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SYMMETRY OF EXTREMAL FUNCTIONS IN MOSER–TRUDINGER INEQUALITIES AND A HÉNON TYPE PROBLEM IN DIMENSION TWO

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Abstract. In this paper, we analyze the symmetry properties of maximizers of a Hénon type functional in dimension two. Namely, we study the symmetry of the functions that realize the maximum

$$\sup_{\substack{u\in H^1(\Omega)\\ ||u||\le 1}} \int_\Omega \left(e^{\gamma u^2}-1\right)|x|^\alpha\,dx,$$

where Ω is the unit ball of $\mathbf{R}^2$ and α, $\gamma>0$. We identify and study the limit functional

$$\sup_{\substack{u\in H^1(\Omega)\\ ||u||\le 1}} \int_{\partial\Omega} \left(e^{\gamma u^2}-1\right)d\sigma,$$

which is the main ingredient to describe the behavior of maximizers as $\alpha\to\infty$. We also consider the limit functional as $\alpha\to 0$ and the properties of its maximizers.

1. Introduction

In this paper we address the question of symmetry properties of extremal functions for certain maximization problems related to Moser–Trudinger type inequalities. These are the natural extension to dimension two of classical inequalities involving the Sobolev space H^1 that hold in dimension $n\ge 3$. Although the validity of these inequalities has long been established,

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the question of symmetry properties of extremal functions is, for some problems, a rather recent research topic which has generated considerable efforts in the last few years (see for example [9], [13], [16], [19] and references therein). Our interest in this kind of topic grew out of the desire to understand in dimension two some symmetry breaking phenomena which were detected in higher dimension for power nonlinearities with Hénon weight (for an idea of the rapidly increasing literature on Hénon type equations see [3], [4], [5], [6], [14], [22], [23], [24], [25], [26]).

Since our results are rather articulated and concern different problems, we will confine ourselves in this Introduction to the description of the main ideas involved in our research, and to the statement of some of our main results.

Throughout this paper Ω denotes the unit disk of $\mathbf{R}^2$, and γ and α are positive real numbers. We first consider the problem

$$S_{\alpha,\gamma} = \sup_{\substack{u\in H^1(\Omega)\\ ||u||\le 1}} \int_\Omega \left(e^{\gamma u^2} - 1\right) |x|^\alpha \, dx, \tag{1.1}$$

where $||u||$ denotes the usual norm in $H^1(\Omega)$. The form of the nonlinearity is the one appearing in Moser–Trudinger inequalities ([2], [7], [8], [20]), while the weight $|x|^\alpha$ is proper of the Hénon equation, originally introduced in [17]. A characteristic feature of the present paper is the fact that we always work in the space $H^1(\Omega)$, and not in $H^1_0(\Omega)$, as in most of the existing literature; as we will see in the course of the paper this will cause relevant differences with known results for Dirichlet boundary conditions.

The numbers $S_{\alpha,\gamma}$ are finite for $\gamma \le 2\pi$ and attained at least for $\gamma < 2\pi$. Since the weight in (1.1) is radially symmetric, it is natural to inquire whether the extremal functions are radial or not, with the aim of obtaining the "symmetry picture" for extremals, as a function of the parameters γ and α. We are thus dealing with the extension in dimension 2 of the problem

$$\Sigma_{\alpha,p} = \sup_{\substack{u\in H^1(B)\\ ||u||\le 1}} \int_B |u|^p |x|^\alpha \, dx, \tag{1.2}$$

where B is the unit ball of $\mathbf{R}^n$, $n \ge 3$. We recall that in this case it is known from [14] that for every $p \in (\frac{2n-2}{n-2}, \frac{2n}{n-2})$, no maximizer of $\Sigma_{\alpha,p}$ is radial provided α is large, while if p is close to 2, then every maximizer is radial for α large. In that paper no information is given for α small. It is interesting to note, also for later reference, that $\frac{2n}{n-2}$ and $\frac{2n-2}{n-2}$ are the critical exponents for the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ and $L^p(\partial\Omega)$ respectively.

In dimension 2, the role of these two critical exponents is played by the limiting exponents γ for the Moser–Trudinger inequality in H^1 and the Moser–Trudinger trace inequality, which are respectively the numbers 2π and π.

A first and natural attempt to attack the question of symmetry breaking is to consider the restriction of problem (1.1) to the space of radial functions, namely

$$S^R_{\alpha,\gamma} = \sup_{\substack{u \in H^1_{rad}(\Omega) \\ ||u|| \le 1}} \int_\Omega \left(e^{\gamma u^2} - 1\right) |x|^\alpha \, dx \tag{1.3}$$

and inquire whether $S_{\alpha,\gamma} > S^R_{\alpha,\gamma}$ for large values of α by means of some asymptotic analysis; this is the approach followed in [25] and [14] for problems with power nonlinearities in higher dimension.

In the present setting, the asymptotic analysis for α large leads to the following facts: if $\gamma \in (\pi, 2\pi)$, then, as $\alpha \to \infty$, both $S_{\alpha,\gamma}$ and $S^R_{\alpha,\gamma}$ decay to zero, but $S_{\alpha,\gamma}$ much more slowly than $S^R_{\alpha,\gamma}$. Therefore, $S_{\alpha,\gamma} > S^R_{\alpha,\gamma}$ for α large, leading to a first symmetry breaking result. On the contrary, if $\gamma < \pi$, then it can be proved that $S_{\alpha,\gamma}$ and $S^R_{\alpha,\gamma}$ are infinitesimal *of the same order*, a fact that apparently leaves few hopes of carrying out the symmetry analysis for these values of γ.

However, and we think that this is one of the main points of interest of our work, the asymptotic analysis mentioned above also reveals the nontrivial fact that (1.1) and (1.2) admit "limit" versions as $\alpha \to \infty$. Indeed, we will prove for example that

$$S_\gamma = \sup_{\substack{u \in H^1(B) \\ ||u|| \le 1}} \int_{\partial\Omega} \left(e^{\gamma u^2} - 1\right) \, d\sigma \tag{1.4}$$

serves as a limit problem for (1.1). Problem (1.4) leads to the Moser–Trudinger trace inequality, studied for example in ([2], [7], [8], [10], [27]). It is known from these papers that S_γ is finite and attained if and only if $\gamma \le \pi$. To our knowledge, the question of symmetry of extremal functions for this type of inequalities is open. For power nonlinearities in higher dimensions, the corresponding classical trace inequality has been studied in [11], [12], [13], [14], [19].

The preceding discussion establishes a link between Hénon type problems and the trace inequality also in dimension two. It is clear, for example, that a symmetry breaking result for maximizers of (1.4) can produce a similar result for maximizers of (1.1) when α is large. Notice that (1.4) makes sense only for $\gamma \le \pi$, which is exactly the range of γ's left out of the above discussion on the levels $S_{\alpha,\gamma}$ and $S^R_{\alpha,\gamma}$.

At this point the question of symmetry for the extremals of (1.1) is transformed into an analogous one for the extremals of (1.4), a problem that is even more interesting since it gives rise to a classical inequality.

Given the unpleasant form of the nonlinearity in (1.4), a very interesting fact is that the symmetry properties of the maximizers are strictly related to rather simple and classical objects. Consider indeed the Steklov problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

It is well known that for this problem, the standard theory applies to yield an unbounded sequence of positive eigenvalues $(\lambda_n)_n$, the first one being simple. Therefore, the principal eigenfunction φ_1 is radial. The (inverses of the) eigenvalues λ_n can be obtained as critical levels of the functional

$$F(u) = \int_{\partial\Omega} u^2 \, d\sigma$$

constrained on the unit sphere of $H^1(\Omega)$.

The importance of the eigenvalues of the Steklov problem in the symmetry properties of extremals for (1.4) results from the observation that the functional F gives, in a suitable sense, the first term of the expansion of the functional in (1.4) around $\gamma = 0$. Thus, it is very likely that the behavior of the eigenvalues of (1.5) will be relevant for problem (1.4) for small values of γ. What is more surprising is that (1.5) provides nontrivial information on the symmetry of extremals for (1.4) *also for larger values* of γ.

The features of problem (1.5) will be used to carry out the analysis for γ small, as is natural, but also to investigate the variational properties of maximizers of S_γ over $H^1_{rad}(\Omega)$ seen as critical points over the whole space $H^1(\Omega)$. These kinds of arguments yield some of our main results.

Theorem A. *Let λ_1, λ_2 be the first two eigenvalues of the Steklov problem* (1.5)*, and let φ_1 be the first eigenfunction, positive, radial and normalized in $H^1(\Omega)$. Then*

i) for every $\gamma \in (0, \pi(\lambda_2 - \lambda_1))$, φ_1 is a nondegenerate local maximum for (1.4) *on the unit sphere of $H^1(\Omega)$;*

ii) for every γ small enough, φ_1 is the unique solution of problem (1.4);

iii) for every $\gamma \in (\pi(\lambda_2 - \lambda_1), \pi]$, all maximizers of (1.4) *are nonradial.*

This is our main result on the Moser–Trudinger trace inequality. The quantity $\lambda_2 - \lambda_1$ is strictly less than one, see Remark 4.2.

Concerning the Hénon problem (1.1), an asymptotic analysis as $\gamma \to 0$ and $\alpha \to \infty$ simultaneously and Theorem A allow us to give the following picture of the symmetry properties of solutions.

Theorem B. *Let λ_1, λ_2 be the first two eigenvalues of the Steklov problem* (1.5)*. Then,*

i) For every γ small enough and for every $\alpha > 0$, problem (1.1) *has a unique solution, which is therefore radial.*

ii) For every $\gamma \in (\pi(\lambda_2 - \lambda_1), 2\pi)$, all maximizers of (1.1) *are nonradial for large α.*

Let us point out immediately that none of these results cover the whole interval of γ's where the problems are defined. Indeed we don't know what happens (for both problems) between "γ small" and the number $\pi(\lambda_2 - \lambda_1)$.

A few comments are in order. First of all, notice that although when $\gamma < \pi$ the levels $S_{\alpha,\gamma}$ and $S^R_{\alpha,\gamma}$ are of the same order as $\alpha \to \infty$, as we said above, the argument used in the proof of Theorem B will allow us to say that the leading coefficient in the expansion of $S_{\alpha,\gamma}$ is larger than the corresponding one for $S^R_{\alpha,\gamma}$, at least in the interval $(\pi(\lambda_2 - \lambda_1), 2\pi)$. This is rather precise information that is not common to obtain.

Furthermore, the above theorems clarify the reason for the loss of symmetry in problem (1.1). Indeed maximizers of $S_{\alpha,\gamma}$ can fail to be radial either because of the absence of a trace inequality in the limit problem (this happens for $\gamma \in (\pi, 2\pi)$) or because of a loss of symmetry in the limit problem itself (as is the case for $\gamma \in (\pi(\lambda_2 - \lambda_1), \pi])$. In the first case we have a different asymptotic behavior for the levels $S_{\alpha,\gamma}$ and $S^R_{\alpha,\gamma}$ as $\alpha \to \infty$, while in the second the difference between levels is much more subtle to detect.

It is also interesting to compare the previous result to the case of power nonlinearities in the unit ball B of $\mathbf{R}^n$, with $n \geq 3$, studied in [14]. As we said, the authors proved there a symmetry breaking result for extremals of problem (1.2) provided $p \in \left(\frac{2n-2}{n-2}, \frac{2n}{n-2}\right)$.

In that case, maximizers over $H^1_{rad}(B)$ for the limiting problem are not local maximizers over $H^1(B)$ for $p > 1+\frac{\lambda_2}{\lambda_1}$, a number analogous to $\pi(\lambda_2-\lambda_1)$ in the preceding Theorem. However, this fact *is useless* to prove symmetry breaking for large α because $1 + \frac{\lambda_2}{\lambda_1}$ is always larger than $\frac{2n-2}{n-2}$, so that it falls in an interval where one has already proved the result, and by a simpler argument of level estimates.

The occurrence of symmetry breaking below the critical exponent for the trace imbedding is thus peculiar to the situation described in the present work.

