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VARIATIONAL METHODS FOR ORDINARY p-LAPLACIAN SYSTEMS WITH POTENTIAL BOUNDARY CONDITIONS

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Abstract. We present existence results for ordinary p-Laplacian systems of the form

$$-(|u'|^{p-2}u')' = f(t,u), \quad \text{in } [0,T], \tag{$*$}$$

submitted to the general potential boundary condition

$$((|u'|^{p-2}u')(0), -(|u'|^{p-2}u')(T)) \in \partial j(u(0), u(T)).$$

Here, $p \in (1,\infty)$ is fixed, $j : \mathbb{R}^N \times \mathbb{R}^N \to (-\infty,+\infty]$ is proper, convex and lower semicontinuous and $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory mapping. Firstly, we deal with the potential case $f(t,u) = \nabla F(t,u)$, with $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ continuously differentiable with respect to the second variable. Secondly, the system will be a nonpotential one. Afterwards, instead of $(*)$ will be the differential inclusions system

$$-(|u'|^{p-2}u')' \in \overline{\partial} F(t,u), \quad \text{in } [0,T],$$

where, this time, F is only locally Lipschitz with respect to the second variable and $\overline{\partial} F(t,x)$ stands for Clarke's generalized gradient of $F(t,\cdot)$ at $x \in \mathbb{R}^N$. Several examples of applications are given.

1. Introduction

Throughout this paper $p \in (1,\infty)$ is fixed and $h_p : \mathbb{R}^N \to \mathbb{R}^N$ denotes the homeomorphism defined by $h_p(x) = |x|^{p-2}x, \quad \forall x \in \mathbb{R}^N$, where $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^N$. The aim of this work is to present existence results for differential systems of type

$$-[h_p(u')]' = f(t,u), \quad \text{in } [0,T] \tag{1.1}$$

associated with the potential multivalued boundary condition

$$(h_p(u')(0), -h_p(u')(T)) \in \partial j(u(0), u(T)). \tag{1.2}$$

Here, $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a measurable mapping satisfying some adequate continuity and growth conditions, while $j : \mathbb{R}^N \times \mathbb{R}^N \to (-\infty,+\infty]$ is

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a convex function, proper (i.e., $D(j) := \{z \in \mathbb{R}^N \times \mathbb{R}^N \ : \ j(z) < +\infty\} \neq \emptyset$), lower semicontinuous (in short, l.s.c.) and ∂j denotes the subdifferential of j in the sense of convex analysis [37]. Recall, for $z \in \mathbb{R}^N \times \mathbb{R}^N$ the set $\partial j(z)$ is defined by

$$\partial j(z) = \{\zeta \in \mathbb{R}^N \times \mathbb{R}^N : j(\xi) - j(z) \geq (\zeta|\xi - z),\ \forall \xi \in \mathbb{R}^N \times \mathbb{R}^N\}.$$

We have denoted by $(\cdot|\cdot)$ the usual inner product in $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$; the same notation will be employed for the inner product in $\mathbb{R}^N$.

By a *solution* of the differential system (1.1) we will understand a function $u : [0, T] \to \mathbb{R}^N$ of class C^1 with $h_p(u')$ absolutely continuous, which satisfies the equality in (1.1) a.e. on $[0, T]$.

It should be noticed that the boundary condition (1.2) recovers the classical ones. For instance, denoting by I_K the indicator function of the nonempty, closed, convex set K, the Dirichlet, Neumann and periodic boundary conditions are obtained by choosing $j = I_K$ with $K = \{(0, 0)\}$, $K = \mathbb{R}^N \times \mathbb{R}^N$ and $K = \{(x, x) : x \in \mathbb{R}^N\}$, respectively. Other possible choices of j yielding various boundary conditions, some of them of special interest, will be given throughout the paper.

Existence results for various differential systems involving the ordinary vector p-Laplacian $[h_p(u')]'$ associated with classical boundary conditions have been obtained recently. See, for example, [11], [28], [29], [30], [31], [38] and the references therein. Due to the generality of the two point boundary condition (1.2), these results made natural and motivated the study of boundary value problems of type (1.1), (1.2). So, making use of a variational method, part of them focus on the case when the right hand side of (1.1) is of potential type. This means that $f(t, u) = \nabla F(t, u)$ with F continuously differentiable with respect to the second variable (see [21], [23], [25]) or, even more general when dealing with a system of differential inclusions, f has the form $f(t, u) = \overline{\partial} F(t, u)$ with F locally Lipschitz with respect to the second variable and $\overline{\partial} F(t, x)$ standing for the generalized Clarke gradient of $F(t, \cdot)$ at $x \in \mathbb{R}^N$ (see [24], [26]). Also, using combined monotonicity-compactness techniques, in [18] and [35] are obtained existence results of Hartman type for nonpotential systems of differential inclusions with vector p-Laplacian, subjected to a boundary condition similar to (1.2), but with a maximal monotone operator instead of ∂j. It is worth to point out that earlier works deal with differential equations with boundary conditions of type (1.2). In this respect, let us remark that paper [4] (also see Section 5.2 in [5]) is devoted to the study of second-order multivalued equations in Hilbert spaces, of the form

$$-[h(u')]' + Au \ni f(t)$$

with two-point boundary conditions of type (1.2), where h is the subdifferential of a convex function φ and A is a maximal monotone operator. Also, in [32] higher order scalar differential equations are considered with boundary conditions in terms of a maximal monotone mapping that is not necessarily a subdifferential. The monotonicity property of the data plays a key role in the approach of [5] and [32]. This will not be the case herein.

The rest of the work is organized as follows. Section 2, presents the functional framework and has a preliminary character. There are given basic results concerning the geometry of the Sobolev space $W^{1,p}([0,T];\mathbb{R}^N)$ and differentiability properties of some energy functionals. Section 3, is concerned with existence of solutions in the potential case. We obtain existence results in a coercive case, existence of mountain pass solutions, as well as existence of infinitely many solutions. Several examples of applications are given. This section is essentially based on the results obtained in Jebelean and Moroşanu [25] and Jebelean [22], [23]. In Section 4, we use the approach from Dincă and Jebelean [12] to provide a uniform nonresonance condition in the nonpotential case. Section 5, is devoted to extensions of the results from Section 3 to potential systems of differential inclusions of type

$$-(|u'|^{p-2}u')' \in \overline{\partial} F(t,u), \quad \text{in } [0,T],$$

where, this time, F is only locally Lipschitz with respect to the second variable. We obtain the existence of solutions in a coercive case as well as the existence of nontrivial solutions when the corresponding energy functional has a mountain pass geometry. This section relies on the results obtained in Jebelean and Moroşanu [24].

2. The functional framework

The Sobolev space $W^{1,p} := W^{1,p}([0,T];\mathbb{R}^N)$ will be considered to be endowed with the norm $\|u\|_\eta = (\|u'\|_{L^p}^p + \eta\|u\|_{L^p}^p)^{\frac{1}{p}}$, where $\eta > 0$ and $\|\cdot\|_{L^p}$ stands for the usual norm on $L^p := L^p([0,T];\mathbb{R}^N)$, i.e.,

$$\|u\|_{L^p} = \Big(\int_0^T |u|^p\Big)^{\frac{1}{p}}.$$

Theorem 2.1. *The space $(W^{1,p}, \|\cdot\|_\eta)$ is uniformly convex.*

Proof. a) The case $p \in [2,+\infty)$. Let $u, v \in W^{1,p}$ be so that $\|u\|_\eta \leq 1$, $\|v\|_\eta \leq 1$ and $\|u - v\|_\eta > \varepsilon > 0$. Using the inequality (see p. 36 in [1])

$$\Big|\frac{z+w}{2}\Big|^p + \Big|\frac{z-w}{2}\Big|^p \leq \frac{1}{2}\left(|z|^p + |w|^p\right), \quad \forall z, w \in \mathbb{R}^N,$$

we obtain

$$\begin{aligned}
&\Big\|\frac{u+v}{2}\Big\|_\eta^p+\Big\|\frac{u-v}{2}\Big\|_\eta^p\\
&=\eta\Big[\int_0^T\Big(\Big|\frac{u+v}{2}\Big|^p+\Big|\frac{u-v}{2}\Big|^p\Big)\Big]+\int_0^T\Big(\Big|\frac{u'+v'}{2}\Big|^p+\Big|\frac{u'-v'}{2}\Big|^p\Big)\\
&\le\frac{\eta}{2}\int_0^T\Big(|u|^p+|v|^p\Big)+\frac12\int_0^T\Big(|u'|^p+|v'|^p\Big)=\frac12\Big(\|u\|_\eta^p+\|v\|_\eta^p\Big)\le 1,
\end{aligned}$$

which yields

$$\Big\|\frac{u+v}{2}\Big\|_\eta^p<1-\Big(\frac{\varepsilon}{2}\Big)^p. \tag{2.1}$$

b) The case $p\in(1,2)$. We begin by noting that if $u,v\in W^{1,p}$, then $|u|^{p'}$, $|u'|^{p'}\in L^{p-1}=L^{p-1}([0,T];\mathbb{R})$, where $1/p+1/p'=1$ and, since $0<p-1<1$, the following inequalities hold true

$$\Big\|\Big|\frac{u+v}{2}\Big|^{p'}+\Big|\frac{u-v}{2}\Big|^{p'}\Big\|_{L^{p-1}}\ge\Big\|\Big|\frac{u+v}{2}\Big|^{p'}\Big\|_{L^{p-1}}+\Big\|\Big|\frac{u-v}{2}\Big|^{p'}\Big\|_{L^{p-1}}, \tag{2.2}$$

$$\Big\|\Big|\frac{u'+v'}{2}\Big|^{p'}+\Big|\frac{u'-v'}{2}\Big|^{p'}\Big\|_{L^{p-1}}\ge\Big\|\Big|\frac{u'+v'}{2}\Big|^{p'}\Big\|_{L^{p-1}}+\Big\|\Big|\frac{u'-v'}{2}\Big|^{p'}\Big\|_{L^{p-1}} \tag{2.3}$$

by virtue of Theorem 2.7 in [1]. Then, following the idea in the proof of Theorem 2 in [10], we use the Minkowski's inequality

$$\big[(a_1+b_1)^r+(a_2+b_2)^r\big]^{\frac1r}\le(a_1^r+a_2^r)^{\frac1r}+(b_1^r+b_2^r)^{\frac1r}$$

with $r=(p-1)^{-1}>1$ and

$$a_1=\eta\int_0^T\Big|\frac{u+v}{2}\Big|^p,\qquad a_2=\eta\int_0^T\Big|\frac{u-v}{2}\Big|^p,$$
$$b_1=\int_0^T\Big|\frac{u'+v'}{2}\Big|^p,\qquad b_2=\int_0^T\Big|\frac{u'-v'}{2}\Big|^p.$$

This together with (2.2) and (2.3) yield

$$\begin{aligned}
\Big(\Big\|\frac{u+v}{2}\Big\|_\eta^{p'}+\Big\|\frac{u-v}{2}\Big\|_\eta^{p'}\Big)^{p-1}&\le\eta\Big[\Big(\int_0^T\Big|\frac{u+v}{2}\Big|^p\Big)^{\frac{1}{p-1}}+\Big(\int_0^T\Big|\frac{u-v}{2}\Big|^p\Big)^{\frac{1}{p-1}}\Big]^{p-1}\\
&+\Big[\Big(\int_0^T\Big|\frac{u'+v'}{2}\Big|^p\Big)^{\frac{1}{p-1}}+\Big(\int_0^T\Big|\frac{u'-v'}{2}\Big|^p\Big)^{\frac{1}{p-1}}\Big]^{p-1}\\
&=\eta\Big(\Big\|\Big|\frac{u+v}{2}\Big|^{p'}\Big\|_{L^{p-1}}+\Big\|\Big|\frac{u-v}{2}\Big|^{p'}\Big\|_{L^{p-1}}\Big)^{p-1}\\
&+\Big(\Big\|\Big|\frac{u'+v'}{2}\Big|^{p'}\Big\|_{L^{p-1}}+\Big\|\Big|\frac{u'-v'}{2}\Big|^{p'}\Big\|_{L^{p-1}}\Big)^{p-1}
\end{aligned}$$

$$\leq \eta \Big\| \Big|\frac{u+v}{2}\Big|^{p'} + \Big|\frac{u-v}{2}\Big|^{p'} \Big\|_{L^{p-1}}^{p-1} + \Big\| \Big|\frac{u'+v'}{2}\Big|^{p'} + \Big|\frac{u'-v'}{2}\Big|^{p'} \Big\|_{L^{p-1}}^{p-1}$$
$$= \eta \int_0^T \Big(\Big|\frac{u+v}{2}\Big|^{p'} + \Big|\frac{u-v}{2}\Big|^{p'} \Big)^{p-1} + \int_0^T \Big(\Big|\frac{u'+v'}{2}\Big|^{p'} + \Big|\frac{u'-v'}{2}\Big|^{p'} \Big)^{p-1}.$$

Next, taking into account the inequality (see p. 36 in [1])

$$\Big(\Big|\frac{z+w}{2}\Big|^{p'} + \Big|\frac{z-w}{2}\Big|^{p'} \Big)^{p-1} \leq \frac{1}{2}\Big(|z|^p + |w|^p\Big), \quad \forall z, w \in \mathbb{R}^N,$$

we infer,

$$\Big(\Big\|\frac{u+v}{2}\Big\|_\eta^{p'} + \Big\|\frac{u-v}{2}\Big\|_\eta^{p'} \Big)^{p-1}$$
$$\leq \frac{\eta}{2} \int_0^T \Big(|u|^p + |v|^p\Big) + \int_0^T \Big(|u'|^p + |v'|^p\Big) = \frac{1}{2}\Big(\|u\|_\eta^p + \|v\|_\eta^p\Big).$$

Therefore, if $u, v \in W^{1,p}$ are so that $\|u\|_\eta \leq 1$, $\|v\|_\eta \leq 1$ and $\|u-v\|_\eta > \varepsilon > 0$, then it holds

$$\Big\|\frac{u+v}{2}\Big\|_\eta^{p'} < 1 - \Big(\frac{\varepsilon}{2}\Big)^{p'}. \tag{2.4}$$

From (2.1) and (2.4), we conclude that in both of the cases *a)* and *b)* there is some $\delta(\varepsilon) > 0$ such that $\|u+v\|_\eta < 2(1-\delta(\varepsilon))$, and the proof is complete. □

Corollary 2.2. *The space $(W^{1,p}, \|\cdot\|_\eta)$ is reflexive and has the Kadec-Klee property, i.e., for any sequence $\{u_n\} \subset W^{1,p}$ such that $u_n \to u$, weakly in $W^{1,p}$, and $\|u_n\|_\eta \to \|u\|_\eta$ one has $u_n \to u$, strongly in $W^{1,p}$.*

Proof. The reflexivity of $W^{1,p}$ is immediate by Theorem 2.1 and the well-known Milman-Pettis theorem (see, e.g., Theorem III.29 in [7]), while the Kadec-Klee property follows from Theorem 2.1 and Proposition III.30 in [7]. □

Theorem 2.3. *The functional $\psi : W^{1,p} \to \mathbb{R}$ defined by*

$$\psi(u) = \frac{1}{p}\|u'\|_{L^p}^p \tag{2.5}$$

is of class C^1 on $W^{1,p}$ and its derivative is given by

$$\langle \psi'(u), v\rangle = \int_0^T (h_p(u')|v'), \quad \forall u, v \in W^{1,p}. \tag{2.6}$$

For the proof we need the following technical result due to Glowinski and Marrocco [19] (also, see Lemma 5.2 in [20]).

Lemma 2.4. *There exist positive constants c_1 and c_2 depending only on p, so that* *(i) if $p \in (1,2]$, then*

$$|h_p(z) - h_p(w)| \leq c_1 |z-w|^{p-1}, \ \ \forall z, w \in \mathbb{R}^N; \tag{2.7}$$

(ii) if $p > 2$, then

$$|h_p(z) - h_p(w)| \leq c_2 |z-w| (|z| + |w|)^{p-2}, \ \ \forall z, w \in \mathbb{R}^N. \tag{2.8}$$

Proof of Theorem 2.3. This follows the ideas in the proof of Theorem 9 in [14] (also, see Theorem 5.3 in [20]). The product space $E = \prod_{i=1}^N L^{p'}([0,T];\mathbb{R})$, $1/p + 1/p' = 1$, is equipped with the norm

$$\|h\|_E = \Big(\sum_{i=1}^N \|h_i\|_{L^{p'}}^{p'} \Big)^{\frac{1}{p'}}, \ \ \forall h = (h_1, ..., h_N) \in E$$

and we define $g = (g_1, ..., g_N) : W^{1,p} \to E$ by putting $g(u) = h_p(u')$, $\forall u \in W^{1,p}$. The mapping g is bounded (i.e., maps bounded subsets of $W^{1,p}$ into bounded subsets of E). Indeed, for $i = \overline{1,N}$, we have

$$\|g_i(u)\|_{L^{p'}}^{p'} = \int_0^T ||u'|^{p-2} u_i'|^{p'} \leq \int_0^T |u'|^{(p-1)p'} = \|u'\|_{L^p}^p \leq \|u\|_\eta^p$$

which implies

$$\|g(u)\|_E \leq N^{\frac{1}{p'}} \|u\|_\eta^{p-1}, \ \ \forall u = (u_1, ..., u_N) \in W^{1,p}.$$

We shall prove that g is continuous. With this aim let us begin by noting that

$$\|h\|_E^{p'} \leq c \int_0^T |h|^{p'}, \ \ \forall h \in E, \tag{2.9}$$

with c a positive constant; this is immediate from the equivalence of the norms on $\mathbb{R}^N$.

If $p \in (1,2]$ and $u, v \in W^{1,p}$, then from (2.9) and (2.7) it follows

$$\|g(u) - g(v)\|_E^{p'} \leq c \int_0^T |h_p(u') - h_p(v')|^{p'} \leq c c_1^{p'} \int_0^T |u' - v'|^p \leq c c_1^{p'} \|u - v\|_\eta^p$$

or

$$\|g(u) - g(v)\|_E \leq k_1 \|u - v\|_\eta^{p-1}, \tag{2.10}$$

with $k_1 > 0$ a constant independent of u and v.

