ASYMPTOTIC BEHAVIOR OF STRONG SOLUTIONS FOR NONLINEAR PARABOLIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENT

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Abstract. In this paper, we discuss the asymptotic behavior of solutions of nonlinear parabolic equations in $\mathbf{R}^N$ involving critical Sobolev exponent. For the semilinear and subcritical problem, it is well-known that the solution which intersects with the "stable set" must be a global one and the solution which enters the "unstable set" should blow up in finite time. But in the critical case, it is not clear that the same result holds or not. In this paper, we show that the same result holds also in the critical case. The proof of our main result requires the method different from that for the subcritical problem and is based on the direct analysis of L^∞-norm of solutions with the aid of the blow up argument and the concentration compactness type argument.

1. Introduction and main results

Let $N \geq 2$ and let Ω be a smooth domain in $\mathbf{R}^N$. In this paper, we are concerned with the asymptotic behavior of solutions of the following nonlinear parabolic equation:

$$\text{(P)} \quad \begin{cases} \partial u/\partial t = \Delta_p u + u|u|^{q-2} & \text{in} \quad \Omega \times (0, T_m), & \text{(E1)} \\ u = 0 & \text{on} \quad \partial\Omega \times (0, T_m), & \text{(E2)} \\ u|_{t=0} = u_0 & \text{in} \quad \Omega, & \text{(E3)} \end{cases}$$

where $p \in (2N/(N+2), N)$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $q \leq p^*$ ($p^* := Np/(N-p)$ denotes the critical exponent of the Sobolev embedding $D_0^{1,p} := \overline{C_0^\infty}^{\|\nabla \cdot\|_p} \hookrightarrow L^q$), $u_0 \in D_0^{1,p}(\Omega) \cap L^2(\Omega) \cap L^\infty(\Omega) =: X$ and T_m denotes the maximal existence time of the solution which satisfies (1.3)-(1.5) below. In

Accepted for publication: February 2008.
AMS Subject Classifications: 35K65, 35B35, 35B33.
[1]This research was partially supported by the Grant-in-Aid for Young Scientists (B) #19740081, The Ministry of Education, Culture, Sports, Science and Technology, Japan

the main part of this paper, we assume that $\Omega = \mathbf{R}^N$ and $q = p^*$. In this case, we regard (E2) as the condition at spatial infinity.

For $\omega \subset \mathbf{R}^N$, we denote the standard $L^r(\omega)$-norm by $\|\cdot\|_{r,\omega}$ and we occasionally omit the subscript ω.

The following functionals J, I defined on X and subsets W, V of X play an important role throughout this paper:

$$\begin{aligned} J(u) &:= \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|u\|_q^q, \\ I(u) &:= -\|\nabla u\|_p^p + \|u\|_q^q, \\ W &:= \Big\{u \in X;\ J(u) < \Big(\frac{1}{p} - \frac{1}{q}\Big)S^{q/(q-p)},\ I(u) < 0\Big\}, \qquad (1.1) \\ V &:= \Big\{u \in X;\ J(u) < \Big(\frac{1}{p} - \frac{1}{q}\Big)S^{q/(q-p)},\ I(u) > 0\Big\}, \qquad (1.2) \end{aligned}$$

where S denotes the best constant of the Sobolev inequality defined by $S := \inf_{u\in D_0^{1,p}\setminus\{0\}} \|\nabla u\|_p^p/\|u\|_q^p$.

In this paper, we consider the solution of (P) which satisfies

$$\Delta_p u,\ u|u|^{q-2} \in L^2(0,T;L^2), \tag{1.3}$$

$$u \in W^{1,2}(0,T;L^2) \cap C([0,T];D_0^{1,p}) \cap L^\infty(0,T;L^\infty) \tag{1.4}$$

for all $T \in (0, T_m)$. We also assume the following blow up alternative:

$$T_m < \infty \text{ implies } \lim_{t\to T_m} \|u(t)\|_\infty = \infty. \tag{1.5}$$

We can construct the solution of (P) which satisfies (1.3)-(1.5) by several methods (see e.g. [19, Theorem 2.1]). This solution is Hölder continuous (see [6, Chapter III, Theorem 1.1]); thus, T_m coincides with the maximal existence time of the *classical* solution if $p = 2$.

The purpose of this paper is to shed some new light on the analysis of the asymptotic behavior of solutions which intersect with W or V in the critical case.

There are large amounts of works concerning the asymptotic behavior of solutions of (P) (see e.g. [4], [7], [10], [12], [13], [15], [20], [25] and references therein). Among them, we recall the following result concerning the semilinear and subcritical problem.

Proposition 1.1. [12, Theorem 3.1 and Theorem 3.2] *Let $p = 2$, $q < 2^*$ and let Ω be a bounded domain in $\mathbf{R}^N$ ($N \geq 3$).*

(a) *Assume that $u(\bar{t}) \in W$ for some $\bar{t}$. Then $T_m = \infty$ and $\|\nabla u(t)\|_2 \to 0$ as $t \to \infty$.*

(b) *Assume that* $u(\bar{t}) \in V$ *for some* $\bar{t}$. *Then* $T_m < \infty$ *and* $\|\nabla u(t)\|_2 \to \infty$ *as* $t \to T_m$.

Owing to these facts, W (respectively, V) is said to be a stable (respectively, an unstable) set.

Now we briefly discuss how to obtain $T_m = \infty$ in Proposition 1.1 (a) (the argument given below is somewhat different from the original one given in [12] but is similar in spirit). The key fact is the following.

