Advances in Differential Equations Volume 13, Numbers 3-4 (2008), 229–272

ON THE WIENER TEST FOR DEGENERATE PARABOLIC EQUATIONS WITH NON-STANDARD GROWTH CONDITION

Igor I. Skrypnik[1]
Institute of Applied Mathematics and Mechanics of NAS of Ukraine
R. Luxemburg str. 74, 83114 Donetsk, Ukraine

(Submitted by: Matania Ben-Artzi)

Abstract. We investigate the continuity of solutions for general nonlinear parabolic equations of the form

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\Big) = 0, \qquad 2 < p_1 \le p(x) \le p_2$$

near a nonsmooth boundary of a cylindrical domain. We prove the sufficient and necessary condition for regularity of a boundary point in terms of the $p(x)$-capacity.

1. Introduction

Let Ω be bounded domain in $\mathbb{R}^n$ and u be a solution of equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\Big) = 0, \tag{1.1}$$

with function $p(\cdot)$, which satisfies the conditions

$$1 < p_1 \le p(x) \le p_2, \tag{1.2}$$

$$|p(x) - p(y)| \le \frac{L}{\ln|x-y|^{-1}}, \tag{1.3}$$

for any $x, y \in \Omega,\ |x-y| < \frac{1}{2}$.

In case $p(x) \equiv$ const quasilinear equation (1.1) and natural generalizations have been thoroughly studied. The review of the literature can be found in [13].

A criterion for regularity of a boundary point for the Laplace equation was proved by N. Wiener [23]. A sufficient condition for the regularity of

Accepted for publication: January 2007.
AMS Subject Classifications: 35B05, 35B65, 35J15.
[1]Partially supported by grant INTAS Ref. Nr. 05-1000008-7921 "Investigation of Global Catastrophes for Nonlinear Processes in Continuum Mechanics"

a boundary point was proved in [11] for general equation (1.1) with $p(x) \equiv$ const. A necessary condition for the regularity of a boundary point, which is also a sufficient condition in the case when $p(x) \equiv$ const, was established in [12]. Equation (1.1) belongs to a wide class of elliptic equations with non-standard growth conditions. This area, which has been actively developed over the last decade, is not only of interest in itself. Equation (1.1) and systems of such equations arise in many problems of mathematical physics such as an electrorheological fluid problem, a thermistor problem, a nonlinear Stokes system, etc. (see [16, 25, 26]).

The interior Hölder continuity of solutions of equation (1.1) was obtained in [3, 4, 7, 10]. A Wiener criterion for the regularity of a boundary point for the equation (1.1) in terms of $p(x)$-capacity was proved in [5].

We consider the following equation in $\Omega_T \equiv \Omega \times (0,T)$

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\Big) = 0. \tag{1.4}$$

Equation (1.4) and system of such equations arise when studying certain classes of non-Newtonian fluids such as electrorheological fluids or fluids with viscosity depending on the temperature [2]. In the case when $p(x) \equiv$ const, the interior regularity for general equation of the form (1.4) was obtained by E. Di Benedetto, see [6] where a review of the literature can be also found.

In the case $p = 2$, the sufficient and necessary condition for the regularity of a boundary point was proved in [27] and [21], respectively.

The Wiener criterion for regularity of a boundary point for general equation (1.4) with $p(x) \equiv \text{const} \neq 2$ was established in [17, 18, 19, 20].

Note that the proofs of the regularity of solutions of equation (1.4) are essentially different from that for equations corresponding to the case of linear growth of coefficients with respect to $\frac{\partial u}{\partial x}$.

The difficulties arising here are well known (see [6]) even in the proof of the interior Hölder continuity of solutions. The local boundedness and the interior Hölder continuity of weak solutions of equation (1.4) with $p = p(x,t)$ were obtained in [15].

In this paper we establish the boundary regularity for a weak solution of equation (1.4). It is worth noticing that in the sufficient part our approach is based on new pointwise estimates for some auxiliary solutions and substantially differs from the ideas of R. Gariepy and W. P. Ziemer [11, 27, 28].

The paper is organized as follows. In Section 2 the assumptions and main results are formulated. Auxiliary integral and pointwise estimates for solution are established in Section 3. A sufficient condition for regularity of a boundary point is proved in Section 4. A necessary condition for regularity of a boundary point, which coincides with the sufficient condition, is proved in Section 5.

2. Formulations of assumptions and main results

We denote $v_+ = \max(v,0)$, $v_- = \max(-v,0)$. The spaces

$$W(\Omega) \equiv \Big\{ f(x) \in W_1^1(\Omega) : \int_\Omega \Big| \frac{\partial f(x)}{\partial x} \Big|^{p(x)} dx < \infty \Big\}$$

with norm

$$\|f\|_{W(\Omega)} = \inf \Big\{ \lambda > 0 : \int_\Omega \Big| \frac{1}{\lambda} \frac{\partial f(x)}{\partial x} \Big|^{p(x)} dx \le 1 \Big\} + \inf \Big\{ \lambda > 0 : \int_\Omega \Big| \frac{1}{\lambda} f(x) \Big|^{p(x)} dx \le 1 \Big\},$$

and

$$L(0,T;W(\Omega)) \equiv \Big\{ f(x,t) \in L_1(0,T;W_1^1(\Omega)) : \iint_{\Omega_T} \Big| \frac{\partial f(x,t)}{\partial x} \Big|^{p(x)} dx\, dt < \infty \Big\}$$

with norm

$$\|f\|_{L(0,T;W(\Omega))} = \inf \Big\{ \lambda > 0 : \iint_{\Omega_T} \Big| \frac{1}{\lambda} \frac{\partial f(x,t)}{\partial x} \Big|^{p(x)} dx\, dt \le 1 \Big\} + \inf \Big\{ \lambda > 0 : \iint_{\Omega_T} \Big| \frac{1}{\lambda} f(x,t) \Big|^{p(x)} dx\, dt \le 1 \Big\}$$

are two Banach spaces, see [10]. It is proved in [24] that if $p(x)$ satisfies (1.2), (1.3), then for any $\varphi(x) \in W(\Omega)$ there exists $\varphi_j(x) \in C^\infty(\Omega)$ such that

$$\lim_{j\to\infty} \|\varphi_j - \varphi\|_{W_1^1(\Omega')} = 0 \qquad \text{and}$$
$$\lim_{j\to\infty} \int_{\Omega'} \Big| \frac{\partial \varphi_j(x)}{\partial x} \Big|^{p(x)} dx = \int_{\Omega'} \Big| \frac{\partial \varphi(x)}{\partial x} \Big|^{p(x)} dx \quad \forall\, \Omega' \Subset \Omega. \tag{2.1}$$

We will also consider the class $\overset{\circ}{W}(\Omega)$, defined as the closure(in the sense of the convergence (2.1)) of the set of functions belonging to $W(\Omega)$ that have compact supports contained in Ω.

Let us consider the Banach spaces

$$V(\Omega_T) \equiv L_\infty(0,T;L_2(\Omega)) \cap L(0,T;W(\Omega)),$$
$$\overset{\circ}{V}(\Omega_T) \equiv L_\infty(0,T;L_2(\Omega)) \cap L(0,T;\overset{\circ}{W}(\Omega))$$

with norm

$$\|f\|_{V(\Omega_T)} \equiv \operatorname*{ess\,sup}_{0<t<T} \|f(\cdot,t)\|_{L_2(\Omega)} + \|f\|_{L(0,T;W(\Omega))}.$$

Definition 2.1. *A function $u(x,t)$ is a weak solution of equation (1.4) in Ω_T, if $u(x,t) \in C(0,T;L_2(\Omega)) \cap L(0,T;W(\Omega))$ and for any interval $(t_1,t_2) \subset [0,T)$*

$$\int_\Omega u(x,t)\varphi(x,t)\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \Big\{-u\varphi_t + \sum_{i=1}^n \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\Big\}dx\,dt = 0 \tag{2.2}$$

for all testing functions $\varphi \in W_2^{1,1}(\Omega_T) \cap L(0,T;\overset{\circ}{W}(\Omega))$.

The existence of weak solution of equation (1.4) follows from the classical monotonicity method of J.-L. Lions [14].

Without loss of generality assume later that $\frac{\partial u(x,t)}{\partial t} \in L_2(\Omega_T)$. In the opposite case we introduce the Steklov averages (see [6]).

Definition 2.2. *A boundary point $(x_0,t_0) \in S_T \equiv \partial\Omega \times (0,T)$ is said to be regular for equation (1.4), if for any solution $u(x,t)$ of this equation, satisfying condition*

$$(u(x,t) - f(x,t)) \in \overset{\circ}{V}(\Omega_T) \tag{2.3}$$

with any $f(x,t) \in C(\overline{\Omega}_T) \cap W_2^{1,1}(\Omega_T) \cap L(0,T;W(\Omega))$, there holds

$$\lim_{(x,t)\to(x_0,t_0)} u(x,t) = f(x_0,t_0), \quad (x,t) \in \Omega_T. \tag{2.4}$$

In order to prove our main result, we need to introduce the notion of the capacity of a compact set, [5].

Definition 2.3. *Let $K \subset B_r(x_0) = \{x : |x - x_0| < r\}$ be a compact set*

$$C_p(K, B_r(x_0)) = \inf\Big\{\sum_{i=1}^n \int_{B_r(x_0)} \frac{1}{p(x)}\Big|\frac{\partial\varphi(x)}{\partial x_i}\Big|^{p(x)} dx,\ \varphi(x) \in \mathfrak{M}(K)\Big\}, \tag{2.5}$$

is called the p-capacity of K with respect to $B_r(x_0)$, where $\mathfrak{M}(K) = \{\varphi(x) \in C_0^\infty(B_r(x_0)),\ \varphi(x) \geq 1,\ x \in K\}$.

The next lemma gives an estimate for the capacity of the closed ball $\overline{B}_\rho(x_0)$ with respect to the concentric ball $B_r(x_0)$ (see. [5, p. 1086]).

Lemma 2.1. *Let r be a sufficiently small positive number. Assume that conditions* (1.2), (1.3) *hold. We put $p_0 = p(x_0)$. If $\rho \le \frac{r}{2}$, then*

$$C_p(\overline{B_\rho(x_0)}, B_r(x_0)) \le c \begin{cases} \rho^{n-p_0}, & \text{if } p_0 < n \\ (\ln \frac{r}{\rho})^{1-n}, & \text{if } p_0 = n \\ r^{n-p_0}, & \text{if } p_0 > n. \end{cases} \tag{2.6}$$

If there are constants $\alpha \ge 1, \beta > 0$ with $\frac{r^\alpha}{4} \le \rho \le \frac{r}{2}$ when $p_0 < n$ and $\exp(-\beta r^{-1}) \le \rho \le \frac{r}{2}$ in the case $p_0 = n$, then

$$C_p(\overline{B_\rho(x_0)}, B_r(x_0)) \ge C \begin{cases} \rho^{n-p_0}, & \text{if } p_0 < n \\ (\ln \frac{r}{\rho})^{1-n}, & \text{if } p_0 = n \\ r^{n-p_0}, & \text{if } p_0 > n, \end{cases} \tag{2.7}$$

with a positive constant C, depending only on $n, p_1, p_2, \alpha, \beta$.

We also need the following lemma, see e.g., [22, p. 237].

Lemma 2.2. *Let $\{\alpha_i\}$ be a bounded numerical sequence such that for all $i = 1, 2, \dots$ the inequality*

$$\alpha_i \le A a^i \alpha_{i+1}^\sigma \tag{2.8}$$

holds with positive numbers A, a and σ belonging to the interval $(0, 1)$. Then the estimate

$$\alpha_1 \le c A^{\frac{1}{1-\sigma}} \tag{2.9}$$

is valid with a constant c, depending only on σ and a.

We extend $p(x)$ to the whole $\mathbb{R}^n$, preserving properties (1.2) and (1.3) and put

$$p_0 = p(x_0), \qquad \Delta(r) = [r^{p_0-n} C_p(\overline{B_r(x_0)} \setminus \Omega, B_{2r}(x_0))]^{\frac{1}{p_0-1}}.$$

Our main result reads as follows.

Theorem 2.1. *Let $u(x,t)$ be a bounded weak solution of equation* (1.4) *such that $u = f$ on S_T in the sense* (2.3)*. Let conditions* (1.2), (1.3) *be fulfilled and $2 < p_1 \le p_2 < n$. A boundary point $(x_0, t_0) \in S_T$ is regular if and only if*

$$\int_0^{1/2} \Delta(r) \, \frac{dr}{r} = \infty. \tag{2.10}$$

The case $\frac{2n}{n+2} < p_1 \le 2$ is still open and much more complicated even in the case of the constant $p(x)$ (see [6] and [19, 20]). It will be the object of future investigations.

Remark 2.1. The necessary condition is valid for a more general equation of the form

$$u_t - \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x,t,u,u_x) = a_0(x,t,u,u_x), \tag{2.11}$$

where the functions $a_i(x,t,u,\xi),\ i=\overline{0,n}$ satisfy the structure conditions

$$\sum_{i=1}^{n} a_i(x,t,u,\xi)\xi_i \ge c_1|\xi|^{p(x)}, \quad |a_i(x,t,u,\xi)| \le c_2(|\xi|^{p(x)-1}+1), \quad i=\overline{0,n}. \tag{2.12}$$

Remark 2.2. The sufficient condition is valid for the equation (2.11) where $a_i(x,t,u,\xi),\ i=\overline{0,n}$ satisfy the structure conditions

$$\begin{aligned} &\sum_{i=1}^{n} a_i(x,t,u,\xi)\xi_i \ge c_1(|\xi|^{p(x)}-1), \\ &|a_i(x,t,u,\xi)| \le c_2(|\xi|^{p(x)-1}+1), \qquad i=\overline{0,n}, \end{aligned} \tag{2.13}$$

and the comparison principle holds.

Below we use the notation

$$M = \operatorname{ess\,sup}\{|u(x,t)| : (x,t) \in \Omega_T\}. \tag{2.14}$$

3. Estimates of auxiliary solutions

In this section we will consider some "potential" type solutions. In the elliptic case ($p(x) \equiv$ const) this type of solutions was considered in [22]. In the parabolic case ($p(x) \equiv$ const) such solutions were studied in [17, 19], where the sufficient condition of regularity of a boundary point was obtained.

Let $x_1 \in E$, $E \subset B_R(x_1),\ 0 < R < 1,\ t_1 \in (0,\infty)$. Set

$$p^- = \inf\{p(x),\ x \in B_{8R}(x_1)\}, \qquad p^+ = \sup\{p(x),\ x \in B_{8R}(x_1)\}. \tag{3.1}$$

For any number $0 < k \le 2M$ define

$$\eta(k,R) = \Big(\frac{2^{s_0}}{k\Delta(R)}\Big)^{p_0-2}, \qquad \Delta(R) = \Big(\frac{C_p(\overline{E}, B_{8R}(x_1))}{R^{n-p_0}}\Big)^{\frac{1}{p_0-1}}, \tag{3.2}$$

where s_0 is a fixed positive number depending only on n, p_1, p_2, L, M, which we will determined later.

Define the sets

$$Q_1 = \Omega_R \times \big(t_1 - \eta(k,R)(8R)^{p_0}, t_1 - \frac{1}{2}\eta(k,R)(8R)^{p_0}\big), \quad \Omega_R = B_{8R}(x_1) \setminus E,$$

$Q_2 = B_{8R}(x_1) \times \big(t_1 - \dfrac{3}{4}\eta(k,R)(8R)^{p_0}, t_1\big).$

Let $\psi(x) \in C_0^\infty(B_{8R}(x_1)),\ 0 \le \psi(x) \le 1,\ \psi(x) = 1,\ x \in B_R(x_1)$. Consider the function $v(x,t) = v(x,t;x_1,t_1)$, which belongs to $k\psi(x) + \overset{\circ}{V}(Q_1)$, and satisfies the initial condition

$$v(x, t_1 - \eta(k,R)(8R)^{p_0}) = 0, \qquad x \in \Omega_R, \tag{3.3}$$

and the integral identity

$$\int_{t_1-\eta(k,R)(8R)^{p_0}} \int_{\Omega_R} \Big\{ -v\varphi_\tau + \sum_{i=1}^n \Big|\frac{\partial v}{\partial x_i}\Big|^{p(x)-2} \frac{\partial v}{\partial x_i}\frac{\partial \varphi}{\partial x_i} \Big\} dx\, d\tau = 0 \tag{3.4}$$

for all $t_1 - \eta(k,R)(8R)^{p_0} < t < t_1 - \frac{1}{2}\eta(k,R)(8R)^{p_0}$ and arbitrary $\varphi(x,t) \in W_2^{1,1}(Q_1) \cap L\big(t_1 - \eta(k,R)(8R)^{p_0}, t_1 - \frac{1}{2}\eta(k,R)(8R)^{p_0}; \overset{\circ}{W}(\Omega_R)\big)$. Define

$$w_0(x) = \begin{cases} 0, & x \in E, \\ v(x, t_1 - \frac{3}{4}\eta(k,R)(8R)^{p_0}), & x \in \Omega_R. \end{cases}$$

Consider the function $w(x,t) = w(x,t,x_1,t_1)$, which belongs to $\overset{\circ}{V}(Q_2)$, and satisfies the condition

$$w\big(x, t_1 - \frac{3}{4}\eta(k,R)(8R)^{p_0}\big) = w_0(x), \qquad x \in B_{8R}(x_1) \tag{3.5}$$

and the integral identity

$$\int_{B_{8R}(x_1)} w(x,t)\varphi(x,t)\,dx - \int_{B_{8R}(x_1)} w_0(x)\varphi(x,0)\,dx \tag{3.6}$$
$$+ \int_{t_1-\frac{3}{4}\eta(k,R)(8R)^{p_0}} \int_{B_{8R}(x_1)} \Big\{ w\varphi_\tau + \sum_{i=1}^n \Big|\frac{\partial w}{\partial x_i}\Big|^{p(x)-2} \frac{\partial w}{\partial x_i}\frac{\partial \varphi}{\partial x_i} \Big\} dx\, d\tau = 0$$

for all $t_1 - \frac{3}{4}\eta(k,R)(8R)^{p_0} < t < t_1$ and arbitrary

$$\varphi(x,t) \in W_2^{1,1}(Q_2) \cap L\big(t_1 - \frac{3}{4}\eta(k,R)(8R)^{p_0}, t_1; \overset{\circ}{W}(B_{8R}(x_1))\big).$$

The existence and uniqueness of the solutions $v(x,t)$ and $w(x,t)$ follows from the classical monotonicity method of J. L. Lions [14].