Next, let $p > 2$ and $u, v \in W^{1,p}$. From (2.9), (2.8) and by Hölder's inequality we obtain

$$\|g(u) - g(v)\|_E^{p'} \leq c \int_0^T |h_p(u') - h_p(v')|^{p'} \leq c c_2^{p'} \int_0^T |u' - v'|^{p'} (|u'| + |v'|)^{p'(p-2)}$$

$$\leq cc_2^{p'}\Big(\int_0^T |u'-v'|^{p'\frac{p}{p'}}\Big)^{\frac{p'}{p}}\Big(\int_0^T (|u'|+|v'|)^{p'(p-2)\frac{p}{p'(p-2)}}\Big)^{\frac{p'(p-2)}{p}}$$

$$= cc_2^{p'}\Big(\int_0^T |u'-v'|^{p}\Big)^{\frac{p'}{p}}\Big(\int_0^T (|u'|+|v'|)^{p}\Big)^{\frac{p'(p-2)}{p}}$$

$$\leq cc_2^{p'}\|u-v\|_\eta^{p'}\Big(\|u\|_\eta+\|v\|_\eta\Big)^{p'(p-2)}$$

which yields

$$\|g(u)-g(v)\|_E \leq k_2\|u-v\|_\eta\Big(\|u\|_\eta+\|v\|_\eta\Big)^{p-2} \tag{2.11}$$

with $k_2>0$ another constant independent of u and v.

Now, the continuity of g easily follows from (2.10) and (2.11). For each $u\in W^{1,p}$, the linear functional $v\mapsto\int_0^T(h_p(u')|v')$ is continuous on $W^{1,p}$. Indeed, by Hölder's inequality, we have

$$\Big|\int_0^T (h_p(u')|v')\Big| \leq \int_0^T |u'|^{p-1}|v'|$$

$$\leq \Big(\int_0^T |u'|^{p'(p-1)}\Big)^{\frac{1}{p'}}\Big(\int_0^T |v'|^{p}\Big)^{\frac{1}{p}} = \|u'\|_{L^p}^{\frac{p}{p'}}\|v'\|_{L^p} \leq \|u\|_\eta^{\frac{p}{p'}}\|v\|_\eta$$

for all $v\in W^{1,p}$.

Let us prove that ψ is Fréchet derivable at $u\in W^{1,p}$ and its derivative $\psi'(u)$ is $h_p(u')$, in the sense of (2.6). Using Fubini's theorem, for arbitrarily chosen $v\in W^{1,p}$, we obtain

$$\psi(u+v)-\psi(u)-\int_0^T (h_p(u')|v')dt = \frac{1}{p}\int_0^T \Big[|u'+v'|^p-|u'|^p-p(g(u)|v')\Big]dt$$

$$= \frac{1}{p}\int_0^T\int_0^1 \Big[\frac{d}{ds}|u'+sv'|^p-p(g(u)|v')\Big]ds\, dt$$

$$= \frac{1}{p}\int_0^1\int_0^T \Big[p(h_p(u'+sv')|v')-p(g(u)|v')\Big]dt\, ds$$

$$= \int_0^1\int_0^T (g(u+sv)-g(u)|v')dt\, ds,$$

then by Hölder's inequality and the equivalence of the norms on $\mathbb{R}^N$ we get

$$\Big|\psi(u+v)-\psi(u)-\int_0^T (h_p(u')|v')\Big|$$

$$\leq \int_0^1 \Big(\int_0^T |(g(u+sv)-g(u)|^{p'}\Big)^{\frac{1}{p'}}\Big(\int_0^T |v'|^p\Big)^{\frac{1}{p}}ds$$

$$\leq k\Big(\int_0^1 \|(g(u+sv)-g(u)\|_E ds\Big)\|v\|_\eta,$$

with $k>0$ a constant. Letting $\|v\|_\eta \to 0$ in the inequality

$$\frac{\left|\psi(u+v)-\psi(u)-\int_0^T(h_p(u')|v')\right|}{\|v\|_\eta} \leq k\int_0^1 \|(g(u+sv)-g(u)\|_E ds,$$

as g is bounded and continuous, (2.6) is obtained by Lebesgue's dominated convergence theorem.

It remains to prove that $\psi' : W^{1,p} \to (W^{1,p})^*$ is continuous. Again Hölder's inequality and the equivalence of the norms on $\mathbb{R}^N$ are employed to show that

$$|\langle\psi'(u)-\psi'(v),w\rangle| \leq k\|(g(u)-g(u)\|_E\|w\|_\eta, \ \ \forall u,v,w\in W^{1,p},$$

hence

$$\|\psi'(u)-\psi'(v)\|_{(W^{1,p})^*} \leq k\|(g(u)-g(u)\|_E$$

and the continuity of g completes the proof. □

The space $C := C([0,T];\mathbb{R}^N)$ will be considered with the usual supremum norm $\|\cdot\|_C$ in C, i.e., $\|u\|_C = \max\{|u(t)|; t\in[0,T]\}$. Let $f:[0,T]\times\mathbb{R}^{N_1}\to\mathbb{R}^{N_2}$ be a Carathéodory function, i.e.,

(i) $f(t,\cdot):\mathbb{R}^{N_1}\to\mathbb{R}^{N_2}$ is continuous, for a.e. $t\in[0,T]$,

(ii) $f(\cdot,x):[0,T]\to\mathbb{R}^{N_2}$ is measurable for all $x\in\mathbb{R}^{N_1}$,

satisfying the growth condition:

(iii) for each $\rho>0$ there is some $\alpha_\rho\in L^1([0,T];\mathbb{R})$ such that

$$|f(t,x)| \leq \alpha_\rho(t), \quad \text{for a.e. } t\in[0,T],\ \forall x\in\mathbb{R}^{N_1} \text{ with } |x|\leq\rho.$$

We denote by $N_f : C([0,T];\mathbb{R}^{N_1}) \to L^1([0,T];\mathbb{R}^{N_2})$ the *Nemytskii operator* associated to f; recall this is defined by

$$N_f(u)(t) = f(t,u(t)),\ \forall t\in[0,T],\ \forall u\in C([0,T];\mathbb{R}^{N_1}).$$

Theorem 2.5. *Under the assumptions (i), (ii) and (iii) the operator N_f is correctly defined, bounded and continuous from $C([0,T];\mathbb{R}^{N_1})$ to $L^1([0,T];\mathbb{R}^{N_2})$.*

This theorem is well-known. However, for the sake of completeness, we sketch a proof below.

Proof of Theorem 2.5. If $u\in C([0,T];\mathbb{R}^{N_1})$, then obviously, it is measurable, hence there is a sequence of simple functions $\{u_n\}$ converging a.e. to u. Each $f(t,u_n)$ is measurable by virtue of (ii). From (i) we have that

$f(t, u_n)$ converges a.e. to $f(t, u)$, which ensures its measurability. Then, let $\|u\|_C \le \rho$. From (iii) there is some $\alpha_\rho \in L^1([0,T];\mathbb{R})$ such that

$$|N_f(u)(t)| = |f(t, u(t))| \le \alpha_\rho(t), \quad \text{for a.e. } t \in [0,T],$$

thus,

$$\|N_f(u)\|_{L^1} \le \int_0^T \alpha_\rho(t)dt,$$

which shows that N_f is correctly defined and bounded. To show that N_f is continuous, let $u_n \to u$ in $C([0,T];\mathbb{R}^{N_1})$. From (i) it follows that

$$N_f(u_n)(t) \to N_f(u)(t), \quad \text{for a.e. } t \in [0,T]. \tag{2.12}$$

Since $\{u_n\}$ is bounded in $C([0,T];\mathbb{R}^{N_1})$, there is some $M > 0$ such that $\|u\|_C \le M$ and $\|u_n\|_C \le M$, $\forall n \in \mathbb{N}$. By (iii) we can find some $\alpha_M \in L^1([0,T];\mathbb{R})$ such that $|N_f(u)(t)| \le \alpha_M(t)$ and $|N_f(u_n)(t)| \le \alpha_M(t)$, $\forall n \in \mathbb{N}$, for a.e. $t \in [0,T]$. The continuity of N_f follows then from (2.12) and Lebesgue's dominated convergence theorem. □

Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be such that:

(I) $F(t,\cdot) : \mathbb{R}^N \to \mathbb{R}$ is of class C^1 on $\mathbb{R}^N$ for a.e. $t \in [0,T]$; we shall denote by $\nabla F(t,x)$ the gradient of $F(t,\cdot)$ at $x \in \mathbb{R}^N$;

(II) $F(\cdot,x) : [0,T] \to \mathbb{R}$ is measurable for all $x \in \mathbb{R}^N$ and $F(\cdot,0) \in L^1([0,T];\mathbb{R})$;

(III) for each $\rho > 0$ there is some $\alpha_\rho \in L^1([0,T];\mathbb{R})$ such that

$$|\nabla F(t,x)| \le \alpha_\rho(t), \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| \le \rho. \tag{2.13}$$

Note that

$$|F(t,x) - F(t,0)| = \Big| \int_0^1 \frac{d}{ds} F(t,sx)ds \Big|$$

$$\le \int_0^1 |(\nabla F(t,sx|x)|ds \le |x| \int_0^1 |\nabla F(t,sx)|ds$$

thus,

$$|F(t,x)| \le \rho\alpha_\rho(t) + |F(t,0)|, \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| \le \rho, \tag{2.14}$$

with α_ρ from (2.13). This together with the embedding $W^{1,p} \subset C$ enable us to define $\mathcal{F}_F : W^{1,p} \to \mathbb{R}$ by

$$\mathcal{F}_F(u) = \int_0^T N_F(u), \ \ \forall u \in W^{1,p}. \tag{2.15}$$

Theorem 2.6. *Under the assumptions (I), (II) and (III) the functional $\mathcal{F}_F$ is of class C^1 on $W^{1,p}$ and its derivative is given by*

$$\langle \mathcal{F}_F'(u), v\rangle = \int_0^T (\nabla F(t,u)|v), \quad \forall u, v \in W^{1,p}. \tag{2.16}$$

Proof. For each $u \in W^{1,p}$, the linear functional

$$v \mapsto \int_0^T (\nabla F(t,u)|v)$$

is continuous on $W^{1,p}$. Indeed, taking $\rho = \|u\|_C$ from (III) and the continuity of the embedding $W^{1,p} \subset C$ we obtain

$$\Big|\int_0^T (\nabla F(t,u)|v)\Big| \le \int_0^T |\nabla F(t,u)||v| \le k\Big(\int_0^T \alpha_\rho(t)dt\Big)\|v\|_\eta, \ \forall v \in W^{1,p},$$

with $k > 0$ a constant.

To prove that $\mathcal{F}_F$ is Fréchet derivable at some arbitrarily chosen $u \in W^{1,p}$ and its derivative $\mathcal{F}_F'(u)$ is $\nabla F(\cdot, u)$ in the sense of (2.16), let us denote

$$d(u,v) := \int_0^T N_F(u+v) - \int_0^T N_F(u) - \int_0^T (\nabla F(t,u)|v).$$

We have,

$$\begin{aligned} d(u,v) &= \int_0^T \int_0^1 \Big(\frac{d}{ds}F(t,u+sv) - (\nabla F(t,u)|v)\Big)ds\ dt \\ &= \int_0^T \int_0^1 \Big(\nabla F(t,u+sv) - \nabla F(t,u)|v\Big)ds\ dt \end{aligned}$$

and using Fubini's theorem and the continuity of the embedding $W^{1,p} \subset C$ we derive

$$|d(u,v)| \le k\Big(\int_0^1 \int_0^T |\nabla F(t,u+sv) - \nabla F(t,u)|dt\ ds\Big)\|v\|_\eta, \tag{2.17}$$

for all $v \in W^{1,p}$. Defining $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$f(t,x) = \nabla F(t,x), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N, \tag{2.18}$$

inequality (2.17) rewrites

$$|d(u,v)| \le k\Big(\int_0^1 \|N_f(u+sv) - N_f(u)\|_{L^1} ds\Big)\|v\|_\eta, \quad \forall v \in W^{1,p}.$$

Then, taking into account that N_f is bounded and continuous (see Theorem 2.5) and using Lebegue's dominated convergence theorem we get

$$\frac{|d(u,v)|}{\|v\|_\eta} \leq k \int_0^1 \|N_f(u+sv) - N_f(u)\|_{L^1} ds \to 0, \quad \text{as } \|v\|_\eta \to 0,$$

and (2.16) is proved.

Finally, with f given by (2.18), the continuity of $\mathcal{F}'_F$ follows from the estimate:

$$\|\mathcal{F}'_F(u) - \mathcal{F}'_F(v)\|_{(W^{1,p})^*} \leq k\|N_f(u) - N_f(v)\|_{L^1}, \ \ \forall u, v \in W^{1,p},$$

the continuity of the embedding $W^{1,p} \subset C$ and Theorem 2.5. □

Remark 2.7. (i) In terms of f defined by (2.18), equality (2.16) rewrites

$$\langle \mathcal{F}'_F(u), v\rangle = \int_0^T (N_f(u)|v), \ \ \forall u, v \in W^{1,p}.$$

(ii) The reader will emphasize that actually $\mathcal{F}_F$ can be defined on the whole space C and Theorem 2.6 still remains true with C instead of $W^{1,p}$, with a slightly modification of the above proof.

Theorem 2.8. *For any $c \in L^\infty([0,T];\mathbb{R}_+)$, the functional $\varphi_c : W^{1,p} \to \mathbb{R}$ defined by*

$$\varphi_c(u) = \frac{1}{p}\Big(\|u'\|^p_{L^p} + \int_0^T c(t)|u|^p\Big)$$

is convex, l.s.c. and of class C^1 on $W^{1,p}$. Its derivative is given by

$$\langle {\varphi'}_c(u), v\rangle = \int_0^T (h_p(u')|v') + \int_0^T c(t)(h_p(u)|v), \ \ \forall u, v \in W^{1,p}.$$

Proof. Clearly, φ_c is convex. Then, the conclusion follows by the lower semicontinuity of the norm (see, e.g., Proposition III.5 (iii) in [7]), Theorem 2.3 and Theorem 2.6 which applies with

$$F(t,x) = \frac{c(t)}{p}|x|^p, \ \ \forall (t,x) \in [0,T] \times \mathbb{R}^N.$$

3. Potential systems

In this section we deal with the boundary value problem

$$\begin{cases} -[h_p(u')]' + \varepsilon h_p(u) = \nabla F(t,u), \quad \text{in } [0,T], \\ (h_p(u')(0), -h_p(u')(T)) \in \partial j(u(0), u(T)), \end{cases} \tag{3.1}$$

where $\varepsilon \geq 0$ is a constant and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory mapping satisfying

(H_1) $F(t,\cdot)$ is of class C^1 on $\mathbb{R}^N$ for a.e. $t \in [0,T]$ and $F(\cdot,0) = 0$;

(H_2) for each $\rho > 0$ there is some $\alpha_\rho \in L^1([0,T];\mathbb{R})$ such that

$$|\nabla F(t,x)| \leq \alpha_\rho(t), \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| \leq \rho. \tag{3.2}$$

Our approach for problem (3.1) is a variational one and relies on Szulkin's critical point theory [41]. We obtain existence of solutions in a coercive case, existence of mountain pass solutions, as well as existence of infinitely many solutions.

3.1. **Preliminaries.** Let $(X, \|\cdot\|)$ be a real Banach space and $I : X \to (-\infty, +\infty]$ be a functional of the type

$$I = \Phi + \psi, \tag{3.3}$$

where $\Phi \in C^1(X;\mathbb{R})$ and ψ is proper, convex and l.s.c. A point $u \in X$ is said to be *a critical point* of I if it satisfies the inequality

$$\langle \Phi'(u), v - u\rangle + \psi(v) - \psi(u) \geq 0, \ \ \forall v \in X. \tag{3.4}$$

A number $c \in \mathbb{R}$ such that $I^{-1}(c)$ contains a critical point is called *a critical value* of I.

Proposition 3.1. (Proposition 1.1 in [41]) *If I satisfies* (3.3)*, each local minimum point of I is necessarily a critical point of I.*

The functional I is said *to satisfy the Palais-Smale* (in short, *(PS)*) *condition* if every sequence $\{u_n\} \subset X$ for which $I(u_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'(u_n), v - u_n\rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \ \ \forall v \in X,$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence. The next theorem extends the well-known Mountain Pass Theorem of Ambrosetti and Rabinowitz [2].

Theorem 3.2. (Theorem 3.2 in [41]) *Suppose that I satisfies* (3.3)*, the (PS) condition and*

(i) $I(0) = 0$ and there exist $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ if $\|u\| = \rho$;

(ii) $I(e) \leq 0$ for some $e \in X$, with $\|e\| > \rho$.

Then, I has a critical value $c \geq \alpha$ which can be characterized by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1];X) \ : \ \gamma(0) = 0,\ \gamma(1) = e\}$.

Also, the following existence result of multiple critical points for I will be used in the sequel.

Theorem 3.3. (Corollary 4.8 in [41]) *Suppose that I satisfies* (3.3) *and (PS), $I(0) = 0$ and Φ, ψ are even. Assume also that*

(i) there exists a subspace X_1 of X of finite codimension and numbers $\alpha,\ \rho > 0$ such that $I(u) \geq \alpha$ if $u \in X_1$ and $\|u\| = \rho$;

(ii) for any positive integer n there is a n-dimensional subspace X_2 of X such that $I(u) \to -\infty, \quad as\ \|u\| \to +\infty,\ u \in X_2$.

Then, I has infinitely many distinct pairs of critical points.

