $p\neq 2^*$. Hence, under the growth condition $(\mathbf{f}_p)$ alone (which is the case we are interested in), the Euler functional (2.4) associated to (4.1), namely

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{A}{2}\int_{\mathbb{R}^N}\frac{u^2}{|y|^{\alpha}}dx-\int_{\mathbb{R}^N}F(u)\,dx\,, \tag{4.2}$$

does not necessarily make sense on X. On the other hand, owing to evident difficulties in bounding criticizing sequences, $X\cap L^p(\mathbb{R}^N)$ is not a good space where to look for critical points of I. Therefore a different approach is needed.

Dealing with the radial case of equation (4.1), such obstacles have been overcome in [18] by considering the nonlinear ODE associated to the problem of radial solutions, whose Euler functional turns out to be well defined on a suitable subspace of $H^1(\mathbb{R})$ provided that a suitable compatibility condition between α and p holds.

Here, instead, we follow the technique used in [10] and [36]. In Subsection 4.1 we deepen the study of the symmetric subspace $X_{\mathrm{s}}=X_{\mathrm{s}}(\mathbb{R}^N,|y|^{-\alpha}\,dx)$ introduced in (2.3) and give a pointwise estimate (Theorem 16) which allows us to deduce some embedding and compactness properties of X_{s} in connection with Lebesgue spaces (Proposition 21 and Theorem 23). Then, in Subsection 4.2, we derive a variational principle for equation (4.1) by proving that $X_{\mathrm{s}}\cap L^p(\mathbb{R}^N)$ is, in some sense, a natural constraint for finding weak solutions (Proposition 29). In Subsection 4.3 we finally give the proof of Theorem 2, by showing, via a Steiner symmetrization argument, that I achieves its infimum over the set of solutions which belong to $X_{\mathrm{s}}\cap L^p(\mathbb{R}^N)$.

Unfortunately, maybe for technical reasons, our arguments need high dimensions and rely on a compatibility condition between α and p which implies $\alpha<2$ and $p<2^*$.

4.1. **The weighted Sobolev space $X_{\rm s}(\mathbb{R}^N,|y|^{-\alpha}\,dx)$.** Let $A,\alpha>0$. The first result of this subsection concerns a pointwise estimate which plays a prominent role in the whole section. However, since quite technical, we displace its proof in the Appendix at the end of the paper. Here we denote $m:=N-k$.

Theorem 16. *Let $N>k\geq 2$. If $u\in X_{\rm s}$ is nonincreasing with respect to z, then*

$$u(y,z)\leq \frac{2^{(m-1)/2}(m+1)}{\sqrt{\sigma_k\sigma_m}\left(2^{(m+1)/2}-1\right)}\frac{1}{A^{1/4}}\frac{\|u\|}{|y|^{(2k-2-\alpha)/4}|z|^{m/2}}$$

for almost every $(y,z)\in\mathbb{R}^k\times\mathbb{R}^m$.

For sake of brevity, hereafter we denote

$$C_{N,k,A}:=\frac{2^{(m-1)/2}(m+1)}{\sqrt{\sigma_k\sigma_m}\left(2^{(m+1)/2}-1\right)}\frac{1}{A^{1/4}}\,. \tag{4.3}$$

Recall from (1.13) that we define

$$p_\alpha:=2\frac{2k-2+\alpha}{2k-2-\alpha}=2+\frac{4\alpha}{2k-2-\alpha}\quad \text{for } k\geq 2 \text{ and } \alpha\in(0,2k-2)$$

and notice that $p_\alpha<2^*$ if and only if $\alpha<2(k-1)/(N-1)$.

Proposition 17. *Let $N>k\geq 2$ and $0<\alpha<2k-2$. If $u\in X_{\rm s}$ is nonincreasing with respect to z, then $u\in L^{p_\alpha}(\mathbb{R}^k\times B_R^c)$ for all $R>0$ and*

$$\|u\|_{L^{p_\alpha}(\mathbb{R}^k\times B_R^c)}\leq C_{N,k,A}^{(p_\alpha-2)/p_\alpha}A^{-1/p_\alpha}\|u\|\,R^{-m\alpha/(2k-2+\alpha)}.$$

Proof. From Theorem 16 we get

$$\begin{aligned} u(y,z)^{p_\alpha} &= u(y,z)^{p_\alpha-2}u(y,z)^2\\ &\leq \frac{C_{N,k,A}^{p_\alpha-2}\|u\|^{p_\alpha-2}}{|y|^{(p_\alpha-2)(2k-2-\alpha)/4}|z|^{(p_\alpha-2)m/2}}u(y,z)^2\\ &= C_{N,k,A}^{p_\alpha-2}\|u\|^{p_\alpha-2}\frac{1}{|z|^{2\alpha m/(2k-2-\alpha)}}\frac{u(y,z)^2}{|y|^\alpha}\end{aligned}$$

for almost every $(y,z)\in\mathbb{R}^k\times\mathbb{R}^m$. Hence

$$\int_{\mathbb{R}^k\times B_R^c}u^{p_\alpha}dx\leq C_{N,k,A}^{p_\alpha-2}\|u\|^{p_\alpha-2}\frac{1}{R^{2\alpha m/(2k-2-\alpha)}}\int_{\mathbb{R}^k\times B_R^c}\frac{u^2}{|y|^\alpha}dx$$

and the result ensues. □

Corollary 18. *Let $N > k \geq 2$ and $0 < \alpha < 2(k-1)/(N-1)$. If $u \in X_{\mathrm{s}}$ is nonincreasing with respect to z, then $u \in L^p(\mathbb{R}^k \times B_R^c)$ for all $R > 0$ and $p \in [p_\alpha, 2^*]$. Moreover, there exists a constant $c_1 = c_1(N,k,A,\alpha,p) > 0$ such that*

$$\|u\|_{L^p(\mathbb{R}^k \times B_R^c)} \leq c_1 \|u\| \, R^{-mN(1/p-1/2^*)\alpha/(2k-2+\alpha-\alpha N)}.$$

Proof. Since $u \in L^{2^*}(\mathbb{R}^N)$, the claim follows from Proposition 17 together with Hölder and Sobolev inequalities. □

We now look for estimates on the strips of $\mathbb{R}^N$ with $|z|$ bounded. To this end we have to assume $m \geq 3$, so that we can use a well known radial lemma [15] in dimension m, that is, there exists a constant $C_m > 0$ such that all the radial mappings $w \in D^{1,2}(\mathbb{R}^N)$ satisfy

$$|w(z)| \leq C_m \|\nabla w\|_{L^2(\mathbb{R}^m)} |z|^{-\frac{m-2}{2}} \quad \text{for almost every } z \in \mathbb{R}^m \tag{4.4}$$

(let us observe that in [15] the authors prove (4.4) for $|z| \geq 1$, but their argument actually works for $z \neq 0$).

Proposition 19. *Let $N \geq k+3 \geq 4$. Then every $u \in X_{\mathrm{s}}$ satisfies*

$$\int_{\mathbb{R}^k} u(y,z)^2 \, dy \leq C_m^2 \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \frac{1}{|z|^{m-2}} \quad \text{for almost every } z \in \mathbb{R}^m. \tag{4.5}$$

In particular, $X_{\mathrm{s}} \hookrightarrow L^2(\mathbb{R}^k \times B_R)$ for all $R > 0$ and

$$\|u\|_{L^2(\mathbb{R}^k \times B_R)} \leq C_m \sqrt{\sigma_m/2} \, \|u\| \, R. \tag{4.6}$$

Proof. Note that $m = N - k \geq 3$ and let $u \in X_{\mathrm{s}}$. As one can easily check, for almost every $y \in \mathbb{R}^k$ the radial mapping $u(y,\cdot)$ has weak derivatives given by $\nabla(u(y,\cdot)) = \nabla_z u(y,\cdot) \in L^2(\mathbb{R}^m)$. On the other hand, $u \in X$ implies $u(y,\cdot) \in L^2(\mathbb{R}^m)$. Hence $u(y,\cdot) \in H^1(\mathbb{R}^m) \subset D^{1,2}(\mathbb{R}^m)$ and by (4.4) we get

$$\begin{aligned} |u(y,z)| &\leq C_m \Big(\int_{\mathbb{R}^m} |\nabla_z u(y,\zeta)|^2 \, d\zeta \Big)^{1/2} |z|^{-(m-2)/2} \\ &\leq C_m \Big(\int_{\mathbb{R}^m} |\nabla u(y,\zeta)|^2 \, d\zeta \Big)^{1/2} |z|^{-(m-2)/2} \end{aligned}$$

for almost every $z \in \mathbb{R}^m$, since $|\nabla_z u|^2 \leq |\nabla_y u|^2 + |\nabla_z u|^2 = |\nabla u|^2$ almost everywhere on $\mathbb{R}^N$. Thus (4.5) ensues. Then $u \in L^2(\mathbb{R}^k \times B_R)$ for all $R > 0$ and (4.6) readily follows from integrating on B_R. □

Corollary 20. *Let $N \geq k+3 \geq 4$. Then $X_{\mathrm{s}} \hookrightarrow L^p(\mathbb{R}^k \times B_R)$ for all $R > 0$ and $p \in [2, 2^*]$.*

Proof. As for Corollary 18, it follows from Proposition 19 by interpolation and Sobolev inequality. □

Proposition 21. *Let $N \geq k+3 \geq 5$ and $0 < \alpha < 2(k-1)/(N-1)$. If $u \in X_{\mathrm{s}}$ is nonincreasing with respect to z, then $u \in L^p(\mathbb{R}^N)$ for all $p \in [p_\alpha, 2^*]$. Moreover, there exists $c_0 = c_0(N,k,A,\alpha,p) > 0$ such that $\|u\|_{L^p(\mathbb{R}^N)} \leq c_0 \|u\|$.*

Proof. It readily ensues from Corollaries 18 and 20. □

The above results lead to the compactness property contained in Theorem 23, which also relies on the following well known lemma.