Finally, we would like to point out that we believe that the value $\pi(\lambda_2 - \lambda_1)$ is the real threshold between symmetry and symmetry breaking, for both problems. However, we have to leave this as a conjecture. Another conjecture suggested by the analysis of maximizers for the Moser–Trudinger inequality, is the existence of some $\bar{\gamma} \in]0, 2\pi[$ such that symmetry breaking occurs for every $\gamma > \bar{\gamma}$ and any $\alpha \geq 0$. This is in contrast with what happens when studying the problem in $H_0^1(\Omega)$. Indeed, then the maximizer is always radial for small α. We believe these conjecture certainly deserve further studies.

The paper is organized as follows. In Section 2 we give the main existence results for problem (1.1) and we recall some fundamental results for Moser–Trudinger type inequalities. In Section 3 we carry out the asymptotic analysis for problem (1.1) and we prove a first symmetry breaking result. Section 4 is entirely devoted to Moser–Trudinger type inequalities, without weight, and we prove Theorem A. Finally, in Section 5 we complete the proof of Theorem B.

2. Existence results

We begin by settling the question of existence of radial and nonradial solutions of the maximization problem

$$S_{\alpha,\gamma} = \sup_{\substack{u \in H^1(\Omega) \\ ||u|| \leq 1}} \int_\Omega \left(e^{\gamma u^2} - 1 \right) |x|^\alpha \, dx, \tag{2.1}$$

where Ω is the unit disk in $\mathbf{R}^2$, α and γ are positive numbers and $||u||$ denotes the $H^1(\Omega)$ norm.

Remark 2.1. All the functionals with exponential nonlinearities studied in this paper enjoy a trivial monotonicity property. This means that in maximization problems, the constraints $||u|| \leq 1$ and $||u|| = 1$ are equivalent. We will freely replace one constraint with the other when there is some convenience in doing so. In particular, when we speak of critical points, we always intend that the constraint $||u|| = 1$ is used, so that the standard theory of Lagrange multipliers applies.

The presence of the weight $|x|^\alpha$ introduces some differences in the behavior of (2.1) according to whether one works with radial functions or in the whole space $H^1(\Omega)$. We recall a fundamental property which is the H^1 version of some Moser–Trudinger inequalities; the proof can be recovered from [2], [7], [8], [18], [27].

Theorem 2.2. *There results*

$$\sup_{\substack{u\in H^1(\Omega)\\ ||u||\le 1}} \int_\Omega e^{\gamma u^2}\,dx < +\infty \tag{2.2}$$

if and only if $\gamma \le 2\pi$, *and*

$$\sup_{\substack{u\in H^1(\Omega)\\ ||u||\le 1}} \int_{\partial\Omega} e^{\gamma u^2}\,d\sigma < +\infty \tag{2.3}$$

if and only if $\gamma \le \pi$. *The suprema are attained whenever they are finite.*

As far as Problem (2.1) is concerned, we can immediately obtain an existence result in the *subcritical* range $\gamma < 2\pi$. We provide a proof for completeness.

Theorem 2.3. *For every* $\gamma < 2\pi$ *and every* $\alpha \ge 0$,

$$S_{\alpha,\gamma} = \sup_{\substack{u\in H^1(\Omega)\\ ||u||\le 1}} \int_\Omega \left(e^{\gamma u^2}-1\right)|x|^\alpha\,dx \tag{2.4}$$

is attained. Moreover, if u *is a maximizer, then* u *can be chosen positive and* $\|u\| = 1$.

Proof. Of course $S_{\alpha,\gamma}$ is finite by Theorem 2.2. Let $(u_n)_n \subset H^1(\Omega)$ be a maximizing sequence, with $||u_n|| \le 1$; then, up to subsequences, we can assume that u_n converges to u weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for every p finite. Using the inequality $|e^x - e^y| \le |x-y|(e^x+e^y)$ we now estimate

$$\left|\int_\Omega\left(e^{\gamma u_n^2}-e^{\gamma u^2}\right)|x|^\alpha\,dx\right| \le \int_\Omega\left|e^{\gamma u_n^2}-e^{\gamma u^2}\right|dx \le \gamma\int_\Omega |u_n^2-u^2|(e^{\gamma u_n^2}+e^{\gamma u^2})\,dx.$$

Taking $p > 1$ such that $p\gamma \le 2\pi$ (which is possible since $\gamma < 2\pi$), we see that

$$\int_\Omega e^{\gamma u_n^2}|u_n^2-u^2|\,dx \le \left(\int_\Omega e^{p\gamma u_n^2}\,dx\right)^{1/p}\left(\int_\Omega |u_n^2-u^2|^{p'}\,dx\right)^{1/p'}, \tag{2.5}$$

and likewise for the term containing $e^{\gamma u^2}$. Observe that the first integral is uniformly bounded by Theorem 2.2, while the second tends to zero as $n \to \infty$ by the strong convergence of u_n in every L^p. Thus

$$S_{\alpha,\gamma} + o(1) = \int_\Omega\left(e^{\gamma u_n^2}-1\right)|x|^\alpha\,dx = \int_\Omega\left(e^{\gamma u^2}-1\right)|x|^\alpha\,dx + o(1).$$

Finally, since $||u|| \leq 1$, we see that u is the required maximum. The last statements of the theorem are obvious since for instance if $||u|| < 1$, then setting $v = u/||u||$, we would have

$$\int_\Omega \left(e^{\gamma v^2} - 1\right) |x|^\alpha \, dx > \int_\Omega \left(e^{\gamma u^2} - 1\right) |x|^\alpha \, dx = S_{\alpha,\gamma},$$

a contradiction. Positivity follows from Remark 2.9. □

Remark 2.4. We may also retain from the previous proof the fact that if $(u_n)_n \subset H^1(\Omega)$ is a maximizing sequence for $S_{\alpha,\gamma}$, then it is compact. Indeed if for some subsequence we have $u_n \rightharpoonup u$, then the proof of the theorem shows that $||u|| = 1$ so that the claim follows.

We now restrict Problem (2.1) to the space of radial functions, namely we consider the problem

$$S^R_{\alpha,\gamma} = \sup_{\substack{u \in H^1_{rad}(\Omega) \\ ||u|| \leq 1}} \int_\Omega \left(e^{\gamma u^2} - 1\right) |x|^\alpha \, dx.$$

We will show that $S^R_{\alpha,\gamma}$ is finite and attained for a much wider interval of γ's, depending on α. Such phenomenon, caused by the presence of the weight, has already been observed by Ni in [21] for the Dirichlet problem, in [14] for the Neumann problem (both with power nonlinearities) and in [6] for nonlinearities with exponential growth in H^1_0.

We begin with an estimate of the growth of radial functions in the spirit of the *radial lemma* of [21].

Lemma 2.5. *There exists $C > 0$ such that for all $u \in H^1_{rad}(\Omega)$,*

$$|u(x)| \leq \Big(C + \frac{1}{\sqrt{2\pi}}(-\log|x|)^{1/2}\Big)||u||, \qquad \forall x \in \Omega \setminus \{0\}. \tag{2.6}$$

Proof. We have

$$\begin{aligned} |u(x)| &\leq |u(1)| + \int_{|x|}^1 |u'(\rho)|\, d\rho \leq |u(1)| + \Big(\int_{|x|}^1 |u'(\rho)|^2 \rho \, d\rho\Big)^{\frac12} \Big(\int_{|x|}^1 \frac{1}{\rho}\, d\rho\Big)^{\frac12} \\ &\leq |u(1)| + \left(\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |u'(\rho)|^2 \rho \, d\rho d\theta\right)^{\frac12} (-\log|x|)^{\frac12} \\ &\leq |u(1)| + \frac{1}{\sqrt{2\pi}}(-\log|x|)^{1/2}||\nabla u||_2. \end{aligned}$$

Since $|u(1)|$ can be controlled by $||u||$, the inequality follows. □

With the aid of the preceding lemma we can now prove that the radial problem has a solution in a wide interval of γ's.

Theorem 2.6. *For every* $\gamma < 2\pi(\alpha+2)$,

$$S^R_{\alpha,\gamma} = \sup_{\substack{u\in H^1_{rad}(\Omega)\\ ||u||\leq 1}} \int_\Omega \left(e^{\gamma u^2}-1\right)|x|^\alpha\,dx \tag{2.7}$$

is finite and attained. Moreover, if u *is a maximizer, then* u *can be chosen positive and* $\|u\| = 1$.

Proof. We first show that $S^R_{\alpha,\gamma}$ is finite. Let $u \in H^1_{rad}(\Omega)$ be such that $||u|| \leq 1$ and let $\varepsilon > 0$. By Lemma 2.5,

$$u(x)^2 \leq \left(C + \frac{1}{\sqrt{2\pi}}(-\log|x|)^{1/2}\right)^2 \leq \left(1+\frac{1}{\varepsilon}\right)C^2 + \frac{1+\varepsilon}{2\pi}(-\log|x|).$$

Set $K_\varepsilon = \left(1+\frac{1}{\varepsilon}\right)C^2$. Then

$$\int_\Omega e^{\gamma u^2}|x|^\alpha\,dx \leq e^{\gamma K_\varepsilon}\int_\Omega e^{\gamma\frac{1+\varepsilon}{2\pi}(-\log|x|)}|x|^\alpha\,dx = e^{\gamma K_\varepsilon}\int_\Omega |x|^{\alpha-\gamma\frac{1+\varepsilon}{2\pi}}\,dx,$$

which is finite exactly when $\gamma < \frac{2\pi}{1+\varepsilon}(\alpha+2)$. Since ε can be taken as small as we please, we obtain the required range of γ's.

To show that $S^R_{\alpha,\gamma}$ is attained we proceed as in the proof of Theorem 2.3 until (2.5), this time keeping the weight $|x|^\alpha$. Then we see that for every maximizing sequence $u_n \in H^1_{rad}(\Omega)$, with $||u_n|| \leq 1$ and $u_n \rightharpoonup u$,

$$\int_\Omega e^{\gamma u_n^2}|u_n^2-u^2||x|^\alpha\,dx \leq e^{\gamma K_\varepsilon}\int_\Omega |u_n^2-u^2||x|^{\alpha-\gamma\frac{1+\varepsilon}{2\pi}}\,dx,$$

where we choose ε such that $\gamma\frac{1+\varepsilon}{2\pi} - \alpha < 2$. In this case there exists $p > 1$ such that $p(\gamma\frac{1+\varepsilon}{2\pi} - \alpha) < 2$, so that

$$\begin{aligned}&\int_\Omega e^{\gamma u_n^2}|u_n^2-u^2||x|^\alpha\,dx\\ &\leq e^{\gamma K_\varepsilon}\Big(\int_\Omega |x|^{p(\alpha-\gamma\frac{1+\varepsilon}{2\pi})}\,dx\Big)^{1/p}\Big(\int_\Omega |u_n^2-u^2|^{p'}\,dx\Big)^{1/p'} = o(1)\end{aligned}$$

as $n \to \infty$ since the first integral is finite and $u_n \to u$ in every L^p. This is (as in the proof of Theorem 2.3) what allows to say that the functional to be maximized is weakly continuous in $H^1_{rad}(\Omega)$, for every $\gamma < 2\pi(\alpha+2)$; therefore its supremum is attained. Positivity is again a consequence of Remark 2.9. □

Remark 2.7. The value $2\pi(\alpha+2)$ is the same as the one found in [6] for the Dirichlet boundary conditions. This is not surprising, since for radial functions the spaces H^1 and H^1_0 "differ only by constant functions", and constant functions have no influence on the finiteness of the integrals in question.

Remark 2.8. It is easy to see that $S_{\alpha,\gamma} = +\infty$ if $\gamma > 2\pi$ and that $S^R_{\alpha,\gamma} = +\infty$ if $\gamma > 2\pi(\alpha+2)$. The questions of the existence of maximizers at $\gamma = 2\pi$ or of radial maximizers at $\gamma = 2\pi(\alpha+2)$, is open and deserves further study. We do not address this kind of questions in the present paper.

