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REGULARITY RESULTS FOR NON SMOOTH PARABOLIC PROBLEMS

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Abstract. In this paper we deal with the study of some regularity properties of weak solutions to non-linear, second-order parabolic equations and systems of the type

$$u_t - \operatorname{div} A(Du) = 0\ , \quad (x,t) \in \Omega \times (-T,0) = \Omega_T,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $T > 0$, $A : \mathbb{R}^{nN} \to \mathbb{R}^N$ satisfies a p-growth condition and $u : \Omega_T \to \mathbb{R}^N$. In particular, we focus our attention on local regularity of the spatial gradient of solutions of problems characterized by weak differentiability and ellipticity assumptions on the vector field $A(z)$. We prove the local Lipschitz continuity of solutions in the scalar case ($N = 1$). We extend this result in some vectorial cases under an additional structure condition. Finally, we prove higher integrability and differentiability of the spatial gradient of solutions for general systems.

1. Introduction

In this paper we deal with the study of some regularity properties of weak solutions to non-linear, second-order parabolic equations and systems of the type

$$u_t - \operatorname{div} A(Du) = 0\ , \quad (x,t) \in \Omega \times (-T,0) = \Omega_T,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $T > 0$, $A : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfies a p-growth condition and $u : \Omega_T \to \mathbb{R}^N$.

These systems can be viewed as generalizations for evolutionary p-Laplacian functionals.

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For stationary case, i.e., elliptic systems, the regularity of solutions has been widely investigated. Uhlenbeck (1977) showed that the gradient of a solution of the p-Laplacian is Hölder continuous when $p \geq 2$. Next the $C^{1,\alpha}$ regularity was extended to every $1 < p < 2$.

The same sort of regularity can be proved, arguing on the regularity of solution for the associated Euler-Lagrange equations, for minimizers of integral functionals of the type

$$\mathcal{F}(u) = \int_\Omega f(\nabla u)dx,$$

provided $f \in C^2(\mathbb{R}^n; \mathbb{R}^+)$ and its gradient satisfy the classical ellipticity condition

$$\nu|z|^{p-2}|\lambda|^2 \leq D^2 f(z)\lambda \otimes \lambda \leq L|z|^{p-2}|\lambda|^2.$$

Recently, it has been observed that if a p-growth condition of the type

$$\nu|z|^p \leq f(z) \leq L(1 + |z|^p) \quad p > 1,$$

holds, then the differentiability and the ellipticity properties required on f can be significantly weakened (see for example [1, 11, 12]).

For the evolutionary case the $C^{0,\alpha}$ regularity for the solutions of degenerate parabolic equations was proved by DiBenedetto [7] for $p \in [2, +\infty)$, and by Chen-DiBenedetto [5] for $1 < p < \infty$).

Next, DiBenedetto-Friedman [9] proved that the gradients of solutions of parabolic systems are Hölder continuous when $p \in \left(\frac{2n}{n+2}, +\infty\right)$ and independently Wiegner [18] when $p \in [2, +\infty)$.

Furthermore, we want to point out that in the same direction of studying the regularity of the spatial gradient of solutions of parabolic systems there has been a great deal of work concerning the integrability properties of weak and very weak solutions to nonhomogeneous parabolic systems.

On the other hand, very recently, parabolic systems with non-standard growth conditions have been interested by many authors. In particular, the case when the power p can be a function of x and t since this problem is somewhat related to extensions of the Navier-Stokes equation for electrorheological fluids (see [3]).

In this paper, we prove the local Lipschitz regularity of solutions of a class of parabolic equations characterized by weak growth, differentiability and ellipticity assumptions.

Next, in dealing with the vectorial case we are going to make a further distinction. We start considering a case in which the vector field A has a particular structure. More precisely for $g \in C^2(R^n)$ we consider

$$A(z) = g(|z|^2)z \quad \forall z \in R^{nN}.$$

For such a special situation we derive an analogous result with respect to scalar case. Instead, in the general vectorial case, without any additional structure condition, we are able to prove a higher integrability result for the spatial gradient of solutions of a general system.

To prove the result, we give first precise regularity estimate for the gradient of a solution of parabolic problems in the regular case and then we carry out this estimate to the general case, by means of an approximation argument.

2. Notations and results

Here, we recall the notations used throughout the paper and summarize the main results obtained. In the following Ω will denote a bounded domain of $\mathbb{R}^n$ and $T > 0$ a real number. We shall deal with the following problem (see Sections 2.1 and 2.2)

$$u_t - \operatorname{div} A(Du) = 0\,, \quad (x,t) \in \Omega \times (-T,0) = \Omega_T. \tag{2.1}$$

2.1. **Basic notations and definitions.** Here we fix the notations usually used when dealing with parabolic problems, for a more detailed treatment we refer to the texts [10, 14, 15] and the reference therein. For $(x_0,t_0) \in \Omega_T = \Omega \times (-T,0)$, we let

$$B_R(x_0) = \{x \in R^n \;:\; |x - x_0| < R\},$$
$$Q_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^2, t_0),$$
$$Q_R(\sigma_1,\sigma_2) = B_{R-\sigma_1 R} \times (t_0 - (1-\sigma_2)R^2, t_0).$$

With the symbol $\partial_p\Omega_T$ we mean the parabolic boundary of Ω_T. The symbol c will always denote a positive constant, possibly varying from line to line; the relevant connections with other quantities will be highlighted when necessary.

Now we recall the basic definitions of the functional spaces where solutions of parabolic equations in divergence form are typically found, these spaces of functions depending on $(t,x) \in \Omega_T$ are characterized by the property to exhibit different regularity in the space and time variables. Let $q, r \geq 1$. A function f measurable in Ω_T belongs to $L^{q,r}(\Omega_T) = L^r(0,T;L^q(\Omega))$ if

$$\|f\|_{q,r;\Omega_T} = \Big(\int_0^T \Big(\int_\Omega \|f\|^q \, dx \Big)^{\frac{r}{q}} d\tau \Big)^{\frac{1}{r}} < \infty.$$

Analogous definition can be given letting other spaces play the role played by L^p in the above definition. We will indeed consider the spaces

$$C_{loc}(-T,0;L^2_{loc}(\Omega)), \qquad L^p_{loc}\left(-T,0;W^{1,p}\left(\Omega,\mathbb{R}^N\right)\right),$$

and others, whose notation is self explanatory. For the Banach spaces $L^{q,\infty}(\Omega_T) \cap L^p\left(-T,0;W^{1,p}(\Omega)\right)$ and $L^{q,\infty}(\Omega_T) \cap L^p\big(-T,0;W_0^{1,p}(\Omega)\big)$ we use the standard notations $V^{q,p}(\Omega_T)$ and $V_0^{q,p}(\Omega_T)$.

Definition 1. *For energy solution of* (2.1) *we mean a function* $u \in L^p(-T,0;$ $W^{1,p}(\Omega,\mathbb{R}^N))$ *such that*

$$\int_{\Omega_T} (u\phi_t - A(Du)D\phi)\, dz = 0, \tag{2.2}$$

for all testing function $\phi \in C_0^\infty\left(\Omega_T,\mathbb{R}^N\right)$.

Following [10] (see also [14]) we recall the definitions of the *Steklov averages* that allows us to restate the Definition 1 in an, sometimes technically convenient, equivalent way. For $f \in L^1(\Omega_T)$ and $0 < h < T$, the *Steklov averages* f_h and $f_{\bar{h}}$ of f are defined for all $t \in (-T,0)$ by

$$f_h(x,t) := \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} f(x,s)\, ds, & \text{if } t \in (-T,-h), \\ 0, & \text{if } t > -h, \end{cases}$$

respectively,

$$f_{\bar{h}}(x,t) := \begin{cases} \dfrac{1}{h}\displaystyle\int_{t-h}^{t} f(x,s)\, ds, & \text{if } t \in (-T+h,0), \\ 0, & \text{if } t < -T+h. \end{cases}$$

These averaging in time give us a good approximations for f and for its different quotients in time. Indeed for a.e. $(x,t) \in \Omega \times (-T,-h)$ and for a.e. $(x,t) \in \Omega \times (-T+h,0)$, the following properties holds

$$\frac{\partial f_h}{\partial t} = \frac{1}{h}\left(f(x,t+h) - f(x,t)\right), \quad \frac{\partial f_{\bar{h}}}{\partial t} = \frac{1}{h}\left(f(x,t) - f(x,t-h)\right).$$

Moreover, it is easily seen that if $f \in L^{q,r}(\Omega_T)$, then as $h \to 0$, $f_h \to f$ in $L^{q,r}(\Omega_{T-\varepsilon})$ for every $\varepsilon \in (0,T)$ and if $f \in C(0,T;L^q(\Omega))$, then as $h \to 0$, $f_h(\cdot,t) \to f(\cdot,t)$ in $L^q(\Omega)$ for every $t \in (0,T-\varepsilon)$ and $\varepsilon \in (0,T)$. Clearly, a similar result holds for $f_{\bar{h}}$.

Using the previous notations, Definition 1 is equivalent to:

Definition 2. $u \in L^p\left(-T,0;W^{1,p}\left(\Omega,\mathbb{R}^N\right)\right)$ *is a solution of* (2.1) *if*

$$\int_\Omega (\partial_t(u_h)\phi - [A(Du)]_h D\phi)\, dz = 0\,, \quad \forall\, \phi \in W_0^{1,p}(\Omega,\mathbb{R}^N). \tag{2.3}$$

The equivalence between the two definitions can be easily proved using the convergence properties of Steklov averages

2.2. **Assumptions and main results.** Here, we summarize the main results obtained in the paper. We will assume that $A : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is a continuos vector field satisfying the following monotonicity and growth assumptions:
(H1)

$$\langle A(z_1) - A(z_2), z_1 - z_2 \rangle \geq \nu \left(\mu^2 + (|z_1| + |z_2|)^2\right)^{\frac{p-2}{2}} |z_1 - z_2|^2, \quad \forall\, z_1, z_2 \in \mathbb{R}^{nN}$$

(H2) $$|A(z)| \leq L(\mu^2 + |z|^2)^{\frac{p-1}{2}}, \quad \forall\, z \in \mathbb{R}^{nN}.$$

Remark 1. Here we do not deal with existence of solutions, anyway it is worth to notice that the hypotheses (H1) and (H2) allows us to apply the Galerkin's approximation procedure combined with Minty's theory on monotone operators in order to obtain existence of generalized solutions of the boundary value problems related to (2.1) (see [14] Chapter V.6), these solutions are clearly energy solutions too.

The main regularity result in the scalar case is:

Theorem 1. *Let $N = 1$, u be an energy solution of* (2.1) *and A satisfies (H1) and (H2) with $p > \frac{2n}{n+2}$, then u is locally Lipschitz continuous and, for $Q_R \subset \Omega_T$, the following estimate holds*

$$\sup_{Q_{R/2}} (\mu^2 + |Du|^2) \leq C \left(\fint_{Q_R} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, dz \right)^{1+\frac{2}{n}} + C.$$

In the vectorial case we are going to make a further distinction. As already pointed out in the previous section, we cannot expect the lipschitz regularity of energy solutions in the general vectorial case. Nevertheless, if we assume a particular structure condition on the vector fields A, then we can extend the previous result. Namely, we suppose that there exists a real function $g \in C^2(\mathbb{R})$ such that

(H3) $$A(z) = g(|z|^2) z \quad \forall\, z \in \mathbb{R}^{nN}.$$

Condition (H3) is not unnatural to obtain higher regularity results in vectorial case, indeed it comes naturally when we consider Euler equations associated to radial symmetric variational problems where such type of results are true (see [17, 2, 11]).

For such special problems the following analogous of the previous theorem holds:

Theorem 2. *Let u be an energy solution of* (2.1) *and A satisfies (H1), (H2) with $p > \frac{2n}{n+2}$ and (H3), then Du is in $L^\infty_{loc}(\Omega_T)$ and, for $Q_R \subset \Omega_T$, the* *following estimate holds,*

$$\sup_{Q_{R/2}} (\mu^2 + |Du|^2) \le C\Big(\fint_{Q_R} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, dz \Big)^{1+\frac{2}{n}} + C.$$

In the general vectorial case we turn our attention to the higher integrability of the gradient following [11] where such type of result are obtained for local minimizers of integral functionals with non standard growth conditions.