Lemma 22. *Let $N \geq k+1 \geq 3$. Then for any $R > 0$ and $p \in (2, 2^*)$, the space $H^1_{\mathrm{r}}(\mathbb{R}^k \times B_R) := \left\{ u \in H^1(\mathbb{R}^k \times B_R) : u(y,z) = u(|y|, z) \right\}$ is compactly embedded into $L^p(\mathbb{R}^k \times B_R)$.*

Proof. See Lemma III.2 of [28] (in a different context, the argument is developed also in Lemma 6.8 of [9], and in [32] with full detail). □

Theorem 23. *Let $N \geq k+3 \geq 5$ and $0 < \alpha < 2(k-1)/(N-1)$. Then any bounded sequence $\{u_n\} \subset X_{\mathrm{s}}$ whose functions are nonincreasing in z is relatively compact in $L^p(\mathbb{R}^N)$ for $p \in [p_\alpha, 2^*)$.*

Proof. Let $\{u_n\} \subset X_{\mathrm{s}}$ be as in the statement. Then $u_n \rightharpoonup u$ in X_{s} (up to a subsequence), where also u is nonincreasing in z by almost everywhere convergence. Moreover, there exists $C > 0$ such that $\|u_n\|, \|u\| \leq C$. Now let $\varepsilon > 0$ and fix $R_\varepsilon > 0$ such that $R_\varepsilon^{mN(1-p/2^*)\alpha/(2k-2+\alpha-\alpha N)} > 2^{p+2} c_1^p C^p / \varepsilon$, where c_1 is the constant of Corollary 18. Thus, by Corollary 18 itself, for all n one obtains

$$\begin{aligned} \int_{\mathbb{R}^k \times B^c_{R_\varepsilon}} |u_n - u|^p \, dx &\leq 2^p \Big(\int_{\mathbb{R}^k \times B^c_{R_\varepsilon}} u_n^p dx + \int_{\mathbb{R}^k \times B^c_{R_\varepsilon}} u^p dx \Big) \\ &\leq \frac{2^p c_1^p}{R_\varepsilon^{mN(1-p/2^*)\alpha/(2k-2+\alpha-\alpha N)}} \left(\|u_n\|^p + \|u\|^p \right) \\ &\leq \frac{2^{p+1} c_1^p C^p}{R_\varepsilon^{mN(1-p/2^*)\alpha/(2k-2+\alpha-\alpha N)}} < \frac{\varepsilon}{2} . \end{aligned}$$

On the other hand, X_{s} is continuously embedded into $H^1_{\mathrm{r}}(\mathbb{R}^k \times B_{R_\varepsilon})$ by Proposition 19, so that Lemma 22 gives

$$\int_{\mathbb{R}^k \times B_{R_\varepsilon}} |u_n - u|^p \, dx < \frac{\varepsilon}{2} \quad \text{for } n \text{ sufficiently large.}$$

By the arbitrariness of ε, the proof is thus complete. □

4.2. **Variational approach .** Let $A, \alpha > 0$. By the results of the previous subsection, here we give our variational principle for finding weak solutions to equation (4.1).

Let us begin by deriving conditions on α, p and $u \in X$ in order that $|u|^{p-1}$ defines an element of the dual space X' of X.

Lemma 24. *Let $N > k \geq 2$ and $0 < \alpha < 2k-2$. There exists $L_1 = L_1(N,k,A,\alpha) > 0$ such that if $u \in X_{\mathrm{s}}$ is nonincreasing with respect to z, then*

$$\int_{\mathbb{R}^k \times B_1^c} u^{p_\alpha - 1} |v| \, dx \leq L_1 \|u\|^{p_\alpha - 1} \|v\| \qquad \text{for all } v \in X .$$

Proof. By Hölder inequality we get

$$\begin{aligned}\int_{\mathbb{R}^k \times B_1^c} u^{p_\alpha - 1} |v| \, dx &= \int_{\mathbb{R}^k \times B_1^c} |y|^{\alpha/2} u^{p_\alpha - 1} \frac{|v|}{|y|^{\alpha/2}} dx \\ &\leq \Big(\int_{\mathbb{R}^k \times B_1^c} |y|^\alpha u^{2(p_\alpha - 1)} dx \Big)^{1/2} \Big(\frac{1}{A} \int_{\mathbb{R}^N} \frac{Av^2}{|y|^\alpha} dx \Big)^{1/2} \\ &\leq \frac{1}{\sqrt{A}} \Big(\int_{\mathbb{R}^k \times B_1^c} |y|^\alpha u^{2(p_\alpha - 1)} dx \Big)^{1/2} \|v\| \; .\end{aligned}$$

Then we write $2(p_\alpha - 1) = 2 + 8\alpha/(2k-2-\alpha)$ and use Theorem 16 to obtain

$$\begin{aligned}&\int_{\mathbb{R}^k \times B_1^c} |y|^\alpha u^{2(p_\alpha - 1)} dx = \int_{\mathbb{R}^k \times B_1^c} \frac{u^2}{|y|^\alpha} |y|^{2\alpha} u^{8\alpha/(2k-2-\alpha)} dx \\ &\leq C_{N,k,A}^{8\alpha/(2k-2-\alpha)} \|u\|^{8\alpha/(2k-2-\alpha)} \int_{\mathbb{R}^k \times B_1^c} \frac{u^2}{|y|^\alpha} \frac{|y|^{2\alpha}}{|y|^{2\alpha} |z|^{4m\alpha/(2k-2-\alpha)}} dx \\ &\leq C_{N,k,A}^{8\alpha/(2k-2-\alpha)} A^{-1} \|u\|^{8\alpha/(2k-2-\alpha)} \int_{\mathbb{R}^k \times B_1^c} \frac{Au^2}{|y|^\alpha} dx \\ &\leq C_{N,k,A}^{8\alpha/(2k-2-\alpha)} A^{-1} \|u\|^{2(p_\alpha - 1)} ,\end{aligned}$$

where $C_{N,k,A}$ is given by (4.3). □

Lemma 25. *Let $N \geq k+3 \geq 4$. If $p \in [2 + 2/N, 2^*]$, then there exists $L_2 = L_2(N,k,A,\alpha,p) > 0$ such that*

$$\int_{\mathbb{R}^k \times B_1} |u|^{p-1} |v| \, dx \leq L_2 \|u\|^{p-1} \|v\| \qquad \text{for all } u \in X_{\mathrm{s}} \text{ and } v \in X .$$

Proof. By Hölder and Sobolev inequalities there exists $S = S(N) > 0$ such that

$$\int_{\mathbb{R}^k \times B_1} |u|^{p-1} |v| \, dx \leq \left(\int_{\mathbb{R}^k \times B_1} |u|^{2^*(p-1)/(2^*-1)} \, dx \right)^{(2^*-1)/2^*} \|v\|_{L^{2^*}(\mathbb{R}^N)}$$
$$\leq S \|u\|^{p-1}_{L^{2^*(p-1)/(2^*-1)}(\mathbb{R}^k \times B_1)} \|v\| \, .$$

The result then follows by Corollary 20, because $p \in [2 + 2/N, 2^*]$ amounts to $2 \leq 2^*(p-1)/(2^*-1) \leq 2^*$. □

Recall from (1.14) that, for $N > k \geq 2$, we denote

$$\bar{p}_\alpha := \begin{cases} 2 + \frac{2}{N} & \text{if } \; 0 < \alpha \leq 2\frac{k-1}{2N+1} \\ p_\alpha & \text{if } \; 2\frac{k-1}{2N+1} \leq \alpha < 2\frac{k-1}{N-1} \end{cases}$$

and notice that $\max\{2 + \frac{2}{N}, p_\alpha\} \leq \bar{p}_\alpha < 2^*$ for all $\alpha \in (0, 2(k-1)/(N-1))$. Since Hölder and Sobolev inequalities straightforwardly imply the existence of $L_3 = L_3(N) > 0$ such that

$$\int_{\mathbb{R}^N} |u|^{2^*-1} |v| \, dx \leq L_3 \|u\|^{2^*-1} \|v\| \qquad \text{for all } u, v \in X \, , \tag{4.7}$$

we finally obtain the following proposition by interpolation.

Proposition 26. *Let $N \geq k + 3 \geq 5$ and $0 < \alpha < 2(k-1)/(N-1)$. For every $p \in [\bar{p}_\alpha, 2^*]$ there exists $L = L(N, k, A, \alpha, p) > 0$ such that if $u \in X_{\mathrm{s}}$ is nonincreasing with respect to z, then*

$$\int_{\mathbb{R}^N} u^{p-1} |v| \, dx \leq L \|u\|^{p-1} \|v\| \qquad \textit{for all } v \in X \, .$$

Proof. Taking into account that $p \in [\bar{p}_\alpha, 2^*]$ implies $p \in [p_\alpha, 2^*]$, fix $\lambda, \mu \in (0,1)$ such that $p = \lambda p_\alpha + \mu 2^*$ and $\lambda + \mu = 1$. Then by (4.7) and Lemma 24, we have

$$\int_{\mathbb{R}^k \times B_1^c} u^{p-1} |v| \, dx = \int_{\mathbb{R}^k \times B_1^c} u^{\lambda(p_\alpha - 1) + \mu(2^*-1)} |v|^\lambda |v|^\mu \, dx$$
$$= \int_{\mathbb{R}^k \times B_1^c} \left(u^{p_\alpha - 1} |v| \right)^\lambda \left(u^{2^*-1} |v| \right)^\mu dx$$
$$\leq \left(\int_{\mathbb{R}^k \times B_1^c} u^{p_\alpha - 1} |v| \, dx \right)^\lambda \left(\int_{\mathbb{R}^N} u^{2^*-1} |v| \, dx \right)^\mu \leq L_3^\mu L_1^\lambda \|u\|^{p-1} \|v\|$$

for all $v \in X$. Therefore, as $p \in [\bar{p}_\alpha, 2^*]$ also implies $p \in [2 + 2/N, 2^*]$, the claim follows by Lemma 25. □

We now let $f \in C(\mathbb{R}, \mathbb{R})$ satisfy $(\mathbf{f}_p)$ and recall from Section 2 that any critical point $u \in X \cap L^p(\mathbb{R}^N)$ of the functional $I \in C^1\left(X \cap L^p(\mathbb{R}^N), \mathbb{R}\right)$ defined in (4.2) satisfies

$$\int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla h + \frac{A}{|y|^\alpha} uh\right) dx = \int_{\mathbb{R}^N} f(u)\, h\, dx \tag{4.8}$$

for all $h \in X \cap L^p(\mathbb{R}^N)$. Hence u is a weak solution to equation (4.1) provided that we can take any $h \in X$ as a test function. The next results show that this holds actually true if u also belongs to X_{s} and is nonincreasing in z.

Lemma 27. *Let $N > k \geq 2$ and $0 < \alpha < 2$. If $1 \leq p \leq 2^*$, then $X_{\mathrm{s}} \cap L^p(\mathbb{R}^N)$ is dense in X_{s}.*

Proof. Let $u \in X_{\mathrm{s}}$ and fix $\phi \in C^\infty(\mathbb{R})$ such that $0 \leq \phi \leq 1$ on $\mathbb{R}$, $\phi(t) = 1$ for $t \leq 1$ and $\phi(t) = 0$ for $t \geq 2$. Then for every $n \in \mathbb{N} \setminus \{0\}$ the mapping $\varphi_n(x) := \phi(|y|/n)\phi(|z|/n)$ satisfies

- $|\varphi_n(x)| \leq 1$ for all $x \in \mathbb{R}^N$
- $\varphi_n(x) = 1$ for all $x \in B_n$
- $\operatorname{supp} \varphi_n \subseteq \bar{B}_{2\sqrt{2}n}$.

Moreover, $\nabla \varphi_n(x) = \phi'\left(\frac{1}{n}|y|\right)\phi\left(\frac{1}{n}|z|\right)\frac{y}{n|y|} + \phi(\frac{1}{n}|y|)\phi'(\frac{1}{n}|z|)\frac{z}{n|z|}$ is such that

- $\nabla \varphi_n(x) = 0$ for $|y| \geq 2n$
- $|\nabla \varphi_n(x)| \leq \frac{2}{n}\|\phi'\|_{L^\infty(\mathbb{R})}$ for all $x \in \mathbb{R}^N$.