Remark 2.9. Maximizers for $S_{\alpha,\gamma}$ or for $S^R_{\alpha,\gamma}$ are classical one–sign solutions of the elliptic problem

$$\begin{cases} -\Delta u + u = \lambda |x|^\alpha u e^{\gamma u^2} & \text{in } \Omega, \\ u > 0 & \text{in } \overline{\Omega}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

as one immediately sees by usual elliptic theory and a completely standard argument based on the strong maximum principle. The Lagrange multiplier $\lambda = \lambda(u)$ satisfies

$$\lambda = \frac{||u||^2}{\int_\Omega u^2 e^{\gamma u^2} |x|^\alpha \, dx}.$$

As a final observation, we notice that for $\gamma < 2\pi$ in $H^1(\Omega)$ and $\gamma < 2\pi(\alpha+2)$ in $H^1_{rad}(\Omega)$, we have

$$S_{\alpha,\gamma} \to 0 \qquad \text{and} \qquad S^R_{\alpha,\gamma} \to 0 \qquad \text{as } \alpha \to \infty.$$

Indeed, choosing $p > 1$ such that $p\gamma < 2\pi$, we have, as $\alpha \to \infty$,

$$S_{\alpha,\gamma} \le \int_\Omega e^{\gamma u^2} |x|^\alpha \, dx \le \Big(\int_\Omega e^{p\gamma u^2} \, dx \Big)^{1/p} \Big(\int_\Omega |x|^{\alpha p'} \, dx \Big)^{1/p'} = o(1)$$

by Theorem 2.2. This proves the claim for $H^1(\Omega)$. The conclusion for radial functions follows from Theorem 2.6. In the next section we will obtain much more precise estimates.

3. Asymptotics of maximizers

We carry out in this section the study of asymptotic properties (as $\alpha \to \infty$) of radial and nonradial maximizers. We will prove in this way a first symmetry breaking result.

We begin with the analysis of radial maximizers. To this aim assume that $u_\alpha \in H^1_{rad}(\Omega)$ is such that

$$S^R_{\alpha,\gamma} = \int_\Omega \left(e^{\gamma u_\alpha^2} - 1 \right) |x|^\alpha \, dx.$$

Here γ is a fixed number such that $\gamma < 2\pi(\alpha+2)$; of course, since we are interested in the behavior for $\alpha \to \infty$, there is no real restriction on γ. We have already observed that $S^R_{\alpha,\gamma} \to 0$ as $\alpha \to \infty$; now we obtain a precise estimate.

As in [14], the asymptotic behavior of $S^R_{\alpha,\gamma}$ and u_α can be expressed in terms of eigenvalues and eigenfunctions of the Steklov problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We just recall that the first eigenvalue, λ_1, is positive and simple, and therefore the first eigenfunction is radial. In the sequel, we denote by φ_1 the first normalized eigenfunction, i.e. $||\varphi_1|| = 1$. As usual, λ_1 can be characterized by

$$\lambda_1 = \min_{u \in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{||u||^2}{\int_{\partial\Omega} u^2\, d\sigma} = \frac{||\varphi_1||^2}{\int_{\partial\Omega} \varphi_1^2\, d\sigma} = \frac{1}{2\pi\varphi_1^2(1)}, \tag{3.2}$$

where we used the fact that φ_1 is radial. We will use repeatedly the relation

$$\varphi_1^2(1) = \frac{1}{2\pi\lambda_1}.$$

Proposition 3.1. *Let $\gamma > 0$ and let u_α be a maximizer for $S^R_{\alpha,\gamma}$ over $H^1_{rad}(\Omega)$. Then, as $\alpha \to \infty$,*

$$(\alpha+2)S^R_{\alpha,\gamma} \quad \to \quad 2\pi\big(e^{\frac{\gamma}{2\pi\lambda_1}} - 1\big), \tag{3.3}$$

$$u_\alpha \quad \to \quad \varphi_1 \quad \textit{in} \quad H^1(\Omega). \tag{3.4}$$

Proof. For any u in the unit ball of $H^1(\Omega)$ we compute, by the Divergence Theorem,

$$\begin{aligned} \int_\Omega \big(e^{\gamma u^2} - 1\big)(\alpha+2)|x|^\alpha\, dx &= \int_\Omega \big(e^{\gamma u^2} - 1\big)\operatorname{div}(|x|^\alpha x)\, dx \\ &= \int_{\partial\Omega} \big(e^{\gamma u^2} - 1\big)\, d\sigma - \int_\Omega \nabla\big(e^{\gamma u^2} - 1\big)\cdot x|x|^\alpha\, dx. \end{aligned}$$

Therefore, by the Hölder inequality, and for α large enough, using the continuous embedding of $H^1(\Omega)$ into any $L^p(\Omega)$ and the fact that u is in the unit ball of $H^1(\Omega)$, we get

$$\begin{aligned} &\Big|\int_\Omega \big(e^{\gamma u^2} - 1\big)(\alpha+2)|x|^\alpha\, dx - \int_{\partial\Omega} \big(e^{\gamma u^2} - 1\big)\, d\sigma\Big| \le 2\gamma \int_\Omega e^{\gamma u^2} u \nabla u \cdot x|x|^\alpha\, dx \\ &\le \Big(\int_\Omega |\nabla u|^2\, dx\Big)^{\frac12} \Big(\int_\Omega u^4\, dx\Big)^{\frac14} \Big(\int_\Omega e^{4\gamma u^2}|x|^\alpha\, dx\Big)^{\frac14} \le C\Big(\int_\Omega e^{4\gamma u^2}|x|^\alpha\, dx\Big)^{\frac14}. \end{aligned}$$

If u is radial, then by Lemma 2.5 we see that

$$|u(x)|^2 \leq 2C^2 + \frac{1}{\pi}(-\log|x|);$$

therefore,

$$\int_\Omega e^{4\gamma u^2}|x|^\alpha\, dx \leq e^{8\gamma C^2}\int_\Omega |x|^{\alpha-\frac{4\gamma}{\pi}}\, dx = o(1)$$

as $\alpha \to \infty$ for every fixed γ, uniformly for u in the unit ball of $H^1_{rad}(\Omega)$.

This uniformity allows us to pass to the suprema for $||u|| \leq 1$ in $H^1_{rad}(\Omega)$, to obtain that

$$\begin{aligned}(\alpha+2)S_{\alpha,\gamma} &= \sup_{\substack{u\in H^1_{rad}(\Omega)\\ ||u||\leq 1}} \int_{\partial\Omega}\big(e^{\gamma u^2}-1\big)\, d\sigma + o(1)\\ &= 2\pi \sup_{\substack{u\in H^1_{rad}(\Omega)\\ ||u||\leq 1}} \big(e^{\frac{\gamma}{2\pi}\int_{\partial\Omega}u^2\, d\sigma}-1\big)\, d\sigma + o(1) = 2\pi\big(e^{\frac{\gamma}{2\pi\lambda_1}}-1\big)+o(1),\end{aligned}$$

as $\alpha \to \infty$.

To prove also (3.4) notice that since u_α is bounded, up to subsequences, we have $u_\alpha \rightharpoonup u$ in $H^1(\Omega)$, for some $u \in H^1(\Omega)$ with $||u|| \leq 1$. Also, $u_\alpha(1) \to u(1)$, since u_α is radial. By the preceding arguments, we deduce that

$$\begin{aligned}2\pi\big(e^{\frac{\gamma}{2\pi\lambda_1}}-1\big)+o(1) &= \int_{\partial\Omega}\big(e^{\gamma u_\alpha^2}-1\big)\, d\sigma = 2\pi\big(e^{\gamma u_\alpha^2(1)}-1\big)\\ &= 2\pi\big(e^{\gamma u^2(1)}-1\big)+o(1).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}2\pi\big(e^{\frac{\gamma}{2\pi\lambda_1}}-1\big)+o(1) &= 2\pi\big(e^{\gamma u^2(1)}-1\big) \leq 2\pi\big(e^{\gamma\frac{u^2(1)}{||u||^2}}-1\big)\\ &\leq \sup_{\substack{v\in H^1_{rad}(\Omega)\\ ||v||\leq 1}} \int_{\partial\Omega}\Big(e^{\gamma v^2}-1\Big)\, d\sigma = 2\pi\left(e^{\frac{\gamma}{2\pi\lambda_1}}-1\right),\end{aligned}$$

which shows that $||u|| = 1$. Hence, the convergence of u_α is strong, and $u = \varphi_1$, since λ_1 is simple. □

We now turn to the case of nonradial maximizers. Here the situation is more complex, and we divide the analysis in two cases according to whether $\gamma > \pi$ or not, namely according to whether the supremum for the "trace functional" in (2.3) is infinite or finite. Let us then assume that $\gamma \in (\pi, 2\pi)$. In this case, by Theorem 2.2, we know that for every $M > 0$ there exists a function $v = v_M \in H^1(\Omega)$, with $||v|| \leq 1$ such that

$$\int_{\partial\Omega} \left(e^{\gamma v^2} - 1\right) d\sigma \geq M.$$

Then we have

$$\begin{aligned}(\alpha+2)S_{\alpha,\gamma} &\geq \int_{\Omega} \left(e^{\gamma v^2} - 1\right)(\alpha+2)|x|^{\alpha}\,dx \\ &= \int_{\partial\Omega} \left(e^{\gamma v^2} - 1\right) d\sigma - 2\gamma \int_{\Omega} v e^{\gamma v^2} \nabla v \cdot x|x|^{\alpha}\,dx \\ &\geq M + o(1) \end{aligned} \tag{3.5}$$

as $\alpha \to \infty$, since v is fixed. This estimate is enough to obtain a first symmetry breaking result.

Theorem 3.2. *For every $\gamma \in (\pi, 2\pi)$ no maximizer for $S_{\alpha,\gamma}$ is radial provided α is large enough.*

Proof. Choose a number $M > 2\pi\left(e^{\frac{\gamma}{2\pi\lambda_1}} - 1\right)$; by (3.5) and Proposition 3.1 we have, as $\alpha \to \infty$,

$$(\alpha+2)S_{\alpha,\gamma} \geq M + o(1) > 2\pi\left(e^{\frac{\gamma}{2\pi\lambda_1}} - 1\right) + o(1) = (\alpha+2)S^R_{\alpha,\gamma} + o(1),$$

namely $S_{\alpha,\gamma} > S^R_{\alpha,\gamma}$ for every α large enough. Therefore the maximum is attained by a function in $H^1(\Omega) \setminus H^1_{rad}(\Omega)$. □

Remark 3.3. We can obtain a better estimate of $S_{\alpha,\gamma}$ for $\gamma \in (\pi, 2\pi)$ by working as in [2, Lemma 3.3.]. We sketch the argument, consisting essentially in concentrating a Moser–type sequence ([20]) around the point $e = (0,1) \in \partial\Omega$. In [2] the authors construct a sequence $w_\varepsilon \in H^1(\Omega)$ such that supp $w_\varepsilon \subset B_1(e) \cap \overline{\Omega}$, $||w_\varepsilon|| = 1$, and w_ε is constant in $D_\varepsilon := B_\varepsilon(e) \cap \overline{\Omega}$, where $w_\varepsilon^2 = \frac{1}{\pi}\log\frac{1}{\varepsilon} + O(1)$ as $\varepsilon \to 0$. Then, we have

$$\begin{aligned} S_{\alpha,\gamma} &\geq \int_{D_\varepsilon} \left(e^{\gamma w_\varepsilon^2} - 1\right)|x|^{\alpha}\,dx \\ &\geq (1-\varepsilon)^{\alpha}\left(e^{\frac{\gamma}{\pi}\log\frac{1}{\varepsilon} + O(1)} - 1\right)|D_\varepsilon| \sim C(1-\varepsilon)^{\alpha}\varepsilon^{2-\frac{\gamma}{\pi}} \end{aligned}$$

as $\varepsilon \to 0$. Choosing $\varepsilon = 1/\alpha$, we obtain $S_{\alpha,\gamma} \geq \frac{C}{\alpha^{2-\frac{\gamma}{\pi}}}$ as $\alpha \to \infty$. Notice that this estimate is useless for $\gamma \leq \pi$, since in this range it is worse than the corresponding one on radial functions (see (3.3)).