3.2. **The variational approach for problem** (3.1)**.** For $\varepsilon \geq 0$, let $\varphi_\varepsilon : W^{1,p} \to \mathbb{R}$ be defined by

$$\varphi_\varepsilon(u) := \frac{1}{p}\Big(\|u'\|^p_{L^p} + \varepsilon\|u\|^p_{L^p}\Big), \quad \forall u \in W^{1,p}. \tag{3.5}$$

By virtue of Theorem 2.8 we have that φ_ε is convex, l.s.c., belongs to $C^1(W^{1,p};\mathbb{R})$ and its derivative is given by

$$\langle {\varphi'}_\varepsilon(u), v\rangle = \int_0^T (h_p(u')|v') + \varepsilon \int_0^T (h_p(u)|v), \quad \forall u, v \in W^{1,p}. \tag{3.6}$$

We also consider the functional $J : W^{1,p} \to (-\infty, +\infty]$, defined by

$$J(u) = j(u(0), u(T)), \quad \forall u \in W^{1,p}. \tag{3.7}$$

Note that, as j is proper, convex and l.s.c., the same hold true for J; the fact that J is l.s.c. is an immediate consequence of the continuity of the embedding $W^{1,p} \subset C$. Then, setting

$$\psi_\varepsilon = \varphi_\varepsilon + J, \tag{3.8}$$

with φ_ε in (3.5) and J in (3.7), it is clear that ψ_ε is proper, convex and l.s.c. on $W^{1,p}$.

Further, let us assume that the Carathéodory mapping $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies (H_1) and (H_2). Having in view (2.14), we may introduce the functional $\Phi_F : W^{1,p} \to \mathbb{R}$ by setting

$$\Phi_F(u) = -\int_0^T F(t,u), \quad \forall u \in W^{1,p}. \tag{3.9}$$

From Theorem 2.6 we have that $\Phi_F \in C^1(W^{1,p};\mathbb{R})$ and its derivative is given by

$$\langle \Phi'_F(u), v\rangle = -\int_0^T (\nabla F(t,u)|v), \quad \forall u, v \in W^{1,p}. \tag{3.10}$$

Now, the functional framework of paragraph 3.1 fits the following choices: $X = W^{1,p}$, $\Phi = \Phi_F$ in (3.9), $\psi = \psi_\varepsilon$ in (3.8) and

$$I = \Phi_F + \psi_\varepsilon. \tag{3.11}$$

Theorem 3.4. *Let the Carathéodory mapping* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfy* (H_1), (H_2) *and let* $u \in W^{1,p}$. *If* u *is a critical point of the functional* I *defined by* (3.11), *in the sense of* (3.4), *i.e.,*

$$\langle \Phi'_F(u), v - u\rangle + \psi_\varepsilon(v) - \psi_\varepsilon(u) \geq 0, \;\; \forall v \in W^{1,p}, \tag{3.12}$$

then u *is a solution of problem* (3.1).

Proof. Assume that u is a critical point of I. In (3.12) we take $v = u + sw$, $s > 0$; then dividing by s and letting $s \to 0^+$, we get

$$\langle \Phi'_F(u), w\rangle + \langle \varphi'_\varepsilon(u), w\rangle + J'(u; w) \geq 0, \;\; \forall w \in W^{1,p}, \tag{3.13}$$

where $J'(u; w)$ is the directional derivative of the convex function J at u in the direction w; this is known to exist. By virtue of (3.7), inequality (3.13) becomes

$$\langle \Phi'_F(u), w\rangle + \langle \varphi'_\varepsilon(u), w\rangle + j'\Big((u(0), u(T)); (w(0), w(T))\Big) \geq 0, \;\; \forall w \in W^{1,p}. \tag{3.14}$$

Since $C_0^\infty := C_0^\infty((0,T); \mathbb{R}^N) \subset W^{1,p}$, from (3.14) we infer

$$\langle \Phi'_F(u), w\rangle + \langle \varphi'_\varepsilon(u), w\rangle = 0, \;\; \forall w \in C_0^\infty,$$

which, taking into account (3.6) and (3.10), yields

$$\int_0^T (h_p(u')|w') = \int_0^T (-\varepsilon h_p(u) + \nabla F(t,u)|w), \;\; \forall w \in C_0^\infty. \tag{3.15}$$

Next, as $u \in W^{1,p}$, we have

$$h_p(u), \; h_p(u') \in L^{p'} \subset L^1, \tag{3.16}$$

with $1/p + 1/p' = 1$. Also, the embedding $W^{1,p} \subset C$ and (H_2) implies

$$\nabla F(\cdot, u) \in L^1. \tag{3.17}$$

From (3.15), (3.16) and (3.17) it follows that

$$h_p(u') \in W^{1,1} \tag{3.18}$$

and

$$-[h_p(u')]' = -\varepsilon h_p(u) + \nabla F(t,u), \quad \text{a.e. } t \in [0,T]. \tag{3.19}$$

Since h_p is a homeomorphism, (3.18) ensures that u is of class C^1. This together with (3.19) shows that u is a solution of the differential system in problem (3.1). To prove that u satisfies the boundary condition, we multiply the equality in (3.19) by $w \in W^{1,p}$, then integrating over $[0,T]$, using the integration by parts formula and (3.14) one obtains, $\forall w \in W^{1,p}$,

$$j'((u(0), u(T)); (w(0), w(T))) \geq (h_p(u')(0)|w(0)) - (h_p(u')(T)|w(T)).$$

Thus,

$$j'((u(0), u(T)); (x, y)) \geq (h_p(u')(0)|x) + (-h_p(u')(T)|y), \ \ \forall x, y \in \mathbb{R}^N,$$

which, by a standard result from convex analysis (see, e.g., Theorem 23.2 in [37]), means that

$$(h_p(u')(0), -h_p(u')(T)) \in \partial j(u(0), u(T))$$

and the proof is complete. □

We introduce the first eigenvalue like constant

$$\lambda_1 = \lambda_1(p, j) := \inf \left\{ \frac{\|u'\|_{L^p}^p}{\|u\|_{L^p}^p} : u \in W^{1,p} \setminus \{0\}, (u(0), u(T)) \in D(j) \right\} \quad (3.20)$$

and by means of λ_1, we define

$$\eta_1 = \eta_1(p, j, \varepsilon) := \lambda_1 + \varepsilon, \quad (3.21)$$

for $\varepsilon \geq 0$. These constants will play a key role in the statement of the existence results which we shall present in the sequel.

Remark 3.5. As $D(J) = \{u \in W^{1,p} : \ (u(0), u(T)) \in D(j)\}$, by (3.21) and (3.20) we have

$$\eta_1 = \inf \left\{ \frac{\varepsilon \|u\|_{L^p}^p + \|u'\|_{L^p}^p}{\|u\|_{L^p}^p} : \ u \in W^{1,p} \setminus \{0\}, \ u \in D(J) \right\}, \quad (3.22)$$

and a straightforward computation shows that if $\eta_1 > 0$, then

$$\frac{\|u\|_{\eta_1}^p}{2p} \leq \varphi_\varepsilon(u) \leq \frac{\|u\|_{\eta_1}^p}{p}, \ \ \forall u \in D(J), \quad (3.23)$$

where J is defined by (3.7)

Note that λ_1 can be either equal to 0 (e.g., if $D(j) = \mathbb{R}^N \times \mathbb{R}^N$) or > 0 (e.g., if $D(j) = \{(0,0)\}$). We denote $d = \{(x, x) : x \in \mathbb{R}^N\}$.

Proposition 3.6. *Let $D(j)$ be a closed cone. Then $\lambda_1 > 0$ iff $D(j) \cap d = \{(0,0)\}$.*

Proof. We assume $D(j) \cap d = \{(0,0)\}$ and suppose, by contradiction, that $\lambda_1 = 0$. Then, there is a sequence $\{u_n\} \subset W^{1,p} \setminus \{0\}$ such that $u_n \in D(J)$ and

$$\frac{\|u_n'\|_{L^p}^p}{\|u_n\|_{L^p}^p} < \frac{1}{n}, \ \ \forall n \in \mathbb{N}.$$

Since $D(J)$ is a cone, clearly

$$v_n := \frac{u_n}{\|u_n\|_{L^p}} \in D(J), \ \ \forall n \in \mathbb{N}.$$

Also, $\|v_n'\|_{L^p} \to 0$, as $n \to \infty$ and $\|v_n\|_1^p = \|v_n'\|_{L^p}^p + 1$, which shows that $\{v_n\}$ is bounded in the reflexive Banach space $(W^{1,p}, \|\cdot\|_1)$. There is a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, and some $v \in W^{1,p}$ such that $v_n \to v$, weakly in $W^{1,p}$. As the closed convex set $D(J)$ is weakly closed, we have $v \in D(J)$. By the compactness of the embedding $W^{1,p} \subset L^p$, it follows that $v_n \to v$, strongly in L^p. We obtain

$$\|v\|_{L^p}^p = \lim_{n\to\infty} \|v_n\|_{L^p}^p = 1,$$

and so,

$$\|v\|_{L^p}^p + \|v'\|_{L^p}^p = \|v\|_1^p \le \liminf_{n\to\infty} \|v_n\|_1^p = 1.$$

Consequently, $\|v'\|_{L^p} = 0$ and so, $v(t) = a$, $\forall t \in [0,T]$, with some $a \in \mathbb{R}^N \setminus \{0\}$. Since $v \in D(J)$ it follows $(a,a) \in D(j)$, which is a contradiction. The proof of the converse implication is not difficult and it is left to the reader. □

For $l \in L^1$ given, let us consider the boundary value problem

$$\begin{cases} -[h_p(u')]' + \varepsilon h_p(u) = \nabla F(t,u) + l(t), & \text{in } [0,T], \\ (h_p(u')(0), -h_p(u')(T)) \in \partial j(u(0), u(T)), \end{cases} \tag{3.24}$$

which is nothing else but a problem of type (3.1). Indeed, it is easy to see that if $F(t,x)$ satisfies (H_1) and (H_2), then $F(t,x) + (l(t)|x)$ still satisfies the same hypotheses. Therefore, Theorem 3.4 can be applied for problem (3.24) with I defined by (3.11), (3.8) and

$$\Phi_F(u) = -\int_0^T F(t,u) - \int_0^T (l(t)|u), \quad \forall u \in W^{1,p} \tag{3.25}$$

instead of (3.9). If the nonlinearity F lies asymptotically on the left of η_1, then problem (3.24) is solvable. In this view, the theorem below extends to the boundary value problem (3.24) known results in the case of the p-Laplacian operator associated with homogeneous Dirichlet [3], [14] and Neumann [10] boundary conditions, or for the ordinary vector p-Laplacian with periodic boundary conditions [11].

Theorem 3.7. *Assume that the Caratéodory mapping $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies (H_1), (H_2) and*

$$\limsup_{|x|\to\infty} \frac{pF(t,x)}{|x|^p} < \eta_1, \quad \text{uniformly for a.e. } t \in [0,T]. \tag{3.26}$$

Then the problem (3.24) has at least one solution for any $l \in L^1$.

Proof. We shall prove that the functional I defined by (3.11), (3.8) and (3.25) is sequentially weakly l.s.c. and coercive on the space $(W^{1,p}, \|\cdot\|_\eta)$, with some $\eta > 0$. Then, by a well-known result from calculus of variations, I is bounded from below and attains its infimum at some $u \in W^{1,p}$, which by Propositions 3.1 and Theorem 3.4 is a solution of problem (3.24).

Let us begin by noting that Φ_F in (3.25) is sequentially weakly continuous. This can be shown as follows. Let $u, v \in W^{1,p}$ be such that $\|u\|_C,\ \|v\|_C \le M$, with some constant $M > 0$. By (H_2) there is some $\alpha_{2M} \in L^1([0,T];\mathbb{R})$ such that

$$|\nabla F(t,x)| \le \alpha_{2M}(t), \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| \le 2M. \tag{3.27}$$

We estimate

$$\begin{aligned}
|\Phi_F(u) - \Phi_F(v)| &\le \Big| \int_0^T (F(t,v) - F(t,u)) \Big| + \Big| \int_0^T (l(t)|v-u) \Big| \\
&= \Big| \int_0^T \int_0^1 \frac{d}{ds} F(t, u + s(v-u)) ds \Big| + \Big| \int_0^T (l(t)|v-u) \Big| \\
&= \Big| \int_0^T \int_0^1 (\nabla F(t, u + s(v-u))|v-u) ds \Big| + \Big| \int_0^T (l(t)|v-u) \Big| \\
&\le \Big(\int_0^T \int_0^1 |\nabla F(t, u + s(v-u))| ds + \|l\|_{L^1} \Big) \|v-u\|_C
\end{aligned}$$

and by (3.27) it follows that

$$|\Phi_F(u) - \Phi_F(v)| \le \Big(\int_0^T \alpha_{2M}(t) dt + \|l\|_{L^1} \Big) \|u-v\|_C. \tag{3.28}$$

By the compactness of the embedding $W^{1,p} \subset C$ and (3.28) we get that Φ_F is sequentially weakly continuous on $W^{1,p}$. Then, by the weak lower semicontinuity of ψ_ε, I is sequentially weakly l.s.c. It remains to prove that I is coercive on the space $(W^{1,p}, \|\cdot\|_\eta)$, with some $\eta > 0$.

From (3.26) there are constants $\sigma > 0$ and $\rho > 0$ such that

$$F(t,x) \le \frac{\eta_1 - \sigma}{p} |x|^p, \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| > \rho; \tag{3.29}$$

if $\eta_1 > 0$ we shall assume that $\sigma < \eta_1$. From (H_1), (H_2), (2.14) and (3.29) we have

$$F(t,x) \le \rho \alpha_\rho(t) + \frac{|\eta_1 - \sigma|}{p} \rho^p + \frac{\eta_1 - \sigma}{p} |x|^p, \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N,$$

which, by (3.25), gives

$$\Phi_F(u) \geq -k - \frac{\eta_1 - \sigma}{p}\|u\|_{L^p}^p - \|l\|_{L^1}\|u\|_C, \ \ \forall u \in W^{1,p}, \tag{3.30}$$

where $k = k(\rho, \sigma) \geq 0$ is a constant. Using (3.30) we estimate I as follows

$$I(u) = \Phi_F(u) + \varphi_\varepsilon(u) + J(u)$$
$$\geq -k - \frac{\eta_1 - \sigma}{p}\|u\|_{L^p}^p + \frac{1}{p}(\|u'\|_{L^p}^p + \varepsilon\|u\|_{L^p}^p) + J(u) - \|l\|_{L^1}\|u\|_C$$
$$= -k + \frac{1}{p}\Big(\|u'\|_{L^p}^p + (\sigma - \eta_1 + \varepsilon)\|u\|_{L^p}^p\Big) + J(u) - \|l\|_{L^1}\|u\|_C, \ \ \forall u \in W^{1,p}.$$

If $\eta_1 = 0$, we infer

$$I(u) \geq -k + \frac{1}{p}\|u\|_\sigma^p + J(u) - \|l\|_{L^1}\|u\|_C, \ \ \forall u \in W^{1,p}, \tag{3.31}$$

while in the case $\eta_1 > 0$, by (3.22) and (3.23), we get

$$I(u) \geq -k + \frac{1}{p}\Big(p\varphi_\varepsilon(u) + (\sigma - \eta_1)\frac{\|u'\|_{L^p}^p + \varepsilon\|u\|_{L^p}^p}{\eta_1}\Big) + J(u) - \|l\|_{L^1}\|u\|_C$$
$$\geq -k + \frac{\sigma}{2p\eta_1}\|u\|_{\eta_1}^p + J(u) - \|l\|_{L^1}\|u\|_C, \ \ \forall u \in D(J). \tag{3.32}$$

By virtue of (3.31) and (3.32) in both cases there are constants $\eta, k_0 > 0$ such that

$$I(u) \geq -k + k_0\|u\|_\eta^p + J(u) - \|l\|_{L^1}\|u\|_C, \ \ \forall u \in D(J).$$

Since j is convex, and l.s.c. it is bounded from below by an affine functional (see e.g., p. 96 in [6]). Therefore, on account of (3.7) we can find constants $k_1, k_2, k_3 \geq 0$ such that

$$I(u) \geq -k + k_0\|u\|_\eta^p - k_1|u(0)| - k_2|u(T)| - k_3 - \|l\|_{L^1}\|u\|_C$$
$$\geq -k + k_0\|u\|_\eta^p - (k_1 + k_2 + \|l\|_{L^1})\|u\|_C - k_3, \ \ \forall u \in D(J). \tag{3.33}$$

From (3.33) and the continuity of the embedding $W^{1,p} \subset C$, one obtains

$$I(u) \geq -k + k_0\|u\|_\eta^p - \tilde{k}\|u\|_\eta - k_3, \ \ \forall u \in D(J),$$

with some constant $\tilde{k} \geq 0$. Consequently,

$$I(u) \to +\infty, \quad \text{as } \|u\|_\eta \to \infty,$$

meaning that I is coercive on $(W^{1,p}, \|\cdot\|_\eta)$ and the proof is complete. □

Coming back to problem (3.1), obviously, one has

Corollary 3.8. *Under the assumptions of Theorem 3.7 problem* (3.1) *has at least one solution.*

Remark 3.9. It is worth to point out that no additional assumptions on j are required in Theorem 3.7 and Corollary 3.8. Hence, any choice of j is allowed. If $a, b \in \mathbb{R}$ are given, then with $j = I_{\{(a,b)\}}$ and $j(x,y) = (h_p(a)|x) - (h_p(b)|y)$, $\forall x, y \in \mathbb{R}^N$, one obtains the nonhomogeneous boundary conditions

$$u(0) = a, \quad u(T) = b, \quad \textit{(Dirichlet)}$$

$$u'(0) = a, \quad u'(T) = b, \quad \textit{(Neumann)}$$

respectively. Taking $j = I_d$ and $j = I_{d'}$ with $d' = \{(x, -x) : x \in \mathbb{R}^N\}$, we get

$$u(0) - u(T) = 0 = u'(0) - u'(T), \quad \textit{(periodic)}$$

$$u(0) + u(T) = 0 = u'(0) + u'(T), \quad \textit{(antiperiodic)}$$

respectively. Also, for given $\alpha, \beta > 0$,

$$j(z) = g(x,y) := \frac{1}{p}\Big(\frac{|x|^p}{\alpha^{p-1}} + \frac{|y|^p}{\beta^{p-1}}\Big), \quad \forall z = (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \tag{3.34}$$

yields the Sturm-Liouville type boundary conditions [18, Example 6]:

$$u(0) - \alpha u'(0) = 0, \quad u(T) + \beta u'(T) = 0. \tag{3.35}$$

3.3. **Mountain Pass type solutions.** In this paragraph we are concerned with the existence of nontrivial solutions for problem (3.1). The main tool in obtaining such results will be Theorem 3.2.