Lemma 1.1. ([26], [27], see also [3, Theorem 1], [21, Theorem 3]) *Let* $p = 2$, $q > N(q-2)/2$ (*which is equivalent to* $q < 2^*$), $q \geq 1$, $u_0 \in L^q$ *and let* Ω *be* $\mathbf{R}^N$ *and a bounded domain in* $\mathbf{R}^N$. *Then for any* $M > 0$, *there exists* $\delta = \delta(M) > 0$ *such that* L^q*-solution of* (P) *on* $[0, \delta]$ *exists for all* $u_0 \in L^q$ *satisfying* $\|u_0\|_q \leq M$.

We recall that u is said to be an L^q-solution of (P) on $[0, T]$ if u satisfies the integral equation associated with (P) and $u \in C([0,T]; L^q)$.

By using Lemma 1.1, we can prove that the existence of $\bar{t} \in (0, T_m)$ satisfying $u(\bar{t}) \in W$ yields $T_m = \infty$. In fact, it is easy to see that once the orbit enters the stable set W, then it remains in W (see Proposition 2.3 (b)). Therefore, by the straightforward calculation using the definition of W, we obtain $\sup_{t \in [\bar{t}, T_m)} \|u(t)\|_q^q < S^{q/(q-p)}$. This fact together with Lemma 1.1 implies that one can extend $u(t)$ as long as one wish; i.e., $T_m = \infty$.

The argument above is based on the fact that the local existence time δ in Lemma 1.1 only depends on L^q-norm of the initial data. This fact is no longer true in the critical problem due to the scale invariance of (P) and $L^q (= L^{2^*})$ norm (observe that Lemma 1.1 does not cover the case $q = 2^*$). Hence, it is not clear that the same result as in Proposition 1.1 (a) holds or not in the critical case.

Indeed, few results are known even for the semilinear problem, see e.g. [8], [9], [13]. Among them, we quote the following.

Proposition 1.2. [13, Theorem 1 and Theorem 2] *Let* $p = 2$, $q = 2^*$ *and let* Ω *be a bounded domain in* $\mathbf{R}^N$ $(N \geq 3)$.

(a) *Assume that* $u(\bar{t}) \in W$ *for some* $\bar{t}$ *and* $T_m = \infty$. *Then* $\|\nabla u(t)\|_2 \to 0$ *as* $t \to \infty$.

(b) *Assume that* $u(\bar{t}) \in V$ *for some* $\bar{t}$. *Then* $T_m < \infty$ *and* $\|u(t)\|_\infty \to \infty$ *as* $t \to T_m$.

There exists a difference between the statement of Proposition 1.1 (a) and Proposition 1.2 (a) concerning the position of the claim "$T_m = \infty$". Indeed,

"$T_m = \infty$" is the conclusion of Proposition 1.1 (a) but is the assumption of Proposition 1.2 (a). Our first result fills this gap for general p-Laplacian case.

Theorem 1.1. *Let $p \in (2N/(N+2), N)$, $q = p^*$ and let $\Omega = \mathbf{R}^N$ in* (P). *Let u be a solution of* (P) *with $u(\bar{t}) \in W$ for some $\bar{t} < T_m$. Then*

$$T_m = \infty. \tag{1.6}$$

Moreover,

$$u(t) \in W \cup \{0\}, \quad \forall t \in [\bar{t}, \infty). \tag{1.7}$$

$$u(t) \to 0 \text{ in } L^r, \quad \forall r \in (2, \infty], \tag{1.8}$$

$$u(t) \to 0 \text{ in } D_0^{1,p}, \quad J(u(t)) \to 0 \quad \text{as } t \to \infty.$$

Remark 1.1. The same result holds for general Ω under some appropriate assumptions on Ω (e.g. the smoothness of Ω and the boundedness of the mean curvature of $\partial\Omega$, see e.g. [7] for related facts). In particular, if Ω is a smooth bounded domain in $\mathbf{R}^N$ and $p = 2$, then the same conclusion holds. Nevertheless, in this case, the argument concerning the behavior of the solution near the boundary $\partial\Omega$ as in [10] is needed in addition to the argument given in this paper. The detailed argument is left for the interested reader.

As for the solution which intersects with V, we have the following.

Theorem 1.2. *Let $p \in (2N/(N+2), N)$, $q = p^*$ and let $\Omega = \mathbf{R}^N$ in* (P). *Also let u be a solution of* (P) *with $u(\bar{t}) \in V$ for some $\bar{t} < T_m$. Then*

$$T_m < \infty, \tag{1.9}$$

$$u(t) \in V, \quad \forall t \in [\bar{t}, T_m), \tag{1.10}$$

$$\|u(t)\|_\infty \to \infty \quad \text{as} \quad t \to T_m. \tag{1.11}$$

Remark 1.2. The standard proof as in [25], [12] or [13] does not work for Theorem 1.2, since their arguments essentially use the boundedness of Ω.

Two remarks concerning Theorem 1.1 and Theorem 1.2 are in order.

Remark 1.3. For $\Omega = \mathbf{R}^N$, the potential well type result as Theorem 1.1 or Theorem 1.2 does not hold for the subcritical case, since S (the best constant of the Sobolev inequality) is well-defined only for $q = p^*$. See [15] for results concerning the semilinear and subcritical case with $\Omega = \mathbf{R}^N$.

Remark 1.4. In [23] and [24], some facts related to the potential well type results are claimed for a class of *weak* solutions (in an appropriate sense) of semilinear parabolic problems involving critical Sobolev exponent with bounded domains. Their method is different from ours. In particular, we give the behavior of L^∞-norm of solutions in Theorem 1.1 which cannot be obtained by their methods.

2. Preliminaries

In this section, we collect technical facts for the later use.

Basic properties of the solution of (P). We recall basic properties of the solution of (P) satisfying (1.3)–(1.5).

Multiplying (E1) by $\partial u(t)/\partial t$ and integrating it over Ω, we obtain the following. This procedure is justified by virtue of (1.3) and (1.4).