We have the following:

Theorem 3. *Let $N \ge 1$, u be an energy solution of* (2.1) *and A satisfies (H1), (H2) with $p \ge 2$. Then, $Du \in L^s_{loc}(\Omega_T)$, where $s = p + \frac{4}{n}$ if $n > 2$ and $s \in [1, \infty)$ is any arbitrary number if $n = 2$. Moreover,*

$$D\Big[\big(\mu^2 + |Du|^2\big)^{\frac{p-2}{4}}\, Du \Big] \in L^2_{loc}(\Omega_T),$$

and for each concentric parabolic cylinders $Q_\rho \subset Q_R$, the following a priori estimates are verified,

$$\int_{Q_\rho} \big(\mu^2 + |Du|^2\big)^{\frac{s}{2}}\ dz \le \Big[\frac{c}{(R-\rho)^2} \int_{Q_R} \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\ dz + \frac{c}{(R-\rho)^2} \Big]^{1+\frac{2}{n}},$$

$$\int_{Q_\rho} \Big| D\Big[\big(\mu^2 + |Du|^2\big)^{\frac{p-2}{4}}\, Du \Big] \Big|^2 dz$$
$$\le \frac{c}{(R-\rho)^2} \int_{Q_R} \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\ dz + \frac{c}{(R-\rho)^2}.$$

3. Preliminaries: the approximation lemma

In this section, we state an approximation lemma for non-smooth, monotone, uniformly elliptic operators with standard p-growth. Our aim is to define an approximating sequence A_ε for A which is regular enough in order to apply the a priory estimates proved in Section 4 to the parabolic system associated to A_ε. This kind of procedure is, by now, quite standard when dealing with non-smooth problems (see for instance [12, 11] for the approximation of convex energy densities of integral functionals).

Lemma 1. *Let A be a vector field satisfying* (H1), (H2)*. There exist c_1, c_2, c_3 and $c_4 \in (0, +\infty)$ (depending upon N, ν, L, p) and $\varepsilon_0 \equiv \varepsilon_0(\nu, L, p) > 0$ such that, for each $0 < \varepsilon < \varepsilon_0$ there exist a constant M_ε, depending on ε, and vector fields $A_\varepsilon : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$, such that $A_\varepsilon(z) \in C^1(\mathbb{R}^{nN})$,*

$$A_\varepsilon \to A \text{ uniformly on } K \quad \forall\ K \text{ compact } \subset \mathbb{R}^{nN}, \tag{3.1}$$

$$|A_\varepsilon(z)| \le c_1(\mu^2 + \varepsilon^2 + |z|^2)^{\frac{p-1}{2}} \qquad \forall\ z \in \mathbb{R}^{nN}, \tag{3.2}$$

$$|D_z A_\varepsilon(z)| \le M_\varepsilon \left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}} \qquad \forall\; z \in \mathbb{R}^{nN}, \tag{3.3}$$

$$\langle D_z A_\varepsilon(z)\lambda,\lambda\rangle \ge c_2(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall\; \lambda, z \in \mathbb{R}^{nN}. \tag{3.4}$$

Proof. The approximating smooth vector fields, $A_\varepsilon : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ are constructed by taking a mollification of A in a ball of radius of order $\frac{1}{\varepsilon}$ and gluing it with the $(p-1)$-growing function $c_\varepsilon(z) = p(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}z$ in a way that preserves the desired inequalities. First we define the field,

$$A_\varepsilon^-(z) = \int_{B_1(0)} A(z+\varepsilon\xi)\rho(\xi)\,d\xi,$$

where ρ is a standard radially symmetric mollifier. Using the hypothesis (H2), we can easily deduce that, $\forall\, z \in \mathbb{R}^{Nn}$,

$$|A_\varepsilon^-(z)| \le \int_{B_1(0)} |A(z+\varepsilon\xi)|\,\rho(\xi)\,d\xi \le \int_{B_1(0)} L(\mu^2+|z+\varepsilon\xi|^2)^{\frac{p-1}{2}}\rho(\xi)\,d\xi \tag{3.5}$$

$$\le \int_{B_1(0)} L2^{\frac{p-1}{2}}(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-1}{2}}\rho(\xi)\,d\xi = L_1(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-1}{2}}.$$

Let $\lambda \in \mathbb{R}^{nN}$, writing down the definition of directional derivative, making use of the hypothesis (H1), we deduce that

$$\begin{aligned}
&\left\langle D_z A_\varepsilon^-(z)\lambda,\lambda\right\rangle = \lim_{t\to 0}\left\langle \frac{A_\varepsilon(z+t\lambda)-A_\varepsilon(z)}{t},\lambda\right\rangle \qquad (3.6)\\
&= \lim_{t\to 0}\int_{B_1(0)} \left\langle \frac{A(z+t\lambda+\varepsilon\xi)-A(z+\varepsilon\xi)}{t^2}, t\lambda\right\rangle\rho(\xi)\ d\xi\\
&\ge \lim_{t\to 0}\int_{B_1(0)} \nu\left(\mu^2+(|z+t\lambda+\varepsilon\xi|+|z+\varepsilon\xi|)^2\right)^{\frac{p-2}{2}}|\lambda|^2\rho(\xi)\ d\xi\\
&= \nu|\lambda|^2\int_{B_1(0)} (\mu^2+2|z+\varepsilon\xi|^2)^{\frac{p-2}{2}}\rho(\xi)\ d\xi.
\end{aligned}$$

Now, we treat separately the case $p \ge 2$ and $p < 2$. If $p \ge 2$, then $\frac{p-2}{2} \ge 0$ and we can write

$$\begin{aligned}
&\int_{B_1(0)} (\mu^2+2|z+\varepsilon\xi|^2)^{\frac{p-2}{2}}\rho(\xi)\ d\xi \qquad (3.7)\\
&= \int_{B_1(0)} (\mu^2+2|z|^2+2\varepsilon^2|\xi|^2+4\varepsilon\langle z,\xi\rangle)^{\frac{p-2}{2}}\rho(\xi)\ d\xi
\end{aligned}$$

$$\geq \int_{(B_1\setminus B_{\frac{1}{2}})\cap\{\langle z,\xi\rangle\geq 0\}} \left(\mu^2+2|z|^2+\frac{\varepsilon^2}{2}\right)^{\frac{p-2}{2}} \rho(\xi)\ d\xi$$

$$\geq \frac{1}{2}\left(\mu^2+2|z|^2+\frac{\varepsilon^2}{2}\right)^{\frac{p-2}{2}} \int_{B_1\setminus B_{\frac{1}{2}}} \rho(\xi)\ d\xi \geq c\left(\mu^2+|z|^2+\varepsilon^2\right)^{\frac{p-2}{2}}.$$

If $p<2$, since $\frac{p-2}{2}<0$, we have

$$\int_{B_1(0)} (\mu^2+2|z+\varepsilon\xi|^2)^{\frac{p-2}{2}}\rho(\xi)\ d\xi \geq \int_{B_1(0)} (\mu^2+4|z|^2+4\varepsilon^2|\xi|^2)^{\frac{p-2}{2}}\rho(\xi)\ d\xi$$

$$\geq \int_{B_1(0)} (\mu^2+4|z|^2+4\varepsilon^2)^{\frac{p-2}{2}}\rho(\xi)\ d\xi \geq c\left(\mu^2+|z|^2+\varepsilon^2\right)^{\frac{p-2}{2}}. \tag{3.8}$$

Combining (3.6) with (3.7) or (3.8), we get

$$\langle D_z A_\varepsilon^-(z)\lambda,\lambda\rangle \geq c_0(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \quad , \qquad \forall\ \lambda, z\in\mathbb{R}^{nN}. \tag{3.9}$$

Without loss of generality we suppose

$$c_0 < \min\left\{p-1,\frac{1}{2}\right\}p. \tag{3.10}$$

The smoothing argument used above, allowed us to prove (3.3) and (3.4) for $A_\varepsilon^-(z)$, while in order to obtain a vector field satisfying (3.2), we introduce the auxiliary fields $B_\varepsilon(z)$ as follows

$$B_\varepsilon(z) := \begin{cases} 0 & \text{if} \quad |z|\leq \frac{1}{2\varepsilon} \\ p\left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}} z & \text{if} \quad |z|>\frac{1}{2\varepsilon}, \end{cases}$$

and define the following cut-off function $\eta_\varepsilon : \mathbb{R}^+\to\mathbb{R}^+$, by

$$\eta_\varepsilon(t) := \begin{cases} 0 & \text{if} \quad 0\leq t\leq \frac{1}{\varepsilon} \\ \sigma\log(t\varepsilon) & \text{if} \quad \frac{1}{\varepsilon}<t<\frac{1}{\varepsilon}e^{\frac{1}{\sigma}} \\ 1 & \text{if} \quad t\geq \frac{1}{\varepsilon}e^{\frac{1}{\sigma}}. \end{cases}$$

We use a convex combination of $A^-(z)$ and $B_\varepsilon(z)$ via η_ε to define the following fields:

$$\overline{A}_\varepsilon(z) := (1-\eta(|z|))A_\varepsilon^-(z)+\eta_\varepsilon(|z|)B_\varepsilon(z).$$

The definition of $\overline{A}_\varepsilon$ is the second step in the construction of the good approximation fields. First, we observe that $\overline{A}_\varepsilon(z)$ converges uniformly to $A(z)$ for $\varepsilon\to 0$ in any compact set $K\subset\mathbb{R}^{Nn}$.

If $|z|\in(\frac{1}{\varepsilon},\frac{1}{\varepsilon}e^{\frac{1}{\sigma}})$, we have, by definition,

$$\overline{A}_\varepsilon(z) := (1-\eta(|z|))A_\varepsilon^-(z)+\eta_\varepsilon(|z|)c_\varepsilon(z),$$

where, $c_\varepsilon(z) = p(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}z$.

Before going further, we observe that η_ε is a lipschitz function, such that in the interval $(\frac{1}{\varepsilon}, \frac{1}{\varepsilon}e^{\frac{1}{\sigma}})$ is strictly increasing and its derivative η'_ε is bounded by $\frac{\sigma}{t}$. Moreover, if we set $\delta := \min\left\{p-1, \frac{1}{2}\right\}$, we can prove

$$\langle D_z c_\varepsilon(z)\lambda;\lambda\rangle \geq \delta p\left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall\, z,\lambda\in\mathbb{R}^{nN}. \tag{3.11}$$

We can compute

$$D_z\overline{A}_\varepsilon(z) = (1-\eta(|z|))\,D_zA^-_\varepsilon(z) - \eta'_\varepsilon(|z|)A^-_\varepsilon(z)\otimes\frac{z}{|z|} + \eta_\varepsilon(|z|)D_zc_\varepsilon(z) + \eta'_\varepsilon(|z|)c_\varepsilon(z)\otimes\frac{z}{|z|},$$

and it follows that

$$\begin{aligned}
&\left\langle D_z\overline{A}_\varepsilon(z)\lambda;\lambda\right\rangle = (1-\eta_\varepsilon(|z|))\left\langle D_zA^-_\varepsilon(z)\lambda;\lambda\right\rangle - \eta'_\varepsilon(|z|)\left\langle A^-_\varepsilon(z)\otimes\frac{z}{|z|}\lambda;\lambda\right\rangle\\
&+\eta_\varepsilon(|z|)\left\langle D_zc_\varepsilon(z)\lambda;\lambda\right\rangle + \eta'_\varepsilon(|z|)\left\langle c_\varepsilon(z)\otimes\frac{z}{|z|}\lambda;\lambda\right\rangle\\
&\geq (1-\eta_\varepsilon(|z|))\,c_0(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \qquad\qquad \text{by (3.9)}\\
&-\sqrt[4]{nN}\,\frac{\sigma}{|z|}L_1(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-1}{2}}|\lambda|^2 \qquad\qquad \text{by (3.5)}\\
&+\eta_\varepsilon(|z|)\delta p\left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \qquad\qquad \text{by (3.11)}\\
&-\sqrt[4]{nN}\,\sigma p\left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \qquad\qquad \text{by definition of } c_\varepsilon\\
&\geq \left(c_0 - \sigma\sqrt[4]{nN}\left(p+L_1(\mu^2\varepsilon^2+\varepsilon^4+1)\right)\right)\left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2\\
&+\eta_\varepsilon(|z|)(\delta p - c_0)\left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2.
\end{aligned}$$