We suppose that $x_1 = 0,\ t_1 = \eta(k,R)(8R)^{p_0}$ and let us write $B_{8R} = B_{8R}(0)$.

We suppose also that the number k satisfies the following additional condition

$$k\Delta(R) \ge 2^{s_0} R^{\frac{\varepsilon}{p_0-2}}, \qquad \varepsilon = \frac{1}{2}\min\Big(\frac{1}{2}, \frac{p_0-2}{p_0-1}, \frac{p_0-2}{2}\Big). \tag{3.7}$$

It follows from (1.2), (1.3), (3.7), that

$$c(\varepsilon, L) \leq \Big(\frac{k\Delta(R)}{2^{s_0}}\Big)^{p^+-p^-} \leq C(M,L), \quad c(\varepsilon, L) \leq k^{p^+-p^-} \leq C(M,L), \tag{3.8}$$

$$c(L) \leq R^{p^+-p^-} \leq C(L), \qquad c(\varepsilon, L) \leq \Delta(R)^{p^+-p^-} \leq C(L). \tag{3.9}$$

Lemma 3.1. *Let conditions* (1.2), (1.3) *and inequality* (3.7) *hold. Then*

$$\operatorname*{ess\,sup}_{0<t<\frac{t_1}{2}} \int_{\Omega_R} v^2(x,t)\,dx + \int_0^{\frac{t_1}{2}} \int_{\Omega_R} \sum_{i=1}^n \Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)} dx\,dt$$
$$\leq C_1 2^{s_0(p_0-2)} k^2 \Delta(R) R^n, \tag{3.10}$$

$$\operatorname*{ess\,sup}_{\frac{t_1}{4}<t<t_1} \int_{B_{8R}} w^2(x,t)\,dx + \int_{\frac{t_1}{4}}^{t_1} \int_{B_{8R}} \sum_{i=1}^n \Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)} dx\,dt$$
$$\leq C_1 2^{s_0(p_0-2)} k^2 \Delta(R) R^n, \tag{3.11}$$

with positive constant C_1*, depending upon* n, p_1, p_2, M, L.

Proof. Let us substitute the function

$$\varphi(x,t) = v(x,t) - kf(x), \qquad f(x) \in \mathfrak{M}(E),$$

in identity (3.4). Integrating for $t \in (0,\tau)$, $\tau \in (0, \frac{t_1}{2})$ and using the Young inequality, we get

$$\int_{\Omega_R} v^2(x,\tau)\,dx + \int_0^\tau \int_{\Omega_R} \sum_{i=1}^n \Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)} dx\,dt$$
$$\leq \gamma k \int_{\Omega_R} v(x,\tau) f(x)\,dx + \gamma t_1 \sum_{i=1}^n \int_{B_{8R}} k^{p(x)} \Big|\frac{\partial f(x)}{\partial x_i}\Big|^{p(x)} dx. \tag{3.12}$$

From (3.8), (3.9) and the definition of the $p(x)$-capacity it follows that

$$t_1 \sum_{i=1}^n \int_{B_{8R}} k^{p(x)} \Big|\frac{\partial f(x)}{\partial x_i}\Big|^{p(x)} dx \leq \gamma 2^{s_0(p_0-2)} k^2 \Delta(R) R^n. \tag{3.13}$$

The Poincaré and Young inequalities, and (3.7), (3.8), (3.9) imply

$$k^2 \int_{B_{8R}} f^2(x)\,dx \leq \gamma k^2 \Big(R^{p^- - n} \sum_{i=1}^n \int_{B_{8R}} \Big|\frac{\partial f(x)}{\partial x_i}\Big|^{p^-} dx\Big)^{\frac{2}{p^-}} R^n \tag{3.14}$$
$$\leq \gamma k^2 \Big(R^{p_0 - n} \sum_{i=1}^n \int_{B_{8R}} \Big|\frac{\partial f(x)}{\partial x_i}\Big|^{p(x)} dx + R^{p^-}\Big)^{\frac{2}{p^-}} R^n \leq \gamma k^2 \Delta(R) R^n.$$

Thus (3.12)–(3.14) yield (3.10).

To prove (3.11), let us substitute the function $\varphi(x,t)=w(x,t)$ in integral identity (3.6). Integrating for $t\in(\frac{1}{4}t_1,\tau)$, for all $\tau\in(\frac{1}{4}t_1,t_1)$, we get

$$\int_{B_{8R}} w^2(x,\tau)\,dx+\gamma\sum_{i=1}^{n}\int_{\frac{1}{4}t_1}^{t_1}\int_{B_{8R}}\Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt\le\int_{\Omega_R} v^2\Big(x,\frac{t_1}{4}\Big)dx. \tag{3.15}$$

Using inequality (3.10), from (3.15) we obtain (3.11). □

Define $g_\mu(x,t)=\min\{g(x,t),\mu\},\ \mu<k$.

Lemma 3.2. *Assume that all the conditions of Lemma* 3.1 *hold, then*

$$\operatorname*{ess\,sup}_{0<t<\frac{t_1}{2}}\int_{\Omega_R} v_\mu^2(x,t)\,dx+\sum_{i=1}^{n}\int_0^{\frac{t_1}{2}}\int_{\Omega_R}\Big|\frac{\partial v_\mu(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt \tag{3.16}$$
$$\le C_2\mu k2^{s_0(p_0-2)}\Delta(R)R^n,$$

$$\operatorname*{ess\,sup}_{\frac{t_1}{4}<t<t_1}\int_{B_{8R}} w_\mu^2(x,t)\,dx+\sum_{i=1}^{n}\int_{\frac{t_1}{4}}^{t_1}\int_{B_{8R}}\Big|\frac{\partial w_\mu(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt \tag{3.17}$$
$$\le C_2\mu k2^{s_0(p_0-2)}\Delta(R)R^n,$$

with positive constant C_2*, depending upon* n,p_1,p_2,L,M*.*

Proof. Testing identity (3.4) by the function $\varphi(x,t)=v_\mu(x,t)-\frac{\mu}{k}v(x,t)$, integrating by $t\in(0,\tau),\ \tau\in(0,\frac{t_1}{2})$, we have

$$\int_{\Omega_R} v_\mu^2(x,t)\,dx+\sum_{i=1}^{n}\int_0^{\tau}\int_{\Omega_R}\Big|\frac{\partial v_\mu(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt$$
$$\le\gamma\frac{\mu}{k}\int_{\Omega_R} v^2(x,\tau)\,dx+\gamma\frac{\mu}{k}\sum_{i=1}^{n}\int_0^{\tau}\int_{\Omega_R}\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt. \tag{3.18}$$

From (3.10), (3.18) inequality (3.16) follows.

To prove (3.17) let us substitute the function $\varphi(x,t)=w_\mu(x,t)$ in the integral identity (3.6). Integrating for $t\in(\frac{1}{4}t_1,\tau)$, for all $\tau\in(\frac{1}{4}t_1,t_1)$, we get

$$\int_{B_{8R}} w_\mu^2(x,\tau)\,dx+\gamma\sum_{i=1}^{n}\int_{\frac{1}{4}t_1}^{\tau}\int_{B_{8R}}\Big|\frac{\partial w_\mu(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt\le\gamma\int_{\Omega_R} v_\mu^2(x,\frac{1}{4}t_1)\,dx. \tag{3.19}$$

Inequality (3.17) follows from (3.16), (3.19). This completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume that conditions of Lemma* 3.1 *hold. Then*

$$\operatorname{ess\,sup}\big\{v(x,t) : (x,t) \in \{4R < |x| < 8R\} \times (0, \frac{t_1}{2})\big\} \le C_3 k 2^{s_0(p_0-2)} \Delta(R), \tag{3.20}$$

$$\operatorname{ess\,sup}\big\{w(x,t) : (x,t) \in \{4R < |x| < 8R\} \times (\frac{t_1}{4}, t_1)\big\} \le C_3 k 2^{s_0(p_0-2)} \Delta(R), \tag{3.21}$$

$$\operatorname{ess\,sup}\big\{w(x,t) : (x,t) \in B_{8R} \times (\frac{5}{8}t_1, \frac{3}{4}t_1)\big\} \le C_3 2^{-s_0 \frac{(n-p_0)(p_0-2)}{p_0+n(p_0-2)}} k\Delta(R), \tag{3.22}$$

with constant C_3*, which depends only upon* n, p_1, p_2, L, M*.*

Proof. We will prove only inequality (3.20). Estimates (3.21), (3.22) can be proved analogously, using initial condition (3.5) and inequality (3.20).

Let us consider two numerical sequences $\rho_i^{(1)} = \frac{8}{3}R(1+2^{-i}), \rho_i^{(2)} = \frac{8}{3}R(3-2^{-i})$, $i = 1, 2, \dots$. Define also the sequence of functions $\xi_i(x) \in C_0^\infty(R^n)$, $0 \le \xi_i(x) \le 1$, $\xi_i(x) = 1$ for $x \in G_i = \{\rho_i^{(1)} < |x| \le \rho_i^{(2)}\}$, $\xi_i(x) = 0$, outside G_{i+1}, $|\frac{\partial \xi_i(x)}{\partial x}| \le c 2^i R^{-1}$. Testing identity (3.4) by function $\varphi(x,t) = v^{l+1}(x,t)\xi_i^s(x), l, s > 0$, integrating by $t \in (0,\tau)$, for all $\tau \in (0, \frac{1}{2}t_1)$ and using the Young inequality, we get

$$\begin{aligned} &\int_{\Omega_R} v^{l+2}(x,\tau)\xi_i^s(x)\,dx + \sum_{i=1}^n \int_0^\tau \int_{\Omega_R} v^l(x,t)\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p^-} \xi_i^s(x)\,dx\,dt \\ &\le \int_{\Omega_R} v^{l+2}(x,\tau)\xi_i^s(x)\,dx + \gamma \sum_{i=1}^n \int_0^\tau \int_{\Omega_R} v^l(x,t)\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)} \xi_i^s(x)\,dx\,dt \\ &\qquad + \gamma \int_0^\tau \int_{\Omega_R} v^l(x,t)\xi_i^s(x)\,dx\,dt \\ &\le \gamma (l+s)^c 2^{ic}(1 + (R^{-1}\mu_i)^{p^+}) \iint_{Q_1} v^l(x,t)\xi_i^{s-p^+}(x)\,dx\,dt. \end{aligned} \tag{3.23}$$

Here, we used the notation $\mu_i = \operatorname{ess\,sup}\{v(x,t) : (x,t) \in G_i \times (0, \frac{t_1}{2})\}$.

The embedding theorem and inequality (3.23) imply

$$\begin{aligned} &\iint_{Q_1} v^l(x,t)\xi_i^s(x)\,dx\,dt \le \gamma\Big(\operatorname*{ess\,sup}_{0<t<\frac{t_1}{2}} \int_{\Omega_R} v^{l\frac{n}{n+p^-}}(x,t)\xi_i^{s\frac{n}{n+p^-}}(x)\,dx\Big)^{\frac{p^-}{n}} \\ &\times \Big(\sum_{i=1}^n \iint_{Q_1} v^{l\frac{n}{n+p^-}-p^-}(x,t)\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p^-} \xi_i^{s\frac{n}{n+p^-}}(x)\,dx\,dt \end{aligned}$$

$$+\sum_{i=1}^{n}\iint_{Q_1} v^{l\frac{n}{n+p^-}}(x,t)\Big|\frac{\partial\xi_i}{\partial x_i}\Big|^{p^-}\xi_i^{s\frac{n}{n+p^-}-p^-}(x)\,dx\,dt\Big)$$

$$\le \gamma(l+s)^c 2^{ic}\mu_i^{(p^--2)\frac{p^-}{n}}[1+(R^{-1}\mu_i)^{p^-}+(R^{-1}\mu_i)^{p^+}]^{\frac{n+p^-}{n}}$$

$$\times\Big(\iint_{Q_1} v^{l\frac{n}{n+p^-}-p^-}(x,t)\xi_i^{s\frac{n}{n+p^-}-p^+}(x)\,dx\,dt\Big)^{\frac{n+p^-}{n}}. \tag{3.24}$$

Define the sequences

$$l_j=(2p^-+n)\Big(\frac{n+p^-}{n}\Big)^j-n-p^-,$$

$$s_j=\Big(2p^++\frac{p^+}{p^-}n\Big)\Big(\frac{n+p^-}{n}\Big)^j-\frac{p^+}{p^-}(n+p^-),$$

$$I_j=\Big(\iint_{Q_1} v^{l_j}(x,t)\xi_i^{s_j}(x)\,dx\,dt\Big)^{(\frac{n}{n+p^-})^j},\qquad j=0,1,2,\dots$$

from (3.24) we get

$$I_j\le\gamma C^{(\frac{n}{n+p^-})^{j-1}}2^{ic(\frac{n}{n+p^-})^j} \tag{3.25}$$

$$\times\big(\mu_i^{(p^--2)\frac{p^-}{n+p^-}}[1+(R^{-1}\mu_i)^{p^-}+(R^{-1}\mu_i)^{p^+}]\big)^{(\frac{n}{n+p^-})^{j-1}}I_{j-1}.$$

From here, using Moser's iteration we obtain

$$\mu_{i-1}^{2p^-+n}\le\gamma 2^{ic}\mu_i^{p^--2}[1+(R^{-1}\mu_i)^{p^-}+(R^{-1}\mu_i)^{p^+}]^{\frac{n+p^-}{p^-}}\int_0^{\frac{t_1}{2}}\int_{G_i}v^{p^-}(x,t)\,dx\,dt. \tag{3.26}$$

It is clear that

$$(R^{-1}\mu_i)^{p^+}\le\gamma(R^{-1}\mu_i)^{p^-},\quad v(x,t)=v_{\mu_i}(x,t),\quad (x,t)\in G_i\times(0,\frac{t_1}{2}).$$

Thus from (3.26), using the Poincaré and Young inequalities, and Lemma 3.2 we conclude that

$$\mu_{i-1}^{2p^-+n}\le\gamma 2^{ic}\mu_i^{p^--2}(1+(R^{-1}\mu_i)^{p^-})^{\frac{n+p^-}{p^-}} \tag{3.27}$$

$$\times(k2^{s_0(p_0-2)}\mu_i\Delta(R)R^{n+p^-}+R^{p^-}mesQ_1).$$

If for some integer i_0

$$\mu_{i_0}\le R\qquad\text{or}\qquad\mu_{i_0}\le\Big(\frac{2^{s_0}}{k\Delta(R)}\Big)^{p_0-1}R^{p_0}, \tag{3.28}$$

then condition (3.7) implies

$$\mu_1 \le \mu_{i_0} \le \gamma k \Delta(R). \tag{3.29}$$

If for all $i \ge 1$ (3.28) is not fulfilled, then from (3.27) it follows that

$$\mu_{i-1}^{2p^- + n} \le \gamma 2^{ic} \mu_i^{2p^- + n - 1} k 2^{s_0(p_0 - 2)} \Delta(R). \tag{3.30}$$

By Lemma 2.2, inequality (3.30) yields

$$\mu_1 \le \gamma k 2^{s_0(p_0-2)} \Delta(R). \tag{3.31}$$

Inequalities (3.29), (3.31) imply (3.20). This completes the proof of Lemma 3.3. □

The main result of this section is the next theorem.

Theorem 3.1. *Assume that all the conditions of Lemma* 3.1 *hold. There exists* $0 < \alpha < 1$ *and an integer* $s_1 > s_0$, *which depend only upon* n, p_1, p_2, L, M, *such that either*

$$mes\Big\{(x,t) \in K_R \times (0, \frac{t_1}{4}) : v(x,t) \le \frac{k\Delta(R)}{2^{s_1}}\Big\} \le (1-\alpha) mes K_R \frac{t_1}{4}, \tag{3.32}$$

or

$$mes\Big\{(x,t) \in K_R \times (\frac{t_1}{4}, \frac{3}{4}t_1) : w(x,t) \le \frac{k\Delta(R)}{2^{s_1}}\Big\} \le (1-\alpha) mes K_R \frac{t_1}{2}, \tag{3.33}$$

where $K_R = \{4R \le |x| \le 7R\}$.