Lemma 3.10. *Assume that $D(j)$ is closed and let ψ_ε be defined by* (3.8). *Then, for each sequence $\{u_n\} \subset D(J)$ such that $u_n \to u$, weakly in $W^{1,p}$, and*

$$\liminf_{n\to\infty} \psi'_\varepsilon(u_n; u - u_n) \geq 0, \tag{3.36}$$

one has that $u_n \to u$, strongly in $W^{1,p}$.

Proof. *(i) Case $\varepsilon > 0$.* Since $\eta_1 > 0$, we may consider the space $W^{1,p}$ endowed with the norm $\|\cdot\|_{\eta_1}$. Also, $u \in D(J)$ because the closed convex set $D(J)$ is weakly closed in $(W^{1,p}, \|\cdot\|_{\eta_1})$, hence $J'(u_n; u - u_n) < +\infty$, for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
\limsup_{n\to\infty}\langle\varphi'_\varepsilon(u_n), u_n - u\rangle &= \limsup_{n\to\infty}[-\psi'_\varepsilon(u_n; u - u_n) + J'(u_n; u - u_n)] \\
&\leq -\liminf_{n\to\infty} \psi'_\varepsilon(u_n; u - u_n) + \limsup_{n\to\infty} J'(u_n; u - u_n) \\
&\leq -\liminf_{n\to\infty} \psi'_\varepsilon(u_n; u - u_n) + \limsup_{n\to\infty}[J(u) - J(u_n)] \\
&\leq -\liminf_{n\to\infty} \psi'_\varepsilon(u_n; u - u_n) + J(u) - \liminf_{n\to\infty} J(u_n),
\end{aligned}$$

and, as J is l.s.c., from (3.36) we get

$$\limsup_{n\to\infty}\langle\varphi_\varepsilon'(u_n),u_n-u\rangle\le 0. \tag{3.37}$$

Let φ_{η_1} be defined by (3.5) with η_1 instead of ε, i.e.,

$$\varphi_{\eta_1}(v)=\frac{1}{p}\|v\|_{\eta_1}^p,\ \ \forall v\in W^{1,p}.$$

Clearly, one has

$$\varphi_{\eta_1}(v)=\varphi_\varepsilon(v)+\frac{\eta_1-\varepsilon}{p}\|v\|_{L^p}^p,\ \ \forall v\in W^{1,p},$$

hence,

$$\langle\varphi_{\eta_1}'(v),w\rangle=\langle\varphi_\varepsilon'(v),w\rangle+(\eta_1-\varepsilon)\int_0^T(h_p(v)|w),\ \ \forall v,w\in W^{1,p}. \tag{3.38}$$

Taking into account the compact embedding $W^{1,p}\subset L^p$ and the inequality

$$\Big|\int_0^T(h_p(u_n)|u_n-u)\Big|\le\|u_n\|_{L^p}^{\frac{p}{p'}}\|u_n-u\|_{L^p},$$

we see that

$$\int_0^T(h_p(u_n)|u_n-u)\to 0,\quad\text{as } n\to\infty. \tag{3.39}$$

From (3.38), (3.37) and (3.39) it follows

$$\limsup_{n\to\infty}\langle\varphi_{\eta_1}'(u_n),u_n-u\rangle\le 0. \tag{3.40}$$

Using the Hölder inequality, standard computations show that

$$0\le(\|u_n\|_{\eta_1}^{p-1}-\|u\|_{\eta_1}^{p-1})(\|u_n\|_{\eta_1}-\|u\|_{\eta_1})\le\langle\varphi_{\eta_1}'(u_n)-\varphi_{\eta_1}'(u),u_n-u\rangle.$$

This, together with (3.40) yields

$$\|u_n\|_{\eta_1}\to\|u\|_{\eta_1},\quad\text{as } n\to\infty. \tag{3.41}$$

Finally, since $(W^{1,p},\|\cdot\|_{\eta_1})$ has the Kadec-Klee property (see Corollary 2.2) and as $u_n\to u$, weakly in $W^{1,p}$, from (3.41) we obtain that $u_n\to u$, strongly in $W^{1,p}$.

(ii) Case $\varepsilon=0$. Let $\varepsilon_0>0$ and $\tilde\psi_{\varepsilon_0}:W^{1,p}\to(-\infty,+\infty]$ be defined by

$$\tilde\psi_{\varepsilon_0}(v)=\psi_0(v)+\frac{\varepsilon_0}{p}\|v\|_{L^p}^p,\ \ \forall v\in W^{1,p}.$$

According to the case *(i)* it suffices to show that

$$\liminf_{n\to\infty}\tilde\psi_{\varepsilon_0}'(u_n;u-u_n)\ge 0.$$

But this is straightforward by virtue of the equality

$$\tilde{\psi}'_{\varepsilon_0}(u_n; u - u_n) = \psi'_0(u_n; u - u_n) + \varepsilon_0 \int_0^T (h_p(u_n)|u - u_n)$$

and (3.39). □

Lemma 3.11. *Assume that $D(j)$ is closed and $\eta_1 > 0$, together with (H_1), (H_2). If there are constants $\theta > p$ and $k, M > 0$ such that*

$$j'(z; z) \leq \theta j(z) + k, \quad \forall z \in D(j) \tag{3.42}$$

and

$$\theta F(t,x) \leq (\nabla F(t,x)|x), \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| > M, \tag{3.43}$$

then the functional I in (3.11) satisfies the (PS) condition on $(W^{1,p}, \|\cdot\|_{\eta_1})$, i.e., every sequence $\{u_n\} \subset W^{1,p}$ for which $I(u_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'_F(u_n), v - u_n\rangle + \psi_\varepsilon(v) - \psi_\varepsilon(u_n) \geq -\varepsilon_n \|v - u_n\|_{\eta_1}, \quad \forall v \in W^{1,p}, \tag{3.44}$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

Proof. From (3.7) and (3.42) it follows

$$J(v) - \frac{1}{\theta} J'(v; v) \geq -k_1, \quad \forall v \in D(J), \tag{3.45}$$

with $k_1 = k/\theta$. By (H_1), (H_2) and (2.14) there is some $\alpha_M \in L^1([0,T]; \mathbb{R})$ such that

$$F(t,x) \leq M\alpha_M(t), \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| \leq M. \tag{3.46}$$

Using (3.9), (3.46), (3.43), (3.2) and (3.10), we obtain

$$\begin{aligned}
-\Phi_F(v) &= \int_{[|v| \leq M]} F(t,v) + \int_{[|v| > M]} F(t,v) \\
&\leq M \int_{[|v| \leq M]} \alpha_M(t) + \frac{1}{\theta} \int_{[|v| > M]} (\nabla F(t,v)|v) \\
&\leq M \int_0^T \alpha_M(t) + \frac{1}{\theta}\Big(\int_0^T (\nabla F(t,v)|v) - \int_{[|v| \leq M]} (\nabla F(t,v)|v)\Big) \\
&\leq M\Big(1 + \frac{1}{\theta}\Big) \int_0^T \alpha_M(t) - \frac{1}{\theta} \langle \Phi'_F(v), v\rangle,
\end{aligned}$$

yielding

$$\Phi_F(v) - \frac{1}{\theta} \langle \Phi'_F(v), v\rangle \geq -k_2, \quad \forall v \in W^{1,p}, \tag{3.47}$$

with $k_2 = k_2(M, \theta)$ a positive constant.

Next, the proof resembles the proof of Lemma 3.5 in [16]. Let $\{u_n\} \subset W^{1,p}$ be a sequence for which $I(u_n) \to c \in \mathbb{R}$ and (3.44) holds true with $\varepsilon_n \to 0$. Clearly, $\{u_n\} \subset D(I) = D(J)$ and there is a constant $k_3 > 0$, such that

$$|I(u_n)| \leq k_3, \quad \forall n \in \mathbb{N}. \tag{3.48}$$

In (3.44) we set $v = u_n + su_n$, $s > 0$, then dividing by s and letting $s \to 0^+$, we obtain

$$\langle \Phi'_F(u_n), u_n \rangle + \psi'_\varepsilon(u_n; u_n) \geq -\varepsilon_n \|u_n\|_{\eta_1}, \quad \forall n \in \mathbb{N}. \tag{3.49}$$

Using (3.48), (3.8) and (3.49) one obtains

$$\begin{aligned} k_3 + \frac{\varepsilon_n}{\theta}\|u_n\|_{\eta_1} &\geq \Phi_F(u_n) + \psi_\varepsilon(u_n) + \frac{\varepsilon_n}{\theta}\|u_n\|_{\eta_1} \\ &= \Phi_F(u_n) + \varphi_\varepsilon(u_n) + J(u_n) + \frac{\varepsilon_n}{\theta}\|u_n\|_{\eta_1} \end{aligned}$$

$$\geq \Phi_F(u_n) - \frac{1}{\theta}\langle \Phi'_F(u_n), u_n\rangle + \varphi_\varepsilon(u_n) - \frac{1}{\theta}\langle \varphi'_\varepsilon(u_n), u_n\rangle + J(u_n) - \frac{1}{\theta}J'(u_n; u_n)$$

and by virtue of (3.47), (3.45), (3.5) (3.6) and (3.23) we deduce

$$k_1 + k_2 + k_3 + \frac{\varepsilon_n}{\theta}\|u_n\|_{\eta_1} \geq \Big(\frac{1}{p} - \frac{1}{\theta}\Big)p\varphi_\varepsilon(u_n) \geq \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|u_n\|^p_{\eta_1}. \tag{3.50}$$

Since $\theta > p$, from (3.50) it follows that $\{\|u_n\|_{\eta_1}\}$ is bounded. By the compactness of the embedding $W^{1,p} \subset C$, the sequence $\{u_n\}$ has a subsequence, again denoted by $\{u_n\}$, such that $u_n \to u$, weakly in $W^{1,p}$ and strongly in C. Similarly to (3.49) we derive

$$\langle \Phi'_F(u_n), u - u_n \rangle + \psi'_\varepsilon(u - u_n; u_n) \geq -\varepsilon_n \|u - u_n\|_{\eta_1}, \quad \forall n \in \mathbb{N}. \tag{3.51}$$

As $u_n \to u$, strongly in C, from (3.10) and (3.2) it follows that

$$\langle \Phi'_F(u_n), u - u_n \rangle \to 0, \quad \text{as } n \to \infty. \tag{3.52}$$

Then, because $\{\|u - u_n\|_{\eta_1}\}$ is bounded and $\varepsilon_n \to 0$, (3.51) and (3.52) yield

$$\liminf_{n\to\infty} \psi'_\varepsilon(u_n; u - u_n) \geq 0$$

and Lemma 3.10 applies showing that $u_n \to u$, strongly in $W^{1,p}$. □

Theorem 3.12. *Let the Carathéodory function* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfy* (H_1) *and* (H_2). *We assume that* $\eta_1 > 0$ *and, in addition,*

(i) the set $D(j)$ *is a closed cone and* $(0,0) \in \partial j(0,0)$;

(ii) $\limsup_{|x|\to 0} \dfrac{pF(t,x)}{|x|^p} < \eta_1$, *uniformly for a.e.* $t \in [0,T]$;

(iii) there are constants $\theta > p$ *and* $k, M > 0$ *such that* (3.42) *holds true and*

$$0 < \theta F(t,x) \leq (\nabla F(t,x)|x),$$

$$\textit{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \textit{ with } |x| > M. \tag{3.53}$$

Then, problem (3.1) *has a nontrivial solution.*

Proof. We shall prove that the functional I defined in (3.11) has the geometry required by Theorem 3.2. First, without loss of generality, we may assume

$$j(0,0) = 0. \tag{3.54}$$

Since $(0,0) \in \partial j(0,0)$, from (3.7) and (3.54) it follows

$$J(u) \geq J(0) = 0, \quad \forall u \in D(J). \tag{3.55}$$

Then, it is clear that

$$I(0) = 0. \tag{3.56}$$

By Lemma 3.11, condition (iii) ensures that I satisfies the (PS) condition on $(W^{1,p}, \|\cdot\|_{\eta_1})$. We fix a constant k_0 such that

$$\|u\|_C \leq k_0 \|u\|_{\eta_1}, \quad \forall u \in W^{1,p}; \tag{3.57}$$

this is known to exist by the continuity of the embedding $W^{1,p} \subset C$. From (ii) there are constants $\sigma \in (0, \eta_1)$ and $\rho > 0$, such that

$$F(t,x) \leq \frac{\eta_1 - \sigma}{p}|x|^p, \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| \leq \rho k_0, \tag{3.58}$$

with k_0 from (3.57).

Let $u \in W^{1,p}$, with $\|u\|_{\eta_1} \leq \rho$, be arbitrarily chosen. From (3.57) and (3.58) we have

$$F(t,u) \leq \frac{\eta_1 - \sigma}{p}|u|^p, \quad \text{for a.e. } t \in [0,T].$$

This implies

$$-\Phi_F(u) \leq \frac{\eta_1 - \sigma}{p}\|u\|_{L^p}^p,$$

which, using (3.5), (3.22) and (3.23), gives

$$\begin{aligned}\Phi_F(u) + \varphi_\varepsilon(u) &\geq -\frac{\eta_1 - \sigma}{p}\,\frac{\varepsilon\|u\|_{L^P}^p + \|u'\|_{L^p}^p}{\eta_1} + \frac{1}{p}\left(\varepsilon\|u\|_{L^P}^p + \|u'\|_{L^p}^p\right) \\ &= \frac{\sigma}{\eta_1}\varphi_\varepsilon(u) \geq \frac{\sigma}{2p\eta_1}\|u\|_{\eta_1}^p, \quad \forall u \in D(J),\ \|u\|_{\eta_1} \leq \rho.\end{aligned}$$

By virtue of (3.55),

$$I(u) = \Phi_F(u) + \varphi_\varepsilon(u) + J(u) \geq \alpha, \quad \text{if } \|u\|_{\eta_1} = \rho, \tag{3.59}$$

with $\alpha = \sigma\rho^p(2p\eta_1)^{-1} > 0$. Then, (3.56) and (3.59) show that condition (i) in Theorem 3.2 is fulfilled.

Our next task is to prove that I satisfies condition (ii) in Theorem 3.2. To this end, let us first observe that by virtue of (3.53), for a.e. $t \in [0,T]$ and each $x \in \mathbb{R}^N$, with $|x| > M$, the mapping $s \mapsto \frac{F(t,sx)}{s^\theta}$ is increasing on $[1,\infty)$. It follows that

$$F(t,sx) \geq s^\theta F(t,x),$$

$$\text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N,\ |x| > M \text{ and } \forall s \geq 1. \tag{3.60}$$

Let $\overline{e} \in C_0^\infty$ be such that $|\overline{e}| > M$ on a set of positive measure. From (3.7) and (3.54), we have

$$J(s\overline{e}) = 0, \quad \forall s \in \mathbb{R}. \tag{3.61}$$

Using (2.14), (3.53) and (3.60), for $s \geq 1$, we obtain

$$\begin{aligned}\int_0^T F(t,s\overline{e}) &= \int_{[|s\overline{e}|\leq M]} F(t,s\overline{e}) + \int_{[|s\overline{e}|>M]} F(t,s\overline{e}) \\ &\geq -\int_0^T M\alpha_M(t)dt + \int_{[|\overline{e}|>M]} F(t,s\overline{e}) \\ &\geq -\int_0^T M\alpha_M(t)dt + s^\theta \int_{[|\overline{e}|>M]} F(t,\overline{e})\end{aligned}$$

i.e.,

$$\int_0^T F(t,s\overline{e}) \geq -k_1 + s^\theta k_2, \tag{3.62}$$

with constants $k_1 = k_1(M) \geq 0$ and $k_2 = k_2(M,\overline{e}) > 0$. By (3.11), (3.61), (3.62) and (3.23) we get

$$I(s\overline{e}) \leq -s^\theta k_2 + k_1 + \frac{s^p}{p}\|\overline{e}\|_{\eta_1}^p \to -\infty, \text{ as } s \to +\infty. \tag{3.63}$$

Now, by (3.63), we can choose s_1 sufficiently large to satisfy $I(s_1\overline{e}) \leq 0$ and $\|s_1\overline{e}\|_{\eta_1} > \rho$, with ρ entering in (3.59). This means that condition (ii) in Theorem 3.2 is fulfilled with $e = s_1\overline{e}$.

Consequently, the functional I has a nontrivial critical point $u \in W^{1,p}$, which by Theorem 3.4 is a nontrivial solution of problem (3.1). □

Remark 3.13. (i) Inequality (3.42) in Lemma 3.11 implies that the domain of j necessarily satisfies $[1,\infty)\ D(j) = D(j)$. This is a consequence of the fact that if $D(j)$ is closed, then the equivalence

$$[1,\infty)D(j) = D(j) \Longleftrightarrow j'(z;z) < +\infty \ \ \forall z \in D(j)$$

holds true. Indeed, the implication " $\Longrightarrow$ " is immediate from the definition of $j'(z;z)$ and the monotonicity of the mapping $(0,\infty) \ni t \mapsto t^{-1}(j(z+tz)-j(z))$. To prove the converse implication we can argue using an idea of C.