Recalling (3.10), if we choose ε such that $\mu^2\varepsilon^2+\varepsilon^4+1<2$ and

$$\sigma < \frac{c_0}{2(p+2L_1)\sqrt[4]{nN}},$$

we can conclude that, with this choice of ε and σ we have

$$\left\langle D_z\overline{A}_\varepsilon(z)\lambda;\lambda\right\rangle \geq \frac{c_0}{2}\left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \quad \forall\,|z|\in\left(\frac{1}{\varepsilon},\frac{1}{\varepsilon}e^{\frac{1}{\sigma}}\right). \tag{3.12}$$

If $|z|\leq\frac{1}{\varepsilon}$, then, $\overline{A}_\varepsilon(z) = A^-_\varepsilon(z)$ and by (3.9), we conclude that (3.12) holds true also for this range of values for $|z|$. Outside the ball of radius $\frac{e^{\frac{1}{\sigma}}}{\varepsilon}$, since

$\frac{e^{\frac{1}{\sigma}}}{\varepsilon} > \frac{1}{2\varepsilon}$, we have $\overline{A}_\varepsilon(z) = p\left(\mu^2 + \varepsilon^2 + |z|^2\right)^{\frac{p-2}{2}} z = c_\varepsilon(z)$ and we can use (3.11) and (3.10) to deduce that:

$$\left\langle D_z\overline{A}_\varepsilon(z)\lambda;\lambda\right\rangle \geq \frac{c_0}{2}\left(\mu^2+\varepsilon^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall\, z\in\mathbb{R}^{Nn}. \tag{3.13}$$

Using the bound (3.5) and the definition of $\overline{A}_\varepsilon(z)$ we easily get for all $z \in \mathbb{R}^{Nn}$

$$\begin{aligned}|\overline{A}_\varepsilon(z)| &\leq |A_\varepsilon^-(z)| + |B_\varepsilon(z)| \leq L_1(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-1}{2}} + p(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}|z| \\ &= L_1(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-1}{2}} + p(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}\left(|z|^2\right)^{\frac{1}{2}} \\ &\leq (L_1+p)(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-1}{2}} = L_2(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-1}{2}}. \end{aligned}\tag{3.14}$$

Using $\overline{A}_\varepsilon(z)$, we can finally define A_ε by a mollification argument. Let $\gamma_\varepsilon := \varepsilon^2$ and define

$$A_\varepsilon(z) = \int_{B_1(0)} \overline{A}_\varepsilon(z+\gamma_\varepsilon\xi)\rho(\xi)\,d\xi,$$

where ρ is again the standard radial symmetric convolution kernel. Clearly, $A_\varepsilon(z) \in C^1(\mathbb{R}^{nN})$ and converges uniformly to $A(z)$ for $\varepsilon \to 0$ in any compact set $K \subset \mathbb{R}^{Nn}$. Moreover, if $|z| > \frac{e^{\sigma^{-1}}}{\varepsilon} + \gamma_\varepsilon$, then,

$$A_\varepsilon(z) = \int_{B_1(0)} c_\varepsilon(z+\gamma_\varepsilon\xi)\rho(\xi)\,d\xi = \int_{\mathbb{R}^{Nn}} p(\mu^2+\varepsilon^2+|z+\gamma_\varepsilon\xi|^2)^{\frac{p-2}{2}}(z+\gamma_\varepsilon\xi)\rho(\xi)\,d\xi,$$

and we can estimate the gradient as follows (with I we denote the identity matrix in $\mathbb{R}^{Nn}\times\mathbb{R}^{Nn}$)

$$\begin{aligned}&|D_zA_\varepsilon(z)| \\ &\leq p|p-2|\left|\int_{B_1(0)}(\mu^2+\varepsilon^2+|z+\gamma_\varepsilon\xi|^2)^{\frac{p-4}{2}}(z+\gamma_\varepsilon\xi)\otimes(z+\gamma_\varepsilon\xi)\,\rho(\xi)d\xi\right| \\ &+ p\left|\int_{B_1(0)}(\mu^2+\varepsilon^2+|z+\gamma_\varepsilon\xi|^2)^{\frac{p-2}{2}}\,\rho(\xi)\,d\xi\right| \\ &\leq c_p'\int_{B_1(0)}(\mu^2+\varepsilon^2+|z+\gamma_\varepsilon\xi|^2)^{\frac{p-2}{2}}\,\rho(\xi)\,d\xi \leq c_p(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}.\end{aligned}\tag{3.15}$$

Now, since $A_\varepsilon(z) \in C^1(\mathbb{R}^{nN})$ its gradient is bounded on compact sets, letting

$$M_\varepsilon' = \sup_{|z|\leq\frac{e^{\sigma^{-1}}}{\varepsilon}+\gamma_\varepsilon}|D_zA_\varepsilon(z)| < +\infty,$$

from (3.15) we infer (3.2) with $M_\varepsilon = \frac{M'_\varepsilon}{(\mu^2+\varepsilon^2)^{\frac{p-2}{2}}} + c_p$ if $p > 2$ and $M_\varepsilon = \frac{M'_\varepsilon}{(\mu^2+\varepsilon^2+\frac{e^{\sigma}-1}{\varepsilon})^{\frac{p-2}{2}}} + c_p$ if $p < 2$. The bound (3.3) follows from (3.13), indeed

$$\begin{aligned}\langle D_z A_\varepsilon(z)\lambda;\lambda\rangle &= \int_{B_1(0)} \langle D_z \overline{A}_\varepsilon(z+\gamma_\varepsilon\xi)\lambda;\lambda\rangle \,\rho(\xi)\,d\xi \\ &\geq \frac{c_0}{2}|\lambda|^2 \int_{B_1(0)} \left(\mu^2+\varepsilon^2+|z+\gamma_\varepsilon\xi|^2\right)^{\frac{p-2}{2}} \rho(\xi)\,d\xi,\end{aligned}$$

and we proceed as we did in order to obtain (3.7) and (3.8). Moreover, (3.1) follows from (3.14) arguing exactly as for (3.5). □

4. The a priori estimates

In this section, we prove precise a priori estimates for weak solutions of equations and systems satisfying well behaved smoothness assumptions. The smoothness assumptions imposed are namely the properties satisfied by the elements of an approximating sequence for A chosen as in Lemma 1. Indeed, we prove regularity estimates that do not depend on the estimates on the derivatives of the vector field defining the principal part of the problem under consideration.

4.1. **The scalar case: Equations.** Here, we deal with an a priori L^∞-estimate for solutions $u \in V^{2,p}(\Omega_T)$ of (2.1) in the regular case.

Lemma 2. *Suppose that $A \in C^1(\mathbb{R}^n)$ satisfies*

$$\begin{aligned}|D_z A(z)| &\leq M\left(\mu^2+|z|^2\right)^{\frac{p-2}{2}} &&\forall\, z\in\mathbb{R}^n,\\ \langle D_z A(z)\lambda,\lambda\rangle &\geq \nu(\mu^2+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 &&\forall\, \lambda,z\in\mathbb{R}^n,\\ |A(z)| &\leq L(\mu^2+|z|^2)^{\frac{p-1}{2}} &&\forall\, z\in\mathbb{R}^n.\end{aligned}\tag{4.1}$$

Let $u \in V^{2,p}(\Omega_T)$, be an energy solution of (2.1). *Then, $Du \in L^\infty_{loc}(\Omega_T)$. Moreover, for any cylinder $Q_{R_0} \subset\subset \Omega_T$ the following a priori estimate is verified with the constant C independent of μ and L*

$$\sup_{Q_{R_0/2}} (\mu^2+|Du|^2) \leq C\left(\fint_{Q_{R_0}} (\mu^2+|Du|^2)^{\frac{p}{2}}\,dz\right)^{1+\frac{2}{n}} + C.$$

Proof. We establish the result under the assumption that $u, u_t, D_x u, D_x^2 u$ are in suitable spaces so that the calculations are justified. To be rigorous we should apply an averaging argument in time via Steklov averages and a discretization in space. This argument is quite standard in parabolic literature and its application leads to minor technical complications (we refer for instance to [8] for an example on how to apply that procedure). Let us write (2.1) using the function $\phi = D_s(\chi_\varepsilon \eta^2 \psi)$ as test function for $s \in \{1, \dots, n\}$

$$\int_{Q_R} u_t D_s(\chi_\varepsilon \eta^2 \psi)\, dx\, dt + \int_{Q_R} \chi_\varepsilon(t) A^i(Du) D_s(2\eta D_i \eta\, \psi + \eta^2 D_i \psi)\, dx\, dt = 0,$$

here $\chi_\varepsilon : \mathbb{R} \to [0,1]$ is defined by

$$\chi_\varepsilon(t) = \begin{cases} 1 & t \in (-\infty, l) \\ \frac{l+\varepsilon-t}{\varepsilon} & t \in [l, l+\varepsilon] \\ 0 & t \in (l+\varepsilon, \infty), \end{cases}$$

and η is a smooth parabolic cut-off function, that is $\eta = 1$ in $Q_R(\sigma_1, \sigma_2)$, $\eta = 0$ in a neighborhood of the boundary of Q_R, $0 \le \eta \le 1$ and

$$|D_x \eta| \le \frac{C}{\sigma_1 R}\,, \quad |D_t \eta| \le \frac{C}{\sigma_2 R^2}.$$

Integrating by parts we deduce

$$\begin{aligned} &\int_{Q_R} u_t D_s(\chi_\varepsilon \eta^2 \psi)\, dx\, dt + \int_{Q_R} \chi_\varepsilon A^i(Du) D_s(2\eta D_i \eta\, \psi)\, dx\, dt \\ &- \int_{Q_R} \chi_\varepsilon D_j A^i(Du) D_j(D_s u) \eta^2 D_i \psi\, dx\, dt = 0. \end{aligned} \tag{4.2}$$

Let $\beta > 0$ and consider $\psi = (\mu^2 + |Du|^2)^\beta D_s u$. Using the notation $V = \mu^2 + |Du|^2$, the first term of (4.2) becomes, integrating by parts,

$$\begin{aligned} \int_{Q_R} u_t D_s(\chi_\varepsilon \eta^2 \psi)\, dx\, dt &= \int_{Q_R} u_t D_s(\chi_\varepsilon \eta^2 (\mu^2 + |Du|^2)^\beta D_s u)\, dx\, dt \\ &= -\int_{Q_R} (D_s u)_t \chi_\varepsilon \eta^2 V^\beta D_s u\, dx\, dt. \end{aligned} \tag{4.3}$$

We observe that

$$\frac{d}{dt}\left(V^{\beta+1} \chi_\varepsilon \eta^2\right) = (2\beta+2) V^\beta \langle Du; (Du)_t \rangle\, \eta^2 \chi_\varepsilon + \chi_\varepsilon 2\eta \eta_t V^{\beta+1} + \eta^2 V^{\beta+1} \chi_{\varepsilon t}. \tag{4.4}$$

Summing up in (4.3) on $s \in \{1, 2, \dots, n\}$, we obtain

$$\sum_{s=1}^n \int_{Q_R} u_t D_s(\chi_\varepsilon \eta^2 V^\beta D_s u)\, dx\, dt = -\int_{Q_R} V^\beta \langle Du; (Du)_t \rangle\, \eta^2 \chi_\varepsilon\, dx\, dt.$$

And, by (4.4) we can write, using the properties of χ_ε and η,

$$\sum_{s=1}^n \int_{Q_R} u_t D_s(\chi_\varepsilon \eta^2 V^\beta D_s u)\, dx\, dt \tag{4.5}$$

$$= \frac{1}{2\beta+2}\int_{Q_R}\left(-\frac{d}{dt}\left(V^{\beta+1}\eta^2\chi_\varepsilon\right)+2\eta\eta_t V^{\beta+1}\chi_\varepsilon+\eta^2V^{\beta+1}\chi_{\varepsilon t}\right)dx\,dt$$
$$= \frac{1}{2\beta+2}\left(\int_{Q_R}2\eta\eta_t V^{\beta+1}\chi_\varepsilon\,dx\,dt-\frac{1}{\varepsilon}\int_l^{l+\varepsilon}\int_{B_R}V^{\beta+1}\eta^2\,dx\,dt\right).$$