Proposition 2.1. *Let u be a solution of* (E1)–(E2). *Then*

$$\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^2(a,b;L^2)} = -J(u(b)) + J(u(a))$$

for any $0 \le a < b < T_m$, where

$$\|f\|_{L^2(a,b;L^2)} := \Big(\int_a^b \|f(t)\|_2^2 dt\Big)^{1/2}.$$

Particularly, $J(u(t))$ is a nonincreasing function in t.

It is easy to see that the comparison argument works for the solution of (P) which satisfies (1.3) and (1.4), i.e., if $u_0(x) \le v_0(x)$, then $u(x,t) \le v(x,t)$, where u (respectively, v) is a solution of (P) with initial data u_0 (respectively, v_0). Then, applying the same argument as in [18, Lemma 2.4], we can verify the following.

Proposition 2.2. *Let u be a solution of* (P) *with $T_m = \infty$. Then there exists $c \ge 0$ such that $\lim_{t\to\infty} J(u(t)) = c$.*

The following corollary follows from Proposition 2.1 and Proposition 2.2.

Corollary 2.1. *Let u be a solution of* (P) *satisfying $J(u(\bar{t})) < 0$ for some $\bar{t}$. Then $T_m < \infty$.*

Let W (respectively, V) be the set defined by (1.1) (respectively, (1.2)). The following facts are easily verified by the argument as in [25] (see also [12, Proposition 2.6] and [13, Lemma 3.5]).

Proposition 2.3. (a) *The set V does not intersect with $W \cup \{0\}$.*

(b) *Assume that* $u(\bar{t}) \in W$ *(respectively,* $u(\bar{t}) \in V$*) for some* $\bar{t}$*. Then* $u(t) \in W \cup \{0\}$ *(respectively,* $u(t) \in V$*) for any* $t \in [\bar{t}, T_m)$*.*

Scaling properties. We check the invariance property of (E1) and J under the scaling with respect to x, t and u.

Let u be a solution of (P),

$$\alpha := \frac{p(N+2)-2N}{p}, \qquad \beta := \frac{N-p}{p} \tag{2.1}$$

and let $\lambda > 0$. For any $x_0 \in \mathbf{R}^N$ and $t_0 \in \mathbf{R}^+$, let us define y, s, v by

$$y := \lambda(x-x_0), \quad s := \lambda^\alpha(t-t_0), \quad \lambda^\beta v(y,s) := u(x,t). \tag{2.2}$$

Since all the statements below can be obtained by direct calculations, we omit the proof.

Proposition 2.4. *Let* $\delta > 0$*. Then* v *satisfies*

$$\frac{\partial v}{\partial s} = \Delta_p v + v|v|^{p^*-2} \ \textit{in } \mathbf{R}^N \times [0,\delta]$$

if and only if u *satisfies*

$$\frac{\partial u}{\partial t} = \Delta_p u + u|u|^{p^*-2} \ \textit{in } \mathbf{R}^N \times \Big[t_0, t_0 + \frac{\delta}{\lambda^\alpha}\Big].$$

Moreover,

$$\begin{aligned}
&\Big\|\frac{\partial v}{\partial s}\Big\|_{L^2(0,\delta;L^2(\mathbf{R}^N))} = \Big\|\frac{\partial u}{\partial t}\Big\|_{L^2(t_0,t_0+\delta/\lambda^\alpha;L^2(\mathbf{R}^N))},\\
&\|\nabla v(s)\|_p = \|\nabla u(t)\|_p,\\
&\|v(s)\|_r = \|u(t)\|_r \quad \text{if and only if } r = p^*,\\
&\|v(s)\|_2 = \lambda^{\alpha/2}\|u(t)\|_2.
\end{aligned}$$

Remark 2.1. In the semilinear and subcritical case, the L^∞-bound of global nonnegative solutions of (P) can be obtained by the blow up argument, see e.g. [10]. We cannot use this argument for the verification of Theorem 1.1. Indeed, the blow up argument employed in [10] needs the nonexistence result of (nonnegative) stationary solutions. But for $q = p^*$, the stationary solution of (E1)–(E2) (belonging to $D_0^{1,p}$) exists by virtue of the scale invariance of the energy functional which follows from Proposition 2.4. For a detailed argument, see [16], [17] (see also [22, Chapter I.4] and [11, Theorem 2.2]). Consequently, we cannot apply the argument given in [10] to the critical problem.

Using Proposition 2.4, we have the following.

Proposition 2.5. *Let $p \in (2N/(N+2), N)$, $\Omega = \mathbf{R}^N$ and $q = p^*$ in* (P). *Assume that u is a solution of* (P) *with $T_m = \infty$. Then there exists t_n such that $t_n \to \infty$ and*

$$\|\nabla u(t_n)\|_p^p = \|u(t_n)\|_{p^*}^{p^*} + o(1)$$

as $n \to \infty$.

Proof. Let $\tau_n \to \infty$ be a sequence such that

$$\lim_{n\to\infty} \|u(\tau_n)\|_2 = \limsup_{t\to\infty} \|u(t)\|_2 (\leq \infty).$$