Theorem 3.2 gives a symmetry breaking result based on level estimates for $\gamma \in (\pi, 2\pi)$. We will explore later the possibility to obtain the same kind of result when $\gamma < \pi$. We close the section with the precise asymptotics for $S_{\alpha,\gamma}$ in that range of γ's.

For every $\gamma \in (0, \pi]$, we set

$$S_\gamma = \sup_{\substack{u \in H^1(\Omega) \\ ||u|| \le 1}} Q^\gamma(u), \tag{3.6}$$

where

$$Q^\gamma(u) = \int_{\partial\Omega} (e^{\gamma u^2} - 1)\, d\sigma.$$

This number, the best constant in the Moser–Trudinger trace inequality, is finite and attained for every $\gamma \le \pi$, by Theorem 2.2.

We are now ready to prove the nonradial counterpart of (3.3).

Proposition 3.4. *Let $\gamma < \pi$. Then, as $\alpha \to \infty$,*

$$(\alpha + 2) S_{\alpha,\gamma} \to S_\gamma.$$

Proof. The strategy is analogous to that used for proving (3.3). Indeed, with the same argument we have, as $\alpha \to \infty$,

$$\begin{aligned}
\int_\Omega \left(e^{\gamma u^2} - 1\right)(\alpha + 2)|x|^\alpha\, dx &= \int_\Omega \left(e^{\gamma u^2} - 1\right) \operatorname{div}\left(|x|^\alpha x\right) dx \\
&= \int_{\partial\Omega} \left(e^{\gamma u^2} - 1\right)\, d\sigma - \int_\Omega \nabla\left(e^{\gamma u^2} - 1\right) \cdot x|x|^\alpha\, dx \\
&= \int_{\partial\Omega} \left(e^{\gamma u^2} - 1\right)\, d\sigma + o(1),
\end{aligned}$$

uniformly for $||u|| \le 1$, by Corollary 6.4. Passing to the suprema, we see that

$$(\alpha + 2) S_{\alpha,\gamma} = S_\gamma + o(1).$$

□

Proposition 3.4 and Proposition 3.1 show that when $\gamma < \pi$ the levels $S_{\alpha,\gamma}$ and $S^R_{\alpha,\gamma}$ *are of the same order* as $\alpha \to \infty$. The question of whether $S_{\alpha,\gamma} > S^R_{\alpha,\gamma}$ or not is thus much more delicate than in the interval $(\pi, 2\pi)$. Also it seems to be rather difficult to compute S_γ in order to compare it to the asymptotic value of $(\alpha + 2) S^R_{\alpha,\gamma}$, not to mention the fact that these numbers could even be equal. The way to overcome this obstacle consists in a study of the "limit" problems, namely Moser–Trudinger type inequalities, which is of interest in its own. This is carried out in the next Section.

4. Extremal functions in Moser–Trudinger type inequalities

This Section is independent of the preceding ones, since we study the properties of the extremal functions in some Moser–Trudinger inequalities. Thus there is no weight in the functionals.

4.1. The Moser–Trudinger trace inequality.

We now analyze the properties of the functions yielding the best constant in the Moser–Trudinger trace inequality, i.e. we study the maximization problem (3.6). We will also consider the radial counterpart of this problem, namely

$$S_\gamma^R = \sup_{\substack{u \in H^1_{rad}(\Omega) \\ ||u|| \leq 1}} Q^\gamma(u). \tag{4.1}$$

Since radial functions are constant on the boundary of Ω, one has, for every γ,

$$\int_{\partial\Omega} e^{\gamma u^2}\, d\sigma = 2\pi e^{\frac{\gamma}{2\pi}\int_{\partial\Omega} u^2\, d\sigma},$$

so that we easily obtain

$$S_\gamma^R = 2\pi(e^{\frac{\gamma}{2\pi\lambda_1}} - 1),$$

λ_1 being defined by (3.2). Clearly, φ_1 is the unique maximizer of S_γ^R whatever $\gamma > 0$. We now prove that there exists a threshold value γ^* at which the nature of φ_1 changes when seen as a critical point of Q^γ in the whole of $H^1(\Omega)$.

Theorem 4.1. *Let λ_1, λ_2 denote the two first eigenvalues of the Steklov problem* (3.1) *and define $\gamma^* = \pi(\lambda_2 - \lambda_1)$. Then*

- (i) *if $\gamma < \gamma^*$, φ_1 is a nondegenerate local maximizer of Q^γ on the unit sphere of $H^1(\Omega)$,*
- (ii) *if $\gamma = \gamma^*$, φ_1 is a degenerate local maximizer of Q^γ on the unit sphere of $H^1(\Omega)$,*
- (iii) *if $\gamma > \gamma^*$, φ_1 is not a local maximizer of Q^γ on the unit sphere of $H^1(\Omega)$.*

Proof. We compute the second variation of Q^γ at the point φ_1, arguing as in [24] to transform the constrained maximization problem into a free maximization problem. Defining $J : H^1(\Omega) \to \mathbf{R}$, $N : H^1(\Omega) \setminus \{0\} \to H^1(\Omega) \setminus \{0\}$ and $F : H^1(\Omega) \to \mathbf{R}$ by

$$J(u) := \int_\Omega |\nabla u|^2 + |u|^2\, dx, \quad N(u) = \frac{u^2}{J(u)}, \quad F(u) = \int_{\partial\Omega} (e^{\gamma N(u)} - 1)\, d\sigma,$$

the maximization problem (3.6) transforms into

$$\sup_{\substack{u\in H^1(\Omega)\\ u\neq 0}} F(u). \tag{4.2}$$

Standard but rather tedious computations give

$$N'(\varphi_1)(v) = \frac{2J(\varphi_1)\varphi_1 v - J'(\varphi_1)(v)\varphi_1^2}{J(\varphi_1)^2},$$

$$N''(\varphi_1)(v,v) = \frac{2J(\varphi_1)v^2 - 2J(v)\varphi_1^2}{J(\varphi_1)^2} - \frac{2J'(\varphi_1)(v)}{J(\varphi_1)}N'(\varphi_1)(v)$$

and

$$F''(\varphi_1)(v,v) = \gamma^2\int_{\partial\Omega} e^{\gamma N(\varphi_1)}(N'(\varphi_1)(v))^2\,d\sigma + \gamma\int_{\partial\Omega} e^{\gamma N(\varphi_1)}N''(\varphi_1)(v,v)\,d\sigma.$$

Therefore, using the fact that φ_1 is a critical point of F, i.e., for all $v \in H^1(\Omega)$,

$$F'(\varphi_1)(v) = \gamma\int_{\partial\Omega} e^{\gamma N(\varphi_1)}N'(\varphi_1)(v)\,d\sigma = 0,$$

and is normalized, we conclude that

$$\begin{aligned} F''(\varphi_1)(v,v) = \gamma^2&\int_{\partial\Omega} e^{\gamma\varphi_1^2}\left(2\varphi_1 v - 2\varphi_1^2 J'(\varphi_1)(v)\right)^2\,d\sigma \\ &+ \gamma\int_{\partial\Omega} e^{\gamma\varphi_1^2}\left(2v^2 - 2\varphi_1^2 J(v)\right)\,d\sigma. \end{aligned}$$

We evaluate $F''(\varphi_1)$ on the tangent space to the sphere $||u|| = 1$ at $u = \varphi_1$, namely we assume that v is orthogonal to φ_1, which reads $J'(\varphi_1)v = 0$. Observe that this implies v has zero mean on the boundary.

Inserting this information, and also the fact that $\varphi_1 = 1/(2\pi\lambda_1)$ on the boundary of Ω, we arrive at

$$\begin{aligned} F''(\varphi_1)(v,v) &= 4\gamma^2\int_{\partial\Omega} e^{\gamma\varphi_1^2}\varphi_1^2 v^2\,d\sigma + 2\gamma\int_{\partial\Omega} e^{\gamma\varphi_1^2}(v^2 - J(v)\varphi_1^2)\,d\sigma \\ &= \frac{2\gamma}{\lambda_1}e^{\frac{\gamma}{2\pi\lambda_1}}\int_{\partial\Omega} v^2\,d\sigma\left(\frac{\gamma}{\pi} + \lambda_1 - \frac{J(v)}{\int_{\partial\Omega} v^2\,d\sigma}\right) \end{aligned}$$

which holds for every v orthogonal to φ_1. From the classical characterization of eigenvalues, we infer that for any such v,

$$\frac{J(v)}{\int_{\partial\Omega} v^2\,d\sigma} \geq \lambda_2,$$

where λ_2 is the second eigenvalue of the Steklov problem (3.1). Taking this last estimate into account, we deduce that $F''(\varphi_1)$ is negative definite if

$\gamma < \pi(\lambda_2 - \lambda_1)$, is negative semidefinite if $\gamma = \pi(\lambda_2 - \lambda_1)$ (the kernel being precisely the eigenspace relative to λ_2), and is indefinite if $\gamma > \pi(\lambda_2 - \lambda_1)$. This proves the statements (i), (ii) and (iii). □

Remark 4.2. It is worth noticing that the threshold value $\gamma^* = \pi(\lambda_2-\lambda_1)$ is strictly smaller than π. Indeed for the Steklov problem (3.1) the eigenvalues are given by $\lambda_k = I'_{k-1}(1)/I_{k-1}(1)$, where I_{k-1} is the modified Bessel function of the first kind of order $k-1$. Tables from [1] give $\lambda_1 \approx 0.4463$ and $\lambda_2 \approx 1.2401$, so that $\lambda_2 - \lambda_1 \approx 0.7938 < 1$.

The next corollary is a straightforward consequence of Theorem 4.1. Notice that it also works for $\gamma = \pi$, the "critical exponent" in the present context.

Corollary 4.3. *For every $\gamma \in (\pi(\lambda_2 - \lambda_1), \pi]$, there results $S_\gamma > S_\gamma^R$ so that the functions yielding the best constant in the Moser–Trudinger trace inequality are not radial.*

4.2. **Uniqueness in the Moser–Trudinger trace inequality.** The purpose of this section is to show that for γ small, no symmetry breaking occurs for the maximizers of Q^γ. To do this it is necessary to study the asymptotic behavior of S_γ and of the functions that attain it.

We recall that we denote by λ_1 the first eigenvalue of the Steklov problem (3.1), and by φ_1 the first positive eigenfunction, normalized in $H^1(\Omega)$. The next proposition is the counterpart of Proposition 3.1.

Proposition 4.4. *Let u_γ be a maximizer of Q^γ over $H^1(\Omega)$, so that $S_\gamma = Q^\gamma(u_\gamma)$. Then, as $\gamma \to 0$,*

$$\frac{1}{\gamma} S_\gamma \quad \to \quad \frac{1}{\lambda_1} \tag{4.3}$$

$$u_\gamma \quad \to \quad \varphi_1 \quad \text{in} \quad H^1(\Omega). \tag{4.4}$$

Proof. Since for all $x \geq 0$ we have $e^x - 1 - x \leq \frac{1}{2} x^2 e^x$, we see that, as $\gamma \to 0$,

$$\begin{aligned} \Big| \frac{1}{\gamma} \int_{\partial\Omega} (e^{\gamma u^2} - 1)\, d\sigma - \int_{\partial\Omega} u^2\, d\sigma \Big| &\leq \frac{\gamma}{2} \int_{\partial\Omega} u^4 e^{\gamma u^2}\, d\sigma \\ &\leq \frac{\gamma}{2} \Big(\int_{\partial\Omega} e^{p\gamma u^2}\, d\sigma \Big)^{1/p} \Big(\int_{\partial\Omega} u^{4p'}\, d\sigma \Big)^{1/p'} = O(\gamma), \end{aligned}$$

uniformly for u in the unit ball of $H^1(\Omega)$, provided $p\gamma < \pi$ for all γ. Since this estimate is uniform, we can pass to the suprema with respect to $||u|| \leq 1$ to obtain that

$$\Big| \frac{1}{\gamma} S_\gamma - \frac{1}{\lambda_1} \Big| = O(\gamma)$$

as $\gamma \to 0$, which implies (4.3).