Again, summing up on $s\in\{1,2,\dots,n\}$ from (4.2), (4.3), (4.5), we obtain

$$\frac{1}{(2\beta+2)}\fint_l^{l+\varepsilon}\left(\int_{B_R}V^{\beta+1}\eta^2\,dx\right)dt \tag{4.6}$$
$$+\sum_{s=1}^n\underbrace{\int_{Q_R}\chi_\varepsilon D_jA^i(Du)D_j(D_su)\eta^2D_i\left(V^\beta D_su\right)\,dx\,dt}_{\mathcal{A}_s}$$
$$=\ \frac{1}{2\beta+2}\int_{Q_R}\chi_\varepsilon 2\eta\eta_t V^{\beta+1}\,dx\,dt$$
$$+\sum_{s=1}^n\underbrace{\int_{Q_R}\chi_\varepsilon A^i(Du)D_s(2\eta D_i\eta\,V^\beta D_su)\,dx\,dt}_{\mathcal{B}_s}.$$

Let us note that we can write

$$\mathcal{A}_s=\underbrace{\int_{Q_R}\chi_\varepsilon D_jA^i(Du)D_j(D_su)\beta V^{\beta-1}D_i\left(|Du|^2\right)D_su\,\eta^2\,dx\,dt}_{\mathcal{A}_s^1} \tag{4.7}$$
$$+\underbrace{\int_{Q_R}\chi_\varepsilon D_jA^i(Du)D_j(D_su)V^\beta D_i\left(D_su\right)\eta^2\,dx\,dt}_{\mathcal{A}_s^2}.$$

Moreover, by the hypotheses on A, we have

$$\begin{aligned}\sum_{s=1}^n\mathcal{A}_s^2 &= \sum_{s=1}^n\int_{Q_R}\chi_\varepsilon\langle D_zA(Du)D(D_su);D(D_su)\rangle V^\beta\eta^2\,dx\,dt\\ &\ge \nu\int_{Q_R}\chi_\varepsilon V^{\frac{p-2}{2}+\beta}|D^2u|^2\eta^2\,dx\,dt,\end{aligned}$$

and, since $\sum_{s=1}^n 2D_j(D_su)D_su=D_j(|Du|^2)$,

$$\sum_{s=1}^n\mathcal{A}_s^1 = \frac{1}{2}\int_{Q_R}\chi_\varepsilon\langle D_zA(Du)D(|Du|^2);D(|Du|^2)\rangle\beta V^{\beta-1}\eta^2\,dx\,dt$$

$$\geq \quad \frac{\beta\nu}{2}\int_{Q_R}\chi_\varepsilon V^{\frac{p-2}{2}+\beta-1}\left|D(|Du|^2)\right|^2\eta^2\,dx\,dt.$$

Now, we use the hypothesis (4.1) to estimate the right hand side of (4.6). We have

$$\mathcal{B}_s = \underbrace{\int_{Q_R}\chi_\varepsilon A^i(Du)2\eta D_i\eta\, D_s(V^\beta D_s u)\,dx\,dt}_{\mathcal{B}_s^1} + \underbrace{\int_{Q_R}\chi_\varepsilon A^i(Du)D_s(2\eta D_i\eta)\,V^\beta D_s u\,dx\,dt}_{\mathcal{B}_s^2}. \tag{4.8}$$

Moreover, by the hypotheses on A, we have

$$\begin{aligned}\sum_{s=1}^N \mathcal{B}_s^2 &= \int_{Q_R}\chi_\varepsilon A^i(Du)2\eta D_s(D_i\eta)\,V^\beta D_s u\,dx\,dt \\ &\quad + \int_{Q_R}\chi_\varepsilon A^i(Du)2D_i\eta D_s(\eta)\,V^\beta D_s u\,dx\,dt \\ &\leq c(p,L,n)\int_{Q_R}\chi_\varepsilon V^{\frac{p}{2}+\beta}2\eta|D^2\eta|\,dx\,dt + c(p,L,n)\int_{Q_R}\chi_\varepsilon V^{\frac{p}{2}+\beta}2|D\eta|^2dx\,dt,\end{aligned}$$

and

$$\begin{aligned}\sum_{s=1}^N \mathcal{B}_s^1 &\leq c(p,L,n)\sum_{s=1}^N\int_{Q_R}\chi_\varepsilon V^{\frac{p-1}{2}}(V^\beta|D_s^2u| \\ &\quad + \beta V^{\beta-1}|D_s u||D_s(|Du|^2)|)2\eta|D\eta|\,dx\,dt \\ &\leq c(p,L,n)\int_{Q_R}\chi_\varepsilon V^{\frac{p-1}{2}+\beta}|D^2u|\,\eta\,|D\eta|\,dx\,dt \\ &\quad + c(p,L,n)\beta\int_{Q_R}\chi_\varepsilon V^{\frac{p}{2}+\beta-1}|D(|Du|^2)|\,2\eta|D\eta|\,dx\,dt.\end{aligned}$$

Finally, from (4.6), letting $\varepsilon\to 0$, we infer that, for $l\in(t_0-R^2,t_0)$

$$\underbrace{\int_{B_R}V^{\beta+1}(l,x)\eta^2(l,x)\,dx}_{I_1(l)} + \underbrace{2\nu\int_{t_0-R^2}^{l}\int_{B_R}V^{\frac{p-2}{2}+\beta}|D^2u|^2\eta^2\,dx\,dt}_{I_2(l)}$$
$$+\underbrace{\beta\nu\int_{t_0-R^2}^{l}\int_{B_R}V^{\frac{p-2}{2}+\beta-1}\left|D(|Du|^2)\right|^2\eta^2\,dx\,dt}_{I_3(l)}$$

$$\le \int_{Q_R} V^{\beta+1} 2\eta\,|\eta_t|\,dx\,dt + 2c(\beta+1)\int_{Q_R} V^{\frac{p}{2}+\beta}\left(2\eta|D^2\eta| + 2|D\eta|^2\right)\,dx\,dt$$
$$+ c(\beta+1)\int_{Q_R} V^{\frac{p-1}{2}+\beta}|D^2u|\,\eta\,|D\eta|\,dx\,dt$$
$$+ c(\beta+1)\beta\int_{Q_R} V^{\frac{p-1}{2}+\beta-1}|Du|\,|D(|Du|^2)|\,2\eta|D\eta|\,dx\,dt.$$

From the previous inequality we deduce

$$\sup_{t\in(t_0-R^2,t_0)}\int_{B_R} V^{\beta+1}\eta^2\,dx + 2\nu\int_{Q_R} V^{\frac{p-2}{2}+\beta}|D^2u|^2\eta^2\,dx\,dt \tag{4.9}$$
$$+ \beta\nu\int_{Q_R} V^{\frac{p-2}{2}+\beta-1}\left|D(|Du|^2)\right|^2\eta^2\,dx\,dt$$
$$= \sup_{t\in(t_0-R^2,t_0)} I_1(t) + \sup_{t\in(t_0-R^2,t_0)} I_2(t) + \sup_{t\in(t_0-R^2,t_0)} I_3(t)$$
$$\le 3\max\left\{\sup_{t\in(t_0-R^2,t_0)} I_1(t)\,,\ \sup_{t\in(t_0-R^2,t_0)} I_2(t)\,,\ \sup_{t\in(t_0-R^2,t_0)} I_3(t)\right\}$$
$$\le 3\int_{Q_R} V^{\beta+1}2\eta\,|\eta_t|\,dx\,dt + c(\beta+1)\int_{Q_R} V^{\frac{p}{2}+\beta}\left(2\eta|D^2\eta| + 2|D\eta|^2\right)\,dx\,dt$$
$$+ c(\beta+1)\int_{Q_R} V^{\frac{p-1}{2}+\beta}|D^2u|\,\eta\,|D\eta|\,dx\,dt$$
$$+ c(\beta+1)\beta\int_{Q_R} V^{\frac{p-1}{2}+\beta-1}|Du|\,|D(|Du|^2)|\,2\eta|D\eta|\,dx\,dt.$$

Now, we estimate the last two terms of the previous inequality. We have by Young inequality

$$\int_{Q_R} V^{\frac{p-1}{2}+\beta}|D^2u|\,\eta\,|D\eta|\,dx\,dt = \int_{Q_R}(V^{\frac{\beta}{2}+\frac{p-2}{4}}|D^2u|\,\eta)(V^{\frac{\beta}{2}+\frac{p}{4}}\,|D\eta|)\,dx\,dt$$
$$\le \varepsilon_1\int_{Q_R} V^{\beta+\frac{p-2}{2}}|D^2u|^2\,\eta^2\,dx\,dt + \frac{1}{\varepsilon_1}\int_{Q_R} V^{\beta+\frac{p}{2}}\,|D\eta|^2\,dx\,dt,$$

and, since $\frac{p}{2}+\beta-1 = (\frac{p-2}{2}+\beta-1)/2 + (\frac{p}{2}+\beta)/2$, we can write

$$\int_{Q_R} V^{\frac{p-1}{2}+\beta-1}|Du|\,|D(|Du|^2)|\,2\eta|D\eta|\,dx\,dt$$
$$\le \varepsilon_2\int_{Q_R} V^{\frac{p-2}{2}+\beta-1}\left|D(|Du|^2)\right|^2\eta^2\,dx\,dt + \frac{1}{\varepsilon_2}\int_{Q_R} V^{\frac{p}{2}+\beta}|D\eta|^2\,dx\,dt.$$

Using the previous estimates, we can write (using the position $\varepsilon_1 = \frac{2(\beta+1)}{c}\nu\tilde{\varepsilon}_1$ and $\varepsilon_2 = \frac{(\beta+1)\beta}{c}\nu\tilde{\varepsilon}_2$)

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)} \int_{B_R} V^{\beta+1}\eta^2 dx + 2\nu(\beta+1)(1-(\beta+1)\tilde{\varepsilon}_1)\int_{Q_R} V^{\frac{p-2}{2}+\beta}|D^2u|^2\eta^2 dxdt \\
&+ \nu\beta(\beta+1)(1-(\beta+1)\beta\tilde{\varepsilon}_2)\int_{Q_R} V^{\frac{p-2}{2}+\beta-1}|D(|Du|^2)|^2\eta^2\, dx\, dt \qquad (4.10)\\
&\le \left(\frac{c^2}{2\nu\tilde{\varepsilon}_1} + \frac{c^2}{\nu\tilde{\varepsilon}_2}\right)\int_{Q_R} V^{\beta+\frac{p}{2}}\,|D\eta|^2\, dx\, dt + \int_{Q_R} V^{\beta+1} 2\eta\,|\eta_t|\, dx\, dt \\
&+ 2c(\beta+1)\int_{Q_R} V^{\frac{p}{2}+\beta}\,\left(2\eta|D^2\eta| + 2|D\eta|^2\right)\, dx\, dt.
\end{aligned}$$

Now, we want to estimate $\int_{Q_R} V^{\beta+1} 2\eta\,|\eta_t|\, dx\, dt$. If $p \ge 2$, we easily get

$$\begin{aligned}
\int_{Q_R} V^{\beta+1}|\eta_t|\, dx\, dt &\le \int_{Q_R\cap\{V\ge 1\}} V^{\beta+\frac{p}{2}}\eta|\eta_t|\, dx\, dt + \int_{Q_R\cap\{V\le 1\}} V^{\beta+1}\eta\,|\eta_t|\, dx\, dt \\
&\le \int_{Q_R} V^{\beta+\frac{p}{2}}\,|\eta_t|\, dx\, dt + c\frac{|Q_R|}{(R-\rho)^2}. \qquad (4.11)
\end{aligned}$$

In the case $1 < p < 2$, we use the fact that the L^∞_{loc} norm of u is bounded independently of M (see for instance [7], Section V.3). Indeed, from integration by parts we have

$$\begin{aligned}
&\int_{Q_R} V^{\beta+1}2\eta|\eta_t|\, dx\, dt = \int_{Q_R} V^{\beta}\mu^2 2\eta\,|\eta_t|\, dx\, dt + \int_{Q_R} V^{\beta}\,\langle Du, Du\rangle\, 2\eta\,|\eta_t|\, dx\, dt \\
&= \int_{Q_R} V^{\beta}\mu^2 2\eta\,|\eta_t|\, dx\, dt - \int_{Q_R} u\,\mathrm{div}\left(V^{\beta} Du\,\eta\,|\eta_t|\right)\, dx\, dt \\
&\le \int_{Q_R} V^{\beta}\mu^2 2\eta\,|\eta_t|\, dx\, dt + c\|u\|_{L^\infty(Q_R)}\int_{Q_R} V^{\beta}|D^2u|\eta\,|\eta_t|\, dx\, dt \\
&+ c\|u\|_{L^\infty(Q_R)}\int_{Q_R} V^{\beta+\frac{1}{2}} D(\eta\,|\eta_t|)\, dx\, dt.
\end{aligned}$$