Hence, $\varphi_n u \to u$ in $L^2(\mathbb{R}^N, |y|^{-\alpha} dx)$ and $\varphi_n \nabla u \to \nabla u$ in $L^2(\mathbb{R}^N)$ by dominated convergence. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla \varphi_n|^2 u^2 dx &= \int_{|y|<2n} |\nabla \varphi_n|^2 u^2 dx \leq \frac{4}{n^2}\left\|\phi'\right\|^2_{L^\infty(\mathbb{R})} \int_{|y|<2n} \frac{u^2}{|y|^\alpha}|y|^\alpha dx \\ &\leq \frac{2^{2+\alpha}}{n^{2-\alpha}}\left\|\phi'\right\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}^N} \frac{u^2}{|y|^\alpha} dx = o(1)_{n\to\infty} ,\end{aligned}$$

so that $\nabla(\varphi_n u) = u\nabla\varphi_n + \varphi_n \nabla u \in L^2(\mathbb{R}^N)$ and $\nabla(\varphi_n u) \to \nabla u$ in $L^2(\mathbb{R}^N)$. This implies $\varphi_n u \to u$ in X_{s} with $\varphi_n u \in X_{\mathrm{s}} \cap L^p(\mathbb{R}^N)$ because of the embedding $X_{\mathrm{s}} \hookrightarrow L^p_{loc}(\mathbb{R}^N)$. □

The claim of the following lemma usually follows from the Principle of Symmetric Criticality [30], but in the present case such a principle does not apply, because we do not know whether I is differentiable (not even well defined) on the whole space X or not.

Lemma 28. *Let $N \geq k+3 \geq 5$, $0<\alpha<2(k-1)/(N-1)$ and $\bar{p}_\alpha \leq p \leq 2^*$. If $u \in X_{\mathrm{s}}$ is nonincreasing in z and satisfies* (4.8) *for all $h \in X_{\mathrm{s}}$, then* (4.8) *holds true for all $h \in X$.*

Proof. Let $u \in X_{\mathrm{s}}$ be nonincreasing in z and define

$$T(u)h := \int_{\mathbb{R}^N} \Big(\nabla u \cdot \nabla h + \frac{A}{|y|^\alpha} uh\Big) dx - \int_{\mathbb{R}^N} f(u)\, h\, dx \quad \text{for } h \in X\,,$$

so that $T(u)\,h = 0$ for all $h \in X_{\mathrm{s}}$ means that (4.8) holds for all $h \in X_{\mathrm{s}}$. The linear functional $T(u)$ is well defined and continuous on X, i.e., $T(u) \in X'$, since for all $h \in X$ one has

$$\int_{\mathbb{R}^N} |f(u)|\,|h|\,dx \leq M \int_{\mathbb{R}^N} u^{p-1}\,|h|\,dx \leq ML\,\|u\|^{p-1}\,\|h\|$$

by Proposition 26 and assumption $(\mathbf{f}_p)$. Hence there exists a unique $\bar{u} \in X$ such that $T(u)\,h = (\bar{u} \mid h)$ for all $h \in X$. By means of obvious changes of variable, it is easy to see that for every $h \in X$, $g_1 \in O(k)$ and $g_2 \in O(N-k)$ one has $(\bar{u} \mid h(g_1\cdot, g_2\cdot)) = \left(\bar{u}\left(g_1^{-1}\cdot, g_2^{-1}\cdot\right) \mid h\right)$ and $T(u)\,h(g_1\cdot, g_2\cdot) = T(u)\,h$, so that $\left(\bar{u}\left(g_1^{-1}\cdot, g_2^{-1}\cdot\right) \mid h\right) = (\bar{u} \mid h)$. This means $\bar{u}\left(g_1^{-1}\cdot, g_2^{-1}\cdot\right) = \bar{u}$ for all $g_1 \in O(k)$ and $g_2 \in O(N-k)$, i.e., $\bar{u} \in X_{\mathrm{s}}$. So, $T(u)\,h = 0$ for all $h \in X_{\mathrm{s}}$ implies $\bar{u} = 0$, which yields in turn $T(u)\,h = 0$ for all $h \in X$. □

Proposition 29. *Let $N \geq k+3 \geq 5$, $0 < \alpha < 2(k-1)/(N-1)$, $\bar{p}_\alpha \leq p \leq 2^*$. Then every z-nonincreasing critical point of $I|_{X_{\mathrm{s}} \cap L^p(\mathbb{R}^N)}$ is a weak solution to equation* (4.1).

Proof. Let $u \in X_{\mathrm{s}}$ be nonincreasing in z and let $h \in X_{\mathrm{s}}$. By Lemma 27 take $\{h_n\} \in X_{\mathrm{s}} \cap L^p(\mathbb{R}^N)$ such that $h_n \to h$ in X_{s}. Then

$$\int_{\mathbb{R}^N} \Big(\nabla u \cdot \nabla h_n + \frac{A}{|y|^\alpha} uh_n\Big) dx \to \int_{\mathbb{R}^N} \Big(\nabla u \cdot \nabla h + \frac{A}{|y|^\alpha} uh\Big) dx$$

and, by Proposition 26,

$$\int_{\mathbb{R}^N} u^{p-1}\,|h-h_n|\,dx \leq L\,\|u\|^{p-1}\,\|h-h_n\| = o(1)_{n\to\infty}\,.$$

Hence, thanks to assumption $(\mathbf{f}_p)$, (4.8) holds for all $h \in X_{\mathrm{s}}$ provided that it holds for all $h \in X_{\mathrm{s}} \cap L^p(\mathbb{R}^N)$, namely, that u is a critical point for I restricted to $X_{\mathrm{s}} \cap L^p(\mathbb{R}^N)$. The conclusion then ensues from Lemma 28. □

4.3. **Proof of Theorem 2 .** In this subsection we assume all the hypotheses of Theorem 2 and, according to Remark 3, we also suppose that f is odd. Denote $L^p := L^p(\mathbb{R}^N)$ for brevity.

According to Proposition 29, the proof of the theorem will be obtained, via a Steiner symmetrization argument, by minimizing the Euler functional (4.2) on the Nehari manifold of $X_{\mathrm{s}} \cap L^p$. Recall from Section 2 that I has first Fréchet derivative given by (2.5) and observe that, by standard arguments exploiting $(\mathbf{f}_p')$, which implies $(\mathbf{f}_p)$ since $f(0) = 0$, I is actually of class C^2 on $X_{\mathrm{s}} \cap L^p$ and has second Fréchet derivative at any $u \in X_{\mathrm{s}} \cap L^p$ given by

$$I''(u) h_1 h_2 = (h_1 \mid h_2) - \int_{\mathbb{R}^N} f'(u) h_1 h_2 dx \qquad \text{for all } h_1, h_2 \in X_{\mathrm{s}} \cap L^p.$$

We now define the Nehari manifold

$$\begin{aligned} \mathcal{N} &:= \left\{ u \in X_{\mathrm{s}} \cap L^p \setminus \{0\} : I'(u) u = 0 \right\} \\ &= \left\{ u \in X_{\mathrm{s}} \cap L^p \setminus \{0\} : \|u\|^2 = \int_{\mathbb{R}^N} f(u) u \, dx \right\}. \end{aligned}$$

The following result is rather standard under the assumed hypotheses, so we just give a sketch of its proof (see also [39, Chapter 4], [9], [14], [32]).

Lemma 30. *For any $u \in X_{\mathrm{s}} \cap L^p \setminus \{0\}$ there exists a unique $\mu > 0$ such that $\mu u \in \mathcal{N}$. Moreover, $I(\mu u) = \max_{t \geq 0} I(tu)$, and $I'(u) u < 0 \Rightarrow \mu < 1$.*

Proof. Fix $u \in X_{\mathrm{s}} \cap L^p$, $u \neq 0$, and define $\phi_u \in C^2(\mathbb{R}, \mathbb{R})$ by setting $\phi_u(t) := I(tu)$ for every $t \in \mathbb{R}$. Then if $t \neq 0$, $tu \in \mathcal{N}$ if and only if $\phi_u'(t) = 0$. Exploiting assumptions $(\mathbf{f})$ and $(\mathbf{f}_p')$ (which implies $f'(0) = 0$), one easily proves the following claims:

(i) if $\mu \neq 0$ is a critical point for ϕ_u, then μ is a strict maximum point for ϕ_u;
(ii) there exists at most one positive critical point for ϕ_u;
(iii) 0 is a strict minimum point for ϕ_u.

Now observe that, as f is odd, F is even and assumption $(\mathbf{F})$ yields $F(s) \geq 0$ for all $s \in \mathbb{R}$, whence the first inequality of $(\mathbf{f})$ also gives $F(s) \geq (F(\bar{s})/\bar{s}^{\vartheta})|s|^{\vartheta}$ for all $|s| \geq \bar{s}$. Fix $\bar{t} > 0$ such that $\left|\left\{x \in \mathbb{R}^N : |u(x)| > \bar{s}/\bar{t}\right\}\right| > 0$, which does exist because $u \neq 0$. So one deduces

$$0 < \int_{\{\bar{t}|u| > \bar{s}\}} |u|^{\vartheta} \, dx < +\infty$$

and we conclude

$$\phi_u(t) \leq \frac{t^2}{2}\|u\|^2 - \int_{\{t|u|>\bar{s}\}} F(tu)\,dx \leq \frac{t^2}{2}\|u\|^2 - t^\vartheta \frac{F(\bar{s})}{\bar{s}^\vartheta}\int_{\{t|u|>\bar{s}\}} |u|^\vartheta\,dx$$
$$\leq \frac{t^2}{2}\|u\|^2 - t^\vartheta \frac{F(\bar{s})}{\bar{s}^\vartheta}\int_{\{\bar{t}|u|>\bar{s}\}} |u|^\vartheta\,dx \to -\infty$$

as $t \to +\infty$. The first part of the lemma then follows from claims (iii) and (ii), since one infers the existence of $t_0 > 0$ such that $\phi_u(t_0) = 0 = \phi_u(0)$ and thus of a unique critical point $\mu > 0$ for ϕ_u. By uniqueness and claim (i), μ satisfies $\phi_u(\mu) = \max_{t\geq 0}\phi_u(t)$. Finally, since $\phi_u'(0) = 0$ and $\phi_u''(0) = \|u\|^2 > 0$ imply $\phi_u'(t) > 0$ for $t > 0$ sufficiently small, if $\phi_u'(1) = I'(u)\,u < 0$, then it must be $\mu < 1$. □

As $\mathcal{N}$ is nonempty, it makes sense to consider the following minimization problem:

$$\nu := \inf_{u\in\mathcal{N}} I(u)\,. \tag{4.9}$$

Lemma 31. *There exists a bounded minimizing sequence for problem* (4.9) *whose functions are nonincreasing with respect to z. Moreover, one has $\nu > 0$.*

Proof. Let $\{u_n\} \subset \mathcal{N}$ be any minimizing sequence. As f is odd, I is even and $|u_n| \in \mathcal{N}$, so that we can assume $u_n \geq 0$. Let u_n^* be the $(N-k,N)$-Steiner symmetrization of u_n. By well known properties of Steiner symmetrizations (see [16] for instance), u_n^* belongs to $X_{\mathrm{s}} \cap L^p$, is nonincreasing with respect to z and has the following properties:

$$\|u_n^*\|_{L^p} = \|u_n\|_{L^p} \tag{4.10}$$
$$I(tu_n^*) \leq I(tu_n) \quad \text{for every } t \in \mathbb{R} \tag{4.11}$$
$$I'(u_n^*)\,u_n^* \leq I'(u_n)\,u_n = 0\,. \tag{4.12}$$