To prove also (4.4), let us take u_γ such that $Q^\gamma(u_\gamma) = S_\gamma$. Then, up to subsequences, $u_\gamma \rightharpoonup u$ in $H^1(\Omega)$. By the first part of the proof we have

$$\frac{1}{\lambda_1} + O(\gamma) = \frac{1}{\gamma} S_\gamma = \frac{1}{\gamma} \int_{\partial\Omega} (e^{\gamma u_\gamma^2} - 1)\, d\sigma = \int_{\partial\Omega} u_\gamma^2 \, d\sigma + O(\gamma)$$

as $\gamma \to 0$. Now, as u_γ converges to u strongly in $L^2(\partial\Omega)$, namely

$$\int_{\partial\Omega} u_\gamma^2 \, d\sigma = \int_{\partial\Omega} u^2 \, d\sigma + o(1)$$

we infer that

$$\int_{\partial\Omega} u^2 \, d\sigma = \frac{1}{\lambda_1}.$$

It easily follows that u converges strongly to φ_1 in $H^1(\Omega)$. Notice also that the whole sequence u_γ tends to φ_1. □

Any maximizer u_γ of Q^γ solves equation

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \mu u e^{\gamma u^2} & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

with

$$\mu = \mu(u_\gamma) = \frac{1}{\int_{\partial\Omega} u_\gamma^2 e^{\gamma u_\gamma^2} \, d\sigma}.$$

Notice that by Lemma 6.1, see the Appendix, we have

$$\mu(u_\gamma) \to \frac{1}{\int_{\partial\Omega} \varphi_1^2 \, d\sigma} = \lambda_1,$$

as $\gamma \to 0$. We are now ready to prove the main result of this section.

Theorem 4.5. *For every γ small enough, the function φ_1 is the unique solution to*

$$\sup_{\substack{u \in H^1(\Omega) \\ ||u|| \le 1}} \int_{\partial\Omega} (e^{\gamma u^2} - 1)\, d\sigma. \tag{4.6}$$

This solution is (radial and) nondegenerate.

Proof. Suppose that for some (sub–)sequence $\gamma \to 0$ there exist two solutions $u_\gamma \neq v_\gamma$ of problem (4.6). We show that this fact leads to a contradiction. Subtracting the equation solved by v_γ from the equation solved by u_γ, we see that setting $w_\gamma = u_\gamma - v_\gamma$, this function satisfies

$$\begin{cases} -\Delta w_\gamma + w_\gamma = 0 & \text{in } \Omega \\ \dfrac{\partial w_\gamma}{\partial \nu} = \mu(u_\gamma) u_\gamma e^{\gamma u_\gamma^2} - \mu(v_\gamma) v_\gamma e^{\gamma v_\gamma^2} & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

We know that $u_\gamma, v_\gamma \to \varphi_1$ in $H^1(\Omega)$ and that $\mu(u_\gamma), \mu(v_\gamma) \to \lambda_1$ as $\gamma \to 0$. We study the asymptotic behavior of the nonlinear boundary condition. To this aim we write it as

$$\mu(u_\gamma) u_\gamma e^{\gamma u_\gamma^2} - \mu(v_\gamma) v_\gamma e^{\gamma v_\gamma^2} = \mu(u_\gamma)(u_\gamma e^{\gamma u_\gamma^2} - v_\gamma e^{\gamma v_\gamma^2}) + (\mu(u_\gamma) - \mu(v_\gamma)) v_\gamma e^{\gamma v_\gamma^2} \tag{4.8}$$

and we analyze the two parts separately.

For $t \in [0,1]$, set $w_{\gamma,t} = v_\gamma + t(u_\gamma - v_\gamma)$; then

$$u_\gamma e^{\gamma u_\gamma^2} - v_\gamma e^{\gamma v_\gamma^2} = \int_0^1 \frac{d}{dt}\big(w_{\gamma,t} e^{\gamma w_{\gamma,t}^2}\big)\, dt = \Big(\int_0^1 (1 + 2\gamma w_{\gamma,t}^2) e^{\gamma w_{\gamma,t}^2}\, dt\Big)(u_\gamma - v_\gamma).$$

Notice that $w_{\gamma,t} \to \varphi_1$ in $H^1(\Omega)$ uniformly with respect to t; therefore, by Lemma 6.1,

$$\int_0^1 (1 + 2\gamma w_{\gamma,t}^2) e^{\gamma w_{\gamma,t}^2}\, dt \to 1 \qquad \text{in any } L^p(\partial\Omega),$$

as $\gamma \to 0$. So we have, for example,

$$\mu(u_\gamma)(u_\gamma e^{\gamma u_\gamma^2} - v_\gamma e^{\gamma v_\gamma^2}) = (\lambda_1 + o(1))(1 + o(1)) w_\gamma \qquad \text{in } L^2(\partial\Omega).$$

We now turn to the second term in (4.8). Notice first that with the same notation as above,

$$\begin{aligned} \int_{\partial\Omega} v_\gamma^2 e^{\gamma v_\gamma^2}\, d\sigma - \int_{\partial\Omega} u_\gamma^2 e^{\gamma u_\gamma^2}\, d\sigma &= \int_{\partial\Omega} \Big(\int_0^1 \frac{d}{dt}\big(w_{\gamma,t}^2 e^{\gamma w_{\gamma,t}^2}\big)\, dt\Big)\, d\sigma \\ &= 2\int_{\partial\Omega} \Big(\int_0^1 w_{\gamma,t}(1 + \gamma w_{\gamma,t}^2) e^{\gamma w_{\gamma,t}^2}\, dt\Big)(u_\gamma - v_\gamma)\, d\sigma. \end{aligned}$$

Applying again Lemma 6.1, we see that

$$\int_0^1 w_{\gamma,t}(1 + \gamma w_{\gamma,t}^2) e^{\gamma w_{\gamma,t}^2}\, dt \to \varphi_1 \qquad \text{in } L^2(\partial\Omega).$$

We thus have

$$\int_{\partial\Omega} v_\gamma^2 e^{\gamma v_\gamma^2}\, d\sigma - \int_{\partial\Omega} u_\gamma^2 e^{\gamma u_\gamma^2}\, d\sigma = 2\int_{\partial\Omega} (\varphi_1 + o(1))(u_\gamma - v_\gamma)\, d\sigma,$$

where $o(1)$ has to be understood in the sense of the $L^2(\partial\Omega)$ norm. Therefore, it follows that

$$\begin{aligned}\mu(u_\gamma)-\mu(v_\gamma) &= \frac{\int_{\partial\Omega} v_\gamma^2 e^{\gamma v_\gamma^2}\,d\sigma - \int_{\partial\Omega} u_\gamma^2 e^{\gamma u_\gamma^2}\,d\sigma}{\int_{\partial\Omega} v_\gamma^2 e^{\gamma u_\gamma^2}\,d\sigma \int_{\partial\Omega} u_\gamma^2 e^{\gamma v_\gamma^2}\,d\sigma}\\ &= \frac{2\int_{\partial\Omega}(\varphi_1+o(1))(u_\gamma-v_\gamma)\,d\sigma}{\frac{1}{\lambda_1^2}+o(1)} = 2\lambda_1^2\int_{\partial\Omega}(\varphi_1+o(1))w_\gamma\,d\sigma.\end{aligned}$$

Finally, by Lemma 6.1, we also have $v_\gamma e^{\gamma v_\gamma^2} = \varphi_1 + o(1)$ in $L^2(\partial\Omega)$. Putting everything together, we conclude that

$$\begin{aligned}&\mu(u_\gamma)u_\gamma e^{\gamma u_\gamma^2} - \mu(v_\gamma)v_\gamma e^{\gamma v_\gamma^2}\\ &= (\lambda_1+o(1))(1+o(1))w_\gamma + 2\lambda_1^2(\varphi_1+o(1))\int_{\partial\Omega}(\varphi_1+o(1))w_\gamma\,d\sigma.\end{aligned}$$

Setting $\psi_\gamma = w_\gamma/||w_\gamma||$, we have proved that ψ_γ satisfies

$$\begin{cases} -\Delta\psi_\gamma + \psi_\gamma = 0 & \text{in } \Omega\\ \dfrac{\partial\psi_\gamma}{\partial\nu} = (\lambda_1+o(1))(1+o(1))\psi_\gamma & \\ \qquad +2\lambda_1^2(\varphi_1+o(1))\displaystyle\int_{\partial\Omega}(\varphi_1+o(1))\psi_\gamma\,d\sigma & \text{on } \partial\Omega,\end{cases} \tag{4.9}$$

with $||\psi_\gamma|| = 1$. Of course, up to subsequences, ψ_γ has a weak limit ψ in $H^1(\Omega)$. The weak limit cannot be identically zero because multiplying the first equation in (4.9) and integrating yields

$$\begin{aligned}1 &= ||\psi_\gamma||^2\\ &= (\lambda_1+o(1))\int_{\partial\Omega}(1+o(1))\psi_\gamma^2\,d\sigma + 2\lambda_1^2\Big(\int_{\partial\Omega}(\varphi_1+o(1))\psi_\gamma\,d\sigma\Big)^2 = o(1),\end{aligned}$$

since if ψ were zero we would have $\psi_\gamma \to 0$ strongly in any $L^p(\partial\Omega)$.

Now, multiplying the first equation in (4.9) by $\phi \in H^1(\Omega)$, integrating and passing to the limit as $\gamma \to 0$, we find that ψ satisfies

$$\int_\Omega \nabla\psi\nabla\phi\,dx + \int_\Omega \psi\phi\,dx = \lambda_1\int_{\partial\Omega}\psi\phi\,d\sigma + 2\lambda_1^2\int_{\partial\Omega}\psi\varphi_1\,d\sigma\int_{\partial\Omega}\phi\varphi_1\,d\sigma.$$

However, notice that, since u_γ and v_γ lie on the unit sphere of $H^1(\Omega)$, we have

$$\langle\psi_\gamma, u_\gamma+v_\gamma\rangle = \langle\frac{u_\gamma-v_\gamma}{||u_\gamma-v_\gamma||}, u_\gamma+v_\gamma\rangle = \frac{1}{||u_\gamma-v_\gamma||}(1-1) = 0,$$

so that, as $\gamma \to 0$, we obtain $2\langle \psi, \varphi_1 \rangle = 0$, namely, ψ is orthogonal to φ_1 in $H^1(\Omega)$. In view of the equation satisfied by φ_1, this also means that $\int_{\partial\Omega} \psi \varphi_1 \, d\sigma = 0$. This implies that the equation solved by ψ is

$$\int_\Omega \nabla\psi \nabla\phi \, dx + \int_\Omega \psi\phi \, dx = \lambda_1 \int_{\partial\Omega} \psi\phi \, d\sigma,$$

for all ϕ in $H^1(\Omega)$, which means that ψ is (proportional to) the first Steklov eigenfunction φ_1. This contradicts the fact that ψ is orthogonal to φ_1. We therefore conclude that $u_\gamma = v_\gamma$ for every γ small. Hence by uniqueness, the maximizer is necessarily radial, so that it coincides with φ_1. Nondegeneracy follows from Theorem 4.1. The proof is complete. □

4.3. **The Moser–Trudinger inequality.** We now focus on the properties of the functions yielding the best constant in the Moser–Trudinger inequality, i.e. we study the problem

$$S_{0,\gamma} = \sup_{\substack{u \in H^1(\Omega) \\ ||u|| \le 1}} R^\gamma(u), \tag{4.10}$$

where

$$R^\gamma(u) = \int_\Omega (e^{\gamma u^2} - 1) \, dx.$$

Again, we consider the radial counterpart of this problem, namely

$$S^R_{0,\gamma} = \sup_{\substack{u \in H^1_{rad}(\Omega) \\ ||u|| \le 1}} R^\gamma(u). \tag{4.11}$$

The analysis of radial maximizers for the trace inequality is straightforward as radial functions are constant on the boundary. The geometry of the functionals R^γ is much more difficult to understand, even on radial functions. Let us first observe that the constants are critical points of R^γ constrained to balls of $H^1(\Omega)$. Indeed, if u is constant, there exists λ such that

$$\begin{cases} -\Delta u + u = \lambda u e^{\gamma u^2} & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and one immediately sees that the Lagrange multiplier is given by $\lambda = e^{-\gamma u^2}$. It is easily seen that when restricted to constant functions in the unit ball of $H^1(\Omega)$, R^γ is maximized by $\varphi = 1/\sqrt{\pi}$. Arguing as in Theorem 4.1, we deduce the variational nature of constant solutions of the associated Euler equation.