From Young inequality we deduce, arguing as in (4.11),

$$\begin{aligned}
\int_{Q_R} V^{\beta+1}2\eta|\eta_t|\, dx\, dt &\le c\int_{Q_R} V^{\beta+\frac{p}{2}}\left(|\eta_t| + \frac{c}{\nu\varepsilon_3}|\eta_t|^2 + |D(\eta\,|\eta_t|)|\right)\, dx\, dt \\
&+ c\varepsilon_3\nu(\beta+1)^2\int_{Q_R} V^{\beta+\frac{p-2}{2}}|D^2u|^2\eta^2\, dx\, dt + c\frac{|Q_R|}{(R-\rho)^4}.
\end{aligned}$$

Then, (4.10) becomes

$$\sup_{t\in(t_0-R^2,t_0)}\int_{B_R}V^{\beta+1}\eta^2dx+(2\nu(\beta+1)(1-(\beta+1)\tilde{\varepsilon}))\int_{Q_R}V^{\frac{p-2}{2}+\beta}|D^2u|^2\eta^2dx\,dt$$
$$+\nu\beta(\beta+1)(1-(\beta+1)\beta\tilde{\varepsilon})\int_{Q_R}V^{\frac{p-2}{2}+\beta-1}\left|D(|Du|^2)\right|^2\eta^2\,dx\,dt \tag{4.12}$$
$$\leq\left(\frac{c}{\nu\tilde{\varepsilon}}\right)\int_{Q_R}V^{\beta+\frac{p}{2}}\left(|D\eta|^2+|\eta_t|^2\right)dx\,dt$$
$$+2c(\beta+1)\int_{Q_R}V^{\frac{p}{2}+\beta}\left(2\eta|D^2\eta|+2|D\eta|^2\right)\,dx\,dt$$
$$+c\int_{Q_R}V^{\beta+\frac{p}{2}}\left(|\eta_t|+|D(\eta\,|\eta_t|)|\right)\,dx\,dt+c\frac{|Q_R|}{(R-\rho)^4}.$$

Recalling that

$$|D(|Du|^2)|^2=|2\langle D_i(D_su);D_su\rangle|^2=2\sum_{i=1}^n\sum_{s=1}^n(D_i(D_su))^2(D_su)^2$$
$$\leq C(n)|D^2u|^2|Du|^2.$$

We can write, for sufficiently small ε,

$$\sup_{t\in(t_0-R^2,t_0)}\int_{B_R}V^{\beta+1}\eta^2\,dx+\int_{Q_R}V^{\frac{p-2}{2}+\beta-1}\left|D(|Du|^2)\right|^2\eta^2\,dx\,dt \tag{4.13}$$
$$\leq C(\nu)(\beta+1)^2\int_{Q_R}V^{\beta+\frac{p}{2}}\left(|D\eta|^2+|\eta_t|^2\right)dx\,dt$$
$$+C(\nu)(\beta+1)\int_{Q_R}V^{\frac{p}{2}+\beta}\left(2\eta|D^2\eta|+2|D\eta|^2+|\eta_t|+|D(\eta\,|\eta_t|)|\right)dx\,dt$$
$$+c\frac{|Q_R|}{(R-\rho)^4}.$$

Let $\gamma=\frac{p}{4}+\frac{\beta}{2}$, since

$$|D(V^\gamma\eta)|^2=\left|\gamma V^{\gamma-1}D\left(|Du|^2\right)\eta+V^\gamma D\eta\right|^2$$
$$\leq 2\gamma^2V^{2\gamma-2}\left|D\left(|Du|^2\right)\right|^2\eta^2+2V^{2\gamma}\left|D\eta\right|^2,$$

we can write, using the previous inequalities,

$$\int_{Q_R}|D(V^\gamma\eta)|^2\,dx\,dt$$
$$\leq 2\gamma^2\int_{Q_R}V^{2\gamma-2}\left|D\left(|Du|^2\right)\right|^2\eta^2\,dx\,dt+\int_{Q_R}V^{2\gamma}\left|D\eta\right|^2\,dx\,dt$$

$$\leq C(\nu)\gamma^2(\beta+1)^2\int_{Q_R} V^{\beta+\frac{p}{2}}\left(|D\eta|^2+|\eta_t|^2\right)dx\,dt \tag{4.14}$$
$$+C(\nu)\gamma^2(\beta+1)\int_{Q_R} V^{\frac{p}{2}+\beta}\left(2\eta|D^2\eta|+2|D\eta|^2+|\eta_t|+|D(\eta\,|\eta_t|)|\right)\,dx\,dt$$
$$+c\gamma^2\frac{|Q_R|}{(R-\rho)^4}+\int_{Q_R} V^{2\gamma}\,|D\eta|^2\,dx\,dt.$$

Then,

$$\sup_{t\in(t_0-R^2,t_0)}\left(\int_{B_R}|V|^{\beta+1}\eta^2\,dx\right)+\int_{Q_R}|D(V^\gamma\eta)|^2\,dx\,dt \tag{4.15}$$
$$\leq C(\beta+1)^4\int_{Q_R} V^{\frac{p}{2}+\beta}\left(|D\eta|^2+|D^2\eta|+|\eta_t|+|D(\eta|\eta_t|)|\right)\,dx\,dt$$
$$+C(\beta+1)^2\frac{|Q_R|}{(R-\rho)^4}\,dx\,dt.$$

Applying the Sobolev-Poincare estimate, we deduce that

$$\int_{t_0-R^2}^{t_0}\int_{B_R}|V|^{2\frac{\beta+1}{n}+2\gamma}\eta^{2\left(\frac{2}{n}+1\right)}\,dx\,dt \tag{4.16}$$
$$\leq\; c\int_{t_0-R^2}^{t_0}\left(\int_{B_R}|V|^{\beta+1}\eta^2\,dx\right)^{\frac{2}{n}}\left(\int_{B_R}|V|^{2\gamma\frac{n}{n-2}}\eta^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\;dt$$
$$\leq\; c\Big[\sup_{t\in(t_0-R^2,t_0)}\left(\int_{B_R}|V|^{\beta+1}\eta^2\,dx\right)^{\frac{2}{n}}\Big]\int_{t_0-R^2}^{t_0}\int_{B_R}|D(V^\gamma\eta)|^2\,dx\,dt$$
$$\leq\; c\Big[\sup_{t\in(t_0-R^2,t_0)}\left(\int_{B_R}|V|^{\beta+1}\eta^2\,dx\right)+\int_{Q_R}|D(V^\gamma\eta)|^2\,dx\,dt\Big]^{1+\frac{2}{n}}.$$

Recalling the properties of η, combining (4.15) and (4.16) we can write

$$\int_{Q_\rho}|V|^{2\frac{\beta+1}{n}+2\gamma}\,dx\,dt \tag{4.17}$$
$$\leq\Big[C(\beta+1)^4\frac{1}{(R-\rho)^4}\int_{Q_R} V^{\frac{p}{2}+\beta}\,dx\,dt+c(\beta+1)^2\frac{|Q_R|}{(R-\rho)^4}\Big]^{1+\frac{2}{n}}.$$

Choosing, for some $\beta_0>0$, $R_i=\frac{R_0}{2}(1+2^{-i})$, $\alpha=1+\frac{2}{n}$, $\beta_i=\alpha^i\beta_0+(\alpha^i-1)=\beta_{i-1}+\frac{2}{n}(\beta_{i-1}+1)$, $\psi_i=\fint_{Q_{R_i}}|V|^{\beta_i+\frac{p}{2}}$, we can write the previous inequality as

$$\psi_{i+1}\leq c^{i+1}(\beta_i+1)^{4\alpha}\psi_i^\alpha+c^{i+1}(\beta_i+1)^{2\alpha}.$$

Iterating, we get

$$\psi_{i+1} \le c^{i+1}(\beta_i+1)^{4\alpha}\Big(c^i(\beta_{i-1}+1)^{4\alpha}\psi_{i-1}^{\alpha}+c^i(\beta_{i-1}+1)^{2\alpha}\Big)^{\alpha}+c^{i+1}(\beta_i+1)^{2\alpha}$$

$$\le c^{i+1}c^{i\alpha}(\beta_i+1)^{4\alpha}(\beta_{i-1}+1)^{4\alpha^2}2^{\alpha-1}\psi_{i-1}^{\alpha^2}$$
$$+ c^{i+1}c^{i\alpha}(\beta_i+1)^{4\alpha}(\beta_{i-1}+1)^{2\alpha}2^{\alpha-1}+c^{i+1}(\beta_i+1)^{2\alpha}$$

$$\le c^{i+1}c^{i\alpha}(\beta_i+1)^{4\alpha}(\beta_{i-1}+1)^{4\alpha^2}2^{\alpha-1}\Big(c^{i-1}(\beta_{i-2}+1)^{4\alpha}\psi_{i-2}^{\alpha}$$
$$+c^{i-1}(\beta_{i-2}+1)^{2\alpha}\Big)^{\alpha^2}+c^{i+1}c^{i\alpha}(\beta_i+1)^{4\alpha}(\beta_{i-1}+1)^{2\alpha}2^{\alpha-1}+c^{i+1}(\beta_i+1)^{2\alpha}$$

$$\le c^{i+1}c^{i\alpha}c^{(i-1)\alpha^2}(\beta_i+1)^{4\alpha}(\beta_{i-1}+1)^{4\alpha^2}(\beta_{i-2}+1)^{4\alpha^3}2^{\alpha-1}2^{\alpha^2-1}\psi_{i-2}^{\alpha^3}$$
$$+c^{i+1}c^{i\alpha}c^{(i-1)\alpha^2}(\beta_i+1)^{4\alpha}(\beta_{i-1}+1)^{4\alpha^2}(\beta_{i-2}+1)^{2\alpha^3}2^{\alpha-1}2^{\alpha^2-1}$$
$$+c^{i+1}c^{i\alpha}(\beta_i+1)^{4\alpha}(\beta_{i-1}+1)^{2\alpha}2^{\alpha-1}+c^{i+1}(\beta_i+1)^{2\alpha}$$

$$\le c^{\sum_{j=0}^{i}(i+1-j)\alpha^j}\prod_{j=0}^{i}(\beta_{i-j}+1)^{4\alpha^{j+1}}k^{\alpha^{i+1}}\psi_0^{\alpha^{i+1}}$$
$$+ic^{\sum_{j=0}^{i}(i+1-j)\alpha^j}\prod_{j=0}^{i}(\beta_{i-j}+1)^{4\alpha^{j+1}}k^{\alpha^{i+1}}$$

$$\le C^{\alpha^{i+2}}\prod_{j=0}^{i}(\beta_{i-j}+1)^{4\alpha^{j+1}}\psi_0^{\alpha^{i+1}}+iC^{\alpha^{i+2}}\prod_{j=0}^{i}(\beta_{i-j}+1)^{4\alpha^{j+1}}.$$

We have used the relations

$$\prod_{j=1}^{i}2^{\alpha^j-1}=\prod_{j=1}^{i}\frac{1}{2}2^{\alpha^j}=\frac{1}{2^i}2^{\sum_{j=1}^{i}\alpha^i}=\frac{1}{2^i}2^{\frac{\alpha}{\alpha-1}(\alpha^i-1)}\le k^{\alpha^{i+1}},$$

$$\sum_{j=0}^{i}(i+1-j)\alpha^j=\sum_{j=0}^{i}\sum_{h=0}^{j}\alpha^h=\sum_{j=0}^{i}\frac{\alpha^{i+1-j}-1}{\alpha-1}$$
$$=\Big(\frac{1}{\alpha-1}\Big)\Big(\sum_{j=0}^{i}\alpha^{j+1}-(i+1)\Big)=\Big(\frac{\alpha^{i+2}-1}{(\alpha-1)^2}\Big)-\frac{i-2}{\alpha-1}.$$