In particular, (4.10) implies $u_n^* \neq 0$ and therefore, by Lemma 30 and (4.12), for every n there exists $\mu_n \in (0,1]$ such that $\mu_n u_n^* \in \mathcal{N}$. Hence for all n one has

$$\nu \leq I(\mu_n u_n^*) \leq I(\mu_n u_n) \leq \max_{t\geq 0} I(tu_n) = I(u_n)\,,$$

where we have used (4.11) for the second inequality, while the last equality follows by Lemma 30 from the fact that $u_n \in \mathcal{N}$. As a consequence, setting $v_n := \mu_n u_n^*$, $\{v_n\} \subset \mathcal{N}$ is a minimizing sequence whose functions are nonincreasing with respect to z. Now we use assumption $(\mathbf{f})$ to obtain

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{\vartheta}\right)\|v_n\|^2 &= \frac{1}{2}\|v_n\|^2-\frac{1}{\vartheta}\int_{\mathbb{R}^N} f(v_n)\,v_n dx \\ &\leq \frac{1}{2}\|v_n\|^2-\int_{\mathbb{R}^N} F(v_n)\,dx = I(v_n)\,,\end{aligned} \tag{4.13}$$

whence, since $\vartheta > 2$ and $I(v_n) \to \nu$, one infers $\nu \geq 0$ and $\{\|v_n\|\}$ bounded. Hence $\{v_n\}$ is bounded in $X_{\mathrm{s}} \cap L^p$ by Proposition 21. Finally, suppose by contradiction that $I(v_n) \to 0$, so that (4.13) gives $\|v_n\| \to 0$. Then from condition $(\mathbf{f}_p)$ and Proposition 21 we get

$$\|v_n\|^2 = \int_{\mathbb{R}^N} f(v_n)\,v_n dx \leq M\int_{\mathbb{R}^N} |v_n|^p\,dx \leq Mc_0^p\,\|v_n\|^p\,,$$

which yields the contradiction $1 \leq Mc_0^p\,\|v_n\|^{p-2} \to 0$ since $p > 2$. □

Theorem 32. *The minimization problem* (4.9) *has a z-nonincreasing solution.*

Proof. Let $\{u_n\} \subset \mathcal{N}$ be any bounded minimizing sequence such that $u_n \geq 0$ is nonincreasing in z, which exists by Lemma 31. By Theorem 23, one deduces that (up to a subsequence)

$$\begin{aligned}&u_n \rightharpoonup u \quad \text{in } X_{\mathrm{s}}\\ &u_n \to u \quad \text{in } L^p \text{ and almost everywhere on } \mathbb{R}^N\,.\end{aligned}$$

Then, passing to the limit as $n \to \infty$ in

$$I(u_n) = \frac{1}{2}\|u_n\|^2-\int_{\mathbb{R}^N} F(u_n)\,dx = \int_{\mathbb{R}^N}\left(\frac{1}{2}f(u_n)\,u_n - F(u_n)\right)dx\,,$$

we obtain

$$\nu = \int_{\mathbb{R}^N}\left(\frac{1}{2}f(u)\,u - F(u)\right)dx \tag{4.14}$$

which gives $u \neq 0$ since $\nu > 0$ (Lemma 31). We now show that $u \in \mathcal{N}$, which completes the proof since (4.14) and $\|u\|^2 = \int_{\mathbb{R}^N} f(u)\,u\,dx$ imply $I(u) = \nu$. Note that u is nonincreasing with respect to z by almost everywhere convergence. With a view to deducing a contradiction, assume $u \notin \mathcal{N}$. As $u_n \in \mathcal{N}$ and $u_n \to u$ in L^p, one has

$$\lim_{n\to\infty}\|u_n\|^2 = \lim_{n\to\infty}\int_{\mathbb{R}^N} f(u_n)\,u_n dx = \int_{\mathbb{R}^N} f(u)\,u\,dx\,.$$

Hence, by weak lower semi-continuity of the norm, the weak convergence $u_n \rightharpoonup u$ in X_{s} gives

$$I'(u)\,u = \|u\|^2 - \int_{\mathbb{R}^N} f(u)\,u\,dx \leq \lim_{n\to\infty} \|u_n\|^2 - \int_{\mathbb{R}^N} f(u)\,u\,dx = 0\,.$$

Since $u \notin \mathcal{N}$ and $u \neq 0$, this implies $I'(u)\,u < 0$, so that, by Lemma 30, there exists $\mu \in (0,1)$ such that $\mu u \in \mathcal{N}$. Thus we have

$$\nu \leq I(\mu u) = \int_{\mathbb{R}^N} \left(\frac{1}{2} f(\mu u)\,\mu u - F(\mu u) \right) dx\,. \tag{4.15}$$

On the other hand, the mapping $\Phi(t) := \int_{\mathbb{R}^N} \left(\frac{1}{2} f(tu)\,tu - F(tu) \right) dx$ is strictly increasing for $t > 0$, since one has

$$\begin{aligned} \Phi'(t) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(t f'(tu)\,u^2 - f(tu)\,u \right) dx \\ &= \frac{1}{2t} \int_{\mathbb{R}^N} \left(f'(tu)\,(tu)^2 - f(tu)\,tu \right) dx > 0 \end{aligned}$$

by assumption (**f**). Hence, by (4.15) and (4.14), we get the contradiction $\nu \leq \Phi(\mu) < \Phi(1) = \nu$. □

Proof of Theorem 2. First we observe that $\mathcal{N}$ is a C^1 manifold. Indeed, letting $J(u) := I'(u)\,u$, one has that J is of class C^1 on $X_{\mathrm{s}} \cap L^p$ and

$$J'(u)\,h = 2\,(u \mid h) - \int_{\mathbb{R}^N} \left(f'(u)\,uh + f(u)\,h \right) dx \qquad \text{for all } u, h \in X_{\mathrm{s}} \cap L^p,$$

whence, by assumption (**f**), for every $u \in \mathcal{N}$ we get

$$\begin{aligned} J'(u)\,u &= 2\,\|u\|^2 - \int_{\mathbb{R}^N} \left(f'(u)\,u^2 + f(u)\,u \right) dx \\ &= \int_{\mathbb{R}^N} \left(f(u)\,u - f'(u)\,u^2 \right) dx < 0. \end{aligned} \tag{4.16}$$

Now we apply Theorem 32 to deduce the existence of $u \in \mathcal{N}$ satisfying $I(u) = \min I(\mathcal{N})$ and such that $u(y,z) = u(|y|,|z|) \geq 0$ is nonincreasing with respect to z. Then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $I'(u) = \lambda J'(u)$ in $(X_{\mathrm{s}} \cap L^p)'$. As $u \in \mathcal{N}$, we have in particular $I'(u)\,u = 0$, i.e., $\lambda J'(u)\,u = 0$, which gives $\lambda = 0$ by (4.16) and hence $I'(u) = 0$ in $(X_{\mathrm{s}} \cap L^p)'$. Thanks to Propositions 29 and 21, the proof is thus complete.

5. Nonexistence results

In this section we derive a Pohožaev-type identity for the (classical) equation

$$-\triangle u+\frac{A}{|y|^{\alpha}}u=f(u)\quad \text{in } (\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k} \tag{5.1}$$

with $N>k\geq 2$, $A,\alpha>0$ and $f\in C(\mathbb{R},\mathbb{R})$ (Proposition 33). After checking that classical solutions can be chosen as test functions for equality (5.1) in the distributional sense on $\mathbb{R}^N$ (Lemma 34), this identity allows us to prove Theorem 4, yielding compatibility conditions on the parameters α and p in order that nontrivial solutions are permitted to equation (5.1) with power nonlinearities $f(u)=|u|^{p-2}u$. A nonexistence result for more general nonlinearities also follows (Theorem 35).

Throughout the section we assume $N>k\geq 2$ and $A>0$. Recall that, for $\alpha>0$, we define

$$q_k:=\frac{2k}{k-1}\,,\quad q_k^{\alpha}:=\frac{2k}{k-2+\alpha}\,,\quad q_{k,N}:=\frac{2kN}{(k-2)\,N+2k}\,.$$

Proposition 33. *Let $\alpha>0$ and $f\in C(\mathbb{R},\mathbb{R})$. Assume that $u\in C^2((\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k})$ is a classical solution to equation* (5.1). *If*

$$\int_{\mathbb{R}^N}\Big(|\nabla u|^2+\frac{u^2}{|y|^{\alpha}}+|F(u)|\Big)dx<+\infty \tag{5.2}$$

and

$$\nabla u\in L^q_{loc}(\mathbb{R}^N)\quad \textit{for some } q\geq q_k, \tag{5.3}$$

then

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{N-\alpha}{2}\int_{\mathbb{R}^N}\frac{Au^2}{|y|^{\alpha}}dx=N\int_{\mathbb{R}^N}F(u)\,dx\,. \tag{5.4}$$

Proof. The proof relies on a standard argument [26], suited to the case under discussion. The starting point are the following identities, which hold true on $(\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k}$:

$$(x\cdot\nabla u)\triangle u=\operatorname{div}\Big[(x\cdot\nabla u)\nabla u-\frac12|\nabla u|^2x\Big]+\frac{N-2}{2}|\nabla u|^2\,,$$
$$(x\cdot\nabla u)\frac{Au}{|y|^{\alpha}}=\operatorname{div}\Big[\frac12\frac{Au^2}{|y|^{\alpha}}x\Big]-\frac{N-\alpha}{2}\frac{Au^2}{|y|^{\alpha}}\,,$$
$$(x\cdot\nabla u)f(u)=\operatorname{div}[F(u)\,x]-NF(u)\,.$$

For $R_2 > R_1 > 0$, set $\Omega := \Omega_{R_1,R_2} := \{x \in B_{R_2} : |y| > R_1\}$. Upon multiplying equation (5.1) by $(x \cdot \nabla u)$, the divergence theorem yields

$$-\int_{\partial\Omega} (x \cdot \nabla u)(\nabla u \cdot \nu)\, d\sigma + \frac{1}{2}\int_{\partial\Omega} \Big(|\nabla u|^2 + \frac{Au^2}{|y|^\alpha} \Big) x \cdot \nu \, d\sigma - \int_{\partial\Omega} F(u)\, x \cdot \nu \, d\sigma$$
$$= \frac{N-2}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{N-\alpha}{2}\int_\Omega \frac{Au^2}{|y|^\alpha} dx - N\int_\Omega F(u)\, dx \tag{5.5}$$

where $\nu(x)$ is the outward normal of $\partial\Omega$ at x and $d\sigma$ is the $(N-1)$-dimensional measure of $\partial\Omega$. We have

$$\partial\Omega = \{x \in \partial B_{R_2} : |y| \geq R_1\} \cup \{x \in B_{R_2} : |y| = R_1\} =: \Sigma_{R_1,R_2} \cup \Gamma_{R_1,R_2}$$

with obvious definitions of Σ_{R_1,R_2} and Γ_{R_1,R_2}. Notice that $\Sigma_{R_1,R_2} \cap \Gamma_{R_1,R_2} = \varnothing$. Note also that $\Gamma_{R_1,R_2} = \{x \in \mathbb{R}^N : |y| = R_1,\, |z| < \sqrt{R_2^2 - R_1^2}\}$ and $\nu(x) = -(y/R_1, 0)$ on Γ_{R_1,R_2}. Since the integral

$$\int_0^{R_2} dR_1 \int_{\{|y|=R_1,\, |z|<\sqrt{R_2^2-R_1^2}\}} \Big(|\nabla u|^2 + |\nabla u|^q + \frac{u^2}{|y|^\alpha} + |F(u)| \Big) d\sigma$$
$$= \int_{B_{R_2}} \Big(|\nabla u|^2 + |\nabla u|^q + \frac{u^2}{|y|^\alpha} + |F(u)| \Big) dx$$

is finite by assumptions (5.2) and (5.3), arguing by contradiction it is easy to see that there exists a sequence $R_{1,n} \to 0$, $R_{1,n} > 0$ such that

$$R_{1,n} \int_{\Gamma_{R_{1,n},R_2}} \Big(|\nabla u|^2 + |\nabla u|^q + \frac{u^2}{|y|^\alpha} + |F(u)| \Big) d\sigma \to 0 \quad \text{as } n \to \infty .$$