Let α and β be numbers given in (2.1). The assumption $p > 2N/(N+2)$ implies $\alpha > 0$. We define λ_n by

$$\lambda_n^\alpha := \frac{1}{\|u(\tau_n)\|_2^2} \tag{2.3}$$

and define y, s, u_n by $y := \lambda_n x$, $s := \lambda_n^\alpha (t - \tau_n)$, $\lambda_n^\beta u_n(y,s) := u(x,t)$. Observe that by Proposition 2.4 and (2.3),

$$\|u_n(0)\|_2^2 = \lambda_n^\alpha \|u(\tau_n)\|_2^2 = 1. \tag{2.4}$$

By the assumption $T_m = \infty$ and by Proposition 2.2, we can deduce the existence of $c \geq 0$ with $\lim_{t\to\infty} J(u(t)) = c$. Then by Proposition 2.1 and by Proposition 2.4,

$$\begin{aligned} \left\|\frac{\partial u_n}{\partial s}\right\|_{L^2(0,\delta;L^2)}^2 &= -J(u_n(\delta)) + J(u_n(0))) \\ &= -J(u(\tau_n + \delta/\lambda_n^\alpha)) + J(u(\tau_n)) \\ &\to -c + c = 0 \end{aligned} \tag{2.5}$$

for any $\delta > 0$, thus

$$\|u_n(\sigma) - u_n(0)\|_2 \leq \int_0^\sigma \left\|\frac{\partial u_n(s)}{\partial s}\right\|_2 ds \leq \sqrt{\delta} \left\|\frac{\partial u_n}{\partial s}\right\|_{L^2(0,\delta;L^2)} \to 0$$

as $n \to \infty$ uniformly in $\sigma \in [0,\delta]$. This relation together with (2.4) yields

$$\|u_n(\sigma)\|_2^2 \leq 2\|u_n(0)\|_2^2 = 2, \quad \forall \sigma \in [0,\delta] \tag{2.6}$$

for sufficiently large n. Again by (2.5), we can find $\eta \in [0,\delta]$ such that

$$\left\|\frac{\partial u_n(\eta)}{\partial s}\right\|_2 \to 0 \text{ as } n \to \infty, \tag{2.7}$$

passing to a subsequence if necessary.

By Proposition 2.4, u_n satisfies (E1)–(E2). The relation obtained from multiplying (E1) by u_n and integrating it over $\mathbf{R}^N$ together with (2.6) and (2.7) yields

$$\begin{aligned}\Big| -\|\nabla u_n(\eta)\|_p^p + \|u_n(\eta)\|_{p^*}^{p^*}\Big| &\leq \Big|\int \frac{\partial u_n(\eta)}{\partial s} u_n(\eta)\Big| \\ &\leq \|u_n(\eta)\|_2 \Big\|\frac{\partial u_n(\eta)}{\partial s}\Big\|_2 \to 0 \qquad (2.8)\end{aligned}$$

as $n \to \infty$. Let $t_n := \tau_n + \eta/\lambda_n^\alpha$. Then (2.8) and Proposition 2.4 give

$$\|\nabla u(t_n)\|_p^p = \|\nabla u_n(\eta)\|_p^p = \|u_n(\eta)\|_{p^*}^{p^*} + o(1) = \|u(t_n)\|_{p^*}^{p^*} + o(1),$$

which implies the conclusion. □

Compactness devices. In order to complete the blow up argument, we have to characterize the limit of a sequence formed by rescaled solutions of (E1)–(E2). In the semilinear case, this characterization follows from the Schauder-type argument (see e.g. [10]). In order to handle the quasilinear case, we employ the following two compactness devices.

Proposition 2.6. *Let* (u_n) *be a family of solutions of* (E1)–(E2) *in* $\mathbf{R}^N \times (0, \delta]$. *Assume that for some* $0 < \underline{c} < \overline{c}$,

$$\|u_n\|_{L^\infty(0,\delta;L^\infty(\mathbf{R}^N))} \in [\underline{c}, \overline{c}], \quad \forall n \in \mathbf{N}. \qquad (2.9)$$

Then we have

$$u_n \to u \text{ in } C_{\mathrm{loc}}(\mathbf{R}^N \times (0, \delta]) \qquad (2.10)$$

as $n \to \infty$, *passing to a subsequence if necessary.*

Proof. Let (u_n) be a family of solutions of (E1)–(E2) with (2.9). Then, for all $m \in \mathbf{N}$, we can easily show that the restriction of u_n to $B_m \times (0, \delta]$ is a bounded local solution of (E1)–(E2) in $B_m \times (0, \delta]$ in the sense of [6, pp. 17], where $B_m = \{x \in \mathbf{R}^N; |x| < m\}$.

For $m \in \mathbf{N}$, let $D_m := \{x \in \Omega; |x| \leq m/2\}$, $I_m := [1/m, \delta]$, $K_m := D_m \times I_m$ and let Γ_m be the parabolic boundary of $B_m \times (0, \delta]$. Put

$$d_p(X, Y : u_n) := \begin{cases} \|u_n\|_{L^\infty(0,\delta;L^\infty(B_m))}^{|p-2|/p} |x-y| + |t-s|^{1/p} & \text{if} \quad p \in (2N/(N+2), 2), \\ |x-y| + \|u_n\|_{L^\infty(0,\delta;L^\infty(B_m))}^{|p-2|/p} |t-s|^{1/p} & \text{if} \quad p \in [2, \infty), \end{cases}$$

where $X = (x, t)$, $Y = (y, s) \in \mathbf{R}^N \times \mathbf{R}$ and let

$$p\text{-}\operatorname{dist}(K_m, \Gamma_m; u_n) := \inf_{X \in K_m, Y \in \Gamma_m} d_p(X, Y : u_n).$$

Then by (2.9),

$$\begin{aligned} & p\text{-}\operatorname{dist}(K_m, \Gamma_m; u_n) \\ \geq\ & \min(1, \|u_n\|^{|p-2|/p}_{L^\infty(0,\delta;L^\infty(B_m))}) \inf_{X \in K_m, Y \in \Gamma_m} (|x-y| + |t-s|^{1/p}) \\ \geq\ & \min(1, \underline{c}^{|p-2|/p})(m/2 + (1/m)^{1/p}) =: C_m (> 0). \end{aligned}$$

This relation together with Theorem 1.1 in Chapter III of [6] yields

$$\begin{aligned} |u_n(X) - u_n(Y)| \ &\leq\ \gamma \|u_n\|_{L^\infty(0,\delta;L^\infty(B_m))} \Big(\frac{d_p(X,Y;u_n)}{p\text{-}\operatorname{dist}(K_m, \Gamma; u_n)} \Big)^a \\ &\leq\ \frac{\gamma \overline{c}}{C_m^a} \max(1, \overline{c}^{|p-2|/p})^a \\ &\quad\ \times (|x-y| + |t-s|^{1/p})^a, \quad \forall X, Y \in K_m \end{aligned}$$

for general constants $a \in (0,1)$ and $\gamma > 0$.