Finally, we obtain for the norm, the estimate

$$\psi_{i+1}^{\frac{1}{\beta_{i+1}+\frac{p}{2}}}\le C^{\frac{\alpha^{i+2}}{\beta_{i+1}+\frac{p}{2}}}\prod_{j=0}^{i}(\beta_{i-j}+1)^{\frac{4\alpha^{j+1}}{\beta_{i+1}+\frac{p}{2}}}\psi_0^{\frac{\alpha^{i+1}}{\beta_{i+1}+\frac{p}{2}}}$$

$$+\, i^{\frac{1}{\beta_{i+1}+\frac{p}{2}}} C^{\frac{\alpha^{i+2}}{\beta_{i+1}+\frac{p}{2}}} \prod_{j=0}^{i} (\beta_{i-j}+1)^{\frac{4\alpha^{j+1}}{\beta_{i+1}+\frac{p}{2}}}.$$

Taking the limit for $i \to \infty$ we get

$$\lim \frac{\alpha^{i+2}}{\beta_{i+1}+\frac{p}{2}} = \lim \frac{\alpha^{i+2}}{\alpha^{i+1}(\beta_0+1)-1+\frac{p}{2}} = \frac{\alpha}{\beta_0+1},$$

$$\log \prod_{j=0}^{i} (\beta_{i-j}+1)^{\frac{4\alpha^{j+1}}{\beta_{i+1}+\frac{p}{2}}} = \sum_{j=0}^{i} \frac{4\alpha^{j+1}}{\beta_{i+1}+\frac{p}{2}} \log(\beta_{i-j}+1)$$
$$= \sum_{j=0}^{i} \frac{4\alpha^{j+1}}{\alpha^{i+1}(\beta_0+1)+\frac{p}{2}-1} \log\left(\alpha^{i-j}(\beta_0+1)\right)$$
$$= \sum_{j=0}^{i} \frac{4\alpha^{j+1}}{\alpha^{i+1}(\beta_0+1)+\frac{p}{2}-1} \log(\beta_0+1) + \sum_{j=0}^{i} \frac{4\alpha^{j+1}(i-j)}{\alpha^{i+1}(\beta_0+1)+\frac{p}{2}-1} \log(\alpha)$$
$$\leq \frac{4\log(\beta_0+1)}{\beta_0+1} \sum_{j=0}^{i} \frac{\alpha^{j+1}}{\alpha^{i+1}} + \frac{4\log\alpha}{\beta_0+1} \sum_{j=0}^{i} \frac{\alpha^{j+1}(i-j)}{\alpha^{i+1}}.$$

Finally, observing that

$$\sum_{j=0}^{i} \frac{\alpha^{j+1}}{\alpha^{i+1}} \leq \alpha \sum_{i=0}^{\infty} \frac{1}{\alpha^{i}} \leq \infty$$

$$\sum_{j=0}^{i} \frac{\alpha^{j+1}(i-j)}{\alpha^{i+1}} \leq \alpha \sum_{i=1}^{\infty} \frac{i-1}{\alpha^{i}} \leq \infty,$$

and

$$\lim_{i\to\infty} i^{\frac{1}{\beta_{i+1}+\frac{p}{2}}} = \lim_{i\to\infty} i^{\frac{1}{\alpha^{i+1}(\beta_0+1)+\frac{p}{2}-1}} = 1,$$

we infer

$$\sup_{Q_{R_0/2}} |V| \leq C\Big(\fint_{Q_{R_0}} |V|^{\beta_0+\frac{p}{2}}\Big)^{\frac{\alpha}{\beta_0+1}} + C. \tag{4.18}$$

Now, we observe that (4.17) holds also for $\beta = 0$, indeed, testing the equation against the function $\phi = D_s(\eta^2 D_s u)$ (which correspond to the choice $\beta = 0$ in the definition of ψ given at the beginning of the proof) and proceeding exactly as we did before to obtain (4.17), we easily obtain

$$\int_{Q_\rho} |V|^{\frac{2}{n}+\frac{p}{2}}\, dx\, dt \leq \Big[C\frac{1}{(R-\rho)^4} \int_{Q_R} V^{\frac{p}{2}}\, dx\, dt + c\frac{|Q_R|}{(R-\rho)^4}\Big]^{1+\frac{2}{n}}.$$

Choosing $\beta_0 = \frac{2}{n}$, an averaging of the last estimate with (4.18) implies the final estimate:

$$\sup_{Q_{R_0/4}} |V| \le C\Big(\fint_{Q_{R_0}} |V|^{\frac{p}{2}}\Big)^{\alpha} + C.$$

□

4.2. **The vectorial case 1: "radial" systems.** In this section, we extend the previous a priori estimate to the vectorial case. In particular, we cannot expect global regularity of solutions to general non-linear parabolic systems. We consider here a class of vector fields for which we are able to prove the desired result. The vector fields under considerations arise naturally when we consider, for instance, the Euler equations related to radial energy functionals.

Let us consider a system of the form

$$u_t^{\alpha} - \big(A_{\alpha}^i(Du)\big)_{x_i} = 0, \tag{4.19}$$

for $\alpha \in \{1, \dots, N\}$, $i \in \{1, \dots, n\}$, $A_{\alpha}^i : \mathbb{R}^{nN} \to \mathbb{R}$.

Lemma 3. *Let $u^i \in V^{2,p}(\Omega_T)$ be a solution of (4.19). Suppose that $A_{\alpha}^i \in C^1(\mathbb{R}^{nN})$ verifies*

$$\langle D_z A(z)\lambda, \lambda\rangle = D_{z_{\alpha}^i} A_{\beta}^j(z)\lambda_{\alpha}^i \lambda_{\beta}^j \ge \nu(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall\ \lambda, z \in \mathbb{R}^{nN}, \tag{4.20}$$

$$|D_z A(z)| \le M\left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} \qquad \forall\ z \in \mathbb{R}^{nN}, \tag{4.21}$$

$$|A(z)| \le L(\mu^2 + |z|^2)^{\frac{p-1}{2}} \qquad \forall\ z \in \mathbb{R}^{nN}, \tag{4.22}$$

where $\nu, M, L > 0$. Let us assume that A can be written as $A(z) = g(|z|^2)z$ with $g \in C^2(\mathbb{R})$. Then, $Du \in L^{\infty}$. Moreover, the following a priori estimate is verified with C independent of μ and M

$$\sup_{Q_{R_0/2}} (\mu^2 + |Du|^2) \le C\Big(\fint_{Q_{R_0}} (\mu^2 + |Du|^2)^{\frac{p}{2}}\Big)^{1+\frac{2}{n}} + C.$$

Proof. Since the proof follows that of Lemma 2, here we just give the details of the steps that involve some differences. Using the function $\phi = D_s(\eta^2\psi)$ as test function in (4.19) with $\psi : (0,T) \times \Omega \to \mathbb{R}^N$, $s \in \{1, \dots, N\}$ and η is a smooth parabolic cut-off function, we get

$$\int_{Q_R} u_t^{\alpha} D_s(\eta^2\psi^{\alpha})\,dx\,dt + \int_{Q_R} A_{\alpha}^i(Du) D_s(2\eta D_i\eta\,\psi^{\alpha} + \eta^2 D_i\psi^{\alpha})\,dx\,dt = 0.$$

Integrating by parts we get

$$\int_{Q_R} A^i_\alpha(Du) D_s(\eta^2 D_i\psi^\alpha)\, dx\, dt = -\int_{Q_R} D_s[A^i_\alpha(Du)]\eta^2 D_i\psi^\alpha\, dx\, dt \tag{4.23}$$
$$= -\int_{Q_R} D_{z^k_j} A^i_\alpha(Du) D_j(D_s u^k)\eta^2 D_i\psi^\alpha\, dx\, dt.$$

Setting $\psi = (\mu^2 + |Du|^2)^\beta D_s u$ and using the notation $V = \mu^2 + |Du|^2$ we can write, again integrating by parts,

$$\int_{Q_R} u^\alpha_t D_s(\eta^2\psi^\alpha)\, dx\, dt = -\int_{Q_R} (D_s u^\alpha)_t \eta^2 V^\beta D_s u^\alpha\, dx\, dt,$$

that, summing up on $s \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,N\}$, gives us

$$\sum_{\alpha=1}^N \sum_{s=1}^n \int_{Q_R} u^\alpha_t D_s(\eta^2\psi^\alpha)\, dx\, dt = -\sum_{\alpha=1}^N - \int_{Q_R} \eta^2 V^\beta \left\langle Du^\alpha; (Du^\alpha)_t\right\rangle_n\, dx\, dt$$
$$= -\int_{Q_R} \eta^2 V^\beta \left\langle Du; (Du)_t\right\rangle_{nN}\, dx\, dt.$$

Recalling (4.4) we can write

$$\sum_{\alpha=1}^N \sum_{s=1}^n \int_{Q_R} u^\alpha_t D_s(\eta^2\psi^\alpha)\, dx\, dt \tag{4.24}$$
$$= -\frac{1}{2\beta+2}\int_{Q_R} \frac{d}{dt}\left(V^{\beta+1}\eta^2\right)\, dx\, dt + \frac{1}{2\beta+2}\int_{Q_R} 2\eta\eta_t V^{\beta+1}\, dx\, dt.$$

From (4.24) and (4.23) we deduce

$$\frac{1}{2\beta+2}\int_{Q_R} \frac{d}{dt}\left(V^{\beta+1}\eta^2\right)\, dx\, dt$$
$$+ \sum_{\alpha=1}^N \sum_{s=1}^n \underbrace{\int_{Q_R} D_{z^k_j} A^i_\alpha(Du) D_j(D_s u^k)\eta^2 D_i(V^\beta D_s u^\alpha)\, dx\, dt}_{\mathcal{A}_{\alpha,s}}$$
$$= \frac{1}{2\beta+2}\int_{Q_R} 2\eta\eta_t V^{\beta+1}\, dx\, dt$$
$$+ \sum_{\alpha=1}^N \sum_{s=1}^n \underbrace{\int_{Q_R} A^i_\alpha(Du) D_s(2\eta D_i\eta\, V^\beta D_s u^\alpha)\, dx\, dt}_{\mathcal{B}_{\alpha,s}}.$$

We observe that the terms denoted by $\mathcal{B}_{\alpha,s}$ can be estimated exactly in the same way we estimated the $\mathcal{B}_s$ in the proof of Lemma 2, using the growth conditions on A, to obtain

$$\sum_{\alpha=1}^{N}\sum_{s=1}^{n}\mathcal{B}_{\alpha,s} \le C\int_{Q_R} V^{\frac{p}{2}+\beta}\left(\eta\left|D^2\eta\right| + |D\eta|^2\right)\,dx\,dt \tag{4.25}$$
$$+ C\int_{Q_R} V^{\frac{p-1}{2}+\beta}\left|D^2 u\right|\eta|D\eta|\,dx\,dt$$
$$+ C\int_{Q_R} V^{\frac{p}{2}+\beta-1} D\left(|Du|^2\right) 2\eta|D\eta|\,dx\,dt.$$

So we only need to comment on $\mathcal{A}_{\alpha,s}$. Again, as in the previous proof, we derive the product $V^\beta D_s u^\alpha$, and we estimate separately the two terms

$$\mathcal{A}^1_{\alpha,s} = \int_{Q_R} D_{z^k_j} A^i_\alpha(Du) D_j(D_s u^k)\eta^2 \beta V^{\beta-1} D_i(|Du|^2) D_s u^\alpha\,dx\,dt,$$
$$\mathcal{A}^2_{\alpha,s} = \int_{Q_R} D_{z^k_j} A^i_\alpha(Du) D_j(D_s u^k)\eta^2 V^{\beta} D_i(D_s u^\alpha)\,dx\,dt.$$

Using the ellipticity condition (4.20), we easily obtain

$$\begin{aligned}\sum_{\alpha=1}^{N}\sum_{s=1}^{n}\mathcal{A}^2_{\alpha,s} &= \sum_{s=1}^{n}\sum_{\alpha=1}^{N}\int_{Q_R}\left\langle D_z A^i_\alpha(Du); D(D_s u)\right\rangle D_i(D_s u^\alpha)\eta^2 V^\beta\,dx\,dt\\ &= \sum_{s=1}^{n}\sum_{\alpha=1}^{N}\int_{Q_R}\left(D_z A(Du) D(D_s u)\right)^i_\alpha D(D_s u)^i_\alpha \eta^2 V^\beta\,dx\,dt\\ &= \sum_{s=1}^{n}\int_{Q_R}\left\langle D_z A(Du) D(D_s u); D(D_s u)\right\rangle \eta^2 V^\beta\,dx\,dt\\ &\ge C\nu\int_{Q_R} V^{\frac{p-2}{2}+\beta}|D^2 u|^2\eta^2\,dx\,dt.\end{aligned} \tag{4.26}$$