Proof. Let $\psi(x) \in C_0^\infty(R^n)$, $0 \le \psi(x) \le 1$, $\psi(x) = 1$ for $|x| \le 5R$, $\psi(x) = 0$ for $|x| \ge 6R$, $|\frac{\partial \psi(x)}{\partial x}| \le cR^{-1}$. Testing identity (3.4) by $\varphi(x) = v(x,t) - k\psi(x)$, integrating for $t \in (0, \frac{t_1}{4})$, we get

$$\int_{\Omega_R} v^2(x, \frac{t_1}{4}) + \gamma \sum_{i=1}^n \int_0^{\frac{t_1}{4}} \int_{\Omega_R} \Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)} dx\, dt \tag{3.34}$$

$$\le 2k \int_{\Omega_R} v(x, \frac{t_1}{4}) \psi(x)\, dx + \gamma k \sum_{i=1}^n \int_0^{\frac{t_1}{4}} \int_{\Omega_R} \Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)-1} \Big|\frac{\partial \psi(x)}{\partial x_i}\Big|\, dx\, dt.$$

In analogous way, testing identity (3.6) by $\varphi(x,t) = w(x,t) - k\psi(x)$, integrating for $t \in (\frac{t_1}{4}, \frac{3}{4}t_1)$ to obtain

$$\int_{B_8 R} w^2(x, \frac{3}{4}t_1)\, dx - \int_{\Omega_R} v^2(x, \frac{t_1}{4})\, dx \le 2k \int_{B_{8R}} w(x, \frac{3}{4}t_1) \psi(x)\, dx \tag{3.35}$$

$$- 2k \int_{\Omega_R} v(x, \frac{t_1}{4}) \psi(x)\, dx + \gamma k \sum_{i=1}^n \int_{\frac{t_1}{4}}^{\frac{3}{4}t_1} \int_{B_{8R}} \Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)-1} \Big|\frac{\partial \psi(x)}{\partial x_i}\Big|\, dx\, dt.$$

Adding (3.34) and (3.35), we get

$$\sum_{i=1}^{n}\int_0^{\frac{t_1}{4}}\int_{\Omega_R}\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt \le \gamma k\int_{B_{8R}} w(x,\frac{3}{4}t_1)\psi(x)\,dx$$
$$+\gamma k\sum_{i=1}^{n}\int_0^{\frac{t_1}{4}}\int_{\Omega_R}\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)-1}\Big|\frac{\partial \psi(x)}{\partial x_i}\Big|\,dx\,dt$$
$$+\gamma k\sum_{i=1}^{n}\int_{\frac{t_1}{4}}^{\frac{3}{4}t_1}\int_{B_{8R}}\Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)-1}\Big|\frac{\partial \psi(x)}{\partial x_i}\Big|\,dx\,dt. \qquad (3.36)$$

From the definition of the $p(x)$-capacity and from inequalities (3.8), (3.9) we have

$$\sum_{i=1}^{n}\int_0^{\frac{t_1}{4}}\int_{\Omega_R}\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt = \sum_{i=1}^{n}\int_0^{\frac{t_1}{4}}\int_{\Omega_R}k^{p(x)}\Big|\frac{1}{k}\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt$$
$$\ge \gamma k^{p_0}t_1 C_p(E) = \gamma 2^{s_0(p_0-2)}k^2\Delta(R)R^n. \qquad (3.37)$$

Lemma 3.3 implies

$$k\int_{B_8R} w(x,\frac{3}{4}t_1)\psi(x)\,dx \le \gamma 2^{-s_0\frac{(n-p_0)(p_0-2)}{p_0+n(p_0-2)}}k^2\Delta(R)R^n. \qquad (3.38)$$

We estimate the second and third integrals from the right-hand side of (3.36) using the Young inequality:

$$k\sum_{i=1}^{n}\int_0^{\frac{t_1}{4}}\int_{\Omega_R}\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)-1}\Big|\frac{\partial \psi(x)}{\partial x_i}\Big|\,dx\,dt \qquad (3.39)$$
$$\le \gamma k^d\Delta^b(R)\sum_{i=1}^{n}\int_0^{\frac{t_1}{4}}\int_{\widetilde{K}_R}(v(x,t)+R^\delta)^{-d}\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt$$
$$+\gamma\int_0^{\frac{t_1}{4}}\int_{\widetilde{K}_R}R^{-p(x)}k^{p(x)-d(p(x)-1)}\Delta^{-b(p(x)-1)}(R)(v(x,t)+R^\delta)^{d(p(x)-1)}\,dx\,dt,$$

where $\widetilde{K}_R=\{5R\le|x|\le 6R\}$, d is an arbitrary number from $(0,1)$, $b=d-1$, $\delta\in(0,1)$ will be defined later. In the same way

$$k\sum_{i=1}^{n}\int_{\frac{t_1}{4}}^{\frac{3}{4}t_1}\int_{B_{8R}}\Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)-1}\Big|\frac{\partial \psi(x)}{\partial x_i}\Big|\,dx\,dt \qquad (3.40)$$
$$\le \gamma k^d\Delta^b(R)\sum_{i=1}^{n}\int_{\frac{t_1}{4}}^{\frac{3}{4}t_1}\int_{\widetilde{K}(R)}(w(x,t)+R^\delta)^{-d}\Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt$$

$$+\gamma\int_{\frac{t_1}{4}}^{\frac{3}{4}t_1}\int_{\widetilde{K}(R)} R^{-p(x)}k^{p(x)-d(p(x)-1)}\Delta^{-b(p(x)-1)}(R)(w(x,t)+R^{\delta})^{d(p(x)-1)}\,dx\,dt.$$

Let $\eta(x)\in C_0^{\infty}(\mathbb{R}^n)$, $0\le\eta(x)\le 1$, $\eta(x)=1$ in $\widetilde{K}(R)$, $\eta(x)=0$ outside $K(R)$, $|\frac{\partial\eta(x)}{\partial x}|\le cR^{-1}$. Substituting the function $\varphi(x,t)=(v(x,t)+R^{\delta})^{1-d}\eta^{p_2}(x)$ in (3.4) and integrating for $t\in(0,\frac{1}{4}t_1)$, we get

$$\int_{\Omega_R}(v(x,\frac{t_1}{4})+R^{\delta})^{2-d}\eta^{p_2}(x)\,dx$$
$$+\sum_{i=1}^{n}\int_0^{\frac{t_1}{4}}\int_{\Omega_R}(v(x,t)+R^{\delta})^{-d}\Big|\frac{\partial v(x,t)}{\partial x_i}\Big|^{p(x)}\eta^{p_2}(x)\,dx\,dt$$
$$\le\gamma\int_0^{\frac{t_1}{4}}\int_{K(R)}(v(x,t)+R^{\delta})^{-d+p(x)}R^{-p(x)}\,dx\,dt+\gamma R^{n+\delta(2-d)}. \tag{3.41}$$

Testing (3.6) by $\varphi(x,t)=(w(x,t)+R^{\delta})^{1-d}\eta^{p_2}(x)$, integrating for $t\in(\frac{1}{4}t_1,\frac{3}{4}t_1)$, we have

$$\int_{B_{8R}}(w(x,\frac{3t_1}{4})+R^{\delta})^{2-d}\eta^{p_2}(x)\,dx$$
$$+\sum_{i=1}^{n}\int_{\frac{t_1}{4}}^{\frac{3t_1}{4}}\int_{B_{8R}}(w(x,t)+R^{\delta})^{-d}\Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)}\eta^{p_2}(x)\,dx\,dt$$
$$\le\gamma\int_{\Omega_R}(v(x,\frac{t_1}{4})+R^{\delta})^{2-d}\eta^{p_2}(x)\,dx$$
$$+\gamma\int_{\frac{t_1}{4}}^{\frac{3t_1}{4}}\int_{B_{8R}}(w(x,t)+R^{\delta})^{-d+p(x)}R^{-p(x)}\,dx\,dt+\gamma R^{n+\delta(2-d)}. \tag{3.42}$$

From (3.36)–(3.42) it follows that

$$2^{s_0(p_0-2)}k^2\Delta(R)R^n\le\gamma 2^{-s_0\frac{(n-p_0)(p_0-2)}{p_0+n(p_0-2)}}k^2\Delta(R)R^n+\gamma R^{n+\delta(2-d)}$$
$$+\gamma k^d\Delta^b(R)\int_0^{\frac{1}{4}t_1}\int_{K(R)}(v(x,t)+R^{\delta})^{-d+p(x)}R^{-p(x)}\,dx\,dt \tag{3.43}$$
$$+\gamma\int_0^{\frac{1}{4}t_1}\int_{K(R)}R^{-p(x)}k^{p(x)-d(p(x)-1)}\Delta^{-b(p(x)-1)}(v(x,t)+R^{\delta})^{d(p(x)-1)}\,dx\,dt$$
$$+\gamma k^d\Delta^b(R)\int_{\frac{1}{4}t_1}^{\frac{3}{4}t_1}\int_{K(R)}(w(x,t)+R^{\delta})^{-d+p(x)}R^{-p(x)}\,dx\,dt$$

$$+\gamma\int_{\frac{4}{t}1}^{\frac{3}{4}t_1}\int_{K(R)} R^{-p(x)}k^{p(x)-d(p(x)-1)}\Delta^{-b(p(x)-1)}(R)(w(x,t)+R^{\delta})^{d(p(x)-1)}\,dx\,dt$$
$$=\sum_{l=1}^{6} J_l.$$

From condition (3.7) we get

$$J_2 \le \gamma 2^{-2s_0(p_0-2)}k^2\Delta(R)R^{n+\delta-\frac{2\varepsilon}{p_0-2}}. \tag{3.44}$$

Using the Young inequality and (3.7)–(3.9),

$$J_3 \le \gamma k^d R^{-p_0}\Delta^b(R)\int_0^{\frac{1}{4}t_1}\int_{K(R)}(v(x,t)+R^{\delta})^{-d+p_0}\,dx\,dt \tag{3.45}$$
$$+\gamma k^d\Delta^b(R)\int_0^{\frac{1}{4}t_1}\int_{K(R)}(v(x,t)+R^{\delta})^{-d}\,dx\,dt$$
$$\le\gamma k^d R^{-p_0}\Delta^b(R)\int_0^{\frac{1}{4}t_1}\int_{K(R)}v(x,t)^{p_0-d}\,dx\,dt+\gamma 2^{-s_0(p_0-2)}k^2\Delta(R)R^{n+\delta-\frac{\varepsilon(p_0-1)}{p_0-2}}.$$

Analogously,

$$J_4 \le \gamma R^{-p_0}k^{p_0-d(p_0-1)}\Delta^{-b(p_0-1)}(R)\int_0^{\frac{1}{4}t_1}\int_{K(R)}(v(x,t)+R^{\delta})^{d(p_0-1)}\,dx\,dt$$
$$+\gamma k^d\Delta^b(R)\int_0^{\frac{1}{4}t_1}\int_{K(R)}(v(x,t)+R^{\delta})^{-d}\,dx\,dt$$
$$\le\gamma R^{-p_0}k^{p_0-d(p_0-1)}\Delta^{-b(p_0-1)}(R)\int_0^{\frac{1}{4}t_1}\int_{K(R)}v^{d(p_0-1)}(x,t)\,dx\,dt$$
$$+\gamma 2^{-s_0(p_0-2)}k^2\Delta(R)R^{n+\delta d(p_0-1)-\frac{\varepsilon(p_0-1)}{p_0-2}}. \tag{3.46}$$

Thus, from (3.43)–(3.46) we have

$$2^{s_0(p_0-2)}k^2\Delta(R)R^n \le \gamma 2^{-s_0\frac{(n-p_0)(p_0-2)}{p_0+n(p_0-2)}}k^2\Delta(R)R^n \tag{3.47}$$
$$+\gamma 2^{-s_0(p_0-2)}k^2\Delta(R)R^n(R^{\delta-\frac{\varepsilon(p_0-1)}{p_0-2}}+R^{\delta-\frac{2\varepsilon}{p_0-2}}+R^{\delta d(p_0-1)-\frac{\varepsilon d(p_0-1)}{p_0-2}})$$
$$+\gamma k^d R^{-p_0}\Delta^b(R)\int_0^{\frac{1}{4}t_1}\int_{K(R)}v^{p_0-d}(x,t)\,dx\,dt$$
$$+\gamma k^d R^{-p_0}\Delta^b(R)\int_{\frac{1}{4}t_1}^{\frac{3}{4}t_1}\int_{K(R)}w^{p_0-d}(x,t)\,dx\,dt$$

$$+\gamma R^{-p_0}k^{p_0-d(p_0-1)}\Delta^{-b(p_0-1)}(R)\int_0^{\frac14 t_1}\int_{K(R)} v^{d(p_0-1)}(x,t)\,dx\,dt$$
$$+\gamma R^{-p_0}k^{p_0-d(p_0-1)}\Delta^{-b(p_0-1)}(R)\int_{\frac14 t_1}^{\frac34 t_1}\int_{K(R)} w^{d(p_0-1)}(x,t)\,dx\,dt=\sum_{l=7}^{12}J_l.$$

We choose δ from the condition

$$\delta=\max\Big(\varepsilon\frac{p_0-1}{p_0-2},\frac{2\varepsilon}{p_0-2}\Big). \tag{3.48}$$

Then we have for s_0:

$$\gamma\big(2^{-s_0(p_0-2)}+2^{-s_0\frac{(n-p_0)(p_0-2)}{p_0+n(p_0-2)}}\big)=\frac12\,2^{s_0(p_0-2)}, \tag{3.49}$$

hence we obtain

$$J_7+J_8\le\frac12\,2^{s_0(p_0-2)}k^2\Delta(R)R^n. \tag{3.50}$$

Let $A(s_1)=\{(x,t)\in K(R)\times(0,\frac14 t_1):v(x,t)\ge\frac{k}{2^{s_1}}\Delta(R)\}$. We represent the integrals J_9,J_{11} on the right-hand side of (3.47) as the sum of the integrals over the sets $(K(R)\times(0,\frac14 t_1))\setminus A(s_1)$ and $A(s_1)$. Using Lemma 3.3 we have

$$J_9+J_{11}\le\gamma k^dR^{-p_0}\Delta^b(R)(2^{-s_1}k\Delta(R))^{p_0-d}\frac14 t_1 mesK(R)$$
$$+\gamma k^{p_0-d(p_0-1)}R^{-p_0}\Delta^{-b(p_0-1)}(R)(2^{-s_1}k\Delta(R))^{d(p_0-1)}\frac14 t_1 mesK(R)$$
$$+\gamma k^dR^{p_0}\Delta^b(R)\iint_{A(s_1)}v^{p_0-d}(x,t)\,dx\,dt$$
$$+\gamma k^{p_0-d(p_0-1)}R^{-p_0}\Delta^{-b(p_0-1)}(R)\iint_{A(s_1)}v^{d(p_0-1)}(x,t)\,dx\,dt$$
$$\le\gamma(2^{-(s_1-s_0)(p_0-d)}+2^{-s_1d(p_0-1)+p_0-2})k^2\Delta(R)R^n+$$
$$+\gamma(2^{s_0(p_0-2)(p_0-d)}+2^{s_0d(p_0-1)(p_0-2)})k^{p_0}R^{-p_0}\Delta^{p_0-1}(R)mesA(s_1). \tag{3.51}$$

Analogously

$$J_{10}+J_{12}\le\gamma(2^{-(s_1-s_0)(p_0-d)}+2^{-s_1d(p_0-1)+p_0-2})k^2\Delta(R)R^n+$$
$$+\gamma(2^{s_0(p_0-2)(p_0-d)}+2^{s_0d(p_0-1)(p_0-2)})k^{p_0}R^{-p_0}\Delta^{p_0-1}(R)mes\widetilde{A}(s_1), \tag{3.52}$$

where $\widetilde{A}(s_1)=\{(x,t)\in K(R)\times(\frac{t_1}{4},\frac34 t_1):w(x,t)\ge\frac{k}{2^{s_1}}\Delta(R)\}$. We choose s_1 from the condition

$$\gamma(2^{-(s_1-s_0)(p_0-d)}+2^{-s_1d(p_0-1)+s_0(p_0-2)})+$$

$$+\gamma(2^{-(s_1-s_0)(p_0-d)}+2^{-s_1d(p_0-1)+s_0(p_0-2)})=\frac{1}{4}\,2^{s_0(p_0-2)}. \tag{3.53}$$

From (3.47)–(3.53) we get

$$mesA(s_1)+mes\widetilde{A}(s_1)\geq c(s_0)mesK(R)t_1. \tag{3.54}$$

This proves Theorem 3.1. □

4. Sufficient condition for regularity of a boundary point

Fix point $(x_0,t_0)\in S_T$ and consider the cylinder

$$Q^{(\varepsilon)}_{8R}\equiv B_{8R}(x_0)\times(t_0-(8R)^{p_0-\varepsilon},t_0),$$
$$0<t_0-(8R)^{p_0-\varepsilon}<t_0<T,\quad \varepsilon=\frac{1}{2}\min\Big(\frac{1}{2},\frac{p_0-2}{p_0-1},\frac{p_0-2}{2}\Big).$$

Define

$$\mu^+=\operatorname*{ess\,sup}_{Q^{(\varepsilon)}_{8R}\cap\Omega_T}u(x,t),\qquad \mu^-=\operatorname*{ess\,inf}_{Q^{(\varepsilon)}_{8R}\cap\Omega_T}u(x,t),\qquad \omega=\mu^+-\mu^-.$$

For an integer s_*, which we define later, let

$$\Theta(R)=\Big(\frac{2^{s_*}}{\omega\Delta(R)}\Big)^{p_0-2},\quad \Delta(R)=\Big(\frac{C_p(\overline{B_R(x_0)}\setminus\Omega,B_{8R}(x_0))}{R^{n-p_0}}\Big)^{\frac{1}{p_0-1}}, \tag{4.1}$$

and consider the cylinder

$$Q(8R,\Theta(R))\equiv B_{8R}(x_0)\times(t_0-\Theta(R)(8R)^{p_0},t_0), \tag{4.2}$$

if

$$\omega\Delta(R)\geq 2^{s_*}(8R)^{\frac{\varepsilon}{p_0-2}}, \tag{4.3}$$

then $Q(8R,\Theta(R))\subset Q^{(\varepsilon)}_{8R}$.

Construct the cylinders of smaller size $Q(8R,\eta(R))$. We have

$$Q(8R,\Theta(R))=\cup Q(8R,\eta(R)),$$
$$Q(8R,\eta(R))\equiv B_{8R}(x_0)\times(t_1-\eta(R)(8R)^{p_0},t_1), \tag{4.4}$$

where $t_0-\Theta(R)(8R)^{p_0}\leq t_1-\eta(R)(8R)^{p_0},\ t_1\leq t_0$,

$$\eta(R)=\Big(\frac{2^{s_0}}{\omega\Delta(R)}\Big)^{p_0-2},\qquad s_0<s_*, \tag{4.5}$$

s_0 defined by (3.49), which depends only on n,p_1,p_2,L,M. Let

$$\mu^+_f=\sup_{Q(8R,\Theta(R))\cap S_T}f(x,t),\qquad \mu^-_f=\inf_{Q(8R,\Theta(R))\cap S_T}f(x,t).$$

In every cylinder $Q(8R, \eta(R))$ we consider the cylinders

$$Q_1 = \Omega_R \times (t_1 - \eta(R)(8R)^{p_0}, t_1 - \frac{1}{2}\eta(R)(8R)^{p_0}),$$
$$Q - 2 = B_{8R}(x_0) \times (t_1 - \frac{3}{4}\eta(R)(8R)^{p_0}, t_1),$$
$$\Omega_R = B_{8R}(x_0) \setminus \Omega$$

and solutions $v(x,t), w(x,t)$, defined in Paragraph 3 for $k = \frac{\omega}{2}$.