Hence,

$$\int_{\Gamma_{R_{1,n},R_2}} \Big(|\nabla u|^2 + \frac{Au^2}{|y|^\alpha} + |F(u)| \Big) x \cdot \nu \, d\sigma$$
$$= -R_{1,n} \int_{\Gamma_{R_{1,n},R_2}} \Big(|\nabla u|^2 + \frac{Au^2}{|y|^\alpha} + |F(u)| \Big) d\sigma \to 0 .$$

Moreover, since $q \geq q_k$ implies $(k-1)(q-2)/2 \geq 1$, by Hölder inequality we get

$$\Big| \int_{\Gamma_{R_{1,n},R_2}} (x \cdot \nabla u)(\nabla u \cdot \nu)\, d\sigma \Big| \leq R_2 \int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^2 d\sigma \tag{5.6}$$
$$\leq R_2 \Big(\int_{\Gamma_{R_{1,n},R_2}} d\sigma \Big)^{1-2/q} \Big(\int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^q \, d\sigma \Big)^{2/q}$$

$$\le R_2\Big(\int_{\{|y|=R_{1,n},\,|z|<R_2\}} d\sigma\Big)^{1-2/q}\Big(\int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^q\,d\sigma\Big)^{2/q}$$
$$= R_2\Big(\frac{\sigma_k\sigma_{N-k}R_2^{N-k}}{N-k}\Big)^{1-2/q}\Big(R_{1,n}^{(k-1)(q-2)/2}\int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^q\,d\sigma\Big)^{2/q} \to 0\,.$$

As we can assume that $\{R_{1,n}\}$ is nonincreasing, the sequence $\{\Sigma_{R_{1,n},R_2}\}$ is nondecreasing and satisfies $\bigcup_n \Sigma_{R_{1,n},R_2} = \{x \in \partial B_{R_2} : y \neq 0\}$. Therefore, evaluating (5.5) for $R_1 = R_{1,n}$ and passing to the limit as $n \to \infty$, one obtains

$$-\int_{\partial B_{R_2}} (x\cdot\nabla u)\,(\nabla u\cdot\nu)\,d\sigma + \int_{\partial B_{R_2}} \Big(\frac{1}{2}\,|\nabla u|^2 + \frac{1}{2}\frac{Au^2}{|y|^\alpha} - F\,(u)\Big)x\cdot\nu\,d\sigma$$
$$= \frac{N-2}{2}\int_{B_{R_2}} |\nabla u|^2\,dx + \frac{N-\alpha}{2}\int_{B_{R_2}} \frac{Au^2}{|y|^\alpha}dx - N\int_{B_{R_2}} F\,(u)\,dx\,. \qquad (5.7)$$

Now, again by (5.2), one infers the existence of a sequence $R_{2,n} \to +\infty$ such that

$$R_{2,n}\int_{\partial B_{R_{2,n}}} \Big(|\nabla u|^2 + \frac{u^2}{|x|^\alpha} + |F\,(u)|\Big)d\sigma \to 0 \quad \text{as } n\to\infty\,.$$

The conclusion then follows by evaluating (5.7) for $R_2 = R_{2,n}$ and passing to the limit as $n \to \infty$. □

Lemma 34. *Let $\alpha > 0$ and $f \in C(\mathbb{R},\mathbb{R})$. Assume that $u \in X(\mathbb{R}^N, |y|^{-\alpha}dx) \cap C^2((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$ is a classical solution to equation (5.1) satisfying $f\,(u)\,u \in L^1(\mathbb{R}^N)$ and $\nabla u \in L^q_{loc}(\mathbb{R}^N)$ for $q = \max\{q_k, q_k^\alpha\}$ or $q = q_{k,N}$. Then*

$$\int_{\mathbb{R}^N} \Big(|\nabla u|^2 + \frac{Au^2}{|y|^\alpha}\Big)dx = \int_{\mathbb{R}^N} f\,(u)\,u\,dx\,. \qquad (5.8)$$

Proof. The argument is analogous to the one of the proof of Proposition 33, so we will use the same notations and will be sketchy in some of the details. Multiplying equation (5.1) by u, using the identity $u\,\triangle u = \operatorname{div}[u\,\nabla u] - |\nabla u|^2$ in $(\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$ and applying the divergence theorem on $\Omega = \Omega_{R_1,R_2}$, we obtain

$$\int_\Omega f\,(u)\,u\,dx = \int_\Omega \Big(|\nabla u|^2 + A\frac{u^2}{|y|^\alpha}\Big)dx - \int_{\Gamma_{R_1,R_2}} u\,(\nabla u\cdot\nu)\,d\sigma - \int_{\Sigma_{R_1,R_2}} u\,(\nabla u\cdot\nu)\,d\sigma\,. \qquad (5.9)$$

Suppose $q = \max\{q_k, q_k^\alpha\}$. By the same computation of (5.6), there exists a nonincreasing sequence $R_{1,n} \to 0$, $R_{1,n} > 0$, such that

$$R_{1,n}^{-[(k-1)(q-2)-2]/q} \int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^2 \, d\sigma \to 0 \quad \text{as } n \to \infty\,.$$

Hence,

$$\begin{aligned} \Big| \int_{\Gamma_{R_1,R_2}} u\,(\nabla u \cdot \nu)\, d\sigma \Big| &\leq \Big(\int_{\Gamma_{R_{1,n},R_2}} u^2 d\sigma \Big)^{1/2} \Big(\int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^2 \, d\sigma \Big)^{1/2} \\ &= \left(R_{1,n}^{[(k-1)(q-2)-2]/q+\alpha} \int_{\Gamma_{R_{1,n},R_2}} \frac{u^2}{|y|^\alpha} d\sigma \right)^{1/2} o\,(1)_{n\to\infty} \end{aligned} \tag{5.10}$$

and thus, as $q \geq q_k^\alpha$ implies $\alpha + [(k-1)\,(q-2)-2]\,/q \geq 1$, (5.9) yields

$$\int_{B_{R_2}} f\,(u)\, u \, dx = \int_{B_{R_2}} \left(|\nabla u|^2 + A \frac{u^2}{|y|^\alpha} \right) dx + \int_{\partial B_{R_2}} u\,(\nabla u \cdot \nu)\, d\sigma\,. \tag{5.11}$$

Now assume $q = q_{k,N}$ and take $R_{1,n} \to 0$ such that

$$R_{1,n} \int_{\Gamma_{R_{1,n},R_2}} \left(|u|^{2^*} + |\nabla u|^q \right) d\sigma \to 0 \quad \text{as } n \to \infty\,.$$

Since $2^*/(2^*-1) = 2N/(N+2) < q_{k,N}$ and

$$(k-1) \left(\frac{N+2}{2N} - \frac{1}{q} \right) = \frac{N-2}{2N} + \frac{1}{q}\,,$$

we infer that there exists a constant $C > 0$ such that

$$\begin{aligned} &\Big| \int_{\Gamma_{R_{1,n},R_2}} u\,(\nabla u \cdot \nu)\, d\sigma \Big| \\ \leq\;& \Big(\int_{\Gamma_{R_{1,n},R_2}} |u|^{2^*} \, d\sigma \Big)^{1/2^*} \Big(\int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^{\frac{2N}{N+2}} \, d\sigma \Big)^{(N+2)/2N} \\ \leq\;& C \Big(\int_{\Gamma_{R_{1,n},R_2}} |u|^{2^*} \, d\sigma \Big)^{1/2^*} \Big(\int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^q \, d\sigma \Big)^{1/q} R_{1,n}^{(k-1)[(N+2)/2N-1/q]} \\ =\;& C \Big(R_{1,n} \int_{\Gamma_{R_{1,n},R_2}} |u|^{2^*} \, d\sigma \Big)^{1/2^*} \Big(R_{1,n} \int_{\Gamma_{R_{1,n},R_2}} |\nabla u|^q \, d\sigma \Big)^{1/q}, \end{aligned}$$

so that one gets (5.11) again by passing to the limit as $n \to \infty$. Finally, in a similar way, since $2^*/(2^*-1) < 2$ and $(N-1)/N = 1/2^* + 1/2$ we obtain

$$\begin{aligned}
&\Big| \int_{\partial B_{R_2}} u\,(\nabla u \cdot \nu)\, d\sigma \Big| \\
\leq\ & \Big(\int_{\partial B_{R_2}} |u|^{2^*}\, d\sigma \Big)^{1/2^*} \Big(\int_{\partial B_{R_2}} |\nabla u|^2\, d\sigma \Big)^{1/2} R_2^{(N-1)/N} \sigma_N^{1/N} \\
=\ & \sigma_N^{1/N} \Big(R_2 \int_{\partial B_{R_2}} |u|^{2^*}\, d\sigma \Big)^{1/2^*} \Big(R_2 \int_{\partial B_{R_2}} |\nabla u|^2\, d\sigma \Big)^{1/2},
\end{aligned}$$

so that, as there exists $R_{2,n} \to +\infty$ such that

$$R_{2,n} \int_{\partial B_{R_{2,n}}} \Big(|u|^{2^*} + |\nabla u|^2 \Big)\, d\sigma \to 0 \quad \text{as } n \to \infty\,,$$

the conclusion follows from (5.11). □

Proof of Theorem 4. Assuming that the assertion of the theorem is false, we can apply both Proposition 33 and Lemma 34, with $f\,(u) = |u|^{p-2}\, u$ and $F\,(u) = |u|^p\,/p$. Thus plugging (5.8) into (5.4) one gets

$$\left(\frac{N-2}{2} - \frac{N}{p} \right) \int_{\mathbb{R}^N} |\nabla u|^2\, dx = \left(\frac{N}{p} - \frac{N-\alpha}{2} \right) \int_{\mathbb{R}^N} \frac{Au^2}{|y|^\alpha} dx$$

which is not possible for $u \neq 0$ if

$$\left(\frac{N-2}{2} - \frac{N}{p} \right) \left(\frac{N}{p} - \frac{N-\alpha}{2} \right) < 0\,.$$

As this inequality is equivalent to $(\alpha, p) \in \mathcal{A}$, we have a contradiction. □

The same argument proving Theorem 4 also yields the following nonexistence result, which applies, for instance, to nonlinearities of the form

$$f(s) = \frac{|s|^{p-2}\, s}{1 + |s|^{p-\vartheta}} \quad \text{for all } s \in \mathbb{R}$$

provided that $p > \vartheta > 0$ are large enough.