In order to estimate $\mathcal{A}^1_{\alpha,s}$, we use the structure condition $A(z) = g(|z|^2)z$. Using the following expression

$$D_{z^k_j} A^i_\alpha(z) = D_{z^k_j}\left(g\left(|z|^2\right) z^\alpha_i\right) = 2g'\left(|z|^2\right) z^k_j z^\alpha_i + g\left(|z|^2\right)\delta^{k\alpha}\delta_{ij},$$

and observing that the hypothesis (4.20) implies that:

$$2g'\left(|z|^2\right) z^k_j z^\alpha_i \lambda_k\lambda_\alpha + g\left(|z|^2\right)|\lambda|^2 \ge \nu\left(\mu^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \quad \forall\,\lambda\in\mathbb{R}^N,$$

we can write

$$
\begin{aligned}
&D_{z_j^k} A^i_\alpha(Du) D_j \left(D_s u^k\right) \eta^2 D_i \left(|Du|^2\right) D_s u^\alpha \\
&= 2g'\left(|Du|^2\right) D_j u^k D_i u^\alpha D_j (D_s u^k) D_i \left(|Du|^2\right) D_s u^\alpha \\
&+ g\left(|Du|^2\right) D_i (D_s u^k) D_i \left(|Du|^2\right) D_s u^k \\
&= 2g'\left(|Du|^2\right) D_j u^k D_i u^\alpha D_j \left(|Du|^2\right) D_i \left(|Du|^2\right) \\
&+ g\left(|Du|^2\right) \left|D\left(|Du|^2\right)\right|^2 \geq \nu \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{2}} \left|D\left(|Du|^2\right)\right|^2 .
\end{aligned}
\tag{4.27}
$$

From (4.27), we deduce the desired estimate

$$
\sum_{\alpha=1}^{N} \sum_{s=1}^{n} \mathcal{A}^1_{\alpha,s} \geq \int_{Q_R} \nu\beta V^{\frac{p-2}{2}+\beta-1} \left|D\left(|Du|^2\right)\right|^2 \, dx\, dt. \tag{4.28}
$$

Once we have the estimates (4.25), (4.26) and (4.28), the result follows with an iteration argument adapting the method used in the scalar case. □

4.3. **The vectorial case 2: General systems.** In this subsection, we establish an a priori estimate for solutions of systems (4.19) without the structure condition. Under these hypotheses, we cannot expect the lipschitz regularity. Anyway, we obtain a higher integrability result.

We will make use of some different quotient arguments, the definitions and the basic properties are recalled (see for example [13]).

Let $f(x)$ be a function defined in an open set $\Omega \subset \mathbb{R}^n$, and let $h \in \mathbb{R}$. We call *different quotient* of f with respect to x_s the function

$$
\Delta_{h,s} f(x) = \frac{f(x + he_s) - f(x)}{h},
$$

where $e_s \in \mathbb{R}^n$ is the unit vector in the x_s direction. The function $\Delta_{h,s} f$ is defined in the set

$$
\Omega_{|h|} := \{x \in \Omega \, : \, \operatorname{dist}(x, \partial\Omega) < |h|\} .
$$

The following properties, that we will use throughout the proofs, are easily verified:

(i) If $f \in W^{1,p}(\Omega)$, then, $\Delta_{h,s} f \in W^{1,p}(\Omega_{|h|})$ and we have $D_i(\Delta_{h,s} f) = \Delta_{h,s}(D_i f)$.

(ii) If f or g has compact support in $\Omega_{|h|}$, then,

$$
\int_\Omega f \Delta_{h,s} g \, dx = - \int_\Omega g \Delta_{-h,s} f \, dx.
$$

(iii) We have $\Delta_{h,s}(fg)(x) = f(x + he_s)\Delta_{h,s} g(x) + g(x)\Delta_{h,s} f(x)$.

Lemma 4. *Let $u^i \in V^{2,p}(\Omega_T)$ be a solution of (4.19). Suppose that $A^i_\alpha \in C^1(\mathbb{R}^{nN})$ verifies*

$$\langle D_z A(z)\lambda, \lambda\rangle = D_{z^i_\alpha} A^j_\beta(z)\lambda^i_\alpha \lambda^j_\beta \geq \nu(\mu^2+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \quad \forall\ \lambda, z \in \mathbb{R}^{nN}, \tag{4.29}$$

$$|D_z A(z)| \leq M\left(\mu^2+|z|^2\right)^{\frac{p-2}{2}} \qquad \forall\ z \in \mathbb{R}^{nN}, \tag{4.30}$$

$$|A(z)| \leq L(\mu^2+|z|^2)^{\frac{p-1}{2}} \qquad \forall\ z \in \mathbb{R}^{nN}, \tag{4.31}$$

$$\langle A(z), z\rangle \geq \nu(\mu^2+|z|^2)^{\frac{p-1}{2}}|z| - c_2 \qquad \forall\ z \in \mathbb{R}^{nN}, \tag{4.32}$$

where $\nu, M, L > 0$. Then, $Du \in L^s_{loc}(\Omega_T)$, where $s = p + \frac{4}{n}$ if $n > 2$ and $s \in [1,\infty)$ is any arbitrary number if $n = 2$. Moreover, $D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du] \in L^2_{loc}(\Omega_T)$ and for each concentric parabolic cylinders $Q_\rho \subset Q_R$, the following a priori estimates are verified with c independent of μ and M

$$\int_{Q_\rho} \left(\mu^2+|Du|^2\right)^{\frac{s}{2}}\, dz \leq \left[c\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\, dz + c\right]^{1+\frac{2}{n}},$$

$$\int_{Q_\rho}\left|D\Big[\left(\mu^2+|Du|^2\right)^{\frac{p-2}{4}} Du\Big]\right|^2 dz \leq c\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\, dz + c.$$

Proof. In order to overcome the lack of apriori regularity of the solution, and at the same time to have the right shape of the test function for our purposes (see the test function used in the scalar case), we choose the test function ϕ involving different quotients and an auxiliary truncation function. We consider the function

$$\phi(t,x) = \Delta_{-h,s}\left[\eta^2(t,x)D_s u^\alpha(t,x)\psi\left(|Du(t,x)|+|Du(t,x+he_s)|\right)\right],$$

where η is a standard parabolic cut-off function and $\psi \in C^\infty((0,\infty))$ is a truncation function such that $0 \leq \psi \leq 1$, $\psi(t) = 1$ if $t \leq \frac{K}{2}$, $\psi(t) = 0$, if $t \geq K$ and $|\psi'| \leq \frac{C}{K}$.

First we observe that

$$\int_{Q_R} u^\alpha_t \phi\, dz = -\int_{Q_R}\Delta_{h,s}(u^\alpha_t)\eta^2(t,x)D_s u^\alpha(t,x)\psi(|Du(t,x)|+|Du(t,x+he_s)|).$$

Taking the limit for $h \to 0$, we obtain

$$\lim_{h\to 0}\int_{Q_R} u^\alpha_t\phi\, dz = -\int_{Q_R} D_s\left(u^\alpha_t\right)\eta^2(t,x)D_s u^\alpha(t,x)\psi\left(2|Du(t,x)|\right)\, dz. \tag{4.33}$$

Summing up on α and s in the right hand side of (4.33), we obtain

$$-\int_{Q_R}\eta^2(t,x)\psi\left(2|Du(t,x)|\right)\langle Du;(Du)_t\rangle_{nN}\, dz \tag{4.34}$$

$$=-\int_{Q_R}\frac{d}{dt}\Big(\eta^2(t,x)g\Big(\frac{|Du(t,x)|^2}{2}\Big)\Big)dz+\int_{Q_R}2\eta(t,x)\eta_t(t,x)g\Big(\frac{|Du(t,x)|^2}{2}\Big)dz$$

where we have set

$$g(t)=\int_0^t\psi(2\sqrt{2\tau})\,d\tau+\mu^2.$$

Moreover, we observe that

$$\lim_{K\to\infty}g(t)=\mu^2+t.$$

Now, we consider the term

$$\begin{aligned}\int_{Q_R}&A^i_\alpha(Du)D_i\phi\,dz=\int_{Q_R}A^i_\alpha(Du)D_i\left(\Delta_{-h,s}\left[\eta^2(t,x)D_su^\alpha(t,x)\right.\right.\\&\qquad\left.\left.\times\psi\left(|Du(t,x)|+|Du(t,x+he_s)|\right)\right]\right)dz\\&=-\int_{Q_R}\Delta_{h,s}\left(A^i_\alpha(Du)\right)D_i\left(\eta^2(t,x)D_su^\alpha(t,x)\right.\\&\qquad\left.\times\psi\left(|Du(t,x)|+|Du(t,x+he_s)|\right)\right)dz\\&=-\int_{Q_R}\Delta_{h,s}\left(A^i_\alpha(Du)\right)\eta^2(t,x)D_{is}u^\alpha(t,x)\\&\qquad\times\psi\left(|Du(t,x)|+|Du(t,x+he_s)|\right)dz\\&-\int_{Q_R}\Delta_{h,s}\left(A^i_\alpha(Du)\right)\eta^2(t,x)D_su^\alpha(t,x)\psi'\left(|Du(t,x)|+|Du(t,x+he_s)|\right)\\&\qquad\times D_i\left(|Du(t,x)|+|Du(t,x+he_s)|\right)dz\\&+2\int_{Q_R}A^i_\alpha(Du)\Delta_{-h,s}\left[\eta D_i\eta(t,x)D_su^\alpha(t,x)\right.\\&\qquad\left.\times\psi\left(|Du(t,x)|+|Du(t,x+he_s)|\right)\right]dz.\end{aligned}$$

Taking the limit for $h\to 0$ in the previous formula, we get

$$\begin{aligned}\lim_{h\to 0}&\int_{Q_R}A^i_\alpha(Du)D_i\phi\,dz\qquad(4.35)\\&=-\int_{Q_R}D_{z_j^\beta}A^i_\alpha(Du)D_{js}u^\beta D_{is}u^\alpha(t,x)\eta^2\psi\left(2|Du(t,x)|\right)dz\\&-\int_{Q_R}D_{z_j^\beta}A^i_\alpha(Du)D_{js}u^\beta D_su^\alpha(t,x)\eta^2\psi'\left(2|Du(t,x)|\right)2D_i\left(|Du(t,x)|\right)dz\\&+2\int_{Q_R}A^i_\alpha(Du)D_s\left[\eta D_i\eta D_su^\alpha(t,x)\right]\psi\left(2|Du(t,x)|\right)dz\end{aligned}$$

$$+ 2\int_{Q_R} A^i_\alpha(Du)\eta D_i\eta D_s u^\alpha(t,x)\psi'\left(2|Du(t,x)|\right) 2D_s\left(|Du|\right)\, dz.$$

Taking ϕ as test function in (4.19), summing up to α and s, taking the limit as $h \to 0$, using (4.33), (4.34), (4.35), and the growth conditions on A, we end up with

$$\begin{aligned}
&\int_{Q_R} \frac{d}{dt}\Big(\eta^2 g\Big(\frac{|Du|^2}{2}\Big)\Big) dz + \int_{Q_R} \left(\mu^2+|Du|^2\right)^{\frac{p-2}{2}} |D^2u|^2\eta^2\psi\left(2|Du|\right)\, dz\\
&\leq M\int_{Q_R} \left(\mu^2+|Du|^2\right)^{\frac{p-2}{2}} |D^2u|^2|Du|\eta^2\psi'\left(2|Du|\right)\, dz\\
&+ 2L\int_{Q_R} \left(\mu^2+|Du|^2\right)^{\frac{p-1}{2}} \eta|D\eta||Du||D^2u|\psi'\left(2|Du|\right)\, dz\\
&+ 2L\int_{Q_R} \left(\mu^2+|Du|^2\right)^{\frac{p-1}{2}} \eta|D\eta||D^2u|\psi\left(2|Du|\right)\, dz\\
&+ 2L\int_{Q_R} \left(\mu^2+|Du|^2\right)^{\frac{p-1}{2}} \left(|D\eta|^2+|D^2\eta|\right)|Du|\psi\left(2|Du|\right) dz\\
&+\int_{Q_R} 2\eta\eta_t g\Big(\frac{|Du|^2}{2}\Big) dz.
\end{aligned}$$