Zălinescu, as follows. Let $z \in D(j)$ and $I = \{t \in [1, \infty) : \; tz \in D(j)\}$. Since $D(j)$ is convex, I is an interval. We have to show that $\alpha := \sup I = +\infty$. Suppose, by contradiction, that $\alpha < +\infty$. Since $D(j)$ is closed, $\alpha \in I$, meaning $\alpha z \in D(j)$. It follows that,

$$\lim_{t \searrow 0} \frac{j(\alpha z + t\alpha z) - j(\alpha z)}{t} = j'(\alpha z; \alpha z) < +\infty$$

and, hence there is some $t > 0$ such that $j(\alpha(1+t)z) < +\infty$ and so $\alpha(1+t) \in I$ contradicting the definition of α.

We also note that in the proof of Theorem 3.12 the assumption that $D(j)$ is a cone was not explicitly used. But, in fact, if $D(j)$ is closed, $(0,0) \in \partial j(0,0)$ and (3.42) holds true, then necessarily $D(j)$ is a cone.

Further properties of convex functions satisfying an inequality of type (3.42) can be found in Zălinescu [42, p. 150].

(ii) Hypotheses $(0,0) \in \partial j(0,0)$ and *(ii)* in Theorem 3.12 ensure that problem (3.1) also admits the trivial solution $u = 0$. Indeed, from (3.58) we have

$$F(t,x) \leq a|x|^p, \quad \text{for a.e. } t \in [0,T], \; \forall x \in \mathbb{R}^N \text{ with } |x| \leq \rho_1$$

where a and ρ_1 are some positive constants. This implies that for arbitrarily chosen $y \in \mathbb{R}^N$, it holds

$$(\nabla F(t,0)|y) = \lim_{s \to 0} \frac{F(t,sy)}{s} = 0,$$

meaning $\nabla F(t,0) = 0$, for a.e. $t \in [0,T]$.

Remark 3.14. Theorem 3.12 extends to the case of the boundary value problem (3.1) the well-known result of Ambrosetti and Rabinowitz for the Laplace operator associated with homogeneous Dirichlet boundary conditions obtained in [2, Theorem 3.10] (see also [36, Theorem 2.15]), as well as Theorem 3.6 in [13] ([14, Theorem 18]) for the p-Laplacian operator. In this view condition *(iii)* appears as being the analogue for problem (3.1) of the celebrated hypothesis (p_5) in [2] ((p_4) in [36]).

3.4. **Infinitely many solutions.** Next, we shall prove that if in Theorem 3.12 the functions F and j are even, then problem (3.1) has infinitely many solutions. The main tool in this task will be Theorem 3.3.

Let $m > 0$ be a constant and $n \in \mathbb{N}$. For each $j \in \{1, ..., n\}$ we denote $A_j = \big((j-1)T/n, jT/n\big)$ and let $e_j \in C_0^\infty$ be such that supp $e_j \subset A_j$ and $|e_j| > m$ on a set of positive measure. Clearly, $e_1, ..., e_n$ are linearly independent and

$$Y_n(m) = \text{span } \{e_1, ..., e_n\} \tag{3.64}$$

is an n-dimensional subspace of $W_0^{1,p} = \{u \in W^{1,p} : u(0) = u(T) = 0\}$. On $Y_n(m)$ we consider the norm

$$\|u\|_\infty = \max\{\ |\xi_j|\ :\ j = \overline{1,n}\}$$

if $u = \sum_{j=1}^n \xi_j e_j$.

Lemma 3.15. *Assume that the Carathéodory mapping $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is even in the second argument and satisfies (H_1), (H_2). If there are constants $\theta > p$ and $M > 0$ such that* (3.53) *is fulfilled, then for each subspace $Y_n(M)$ there are constants $c_1 = c_1(M, n; e_1, ..., e_n) > 0$ and $c_2 = c_2(M) \in \mathbb{R}$ such that*

$$\int_0^T F(t,u) \geq c_1\ \|u\|_\infty^\theta - c_2,\ \ \forall u \in Y_n(M),\ \|u\|_\infty \geq 1.$$

Proof. By virtue of (3.53), for a.e. $t \in [0,T]$ and each $x \in \mathbb{R}^N$, with $|x| > M$, the mapping

$$s \mapsto \frac{F(t,sx)}{s^\theta}$$

is increasing on $[1,\infty)$. Since F is even in the second argument, we infer

$$F(t,\xi x) = F(t,|\xi|x) \geq |\xi|^\theta F(t,x) \tag{3.65}$$

for a.e. $t \in [0,T]$, $\forall x \in \mathbb{R}^N$, $|x| > M$ and $\forall \xi \in \mathbb{R}$ with $|\xi| \geq 1$.

If $u = \sum_{j=1}^n \xi_j e_j$ is with $\|u\|_\infty = |\xi_{j_0}| \geq 1$, then using (2.14), (3.53) and (3.65) we successively have

$$\int_0^T F(t,u) = \int_0^T F(t, \sum_{j=1}^n \xi_j e_j) = \sum_{i=1}^n \int_{I_i} F(t, \sum_{j=1}^n \xi_j e_j) = \sum_{i=1}^n \int_{I_i} F(t, \xi_i e_i)$$

$$= \sum_{i=1}^n \Big[\int_{\{t \in I_i\ :\ |\xi_i e_i| \leq M\}} F(t,\xi_i e_i) + \int_{\{t \in I_i\ :\ |\xi_i e_i| > M\}} F(t,\xi_i e_i) \Big]$$

$$\geq -M \int_0^T \alpha_M(t) + \int_{\{t \in I_{j_0}\ :\ |\xi_{j_0} e_{j_0}| > M\}} F(t, \xi_{j_0} e_{j_0})$$

$$\geq -M \int_0^T \alpha_M(t) + \int_{\{t \in I_{j_0}\ :\ |e_{j_0}| > M\}} F(t, \xi_{j_0} e_{j_0})$$

$$\geq -M \int_0^T \alpha_M(t) + |\xi_{j_0}|^\theta \int_{\{t \in I_{j_0}\ :\ |e_{j_0}| > M\}} F(t, e_{j_0}) \geq -c_2 + c_1\ \|u\|_\infty^\theta$$

with

$$c_2 = M\int_0^T \alpha_M(t), \quad \text{and} \quad c_1 = \min_{i=\overline{1,n}} \int_{\{t\in I_i\,:\,|e_i|>M\}} F(t,e_i) > 0.$$

□

We set

$$V := \left\{ v \in W^{1,p} \;:\; \int_0^T v = 0 \right\} \tag{3.66}$$

and let the constants $\overline{\lambda}_1 = \overline{\lambda}_1(p,j)$ and $\overline{\eta}_1 = \overline{\eta}_1(p,j,\varepsilon)$ be defined by

$$\overline{\lambda}_1 = \inf\left\{ \frac{\|u'\|^p_{L^p}}{\|u\|^p_{L^p}} : u \in V\setminus\{0\}, (u(0),u(T)) \in D(j) \right\}, \tag{3.67}$$

$$\overline{\eta}_1 = \overline{\lambda}_1 + \varepsilon. \tag{3.68}$$

Unlike λ_1 (see (3.20)) which can vanish, always $\overline{\lambda}_1 > 0$ by virtue of the Wirtinger inequality. Also, (3.68) and (3.5) yield

$$\frac{\|u\|^p_{L^p}}{p} \leq \frac{\varphi_\varepsilon(u)}{\overline{\eta}_1}, \quad \forall u \in V \cap D(J). \tag{3.69}$$

Theorem 3.16. *Let the Carathéodory function $F : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ be even with respect to the second argument, satisfy (H_1), (H_2) and $\eta_1 > 0$. We assume that*

(i) the set $D(j)$ is a subspace of $\mathbb{R}^N \times \mathbb{R}^N$ and j is even;

(ii) $\limsup_{|x|\to 0} \frac{pF(t,x)}{|x|^p} < \overline{\eta}_1$, uniformly for a.e. $t \in [0,T]$;

(iii) there are constants $\theta > p$ and $k, M > 0$ such that (3.42) *and* (3.53) *hold true.*

Then, problem (3.1) *has infinitely many distinct pairs of solutions.*

Proof. Taking into account Theorem 3.4, basically the proof reduces to verifying that Theorem 3.3 is applicable here. As in the proof of Theorem 3.12, without loss of generality, we may assume (3.54). This implies that (3.55) holds true because $(0,0) \in \partial j(0,0)$. Also, clearly one has (3.56) and Φ_F, ψ_ε are even. By (iii) and Lemma 3.11 we have that I satisfies the (PS) condition on $(W^{1,p}, \|\cdot\|_{\eta_1})$.

To check that condition (i) in Theorem 3.3 is fulfilled, we split $W^{1,p} = \mathbb{R}^N \oplus V$. Using the reasoning in the proof of Theorem 3.12 we find that by (i) there are constants $\sigma \in (0,\overline{\eta}_1)$ and $\rho > 0$, such that

$$-\Phi_F(u) \leq \frac{\overline{\eta}_1 - \sigma}{p}\|u\|^p_{L^p}, \quad \forall u \in W^{1,p},\ \|u\|_{\eta_1} \leq \rho. \tag{3.70}$$

Using (3.70), (3.69) and (3.23) we obtain

$$\Phi_F(u) + \varphi_\varepsilon(u) \geq \frac{\sigma}{\overline{\eta}_1}\varphi_\varepsilon(u) \geq \frac{\sigma}{2p\overline{\eta}_1}\|u\|^p_{\eta_1}, \tag{3.71}$$

provided $u \in V \cap D(J)$, $\|u\|_{\eta_1} \leq \rho$. Taking $\alpha = \sigma\rho^p(2p\overline{\eta}_1)^{-1} > 0$, the estimates (3.71) and (3.55) yield

$$I(u) = \Phi_F(u) + \varphi_\varepsilon(u) + J(u) \geq \alpha, \quad \text{if } u \in V,\ \|u\|_{\eta_1} = \rho,$$

which means that condition (i) in Theorem 3.3 is satisfied with $X_1 = V$.

Next, let the n-dimensional subspace $Y_n(M)$ be defined by means of (3.64) with M entering (3.53). Since on $Y_n(M)$ the norms $\|\cdot\|_{\eta_1}$ and $\|\cdot\|_\infty$ are equivalent, there is a constant $k = k(Y_n(M)) > 0$ such that

$$\|u\|_{\eta_1} \leq k\|u\|_\infty\,, \ \ \forall u \in Y_n(M). \tag{3.72}$$

This, together with (3.8) and (3.23), implies

$$\psi_\varepsilon(u) \leq \frac{k^p}{p}\|u\|^p_\infty\,, \ \ \forall u \in Y_n(M), \tag{3.73}$$

because $J(u) = 0$, $\forall u \in W^{1,p}_0$. Then, by virtue of Lemma 3.15 and using (3.11), (3.9) and (3.73) we obtain

$$I(u) \leq -c_1\|u\|^\theta_\infty + \frac{k^p}{p}\|u\|^p_\infty + c_2\,, \ \ \forall u \in Y_n(M),\ \|u\|_\infty \geq 1.$$

This, together with (3.72) shows that

$$I(u) \to -\infty, \quad \text{ as } \|u\|_{\eta_1} \to +\infty,\ \ u \in Y_n(M)$$

meaning that condition (ii) in Theorem 3.3 is fulfilled with $X_2 = Y_n(M)$ and the proof is complete. □

Remark 3.17. Lemma 3.15 was employed in the above proof to show that under the assumptions of Theorem 3.16 condition (ii) in Theorem 3.3 is satisfied. In the one-dimensional case ($N = 1$) this lemma can be avoided by a specific reasoning, as it is shown in [22].

Remark 3.18. Theorem 3.15 extends to the boundary value problem (3.1) the multiplicity result obtained in [14, Theorem 20] for the p-Laplacian operator associated with homogeneous Dirichlet boundary conditions (also, see [34, Theorem 4.1]). On account of the boundary condition (1.2), Theorem 3.15 also generalizes in certain respects well-known multiplicity results such as [2, Theorem 3.13], [36, Theorem 9.38], [39, Ch.II, Theorem 6.6]), [41, Corollary 5.9 and 5.10].

3.5. **Applications.** We shall show in several examples how Theorems 3.12 and 3.15 can be applied to derive existence and multiplicity results for the ordinary p-Laplacian system

$$-[h_p(u')]' + \varepsilon h_p(u) = \nabla F(t,u), \quad \text{in } [0,T], \tag{3.74}$$

associated with various concrete boundary conditions.

Let $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a convex and Gâteaux differentiable function, with $dg(0,0) = (0,0)$, where dg denotes the differential of g. Also, given a nonempty closed convex cone $K \subset \mathbb{R}^N \times \mathbb{R}^N$, we denote by $N_K(z)$ the normal cone to K at $z \in K$, i.e.,

$$N_K(z) = \{\xi \in \mathbb{R}^N \times \mathbb{R}^N : \ (\xi|\zeta - z) \le 0, \ \forall \zeta \in K\}, \ \ \forall z \in K.$$

The differential system (3.74) is considered to be associated with the boundary condition

$$\begin{aligned} &(u(0), u(T)) \in K, \\ &(h_p(u')(0), -h_p(u')(T)) - dg(u(0), u(T)) \in N_K(u(0), u(T)). \end{aligned} \tag{3.75}$$

We set

$$\eta_{1,K} = \eta_{1,K}(p,\varepsilon) := \varepsilon + \inf\left\{ \frac{\|u'\|_{L^p}^p}{\|u\|_{L^p}^p} : \ u \in W^{1,p} \setminus \{0\}, \ (u(0), u(T)) \in K \right\}.$$

Theorem 3.19. *Let the Carathéodory function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H_1) and (H_2). We assume that $\eta_{1,K} > 0$ and, in addition,*

(i) $\limsup_{|x|\to 0} \dfrac{pF(t,x)}{|x|^p} < \eta_{1,K}$, *uniformly for a.e.* $t \in [0,T]$;

(ii) *there are constants $\theta > p$ and $k, M > 0$ such that* (3.53) *holds and*

$$\langle dg(z), z\rangle \le \theta g(z) + k, \quad \forall z \in K. \tag{3.76}$$

Then problem (3.74), (3.75) *has a nontrivial solution.*

Proof. Since

$$N_K(z) = \partial I_K(z), \quad \forall z \in K,$$

Theorem 3.12 applies with $j(z) = g(z) + I_K(z)$, $\forall z \in \mathbb{R}^N \times \mathbb{R}^N$. □

Remark 3.20. Note that (3.75) allows various choices of g and K, which, among others, recover classical boundary conditions. For instance, if $g = 0$, then the homogeneous Dirichlet and Neumman boundary conditions are obtained by choosing $K = \{(0,0)\}$, respectively $K = \mathbb{R}^N \times \mathbb{R}^N$; taking $K = d$ and $K = d'$ we get the periodic, respectively antiperiodic boundary conditions (see Remark 3.9). In these four cases (3.76) is automatically satisfied with any $\theta \in \mathbb{R}$ and $k = 0$. The Sturm-Liouville boundary conditions (3.35) are obtained with $K = \mathbb{R}^N \times \mathbb{R}^N$ and g defined by (3.34); in this case (3.76) is satisfied with any $\theta \geq p$ and $k = 0$. Therefore, sufficient conditions ensuring the existence of nontrivial solutions of system (3.74) associated with one of these boundary conditions can easily be stated by means of Theorem 3.19. On the other hand, on account of Proposition 3.6, in the Dirichlet and antiperiodic cases ε is allowed to be $= 0$, while in the Neumann and periodic cases ε must be > 0.

If A is a symmetric, positive $(2N \times 2N)$-matrix, $g(z) = 2^{-1}(Az|z)$, $\forall z \in \mathbb{R}^N \times \mathbb{R}^N = K$, then (3.75) reads

$$\begin{pmatrix} (h_p(u')(0) \\ -h_p(u')(T) \end{pmatrix} = A \begin{pmatrix} u(0) \\ u(T) \end{pmatrix}. \tag{3.77}$$

Condition (3.76) is fulfilled with any $\theta \geq 2$ and $k = 0$. As $\eta_{1,\mathbb{R}^N \times \mathbb{R}^N} = \varepsilon$, from Theorem 3.19 we have the following

Corollary 3.21. *Let the Carathéodory function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H_1) and (H_2). We assume that $\varepsilon > 0$ and, in addition,*

(i) $\limsup_{|x| \to 0} \dfrac{pF(t,x)}{|x|^p} < \varepsilon,$ *uniformly for a.e. $t \in [0,T]$;*

(ii) there are constants θ and M, $\theta \geq 2$, $\theta > p$, $M > 0$, such that (3.53) *holds.*

Then problem (3.74), (3.77) *has a nontrivial solution.*

If instead of the convex set K is a subspace $S \subset \mathbb{R}^N \times \mathbb{R}^N$, then (3.75) becomes,

$$(u(0), u(T)) \in S, \quad (h_p(u')(0), -h_p(u')(T)) - dg(u(0), u(T)) \in S^{\perp}, \tag{3.78}$$

where $S^{\perp}$ is the orthogonal complement of S, i.e.,

$$S^{\perp} = \{\xi \in \mathbb{R}^N \times \mathbb{R}^N : \ (\xi|\zeta) = 0, \ \forall \zeta \in S\}.$$

We denote

$$\begin{aligned} \overline{\eta}_{1,S} &= \overline{\eta}_{1,S}(p,\varepsilon) \\ &:= \varepsilon + \inf\Big\{ \frac{\|u'\|^p_{L^p}}{\|u\|^p_{L^p}} : u \in W^{1,p} \setminus \{0\}, \ \int_0^T u = 0, \ (u(0), u(T)) \in S \Big\}. \end{aligned}$$

Theorem 3.22. *Let the Carathéodory function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be even with respect to the second argument, satisfy (H_1), (H_2) and $\eta_{1,S} > 0$. We assume that;*

(i) the convex function $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is even and Gâteaux differentiable;

(ii) $\limsup_{|x| \to 0} \dfrac{pF(t,x)}{|x|^p} < \overline{\eta}_{1,S},$ *uniformly for a.e. $t \in [0,T]$;*

(*iii*) *there are constants* $\theta > p$ *and* $k, M > 0$ *such that* (3.53) *holds and*

$$\langle dg(z), z\rangle \leq \theta g(z) + k, \quad \forall z \in S. \tag{3.79}$$

Then problem (3.74), (3.78) *has infinitely many distinct pairs of solutions.*

Proof. Theorem 3.16 applies with $j(z) = g(z) + I_S(z)$, $\forall z \in \mathbb{R}^N \times \mathbb{R}^N$. $\square$

Let $a, b \in \mathbb{R}$ be two *distinct* real numbers and $r_1, r_2 \in (1, \infty)$. Choosing $S = S_{a,b} := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N \ : \ ax = by\}$ and

$$g(x, y) = \frac{|x|^{r_1}}{r_1} + \frac{|y|^{r_2}}{r_2}, \quad \forall x, y \in \mathbb{R}^N,$$

the boundary conditions (3.78) become

$$\begin{aligned} &a\, u(0) = b\, u(T),\\ &b[h_p(u')(0) - h_{r_1}(u(0))] = a\,[h_p(u')(T) + h_{r_2}(u(T))]. \end{aligned} \tag{3.80}$$

In this case, since $a \neq b$, by virtue of Proposition 3.6 one has $\eta_{1,S_{a,b}} > 0$ and it is easy to see that (3.79) is fulfilled with any $\theta \geq \max\{r_1, r_2\}$ and $k = 0$. Theorem 3.22 can be applied with $\varepsilon = 0$, yielding

Corollary 3.23. *Assume that the Carathéodory mapping* $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ *is even in the second argument, satisfies* (H_1), (H_2), *together with*

(*i*) $\limsup_{x\to 0} \dfrac{pF(t, x)}{|x|^p} < \overline{\eta}_{1,S_{a,b}}$; *uniformly for a.e.* $t \in [0, T]$;

(*ii*) *there are constants* $M > 0$ *and* θ *with* $\theta \geq \max\{r_1, r_2\}$, $\theta > p$, *such that* (3.53) *holds true.*

Then, the system

$$-[h_p(u')]' = \nabla F(t, u), \quad \textit{in } [0, T]$$

has infinitely many distinct pairs of solutions which satisfy (3.80).