This fact implies that $(u_n) \subset C(K_m)$ forms an equicontinuous family and the conclusion follows from the Ascoli-Arzela theorem together with the standard diagonal argument. □

Lemma 2.1. *In addition to the assumption of Proposition* 2.6*, assume further that*

$$\Big\| \frac{\partial u_n}{\partial t} \Big\|_{L^2(0,\delta;L^2)} \to 0 \ \textit{as} \ \ n \to \infty. \tag{2.11}$$

Then u appears in the conclusion of Proposition 2.6 *is t-independent.*

Proof. Take any $t_1, t_2 \in (0, \delta]$ with $t_1 < t_2$ and $R > 0$. By (2.10) and (2.11),

$$\begin{aligned} & \|u(t_1) - u(t_2)\|_{2,B_R} \\ \leq\ & \|u(t_1) - u_n(t_1)\|_{2,B_R} + \Big\| \int_{t_1}^{t_2} dt \frac{\partial u_n}{\partial t} \Big\|_{2,B_R} + \|u(t_2) - u_n(t_2)\|_{2,B_R} \\ \leq\ & 2 \max_{[t_1,t_2] \times \overline{B_R}} |u_n - u| |B_R|^{1/2} + \sqrt{t_2 - t_1} \Big\| \frac{\partial u_n}{\partial t} \Big\|_{L^2(0,\delta:L^2)} \to 0 \end{aligned}$$

as $n \to \infty$, where B_R denotes the ball with radius R centered at the origin. This fact implies that $u(t_1) = u(t_2)$ in $L^2(B_R)$ for any $R > 0$ and for any t_1, $t_2 \in (0, \delta]$. □

Proposition 2.7. *Let $(u_n) \subset D_0^{1,p} \cap L^\infty$ be a bounded sequence such that*

$$f_n := \Delta_p u_n + u_n |u_n|^{p^*-2} \to 0 \ \textit{in} \ L^2 \tag{2.12}$$

as $n \to \infty$*. Then there exists* $u \in D_0^{1,p}$ *such that* $u_n \rightharpoonup u$ *weakly in* $D_0^{1,p}$ *along some subsequence and the alternative*

$$u = 0 \qquad or \qquad \|u\|_{p^*}^{p^*} \geq \|\nabla u\|_p^p \geq S^{p^*/(p^*-p)} \tag{2.13}$$

holds, where

$$S := \inf_{u \in D_0^{1,p} \setminus \{0\}} \|\nabla u\|_p^p / \|u\|_q^p.$$

Proof. Let η be a smooth function from $[0, \infty)$ to $\mathbf{R}$ satisfying $\eta(t) = 0$ for $t \geq 2$, $\eta(t) = 1$ for $t \leq 1$ and $0 \leq \eta \leq 1$ in $[0, \infty)$. For $a \in \mathbf{R}^+$, define $\varphi_a \in C_0^\infty(\mathbf{R}^N)$ by

$$\varphi_a(x) := \eta(|x|/a). \tag{2.14}$$

It is easy to show that there exists $C > 0$ independent of $a (\geq 1)$ satisfying

$$\|\nabla \varphi_a\|_N \leq C\|\eta'\|_\infty, \quad \forall a \geq 1. \tag{2.15}$$

By the assumption of the proposition, there exist C_1, $C_2 > 0$ such that

$$\|\nabla u_n\|_p < C_1, \tag{2.16}$$

$$\|u_n\|_\infty < C_2. \tag{2.17}$$

By (2.16) and [22, pp.44] (see also [1, 2, 5, 14, 16, 17]), we can find $u \in D_0^{1,p}$, a set $S \subset \mathbf{R}^N$ which is at most countable, families of nonnegative numbers $(\mu_{x_j})_{x_j \in S}$ and $(\nu_{x_j})_{x_j \in S}$, nonnegative Radon measures μ and ν such that

$$u_n \rightharpoonup u \text{ weakly in } D_0^{1,p} \text{ and in } L^{p^*} \tag{2.18}$$

$$|\nabla u_n|^p \rightharpoonup \mu \geq |\nabla u|^p + \sum_{x_j \in S} \mu_{x_j} \delta_{x_j} \text{ weakly in } M(\mathbf{R}^N) \tag{2.19}$$

$$|u_n|^{p^*} \rightharpoonup \nu = |u|^{p^*} + \sum_{x_j \in S} \nu_{x_j} \delta_{x_j} \text{ weakly in } M(\mathbf{R}^N) \tag{2.20}$$

$$\nu_{x_j} = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int |u_n|^{p^*} \varphi_{\varepsilon, x_j} \tag{2.21}$$

along an appropriate subsequence, where $M(\mathbf{R}^N)$ denotes the set consists of Radon measures on $\mathbf{R}^N$, δ_{x_j} the delta measure supported at x_j, $\varphi_{\varepsilon, x_j}(\cdot) := \varphi_\varepsilon(\cdot - x_j)$ and $\varphi_\varepsilon(\cdot)$ a cut-off function defined by (2.14) with $a = \varepsilon$.

By (2.17) and (2.21),

$$|\nu_{x_j}| \leq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left| \int |u_n|^{p^*} \varphi_{\varepsilon, x_j} \right| \leq \lim_{\varepsilon \to 0} C_2^{p^*} |B(x_j; 2\varepsilon)| = 0, \tag{2.22}$$

where $B(x_j; 2\varepsilon)$ denotes the ball with radius 2ε centered at x_j.