If $\mu^+ - \frac{\omega}{2} \ge \mu_f^+$, then

$$u(x,t) \le \mu^+ - v(x,t), \tag{4.6}$$

on the parabolic boundary of Q_1 and

$$u(x,t) \le \mu^+ - w(x,t) \tag{4.7}$$

on the parabolic boundary of Q_2.

The comparison principle and Theorem 3.1 imply that at least one of the following inequalities holds true

$$mes\Big\{(x,t) \in Q(8R, \eta(R)) \setminus Q(8R, \tfrac{3}{4}\eta(R)) \cap \Omega_T : u(x,t) \ge \mu^+ - \frac{\omega}{2^{s_1}}\Delta(R)\Big\}$$
$$\le (1-\alpha) mes(Q(8R, \eta(R)) \setminus Q(8R, \tfrac{3}{4}\eta(R))) \tag{4.8}$$

or

$$mes\Big\{(x,t) \in Q(8R, \tfrac{3}{4}\eta(R)) \setminus Q(8R, \tfrac{1}{4}\eta(R)) \cap \Omega_T : u(x,t) \ge \mu^+ - \frac{\omega}{2^{s_1}}\Delta(R)\Big\}$$
$$\le (1-\alpha) mes(Q(8R, \tfrac{3}{4}\eta(R)) \setminus Q(8R, \tfrac{1}{4}\eta(R))). \tag{4.9}$$

If $\mu^- + \frac{\omega}{2} \le \mu_f^-$, then

$$u(x,t) \ge \mu^- + v(x,t), \tag{4.10}$$

on the parabolic boundary of Q_1 and

$$u(x,t) \ge \mu^- + w(x,t) \tag{4.11}$$

on the parabolic boundary of Q_2. The comparison principle and Theorem 3.1 imply that at least one of the following inequalities holds true

$$mes\Big\{(x,t) \in Q(8R, \eta(R)) \setminus Q(8R, \tfrac{3}{4}\eta(R)) \cap \Omega_T : u(x,t) \le \mu^- + \frac{\omega}{2^{s_1}}\Delta(R)\Big\}$$
$$(1-\alpha) mes(Q(8R, \eta(R)) \setminus Q(8R, \tfrac{3}{4}\eta(R))) \tag{4.12}$$

or

$$mes\Big\{(x,t) \in Q(8R, \tfrac{3}{4}\eta(R)) \setminus Q(8R, \tfrac{1}{4}\eta(R)) \cap \Omega_T : u(x,t) \le \mu^- + \frac{\omega}{2^{s_1}}\Delta(R)\Big\}$$
$$(1-\alpha) mes(Q(8R, \tfrac{3}{4}\eta(R)) \setminus Q(8R, \tfrac{1}{4}\eta(R))). \tag{4.13}$$

Combining inequalities (4.8), (4.9), (4.12), (4.13) we obtain:

Theorem 4.1. *Assume that all the condition of Theorem* 2.1 *hold. Suppose also that inequality* (4.3) *holds. If*

$$\mu^+ - \frac{\omega}{2} \geq \mu_f^+, \tag{4.14}$$

then

$$mes\Big\{(x,t) \in Q(8R,\eta(R)) \setminus Q(8R,\tfrac{1}{4}\eta(R)) \cap \Omega_T : u(x,t) \geq \mu^+ - \frac{\omega}{2^{s_1}}\Delta(R)\Big\}$$
$$\leq (1-\frac{\alpha}{3}) mes(Q(8R,\eta(R)) \setminus Q(8R,\tfrac{1}{4}\eta(R))), \tag{4.15}$$

for all cylinders $Q(8R,\eta(R)) \subset Q(8R,\Theta(R))$. *If*

$$\mu^- + \frac{\omega}{2} \leq \mu_f^-, \tag{4.16}$$

then

$$mes\Big\{(x,t) \in Q(8R,\eta(R)) \setminus Q(8R,\tfrac{1}{4}\eta(R)) \cap \Omega_T : u(x,t) \leq \mu^- + \frac{\omega}{2^{s_1}}\Delta(R)\Big\}$$
$$\leq (1-\frac{\alpha}{3}) mes(Q(8R,\eta(R)) \setminus Q(8R,\tfrac{1}{4}\eta(R))), \tag{4.17}$$

for all cylinders $Q(8R,\eta(R)) \subset Q(8R,\Theta(R))$, *with number* $s_1 < s_*$ *defined in* (3.53), $0<\alpha<1$ *defined in* (3.54), *which depend only on* n, p_1, p_2, L, M, s_0.

Now we are in the situation which is analogous to [15, Proposition 4.10, p. 805] and [6, the second alternative, Chapter III]. Next theorem follows from Theorem 3.1.

Theorem 4.2. *Assume that all the conditions of Theorem* 2.1 *hold. If* (4.3), (4.14) *hold, then*

$$u(x,t) \leq \mu^+ - \frac{\omega}{2^{s_*}}\Delta(R), \qquad \forall\,(x,t) \in Q\Big(\frac{R}{2},\Theta(R)\Big) \cap \Omega_T. \tag{4.18}$$

If (4.3), (4.16) *hold, then*

$$u(x,t) \geq \mu^- + \frac{\omega}{2^{s_*}}\Delta(R), \qquad \forall\,(x,t) \in Q\Big(\frac{R}{2},\Theta(R)\Big) \cap \Omega_T, \tag{4.19}$$

where s_* *is sufficiently large positive number, which depends only on* n, p_1, p_2, M, L.

We give a sketch of the proof of Theorem 4.2. Let us assume, for example, that inequality (4.14) is true. Then the levels

$$k_s = \mu^+ - \frac{\omega}{2^s}\Delta(R), \qquad \forall\, s \geq s_1 \tag{4.20}$$

satisfy the condition

$$k_s \geq \mu_f^+. \tag{4.21}$$

Since $(u-k_s)_+$ vanishes on $Q(8R,\Theta(R))\cap S_T$, we may extend it to the whole $Q(8R,\Theta(R))$ by setting it to be zero outside Ω_T within the box $Q(8R,\Theta(R))$. Also we take a cutoff function $\zeta(x,t)$ vanishing on the parabolic boundary $Q(8R,\Theta(R))$. Taking into account these remarks, we test the integral identity (2.2) by $\varphi=(u-k_s)_+\zeta^{p_2}(x,t)$, The next lemma is proved in the same way as [6, Chapter II] or inequality (4.18) [15].

Lemma 4.1. *Let $u(x,t)$ be a bounded weak solution of equation* (1.4) *in Ω_T. Then for every cylinder $Q(\rho,\Theta)\subset Q(8R,\Theta(R))$ and for every level k_s satisfying* (4.21) *the following inequality holds*

$$\sup_{t_0-\Theta\rho^{p_0}\le t\le t_0}\int_{B_\rho(x_0)}(u-k_s)_+^2\zeta^{p_2}(x,t)\,dx \tag{4.22}$$
$$+\sum_{i=1}^n\iint_{Q(\rho,\Theta)}\Big|\frac{\partial}{\partial x_i}(u-k_s)_+\Big|^{p^-}\zeta^{p_2}(x,t)\,dx\,dt$$
$$\le\int_{B_\rho(x_0)}(u-k_s)_+^2\zeta^{p_2}(x,t_0-\Theta\rho^{p_0})\,dx+\gamma\iint_{Q(\rho,\Theta)}(u-k_s)_+^2\Big|\frac{\partial\zeta(x,t)}{\partial t}\Big|\,dx\,dt$$
$$+\gamma\sum_{i=1}^n\iint_{Q(\rho,\Theta)}(u-k_s)_+^{p^+}\Big|\frac{\partial\zeta(x,t)}{\partial x_i}\Big|^{p^+}dx\,dt+\gamma\int_{t_0-\Theta\rho^{p_0}}^{t_0}mesA_{k_s,\rho}(t)\,dt,$$

where $p^-=\inf_{B_\rho(x_0)}p(x)$, $p^+=\sup_{B_\rho(x_0)}p(x)$,

$$A_{k_s,\rho}(t)=\{x\in B_\rho(x_0):u(x,t)>k_s\}.$$

Let us consider the logarithmic function

$$\Psi(x,t)=\ln_+\frac{H_s}{H_s-(u-k_s)_++a},\quad 0<a<H_s,$$
$$H_s=\operatorname*{ess\,sup}_{Q(\rho,\Theta)}(u-k_s)_+.$$

In the cylinder $Q(\rho,\Theta)$ we take a cutoff function ζ independent of $t\in(t_0-\Theta\rho^{p_0},t_0)$.

Similarly to Proposition 3.2 [6, Chapter II] or Lemma 4.6 [15] we get:

Lemma 4.2. *Let $u(x,t)$ be a bounded weak solution of equation* (1.4) *in Ω_T, then for every cylinder $Q(\rho,\Theta)\subset Q(8R,\Theta(R))$ and for every level k_s satisfying* (4.21) *next inequality holds*

$$\sup_{t_0-\Theta\rho^p\le t\le t_0}\int_{B_\rho(x_0)}\Psi^2(x,t)\zeta^{p_2}(x)\,dx\le\int_{B_\rho(x_0)}\Psi^2(x,t_0-\Theta\rho^p)\zeta^{p_2}(x)\,dx$$

$$+\gamma\sum_{i=1}^{n}\iint_{Q(\rho,\Theta)}\Psi|\Psi_u|^{2-p^+}\Big|\frac{\partial\zeta(x)}{\partial x_i}\Big|^{p^+}dx\,dt+\gamma\int_{t_0-\Theta\rho^{p_0}}^{t_0} mes A_{k_s,\rho}(t)\,dt. \tag{4.23}$$

Application of Theorem 4.1 yields

Lemma 4.3. *Let* $Q(8R,\eta(R))\subset Q(8R,\Theta(R))$ *be fixed and let* (4.3), (4.14) *hold. There exists a time level*

$$t_*\in\Big[t_1-\eta(R)(8R)^{p_0},t_1-\frac{1}{4}\eta(R)(8R)^{p_0}-\frac{\alpha}{8}\eta(R)(8R)^{p_0}\Big],$$

such that

$$mes\{x\in B_{8R}(x_0):u(x,t_*)\geq\mu^+-\frac{\omega}{2^{s_1}}\Delta(R)\}\leq\frac{1-\frac{\alpha}{3}}{1-\frac{\alpha}{6}}mesB_{8R}(x_0). \tag{4.24}$$

Proof. If not, for all $t\in[t_1-\eta(R)(8R)^{p_0},t_1-\frac{1}{4}\eta(R)(8R)^{p_0}-\frac{\alpha}{8}\eta(R)(8R)^{p_0}]$

$$mes\{x\in B_{8R}(x_0):u(x,t)\geq\mu^+-\frac{\omega}{2^{s_1}}\Delta(R)\}>\frac{1-\frac{\alpha}{3}}{1-\frac{\alpha}{6}}mesB_{8R}(x_0)$$

and

$$\begin{aligned}&mes\{(x,t)\in Q(8R,\eta(R))\setminus Q(8R,\frac{1}{4}\eta(R)):u(x,t)\geq\mu^+-\frac{\omega}{2^{s_1}}\Delta(R)\}\\&\geq\int_{t_1-\eta(R)(8R)^{p_0}}^{t_1-\frac{1}{4}(1-\frac{\alpha}{2})\eta(R)(8R)^{p_0}} mes\Big\{x\in B_{8R}(x_0):u(x,t)\geq\mu^+-\frac{\omega}{2^{s_1}}\Delta(R)\Big\}dt\\&>(1-\frac{\alpha}{3})mes(Q(8R,\eta(R))\setminus Q(8R,\frac{1}{4}\eta(R))),\end{aligned}$$

contradicting (4.15). □

Lemma 4.4. *Assume that* (4.3), (4.14) *hold. There exists a positive integer* $s_2>s_1$*, which depend only on* n,p_1,p_2,L,M,s_0,s_1 *such that*

$$mes\Big\{x\in B_{8R}(x_0):u(x,t)\geq\mu^+-\frac{\omega}{2^{s_2}}\Delta(R)\Big\}\leq\Big(1-\frac{1}{2}\Big(\frac{\alpha}{3}\Big)^2\Big)mesB_{8R}(x_0) \tag{4.25}$$

for all $t\in[t_1-\frac{1}{4}(1-\frac{\alpha}{2})\eta(R)(8R)^{p_0},t_1]$.

Proof. Consider the logarithmic inequality (4.23) written over the box $B_{8R}(x_0)\times(t_*,t_1)$ for the function $(u-k_{s_1})_+$ for the level $k_{s_1}=\mu^+-\frac{\omega}{2^{s_1}}\Delta(R)$. As for the number a in definition of the function Ψ, we take

$$a=\frac{\omega}{2^{s_1+j}}\Delta(R),\quad j>0\quad\text{to be chosen.}$$

The cutoff function $\zeta(x)$ is taken so that $\zeta(x) = 1$ in the ball $B_{8R(1-\sigma)}(x_0)$, $\sigma \in (0,1)$, $\zeta(x) = 0$ outside $B_{8R}(x_0)$, and $|\frac{\partial \zeta(x)}{\partial x}| \le \gamma(\sigma R)^{-1}$. With these choices, inequality (4.23) yields, for all $t \in (t_*, t_1)$,

$$\int_{B_{8R(1-\sigma)}(x_0)} \Psi^2(x,t)\,dx \le \int_{B_{8R}(x_0)} \Psi^2(x,t_*)\,dx$$

$$+\gamma \sum_{i=1}^{n} \int_{t_*}^{t} \int_{B_{8R}(x_0)} \Psi |\Psi'_u|^{2-p^+} \Big| \frac{\partial \zeta(x)}{\partial x_i} \Big|^{p^+} dx\,dt + \gamma(t_1 - t_*) mes B_{8R}(x_0)$$

$$\le j^2 \ln^2 2 \frac{1-\frac{\alpha}{3}}{1-\frac{\alpha}{6}} mes B_{8R}(x_0) + \gamma(\sigma R)^{-p_0} j \Big(\frac{\omega \Delta(R)}{2^{s_1}}\Big)^{p_0-2} (t_1 - t_*) mes B_{8R}(x_0)$$

$$+\gamma(t_1 - t_*) mes B_{8R}(x_0) \le \Big(j^2 \ln^2 2 \frac{1-\frac{\alpha}{3}}{1-\frac{\alpha}{6}} + \gamma j \sigma^{-p_0}\Big) mes B_{8R}(x_0), \quad (4.26)$$

where we use $s_1 > s_0$, $\Psi \le j \ln 2$, $|\Psi'_u|^{2-p^+} \le \gamma(\frac{\omega\Delta(R)}{2^{s_1}})$ and (3.8), (3.9), (4.3). The left hand side of (4.26) is estimated over the smaller set

$$\Big\{x \in B_{8R(1-\sigma)}(x_0) : u(x,t) \ge \mu^+ - \frac{\omega \Delta(R)}{2^{s_1+j}}\Big\}.$$

On such set we estimate

$$\Psi^2 \ge (j-1)^2 \ln^2 2.$$

Therefore, (4.26) implies

$$mes \Big\{x \in B_{8R}(x_0) : u(x,t) \ge \mu^+ - \frac{\omega \Delta(R)}{2^{s_1+j}}\Big\}$$

$$\le \Big[\Big(\frac{j}{j-1}\Big)^2 \frac{1-\frac{\alpha}{3}}{1-\frac{\alpha}{6}} + \frac{\gamma}{\sigma^{p_0} j} + n\sigma\Big] mes B_{8R}(x_0) \quad (4.27)$$

for all $t \in [t_*, t_1]$. Choose σ so small that $\sigma n \le \frac{1}{4}(\frac{\alpha}{3})^2$ and then j so large that

$$\Big(\frac{j}{j-1}\Big)^2 \le \Big(1 - \frac{\alpha}{6}\Big)(1 + \frac{\alpha}{3}) \quad \text{and} \quad \frac{\gamma}{\sigma^{p_0} j} \le \frac{1}{4}\Big(\frac{\alpha}{3}\Big)^2.$$

Then for such a choice of j the lemma follows with $s_2 = s_1 + j$. □

Now if we recall Theorem 4.1, we get that (4.15) holds for all cylinders of the type $Q(8R, \eta(R))$, the conclusion of Lemma 4.4 holds true for all time levels satisfying

$$t \ge t_0 - (1 - 2^{(s_0 - s_*)(p_0 - 2)})\Theta(R)(8R)^{p_0}.$$

If the number s_* chosen sufficiently large, we deduce

Corollary 4.1. *Let* (4.3), (4.14) *hold. For all* $t \in [t_0 - \frac{1}{2}\Theta(R)(8R)^{p_0}, t_0]$,

$$mes\Big\{x \in B_{8R}(x_0) : u(x,t) \geq \mu^+ - \frac{\omega\Delta(R)}{2^{s_2}}\Big\} \leq \Big(1 - \frac{1}{2}\Big(\frac{\alpha}{3}\Big)^2\Big) mes B_{8R}(x_0). \tag{4.28}$$

Lemma 4.5. *Let* (4.3), (4.14) *hold. For every* $\nu \in (0,1)$ *there exists a positive number* $s_* > s_2$*, depending only on* $n, p_1, p_2, L, M, s_0, s_1$*, such that*

$$mes\Big\{Q\Big(8R, \frac{1}{2}\Theta(R)\Big) : u(x,t) \geq \mu^+ - \frac{\omega}{2^{s_*}}\Delta(R)\Big\} \leq \nu mes Q\Big(8R, \frac{1}{2}\Theta(R)\Big). \tag{4.29}$$

Proof. Consider the local estimate (4.22) for the functions $(u-k_s)_+$, $s_2 \leq s \leq s_*$ and s_* is to be chosen. The levels k_s defined in (4.20) and satisfy (4.21). We take a cutoff function $\zeta(x,t)$ that equals one on $Q(8R, \frac{1}{2}\Theta(R))$, vanishes on the parabolic boundary $Q(8R,\Theta(R))$ and such that

$$\Big|\frac{\partial\zeta(x,t)}{\partial x}\Big| \leq \gamma R^{-1}, \qquad \Big|\frac{\partial\zeta(x,t)}{\partial t}\Big| \leq \gamma\Theta^{-1}(R)R^{-p_0},$$

we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} \iint_{Q(8R,\frac{1}{2}\Theta(R))} \Big|\frac{\partial}{\partial x_i}(u-k_s)_+\Big|^{p^-} dx\,dt \\
&\leq \gamma \sum_{i=1}^{n} \iint_{Q(8R,\Theta(R))} (u-k_s)_+^{p^+} \Big|\frac{\partial\zeta}{\partial x_i}\Big|^{p^+} dx\,dt \\
&+\gamma \iint_{Q(8R,\Theta(R))} (u-k_s)_+^2 \Big|\frac{\partial\zeta}{\partial t}\Big|^2 dx\,dt + \gamma mes Q(8R,\Theta(R)) \\
&\leq \frac{\gamma}{R^{p_0}}\Big(\frac{\omega\Delta(R)}{2^s}\Big)^{p_0} mes Q(8R,\Theta(R)),
\end{aligned} \tag{4.30}$$

here we use also (3.8), (3.9), (4.3).