As a simple example of a Carathéodory function, even in the second argument, satisfying (H_1), (H_2) and (3.53) is

$$F(t, x) = \frac{m(t)}{q}|x|^q, \ \ \forall t \in [0, T], \ x \in \mathbb{R}^N, \tag{3.81}$$

with $m \in L^\infty([0, T]; \mathbb{R})$, $m > 0$ a.e. on $[0, T]$, and $q > p$. It is easily seen that

$$\limsup_{x\to 0} \frac{pF(t, x)}{|x|^p} = 0 \ \ (< \overline{\eta}_{1,S_{a,b}}), \ \ \text{uniformly for a.e. } t \in [0, T]$$

and, as

$$\nabla F(t, x) = m(t)|x|^{q-2}x,$$

the above existence and multiplicity results can be employed for the system

$$-[h_p(u')]' + \varepsilon h_p(u) = m(t)|u|^{q-2}u, \quad \text{in } [0,T],$$

associated with various concrete boundary conditions.

4. A nonpotential system

In this section we are concerned with the nonpotential system

$$-[h_p(u')]' = f(t,u) + l(t), \quad \text{in } [0,T], \tag{4.1}$$

associated with the boundary condition (1.2).
Here, $l \in L^1$ and $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory mapping satisfying the growth condition:

(H) for each $\rho > 0$ there is some $\alpha_\rho \in L^1([0,T];\mathbb{R})$ such that

$$|f(t,x)| \leq \alpha_\rho(t), \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| \leq \rho.$$

It is our purpose to provide a sufficient simple condition of uniform nonresonance on the right hand term of the system (4.1) which ensures the solvability of the boundary value problem (4.1), (1.2). With this aim we shall use the a priori estimates method.

4.1. **A fixed point approach.** Let $\omega : L^p \times L^p \to \mathbb{R}$ be defined by

$$\omega(u,v) = \int_0^T (h_p(u) - h_p(v)|u-v), \ \ \forall u,v \in L^p.$$

The following technical result will be needed in the sequel.

Lemma 4.1. *(i) If $p \in (1,2]$, then*

$$\|u-v\|_{L^p}^2 \leq \Big(\|u\|_{L^p} + \|v\|_{L^p}\Big)^{2-p} \omega(u,v), \ \ \forall u,v \in L^p. \tag{4.2}$$

(ii) If $p \in (2,\infty)$, then there is a positive constant k, such that

$$k\|u-v\|_{L^p}^p \leq \omega(u,v), \ \ \forall u,v \in L^p. \tag{4.3}$$

Proof. According to Glowinski and Marrocco [19] (also, see [20, p.105-106]), if $p \in (1,2]$, then

$$|y-z|^2 \leq (|y| + |z|)^{2-p}(h_p(y) - h_p(z)|y-z), \ \ \forall y,z \in \mathbb{R}, \tag{4.4}$$

while if $p \in (1,\infty)$, then there is a constant $k > 0$, such that

$$k|y-z|^p \leq (h_p(y) - h_p(z)|y-z), \ \ \forall y,z \in \mathbb{R}. \tag{4.5}$$

(i) For arbitrary $u,v \in L^p$, by virtue of (4.4) it holds

$$\|u-v\|_{L^p}^p = \int_0^T |u-v|^{2\frac{p}{2}} \leq \int_0^T (|u| + |v|)^{(2-p)\frac{p}{2}} (h_p(u) - h_p(v)|u-v)^{\frac{p}{2}}.$$

The case $p = 2$ is immediate. If $p \in (1, 2)$, then since $(|u| + |v|)^{(2-p)\frac{p}{2}} \in L^{\frac{2}{2-p}}([0, T]; \mathbb{R})$, $(h_p(u) - h_p(v)|u - v)^{\frac{p}{2}} \in L^{\frac{2}{p}}([0, T]; \mathbb{R})$, and $\big(\frac{2}{2-p}\big)^{-1} + \big(\frac{2}{p}\big)^{-1} = 1$, by Hölder's inequality we obtain

$$\|u - v\|_{L^p}^p \leq \Big(\int_0^T (|u| + |v|)^p\Big)^{\frac{2-p}{2}} \omega(u, v)^{\frac{p}{2}}$$

$$= \|\,|u| + |v|\,\|_{L^p}^{\frac{p(2-p)}{2}} \omega(u, v)^{\frac{p}{2}} \leq (\|u\|_{L^p} + \|v\|_{L^p})^{\frac{p(2-p)}{2}} \omega(u, v)^{\frac{p}{2}}$$

which yields (4.2).

(ii) Inequality (4.5) gives

$$k\|u - v\|_{L^p}^p = k \int_0^T |u - v|^p \leq \omega(u, v), \quad \forall u, v \in L^p.$$

□

For arbitrarily given $e \in L^1$, let the potential system

$$-[h_p(u')]' + h_p(u) = e(t), \quad \text{in } [0, T], \tag{4.6}$$

be associated with the boundary condition (1.2).

Proposition 4.2. *For each $e \in L^1$ the problem* (4.6), (1.2) *has a unique solution.*

Proof. The *existence* part is immediate from Theorem 3.7, which applies with $\varepsilon = 1$, $F \equiv 0$ and $l = e$.

To prove the *uniqueness*, let u and v be two solutions of (4.6), (1.2). As ∂j is (maximal) monotone, by virtue of (1.2) one obtains

$$\gamma(u, v) := \Big(h_p(u')(0) - h_p(v')(0)|u(0) - v(0)\Big)$$

$$+\Big(-h_p(u')(T) + h_p(v')(T)|u(T) - v(T)\Big) \geq 0. \tag{4.7}$$

On the other hand, multiplying the equality

$$-[h_p(u')]' + [h_p(v')]' + h_p(u) - h_p(v) = 0$$

by $u - v$, integrating over $[0, T]$ and using the integration by parts formula we derive

$$\gamma(u, v) + \omega(u', v') + \omega(u, v) = 0,$$

which by (4.7) and Lemma 4.1 implies that $u = v$. □

The above existence and uniqueness result enables us to define the operator $S : L^1 \to W^{1,p}$ by

$$S(e) := \textit{the unique solution of } (4.6),\ (1.2), \quad \forall e \in L^1.$$

The space $W^{1,p}$ will be equipped with the norm $\|u\|_1$.

Proposition 4.3. *There are positive constants* α_1 *and* α_2 *depending on* p, *such that*
(i) *if* $p \in (1, 2]$, *then*

$$\|S(e_1) - S(e_2)\|_1 \leq \alpha_1(\|S(e_1)\|_1 + \|S(e_2)\|_1)^{2-p}\|e_1 - e_2\|_{L^1}, \ \ \forall e_1, e_2 \in L^1;$$

(ii) *if* $p \in (2, \infty)$, *then*

$$\|S(e_1) - S(e_2)\|_1 \leq \alpha_2\|e_1 - e_2\|_{L^1}^{\frac{1}{p-1}}, \ \ \forall e_1, e_2 \in L^1.$$

Proof. Let us denote $u = S(e_1)$ and $v = S(e_2)$. Similarly, to the uniqueness part of the proof of Proposition 4.2, using (4.7), from the equality

$$-[h_p(u')]' + [h_p(v')]' + h_p(u) - h_p(v) = e_1(t) - e_2(t)$$

we get

$$\gamma(u, v) + \omega(u', v') + \omega(u, v) = \int_0^T (e_1(t) - e_2(t)|u - v),$$

hence,

$$\omega(u', v') + \omega(u, v) \leq \int_0^T (e_1(t) - e_2(t)|u - v) \leq \|u - v\|_C\|e_1 - e_2\|_{L^1}$$

and by the continuity of the embedding $W^{1,p} \subset C$, there is a positive constant k_1, such that,

$$\omega(u', v') + \omega(u, v) \leq k_1\|u - v\|_1 \ \|e_1 - e_2\|_{L^1}. \tag{4.8}$$

(i) Multiplying (4.8) by $(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p} + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}$ one obtains

$$(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p}\omega(u', v') + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}\omega(u, v)$$
$$+(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p}\omega(u, v) + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}\omega(u', v')$$
$$\leq k_1[(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p} + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}]\|u - v\|_1 \ \|e_1 - e_2\|_{L^1}$$

which yields

$$(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p}\omega(u', v') + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}\omega(u, v)$$
$$\leq k_1[(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p} + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}]\|u - v\|_1 \ \|e_1 - e_2\|_{L^1}.$$

From Lemma 4.1 (i) it follows

$$\|u' - v'\|_{L^p}^2 + \|u - v\|_{L^p}^2$$
$$\leq k_1[(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p} + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}]\|u - v\|_1 \ \|e_1 - e_2\|_{L^1}$$

and by the equivalence of the norms on $\mathbb{R}^2$, we can find a positive constant k_2 such that,

$$k_2(\|u' - v'\|_{L^p}^p + \|u - v\|_{L^p}^p)^{\frac{2}{p}}$$

$$\leq k_1[(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p} + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}]\|u-v\|_1 \, \|e_1 - e_2\|_{L^1},$$

yielding

$$\|u-v\|_1 \leq k_3[(\|u'\|_{L^p} + \|v'\|_{L^p})^{2-p} + (\|u\|_{L^p} + \|v\|_{L^p})^{2-p}] \, \|e_1 - e_2\|_{L^1},$$

with $k_3 = k_1/k_2$. Then, since for $r > 0$, one has

$$a^r + b^r \leq 2(a+b)^r, \quad \forall a, b \geq 0,$$

and using again the equivalence of the norms on $\mathbb{R}^2$, we infer

$$\begin{aligned} \|u-v\|_1 &\leq 2k_3(\|u'\|_{L^p} + \|u\|_{L^p} + \|v'\|_{L^p} + \|v\|_{L^p})^{2-p}\|e_1 - e_2\|_{L^1} \\ &\leq \alpha_1[(\|u'\|_{L^p} + \|u\|_{L^p})^{\frac{1}{p}} + (\|v'\|_{L^p} + \|v\|_{L^p})^{\frac{1}{p}}]^{2-p}\|e_1 - e_2\|_{L^1} \\ &= \alpha_1(\|u\|_1 + \|v\|_1)^{2-p}\|e_1 - e_2\|_{L^1}, \end{aligned}$$

with α_1 a positive constant depending on p.

(ii) From (4.8) and Lemma 4.1 (ii) we have

$$k\|u-v\|_1^p \leq k_1\|u-v\|_1 \, \|e_1 - e_2\|_{L^1}$$

which gives

$$\|u-v\|_1 \leq \alpha_2\|e_1 - e_2\|_{L^1}^{\frac{1}{p-1}},$$

with $\alpha_2 = (k_1/k)^{\frac{1}{p-1}}$. □

Theorem 4.4. *The operator S is continuous and bounded from L^1 to $W^{1,p}$.*

Proof. We shall prove that S is bounded. Then the continuity of S is straightforward by virtue of Proposition 4.3.

If $p \in (1, 2]$ then for arbitrary $e \in L^1$ with $\|e\|_{L^1} \leq M$, from Proposition 4.3 (i) we successively have

$$\begin{aligned} \|S(e)\|_1 &\leq \|S(e) - S(0)\|_1 + \|S(0)\|_1 \\ &\leq \alpha_1(\|S(e)\|_1 + \|S(0)\|_1)^{2-p}\|e\|_{L^1} + \|S(0)\|_1 \\ &\leq \alpha_1 M 2^{2-p}(\|S(e)\|_1^{2-p} + \|S(0)\|_1^{2-p}) + \|S(0)\|_1 = k_1\|S(e)\|_1^{2-p} + k_2, \end{aligned}$$

showing that $\|S(e)\|_1 \leq k_3$, with some constant $k_3 > 0$ depending on M.

In the case $\in (2, \infty)$ by Proposition 4.3 (ii) it follows

$$\|S(e)\|_1 \leq \|S(e) - S(0)\|_1 + \|S(0)\|_1 \leq \alpha_2\|e\|_{L^1}^{\frac{1}{p-1}} + \|S(0)\|_1 \,,$$

which clearly implies that S is bounded. □

Now, assume that $l \in L^1$, $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory and satisfies the growth condition (H). We define $g : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$g(t, x) = f(t, x) + h_p(x) + l(t), \ \ \forall t \in [0, T], \ x \in \mathbb{R}^N.$$

It is easy to see that g is also a Carathéodory mapping and satisfies (H). By Theorem 2.5 the Nemytskii operator $N_g : C \to L^1$, $N_g(u) = g(t,u)$, is bounded and continuous. Then, let $Q : W^{1,p} \to W^{1,p}$ be defined by

$$Q = S \circ N_g \circ i, \tag{4.9}$$

where $i : W^{1,p} \to C$ is the identity map.

Theorem 4.5. *The operator Q is completely continuous.*

Proof. It follows by Theorem 4.4 and the compactness of i. □

The following theorem reduces problem (4.1), (1.2) to a fixed point problem for Q.

Theorem 4.6. *A function $u \in W^{1,p}$ is a solution of problem* (4.1)*,* (1.2) *iff it is a fixed point of Q.*

Proof. It is straightforward. □

4.2. **Solvability by a priori estimates.** The constant λ_1 being defined by (3.20), we can state the following existence result.

Theorem 4.7. *Assume that $(0,0) \in \partial j(0,0)$ and the Caratéodory mapping $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the growth condition (H). If*

$$\limsup_{|x|\to\infty} \frac{(f(t,x)|x)}{|x|^p} < \lambda_1, \quad \textit{uniformly for a.e. } t \in [0,T], \tag{4.10}$$

then problem (4.1)*,* (1.2) *has at least one solution for each $l \in L^1$.*

Proof. Taking into account Theorem 4.6, we shall prove that the operator $Q : W^{1,p} \to W^{1,p}$ given by (4.9) has a fixed point. With this aim, according to Schaefer's theorem (see, e.g., [27, Corollary 4.4.12]), it suffices to show that the set

$$\mathcal{M} = \{u \in W^{1,p} \ : \ \exists \lambda \in (0,1] \ \text{ such that } u = \lambda Q(u)\}$$

is bounded in $(W^{1,p}, \|\cdot\|_1)$.