This relation together with (2.18) and (2.20) yields

$$u_n \to u \text{ strongly in } L^{p^*}_{\rm loc}. \tag{2.23}$$

Take any $R \geq 1$ and let φ_R be a cut-off function defined by (2.14) with $a = R$. We shall calculate

$$\int f_n u_n \varphi_R = \int \Delta_p u_n (u_n \varphi_R) + \int u_n |u_n|^{p^*-2} u_n \varphi_R.$$

The integration by parts yields

$$\int |\nabla u_n|^p \varphi_R = \int |u_n|^{p^*} \varphi_R - \int |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi_R u_n - \int f_n u_n \varphi_R. \tag{2.24}$$

Since $(p-1)/p + 1/p^* + 1/N = 1$, the Hölder inequality implies that

$$\left| \int |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi_R u_n \right| \leq \|\nabla u_n\|_p^{p-1} \|u_n\|_{p^*,A_R} \|\nabla \varphi_R\|_{N,B_{2R}}, \tag{2.25}$$

where $A_R = \{x;\, R < |x| < 2R\})$ and $B_{2R} = \{x \in \mathbf{R}^N;\, |x| < 2R\}$.

By (2.16) and (2.15), we can find $C_3 > 0$ which does not depend on R such that

$$\|\nabla u_n\|_p^{p-1} \|\nabla \varphi_R\|_{N,B_{2R}} \quad \leq \quad C_3. \tag{2.26}$$

Then (2.23)–(2.26) yield

$$\left| \int |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi_R u_n \right| \leq C_3 \|u_n\|_{p^*,A_R} \overset{n\to\infty}{\to} C_3 \|u\|_{p^*,A_R} \overset{R\to\infty}{\to} 0. \tag{2.27}$$

Moreover, (2.17) and (2.12) lead

$$\left| \int f_n u_n \varphi_R \right| \quad \leq \quad \|f_n\|_2 \|\varphi_R\|_2 \|u_n\|_\infty \overset{n\to\infty}{\to} 0, \ \forall R > 0. \tag{2.28}$$

Hence by (2.24), (2.27) and (2.28), we find

$$\lim_{R\to\infty} \lim_{n\to\infty} \int |\nabla u_n|^p \varphi_R = \lim_{R\to\infty} \lim_{n\to\infty} \int |u_n|^{p^*} \varphi_R. \tag{2.29}$$

Finally, by (2.19), (2.20) and (2.22), we see that

$$\begin{aligned} \int |\nabla u_n|^p \varphi_R \overset{n\to\infty}{\to} \mu(\varphi_R) &\geq \int |\nabla u|^p \varphi_R + \sum_{x_j \in \mathcal{S}} \mu_{x_j} \varphi_R(x_j) \\ &\geq \int |\nabla u|^p \varphi_R \overset{R\to\infty}{\to} \|\nabla u\|_p^p, \end{aligned} \tag{2.30}$$

$$\int |u_n|^{p^*}\varphi_R \overset{n\to\infty}{\to} \nu(\varphi_R) = \int |u|^{p^*}\varphi_R + \sum_{x_j\in\mathcal{S}} 0\times\varphi_R(x_j) \overset{R\to\infty}{\to} \|u\|_{p^*}^{p^*}. \tag{2.31}$$

Combining (2.29)-(2.31), we obtain

$$\|\nabla u\|_p^p \le \lim_{R\to\infty}\lim_{n\to\infty}\int |\nabla u_n|^p\varphi_R = \lim_{R\to\infty}\lim_{n\to\infty}\int |u_n|^{p^*}\varphi_R = \|u\|_{p^*}^{p^*}.$$

This relation and the Sobolev inequality yield the alternative (2.13). □

3. Proof of Theorem 1.1

Now we are in the position to give the proof of Theorem 1.1. We divide the statement of Theorem 1.1 into two parts. The first part, Proposition 3.1, corresponds to (1.6), (1.7) and the L^∞ case of (1.8) in Theorem 1.1. The latter part, Proposition 3.2, involves the rest of the statement in Theorem 1.1.

Proposition 3.1. *Assume that there exists $\bar{t}$ such that $u(\bar{t})\in W$. Then*

$$\lim_{t\to T_m}\|u(t)\|_\infty = 0. \tag{3.1}$$

In particular,

$$T_m=\infty \quad\text{and}\quad u(t)\in W\cup\{0\} \text{ for all } t\in[\bar{t},\infty). \tag{3.2}$$

Proof. Proposition 2.3 (b) together with the assumption of Proposition 3.1 implies that $u(t)\in W\cup\{0\}$ for all $t\in[\bar{t},T_m)$.

Hereafter, we assume that

$$u(t)\in W \text{ for all } t\in[\bar{t},T_m), \tag{3.3}$$

since the other case is easy to deal with.

This fact together with the definition of W yields

$$J(u(t)) = \frac{1}{p}\|\nabla u(t)\|_p^p - \frac{1}{p^*}\|u(t)\|_{p^*}^{p^*} \ge \left(\frac{1}{p}-\frac{1}{p^*}\right)\|u(t)\|_{p^*}^{p^*} > 0 \tag{3.4}$$

for all $t\in[\bar{t},T_m)$. From Proposition 2.1 and (3.4), we can find $c\ge 0$ such that

$$\lim_{t\to T_m} J(u(t)) = c. \tag{3.5}$$

Now suppose that (3.1) does not hold, i.e., suppose that there exists $\eta \in (0,\infty]$ such that

$$\limsup_{t\to T_m} \|u(t)\|_\infty = \eta. \tag{3.6}$$

We shall deduce some contradiction.

Step 1. Construction of rescaled solutions. The following claim easily follows from (3.6).