Next we use Lemma 2.2 [6, Chapter I] applied to the function $u(\cdot,t)$ for all time levels $t_0 - \frac{\Theta(R)}{2}(8R)^{p_0} \leq t \leq t_0$ and for the levels

$$k = k_s, \quad l = k_{s+1}, \quad l - k = \frac{\omega}{2^{s+1}}\Delta(R),$$

where k_s defined in (4.20). Notice that by virtue of Corollary 4.1 we have

$$mes\Big\{x \in B_{8R}(x_0) : u(x,t) < \mu^+ - \frac{\omega}{2^s}\Delta(R)\Big\} \geq \frac{1}{2}\Big(\frac{\alpha}{3}\Big)^2 mes B_{8R}(x_0) \tag{4.31}$$

for all $t \in [t_0 - \frac{1}{2}\Theta(R)(8R)^{p_0}, t_0]$. Applying Lemma 2.2 [6, Chapter I] in this setting we have

$$\frac{\omega}{2^{s+1}}\Delta(R) mes A_{s+1}(t) \le \frac{4\gamma}{\alpha^2}\frac{R^{n+1}}{mes B_{8R}(x_0)}\int_{A_s(t)\setminus A_{s+1}(t)}\Big|\frac{\partial u}{\partial x}\Big|\,dx,$$

for all $t \in [t_0 - \frac{1}{2}\Theta(R)(8R)^{p_0}, t_0]$, where $A_s(t) = \{x \in B_{8R}(x_0) : u(x,t) \ge \mu^+ - \frac{\omega}{2^s}\Delta(R)\}$. From this, integrating over such a time interval we get

$$\frac{\omega}{2^{s+1}}\Delta(R) mes A_{s+1} \le \gamma\alpha^{-2}R\Big(\iint_{A_s}\Big|\frac{\partial u}{\partial x}\Big|^{p^-}\,dx\,dt\Big)^{\frac{1}{p^-}}(mes(A_s\setminus A_{s+1}))^{1-\frac{1}{p^-}},$$

here $A_s = \int_{t_0-\frac{1}{2}\Theta(R)(8R)^{p_0}}^{t_0} mes A_s(t)\,dt$. Take the $\frac{p^-}{p^- -1}$ power, estimate the integral on the right hand side by (4.30) and divide through by $(\frac{\omega\Delta(R)}{2^{s+1}})^{\frac{p^-}{p^- -1}}$. This gives

$$(mes A_{s+1})^{\frac{p^-}{p^- -1}} \le \gamma\alpha^{-\frac{2p^-}{p^- -1}}\Big(mes Q\Big(8R, \frac{1}{2}\Theta(R)\Big)\Big)^{\frac{1}{p^- -1}} mes(A_s\setminus A_{s+1}). \tag{4.32}$$

These inequalities are valid for all $s_2 \le s \le s_*$. We add them for $s = s_2, s_2+1, \dots, s_*$. The right hand side can be dominated by a convergent series bounded above by $mes Q(8R, \Theta(R))$. Therefore,

$$(s_* - s_2)(mes A_{s_*})^{\frac{p^-}{p^- -1}} \le \gamma\alpha^{-\frac{2p^-}{p^- -1}}(mes Q(8R,\Theta(R)))^{\frac{p^-}{p^- -1}}. \tag{4.33}$$

To prove the lemma we divide by $(s_* - s_2)$ and take s_* so large that

$$\gamma\alpha^{-2}(s_* - s_2)^{-\frac{p_0-1}{p_0}} \le \nu. \qquad \square$$

Now we are ready to prove Theorem 4.2. We will apply Lemma 4.1 over the boxes $Q_j = Q(R_j, \Theta(R))$ to the functions $(u - k_j)_+$, where for all $j = 0, 1, 2, \dots$

$$R_j = 4R(1 + 2^{-j}), \quad k_j = \mu^+ - \frac{\omega\Delta(R)}{2^{s_*+1}} - \frac{\omega\Delta(R)}{2^{s_*+1+j}}.$$

The levels k_j satisfy condition (4.21) for every $j \ge 0$.

The cutoff functions ζ_j are taken to satisfy

$$0 \le \zeta_j \quad \forall\,(x,t) \in Q_j,$$
$$\zeta_j \equiv 1 \quad \text{in} \quad Q_{j+1},$$
$$\zeta_j = 0 \quad \text{on the parabolic boundary of } Q_j,$$
$$\Big|\frac{\partial\zeta_j}{\partial x}\Big| \le \gamma 2^j R^{-1}, \quad \Big|\frac{\partial\zeta_j}{\partial t}\Big| \le \gamma 2^{jp_0}\Theta^{-1}R^{-p_0}.$$

With these choices, inequality (4.22) takes the form

$$\sup_{t_0-R_j^{p_0}\Theta(R)\le t\le t_0}\int_{B_{R_j}(x_0)}(u-k_j)_+^2\zeta_j^{p_2}(x,t)\,dx$$
$$+\sum_{i=1}^n\iint_{Q_j}\Big|\frac{\partial}{\partial x_i}(u-k_j)_+\Big|^{p_j^-}\zeta_j^{p_2}(x,t)\,dx\,dt\le\gamma\iint_{Q_j}(u-k_j)_+^2\Big|\frac{\partial\zeta_j}{\partial t}\Big|\,dx\,dt$$
$$+\gamma\sum_{i=1}^n\iint_{Q_j}(u-k_j)_+^{p_j^+}\Big|\frac{\partial\zeta_j}{\partial x_i}\Big|^{p_j^+}dx\,dt+\gamma mesA_j$$
$$\le\gamma\frac{2^{jp_2}}{R^{p_0}}\Big(\frac{\omega\Delta(R)}{2^{s_*}}\Big)^{p_0}mesA_j, \tag{4.34}$$

where $A_j=\{(x,t)\in Q_j: u(x,t)>k_j\}$,

$$p_j^-=\inf_{B_{R_j}(x_0)}p(x),\qquad p_j^+=\sup_{B_{R_j}(x_0)}p(x),$$

we use also inequalities (3.8), (3.9), (4.3).

Using the parabolic embedding theorem, we get from (4.34)

$$\frac{1}{2^{p_j^-(j+2)}}\Big(\frac{\omega\Delta(R)}{2^{s_*}}\Big)^{p_j^-}mesA_{j+1}\le\iint_{Q_{j+1}}(u-k_j)_+^{p_j^-}\,dx\,dt$$
$$\le\gamma mesA_j^{\frac{p_j^-}{p_j^-+n}}\|(u-k_j)_+\zeta_j\|_{V_{p_j^-}(Q_j)}$$
$$\le\gamma\frac{2^{jp_2}}{R^{p_0}}\Big(\frac{\omega\Delta(R)}{2^{s_*}}\Big)^{p_0+(p_j^--2)\frac{p_j^-}{p_j^-+n}}(mesA_j)^{1+\frac{p_j^-}{p_j^-+n}}. \tag{4.35}$$

Thus setting $y_j=\frac{mesA_j}{mesQ_j}$, we have the recursive inequalities

$$y_{j+1}\le\gamma2^{2jp_2}y_j^{1+\frac{p_j^-}{p_j^-+n}}\le\gamma2^{2jp_2}y_j^{1+\frac{p_0}{p_0+n}}.$$

It follows from these with the aid of Lemma 4.2 [6, Chapter I] that y_j tends to zero as $j\to\infty$ provided

$$y_0\le\gamma^{-\frac{p_0+n}{p_0}}2^{-2p_2(\frac{p_0+n}{p_0})^2}\equiv\nu. \tag{4.36}$$

This proves Theorem 4.2 in case when (4.3), (4.14) hold.

If inequality (4.16) is true. Then the levels $k_s=\mu^-+\frac{\omega}{2^s}\Delta(R)$, $s\ge s_1$ satisfy the condition $k_s\le\mu_f^-$. Since $(k_s-u)_+$ vanishes on $Q(8R,\Theta(R))\cap S_T$, we setting it to be zero outside Ω_T within the box $Q(8R, \Theta(R))$. Thus, under the conditions (4.3), (4.16), Theorem 4.2 can be proved analogously.

Condition (2.10) is equivalent to

$$\sum_{i=1}^{\infty} \Delta(R_i) = \infty, \tag{4.37}$$

$R_i = \frac{R_0}{C^i},\ i = 1, 2, \dots,\ R_0 > 0,\ C > 1$ are fixed numbers.

It follows from (4.20) that for any $i \geq 1$ there exists a number $j \geq 1$ such that

$$\frac{\Delta(R_{i+j})}{\Delta(R_i)} \geq \Big(\frac{1}{2}\Big)^j. \tag{4.38}$$

Let

$$C = \frac{2^{\frac{p_0-2}{p_0}+\frac{1}{\varepsilon}}}{(1-\frac{1}{2^{s_*}})^{\frac{p_0-2}{\varepsilon}}}, \qquad R_0 = 8R. \tag{4.39}$$

Define the sequence $\{i_j\}$ such as

$$\frac{\Delta(R_{i_{j+1}})}{\Delta(R_{i_j})} \geq \Big(\frac{1}{2}\Big)^{i_{j+1}-i_j} \tag{4.40}$$

and for any $i_j \leq i \leq i_{j+1} - 1$

$$\frac{\Delta(R_i)}{\Delta(R_{i_j})} \leq \Big(\frac{1}{2}\Big)^{i-i_j}. \tag{4.41}$$

From (4.37), it follows that $\sum_{i_j=i_0}^{\infty} \Delta(R_{i_j}) = \infty$, in opposite case, from (4.41)

$$\sum_{i=1}^{\infty} \Delta(R_i) \leq c \sum_{i_j=i_0}^{\infty} \sum_{k=i_j}^{i_{j+1}-1} \Delta(R_k) \leq 2c \sum_{i_j=i_0}^{\infty} \Delta(R_{i_j}) < \infty, \tag{4.42}$$

this contradicts (4.37). Therefore, we can assume, using a subsequence, that

$$\frac{\Delta(R_{i+1})}{\Delta(R_i)} \geq \frac{1}{2}. \tag{4.43}$$

It follows from (4.43) that

$$Q(R_{i+1}, \Theta(R_{i+1})) \subset Q(R_i, \Theta(R_i)), \qquad i = 1, 2, \dots. \tag{4.44}$$

If inequality (4.3) is not satisfied, then

$$\operatorname*{ess\,osc}_{Q^{(\varepsilon)}_{\frac{R}{2}} \cap \Omega_T} u(x,t) \leq \Big(1 - \frac{1}{2^{s_*}} \Delta(R)\Big)\omega + R^{\frac{\varepsilon}{p_0-2}}, \tag{4.45}$$

in opposite case, from Theorem 4.2

$$\operatorname*{ess\,osc}_{Q(R_{i+1},\Theta(R_{i+1}))} u(x,t) \le \omega_{i+1}$$
$$= \max\Big\{\Big(1 - \frac{1}{2^{s_*}}\Delta(R_i)\Big)\omega_i, 2 \operatorname*{osc}_{Q(R_i,\Theta(R_i))\cap S_T} f(x,t)\Big\}, \quad i = 1, 2, \dots. \tag{4.46}$$

Let

$$Q(\rho) = B_\rho(x_0) \times (t_0 - M^{2-p_0}\rho^{p_0}, t_0), \tag{4.47}$$

from (4.45), (4.46) using [6, Proposition 3.1, p. 44–45], we get for all $0 < \rho \le R_0$

$$\operatorname*{ess\,osc}_{Q(\rho)\cap\Omega_T} u(x,t) \le \gamma \exp\Big(-c\int_\rho^{1/2} \Delta(r)\frac{dr}{r}\Big) + \gamma\Big(\frac{\rho}{R_0}\Big)^{\frac{\varepsilon}{p_0-2}} + \gamma \operatorname*{osc}_{Q_\rho^{(\varepsilon)}\cap S_T} f(x,t). \tag{4.48}$$

This proves the sufficient condition for regularity of a boundary point.

5. Necessary condition for regularity of a boundary point

In this paragraph we develop the method of [12], where the necessary condition for regularity of a boundary point of elliptic equations ($p(x) \equiv$ const), was obtained. In the parabolic case ($p(x) \equiv$ const) analogous results were obtained in [18, 20].

We assume that (2.10) is false, i.e.,

$$\int_0 \Big(\frac{C_p(\overline{B_r(x_0)} \setminus \Omega, B_{2r}(x_0))}{r^{n-p_0}}\Big)^{\frac{1}{p_0-1}} \frac{dr}{r} < \infty. \tag{5.1}$$

Let $f(x) \in C_0^\infty(\mathbb{R}^n)$, $f(x) \equiv 1$ in some neighbourhood of x_0, $0 \le f(x) \le 1$,

$$\int_{\mathbb{R}^n} f^2(x)\,dx + \int_{\mathbb{R}^n} \Big|\frac{\partial f(x)}{\partial x}\Big|^{p(x)} dx \le \varepsilon^2,$$

where $\varepsilon \in (0,1)$ will be defined later.

Let $g(t) \in C_0^\infty(0,T)$, $g(t) \equiv 1$ for $t \in (\frac{t_0}{2}, \frac{T+t_0}{2})$, $0 \le g(t) \le 1, |\frac{dg(t)}{dt}| \le \gamma(t_0)$. We consider $u(x,t)$ as the solution of equation (1.4), which satisfies the conditions

$$\begin{aligned} &u(x,t) = f(x)g(t), \quad (x,t) \in S_T \\ &u(x,0) = 0, \quad x \in \Omega. \end{aligned} \tag{5.2}$$

We will prove the irregularity of a boundary point (x_0, t_0) under condition (5.1). Let δ, R be arbitrary positive numbers such that $(t_0 - \frac{R^{p_0}}{\delta^{p_0-2}}, t_0 + \frac{R^{p_0}}{\delta^{p_0-2}}) \subset (0,T)$, $R < 1$. Define

$$B_R = B_R(0), \qquad Q \equiv B_R \times \Big(t_0 - \frac{R^{p_0}}{\delta^{p_0-2}}, t_0 + \frac{R^{p_0}}{\delta^{p_0-2}}\Big).$$

Assume that the functions $\xi(x), \eta(x), \zeta(x)$ are defined in $\mathbb{R}^n$ and satisfy the following conditions

1) $\xi(x), \eta(x) \in W(B_R), \quad \xi(x)\zeta(x) \in \overset{\circ}{W}(\Omega)$;
2) $0 \le \xi(x) \le 1,\ \xi(x) = 1$ in $B_{\frac{R}{2}},\ \xi(x) = 0$ outside $B_R,\ \Big|\dfrac{\partial \xi(x)}{\partial x}\Big| \le cR^{-1}$;
3) $0 \le \eta(x) \le 1, \quad 0 \le \zeta(x) \le 1, \quad [1-\eta(x)][1-\zeta(x)] \equiv 0.$

We define $\Theta(t)$ as a cut-off function for the interval $(t_0 - \frac{R^{p_0}}{\delta^{p_0-2}}, t_0 + \frac{R^{p_0}}{\delta^{p_0-2}})$, which is equal to one on $(t_0 - \frac{4}{9}\frac{R^{p_0}}{\delta^{p_0-2}}, t_0 + \frac{4}{9}\frac{R^{p_0}}{\delta^{p_0-2}})$, $\big|\frac{d\Theta(t)}{dt}\big| \le c\delta^{p_0-2}R^{-p_0}$.