Theorem 4.6. *Let $\varphi = 1/\sqrt{\pi}$, let μ_2 denote the second eigenvalue of $-\Delta$ in $H^1(\Omega)$ and define $\gamma_* = \pi\mu_2/2$. Then,*

(i) *if $\gamma < \gamma_*$, φ is a nondegenerate local maximizer of R^γ on the unit sphere of $H^1(\Omega)$,*
(ii) *if $\gamma = \gamma_*$, φ is a degenerate local maximizer of R^γ on the unit sphere of $H^1(\Omega)$,*
(iii) *if $\gamma > \gamma_*$, φ is not a local maximizer of R^γ on the unit sphere of $H^1(\Omega)$.*

Proof. We follow the same lines of the proof of Theorem 4.1. Set

$$G(u) = \int_\Omega \left(e^{\gamma N(u)} - 1\right) dx,$$

with $N(u) = u^2/J(u) = u^2/||u||^2$.

First of all, if φ were degenerate as a critical point of G restricted to the unit sphere of $H^1(\Omega)$, there would exist a solution of the equation

$$-\Delta v = 2\gamma\varphi^2 v = \frac{2\gamma}{\pi} v$$

with Neumann boundary conditions. Since v must be orthogonal to φ, this is impossible if $\frac{2\gamma}{\pi} < \mu_2$.

Next, computing the second derivative of G, with the same arguments as in Theorem 4.1, we see that

$$G''(\varphi)(v,v) = 2\gamma e^{\frac{\gamma}{\pi}} \int_\Omega v^2\, dx \left(\frac{2\gamma}{\pi} + 1 - \frac{J(v)}{\int_\Omega v^2\, dx}\right).$$

Since for all v orthogonal to φ we have

$$\frac{J(v)}{\int_{\partial\Omega} v^2\, d\sigma} = \frac{\int_\Omega |\nabla v|^2\, dx + \int_\Omega v^2\, dx}{\int_\Omega v^2\, dx} \geq \mu_2 + 1,$$

we conclude easily as in the proof of Theorem 4.1. □

A natural conjecture is that $S_{0,\gamma}$ is achieved by φ for γ small. Indeed, as γ goes to 0, arguing as in Theorem 4.5, we can prove uniqueness of the maximizer, and Theorem 4.6 shows φ is a local maximizer for γ small. We now actually prove that for small γ, R^γ constrained to the unit sphere of $H^1(\Omega)$ has a unique positive critical point. Hence this critical point is a constant. In particular, it follows that $S_{0,\gamma}$ is achieved by φ for γ small.

Theorem 4.7. *Let B be the unit ball of $H^1(\Omega)$ centered at zero. Then, there exists γ_1 such that for $\gamma \leq \gamma_1$, R^γ constrained to ∂B has a unique positive critical point. Hence, for $\gamma \leq \gamma_1$, R^γ has a unique positive maximizer, which is a constant function.*

Proof. If u is a critical point of R^γ constrained to ∂B, then there exists λ such that

$$\begin{cases} -\Delta u + u = \lambda u e^{\gamma u^2} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that $\gamma < 2\pi$. We claim that u is bounded in L^∞, uniformly in γ over compact subintervals of $[0, 2\pi)$. Observe first that integrating the equation, we clearly deduce (by positivity of u) that $\lambda < 1$. Since $\|u\|_{H^1} = 1$, u is a priori bounded in any L^p. We then infer from the Hölder inequality and Theorem 2.2 that $\lambda u e^{\gamma u^2} \in L^s$ for some $s > 1$. By elliptic regularity, it follows that $u \in W^{2,s}(\Omega)$ and by the Sobolev imbedding, the claim is proved.

Now, we can complete the proof. Indeed, decompose u as $\bar{u} + \phi$, with $\bar{u} = \frac{1}{\pi}\int_\Omega u\, dx$. Then multiplying the Euler-Lagrange equation by ϕ, we get

$$\int_\Omega |\nabla\phi|^2 + |\phi|^2\, dx = \lambda \int_\Omega u e^{\gamma u^2} \phi.$$

As u is a priori bounded in $L^\infty \cap L^1$, we observe that

$$\begin{aligned} \int_\Omega u e^{\gamma u^2} \phi &= \int_\Omega \left(\int_0^1 (1 + 2\gamma(\bar{u} + t\phi)^2) e^{\gamma(\bar{u}+t\phi)^2}\, dt \right) |\phi|^2\, dx \\ &\le (1 + 2\gamma C) e^{\gamma C} \int_\Omega |\phi|^2\, dx. \end{aligned}$$

The constant in the right-hand side in the last inequality being smaller than $\mu_2 + 1$ (recall that μ_2 is the second eigenvalue of the Neumann problem) for γ small enough, we conclude that ϕ is zero which means that u is constant. □

5. The symmetry picture for the Hénon problem

The results obtained in the preceding sections will now be integrated to describe the loss of symmetry for maximizers of the Hénon problem. It will turn out that a rupture of symmetry takes place for two different reasons: first the absence of a trace inequality for the limiting problem (as is the case for $\gamma \in (\pi, 2\pi)$); secondly because the limiting problem itself breaks symmetry, as it happens for some other values of γ.

The results are completed by the analysis of the problem for small γ. We will prove indeed that if γ is small enough, then for *every* α there exists only one maximizer in $H^1(\Omega)$, which is therefore a radial function. This is in sharp contrast with the Dirichlet case ([25]) where symmetry breaking always occurs provided α is large enough.

A phenomenon similar to the present one has been detected instead in [14], for the Neumann case with power nonlinearity. In that paper however, the symmetry result is proved only for large α. We point out that the proof presented here can be adapted to the settings of [14], yielding a uniform symmetry result in α for powers with small growth.

The proof of the symmetry result is rather long and tedious because of the number of estimates required, although the computations are similar to those already described in the previous section. We have confined some of the most technical results in the appendix, and at some points we will only sketch the proof of particular cases.

We begin with the analysis of the loss of symmetry. The eigenvalues in the next statement are those of the Steklov problem (3.1).

Theorem 5.1. *For every $\gamma \in (\pi(\lambda_2 - \lambda_1), 2\pi)$, no maximizer for $S_{\alpha,\gamma}$ is radial provided α is large enough.*

Proof. If $\gamma \in (\pi, 2\pi)$, the statement is Theorem 3.2. In this case it is the absence of a trace inequality beyond $\gamma = \pi$ that leads to the result.

If $\gamma \in (\pi(\lambda_2 - \lambda_1), \pi)$ it follows from Propositions 3.1 and 3.4 that $(\alpha + 2)S^R_{\alpha,\gamma} \to S^R_\gamma$ and $(\alpha+2)S_{\alpha,\gamma} \to S_\gamma$ as $\alpha \to \infty$. By Corollary 4.3, we know that $S_\gamma > S^R_\gamma$ for $\gamma > \pi(\lambda_2 - \lambda_1)$, so that $S_{\alpha,\gamma} > S^R_{\alpha,\gamma}$ for every α large. This time the loss of symmetry is inherited from the same phenomenon in the limiting problem.

The case $\gamma = \pi$ is a little more subtle since Proposition 3.4 does not allow us to say that $(\alpha+2)S_{\alpha,\pi} \to S_\pi$. In this case we proceed as follows.

First of all, notice that $S_\gamma \to S_\pi$ as $\gamma \to \pi^-$. Indeed, let u_π be a maximizer for S_π and let $\gamma < \pi$. Then, as $\gamma \to \pi^-$, we infer from monotone convergence that

$$S_\pi \geq S_\gamma \geq \int_{\partial\Omega} \left(e^{\gamma u_\pi^2} - 1\right) d\sigma = S_\pi + o(1).$$

From Theorem 4.1, we know that $S_\pi > S^R_\pi$. The continuity of S_γ just proved allow us to choose $\delta > 0$ so small that $S_{\pi-\delta} > S^R_\pi$. Then, by Propositions 3.1 and 3.4, we get

$$\liminf_{\alpha\to\infty}(\alpha+2)S_{\alpha,\pi} \geq \lim_{\alpha\to\infty}(\alpha+2)S_{\alpha,\pi-\delta} = S_{\pi-\delta} > S^R_\pi = \lim_{\alpha\to\infty}(\alpha+2)S^R_{\alpha,\pi}.$$

At last, we conclude that $S_{\alpha,\pi} > S^R_{\alpha,\pi}$, for every α large, so that the proof is complete. □

We now turn to the analysis of the case γ small.

Theorem 5.2. *For every γ small enough and and for every $\alpha > 0$, the problem*

$$\sup_{\substack{u\in H^1(\Omega)\\ ||u||\leq 1}} \int_\Omega \left(e^{\gamma u^2} - 1\right) |x|^\alpha \, dx \tag{5.1}$$

has a unique solution (which is therefore a radial function).

We are going to prove the result by contradiction, assuming that, there exist sequences $\gamma_n \to 0$ and $\alpha_n > 0$ such that (5.1) admits at least two solutions. Two alternatives may present: either $\alpha_n \to +\infty$, or $\alpha_n \to \alpha \geq 0$. We carry out the details only in the first case, while for the second one (which is simpler) we will only indicate the main steps. We therefore assume from now on and until further notice that $\gamma_n \to 0$ and $\alpha_n \to +\infty$. We will proceed in several steps, starting with a series of preliminaries. In our first lemma, we identify the asymptotic behavior of the functional in (5.1).

Lemma 5.3. *Assume that $\gamma_n \to 0$ and $\alpha_n \to +\infty$ as $n \to \infty$. Then, we have*

$$\frac{\alpha_n + 2}{\gamma_n} \int_\Omega \left(e^{\gamma_n u^2} - 1\right) |x|^{\alpha_n}\, dx = \int_{\partial\Omega} u^2\, d\sigma + o(1),$$

as $n \to \infty$, uniformly for $||u|| \leq 1$.

Proof. By the Divergence Theorem, applied as in Proposition 3.1, we have

$$\begin{aligned} &\frac{\alpha_n + 2}{\gamma_n} \int_\Omega \left(e^{\gamma_n u^2} - 1\right) |x|^{\alpha_n}\, dx \\ &= \frac{1}{\gamma_n} \int_{\partial\Omega} \left(e^{\gamma_n u^2} - 1\right) d\sigma - \frac{1}{\gamma_n} \int_\Omega \nabla \left(e^{\gamma_n u^2} - 1\right) \cdot x|x|^{\alpha_n}\, dx. \end{aligned}$$

Arguing as in Proposition 4.4, we infer that

$$\frac{1}{\gamma_n} \int_{\partial\Omega} \left(e^{\gamma_n u^2} - 1\right) d\sigma = \int_{\partial\Omega} u^2\, d\sigma + o(1), \quad \text{as } n \to \infty,$$

uniformly for $||u|| \leq 1$, while

$$\frac{1}{\gamma_n} \int_\Omega \nabla \left(e^{\gamma_n u^2} - 1\right) \cdot x|x|^{\alpha_n}\, dx = 2 \int_\Omega u e^{\gamma_n u^2} \nabla u \cdot x|x|^{\alpha_n}\, dx = o(1),$$

as $n \to \infty$, uniformly for $||u|| \leq 1$, by Lemma 6.3. $\square$

We now turn to the asymptotic properties of maximizers and of their levels, like in Proposition 4.4. In the next statement we use the definition of $S_{\alpha,\gamma}$ from (2.4) and, as usual, we denote by λ_1 and φ_1 the first eigenvalue and eigenfunction of the Steklov problem as defined in (3.1).