Claim 1. *There exists* (t_n) *such that*

$$t_n \to T_m, \tag{3.7}$$

$$\|u(t_n)\|_\infty \to \eta \tag{3.8}$$

as $n \to \infty$ *and*

$$\|u(t_n)\|_\infty \ge \sup_{t\in[0,t_n]} \|u(t)\|_\infty/2, \quad \forall n \in \mathbf{N}. \tag{3.9}$$

Let $(x_n) \subset \mathbf{R}^N$ be a sequence which satisfies

$$\frac{1}{2}\|u(t_n)\|_\infty \le |u(x_n,t_n)|. \tag{3.10}$$

Let (λ_n), y, s, (u_n) be

$$\|u(t_n)\|_\infty = \lambda_n^\beta \tag{3.11}$$

and (see (2.1) for α and β)

$$y = \lambda_n(x-x_n),\ s = \lambda_n^\alpha(t-t_n),\ \lambda_n^\beta u_n(y,s) = u(x,t). \tag{3.12}$$

It is easy to see that (3.12) and (3.11) yield

$$\|u_n(0)\|_\infty = \frac{\|u(t_n)\|_\infty}{\lambda_n^\beta} = 1. \tag{3.13}$$

Step 2. Convergence of rescaled solutions.

Claim 2. *There exists* $\gamma > 0$*, which is independent of* n*, and a subsequence of* (u_n) *(still denoted by the same symbol) such that*

$$\left\|\frac{\partial u_n}{\partial s}\right\|_{L^2(-1,0;L^2(\mathbf{R}^N))} \to 0 \text{ as } n \to \infty \tag{3.14}$$

and

$$\|u_n\|_{L^\infty(-1,0;L^\infty(\mathbf{R}^N))} \in \left[\tfrac{1}{2}, 2\right], \tag{3.15}$$

$$|u_n(0,0)| \ge \frac{1}{2}, \tag{3.16}$$

$$\|\nabla u_n(s)\|_p^p \le S^{q/(q-p)} - \gamma, \quad \forall s \in [-1,0] \tag{3.17}$$

for all n.

Proof of Claim 2. It is easy to see that (3.10)–(3.12) yield

$$|u_n(0,0)| = \frac{|u(x_n,t_n)|}{\lambda_n^\beta} = \frac{|u(x_n,t_n)|}{\|u(t_n)\|_\infty} \ge \frac{1}{2},$$

i.e., (3.16). Moreover, (3.12), (3.11) and (3.9) imply that

$$\begin{aligned}\|u_n(s)\|_\infty &= \frac{\|u(t_n+s/\lambda_n^\alpha)\|_\infty}{\lambda_n^\beta} = \frac{\|u(t_n+s/\lambda_n^\alpha)\|_\infty}{\|u(t_n)\|_\infty} \\ &\le \frac{\sup_{\sigma\in[0,t_n]}\|u(\sigma)\|_\infty}{\|u(t_n)\|_\infty} \le \frac{2\|u(t_n)\|_\infty}{\|u(t_n)\|_\infty} = 2\end{aligned}$$

for any $s \in [-1,0]$; thus (3.15).

Now we show (3.14). By the assumption $p > 2N/(N+2)$, we have $\alpha > 0$. Assume first that $\eta = \infty$. It is easy to see that

$$t_n - \frac{1}{\lambda_n^\alpha} \uparrow T_m \tag{3.18}$$

by virtue of (3.7) and $\alpha > 0$.

If $\eta < \infty$, (1.5) gives $T_m = \infty$. Hence, by (3.8), (3.11) and (3.7),

$$t_n - \frac{1}{\lambda_n^\alpha} = t_n - \frac{1}{\eta^{\alpha/\beta}} + o(1) = T_m - \frac{1}{\eta^{\alpha/\beta}} + o(1) = \infty. \tag{3.19}$$

Then Proposition 2.1, Proposition 2.4, (3.18), (3.19) and (3.5) yield

$$\begin{aligned}\left\|\frac{\partial u_n}{\partial s}\right\|_{L^2(-1,0;L^2)} &= -J(u_n(0)) + J(u_n(-1)) \\ &= -J(u(t_n)) + J(u(t_n - 1/\lambda_n^\alpha)) \to -c + c = 0,\end{aligned}$$

i.e., (3.14).

Note that by (3.3) and by (3.18) (when $\eta = \infty$) or (3.19) (when $\eta < \infty$),

$$t_n + s/\lambda_n^\alpha > \bar{t} \quad \text{and} \quad u(t_n + s/\lambda_n^\alpha) \in W \tag{3.20}$$

hold for sufficiently large n and all $s \in [-1,0]$. The fact $u(\bar{t}) \in W$ together with (3.20), Proposition 2.1, the definition of W and Proposition 2.4 yields the existence of $\varepsilon > 0$ such that

$$\begin{aligned}&\left(\frac{1}{p} - \frac{1}{p^*}\right)S^{p^*/(p^*-p)} - \varepsilon = J(u(\bar{t})) \ge J(u(t_n + s/\lambda^\alpha)) \\ \ge\ &\left(\frac{1}{p} - \frac{1}{p^*}\right)\|\nabla u(t_n + s/\lambda^\alpha)\|_p^p = \left(\frac{1}{p} - \frac{1}{p^*}\right)\|\nabla u_n(s)\|_p^p, \ \forall s \in [-1,0],\end{aligned}$$

thus (3.17). □

By (3.14), (3.15), Proposition 2.6 and Lemma 2.1, we can find v independent of s such that

$$u_n \to v \text{ in } C_{\mathrm{loc}}(\mathbf{R}^N \times (-1,0]) \tag{3.21}$$

passing to a subsequence if necessary.