Define $\psi(x) = \xi(x)\zeta(x),\ \sigma(x) = \psi(x)\eta(x)$. Let $0 < l < M$ and

$$L = Q \cap \Omega_T \cap \{u > l\}, \qquad L(\tau) = \{(x,t) \in L : t = \tau\},$$
$$E = L \cap \{\eta(x) < 1\}, \qquad F = L \cap \{\eta(x) = 1\}.$$

Denote

$$\lambda = \min\Big\{\frac{2}{p_0+1}, \frac{1}{p_0-1}, 2(p_0-2), \frac{1}{n}\Big\}. \tag{5.3}$$

Lemma 5.1. *Let k be a positive number, such that $k > p_2$. Assume also that the number δ satisfies*

$$\delta \ge C(t_0) R^{\frac{p_0}{p_0-2}}, \qquad C(t_0) = \Big\{\frac{1}{\min(t_0, T-t_0)}\Big\}^{\frac{1}{p_0-2}}. \tag{5.4}$$

Then

$$\operatorname*{ess\,sup}_{0<t<T} \int_{L(t)} G\Big(\frac{u(x,t)-l}{\delta}\Big)\psi^k(x)\Theta^k(t)\,dx \tag{5.5}$$
$$+\delta^{p_0-2}\sum_{i=1}^{n} \iint_L \Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p^-} \psi^k(x)\Theta^k(t)\,dx\,dt$$
$$\le C_4 \frac{\delta^{p_0-2}}{R^{p_0}} \iint_E \Big[\Big(1+\frac{u-l}{\delta}\Big)^{1-\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{\lambda}\Big]^{p_0-1} \psi^{k-p^+}(x)\Theta^{k-1}(t)\,dx\,dt$$
$$+C_4\frac{1}{\delta^{p_0-1}}R^{p_0}\sum_{i=1}^{n}\int_{B_R}\Big|\frac{\partial}{\partial x_i}(\zeta(x)\eta(x))\Big|^{p(x)}dx + C_4\Big(\frac{1}{\delta^{p_0-2}}+\frac{1}{\delta^{p_0-1}}\Big)R^{n+p_0},$$

where constant C_4 depends only on k, n, p_1, p_2, L, M,

$$w(x,t) = \frac{1}{\delta}\Big(\int_l^{u(x,t)} \Big(1+\frac{s-l}{\delta}\Big)^{-\frac{1}{p_0}+\frac{\lambda}{2p_0}} \Big(\frac{s-l}{\delta}\Big)^{-\frac{\lambda}{p_0}} ds\Big)_+,$$

$$G(s) = \begin{cases} s & \text{for } s \ge 1,\\ s^{2-\lambda} & \text{for } 0 \le s \le 1,\end{cases}$$

$$p^- = \inf_{B_R} p(x), \qquad p^+ = \sup_{B_R} p(x).$$

Proof. We substitute the test function

$$\varphi(x,t) = \Big(\int_l^{u(x,t)} \Big(1 + \frac{s-l}{\delta}\Big)^{-1+\frac{\lambda}{2}} \Big(\frac{s-l}{\delta}\Big)^{-\lambda} ds \Big)_+ \psi^k(x)\Theta^k(t).$$

into the integral identity (2.2). Note that

$$\int_l^{u(x,t)} \Big(1 + \frac{s-l}{\delta}\Big)^{-1+\frac{\lambda}{2}} \Big(\frac{s-l}{\delta}\Big)^{-\lambda} ds \le c\delta. \tag{5.6}$$

We have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<T} \int_{L(t)} \Big\{ \int_l^{u(x,t)} \int_l^v \Big(1 + \frac{s-l}{\delta}\Big)^{-1+\frac{\lambda}{2}} \Big(\frac{s-l}{\delta}\Big)^{-\lambda} ds\, dv \Big\} \psi^k(x)\Theta^k(t)\, dx \\
&\quad + \sum_{i=1}^n \iint_L \Big(1 + \frac{u-l}{\delta}\Big)^{-1+\frac{\lambda}{2}} \Big(\frac{u-l}{\delta}\Big)^{-\lambda} \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)} \psi^k(x)\Theta^k(t)\, dx\, dt \\
&\quad\quad \le \gamma\delta \sum_{i=1}^n \iint_L \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-1} \Big|\frac{\partial}{\partial x_i}\psi(x)\Big| \psi^{k-1}(x)\Theta^k(t)\, dx\, dt \\
&\quad\quad + \gamma \iint_L \int_l^{u(x,t)} \int_l^v \Big(1 + \frac{s-l}{\delta}\Big)^{-1+\frac{\lambda}{2}} \Big(\frac{s-l}{\delta}\Big)^{-\lambda} ds\, dv \\
&\quad\quad\quad \times \Big|\frac{d\Theta(t)}{dt}\Big| \psi^k(x)\Theta^{k-1}(t)\, dx\, dt = I_1 + I_2.
\end{aligned} \tag{5.7}$$

Now,

$$\begin{aligned}
\int_l^u \int_l^v \Big(1 + \frac{s-l}{\delta}\Big)^{-1+\frac{\lambda}{2}} \Big(\frac{s-l}{\delta}\Big)^{-\lambda} ds\, dv &= \int_l^u \Big(1 + \frac{s-l}{\delta}\Big)^{-1+\frac{\lambda}{2}} (u-s)\, ds \\
&\ge \frac{u-l}{2} \int_l^{\frac{u-l}{2}} \Big(1 + \frac{s-l}{\delta}\Big)^{-1+\frac{\lambda}{2}} \Big(\frac{s-l}{\delta}\Big)^{-\lambda} ds \\
&= \frac{u-l}{2}\delta \int_0^{\frac{u-l}{2\delta}} (1+s)^{-1+\frac{\lambda}{2}} s^{-\lambda}\, ds \ge \gamma\delta^2 G\Big(\frac{u(x,t)-l}{\delta}\Big).
\end{aligned} \tag{5.8}$$

From (1.3) and (5.4) it follows that

$$\begin{gathered}
c(p_0, L, M) \le \Big(\frac{u-l}{\delta}\Big)^{p^+-p^-} \le C(p_0, L, M), \\
c(L) \le \delta^{p^+-p^-} \le C(L), \qquad c(L) \le R^{p^+-p^-} \le C(L).
\end{gathered} \tag{5.9}$$

We represent the integral on the right-hand side of (5.7) as the sum of the integrals over the sets E and F and take into account that $L = E \cup F$.

First we estimate the integral over E, taking into account that $\zeta(x) = 1$ and $\psi(x) = \xi(x)$ for $x \in E$. Using the Young inequality and (5.9) we obtain

$$I_1 \le \frac{1}{2}\sum_{i=1}^{n} \iint_E \Big(1+\frac{u-l}{\delta}\Big)^{-1+\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{-\lambda}\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\psi^k(x)\Theta^k(t)\,dx\,dt$$
$$+\gamma\iint_E\Big[\Big(1+\frac{u-l}{\delta}\Big)^{1-\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{\lambda}\Big]^{p(x)-1}\Big(\frac{\delta}{R}\Big)^{p(x)}\psi^{k-p^+}(x)\Theta^k(t)\,dx\,dt$$
$$\le \frac{1}{2}\sum_{i=1}^{n} \iint_E \Big(1+\frac{u-l}{\delta}\Big)^{-1+\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{-\lambda}\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\psi^k(x)\Theta^k(t)\,dx\,dt$$
$$+\gamma\Big(\frac{\delta}{R}\Big)^{p_0}\iint_E\Big[\Big(1+\frac{u-l}{\delta}\Big)^{1-\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{\lambda}\Big]^{p_0-1}\psi^{k-p^+}(x)\Theta^k(t)\,dx\,dt. \quad (5.10)$$

Further, by virtue of the choice of λ and (5.6), we have

$$I_2 \le \gamma\frac{\delta^{p_0-1}}{R^{p_0}}\iint_E (u-l)\psi^k(x)\Theta^{k-1}(t)\,dx\,dt \quad (5.11)$$
$$\le \gamma\Big(\frac{\delta}{R}\Big)^{p_0}\int_E\Big[\Big(1+\frac{u-l}{\delta}\Big)^{1-\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{\lambda}\Big]^{p_0-1}\psi^{k-p^+}(x)\Theta^{k-1}(t)\,dx\,dt.$$

Testing (2.2) by the function $\varphi(x,t) = (u-l)\sigma^k(x)\Theta^k(t)$, using the Young inequality, we obtain

$$\operatorname*{ess\,sup}_{0<t<T}\int_{L(t)}(u-l)^2\sigma^k(x)\Theta^k(t)\,dx + \sum_{i=1}^{n}\iint_L\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\sigma^k(x)\Theta^k(t)\,dx\,dt$$
$$\le \gamma\iint_L (u-l)^2\Big|\frac{d\Theta(t)}{dt}\Big|\sigma^k(x)\Theta^{k-1}(t)\,dx\,dt$$
$$+\gamma\iint_L (u-l)^{p(x)}R^{-p(x)}[\zeta(x)\eta(x)\Theta(t)]^k\,dx\,dt$$
$$+\gamma\sum_{i=1}^{n}\iint_L (u-l)^{p(x)}\Big|\frac{\partial}{\partial x_i}(\zeta(x)\eta(x))\Big|^{p(x)}\,dx\,dt. \quad (5.12)$$

Using (5.9), the Young and Poincaré inequalities

$$\int_{B_R}[\zeta(x)\eta(x)]^k\,dx \le \gamma R^{p_0}\sum_{i=1}^{n}\int_{B_R}\Big|\frac{\partial}{\partial x_i}(\zeta(x)\eta(x))\Big|^{p^-}\,dx$$
$$\le \gamma R^{p_0}\sum_{i=1}^{n}\int_{B_R}\Big|\frac{\partial}{\partial x_i}(\zeta(x)\eta(x))\Big|^{p(x)}\,dx + \gamma R^{n+p_0}. \quad (5.13)$$

Since $\sigma(x)=\psi(x)$ on F,

$$\delta\sum_{i=1}^{n}\iint_F\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-1}\Big|\frac{\partial\psi(x)}{\partial x_i}\Big|\psi^{k-1}(x)\Theta^k(t)\,dx\,dt$$

$$\leq\gamma\delta\Big\{\sum_{i=1}^{n}\iint_F\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\psi^k(x)\Theta^k(t)\,dx\,dt \tag{5.14}$$

$$+\gamma\iint_F R^{-p(x)}[\zeta(x)\Theta(t)]^{k-p^+}\,dx\,dt+\gamma\sum_{i=1}^{n}\iint_F\Big|\frac{\partial\zeta(x)}{\partial x_i}\Big|^{p^+}\Theta^k(t)\,dx\,dt\Big\}.$$

Combining inequalities (5.7)–(5.14), using (5.4), we get

$$\int_{L(t)}G\Big(\frac{u(x,t)-l}{\delta}\Big)\psi^k(x)\Theta^k(t)\,dx$$

$$+\delta^{-2}\sum_{i=1}^{n}\iint_L\Big(1+\frac{u-l}{\delta}\Big)^{-1+\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{-\lambda}\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\psi^k(x)\Theta^k(t)\,dx\,dt$$

$$\leq\gamma\frac{\delta^{p_0-2}}{R^{p_0}}\iint_E\Big[\Big(1+\frac{u-l}{\delta}\Big)^{1-\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{\lambda}\Big]^{p_0-1}\psi^{k-p^+}(x)\Theta^k(t)\,dx\,dt$$

$$+\gamma\sum_{i=1}^{n}\frac{R^{p_0}}{\delta^{p_0-1}}\int_{B_R}\Big|\frac{\partial}{\partial x_i}(\zeta(x)\eta(x))\Big|^{p(x)}dx+\gamma\frac{R^{n+p_0}}{\delta^{p_0-1}}. \tag{5.15}$$

From (5.9)

$$\delta^{p(x)}\Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)}=\Big(1+\frac{u-l}{\delta}\Big)^{-(\frac{1}{p_0}-\frac{\lambda}{2p_0})p(x)}\Big(\frac{u-l}{\delta}\Big)^{-\frac{\lambda p(x)}{p_0}}\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}$$

$$\leq\gamma\Big(1+\frac{u-l}{\delta}\Big)^{-1+\frac{\lambda}{2}}\Big(\frac{u-l}{\delta}\Big)^{-\lambda}\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}. \tag{5.16}$$

Therefore (5.9) and the Young inequality imply

$$\delta^{p^-}\sum_{i=1}^{n}\iint_L\Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p^-}\psi^k(x)\Theta^k(t)\,dx\,dt$$

$$\leq\gamma\sum_{i=1}^{n}\iint_L\delta^{p(x)}\Big|\frac{\partial w(x,t)}{\partial x_i}\Big|^{p(x)}\psi^k(x)\Theta^k(t)\,dx\,dt+\gamma\frac{R^{n+p_0}}{\delta^{p_0-2}}. \tag{5.17}$$

Inequalities (5.15), (5.17) imply (5.5). This completes the proof of Lemma 5.1. □

Lemma 5.2. *Assume that all the conditions of Lemma* 5.1 *are fulfilled. Then*

$$\operatorname*{ess\,sup}_{0<t<T} \int_{L(t)} (u-l)\sigma^k(x)\Theta^k(t)\,dx$$
$$\leq C_5 R^{p_0}\Big(1+\frac{1}{\delta^{p_0-2}}\Big)\sum_{i=1}^{n}\int_{B_R}\Big|\frac{\partial}{\partial x_i}(\zeta(x)\eta(x))\Big|^{p(x)}dx + C_5\frac{R^{n+p_0}}{\delta^{p_0-2}}, \tag{5.18}$$

with a positive constant C_5*, which depends only on* n, p_1, p_2, L, M*.*

Proof. To prove inequality (5.18) we substitute the test function $\varphi(x,t) = \frac{(u-l)_+}{u-l+\varepsilon}\sigma^k(x)\Theta^k(t)$, $\varepsilon > 0$, into (2.2) and use estimate (5.12). Sending $\varepsilon \to 0$, we get (5.18). □

Assume that $R_0 \in (0,1)$, $R_j = \frac{R_0}{2^j}$, $j = 1,2,\ldots$. Denote $B_j = B_{R_j}(x_0)$. We choose a sequence of functions $\{\xi_j(x)\}$, $j = 1,2,\ldots$, where $\xi_j(x)$ is a cut-off function for B_j, $\xi_j(x) = 1$ in B_{j+1}. We define functions $g_j(x) \in C_0^\infty(B_{j-1})$, so that $g_j(x) = 1$ for $x \in B_j \setminus \Omega$ and

$$\sum_{i=1}^{n}\int_{B_{j-1}}\Big|\frac{\partial g_j(x)}{\partial x_i}\Big|^{p(x)}dx \leq c\,C_p(\overline{B}_j \setminus \Omega, B_{j-1}). \tag{5.19}$$

Let $g'_j(x) = \min\{1,(g_j(x))_+\}$. Define the sequences of functions $\zeta_j(x)$, $\eta_j(x)$, $\Theta_j(t)$,

$$\eta_j(x) = \min\{1, 3g'_j(x) + 3g'_{j-1}\}, \qquad \zeta_j(x) = \min\{1,(2-3g'_j(x))_+\},$$
$$\psi_j(x) = \xi_j(x)\zeta_j(x), \qquad \sigma_j(x) = \psi_j(x)\eta_j(x),$$

$\Theta_j(t)$ is a cut-off function for interval $(t_0 - \frac{R_j^{p_0}}{\delta_j^{p_0-2}}, t_0 + \frac{R_j^{p_0}}{\delta_j^{p_0-2}})$, which is equal to one on $(t_0 - \frac{4}{9}\frac{R_j^{p_0}}{\delta_j^{p_0-2}}, t_0 + \frac{4}{9}\frac{R_j^{p_0}}{\delta_j^{p_0-2}})$. Let us introduce the sets Q_j, L_j, E_j, F_j, $\widetilde{Q}_j$, $\widetilde{L}_j$

$$Q_j = B_j \times \Big(t_0 - \frac{R_j^{p_0}}{\delta_j^{p_0-2}}, t_0 + \frac{R_j^{p_0}}{\delta_j^{p_0-2}}\Big),$$
$$\widetilde{Q}_j = B_j \times \Big(t_0 - \frac{R_j^{p_0}}{(l-l_j)^{p_0-2}}, t_0 + \frac{R_j^{p_0}}{(l-l_j)^{p_0-2}}\Big),$$
$$L_j = Q_j \cap \Omega_T \cap \{u > l_j\}, \quad E_j = L_j \cap \{\eta(x<1)\}, \quad F_j = L_j \cap \{\eta_j(x) = 1\},$$
$$\widetilde{L}_j = \widetilde{Q}_j \cap \Omega_T \cap \{u > l_j\}.$$

Let us specify the choice of δ_j, l_j. We set $l_0 = 0$ and assume that the values $l_1, \dots, l_j$ and $\delta_0, \dots, \delta_{j-1}$ have already been chosen, fix $a \in (0,1)$, which will be defined later, depending only on n, p_1, p_2, L, M. We define

$$A_j(l) = \frac{(l-l_j)^{p_0-2}}{R_j^{n+p_0}} \iint_{\widetilde{L}_j} \Big(\frac{u-l_j}{l-l_j}\Big)^{(1+\frac{\lambda}{2})(p_0-1)} \psi_j^{k-p_2}(x)\Theta_j^{k-p_2}(t)\,dx\,dt$$
$$+\operatorname*{ess\,sup}_{0<t<T} \frac{1}{R_j^n} \int_{\widetilde{L}_j(t)} G\Big(\frac{u-l_j}{l-l_j}\Big)\psi_j^k(x)\Theta_j^k(t)\,dx. \tag{5.20}$$

Consider the equation

$$A_j(l) = a. \tag{5.21}$$

If there exists a solution $l(a)$ of this equation for $l > l_j + C(t_0)R_j^{\frac{p_0}{p_0-2}}$, then we assume that $l_{j+1} = l(a)$. Otherwise

$$l_{j+1} = l_j + C(t_0)R_j^{\frac{p_0}{p_0-2}}. \tag{5.22}$$

In both cases we put $\delta_j = l_{j+1} - l_j$.

Lemma 5.3. *Suppose that conditions* (1.2), (1.3), (5.1) *are satisfied. Then there exist* $R_0 \in (0,1)$, *positive numbers* C_6, k, a, *depending only on* n, p_1, p_2, L, M, *such that for all* $j \geq 1$ *the following estimate is true*

$$\delta_j \leq \frac{1}{2}\delta_{j-1} + C(t_0)R_j^{\frac{p_0}{p_0-2}} + C_6\Big(\frac{C_p(\overline{B}_j \setminus \Omega, B_{j-1})}{R_j^{n-p_0}}\Big)^{\frac{1}{p_0-1}}. \tag{5.23}$$

Proof. We fix $j \geq 1$ and assume that

$$\delta_j > \frac{1}{2}\delta_{j-1}, \qquad \delta_j > C(t_0)R_j^{\frac{p_0}{p_0-2}}, \tag{5.24}$$

because, otherwise estimate (5.23) is trivial.