Using Young's inequality and absorbing on the left hand side, we obtain

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)}\int_{B_R}\eta^2 g\Big(\frac{|Du|^2}{2}\Big)dx + \int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p-2}{2}} |D^2u|^2\eta^2\psi\left(2|Du|\right) dz\\
&\leq C(M+L)\int_{Q_R}(\mu^2+|Du|^2)^{\frac{p-2}{2}}\Big(|D^2u|^2\eta^2\\
&\qquad\qquad + \left(\mu^2+|Du|^2\right)|D\eta|^2\Big)|Du|\psi'\left(2|Du|\right)\, dz\\
&+ C\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\left(|D\eta|^2+|D^2\eta|\right)\psi\left(2|Du(t,x)|\right)\, dz\\
&+\int_{Q_R} 2\eta\eta_t g\left(\frac{|Du|^2}{2}\right)\, dz.
\end{aligned}$$

Now, we use the fact that $|\psi'|\leq \mathbf{1}_{[K/2,K]}$ in order to get rid of the dependence on M in the previous estimate. We can write

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)}\int_{B_R}\eta^2 g\Big(\frac{|Du|^2}{2}\Big)dx + \int_{Q_R}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\eta^2\psi(2|Du|)dz\\
&\leq C(M+L)\int_{Q_R\cup\{K/4\leq|Du|\leq K/2\}}\left(\mu^2+|Du|^2\right)^{\frac{p-2}{2}}\Big(|D^2u|^2\eta^2
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \left(\mu^2+|Du|^2\right)|D\eta|^2\Big)|Du|\,dz\\
&+C\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\left(|D\eta|^2+|D^2\eta|\right)\psi\left(2|Du(t,x)|\right)\,dz\\
&+\int_{Q_R}2\eta|\eta_t|g\left(\frac{|Du|^2}{2}\right)\,dz.
\end{aligned}
$$

Letting $K\to+\infty$, and since

$$
\begin{aligned}
&\lim_{K\to\infty}\int_{Q_R}2\eta|\eta_t|g\Big(\frac{|Du|^2}{2}\Big)dz\le c\int_{Q_R}(\mu^2+|Du|^2)\eta|\eta_t|\,dz\\
&\le c\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}|\eta_t|+\frac{c'}{(R-\rho)^2},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)}\int_{B_R}\eta^2\left(\mu^2+|Du|^2\right)\,dx+\int_{Q_R}\left|D\Big[\eta\left(\mu^2+|Du|^2\right)^{\frac{p-2}{4}}Du\Big]\right|^2dz\\
&+\int_{Q_R}\left|D\left[\eta\left(\mu^2+|Du|^2\right)^{\frac{p}{4}}\right]\right|^2dz\\
&\le c\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\left(|D\eta|^2+|D^2\eta|+|\eta_t|\right)\,dz+\frac{c'}{(R-\rho)^2}.
\end{aligned}
\tag{4.36}
$$

Using the notation $V:=\mu^2+|Du|^2$ and applying the Sobolev-Poincare estimate, we deduce that

$$
\begin{aligned}
&\int_{t_0-R^2}^{t_0}\int_{B_R}|V|^{\frac{2}{n}+\frac{p}{2}}\eta^{2\left(\frac{2}{n}+1\right)}\,dx\,dt \qquad (4.37)\\
&\le\ c\int_{t_0-R^2}^{t_0}\Big(\int_{B_R}|V|\eta^2\,dx\Big)^{\frac{2}{n}}\Big(\int_{B_R}|V|^{\frac{p}{2}\frac{n}{n-2}}\eta^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}\,dt\\
&\le c\Big[\sup_{t\in(t_0-R^2,t_0)}\Big(\int_{B_R}|V|\eta^2\,dx\Big)^{\frac{2}{n}}\Big]\int_{t_0-R^2}^{t_0}\int_{B_R}|D(V^{\frac{p}{4}}\eta)|^2\,dx\,dt\\
&\le c\Big[\sup_{t\in(t_0-R^2,t_0)}\Big(\int_{B_R}|V|\eta^2\,dx\Big)+\int_{Q_R}|D(V^{\frac{p}{4}}\eta)|^2\,dx\,dt\Big]^{1+\frac{2}{n}}.
\end{aligned}
$$

From (4.36), (4.37) we infer

$$
\int_{Q_\rho}|V|^{\frac{2}{n}+\frac{p}{2}}\,dz\le\Big[\frac{c}{(R-\rho)^2}\int_{Q_R}V^{\frac{p}{2}}\,dz+\frac{c'}{(R-\rho)^2}\Big]^{1+\frac{2}{n}}.
\tag{4.38}
$$

□

5. Proofs of the main results

The proofs of the regularity results are based on the approximation procedure via regular problems associated to the vector fields A_ε defined in Lemma 1.

Let $u_\varepsilon \in V^{2,p}(\Omega_T)$ be a solution of the following first boundary problem

$$\begin{cases}(u^\varepsilon)_t - \operatorname{div}(A_\varepsilon(Du_\varepsilon)) = 0 & \text{in } Q_R\\ u^\varepsilon = u & \text{on } \partial_p Q_R.\end{cases} \tag{5.1}$$

For the solvability of (5.1) we refer to [14] Chapter 5 n.6. For the computational simplicity we assume u_ε and u have t derivative with sufficient integrability, this allows us to write the computation formally. To proceed in a rigorous way, one should use a smoothing argument in time, namely approximate the solutions via Steklov averages and use the Definition 2. As already noticed this is a standard argument that yields only technical minor changes.

By taking $u^\varepsilon - u$ as test function in (5.1) and (2.1), we get

$$\begin{aligned}\int_{Q_R}(u^\varepsilon-u)_t(u^\varepsilon-u)\,dz + \int_{Q_R} A_\varepsilon(Du^\varepsilon)(Du^\varepsilon - Du)dz \\ - \int_{Q_R} A(Du)(Du^\varepsilon - Du)dz = 0.\end{aligned} \tag{5.2}$$

Using the growth conditions on A_ε and A, since

$$\int_{Q_R}(u^\varepsilon-u)_t(u^\varepsilon-u)\,dz \geq 0,$$

we obtain

$$\begin{aligned}
\int_{Q_R}|Du^\varepsilon|^p dz &\leq \int_{Q_R}(u^\varepsilon-u)_t(u^\varepsilon-u)dz + c\int_{Q_R}(A_\varepsilon(Du^\varepsilon)-A_\varepsilon(0))(Du^\varepsilon)dz\\
&\leq c\int_{Q_R}(|A(Du)||Du| + |A_\varepsilon(Du^\varepsilon)||Du| + (|A(Du)|+|A_\varepsilon(0)|)|Du^\varepsilon|+)\,dz\\
&\leq c\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p-1}{2}}|Du|\,dz\\
&\quad + c\int_{Q_R}(|A_\varepsilon(Du_\varepsilon)||Du| + (|A(Du)|+|A_\varepsilon(0)|)|Du^\varepsilon|)\,dz + c\\
&\leq c\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}dz + c\int_{Q_R}\left(\mu^2+\varepsilon^2+|Du^\varepsilon|^2\right)^{\frac{p-1}{2}}|Du|\,dz
\end{aligned}$$

$$+ c\int_{Q_R} \left(\mu^2 + |Du|^2\right)^{\frac{p-1}{2}} |Du^\varepsilon| \, dz + c. \tag{5.3}$$

From (5.3) using Young inequality we infer the uniform bound for L^p-norm of $\{Du_\varepsilon\}$:

$$\int_{Q_R} |\mu^2 + \varepsilon^2 + Du^\varepsilon|^p \, dz \le c\int_{Q_R} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} dz + c. \tag{5.4}$$

Now, let us observe that we can rewrite (5.2) as follows

$$\begin{aligned} \int_{Q_R} (u^\varepsilon - u)_t (u^\varepsilon - u) dz + \int_{Q_R} (A(Du^\varepsilon) - A(Du))(Du^\varepsilon - Du) dz \\ = -\int_{Q_R} (A_\varepsilon(Du^\varepsilon) - A(Du^\varepsilon))(Du^\varepsilon - Du) dz. \end{aligned} \tag{5.5}$$

The proofs of the main theorems go as follows:

Proof of Theorem 1. By the uniform convergence of A_ε to A on compact sets, ensured by Lemma 1, we have immediately from (5.4)

$$\int_{Q_R} (A_\varepsilon(Du^\varepsilon) - A(Du^\varepsilon))(Du^\varepsilon - Du) \, dz \to 0. \tag{5.6}$$

Combining (5.6) with (5.5) and using the monotonicity condition on A, we see that Du^ε converges a.e. to Du (up to a subsequence). Indeed

$$\lim_{\varepsilon \to 0} \nu \int_{Q_R} \left(\mu^2 + |Du^\varepsilon|^2 + |Du|^2\right)^{\frac{p-2}{2}} |Du^\varepsilon - Du| = 0. \tag{5.7}$$

Since A_ε verify the hypotheses of Lemma 2 we can write, from (5.4)

$$\sup_{Q_{R_0/2}} (\mu^2 + \varepsilon^2 + |Du^\varepsilon|^2) \le C\Big(⨍_{Q_{R_0}} (\mu^2 + |Du|^2)^{\frac{p}{2}} \, dz\Big)^\beta + C. \tag{5.8}$$

By virtue of (5.7) we can pass to the limit in (5.8) to get the desired result

$$\sup_{Q_{R_0/2}} (\mu^2 + |Du|^2) \le C\Big(⨍_{Q_{R_0}} (\mu^2 + |Du|^2)^{\frac{p}{2}} \, dz\Big)^\beta + C.$$

□

Proof of Theorem 2. We observe that we can construct A_ε satisfying the structure condition (H3) (see [11]). As in the previous proof A_ε verify the hypotheses of Lemma 3 and we can proceed exactly as before. □

Proof of Theorem 3. Applying (5.3) and since A_ε verify the hypotheses of Lemma 4 we can write

$$\int_{Q_R} |Du^\varepsilon|^p\,dz + \int_{Q_\rho} \left(\mu^2+|Du^\varepsilon|^2\right)^{\frac{s}{2}}\,dz + \int_{Q_\rho} \left|D\left[\left(\mu^2+|Du^\varepsilon|^2\right)^{\frac{p}{4}}\right]\right|^2 dz$$
$$\le c\int_{Q_R} \left(\mu^2+|Du|^2\right)^{\frac{p}{2}}dz + c.$$

From the previous estimates we deduce that, up to subsequences, $(\mu^2+|Du^\varepsilon|^2)^{\frac{p}{4}} \rightharpoonup f$ weakly in $W^{1,2}(Q_\rho)$ and, using an Aubin-Lions type argument, that $Du^\varepsilon \rightharpoonup g$ weakly in $L^s(Q_\rho)$. Therefore, for any $q<s$, we have

$$Du^\varepsilon \to g \quad a.e.\ x\in Q_\rho \text{ and in } L^q(Q_\rho).$$

Since A_ε is uniform convergent on compact sets, using the uniform continuity we get

$$A_\varepsilon(Du^\varepsilon)\to A(g) \quad a.e. \text{ in } Q_\rho.$$

Moreover, Du^ε in equi-p-integrable, then using the growth condition, from the Lebesgue convergence theorem, we infer that

$$A_\varepsilon(Du^\varepsilon)\to A(g) \ \text{ in } L^{p'}(Q_\rho).$$

Thus,

$$\int_{Q_R} (A_\varepsilon(Du^\varepsilon)-A(Du^\varepsilon))(Du^\varepsilon-Du)\,dz \to 0.$$

As in the previous proofs, from (5.5) we deduce that Du^ε converges a.e. to Du (up to a subsequence).

From Lemma 4 we can write

$$\int_{Q_\rho} \left(\mu^2+\varepsilon+|Du^\varepsilon|^2\right)^{\frac{s}{2}}\,dz \le \left[c\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\,dz + c\right]^{1+\frac{2}{n}},$$

and

$$\int_{Q_\rho} \left|D\left[\left(\mu^2+\varepsilon+|Du^\varepsilon|^2\right)^{\frac{p}{4}}\right]\right|^2 dz \le c\int_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\,dz + c.$$

The result follows passing to the limit in the previous inequalities. □

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