Theorem 35. *Let $0 < \alpha < N$ and $f \in C\,(\mathbb{R}, \mathbb{R})$. Assume that there exists $\vartheta \geq \max\{2^*, 2_\alpha\}$ such that $\vartheta F\,(s) \leq f\,(s)\, s$ for all $s \in \mathbb{R}$. If $\alpha = 2$, assume in particular $\vartheta > 2^*$ $(= 2_\alpha)$. Then equation* (5.1) *has no nonzero classical solution $u \in C^2((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}) \cap X(\mathbb{R}^N, |y|^{-\alpha}\, dx)$ satisfying $f\,(u)\, u \in L^1(\mathbb{R}^N)$ and $\nabla u \in L^q_{loc}\left(\mathbb{R}^N\right)$ for $q = \max\{q_k, q_k^\alpha\}$ or $q = \max\{q_k, q_{k,N}\}$.*

Proof. One proceeds as in the proof of Theorem 4. Indeed, from (5.8) we get

$$\frac{1}{\vartheta}\int_{\mathbb{R}^N}\left(|\nabla u|^2+\frac{Au^2}{|y|^\alpha}\right)dx\geq\int_{\mathbb{R}^N}F(u)\,dx$$

so that (5.4) gives

$$(N-2)(\vartheta-2^*)\int_{\mathbb{R}^N}|\nabla u|^2\,dx+(N-\alpha)(\vartheta-2_\alpha)\int_{\mathbb{R}^N}\frac{Au^2}{|y|^\alpha}dx\leq 0$$

with $2^*\neq 2_\alpha$ if $\alpha\neq 2$ and $\vartheta>2^*=2_\alpha$ if $\alpha=2$. □

6. Behavior of weak solutions

This section is devoted to the proof of Theorem 5, which relies on a comparison argument in the spirit of the maximum principle. Accordingly, we assume all the hypotheses of the theorem and let $u\in X_{\mathrm{s}}(\mathbb{R}^N,|y|^{-\alpha}\,dx)$ be any z-nonincreasing weak solution of equation (1.4).

We need the following result from [19, Theorem 2 and Lemma 9].

Proposition 36. *Assume that $v\in D^{1,2}(\mathbb{R}^N)$ is nonnegative and satisfies*

$$\int_{\mathbb{R}^N}\nabla v\cdot\nabla\varphi\,dx\leq\int_{\mathbb{R}^N}\phi(x,v)\,\varphi\,dx\qquad\text{for all }\varphi\in C_c^\infty\left(\mathbb{R}^N\right),\ \varphi\geq 0,$$

where $\phi:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that

$$0\leq\phi(x,s)\leq b(x)\,s^{a-1}\qquad\text{for all }s\geq 0\text{ and almost every }x\in\mathbb{R}^N$$

with $a\in(2,2^*)$ and $b\in L^{\frac{2^*}{2^*-a}}(\mathbb{R}^N)$. Then v is bounded in a neighborhood of the origin and satisfies $\limsup_{|x|\to\infty}|x|^{N-2}v(x)<\infty$.

Since $\alpha<2\leq k$ implies $C_c^\infty(\mathbb{R}^N)\subseteq X$, every weak solution actually solves equation (1.4) in the distributional sense on $\mathbb{R}^N$. The boundedness of u and a first decay estimate then follow from Proposition 36, as we show in the following lemma.

Lemma 37. *$u\in L^\infty(\mathbb{R}^N)$ and $\limsup_{|x|\to\infty}|x|^{N-2}u(x)<\infty$.*

Proof. First recall from Proposition 21 that $u\in L^q(\mathbb{R}^N)$ for all $q\in[p_\alpha,2^*]$ and observe that $p\in(2+2p_\alpha/N,2^*)$ is equivalent to $p_\alpha<2^*(p-2)/(2^*-2)<2^*$, so that we can fix $a\in(2,2^*)$ such that $u\in L^{\frac{2^*(p-a)}{2^*-a}}\left(\mathbb{R}^N\right)$. Then, with a view to applying Proposition 36, define

$$\phi(x,s):=\left|f\left(u(x)^{\frac{p-a}{p-1}}|s|^{\frac{a-1}{p-1}}\right)\right|\quad\text{for all }s\in\mathbb{R}\text{ and almost every }x\in\mathbb{R}^N.$$

Since $\phi(x,u(x)) = |f(u(x))|$ for almost every $x \in \mathbb{R}^N$, by definition of weak solution one has

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi \, dx &\leq \int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^N} \frac{A}{|y|^\alpha} u\varphi \, dx = \int_{\mathbb{R}^N} f(u) \varphi \, dx \\ &\leq \int_{\mathbb{R}^N} \phi(x,u) \varphi \, dx\end{aligned}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi \geq 0$. Moreover $(\mathbf{f}_p)$ yields

$$\phi(x,s) \leq M \left| u(x)^{\frac{p-a}{p-1}} s^{\frac{a-1}{p-1}} \right|^{p-1} = M u(x)^{p-a} s^{a-1}$$

for all $s \geq 0$ and almost every $x \in \mathbb{R}^N$. Therefore, by Proposition 36, we conclude that u is bounded in a neighborhood of the origin and satisfies $\limsup_{|x|\to\infty} |x|^{N-2} u(x) < \infty$. Thus u is bounded in a neighborhood of $\{0\} \times \mathbb{R}^{N-k}$ by z-nonincreasing and, as $u \in C^2(\mathbb{R}^N \setminus (\{0\} \times \mathbb{R}^{N-k}))$ by standard elliptic regularity theory, the proof of the lemma is complete. □

In order to improve the asymptotic estimate of Lemma 37 and deduce (1.16), we need some preliminaries. First, let $\beta < 2\sqrt{A}/(2-\alpha)$ and fix $\varepsilon \in (0,A)$ such that

$$\beta_\varepsilon := \frac{2\sqrt{A-\varepsilon}}{2-\alpha} > \beta. \tag{6.1}$$

Notice that $\beta_\varepsilon^2 (1-\alpha/2)^2 = A - \varepsilon$. Second, by $(\mathbf{f}_p)$ and Lemma 37, let $C_1 > 0$ and $R_1 > 0$ be such that

$$f(u) \leq M u^{p-2} u \leq \frac{C_1}{|x|^{(N-2)(p-2)-\alpha}} \frac{u}{|x|^\alpha} \qquad \text{for almost every } |x| \geq R_1 . \tag{6.2}$$

Finally, thanks to the fact that $p \in (2 + 2p_\alpha/N, 2^*)$ implies $(N-2)(p-2) > \alpha$, fix $R_\varepsilon \geq R_1$ such that $|x| \geq R_\varepsilon$ implies $C_1 |x|^{\alpha-(N-2)(p-2)} \leq \varepsilon$ and set

$$\Omega_\varepsilon := \mathbb{R}^N \setminus \bar{B}_{R_\varepsilon} .$$

Lemma 38. *For all nonnegative $h \in X$ with $\operatorname{supp} h \subset \Omega_\varepsilon$ one has*

$$\int_{\Omega_\varepsilon} \nabla u \cdot \nabla h \, dx \leq -(A-\varepsilon) \int_{\Omega_\varepsilon} \frac{uh}{|x|^\alpha} dx . \tag{6.3}$$

Proof. By (6.2) and the definition of Ω_ε we have

$$f(u) \leq \frac{C_1}{|x|^{(N-2)(p-2)-\alpha}} \frac{u}{|x|^\alpha} \leq \varepsilon \frac{u}{|x|^\alpha} \qquad \text{and} \qquad \frac{A}{|y|^\alpha} \geq \frac{A}{|x|^\alpha}$$

for almost every $x \in \Omega_\varepsilon$. Hence, by definition of weak solution, one obtains

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla u \cdot \nabla h \, dx &= \int_{\Omega_\varepsilon} f(u) \, h \, dx - A \int_{\Omega_\varepsilon} \frac{uh}{|y|^\alpha} dx \\ &\leq \varepsilon \int_{\Omega_\varepsilon} \frac{uh}{|x|^\alpha} dx - A \int_{\Omega_\varepsilon} \frac{uh}{|x|^\alpha} dx = -(A-\varepsilon) \int_{\Omega_\varepsilon} \frac{uh}{|x|^\alpha} dx\end{aligned}$$

for every h as in the statement of the lemma. □

In order to develop our comparison argument, define

$$v_\varepsilon(x) := e^{-\beta_\varepsilon |x|^{1-\alpha/2}} \qquad \text{for all } x \in \mathbb{R}^N.$$

Lemma 39. *$v_\varepsilon \in D^{1,2}(\mathbb{R}^N)$ and for all nonnegative $h \in D_0^{1,2}(\Omega_\varepsilon)$ one has*

$$\int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla h \, dx \geq -(A-\varepsilon) \int_{\Omega_\varepsilon} \frac{v_\varepsilon h}{|x|^\alpha} dx \,. \tag{6.4}$$

Proof. For every $x \in \mathbb{R}^N \setminus \{0\}$ straightforward computations give

$$|\nabla v_\varepsilon(x)| = \beta_\varepsilon \left(1 - \frac{\alpha}{2}\right) |x|^{-\alpha/2} v_\varepsilon(x) = \sqrt{A-\varepsilon}\, |x|^{-\alpha/2} v_\varepsilon(x)$$

and

$$\triangle v_\varepsilon(x) = \sqrt{A-\varepsilon} \left(\sqrt{A-\varepsilon} - \left(N - 1 - \frac{\alpha}{2}\right) |x|^{\alpha/2-1}\right) |x|^{-\alpha} v_\varepsilon(x) \,.$$

Hence, $|\nabla v_\varepsilon|^2 = (A-\varepsilon) |x|^{-\alpha} v_\varepsilon^2 \in L^1(\mathbb{R}^N)$ and for all $h \in C_c^\infty(\Omega_\varepsilon)$ one has

$$\int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla h \, dx = -\sqrt{A-\varepsilon} \int_{\Omega_\varepsilon} \left(\sqrt{A-\varepsilon} - \frac{N-1-\alpha/2}{|x|^{1-\alpha/2}}\right) \frac{v_\varepsilon h}{|x|^\alpha} dx \,. \tag{6.5}$$

As $|x|^{-\alpha} v_\varepsilon, |x|^{-1-\alpha/2} v_\varepsilon \in L^{2^*/(2^*-1)}(\Omega_\varepsilon)$, an obvious density argument shows that (6.5) also holds for all $h \in D_0^{1,2}(\Omega_\varepsilon)$ and the conclusion then follows since v_ε is positive and $1 + \alpha/2 < 2 < N$. □

We can now conclude the proof of Theorem 5.