It follows from the second inequality in (5.24) that $l_{j+1} = l(a)$, where $l(a)$ is a solution of equation (5.21). Let us estimate the terms on the right-hand side of (5.20) for $l = l_{j+1}$. In this case $Q_j = \widetilde{Q}_j,\ L_j = \widetilde{L}_j$. We represent L_j as

$$L_j = L_j' \cup L_j'',$$

where

$$L_j' = L_j \cap \Big\{\frac{u-l_j}{\delta_j} < \varepsilon_1\Big\}, \qquad L_j'' = L_j \setminus L_j'$$

with some $\varepsilon_1 \in (0,1)$. We have

$$\frac{\delta_j^{p_0-2}}{R_j^{n+p_0}} \iint_{L_j} \Big(\frac{u-l_j}{\delta_j}\Big)^{(1+\frac{\lambda}{2})(p_0-1)} \psi_j^{k-p_2}(x)\Theta_j^{k-p_2}(t)\,dx\,dt$$

$$\leq \varepsilon_1^{(1+\frac{\lambda}{2})(p_0-1)} \frac{\delta_j^{p_0-2}}{R_j^{n+p_0}} \Big\{ mesE'_j + \iint_{F_j} \psi_j^{k-p_2}(x)\Theta_j^{k-p_2}(t)\,dx\,dt \Big\}$$

$$+\frac{\delta_j^{p_0-2}}{R_j^{n+p_0}} \iint_{L''_j} \Big(\frac{u-l_j}{\delta_j}\Big)^{(1+\frac{\lambda}{2})(p_0-1)} \psi_j^{k-p_2}(x)\Theta_j^{k-p_2}(t)\,dx\,dt. \tag{5.25}$$

By definition $\zeta_{j-1}(x) = 1,\ \xi_{j-1}(x) = 1,\ \Theta_{j-1}(t) = 1$ on E_j, therefore,

$$mesE'_j \leq \iint_{E'_j} \Big(\frac{u-l_{j-1}}{\delta_{j-1}}\Big)^{2-\lambda} \psi_{j-1}^k(x)\Theta_{j-1}^k(t)\,dx\,dt \tag{5.26}$$

$$\leq \frac{2R_j^{p_0}}{\delta_j^{p_0-2}} \operatorname*{ess\,sup}_{0<t<T} \int_{E_j(t)} G\Big(\frac{u-l_{j-1}}{\delta_{j-1}}\Big)\psi_{j-1}^k(x)\Theta_{j-1}^k(t)\,dx$$

$$\leq \frac{2R_j^{p_0}}{\delta_j^{p_0-2}} \operatorname*{ess\,sup}_{0<t<T} \int_{L_{j-1}(t)} G\Big(\frac{u-l_{j-1}}{\delta_{j-1}}\Big)\psi_{j-1}^k(x)\Theta_{j-1}^k(t)\,dx \leq 2^{n+1}\frac{R_j^{n+p_0}}{\delta_j^{p_0-2}}\,a.$$

Here we use also an estimate $A_{j-1}(l_j) \leq a$, which follows from the definition of sequence $\{l_j\}$. We define

$$w_j(x,t) = \frac{1}{\delta_j}\Big(\int_{l_j}^{u(x,t)} \Big(1+\frac{s-l_j}{\delta_j}\Big)^{-\frac{1}{p_0}+\frac{\lambda}{2p_0}} \Big(\frac{s-l_j}{\delta_j}\Big)^{-\frac{\lambda}{p_0}}\,ds\Big)_+. \tag{5.27}$$

This function satisfies the estimates

$$c(\varepsilon_1)\Big(\frac{u-l_j}{\delta_j}\Big)^{1-\frac{1}{p_0}-\frac{\lambda}{2p_0}} \leq w_j(x,t) \leq C(\varepsilon_1)\Big(\frac{u-l_j}{\delta_j}\Big)^{1-\frac{1}{p_0}-\frac{\lambda}{2p_0}}, \quad (x,t)\in L''_j. \tag{5.28}$$

Let q be defined by

$$\Big(1-\frac{1}{p_0}-\frac{\lambda}{2p_0}\Big)q = \Big(1+\frac{\lambda}{2}\Big)(p_0-1). \tag{5.29}$$

Using the Young inequality, we estimate the last term in (5.25) as follows

$$\iint_{L''_j} \Big(\frac{u-l_j}{\delta_j}\Big)^{(1+\frac{\lambda}{2})(p_0-1)} \psi_j^{k-p_2}(x)\Theta_j^{k-p_2}(t)\,dx\,dt$$

$$\leq \gamma(\varepsilon_1)\iint_{L''_j} w_j^q(x,t)\psi_j^{k-p_2}(x)\Theta_j^{k-p_2}(t)\,dx\,dt$$

$$\leq \varepsilon_2 \iint_{L''_j} w_j^{\rho(\lambda)}(x,t)\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\,dt$$

$$+\gamma(\varepsilon_1,\varepsilon_2)\iint_{L''_j} w_j^{(q-\rho(\lambda))z+\rho(\lambda)}(x,t)[\psi_j(x)\Theta_j(t)]^{(k-2p_2)z+p_2}\,dx\,dt, \tag{5.30}$$

where ε_2 is an arbitrary number from the interval $(0,1)$, $z>1$, is determined from the condition

$$(q-\rho(\lambda))z+\rho(\lambda)=\overline{q}=p_0\frac{n+\rho(\lambda)}{n},\qquad \rho(\lambda)=\frac{2p_0}{2p_0-2-\lambda}. \tag{5.31}$$

The condition for λ guarantees that $z>1$. We have

$$\begin{aligned}&\iint_{L''_j} w_j^{\rho(\lambda)}(x,t)\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\,dt\\ &\le \gamma(\varepsilon_1)\Big\{\iint_{E_j\cap(\frac{u-l_j}{\delta_j}\ge\varepsilon_1)}\frac{u-l_j}{\delta_j}\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\,dt\\ &\quad+\iint_{F_j\cap(\frac{u-l_j}{\delta_j}\ge\varepsilon_1)}\frac{u-l_j}{\delta_j}\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\,dt\Big\}.\end{aligned} \tag{5.32}$$

From the first inequality in (5.24), using the conditions $\psi_{j-1}(x)=1$, $\Theta_{j-1}(t)=1$ on E_j, we estimate the first integral in the right-hand side of (5.32)

$$\begin{aligned}&\iint_{E_j\cap(\frac{u-l_j}{\delta_j}\ge\varepsilon_1)}\frac{u-l_j}{\delta_j}\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\,dt\\ &\le 2\iint_{E_j\cap(\frac{u-l_j}{\delta_j}\ge\varepsilon_1)}\frac{u-l_{j-1}}{\delta_{j-1}}\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\,dt\\ &\le \frac{2R_j^{p_0}}{\delta_j^{p_0-2}}\operatorname*{ess\,sup}_{0<t<T}\int_{E_j\cap(\frac{u-l_j}{\delta_j}\ge\varepsilon_1)}\frac{u-l_{j-1}}{\delta_{j-1}}\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\\ &\le \frac{\gamma(\varepsilon_1)R_j^{p_0}}{\delta_j^{p_0-2}}\operatorname*{ess\,sup}_{0<t<T}\int_{E_j\cap(\frac{u-l_j}{\delta_j}\ge\varepsilon_1)}G\Big(\frac{u-l_{j-1}}{\delta_{j-1}}\Big)\psi_{j-1}^{k}(x)\Theta_{j-1}^{k}(t)\,dx\\ &\le \frac{\gamma(\varepsilon_1)R_j^{p_0}}{\delta_j^{p_0-2}}\operatorname*{ess\,sup}_{0<t<T}\int_{L_{j-1}(t)}G\Big(\frac{u-l_{j-1}}{\delta_{j-1}}\Big)\psi_{j-1}^{k}(x)\Theta_{j-1}^{k}(t)\,dx\le\frac{\gamma(\varepsilon_1)R_j^{n+p_0}}{\delta_j^{p_0-2}}\,a,\end{aligned} \tag{5.33}$$

where we also used the inequality $A_{j-1}(l_j)\le a$, which follows from the definition of the sequence $\{l_j\}$.

Now using Lemma 5.2, inequality (5.33), we have from (5.32)

$$\begin{aligned}&\varepsilon_2\iint_{L''_j} w_j^{\rho(\lambda)}(x,t)\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\,dt\le\gamma(\varepsilon_1)\varepsilon_2\frac{R_j^{n+p_0}}{\delta_j^{p_0-2}}\\ &\times\Big\{a+\Big(\frac{1}{\delta_j}+\frac{1}{\delta_j^{p_0-1}}\Big)R_j^{p_0-n}\sum_{i=1}^{n}\int_{B_j}\Big|\frac{\partial}{\partial x_i}(\zeta_j(x)\eta_j(x))\Big|^{p(x)}dx+\frac{R_j^{p_0}}{\delta_j^{p_0-1}}\Big\}.\end{aligned} \tag{5.34}$$

From (5.9), (5.28) and (5.31), we get

$$w_j^{(q-\rho(\lambda))z+\rho(\lambda)} \le c(\varepsilon_1) w_j^{p_j^- \frac{n+\rho(\lambda)}{n}}(x,t), \qquad (x,t)\in L_j'', \tag{5.35}$$

where $p_j^- = \inf_{B_j} p(x)$. Thus using the embedding theorem

$$\begin{aligned}
&\iint_{L_j''} w_j^{(q-\rho(\lambda))z+\rho(\lambda)}(x,t)[\psi_j(x)\Theta_j(t)]^{(k-2p_2)z+p_2}\,dx\,dt\\
&\le \gamma(\varepsilon_1)\iint_{L_j''} w_j^{p_j^- \frac{n+\rho(\lambda)}{n}}(x,t)[\psi_j(x)\Theta_j(t)]^{(k-2p_2)z+p_2}\,dx\,dt\\
&\le \gamma(\varepsilon_1)\Big(\operatorname*{ess\,sup}_{0<t<T}\int_{L_j''(t)} w_j^{\rho(\lambda)}(x,t)\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\Big)^{\frac{p_j^-}{n}}\times\\
&\times\int_0^T\Big(\int_{L_j(t)} w_j^{\frac{np_j^-}{n-p_j^-}}(x,t)[\psi_j(x)\Theta_j(t)]^{(k-2p_2)z\frac{n}{n-p_j^-}+p_2}\,dx\Big)^{\frac{n-p_j^-}{n}}\,dt\\
&\le \gamma(\varepsilon_1)\Big(\operatorname*{ess\,sup}_{0<t<T}\int_{L_j''(t)} w_j^{\rho(\lambda)}(x,t)\psi_j^{p_2}(x)\Theta_j^{p_2}(t)\,dx\Big)^{\frac{p_j^-}{n}}\times\\
&\times\sum_{i=1}^n\iint_{L_j}\Big|\frac{\partial}{\partial x_i}\Big\{w_j(x,t)[\psi_j(x)\Theta_j(t)]^{(k-2p_2)\frac{z}{p_j^-}+p_2\frac{n-p_j^-}{np_j^-}}\Big\}\Big|^{p_j^-}\,dx\,dt.
\end{aligned}\tag{5.36}$$

First integral in the right-hand side of (5.36) we estimate analogously to (5.34) and fix the value of k:

$$(k-2p_2)z - p_2\frac{p_j^-}{n} > k. \tag{5.37}$$

From Lemma 5.1 and inequality (5.26) we have

$$\begin{aligned}
&\delta_j^{p_0-2}\sum_{i=1}^n\iint_{L_j}\Big|\frac{\partial}{\partial x_i}\Big\{w_j(x,t)[\psi_j(x)\Theta_j(t)]^{(k-2p_2)\frac{z}{p_j^-}+p_2\frac{n-p_j^-}{np_j^-}}\Big\}\Big|^{p_j^-}\,dx\,dt\\
&\le \gamma\frac{\delta_j^{p_0-2}}{R_j^{p_0}}\iint_{E_j}\Big(1+\frac{u-l_j}{\delta_j}\Big)^{(1+\frac{\lambda}{2})(p_0-1)}\psi_j^{k-p_2}(x)\Theta_j^{k-p_2}(t)\,dx\,dt\\
&+\gamma\frac{R_j^{p_0}}{\delta_j^{p_0-1}}\sum_{i=1}^n\int_{B_j}\Big|\frac{\partial}{\partial x_i}(\zeta_j(x)\eta_j(x))\Big|^{p(x)}\,dx+\gamma\Big(\frac{1}{\delta_j^{p_0-2}}+\frac{1}{\delta_j^{p_0-1}}\Big)R_j^{n+p_0}
\end{aligned}\tag{5.38}$$

$$\le \gamma R_j^n a + \gamma \frac{R_j^{p_0}}{\delta_j^{p_0-1}} \sum_{i=1}^n \int_{B_j} \Big| \frac{\partial}{\partial x_i} (\zeta_j(x)\eta_j(x)) \Big|^{p(x)} dx + \gamma \Big(\frac{1}{\delta_j^{p_0-2}} + \frac{1}{\delta_j^{p_0-1}} \Big) R_j^{n+p_0}.$$

Using estimates (5.26)–(5.38), we get from (5.25)

$$\frac{\delta_j^{p_0-2}}{R_j^{n+p_0}} \iint_{L_j} \Big(\frac{u-l_j}{\delta_j} \Big)^{(1+\frac{\lambda}{2})(p_0-1)} \psi_j^{k-p_2}(x) \Theta_j^{k-p_2}(t)\, dx\, dt$$
$$\le (\varepsilon_1^{(1+\frac{\lambda}{2})(p_0-1)} + \varepsilon_2) a + \gamma(\varepsilon_1, \varepsilon_2) \Big(1 + \frac{1}{\delta_j^{p_0-1}} + \frac{1}{\delta_j^{p_0-2}} \Big) R_j^{p_0}$$
$$+ \gamma(\varepsilon_1, \varepsilon_2) \Big(1 + \frac{1}{\delta_j^{p_0-1}} \Big) R_j^{p_0-n} \sum_{i=1}^n \int_{B_j} \Big| \frac{\partial}{\partial x_i} (\zeta_j(x)\eta_j(x)) \Big|^{p(x)} dx$$
$$+ \gamma(\varepsilon_1, \varepsilon_2) \Big\{ a + \frac{1}{\delta_j^{p_0-1}} R_j^{p_0-n} \sum_{i=1}^n \int_{B_j} \Big| \frac{\partial}{\partial x_i} (\zeta_j(x)\eta_j(x)) \Big|^{p(x)} dx$$
$$+ \Big(\frac{1}{\delta_j^{p_0-1}} + \frac{1}{\delta_j^{p_0-2}} \Big) R_j^{p_0} \Big\}^{1+\frac{p_j^-}{n}}. \tag{5.39}$$

Now let us estimate the second term in (5.21). From Lemma 5.1 we have

$$\operatorname*{ess\,sup}_{0<t<T} \frac{1}{R_j^n} \int_{L_j(t)} G\Big(\frac{u-l_j}{\delta_j} \Big) \psi_j^k(x) \Theta_j^k(t)\, dx \tag{5.40}$$
$$\le \gamma \frac{\delta_j^{p_0-2}}{R_j^{n+p_0}} \iint_{E_j} \Big[\Big(1 + \frac{u-l_j}{\delta_j} \Big)^{1-\frac{\lambda}{2}} \Big(\frac{u-l_j}{\delta_j} \Big)^{\lambda} \Big]^{p_0-1} \psi_j^{k-p_2}(x) \Theta_j^{k-p_2}(t)\, dx\, dt$$
$$+ \gamma \frac{1}{\delta_j^{p_0-1}} R_j^{p_0-n} \sum_{i=1}^n \int_{B_j} \Big| \frac{\partial}{\partial x_i} (\zeta_j(x)\eta_j(x)) \Big|^{p(x)} dx + \gamma \Big(\frac{1}{\delta_j^{p_0-2}} + \frac{1}{\delta_j^{p_0-1}} \Big) R_j^{p_0}.$$

We estimate the first term in the right-hand side of (5.40), representing $E_j = E_j' \cap E_j''$, where $E_j' = E_j \cap \{ \frac{u-l_j}{\delta_j} < \varepsilon_1 \}$, $E_j'' = E_j \setminus E_j'$. Then we have

$$\iint_{E_j} \Big[\Big(1 + \frac{u-l_j}{\delta_j} \Big)^{1-\frac{\lambda}{2}} \Big(\frac{u-l_j}{\delta_j} \Big)^{\lambda} \Big]^{p_0-1} \psi_j^{k-p_2}(x) \Theta_j^{k-p_2}(t)\, dx\, dt$$
$$\le \varepsilon_1^{\lambda(p_0-1)} mes E_j' + \varepsilon_1^{-(p_0-1)(1-\frac{\lambda}{2})}$$
$$\iint_{E_j''} \Big(\frac{u-l_j}{\delta_j} \Big)^{(1+\frac{\lambda}{2})(p_0-1)} \psi_j^{k-p_2}(x) \Theta_j^{k-p_2}(t)\, dx\, dt. \tag{5.41}$$