Combining (3.21) with (3.16), we see that $|v(0_y)| = |v(0_y, 0_s)| \geq 1/2$; thus

$$v \not\equiv 0. \tag{3.22}$$

Here we note that by Proposition 2.4, (3.14) and (3.15),

$$\Delta_p u_n(\sigma) + u_n(\sigma)|u_n(\sigma)|^{q-2} = \frac{\partial u_n(\sigma)}{\partial s} \to 0 \text{ in } L^2, \tag{3.23}$$

$$\|u_n(\sigma)\|_\infty \leq 2, \quad \forall n \in \mathbf{N} \tag{3.24}$$

for some $\sigma \in [-1, 0]$. From (3.17), (3.23), (3.24) and Proposition 2.7, there exists $w \in D_0^{1,p}$ such that

$$u_n(\sigma) \rightharpoonup w \text{ weakly in } D_0^{1,p}, \tag{3.25}$$

$$w = 0 \quad \text{or} \quad \|\nabla w\|_p^p \geq S^{p^*/(p^*-p)}. \tag{3.26}$$

Observe that (3.25) together with (3.21) and (3.22) gives $w = v \not\equiv 0$. Then (3.26), (3.25) and (3.17) yield

$$S^{p^*/(p^*-p)} \leq \|\nabla w\|_p^p \leq \lim_{n\to\infty} \|\nabla u_n(\sigma)\|_p^p < S^{p^*/(p^*-p)} - \gamma,$$

a contradiction.

Consequently we obtain $\lim_{t\to T_m} \|u(t)\|_\infty = 0$. This fact together with (1.5) and (3.3) implies (3.2). □

Proposition 3.2. *Assume that $u(\bar{t}) \in W$ for some $\bar{t}$. Then there exists $\eta \geq 0$ such that*

$$\|u(t)\|_2 \downarrow \eta \quad \textit{as } t \to \infty, \tag{3.27}$$

$$\lim_{t\to\infty} J(u(t)) = 0. \tag{3.28}$$

In particular,

$$\lim_{t\to\infty} \|u(t)\|_r = 0, \quad \forall r \in (2, \infty], \tag{3.29}$$

$$\lim_{t\to\infty} \|\nabla u(t)\|_p = 0. \tag{3.30}$$

Proof. Recall that we already obtain (3.1) and (3.2) (see Proposition 3.1). The definition of W together with (3.2) yields $I(u(t)) \leq 0$ for all $t \in [\bar{t}, \infty)$. Hence, by the relation obtained from multiplying (E1) by u and integrating it over $\mathbf{R}^N$, we have

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|_2^2 = I(u(t)) < 0,$$

thus (3.27).

It is easy to see that (3.29) follows from (3.27) and (3.1).

Proposition 2.2 and (3.2) imply that there exists $c \geq 0$ satisfying

$$c = \lim_{t\to\infty} J(u(t)). \tag{3.31}$$

Then by Proposition 2.5, we have

$$c = \lim_{n\to\infty} J(u(t_n)) = \lim_{n\to\infty}\left(\frac{1}{p} - \frac{1}{p^*}\right)\|u(t_n)\|_{p^*}^{p^*} \tag{3.32}$$

along some $t_n \to \infty$. Observe that (3.29) with $r = p^*$ yields

$$\lim_{t\to\infty} \|u(t)\|_{p^*}^{p^*} = 0.$$

This fact together with (3.31) and (3.32) implies (3.28). By (3.28) and (3.29) with $r = p^*$,

$$\|\nabla u(t)\|_p^p \quad = \quad pJ(u(t)) + \frac{p}{p^*}\|u(t)\|_{p^*}^{p^*} \to 0,$$

i.e. (3.30) holds. □

4. Proof of Theorem 1.2

Assume that there exists $\bar{t}$ such that $u(\bar{t}) \in V$. Observe that (1.10) follows from Proposition 2.3 (b).

Assume further that $J(u(t)) < 0$ for some $t \in [\bar{t}, T_m)$. Then Corollary 2.1 together with (1.5) yields (1.9) and (1.11).

Hence, hereafter we shall consider the case where $J(u(t)) \geq 0$ for all $t \in [\bar{t}, T_m)$. Suppose that (1.9) is false, i.e., suppose that $T_m = \infty$. Then, by Proposition 2.5, we can find $t_n \to \infty$ such that

$$\left(\frac{1}{p} - \frac{1}{p^*}\right)\|u(t_n)\|_{p^*}^{p^*} + o(1) = J(u(t_n)). \tag{4.1}$$

Note that (1.10) yields

$$u(t_n) \in V \tag{4.2}$$

for large n. Hence by the definition of V and by Proposition 2.1, we can find $\varepsilon' > 0$ such that

$$J(u(t_n)) \leq \Big(\frac{1}{p} - \frac{1}{p^*}\Big) S^{p^*/(p^*-p)} - \varepsilon' \tag{4.3}$$

for large n.

It is easy to see that (4.1) and (4.3) yield $\|u(t_n)\|_{p^*}^{p^*-p} \leq S - \varepsilon$ for some $\varepsilon > 0$. Therefore by the Sobolev inequality,

$$\begin{aligned} I(u(t_n)) &= -\|\nabla u(t_n)\|_p^p + \|u(t_n)\|_{p^*}^{p^*} \leq -S\|u(t_n)\|_{p^*}^p + \|u(t_n)\|_{p^*}^{p^*} \\ &= (-S + \|u(t_n)\|_{p^*}^{p^*-p})\|u(t_n)\|_{p^*}^p \leq -\varepsilon\|u(t_n)\|_{p^*}^p. \end{aligned}$$

This relation together with (4.3) yields $u(t_n) \in W \cup \{0\}$ and thus $u(t_n) \in (W \cup \{0\}) \cap V$ by (4.2). This is absurd in view of Proposition 2.3 (a). Therefore (1.9) follows. It is easy to see that (1.11) follows from (1.5). □

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