Lemma 5.4. *Assume that $\gamma_n \to 0$, $\alpha_n \to +\infty$ as $n \to \infty$ and let u_n be a maximizer for S_{α_n,γ_n}. Then, as $n \to \infty$, we have*

$$\frac{\alpha_n + 2}{\gamma_n} S_{\alpha_n,\gamma_n} \to \frac{1}{\lambda_1} \tag{5.2}$$

$$u_n \to \varphi_1 \quad \text{in} \quad H^1(\Omega). \tag{5.3}$$

Proof. The uniformity proved in Lemma 5.3 shows, by passing to the suprema, the validity of (5.2). To prove (5.3), notice that, up to subsequences, $u_n \rightharpoonup u$ in $H^1(\Omega)$ and strongly in $L^p(\partial\Omega)$ for every finite p. Then, from (5.2) and Lemma 5.3,

$$\frac{1}{\lambda_1}+o(1)=\frac{\alpha_n+2}{\gamma_n}\int_\Omega \left(e^{\gamma_n u_n^2}-1\right)|x|^{\alpha_n}\,dx=\int_{\partial\Omega} u_n^2\,d\sigma+o(1)=\int_{\partial\Omega} u^2\,d\sigma+o(1),$$

which shows that $\int_{\partial\Omega} u^2\,d\sigma=\frac{1}{\lambda_1}$. The convergence of u_n to u must be strong, since otherwise the extremality of λ_1 would be contradicted. Thus $u_n \to \varphi_1$ strongly in $H^1(\Omega)$. □

If u_n is any maximizer for S_{α_n,γ_n}, then it solves the problem

$$\begin{cases} -\Delta u+u=\lambda u e^{\gamma_n u^2}|x|^{\alpha_n} & \text{in } \Omega \\ \dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

where the Lagrange multiplier $\lambda=\lambda(u_n)$ satisfies

$$\lambda(u_n)=\frac{1}{\int_\Omega u_n^2 e^{\gamma_n u_n^2}|x|^{\alpha_n}\,dx}.$$

The next lemma clarifies the behavior of $\lambda(u_n)$.

Lemma 5.5. *Assume that $\gamma_n \to 0$ and $\alpha_n \to +\infty$ as $n\to\infty$. Then, as $n\to\infty$, we have*

$$\frac{\lambda(u_n)}{\alpha_n+2}\to\lambda_1. \tag{5.5}$$

Proof. By the Divergence Theorem, we compute

$$(\alpha_n+2)\int_\Omega u_n^2 e^{\gamma_n u_n^2}|x|^{\alpha_n}\,dx=\int_{\partial\Omega} u_n^2 e^{\gamma_n u_n^2}\,d\sigma-\int_\Omega \nabla\left(u_n^2 e^{\gamma_n u_n^2}\right)\cdot x|x|^{\alpha_n}\,dx.$$

The last integral is $o(1)$ by Lemma 6.4, while by Lemma 6.1,

$$\int_{\partial\Omega} u_n^2 e^{\gamma_n u_n^2}\,d\sigma=\frac{1}{\lambda_1}+o(1),$$

where we have of course used the fact that $u_n\to\varphi_1$ in $H^1(\Omega)$. This proves (5.5). □

Suppose now that u_n and v_n are two different maximizers for S_{α_n,γ_n}. From Lemma 5.4 we know that u_n and v_n both converge to φ_1 strongly in $H^1(\Omega)$.

Setting

$$\psi_n=\frac{u_n-v_n}{||u_n-v_n||},$$

we have that $\psi_n \rightharpoonup \psi$ in $H^1(\Omega)$, up to subsequences. Notice that

$$\langle \psi, \varphi_1 \rangle = \lim_{n\to\infty} \frac{1}{2} \langle \psi_n, u_n + v_n \rangle = \lim_{n\to\infty} \frac{1}{2} \langle \frac{u_n - v_n}{||u_n - v_n||}, u_n + v_n \rangle = 0 \tag{5.6}$$

by the normalization of u_n and v_n, which also means, in view of the equation solved by φ_1, that

$$\int_{\partial\Omega} \psi \varphi_1 \, d\sigma = 0. \tag{5.7}$$

The functions ψ_n satisfy

$$\begin{cases} -\Delta \psi_n + \psi_n = \frac{\lambda(u_n)}{||u_n - v_n||} u_n e^{\gamma_n u_n^2} |x|^{\alpha_n} - \frac{\lambda(v_n)}{||u_n - v_n||} v_n e^{\gamma_n v_n^2} |x|^{\alpha_n} & \text{in } \Omega \\ \frac{\partial \psi_n}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

We now write the preceding problem in a suitable form in order to study its limit behavior as $n \to \infty$. In its weak form, (5.8) is equivalent to

$$\begin{aligned} \langle \psi_n, \phi \rangle &= \frac{\lambda(u_n) - \lambda(v_n)}{||u_n - v_n||} \int_\Omega u_n e^{\gamma_n u_n^2} \phi |x|^{\alpha_n} \, dx \\ &\quad - \frac{\lambda(v_n)}{||u_n - v_n||} \int_\Omega \big(v_n e^{\gamma_n v_n^2} - u_n e^{\gamma_n u_n^2} \big) \phi |x|^{\alpha_n} \, dx =: A_n - B_n \end{aligned} \tag{5.9}$$

for all $\phi \in H^1(\Omega)$.

We now proceed to estimate the terms A_n and B_n, by means to the following lemma. For further use, its statement is a little more general than what we need here.

Lemma 5.6. *Assume that $\gamma_n \to 0$, $\alpha_n \to +\infty$ and $w_n \rightharpoonup w$ in $H^1(\Omega)$ as $n \to \infty$. Then, as $n \to \infty$,*

$$\langle \psi_n, w_n \rangle \to \lambda_1 \int_{\partial\Omega} \psi w \, d\sigma.$$

Proof. Let us first prove that

$$\langle \psi_n, \phi \rangle \to \lambda_1 \int_{\partial\Omega} \psi \phi \, d\sigma,$$

for any fixed $\phi \in H^1(\Omega)$. We will proceed in two steps.

Step 1: $A_n \to 0$ as $n \to \infty$. First of all, we write, using Lemma 5.5,

$$\begin{aligned} A_n = (\lambda_1^2 + o(1)) \Big(\int_\Omega \frac{v_n^2 e^{\gamma_n v_n^2} - u_n^2 e^{\gamma_n u_n^2}}{||u_n - v_n||} (\alpha_n + 2) |x|^{\alpha_n} \, dx \Big) \\ \times \Big(\int_\Omega u_n e^{\gamma_n u_n^2} \phi (\alpha_n + 2) |x|^{\alpha_n} \, dx \Big) \end{aligned}$$

and we observe that the last integral is

$$\int_{\partial\Omega} u_n e^{\gamma_n u_n^2}\phi \, d\sigma - \int_\Omega \nabla\big(u_n e^{\gamma_n u_n^2}\phi\big)\cdot x|x|^{\alpha_n}\, dx = \int_{\partial\Omega} \varphi_1 \phi \, d\sigma + o(1)$$

by Lemma 6.1 and Corollary 6.4. Thus, we only have to show that the first integral tends to zero. To see this we write it as

$$\int_\Omega \frac{v_n^2 - u_n^2}{||u_n - v_n||} e^{\gamma_n v_n^2}(\alpha_n+2)|x|^{\alpha_n}\, dx - \int_\Omega u_n^2 \frac{e^{\gamma_n u_n^2} - e^{\gamma_n v_n^2}}{||u_n - v_n||}(\alpha_n+2)|x|^{\alpha_n}\, dx$$
$$=: C_n - D_n \tag{5.10}$$

and we study C_n and D_n separately. We have, with the usual arguments, Lemma 6.1 and Corollary 6.4,

$$\begin{aligned} C_n &= -\int_{\partial\Omega} \psi_n(u_n+v_n)e^{\gamma_n v_n^2}\, d\sigma + \int_\Omega \nabla\big(\psi_n(u_n+v_n)e^{\gamma_n v_n^2}\big)\cdot x|x|^{\alpha_n}\, dx \\ &= -2\int_{\partial\Omega} \psi\varphi_1\, d\sigma + o(1) = o(1) \end{aligned}$$

since φ_1 and ψ satisfy (5.7).

To estimate D_n, we use the inequality $|e^x - e^y| \le |x-y|(e^x+e^y)$ to get

$$\begin{aligned} |D_n| &\le \gamma_n \int_\Omega \psi_n u_n^2(u_n+v_n)\big(e^{\gamma_n u_n^2} + e^{\gamma_n v_n^2}\big)(\alpha_n+2)|x|^{\alpha_n}\, dx \\ &= \gamma_n \int_{\partial\Omega} \psi_n u_n^2(u_n+v_n)\big(e^{\gamma_n u_n^2} + e^{\gamma_n v_n^2}\big)\, d\sigma \\ &- \gamma_n \int_\Omega \nabla\Big(\psi_n u_n^2(u_n+v_n)\big(e^{\gamma_n u_n^2} + e^{\gamma_n v_n^2}\big)\Big)\cdot x|x|^{\alpha_n}\, dx = o(1), \end{aligned}$$

with the same arguments already used repeatedly. Putting all the pieces together shows that $A_n \to 0$, as we claimed.

Step 2: $B_n \to -\lambda_1 \int_{\partial\Omega}\psi\phi\, d\sigma$ as $n\to\infty$. Let us write

$$\begin{aligned} B_n = \frac{(\lambda_1 + o(1))(\alpha_n+2)}{||u_n - v_n||}\Big[&\int_\Omega (v_n - u_n)e^{\gamma_n v_n^2}\phi|x|^{\alpha_n}\, dx \\ &- \int_\Omega u_n\big(e^{\gamma_n u_n^2} - e^{\gamma_n v_n^2}\big)\phi|x|^{\alpha_n}\, dx\Big] \\ = -(\lambda_1+o(1))\Big[&\int_\Omega \psi_n e^{\gamma_n v_n^2}\phi(\alpha_n+2)|x|^{\alpha_n}\, dx \\ + &\int_\Omega u_n \frac{e^{\gamma_n u_n^2} - e^{\gamma_n v_n^2}}{||u_n - v_n||}\phi(\alpha_n+2)|x|^{\alpha_n}\, dx\Big] =: -(\lambda_1+o(1))(E_n+F_n). \end{aligned}$$

Now, arguing as above, we get

$$E_n = \int_{\partial\Omega} \psi_n e^{\gamma_n v_n^2} \phi \, d\sigma - \int_{\Omega} \nabla\big(\psi_n e^{\gamma_n v_n^2} \phi\big) \cdot x|x|^{\alpha_n} \, dx = \int_{\partial\Omega} \psi\phi \, d\sigma + o(1)$$

while F_n can be treated like D_n in Step 1 to show that it tends to zero as $n \to \infty$. This concludes this step.

Conclusion. It follows from the preceding steps that for any fixed ϕ,

$$\langle \psi_n, \phi \rangle \to \lambda_1 \int_{\partial\Omega} \psi\phi \, d\sigma,$$

as $n \to \infty$. A systematic inspection of the preceding estimates with the help of Corollary 6.4 guarantees the required uniformity in the limits so that using also the fact that the convergence of the sequence $(w_n)_n$ is strong in $L^p(\Omega)$ and in $L^p(\partial\Omega)$ for every finite p, the conclusion follows. □

The preceding lemma allows us to pass to the limit in (5.9). Hence, we deduce that ψ satisfies

$$\langle \psi, \phi \rangle = \lambda_1 \int_{\partial\Omega} \psi\phi \qquad \text{for all } \phi \in H^1(\Omega),$$

namely that ψ solves the Steklov problem

$$\begin{cases} -\Delta\psi + \psi = 0 & \text{in } \Omega \\ \dfrac{\partial\psi}{\partial\nu} = \lambda_1 \psi & \text{on } \partial\Omega. \end{cases}$$

Since λ_1 is simple and ψ is orthogonal to φ_1 by (5.6), this leads to a contradiction as soon as we prove that $\psi \not\equiv 0$. This is an obvious consequence of Lemma 5.6.