Let $u \in W^{1,p}$ be such that $Q(u) = \lambda^{-1}u$, with some $\lambda \in (0,1]$. By virtue of (4.9) and the definition of S this means

$$-[h_p(u')]' + h_p(u) = \lambda^{p-1}(f(t,u) + h_p(u) + l(t)), \ \text{ for a.e. } t \in [0,T] \tag{4.11}$$

and

$$(h_p(\lambda^{-1}u')(0), -h_p(\lambda^{-1}u')(T)) \in \partial j(\lambda^{-1}u(0), \lambda^{-1}u(T)). \tag{4.12}$$

Multiplying (4.11) by u, integrating over $[0,T]$ and using integration by parts formula, we infer

$$-(h_p(u')(T)|u(T)) + (h_p(u')(0)|u(0)) + \int_0^T |u'|^p + \int_0^T |u|^p$$
$$= \lambda^{p-1}\Big(\int_0^T (f(t,u)|u) + \int_0^T |u|^p + \int_0^T (l(t)|u)\Big).$$

The monotonicity of ∂j, (4.12) and the hypothesis $(0,0) \in \partial j(0,0)$ yield

$$-(h_p(u')(T)|u(T)) + (h_p(u')(0)|u(0)) \geq 0$$

and so

$$\int_0^T |u'|^p + \int_0^T |u|^p \leq \lambda^{p-1}\Big(\int_0^T (f(t,u)|u) + \int_0^T |u|^p + \int_0^T (l(t)|u)\Big)$$
$$\leq \lambda^{p-1}\Big(\int_0^T (f(t,u)|u) + \int_0^T (l(t)|u)\Big) + \int_0^T |u|^p,$$

which implies

$$\|u'\|^p_{L^p} \leq \int_0^T (f(t,u)|u) + \|l\|_{L^1}\|u\|_C. \tag{4.13}$$

We fix a constant k_0 such that

$$\|v\|_C \leq k_0\|v\|_1, \quad \forall v \in W^{1,p};$$

this is known to exist by the continuity of the embedding $W^{1,p} \subset C$. Then (4.13) yield

$$\|u'\|^p_{L^p} \leq \int_0^T (f(t,u)|u) + k_1\|u\|_1, \tag{4.14}$$

with $k_1 = k_0\|l\|_{L^1}$. Further, from (4.10) there are positive constants σ and ρ such that

$$\frac{(f(t,x)|x)}{|x|^p} \leq \lambda_1 - \sigma\ , \quad \text{for a.e. } t \in [0,T],\ \ \forall x \in \mathbb{R}^N \text{ with } |x| > \rho \tag{4.15}$$

and by virtue of the hypothesis (H) there is some $\alpha_\rho \in L^1([0,T];\mathbb{R})$, $\alpha_\rho \geq 0$, such that

$$(f(t,x)|x) \leq \rho\alpha_\rho(t), \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N \text{ with } |x| \leq \rho. \tag{4.16}$$

At this stage we shall treat separately the cases $\lambda_1 > 0$ and $\lambda_1 = 0$.
The case $\lambda_1 > 0$. Without loss of generality we may assume that $\sigma \in (0,\lambda_1)$. From (4.15) and (4.16) it follows

$$(f(t,x)|x) \leq (\lambda_1 - \sigma)|x|^p + \rho\alpha_\rho(t), \quad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^N,$$

which, on account of (4.14), gives

$$\|u'\|_{L^p}^p \le (\lambda_1 - \sigma)\|u\|_{L^p}^p + k_1\|u\|_1 + k_2, \tag{4.17}$$

with $k_2 = \rho\int_0^T \alpha_\rho(t)\mathrm{d}t$. Since $(\lambda^{-1}u(0), \lambda^{-1}u(T)) \in D(j)$, by (3.20) it holds

$$\lambda_1\|u\|_{L^p}^p \le \|u'\|_{L^p}^p. \tag{4.18}$$

Combining (4.17) and (4.18) one obtains

$$\sigma\|u\|_{L^p}^p \le k_1\|u\|_1 + k_2, \tag{4.19}$$

then using (4.17) and (4.19) we infer

$$\begin{aligned}\min\{1,\sigma\}\|u\|_1^p &\le \|u'\|_{L^p}^p + \sigma\|u\|_{L^p}^p \le (\lambda_1-\sigma)\|u\|_{L^p}^p + 2k_1\|u\|_1 + 2k_2\\ &\le (\lambda_1-\sigma)\Big(\frac{k_1}{\sigma}\|u\|_1 + \frac{k_2}{\sigma}\Big) + 2k_1\|u\|_1 + 2k_2.\end{aligned}$$

This shows that there are nonnegative constants $k_3, k_4 \in \mathbb{R}$ such that

$$\|u\|_1^p \le k_3\|u\|_1 + k_4$$

meaning $\|u\|_1 \le k_5$, with some constant k_5 independent on λ.

The case $\lambda_1 = 0$. Using (4.16) and (4.15), with k_2 as above, we successively have:

$$\begin{aligned}\int_0^T (f(t,u)|u) &= \int_{[|u|>\rho]}(f(t,u)|u) + \int_{[|u|\le\rho]}(f(t,u)|u)\\ &\le \int_{[|u|>\rho]}(f(t,u)|u) + k_2 \le -\sigma\int_{[|u|>\rho]}|u|^p + k_2\\ &= -\sigma\|u\|_{L^p}^p + \sigma\int_{[|u|\le\rho]}|u|^p + k_2 \le -\sigma\|u\|_{L^p}^p + T\sigma\rho^p + k_2.\end{aligned}$$

From (4.14) it follows

$$\|u'\|_{L^p}^p \le -\sigma\|u\|_{L^p}^p + k_1\|u\|_1 + k$$

with $k = T\sigma\rho^p + k_2$. This gives

$$\min\{1,\sigma\}\|u\|_1^p \le \|u'\|_{L^p}^p + \sigma\|u\|_{L^p}^p \le k_1\|u\|_1 + k,$$

showing that $\|u\|_1 \le \tilde{k}$, with some constant $\tilde{k}$ independent on λ and the proof is complete. □

Example 4.8. Let $j : \mathbb{R}^2 \times \mathbb{R}^2 \to (-\infty, +\infty]$ be convex, proper and l.s.c. If $(0,0) \in \partial j(0,0)$, then by Theorem 4.7 the nonpotential system

$$\begin{cases} (|u'|^{p-2}u_1')' = |u|^{p-2}u_1 + \sin u_2 + t,\\ (|u'|^{p-2}u_2')' = |u|^{p-2}u_2 + \cos u_1 + 1\end{cases}$$

has at least one solution $u = (u_1, u_2) : [0,T] \to \mathbb{R}^2$ which satisfies the potential boundary condition (1.2).

5. Discontinuous potential systems

Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a mapping satisfying:

($\tilde{H}_1$) for each $\rho > 0$ there is some $\alpha_\rho \in L^1([0,T];\mathbb{R})$ such that, for all $t \in [0,T]$ and $x, y \in B_\rho := \{\eta \in \mathbb{R}^N \ : \ |\eta| \le \rho\}$, it holds

$$|F(t,x) - F(t,y)| \le \alpha_\rho(t)|x-y|;$$

($\tilde{H}_2$) $F(\cdot,x) : [0,T] \to \mathbb{R}$ is measurable for each $x \in \mathbb{R}^N$ and $F(\cdot,0) = 0$.

We consider the discontinuous boundary value problem

$$\begin{cases} -[h_p(u')]' + \varepsilon h_p(u) \in \overline{\partial} F(t,u), & \text{in } [0,T], \\ (h_p(u')(0), -h_p(u')(T)) \in \partial j(u(0), u(T)). \end{cases} \tag{5.1}$$

Here, $\varepsilon \ge 0$ is a constant and $\overline{\partial} F(t,x)$ denotes the generalized Clarke gradient of $F(t,\cdot)$ at $x \in \mathbb{R}^N$.

By a *solution* of the differential inclusions system in problem (5.1) we will understand a function $u : [0,T] \to \mathbb{R}^N$ of class C^1 with $h_p(u')$ absolutely continuous, which satisfies

$$-[h_p(u')(t)]' + \varepsilon h_p(u(t)) \in \overline{\partial} F(t,u(t)), \quad \text{for a.e. } t \in [0,T].$$

In this section we extend the existence results obtained in Corollary 3.8 and Theorem 3.12 to the boundary value problem (5.1). The approach is a variational one and relies on the nonsmooth critical point theory developed in Motreanu and Panagiotopoulos [33].

5.1. **Preliminaries.** Let $(X, \|\cdot\|)$ be a real Banach space and $\Phi : X \to \mathbb{R}$ be a locally Lipschitz function on X. Recall that *the generalized directional derivative* of Φ at $u \in X$ in the direction of $v \in X$ is defined by

$$\Phi^0(u;v) = \limsup_{w \to u, s \searrow 0} \frac{\Phi(w+sv) - \Phi(w)}{s}$$

and *the generalized gradient* (in the sense of Clarke [9]) of Φ at $u \in X$ is defined as being the subset of X^*

$$\overline{\partial}\Phi(u) = \{\eta \in X^* : \ \Phi^0(u;v) \ge \langle \eta, v\rangle, \ \forall v \in X\},$$

where $\langle \cdot,\cdot\rangle$ stands for the duality pairing between X^* and X.

Let $I : X \to (-\infty, +\infty]$ be a functional having the following structure:

$$I = \Phi + \psi, \tag{5.2}$$

where $\Phi : X \to \mathbb{R}$ is locally Lipschitz and $\psi : X \to (-\infty,+\infty]$ is a proper, convex and l.s.c. function. An element $u \in X$ is said to be *a critical point* of the functional I if the inequality below holds

$$\Phi^0(u; v-u) + \psi(v) - \psi(u) \ge 0, \ \forall\, v \in X.$$

A number $c \in \mathbb{R}$ such that $I^{-1}(c)$ contains a critical point will be called *critical value* of I.

Proposition 5.1. (Proposition 2.1 in [15]). *Each local minimum point of I is necessarily a critical point of I.*

The functional I is said *to satisfy the Palais-Smale* (in short, (PS)) *condition* if every sequence $\{u_n\} \subset X$ for which $I(u_n)$ is bounded and

$$\Phi^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$, possesses a convergent subsequence.

Theorem 5.2. (Corollary 3.2 in [33]). *Suppose that I satisfies the (PS) condition and*

(i) $I(0) = 0$ and there exist $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ if $\|u\| = \rho$;

(ii) $I(e) \leq 0$ for some $e \in X$, with $\|e\| > \rho$.

Then, the number $c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\}$, is a critical value of I with $c \geq \alpha$.

The above theorem extends the well-known Mountain Pass Theorem of Ambrosetti and Rabinowitz [2], as well as Theorem 3.2, to the class of the functionals I having the structure (5.2).

5.2. **The variational approach for problem** (5.1)**.** As in Section 3, the functions $\varphi_\varepsilon : W^{1,p} \to \mathbb{R}$ and $J : W^{1,p} \to (-\infty, +\infty]$ are defined by (3.5) and (3.7), respectively. Then, $\psi_\varepsilon : W^{1,p} \to (-\infty, +\infty]$ given by (3.8) is proper, convex and l.s.c. on $W^{1,p}$.

Further, let us assume that the mapping $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies $(\tilde{H}_1)$ and $(\tilde{H}_2)$. Note that for each $\rho > 0$ one has

$$|F(t,x)| \leq \rho \alpha_\rho(t),\ \forall t \in [0,T],\ \forall x \in \mathbb{R}^N,\ \ |x| \leq \rho, \tag{5.3}$$

with $\alpha_\rho \in L^1([0,T];\mathbb{R})$ from $(\tilde{H}_1)$. Viewing (5.3) we can define $\mathcal{F} : C \to \mathbb{R}$, by putting

$$\mathcal{F}(v) = -\int_0^T F(t,v), \quad \forall v \in C$$

and on account of the embedding $W^{1,p} \subset C$, we introduce the functional

$$\Phi_F = \mathcal{F}|_{W^{1,p}}. \tag{5.4}$$

Hypothesis $(\tilde{H}_1)$ ensures that $\mathcal{F}$ is Lipschitz continuous on the bounded subsets of $(C, \|\cdot\|_C)$. Then, by the continuity of the embedding $W^{1,p} \subset C$ it is clear that Φ_F in (5.4) is locally Lipschitz on $(W^{1,p}, \|\cdot\|_\eta)$. Also, note that since $W^{1,p}$ is dense in C, it holds (see [9], p. 47):

$$\overline{\partial}\Phi_F(u) = \overline{\partial}\mathcal{F}(u), \quad \forall u \in W^{1,p}. \tag{5.5}$$

Proposition 5.3. *Assume that* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* $(\tilde{H}_1)$, $(\tilde{H}_2)$ *and let* Φ_F *be defined by* (5.4) *and* $u \in W^{1,p}$. *If* $l \in \overline{\partial}\Phi_F(u)$, *then there is some* $u_l \in L^1$ *such that* $u_l(t) \in \overline{\partial}F(t,u(t))$ *for a.e.* $t \in [0,T]$ *and*

$$\langle l, v\rangle = -\int_0^T (u_l|v), \quad \forall v \in W^{1,p}. \tag{5.6}$$

Proof. From (5.5) we infer that

$$l \in \overline{\partial}\mathcal{F}(u). \tag{5.7}$$

Next, using hypothesis $(\tilde{H}_1)$ for $\rho = \rho(u) := 2\|u\|_C + 1$, there is some $\alpha_\rho \in L^1([0,T];\mathbb{R})$ such that

$$|F(t,x) - F(t,y)| \leq \alpha_\rho(t)|x-y|, \ \forall t \in [0,T] \text{ and } x, y \in B_\rho.$$

Putting $\delta = \delta(u) := \|u\|_C + 1$, it is easy to see that

$$|F(t,x) - F(t,y)| \leq \alpha_\rho(t)|x-y|, \ \forall t \in [0,T] \text{ and } x, y \in u(t) + B_\delta. \tag{5.8}$$

Since C is a closed subspace of L^∞ hypothesis $(\tilde{H}_2)$ and (5.8) enables us to apply Theorem 2.7.3 in [9]. Thus, we obtain

$$\overline{\partial}\mathcal{F}(u) \subset -\int_0^T \overline{\partial}F(t,u),$$

which, together with (5.7) shows that there is a measurable function $u_l : [0,T] \to \mathbb{R}^N$, such that $u_l(t) \in \overline{\partial}F(t,u(t))$ for a.e. $t \in [0,T]$ and (5.6) holds true. To see that $u_l \in L^1$, from (5.8) we derive

$$(u_l(t)|x) \leq F^0(t,u(t);x) \leq \alpha_\rho(t)|x|, \quad \text{for a.e. } t \in [0,T], \forall x \in \mathbb{R}^N.$$

This yields $|u_l(t)| \leq \alpha_\rho(t)$ for a.e. $t \in [0,T]$, and the proof is complete. □

The functional framework of paragraph 5.1 fits the following choices: $X = W^{1,p}$, $\Phi = \Phi_F$ in (5.4), $\psi = \psi_\varepsilon$ in (3.8) and

$$I = \Phi_F + \psi_\varepsilon. \tag{5.9}$$

Theorem 5.4. *Assume that* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* $(\tilde{H}_1)$, $(\tilde{H}_2)$ *and let* $u \in W^{1,p}$. *If* u *is a critical point of the functional* I *in* (5.9), *i.e.,*

$$\Phi_F^0(u; v-u) + \psi_\varepsilon(v) - \psi_\varepsilon(u) \geq 0, \ \ \forall v \in W^{1,p}, \tag{5.10}$$

then u *is a solution of problem* (5.1).

Proof. In (5.10) we take $v = u + sw$, $s > 0$; then dividing by s and letting $s \to 0^+$, we get

$$\Phi_F^0(u;w) + \langle \varphi_\varepsilon'(u), w\rangle + J'(u;w) \geq 0, \ \ \forall w \in W^{1,p}, \tag{5.11}$$

where $J'(u;w)$ is the directional derivative of the convex function J at u in the direction w. By virtue of (3.7), inequality (5.11) becomes

$$\Phi_F^0(u;w) + \langle \varphi'_\varepsilon(u), w\rangle + j'\Big((u(0),u(T));(w(0),w(T))\Big) \geq 0,\ \forall w \in W^{1,p}. \tag{5.12}$$

Since $C_0^\infty \subset W^{1,p}$, from (5.12) we infer

$$\Phi_F^0(u;w) + \langle \varphi'_\varepsilon(u), w\rangle \geq 0, \ \ \forall w \in C_0^\infty.$$

The function $\Phi_F^0(u;\cdot)$ being subadditive and positively homogeneous, by the Hahn-Banach theorem, there exists a linear functional $l : W^{1,p} \to \mathbb{R}$, extending $-\varphi'_\varepsilon(u)|_{C_0^\infty}$, with

$$\Phi_F^0(u;w) \geq \langle l, w\rangle, \ \ \forall w \in W^{1,p}. \tag{5.13}$$

We can find a constant $k > 0$ (depending on u) such that

$$\Phi_F^0(u;w) \leq k\|w\|_\eta, \ \ \forall w \in W^{1,p},$$

which, together with (5.13), yields

$$|\langle l, w\rangle| \leq k\|w\|_\eta, \ \ \forall w \in W^{1,p},$$

showing that $l \in (W^{1,p})^*$ - the dual space of $W^{1,p}$. From (5.13) we infer that $l \in \overline{\partial}\Phi_F(u)$. Using Proposition 5.3 we deduce that there is some $u_l \in L^1$ such that

$$u_l(t) \in \overline{\partial}F(t,u(t)), \text{ a.e. } t \in [0,T] \tag{5.14}$$

and (5.6) holds true. Therefore, one has

$$\langle \varphi'_\varepsilon(u), w\rangle = \int_0^T (u_l|w), \ \ \forall w \in C_0^\infty$$

and taking into account (3.6) it follows

$$\int_0^T (h_p(u')|w') = \int_0^T (-\varepsilon h_p(u) + u_l|w), \ \ \forall w \in C_0^\infty. \tag{5.15}$$

As $u \in W^{1,p}$, we have $h_p(u)$, $h_p(u') \in L^{p'}$, with $1/p + 1/p' = 1$, and (5.15) reads:

$$h_p(u') \in W^{1,1} \tag{5.16}$$

and

$$-[h_p(u')(t)]' = -\varepsilon h_p(u(t)) + u_l(t), \quad \text{for a.e. } t \in [0,T]. \tag{5.17}$$

Since h_p is a homeomorphism, (5.16) ensures that u is of class C^1. This together with (5.17) and (5.14) show that u is a solution of the differential inclusions system in problem (5.1).