Proof of Theorem 5. Define $\Omega_0 := \left\{x \in \mathbb{R}^N : R_\varepsilon < |x| < R_\varepsilon + 1\right\} \subset \Omega_\varepsilon$ and introduce the mappings $w := \|u/v_\varepsilon\|_{L^\infty(\Omega_0)} v_\varepsilon - u \in D^{1,2}(\mathbb{R}^N)$ and $\bar{w} := \chi_{\Omega_\varepsilon} w_-$, defined almost everywhere in $\mathbb{R}^N$. Clearly $\bar{w}$ is nonnegative and $\operatorname{supp} \bar{w} \subset \Omega_\varepsilon$. Moreover, from

$$0 \leq w_- = -\chi_{\{w<0\}} w = \chi_{\{w<0\}} \left(u - \|u/v_\varepsilon\|_{L^\infty(\Omega_0)} v_\varepsilon\right) \leq u$$

we derive that $\bar{w} \in L^2(\mathbb{R}^N, |y|^{-\alpha} \, dx)$. Finally, taking into account that w_- vanishes almost everywhere in Ω_0, it is not difficult to check that $\bar{w} \in D_0^{1,2}(\Omega_\varepsilon)$, which also gives $\bar{w} \in X$. So we can apply both Lemmas 38 and 39 with $h = \bar{w}$, so that, upon multiplying (6.4) by $\|u/v_\varepsilon\|_{L^\infty(\Omega_0)}$ and subtracting (6.3), we obtain

$$\int_{\Omega_\varepsilon} \nabla w \cdot \nabla \bar{w}\, dx \geq -(A-\varepsilon) \int_{\Omega_\varepsilon} \frac{w\bar{w}}{|x|^\alpha} dx\,,$$

that is,

$$-\int_{\Omega_\varepsilon} |\nabla w_-|^2\, dx \geq (A-\varepsilon) \int_{\Omega_\varepsilon} \frac{w_-^2}{|x|^\alpha} dx\,.$$

This implies $\|w_-\|_{D_0^{1,2}(\Omega_\varepsilon)} = 0$, which means $u \leq \|u/v_\varepsilon\|_{L^\infty(\Omega_0)} v_\varepsilon$ almost everywhere in Ω_ε. So, recalling (6.1), we get

$$\limsup_{|x|\to\infty} e^{\beta |x|^{1-\alpha/2}} u(x) \leq \limsup_{|x|\to\infty} e^{\beta_\varepsilon |x|^{1-\alpha/2}} u(x) < \infty$$

and this, together with the first part of Lemma 37, completes the proof.

7. Appendix

In this appendix we give the proof of Theorem 16, which relies on the study of the function (7.1) below, whose consideration is quite standard when dealing with cylindrical mappings (see for example [27], [28]).

Denote $m := N-k$ as in Subsection 4.1 and begin by assuming $N > k \geq 1$. Define

$$D_{\mathrm{s}}^{1,2}\left(\mathbb{R}^N\right) := \left\{u \in D^{1,2}\left(\mathbb{R}^N\right) : u(y,z) = u(|y|, |z|)\right\}$$

and, for any $u \in D_{\mathrm{s}}^{1,2}(\mathbb{R}^N)$, fix an auxiliary mapping $\tilde{u} : \mathbb{R}^2 \to \mathbb{R}$ such that $u(y,z) = \tilde{u}(|y|, |z|)$ for almost every $(y,z) \in \mathbb{R}^k \times \mathbb{R}^m$.

Lemma 40. *Let $\Lambda_1, \Lambda_2 \subset (0,+\infty)$ be any two open intervals, both bounded and bounded away from zero. Then $\tilde{u} \in W^{1,1}(\Lambda_1 \times \Lambda_2)$.*

Proof. Denote $\lambda_i := \inf \Lambda_i > 0$ for $i = 1, 2$ and set $\Lambda_1^{(k)} := \{y \in \mathbb{R}^k : |y| \in \Lambda_1\}$, $\Lambda_2^{(m)} := \{z \in \mathbb{R}^m : |z| \in \Lambda_2\}$. Since $u \in L^1_{loc}(\mathbb{R}^N)$, by Fubini's theorem we readily have

$$\begin{aligned} \int_{\Lambda_1 \times \Lambda_2} |\tilde{u}(r,t)|\, dr dt &\leq \frac{1}{\lambda_2^{m-1}} \int_{\Lambda_1} dr \int_{\Lambda_2} t^{m-1} |\tilde{u}(r,t)|\, dt \\ &= \frac{1}{\sigma_m \lambda_2^{m-1}} \int_{\Lambda_1} dr \int_{\Lambda_2^{(m)}} |\tilde{u}(r,|z|)|\, dz \\ &\leq \frac{1}{\sigma_k \sigma_m \lambda_1^{k-1} \lambda_2^{m-1}} \int_{\Lambda_1^{(k)} \times \Lambda_2^{(m)}} |\tilde{u}(|y|, |z|)|\, dx < \infty\,. \end{aligned}$$

Now we exploit the density of $C_c^\infty(\mathbb{R}^N) \cap D_{\mathrm{s}}^{1,2}(\mathbb{R}^N)$ in $D_{\mathrm{s}}^{1,2}(\mathbb{R}^N)$ (which follows from standard convolution and regularization arguments) in order to infer that, by the symmetries of u, there exist $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla_y u(y,z) \cdot y = \varphi(|y|, |z|)$ and $\nabla_z u(y,z) \cdot z = \psi(|y|, |z|)$ for almost every $(y,z) \in \mathbb{R}^k \times \mathbb{R}^m$. Then we have

$$\begin{aligned} \int_{\Lambda_1 \times \Lambda_2} \frac{|\varphi(r,t)|}{r} drdt &= \frac{1}{\sigma_k \sigma_m} \int_{\Lambda_1^{(k)} \times \Lambda_2^{(m)}} \frac{|\nabla_y u(y,z) \cdot y|}{|y|^k |z|^{m-1}} dx \\ &\leq \frac{1}{\sigma_k \sigma_m \lambda_1^{k-1} \lambda_2^{m-1}} \int_{\Lambda_1^{(k)} \times \Lambda_2^{(m)}} |\nabla_y u(y,z)| \, dx < \infty . \end{aligned}$$

Finally, letting $\phi \in C_c^\infty(\Lambda_1 \times \Lambda_2)$ and setting $\Phi(x) := \phi(|y|, |z|) \in C_c^\infty(\Lambda_1^{(k)} \times \Lambda_2^{(m)})$, we get

$$\begin{aligned} &\int_{\Lambda_1 \times \Lambda_2} \tilde{u}(r,t) \frac{\partial \phi}{\partial r}(r,t) \, drdt \\ &= \frac{1}{\sigma_k \sigma_m} \int_{\Lambda_1^{(k)} \times \Lambda_2^{(m)}} \frac{\tilde{u}(|y|, |z|)}{|y|^{k-1} |z|^{m-1}} \frac{\partial \phi}{\partial r}(|y|, |z|) \, dx \\ &= \frac{1}{\sigma_k \sigma_m} \int_{\Lambda_1^{(k)} \times \Lambda_2^{(m)}} \frac{u(x)}{|y|^{k-1} |z|^{m-1}} \sum_{j=1}^k \frac{\partial \Phi}{\partial y_j}(x) \frac{y_j}{|y|} dx \\ &= -\frac{1}{\sigma_k \sigma_m} \sum_{j=1}^k \int_{\Lambda_1^{(k)} \times \Lambda_2^{(m)}} \frac{\partial u}{\partial y_j}(x) \frac{y_j}{|y|^k |z|^{m-1}} \Phi(x) \, dx \\ &= -\frac{1}{\sigma_k \sigma_m} \int_{\Lambda_1^{(k)} \times \Lambda_2^{(m)}} \frac{\nabla_y u(x) \cdot y}{|y|^k |z|^{m-1}} \Phi(x) \, dx \\ &= -\frac{1}{\sigma_k \sigma_m} \int_{\Lambda_1^{(k)} \times \Lambda_2^{(m)}} \frac{\varphi(|y|, |z|)}{|y|^k |z|^{m-1}} \phi(|y|, |z|) \, dx \\ &= -\int_{\Lambda_1 \times \Lambda_2} \frac{\varphi(r,t)}{r} \phi(r,t) \, drdt . \end{aligned}$$

Similarly, one obtains $\frac{\partial \tilde{u}}{\partial t}(r,t) = t^{-1} \psi(r,t) \in L^1(\Lambda_1 \times \Lambda_2)$ and the proof is thus complete. □

For any $z \in \mathbb{R}^m$, $z \neq 0$, define

$$\bar{u}(r) := \int_{|z|/2}^{|z|} \tilde{u}(r,t) \, t^{\frac{m-1}{2}} dt \quad \text{for all } r > 0 . \tag{7.1}$$

Let $a, b \in \mathbb{R}$ be such that $0 < a < b$.

Lemma 41. *$\bar{u} \in W^{1,1}((a,b))$ and*

$$\bar{u}'(r) = \int_{|z|/2}^{|z|} \frac{\partial \tilde{u}}{\partial r}(r,t)\, t^{\frac{m-1}{2}} dt \quad \text{for almost every } r \in (a,b). \tag{7.2}$$

Proof. Set $\Lambda_1 := (a,b)$ and $\Lambda_2 := (|z|/2, |z|)$. From Lemma 40 one readily infers that $\bar{u}, \bar{u}' \in L^1(\Lambda_1)$, where $\bar{u}'$ is given by (7.2). Now fix any $\varphi \in C_c^\infty(\Lambda_1)$. Then, for every $\psi \in C_c^\infty(\Lambda_2)$, the product $\varphi(r)\psi(t)$ belongs to $C_c^\infty(\Lambda_1 \times \Lambda_2)$ and we have

$$\int_{\Lambda_2} \psi(t) \Big(\int_{\Lambda_1} \tilde{u}(r,t)\, \varphi'(r)\, dr \Big) dt = \int_{\Lambda_1 \times \Lambda_2} \tilde{u}(r,t)\, \varphi'(r)\, \psi(t)\, dr dt$$
$$= - \int_{\Lambda_1 \times \Lambda_2} \frac{\partial \tilde{u}}{\partial r}(r,t)\, \varphi(r)\, \psi(t)\, dr dt = - \int_{\Lambda_2} \psi(t) \Big(\int_{\Lambda_1} \frac{\partial \tilde{u}}{\partial r}(r,t)\, \varphi(r)\, dr \Big) dt$$

again by Lemma 40. Hence $\int_{\Lambda_1} \tilde{u}(r,t)\, \varphi'(r)\, dr = - \int_{\Lambda_1} \frac{\partial \tilde{u}}{\partial r}(r,t)\, \varphi(r)\, dr$ for almost every $r \in \Lambda_2$, whence, upon multiplying by $t^{(m-1)/2}$ and integrating over Λ_2, one readily deduces that $\bar{u}'$ is the weak derivative of $\bar{u}$ on Λ_1. $\square$

For any $\alpha > 0$, we now define $v(r) := r^{k-1-\alpha/2} \bar{u}(r)^2$ for all $r > 0$. Then, by Lemma 41, $v \in W^{1,1}((a,b))$ and

$$v'(r) = \Big(k - 1 - \frac{\alpha}{2}\Big) r^{k-2-\frac{\alpha}{2}} \bar{u}(r)^2 + 2 r^{k-1-\frac{\alpha}{2}} \bar{u}(r)\, \bar{u}'(r) \tag{7.3}$$

for almost every $r \in (a,b)$. Moreover, $v \in C([a,b])$ and

$$v(b) - v(a) = \int_a^b v'(r)\, dr. \tag{7.4}$$

Lemma 42. *Assume $\int_{(B_b \setminus B_a) \times \mathbb{R}^m} |y|^{-\alpha} u^2 dx < \infty$. Then*

$$v(b) - v(a) \leq \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}}{\sigma_k \sigma_m} \Big(\int_{(B_b \setminus B_a) \times \mathbb{R}^m} \frac{u^2}{|y|^\alpha} dx \Big)^{1/2} |z| \tag{7.5}$$

or

$$v(b) - v(a) \geq - \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}}{\sigma_k \sigma_m} \Big(\int_{(B_b \setminus B_a) \times \mathbb{R}^m} \frac{u^2}{|y|^\alpha} dx \Big)^{1/2} |z| \tag{7.6}$$

according as $\alpha \geq 2k-2$ or $\alpha < 2k-2$.