Lemma 5.7. *Assume that $\gamma_n \to 0$ and $\alpha_n \to +\infty$ as $n \to \infty$. Then the limit ψ of the sequence $(\psi_n)_n$ does not vanish identically.*

Proof. Assume by contradiction that $\psi_n \rightharpoonup 0$ as $n \to \infty$. Choosing $w_n = \psi_n$ in Lemma 5.6, we get

$$1 = ||\psi_n||^2 \to \lambda_1 \int_{\partial\Omega} \psi^2 \, d\sigma = 0,$$

which is a contradiction. □

Proof of Theorem 5.2. The results obtained so far show that assuming the existence of two distinct maximizers u_n and v_n leads to a contradiction when $\gamma_n \to 0$ and $\alpha_n \to \infty$.

The proof will be complete if we show that in the case $\gamma_n \to 0$ and $\alpha_n \to \alpha \geq 0$ the presence of two distinct maximizers is also impossible.

This case is considerably simpler than the previous one since the passage to the limit $\alpha_n \to \alpha$ is trivial: there is no need to use the Divergence Theorem argument, since $|x|^{\alpha_n} \to |x|^\alpha$ in every $L^p(\Omega)$ (and in $L^\infty(\Omega)$ if $\alpha \neq 0$).

As a consequence the limit problem will not carry Steklov–type boundary conditions but will again be a Neumann problem. More precisely, it is easy to see, repeating (and simplifying) the steps carried out earlier that

$$\frac{1}{\gamma_n}\int_\Omega \left(e^{\gamma_n u^2} - 1\right)|x|^{\alpha_n}\,dx \to \int_\Omega u^2 |x|^\alpha\,dx$$

uniformly for $||u|| \leq 1$. We denote by λ_α the first (simple) eigenvalue of the problem

$$\begin{cases} -\Delta u + u = \lambda |x|^\alpha u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and by ϕ_α the first eigenfunction, positive, normalized in $H^1(\Omega)$ and radial.

Let u_n be a maximizer for S_{α_n,γ_n}; then, as in Lemma 5.4 one can easily prove that

$$\frac{1}{\gamma_n} S_{\alpha_n,\gamma_n} \to \frac{1}{\lambda_\alpha} \tag{5.11}$$

$$u_n \to \phi_\alpha \quad \text{in} \quad H^1(\Omega). \tag{5.12}$$

If u_n and v_n are two different maximizers for S_{α_n,γ_n}, we set

$$\psi_n = \frac{u_n - v_n}{||u_n - v_n||},$$

so that $\psi_n \rightharpoonup \psi$ in $H^1(\Omega)$, up to subsequences. Also in this case

$$\langle \psi, \phi_\alpha \rangle = \lim_{n\to\infty} \frac{1}{2}\langle \frac{u_n - v_n}{||u_n - v_n||}, u_n + v_n \rangle = 0, \tag{5.13}$$

which also implies $\int_\Omega \psi \varphi_1 |x|^\alpha\,dx = 0$. The functions ψ_n satisfy (5.8), or in the weak form,

$$\begin{aligned} \langle \psi_n, \phi \rangle &= \frac{\lambda(u_n) - \lambda(v_n)}{||u_n - v_n||}\int_\Omega u_n e^{\gamma_n u_n^2}\phi |x|^{\alpha_n}\,dx \\ &- \frac{\lambda(v_n)}{||u_n - v_n||}\int_\Omega \left(v_n e^{\gamma_n v_n^2} - u_n e^{\gamma_n u_n^2}\right)\phi |x|^{\alpha_n}\,dx \end{aligned} \tag{5.14}$$

for all $\phi \in H^1(\Omega)$. The analysis of the asymptotic behavior of the two terms in the right–hand–side of (5.14) can be carried out along the same steps followed in the case $\alpha_n \to \infty$. All the computations are much simpler, since there is no need to use the Divergence argument and therefore there are no remainders to estimate. Carrying out the computations, it is easy to see that the limit ψ (which does not vanish identically, as in the preceding case) satisfies

$$\begin{cases} -\Delta\psi + \psi = \lambda_\alpha |x|^\alpha u & \text{in } \Omega \\ \dfrac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

This leads again to a contradiction since ψ is orthogonal to ϕ_α and λ_α is simple. The proof is now complete. □

6. APPENDIX

In this appendix we prove some technical results that have been used several times in the course of the paper.

Lemma 6.1. *Let u_n, v_n in $H^1(\Omega)$ be such that $||u_n||, ||v_n|| \le 1$. Assume that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $H^1(\Omega)$.*

If $\gamma_n \to \gamma < 2\pi$ in $\mathbf{R}$, then for every $p \in [1, +\infty)$ and for every $q \in [1, \frac{2\pi}{\gamma})$ there results

$$v_n^p e^{\gamma_n u_n^2} \to v^p e^{\gamma u^2} \quad \textit{strongly in } L^q(\Omega). \tag{6.1}$$

If $\gamma_n \to \gamma < \pi$ in $\mathbf{R}$, then for every $p \in [1, +\infty)$ and for every $q \in [1, \frac{\pi}{\gamma})$ there results

$$v_n^p e^{\gamma_n u_n^2} \to v^p e^{\gamma u^2} \quad \textit{strongly in } L^q(\partial\Omega). \tag{6.2}$$

Proof. We only prove (6.1) since the argument works also for (6.2). For definiteness we suppose that $q\gamma_n < 2\pi$ for every n. We write

$$\begin{aligned} \int_\Omega \left| v_n^p e^{\gamma_n u_n^2} - v^p e^{\gamma u^2} \right|^q dx \le C\Big(& \int_\Omega \left| v_n^p e^{\gamma_n u_n^2} - v^p e^{\gamma_n u_n^2} \right|^q dx \\ & + \int_\Omega \left| v^p e^{\gamma_n u_n^2} - v^p e^{\gamma_n u^2} \right|^q dx + \int_\Omega \left| v^p e^{\gamma_n u^2} - v^p e^{\gamma u^2} \right|^q dx \Big) \\ & =: C(I_1 + I_2 + I_3) \end{aligned}$$

and we analyze the three integrals separately as $n \to \infty$.

We have, choosing some $\theta > 1$ such that $\theta q \gamma_n < 2\pi$,

$$I_1 = \int_\Omega |v_n^p - v^p|^q e^{q\gamma_n u_n^2} dx \le \Big(\int_\Omega e^{\theta q \gamma_n u_n^2} dx \Big)^{1/\theta} \Big(\int_\Omega |v_n^p - v^p|^{q\theta'} \Big)^{1/\theta'} \le Co(1),$$

by Theorem 2.2 and the compact embedding of $H^1(\Omega)$ into $L^r(\Omega)$ for every finite r.

Now, by convexity of the exponential we can write

$$|e^{\gamma_n u_n^2} - e^{\gamma_n u^2}| \quad \le \quad \gamma_n |u_n^2 - u^2| \int_0^1 e^{\gamma_n (t u_n^2 + (1-t)u^2)} \, dt$$

$$\begin{aligned} &\leq \gamma_n|u_n^2-u^2|\int_0^1 (te^{\gamma_n u_n^2}+(1-t)e^{\gamma_n u^2})\,dt \\ &\leq \gamma_n|u_n^2-u^2|\big(e^{\gamma_n u_n^2}+\gamma_n e^{\gamma_n u^2}\big). \end{aligned}$$

Therefore,

$$I_2 \leq C\gamma_n^q\int_\Omega |v|^{qp}|u_n^2-u^2|^q e^{q\gamma_n u_n^2}+C\gamma_n^q\int_\Omega |v|^{qp}|u_n^2-u^2|^q e^{q\gamma_n u^2}=o(1)$$

as for I_1.

Finally, by the elementary inequality

$$|e^{\alpha x}-e^{\beta x}|\leq|\alpha-\beta|xe^{\max(\alpha,\beta)x},\qquad \forall\alpha,\beta,x\geq 0,$$

we have

$$I_3=\int_\Omega |v|^{qp}|e^{\gamma_n u^2}-e^{\gamma u^2}|^q\,dx\leq|\gamma_n-\gamma|^q\int_\Omega |v|^{qp}|u|^{2q}e^{q\max(\gamma_n,\gamma)u^2}=o(1)$$

because $\gamma_n\to\gamma$ and $q\max(\gamma_n,\gamma)<2\pi$. This completes the proof. □

As a consequence, we note that if $w_n:[0,1]\times\Omega\to\mathbf{R}$ is a sequence such that for some $w:\Omega\to\mathbf{R}$ and for some $p\geq 1$

$$w_n(t,\cdot)\to w \quad \text{ in } L^p(\Omega), \text{ uniformly for } t\in[0,1],$$

then $\int_0^1 w_n(t,\cdot)\,dt\to w$ in $L^p(\Omega)$.

The next results have also been used several times.

Lemma 6.2. *Let $a,b\geq 1$. Then, there exists $p>1$ such that $\nabla\big(|u|^a|v|^be^{\gamma w^2}\big)$ is bounded in $L^p(\Omega)$, uniformly for u,v,w in the unit ball of $H^1(\Omega)$ and γ in compact subsets of $[0,\pi)$.*

Proof. Since

$$\begin{aligned} \left|\nabla\left(|u|^a|v|^be^{\gamma w^2}\right)\right|^p \leq C_p\Big(&|u|^{p(a-1)}|v|^{pb}e^{p\gamma w^2}|\nabla u|^p+|u|^{pa}|v|^{p(b-1)}e^{p\gamma w^2}|\nabla v|^p \\ &+|u|^{pa}|v|^{pb}|w|^pe^{p\gamma w^2}|\nabla w|^p\Big), \end{aligned}$$

we carry out the proof for the first term only, the others being handled in a similar way.

Fix $p=1+\varepsilon$, with ε small to be determined. Applying the Hölder inequality with exponents $4/\varepsilon,\ 4/\varepsilon,\ 2/(1-2\varepsilon),\ 2/(1+\varepsilon)$ we obtain

$$\begin{aligned} &\int_\Omega |u|^{(1+\varepsilon)(a-1)}|v|^{(1+\varepsilon)b}e^{(1+\varepsilon)\gamma w^2}|\nabla u|^{(1+\varepsilon)}\,dx \\ &\leq\Big(\int_\Omega |u|^{\frac{4}{\varepsilon}(1+\varepsilon)(a-1)}\,dx\Big)^{\frac{\varepsilon}{4}}\Big(\int_\Omega |v|^{\frac{4}{\varepsilon}(1+\varepsilon)b}\,dx\Big)^{\frac{\varepsilon}{4}} \end{aligned}$$

$$\times \left(\int_\Omega e^{\frac{2+2\varepsilon}{1-2\varepsilon}\gamma w^2}\, dx \right)^{\frac{1-2\varepsilon}{2}} \left(\int_\Omega |\nabla u|^2\, dx \right)^{\frac{1+\varepsilon}{2}} dx.$$

The first two integrals are uniformly bounded due to the embedding of $H^1(\Omega)$ into $L^q(\Omega)$ for any finite q, while the third is uniformly bounded as soon as $\frac{2+2\varepsilon}{1-2\varepsilon}\gamma \le 2\pi$, by Theorem 2.2, which we can always achieve by taking ε small. □

Lemma 6.3. *Let $p > 1$. Then,*

$$\int_\Omega f|x|^\alpha\, dx = o(1)$$

as $\alpha \to \infty$, uniformly for f in a bounded subset of $L^p(\Omega)$.

Proof. By the Hölder inequality,

$$\left| \int_\Omega f|x|^\alpha\, dx \right| \le \left(\int_\Omega |f|^p\, dx \right)^{1/p} \left(\int_\Omega |x|^{\alpha p'}\, dx \right)^{1/p'} \le Co(1).$$ □

Corollary 6.4. *Let $a, b \ge 1$. Then,*

$$\int_\Omega \nabla \left(|u|^a |v|^b e^{\gamma w^2} \right) \cdot x|x|^\alpha\, dx = o(1)$$

as $\alpha \to \infty$, uniformly for u, v, w in the unit ball of $H^1(\Omega)$ and γ in compact subsets of $[0, \pi)$.

Proof. Apply Lemma 6.3 to $f = \nabla(|u|^a|v|^b e^{\gamma w^2}) \cdot x$, which is bounded in $L^p(\Omega)$ for some $p > 1$ by Lemma 6.2. □

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