Next, we prove that u satisfies the boundary condition. Viewing (5.14), we have

$$(u_l(t)|w(t)) \leq F^0(t,u(t);w(t)), \quad \text{for a.e. } t \in [0,T],\ \forall w \in W^{1,p}. \tag{5.18}$$

Using (5.17) (multiplying by $w \in W^{1,p}$ and integrating from 0 to T) and (5.18), it follows that, $\forall w \in W^{1,p}$,

$$\langle \varphi'_\varepsilon(u), w\rangle \leq \int_0^T F^0(t,u,w) + (h_p(u')(T)|w(T)) - (h_p(u')(0)|w(0)).$$

Then, taking into account the inequality (see [9], p. 80):

$$\Phi_F^0(u;w) \leq \int_0^T (-F)^0(t,u;w)$$

and (5.12), we get

$$\int_0^T (-F)^0(t,u;w) - \int_0^T F^0(t,u;w) + j'((u(0),u(T));(w(0),w(T))) \\ \geq (h_p(u')(0)|w(0)) - (h_p(u')(T)|w(T)), \quad \forall w \in W^{1,p}. \tag{5.19}$$

Now, let $x, y \in \mathbb{R}^N$ be arbitrarily chosen and, for each $n \in \mathbb{N}$, let $w_n \in W^{1,p}$ be defined by

$$w_n(t) = \begin{cases} (1-nt)x & t \in [0,\frac{1}{n}), \\ 0 & t \in [\frac{1}{n}, T-\frac{1}{n}], \\ (n(t-T)+1)y & t \in (T-\frac{1}{n}, T]. \end{cases}$$

From hypothesis $(\tilde{H}_1)$ there is some $\alpha_\rho \in L^1([0,T];\mathbb{R})$ such that (5.8) holds true. We infer that

$$|F^0(t,u(t);\eta)| \leq \alpha_\rho(t)|\eta|, \quad \forall t \in [0,T],\ \forall \eta \in \mathbb{R}^N$$

and taking $\eta = w_n(t)$, one obtains

$$|F^0(t,u(t);w_n(t))| \leq \alpha_\rho(t)\max\{|x|,|y|\}, \quad \forall t \in [0,T],\ \forall n \in \mathbb{N}. \tag{5.20}$$

On the other hand, it is easy to see that

$$F^0(t,u(t);w_n(t)) \to F^0(t,u(t);0) = 0, \quad \forall t \in (0,T).$$

This, together with (5.20), yields

$$\int_0^T F^0(t,u;w_n) \to 0, \quad \text{as } n \to \infty, \tag{5.21}$$

by virtue of Lebesgue's dominated convergence theorem. Similarly, one has

$$\int_0^T (-F)^0(t,u;w_n) \to 0, \quad \text{as } n \to \infty. \tag{5.22}$$

In (5.19) we take $w = w_n$ and letting $n \to \infty$, from (5.21), (5.22) we derive

$$j'((u(0), u(T)); (x, y)) \geq (h_p(u')(0)|x) - (h_p(u')(T)|y),$$

which, as x, y were arbitrarily chosen in $\mathbb{R}^N$, means that

$$(h_p(u')(0), -h_p(u')(T)) \in \partial j(u(0), u(T))$$

holds true. □

5.3. **Existence results.** The following result extends Corollary 3.8 to problem (5.1).

Theorem 5.5. *Assume that the mapping $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies $(\tilde{H}_1)$, $(\tilde{H}_2)$ and*

$$\limsup_{|x| \to \infty} \frac{pF(t, x)}{|x|^p} < \eta_1, \quad \textit{uniformly for a.e. } t \in [0, T].$$

Then the problem (5.1) *has at least one solution.*

Proof (sketch). This follows the outline of the proof of Theorem 3.7. On account of the compactness of the embedding $W^{1,p} \subset C$ and of the continuity of $\mathcal{F}$, the functional Φ_F in (5.4) is sequentially weakly continuous on $W^{1,p}$. Therefore, I in (5.9) is sequentially lower semicontinuous by the weak lower semicontinuity of ψ_ε in (3.8). Then, by the reasoning in the proof of Theorem 3.7 (with $l = 0$), I is coercive on $(W^{1,p}, \|\cdot\|_\eta)$, with some $\eta > 0$. This implies that I is bounded from below and attains its infimum at some $u \in W^{1,p}$, which by Proposition 5.1 and Theorem 5.4 is a solution of problem (5.1). □

Towards the application of Theorem 5.2 to the functional I in (5.9) we have to know when I satisfies the *(PS)* condition. The lemma below provides such a useful sufficient condition.

Lemma 5.6. *Assume that $D(j)$ is closed and $\eta_1 > 0$, together with $(\tilde{H}_1)$, $(\tilde{H}_2)$. If there are constants $\theta > p$ and $k, M > 0$ such that*

$$j'(z; z) \leq \theta j(z) + k, \quad \forall z \in D(j) \tag{5.23}$$

and

$$\theta F(t, x) \leq (\xi|x), \quad \textit{for a.e. } t \in [0, T], \ \forall |x| > M, \ \forall \xi \in \overline{\partial} F(t, x), \tag{5.24}$$

then the functional I in (5.9) *satisfies the (PS) condition on $(W^{1,p}, \|\cdot\|_{\eta_1})$, i.e., every sequence $\{u_n\} \subset W^{1,p}$ for which $I(u_n)$ is bounded and*

$$\Phi_F^0(u_n; , v - u_n) + \psi_\varepsilon(v) - \psi_\varepsilon(u_n) \geq -\varepsilon_n \|v - u_n\|_{\eta_1}, \quad \forall v \in W^{1,p}, \tag{5.25}$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$, possesses a convergent subsequence.

Proof. From (5.23) and (3.7) it follows

$$J(v) - \frac{1}{\theta} J'(v;v) \geq -k_1, \ \ \forall v \in D(J), \tag{5.26}$$

with $k_1 = k/\theta$. We claim that

$$\Phi_F(v) - \frac{1}{\theta}\Phi_F^0(v;v) \geq -k_2, \ \ \forall v \in W^{1,p}, \tag{5.27}$$

with $k_2 = k_2(M, \theta)$ a positive constant.

To see this, let $\overline{\rho} = M + 1$. Using hypothesis $(\tilde{H}_1)$, a straightforward computation shows that there exists some $\alpha_{\overline{\rho}} \in L^1([0,T];\mathbb{R})$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}^N$ with $|x| < \overline{\rho}$, it holds

$$|(\xi|y)| \leq \alpha_{\overline{\rho}}(t)|y|, \ \ \forall \xi \in \overline{\partial}F(t,x), \ \forall y \in \mathbb{R}^N. \tag{5.28}$$

Let $v \in W^{1,p}$ and $l \in \overline{\partial}\Phi_F(v)$ be arbitrarily chosen. By Proposition 5.3 there is some $v_l \in L^1$ such that

$$v_l(t) \in \overline{\partial}F(t, v(t)), \ \text{for a.e. } t \in [0,T] \tag{5.29}$$

and

$$\langle l, w\rangle = -\int_0^T (v_l|w), \ \ \forall w \in W^{1,p}. \tag{5.30}$$

Using (5.3), (5.29), (5.24), (5.28) and (5.30), we estimate

$$\begin{aligned}
-\Phi_F(v) &= \int_{[|v|<\overline{\rho}]} F(t,v) + \int_{[|v|\geq\overline{\rho}]} F(t,v) \leq \int_{[|v|<\overline{\rho}]} \overline{\rho}\alpha_{\overline{\rho}}(t) + \int_{[|v|\geq\overline{\rho}]} F(t,v) \\
&\leq \int_{[|v|<\overline{\rho}]} \overline{\rho}\alpha_{\overline{\rho}}(t) + \frac{1}{\theta}\int_{[|v|\geq\overline{\rho}]} (v_l|v) \\
&= \int_{[|v|<\overline{\rho}]} \overline{\rho}\alpha_{\overline{\rho}}(t) + \frac{1}{\theta}\Big(\int_0^T (v_l|v) - \int_{[|v|<\overline{\rho}]} (v_l|v)\Big) \\
&\leq \int_0^T \Big(\overline{\rho}\alpha_{\overline{\rho}}(t) + \frac{1}{\theta}\overline{\rho}\alpha_{\overline{\rho}}(t)\Big) - \frac{1}{\theta}\langle l, v\rangle
\end{aligned}$$

yielding

$$\Phi_F(v) \geq \frac{1}{\theta}\langle l, v\rangle - k_2, \ \ \forall l \in \overline{\partial}\Phi_F(v)$$

with $k_2 = k_2(M,\theta)$ a positive constant. Therefore, we get

$$\Phi_F(v) \geq \frac{1}{\theta}\max\{\langle l, v\rangle : \ l \in \overline{\partial}\Phi_F(v)\} - k_2 = \frac{1}{\theta}\Phi_F^0(v;v) - k_2, \ \forall v \in W^{1,p},$$

and (5.27) is proved, as claimed.

Next, let $\{u_n\} \subset W^{1,p}$ be a sequence for which $I(u_n)$ is bounded and (5.25) holds true with $\varepsilon_n \to 0$. Clearly, $\{u_n\} \subset D(I) = D(J)$ and there is a constant $k_3 > 0$, such that

$$|I(u_n)| \le k_3, \quad \forall n \in \mathbb{N}. \tag{5.31}$$

In (5.25) we set $v = u_n + su_n$, $s > 0$, then dividing by s and letting $s \to 0^+$, we obtain

$$\Phi_F^0(u_n; u_n) + \psi'_\varepsilon(u_n; u_n) \ge -\varepsilon_n \|u_n\|_{\eta_1}, \quad \forall n \in \mathbb{N}. \tag{5.32}$$

Using (5.31), (3.8) and (5.32) one obtains

$$k_3 + \frac{\varepsilon_n}{\theta}\|u_n\|_{\eta_1} \ge \Phi_F(u_n) + \psi_\varepsilon(u_n) + \frac{\varepsilon_n}{\theta}\|u_n\|_{\eta_1}$$

$$= \Phi_F(u_n) + \varphi_\varepsilon(u_n) + J(u_n) + \frac{\varepsilon_n}{\theta}\|u_n\|_{\eta_1}$$

$$\ge \Phi_F(u_n) - \frac{1}{\theta}\Phi_F^0(u_n; u_n) + \varphi_\varepsilon(u_n) - \frac{1}{\theta}\langle \varphi'_\varepsilon(u_n), u_n\rangle + J(u_n) - \frac{1}{\theta}J'(u_n; u_n)$$

and by virtue of (5.27), (5.26), (3.5), (3.6) and (3.23) we deduce

$$k_1 + k_2 + k_3 + \frac{\varepsilon_n}{\theta}\|u_n\|_{\eta_1} \ge \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|u_n\|_{\eta_1}^p. \tag{5.33}$$

Since $\theta > p$, from (5.33) it follows that $\{\|u_n\|_{\eta_1}\}$ is bounded. By the compactness of the embedding $W^{1,p} \subset C$, the sequence $\{u_n\}$ has a subsequence, again denoted by $\{u_n\}$, such that $u_n \to u$, weakly in $W^{1,p}$ and strongly in C. Similarly to (5.32) we derive

$$\Phi_F^0(u_n; u - u_n) + \psi'_\varepsilon(u_n; u - u_n) \ge -\varepsilon_n\|u - u_n\|_{\eta_1}, \quad \forall n \in \mathbb{N}.$$

This implies

$$\mathcal{F}^0(u_n; u - u_n) + \psi'_\varepsilon(u_n; u - u_n) \ge -\varepsilon_n\|u - u_n\|_{\eta_1}, \quad \forall n \in \mathbb{N}. \tag{5.34}$$

As $\{u_n\}$ is bounded in $W^{1,p}$ and $u_n \to u$, strongly in C, we infer from (5.34) and the upper semicontinuity of $\mathcal{F}^0$ that (3.36) holds true and Lemma 3.10 applies showing that $u_n \to u$, strongly in $W^{1,p}$. □

Theorem 5.7. *Let the mapping $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfy $(\tilde{H}_1)$ and $(\tilde{H}_2)$. We assume that $\eta_1 > 0$ and, in addition,*

(i) the set $D(j)$ is a closed cone and $(0,0) \in \partial j(0,0)$;

(ii) $\limsup_{|x|\to 0} \dfrac{pF(t,x)}{|x|^p} < \eta_1,$ *uniformly for a.e. $t \in [0,T]$;*

(iii) there are constants $\theta > p$ and $k, M > 0$ such that (5.23) holds true and

$$0 < \theta F(t,x) \le (\xi|x),$$

$$\text{for a.e. } t \in [0,T],\ \forall |x| > M, \forall \xi \in \overline{\partial} F(t,x). \tag{5.35}$$

Then problem (5.1) *has a nontrivial solution.*

Proof. We show that Theorem 5.2 is applicable here.

Without loss of generality, we may assume that $j(0,0) = 0$. By Lemma 5.6, condition (iii) ensures that I in (5.9) satisfies the *(PS)* condition. Clearly on has $I(0) = 0$. Then, basically employing the reasoning in the proof of Theorem 3.12, from (i) and (ii) we can find constants α, $\rho > 0$ such that

$$I(u) \geq \alpha, \quad \text{if } \|u\|_{\eta_1} = \rho. \tag{5.36}$$

Consequently, condition (i) in Theorem 5.2 is fulfilled. To prove that condition (ii) in Theorem 5.2 is satisfied, let us firstly note that by virtue of (5.35), for a.e. $t \in [0,T]$ and each $x \in \mathbb{R}^N$, with $|x| > M$, the mapping

$$r(s) = \frac{F(t,sx)}{s^\theta}$$

is increasing on $[1,\infty)$. Indeed, $s \mapsto F(t,sx)$ is locally Lipschitz on $(0,+\infty)$ and a straightforward computation gives

$$\begin{aligned}\overline{\partial}_s\Big(s^{-\theta}F(t,sx)\Big) &\subset \overline{\partial}_s\Big(s^{-\theta}\Big)F(t,sx) + s^{-\theta}\overline{\partial}_s\Big(F(t,sx)\Big)\\ &= s^{-\theta-1}\Big(-\theta F(t,sx) + (\overline{\partial}F(t,sx)|sx)\Big), \ \forall s > 0,\end{aligned} \tag{5.37}$$

where $\overline{\partial}_s$ stands for the generalized Clarke gradient with respect to s. For $1 \leq s_1 < s_2$, by Lebourg's mean value theorem there is some $\tau \in (s_1,s_2)$ such that $r(s_2) - r(s_1) = \xi(s_2 - s_1)$, with some $\xi \in \overline{\partial}_s\big(s^{-\theta}F(t,sx)\big)|_{s=\tau}$; as $\xi \geq 0$ by (5.35) and (5.37), it follows that r is increasing. Therefore, we have

$$F(t,sx) \geq s^\theta F(t,x), \quad \text{for a.e. } t \in [0,T], \forall |x| > M, \ \forall s \geq 1. \tag{5.38}$$

Let $\overline{e} \in C_0^\infty$ be such that $|\overline{e}| > M$ on a set of positive measure. Using (5.3), (5.35) and (5.38), for $s \geq 1$, we obtain

$$\int_0^T F(t,s\overline{e}) = \int_{[|s\overline{e}|\leq M]} F(t,s\overline{e}) + \int_{[|s\overline{e}|>M]} F(t,s\overline{e})$$

$$\geq -\int_0^T M\alpha_M(t) + \int_{[|\overline{e}|>M]} F(t,s\overline{e}) \geq -\int_0^T M\alpha_M(t) + s^\theta \int_{[|\overline{e}|>M]} F(t,\overline{e}),$$

i.e.,

$$\int_0^T F(t,s\overline{e}) \geq -k_1 + s^\theta k_2, \tag{5.39}$$

with constants $k_1 = k_1(M) \geq 0$ and $k_2 = k_2(M,\overline{e}) > 0$. As $J(s\overline{e}) = j(0,0) = 0$, from (5.9), (5.39) and (3.23), we get

$$I(s\overline{e}) \leq -s^\theta k_2 + k_1 + \frac{s^p}{p}\|\overline{e}\|_{\eta_1}^p \to -\infty, \text{ as } s \to +\infty. \tag{5.40}$$

Now, by (5.40), we can choose s_0 sufficiently large to satisfy $I(s_0\overline{e}) \leq 0$ and $\|s_0\overline{e}\|_{\eta_1} > \rho$, with ρ entering in (4.18). This means that condition (ii) in Theorem 2.2 is fulfilled with $e = s_0\overline{e}$.

The functional I has a nontrivial critical point $u \in W^{1,p}$, which by Theorem 5.4 is a nontrivial solution of problem (5.1). □

We conclude this section with a simple example of application of Theorem 5.7. to existence of Filipov solutions [17] (also, see [8], [40]) for differential inclusions. For the sake of simplicity we restrict ourselves to an autonomous case and we consider the following one-dimensional discontinuous boundary value problem:

$$\begin{cases} -(|u'|^{p-2}u')' + \varepsilon|u|^{p-2}u \in [\underline{f}(u), \overline{f}(u)] \quad \text{in } [0,T], \\ ((|u'|^{p-2}u')(0), -(|u'|^{p-2}u')(T)) \in \partial j_1(u(0), u(T)), \end{cases} \tag{5.41}$$

where $\varepsilon \geq 0$, $p \in (1, \infty)$ are fixed, $j_1 : \mathbb{R}\times\mathbb{R} \to (-\infty, +\infty]$ is a proper, convex and l.s.c. function and $f : \mathbb{R} \to \mathbb{R}$ is a locally bounded measurable mapping. As usual, here we have denoted $\underline{f}(s) = \lim_{\delta\searrow 0} \operatorname{essinf}\{f(r) : |s - r| < \delta\}$, $\overline{f}(s) = \lim_{\delta\searrow 0} \operatorname{esssup}\{f(r) : |s - r| < \delta\}$, for $s \in \mathbb{R}$. Letting

$$F_1(s) = \int_0^s f(r)dr,$$

we can state:

Theorem 5.8. *Let $N = 1$ and assume that $\eta_1 > 0$. If*
(i) the set $D(j_1)$ is a closed cone in $\mathbb{R} \times \mathbb{R}$ and $(0, 0) \in \partial j_1(0, 0)$;
(ii) $\displaystyle\limsup_{|s|\to 0} \frac{pF_1(s)}{|s|^p} < \eta_1$;
(iii) there are constants $\theta > p$ and $k, M > 0$ such that $j_1'(z; z) \leq \theta j_1(z) + k$, $\forall z \in D(j_1)$ and

$$0 < \theta F_1(s) \leq sf(s), \ \forall\, |s| > M, \tag{5.42}$$

then problem (5.41) has a nontrivial solution.

Proof. According to [8],

$$\overline{\partial} F_1(s) \subset [\underline{f}(s), \overline{f}(s)], \ \forall s \in \mathbb{R}. \tag{5.43}$$

Then, using (5.42) and (5.43), a straightforward computation shows that $0 < \theta F_1(s) \leq s\,\xi$, $\forall |s| > M$, $\forall \xi \in \overline{\partial} F_1(s)$. Theorem 5.7 applies with $j = j_1$ and $F = F_1$. □

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