Proof. First, according as $\alpha \geq 2k-2$ or $\alpha < 2k-2$, i.e., $k - 1 - \alpha/2 \leq 0$ or $k - 1 - \alpha/2 > 0$, by (7.3) and (7.2) we have

$$v'(r) \leq 2 r^{k-1-\frac{\alpha}{2}} \bar{u}(r)\, \bar{u}'(r) \leq 2 r^{k-1-\frac{\alpha}{2}} |\bar{u}(r)| \left|\bar{u}'(r)\right|$$

$$\leq \quad 2r^{k-1-\frac{\alpha}{2}} \int_{|z|/2}^{|z|} |\tilde{u}(r,t)|\, t^{\frac{m-1}{2}} dt \int_{|z|/2}^{|z|} \left| \frac{\partial \tilde{u}}{\partial r}(r,t) \right| t^{\frac{m-1}{2}} dt$$

or

$$\begin{aligned} v'(r) &\geq 2r^{k-1-\frac{\alpha}{2}} \bar{u}(r)\, \bar{u}'(r) \geq -2r^{k-1-\frac{\alpha}{2}} |\bar{u}(r)| \left|\bar{u}'(r)\right| \\ &\geq -2r^{k-1-\frac{\alpha}{2}} \int_{|z|/2}^{|z|} |\tilde{u}(r,t)|\, t^{\frac{m-1}{2}} dt \int_{|z|/2}^{|z|} \left| \frac{\partial \tilde{u}}{\partial r}(r,t) \right| t^{\frac{m-1}{2}} dt \end{aligned}$$

for almost every $r \in (a,b)$. Then by Hölder inequalities, we compute

$$\begin{aligned} & \int_a^b r^{k-1-\frac{\alpha}{2}} \Big(\int_{|z|/2}^{|z|} |\tilde{u}(r,t)|\, t^{\frac{m-1}{2}} dt \Big) \Big(\int_{|z|/2}^{|z|} \big| \frac{\partial \tilde{u}}{\partial r}(r,t) \big| t^{\frac{m-1}{2}} dt \Big) dr \\ = & \int_a^b r^{\frac{k-1}{2}-\frac{\alpha}{2}} \Big(\int_{|z|/2}^{|z|} |\tilde{u}(r,t)|\, t^{\frac{m-1}{2}} dt \Big) r^{\frac{k-1}{2}} \Big(\int_{|z|/2}^{|z|} \big| \frac{\partial \tilde{u}}{\partial r}(r,t) | t^{\frac{m-1}{2}} dt \Big) dr \\ \leq & \Big(\int_a^b r^{k-1-\alpha} \Big(\int_{|z|/2}^{|z|} |\tilde{u}(r,t)| t^{\frac{m-1}{2}} dt \Big)^2 dr \Big)^{1/2} \cdot \\ & \cdot \Big(\int_a^b r^{k-1} \Big(\int_{|z|/2}^{|z|} \big| \frac{\partial \tilde{u}}{\partial r}(r,t) | t^{\frac{m-1}{2}} dt \Big)^2 dr \Big)^{1/2} \\ \leq & \Big(\int_a^b r^{k-1-\alpha} \frac{|z|}{2} \Big(\int_{|z|/2}^{|z|} \tilde{u}(r,t)^2 t^{m-1} dt \Big) dr \Big)^{1/2} \cdot \\ & \cdot \Big(\int_a^b r^{k-1} \frac{|z|}{2} \Big(\int_{|z|/2}^{|z|} \big| \frac{\partial \tilde{u}}{\partial r}(r,t) |^2 t^{m-1} dt \Big) dr \Big)^{1/2} \\ \leq & \frac{|z|}{2} \Big(\int_{(a,b)\times(0,+\infty)} \frac{\tilde{u}(r,t)^2}{r^\alpha} r^{k-1} t^{m-1} dr dt \Big)^{1/2} \cdot \\ & \cdot \Big(\int_{(a,b)\times(0,+\infty)} \big| \frac{\partial \tilde{u}}{\partial r}(r,t) \big|^2 r^{k-1} t^{m-1} dr dt \Big)^{1/2} \\ = & \frac{|z|}{2\sigma_k \sigma_m} \Big(\int_{(B_b \setminus B_a)\times \mathbb{R}^m} \frac{u^2}{|y|^\alpha} dx \Big)^{1/2} \Big(\int_{(B_b \setminus B_a)\times \mathbb{R}^m} \big| \frac{\partial \tilde{u}}{\partial r}(|y|,|z|) \big|^2 dx \Big)^{1/2} \\ \leq & \frac{|z|}{2\sigma_k \sigma_m} \Big(\int_{(B_b \setminus B_a)\times \mathbb{R}^m} \frac{u^2}{|y|^\alpha} dx \Big)^{1/2} \Big(\int_{\mathbb{R}^N} \big| \nabla_y u\big(x\big) \cdot \frac{y}{|y|} \big|^2 dx \Big)^{1/2} \\ \leq & \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}}{2\sigma_k \sigma_m} \Big(\int_{(B_b \setminus B_a)\times \mathbb{R}^m} \frac{u^2}{|y|^\alpha} dx \Big)^{1/2} |z| \end{aligned}$$

and the conclusion finally follows from (7.4). □

In order to give the proof of Theorem 16, we now assume $N > k \geq 2$ and let $A, \alpha > 0$.

Proof of Theorem 16. Let $u \in X_{\rm s}(\mathbb{R}^N, |y|^{-\alpha}\, dx)$ be nonincreasing with respect to z. First we observe that

$$\begin{aligned}\int_0^{+\infty} r^{k-1-\alpha}\bar u(r)^2 dr &\leq \frac{|z|}{2}\int_0^{+\infty} dr \int_{|z|/2}^{|z|} r^{k-1}\frac{\tilde u(r,t)^2}{r^\alpha} t^{m-1} dt \\ &\leq \frac{|z|}{2\sigma_k\sigma_m}\int_{\mathbb{R}^N}\frac{u^2}{|y|^\alpha}dx < \infty\,.\end{aligned}$$

Hence, according as $\alpha \geq 2$ or $\alpha < 2$, there exists $r_{1,n} \to 0^+$ or $r_{2,n} \to +\infty$ such that $v(r_{i,n}) \to 0$ as $n \to \infty$. Indeed, $\lambda := \liminf_{r\to 0^+} r^{k-1-\alpha/2}\bar u(r)^2 > 0$ implies $r^{k-1-\alpha}\bar u(r)^2 \geq \lambda r^{-\alpha/2}/2$ for r close to 0, and a contradiction then ensues if $\alpha \geq 2$; similarly for $\alpha < 2$ and r near $+\infty$. Moreover, since

$$\begin{aligned}\bar u\Big(r\Big)^{2^*} &\leq \Big(\frac{|z|}{2}\Big)^{2^*-1}\int_{|z|/2}^{|z|}\tilde u(r,t)^{2^*} t^{\frac{m-1}{2}2^*} dt \\ &= \Big(\frac{|z|}{2}\Big)^{2^*-1}\int_{|z|/2}^{|z|}\tilde u(r,t)^{2^*} t^{m-1+\frac{(m-1)(2^*-2)}{2}} dt \\ &\leq \frac{|z|^{2^*(m+1)/2-1}}{2^{2^*-1}}\int_{|z|/2}^{|z|}\tilde u(r,t)^{2^*} t^{m-1} dt\end{aligned}$$

implies

$$\begin{aligned}\int_0^{+\infty} r^{k-1}\bar u(r)^{2^*} dr &\leq \frac{|z|^{2^*(m+1)/2-1}}{2^{2^*-1}}\int_0^{+\infty} dr\int_{|z|/2}^{|z|} r^{k-1}\tilde u(r,t)^{2^*} t^{m-1} dt \\ &\leq \frac{|z|^{2^*(m+1)/2-1}}{2^{2^*-1}\sigma_k\sigma_m}\int_{\mathbb{R}^N} u^{2^*} dx < \infty\,,\end{aligned}$$

the same argument yields that a sequence $r_{2,n} \to +\infty$ such that $v(r_{2,n}) \to 0$ as $n \to \infty$ also exists for $\alpha \geq 4k/N - 2$. Thus it exists for every $\alpha > 0$, because $k < N$ implies $4k/N - 2 < 2$. Now suppose $\alpha \geq 2k - 2$. Then one has $\alpha \geq 2$ (recall that $k \geq 2$) and we can apply Lemma 42 with $b = r > 0$ arbitrary and $a = r_{1,n}$ (with n large enough to assure $a < b$). Thus (7.5) gives

$$v(r) - v(r_{1,n}) \leq \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}}{\sigma_k\sigma_m}\Big(\int_{(B_r\setminus B_{r_{1,n}})\times\mathbb{R}^m}\frac{u^2}{|y|^\alpha}dx\Big)^{1/2}|z|$$

$$\leq \quad \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}}{\sigma_k \sigma_m \sqrt{A}} \Big(\int_{\mathbb{R}^N} \frac{Au^2}{|y|^\alpha} dx \Big)^{1/2} |z| \leq \frac{\|u\|^2}{\sigma_k \sigma_m \sqrt{A}} |z|$$

and letting $n \to \infty$ we get

$$r^{k-1-\alpha/2} \bar{u}(r)^2 \leq \frac{\|u\|^2}{\sigma_k \sigma_m \sqrt{A}} |z| \quad \text{for all } r > 0 \,. \tag{7.7}$$

Similarly, if $\alpha < 2k - 2$, for $a = r > 0$ arbitrary and $b = r_{2,n}$ with n large, (7.6) yields

$$v(r_{2,n}) - v(r) \geq -\frac{\|u\|^2}{\sigma_k \sigma_m \sqrt{A}} |z| \ ,$$

so that, as $n \to \infty$, we get (7.7) again. On the other hand, by nonincreasing, for all $r > 0$ one has

$$\bar{u}(r) \geq \tilde{u}(r, |z|) \int_{|z|/2}^{|z|} t^{\frac{m-1}{2}} dt = \frac{2^{(m+1)/2} - 1}{2^{(m-1)/2}(m+1)} \tilde{u}(r, |z|) \, |z|^{\frac{m+1}{2}} \,.$$

Hence we conclude

$$r^{k-1-\alpha/2} \Big(\frac{2^{(m+1)/2} - 1}{2^{(m-1)/2}(m+1)} \tilde{u}(r, |z|) \, |z|^{\frac{m+1}{2}} \Big)^2 \leq \frac{\|u\|^2}{\sigma_k \sigma_m \sqrt{A}} |z|$$

and the result ensues.

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