Estimates of the terms in the right-hand side of (5.41) were obtained in (5.26), (5.30), (5.32)–(5.38). Thus from $A_j(l_{j+1}) = a$ and (5.39), (5.41), we have

$$a \le \gamma(\varepsilon_1^{\lambda(p_0-1)} + \varepsilon_1^{(1+\frac{\lambda}{2})(p_0-1)} + \varepsilon_2)\, a$$
$$+\gamma(\varepsilon_1,\varepsilon_2)(1+\delta_j^{1-p_0})R_j^{p_0-n}\sum_{i=1}^{n}\int_{B_j}\Big|\frac{\partial}{\partial x_i}(\zeta_j(x)\eta_j(x))\Big|^{p(x)}dx$$
$$+\gamma(\varepsilon_1,\varepsilon_2)(1+\delta_j^{1-p_0}+\delta_j^{2-p_0})R_j^{p_0}$$
$$+\gamma(\varepsilon_1,\varepsilon_2)\Big\{a+\delta_j^{1-p_0}R_j^{p_0-n}\sum_{i=1}^{n}\int_{B_j}\Big|\frac{\partial}{\partial x_i}(\zeta_j(x)\eta_j(x))\Big|^{p(x)}dx$$
$$+(\delta_j^{1-p_0}+\delta_j^{2-p_0})R_j^{p_0}\Big\}^{1+\frac{p_j^-}{n}}. \tag{5.42}$$

Choosing $\varepsilon_1, \varepsilon_2$ from the condition

$$\varepsilon_1^{\lambda(p_0-1)} + \varepsilon_1^{(1+\frac{\lambda}{2})(p_0-1)} + \varepsilon_2 = \tfrac{1}{8}, \tag{5.43}$$

fixing the values of $\varepsilon_1, \varepsilon_2$, we determine the number a from the equality

$$\gamma(\varepsilon_1,\varepsilon_2)a2^L = \tfrac{1}{8}, \tag{5.44}$$

where L is the constant from (1.3). From the definition of $\zeta_j(x), \eta_j(x)$ we have

$$\sum_{i=1}^{n}\int_{B_j}\Big|\frac{\partial}{\partial x_i}(\zeta_j(x)\eta_j(x))\Big|^{p(x)}dx \le \gamma C_p(\overline{B}_{j-1}\setminus\Omega, B_{j-2}). \tag{5.45}$$

Using assumption (5.1) we choose R_0 such that

$$\gamma(\varepsilon_1,\varepsilon_2)C_p(\overline{B_{R_0}(x_0)}\setminus\Omega, B_{2R_0}(x_0)) + \gamma(\varepsilon_1,\varepsilon_2)R_0^{p_0} + R_0 \le \frac{1}{8}a. \tag{5.46}$$

It follows from (5.44), (5.46) that

$$c(L) \le \Big(\frac{1}{a}\Big)^{p_0-p_j^-} \le C(L), \tag{5.47}$$

hence (5.42) implies at least one of estimates

$$\delta_j^{1-p_0}(R_j^{p_0-n}C_p(\overline{B}_{j-1}\setminus\Omega, B_{j-2}) + R_j^{p_0}) \ge \frac{1}{2}\frac{a}{\gamma(\varepsilon_1,\varepsilon_2)}, \tag{5.48}$$

$$\delta_j^{1-p_0}(R_j^{p_0-n}C_p(\overline{B}_{j-1}\setminus\Omega, B_{j-2}) + R_j^{p_0}) \ge \Big(\frac{1}{2}\frac{a}{\gamma(\varepsilon_1,\varepsilon_2)}\Big)^{\frac{n}{n+p_0}}. \tag{5.49}$$

This proves inequality (5.23), which completes the proof of Lemma 5.3. □

Theorem 5.1. *Suppose that conditions of Lemma 5.3 are satisfied. Then*

$$\bar{l} = \lim_{j\to\infty} l_j \le C_7\Big(\frac{1}{R_0^n}\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u_+(x,t)\,dx\Big)^{\frac{1}{(1+\frac{\lambda}{2})(p_0-1)}}$$
$$+C_7 R_0^{\frac{p_0}{p_0-2}} + C_7\int_0^{2R_0}\Big(\frac{C_p(\overline{B_r(x_0)}\setminus\Omega, B_{2r}(x_0))}{r^{n-p_0}}\Big)^{\frac{1}{p_0-1}}\frac{dr}{r}, \qquad (5.50)$$

with positive constant C_7, depending only on n, p_1, p_2, L, M.

Proof. Summing up inequalities (5.23), we establish that, for any $J \ge 1$, the following estimate is true

$$\sum_{j=1}^{J}\delta_j \le \frac{1}{2}\sum_{j=0}^{J-1}\delta_j + \gamma\sum_{j=1}^{J}R_j^{\frac{p_0}{p_0-2}} + \gamma\sum_{j=1}^{J}\Big(\frac{C_p(\overline{B}_{j-1}\setminus\Omega, B_{j-2})}{R_j^{n-p}}\Big)^{\frac{1}{p_0-1}}. \quad (5.51)$$

Using this inequality we obtain

$$l_J \le \gamma\Big\{\delta_0 + R_0^{\frac{p_0}{p_0-2}} + \int_0^{2R_0}\Big(\frac{C_p(\overline{B_r(x_0)}\setminus\Omega, B_{2r}(x_0))}{r^{n-p_0}}\Big)^{\frac{1}{p_0-1}}\frac{dr}{r}\Big\}. \quad (5.52)$$

Let us estimate δ_0. In case when l_1 is defined by (5.22), inequality (5.50) follows from (5.52).

If l_1 is defined by $A_0(l_1) = a$, then at least one of the following inequalities is true:

$$\frac{\delta_0^{p_0-2}}{R_0^{n+p}}\iint_{L_0}\Big(\frac{u_+}{\delta_0}\Big)^{(1+\frac{\lambda}{2})(p_0-1)}dx\,dt \ge \frac{a}{2}, \qquad (5.53)$$

$$\operatorname*{ess\,sup}_{0<t<T}\frac{1}{R_0^n}\int_{L_0(t)}\frac{u_+}{\delta_0}\,dx\,dt \ge \frac{a}{2}. \qquad (5.54)$$

Hence, using the boundedness of $u(x,t)$, we get

$$\delta_0 \le \gamma\Big(\frac{1}{R_0^n}\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u_+(x,t)\,dx\Big)^{\frac{1}{(1+\frac{\lambda}{2})(p_0-1)}}, \qquad (5.55)$$

which proves estimate (5.50). □

Definition 5.1. *For an arbitrary compact $E \subset Q_r \equiv B_r(x_0)\times(t_0-r^p, t_0+r^p)$ we define the parabolic capacity $\Gamma_p(E)$ by the equality*

$$\Gamma_p(E) = \inf\Big\{\operatorname*{ess\,sup}_{t}\int_{B_r(x_0)}\frac{1}{p(x)}|\varphi(x,t)|^{p(x)}\,dx$$
$$+\sum_{i=1}^{n}\iint_{Q_r}\frac{1}{p(x)}\Big|\frac{\partial\varphi(x,t)}{\partial x_i}\Big|^{p(x)}dx\,dt,\ \varphi\in\mathfrak{M}(E)\Big\},$$

where $\mathfrak{M}(E) = \{\varphi(x,t)\in C_0^\infty(Q_r) : \varphi(x,t)\ge 1,\ (x,t)\in E\}$.

Theorem 5.2. *Let all the conditions of Theorem* 5.1 *be satisfied. Then next inequality is true*

$$\inf\Big\{l : \int_0^1 \frac{\Gamma_p(\overline{Q_r \cap \{u > l\}})}{r^{n+1}}\,dr < \infty\Big\} \le \bar{l}, \tag{5.56}$$

where $\bar{l}$ *defined in Theorem* 5.1.

Proof. We need to prove the inequality

$$\int_0^1 \frac{\Gamma_p(\overline{Q_r \cap \{u > l+\varepsilon\}})}{r^{n+1}}\,dr < \infty \tag{5.57}$$

for every $\varepsilon \in (0,1)$. We define the function by $w_\varepsilon(x,t)$

$$w_\varepsilon(x,t) = \frac{1}{\varepsilon}\Big(\int_{\bar{l}+\varepsilon}^{u(x,t)} \Big(\frac{s-\bar{l}}{\varepsilon}\Big)^{-\frac{\lambda}{p_0}+\frac{\lambda}{2p_0}} \Big(\frac{s-\bar{l}-\varepsilon}{\varepsilon}\Big)^{-\frac{\lambda}{p_0}}\Big)_+. \tag{5.58}$$

Note that for $u(x,t) > \bar{l} + 2\varepsilon$

$$w_\varepsilon(x,t) \ge \frac{2^{1-\frac{1}{p_0}-\frac{\lambda}{2p_0}} - 1}{1 - \frac{1}{p_0} - \frac{\lambda}{2p_0}} = \mu. \tag{5.59}$$

Let $\{R_j\}, \{\xi_j(x)\}, \{\zeta_j(x)\}, \{\eta_j(x)\}$ $j = 1, 2, \ldots$ be the sequences defined earlier.

Let $\overline{\Theta}_j(t) \in C^\infty(R^1),\ 0 \le \overline{\Theta}_j(t) \le 1,$

$$\overline{\Theta}_j(t) = 1 \text{ for } |t - t_0| < \frac{4}{9}\frac{R_j^{p_0}}{\varepsilon^{p_0-2}},$$

$$\overline{\Theta}_j(t) = 0 \text{ for } |t - t_0| > \frac{R_j^{p_0}}{\varepsilon^{p_0-2}},$$

$$\Big|\frac{d\overline{\Theta}_j(t)}{dt}\Big| \le c\frac{\varepsilon^{p_0-2}}{R_j^{p_0}}.$$

Denote $Q'_j = \{B_j \cap \Omega\} \times (t_0 - R_j^{p_0}, t_0 + R_j^{p_0}),\ Q''_j = Q'_j \setminus G_j, G_j = \{x \in B_j \cap \Omega : g_j(x) > \frac{1}{3}\} \times (t_0 - R_j^{p_0}, t_0 + R_j^{p_0})$. From the definition of parabolic p-capacity and function $g_j(x)$, we have

$$\Gamma_p(\overline{G}_{j+1}, Q'_j) \le \gamma R_j^{p_0}[C_p(\overline{B}_j \setminus \Omega, B_{j-1}) + R_j^n]. \tag{5.60}$$

For $\varepsilon^{p_0-2} \le \frac{4}{9}$ we have

$$\Gamma_p(\overline{Q_{j+1} \cap \{u > \bar{l} - 2\varepsilon\}})$$

$$\le \gamma\mu^{-p_0}\Big\{\operatorname*{ess\,sup}_{0<t<T} \int_{B_j} |w_\varepsilon \zeta_j^{\frac{k}{p(x)}}(x)\xi_j^{\frac{k}{p(x)}}(x)\overline{\Theta}_j^{\frac{k}{p(x)}}(t)|^{p(x)}\,dx$$

$$+\sum_{i=1}^{n}\iint_{Q_j}\Big|\frac{\partial}{\partial x_i}(w_\varepsilon(x,t)\xi_j^{\frac{k}{p(x)}}(x)\zeta_j^{\frac{k}{p(x)}}(x)\overline{\Theta}_j^{\frac{k}{p(x)}}(t))\Big|^{p(x)}\,dx\,dt\Big\}. \qquad (5.61)$$

We suppose that j is so large, that

$$\frac{R_j^{p_0}}{\varepsilon^{p_0-2}} \le \min\{t_0, T-t_0\}. \qquad (5.62)$$

Using Lemma 5.1, we have

$$\begin{aligned}&\operatorname*{ess\,sup}_{0<t<T}\int_{\overline{L}_j(t)} G\Big(\frac{u-\bar{l}-\varepsilon}{\varepsilon}\Big)\xi_j^k(x)\zeta_j^k(x)\overline{\Theta}_j^k(t)\,dx \qquad (5.63)\\ &+\varepsilon^{p_0-2}\sum_{i=1}^{n}\iint_{\overline{L}_j}\Big|\frac{\partial w_\varepsilon}{\partial x_i}\Big|^{p(x)}\xi_j^k(x)\zeta_j^k(x)\overline{\Theta}_j^k(t)\,dx\,dt\\ &\le\gamma\frac{\varepsilon^{p_0-2}}{R_j^{p_0}}\int_{\overline{E}_j}\Big[\Big(1+\frac{u-\bar{l}-\varepsilon}{\varepsilon}\Big)^{1-\frac{\lambda}{2}}\Big(\frac{u-\bar{l}-\varepsilon}{\varepsilon}\Big)^{\lambda}\Big]^{p_0-1}\\ &\times\xi_j^{k-p_2}(x)\zeta_j^{k-p_2}(x)\overline{\Theta}_j^{k-1}(t)\,dx\,dt\\ &+\gamma\frac{R_j^{p_0}}{\varepsilon^{p_0-1}}\sum_{i=1}^{n}\int_{B_j}\Big|\frac{\partial}{\partial x_i}(\zeta_j(x)\eta_j(x))\Big|^{\varphi(x)}\,dx+\gamma\Big(\frac{1}{\varepsilon^{p_0-1}}+\frac{1}{\varepsilon^{p_0-2}}\Big)R_j^{n+p_0},\end{aligned}$$

where $\overline{L}_j=\overline{Q}_j\cap\Omega_T\cap\{u>\bar{l}+\varepsilon\}$, $\overline{Q}_j=B_j\times(t_0-\frac{R_j^{p_0}}{\varepsilon^{p_0-2}}, t_0+\frac{R_j^{p_0}}{\varepsilon^{p_0-2}})$, $\overline{E}_j=\overline{L}_j\cap\{\eta_j(x)<1\}$. We suppose also $j\ge J(\varepsilon)$ such that next inequality holds

$$\delta_j\le\varepsilon. \qquad (5.64)$$

The number $J(\varepsilon)$ exists due to the convergence $\sum_{j=1}^{\infty}\delta_j<\infty$ which follows from Theorem 5.1.

Let us estimate first term in (5.63), using inequality $A_j(l_{j+1})\le a$, which follows from the definition of l_{j+1}. We have

$$\begin{aligned}&\iint_{L_j}(u-l_j)^{(1+\frac{\lambda}{2})(p_0-1)}\xi_j^{k-p_2}(x)\zeta_j^{k-p_2}(x)\overline{\Theta}_j^{k-p_2}(t)\,dx\,dt\\ &\le\gamma R_j^{n+p_0}\delta_j^{1+\frac{\lambda}{2}(p_0-1)}. \qquad (5.65)\end{aligned}$$

From the definition of $L_j,\overline{L}_j,\Theta_j(t),\overline{\Theta}_j(t)$ and inequality (5.65) we have

$$\overline{L}_j\subset L_j,\qquad \overline{\Theta}_j(t)\le\Theta_j(t). \qquad (5.66)$$

Also we have for $(x,t)\in\overline{L}_j$

$$\Big(1+\frac{u-\bar{l}-\varepsilon}{\varepsilon}\Big)^{1-\frac{\lambda}{2}}\Big(\frac{u-\bar{l}-\varepsilon}{\varepsilon}\Big)^{\lambda}\le\gamma(\varepsilon)(u-l_j)^{1+\frac{\lambda}{2}}. \qquad (5.67)$$

Using (5.64)–(5.67), we get

$$\iint_{\overline{L}_j} \Big[\Big(1+\frac{u-\overline{l}-\varepsilon}{\varepsilon}\Big)^{1-\frac{\lambda}{2}}\Big(\frac{u-\overline{l}-\varepsilon}{\varepsilon}\Big)^{\lambda}\Big]^{p_0-1}\xi_j^{k-p_2}(x)\zeta_j^{k-p_2}(x)\overline{\Theta}_j^{k-1}(t)\,dx\,dt \\ \le \gamma(\varepsilon)R_j^{n+p_0}\delta_j^{1+\frac{\lambda}{2}(p_0-1)}. \tag{5.68}$$

From (5.61), (5.63), (5.68) we obtain

$$\Gamma_p(\overline{Q_j''\cap\{u>\overline{l}+2\varepsilon\}}) \le \gamma(\varepsilon)\{\delta_j^{1+\frac{\lambda}{2}(p_0-1)}R_j^n + R_j^p C_p(\overline{B}_{j-1}\backslash\Omega, B_{j-2}) + R_j^{n+p_0}\}. \tag{5.69}$$

Using definition Γ_p and (5.60), (5.69) we get

$$\sum_{j=J(\varepsilon)}^{\infty}\frac{\Gamma_p(\overline{Q_j'\cap\{u>\overline{l}+2\varepsilon\}})}{R_j^n} \le \gamma(\varepsilon)\Big\{\sum_{j-J(\varepsilon)}^{\infty}\delta_j^{1+\frac{\lambda}{2}(p_0-1)} \\ + \sum_{j=J(\varepsilon)}^{\infty}\frac{C_p(\overline{B}_{j-1}\setminus\Omega, B_{j-2})}{R_j^{n-p_0}} + \sum_{j=J(\varepsilon)}^{\infty}R_j^{p_0}\Big\}. \tag{5.70}$$

The right-hand side of (5.70) is finite which implies (5.57). This completes the proof of Theorem 5.2. □

Testing integral identity (2.2) by $\varphi(x,t)=u(x,t)-f(x)g(t)$, we get

$$\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u^2(x,t)\,dx + \int_0^T\int_\Omega\Big|\frac{\partial u(x,t)}{\partial x}\Big|^{p(x)}dx\,dt \\ \le \gamma(t_0)\Big\{\int_\Omega f^2(x)\,dx + \int_\Omega\Big|\frac{\partial f}{\partial x}\Big|^{p(x)}dx\Big\} \le \gamma(t_0)\varepsilon^2. \tag{5.71}$$

This inequality implies that

$$\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u_+(x,t)\,dx \le \gamma(t_0)\varepsilon. \tag{5.72}$$

Using Theorem 5.1, we have

$$\overline{l} \le \gamma(t_0)\Big\{\Big(\frac{\varepsilon}{R_0^n}\Big)^{\frac{1}{(1+\frac{\lambda}{2})(p_0-1)}} + R_0^{p_0} + \int_0^{2R_0}\Big\{\frac{C_p(\overline{B_r(x_0)}\setminus\Omega, B_{2r}(x_0))}{r^{n-p_0}}\Big\}^{\frac{1}{p_0-1}}\frac{dr}{r}\Big\}. \tag{5.73}$$

Using condition (5.1) we choose R_0 such that

$$\gamma(t_0)\Big\{R_0^{p_0} + \int_0^{2R_0}\Big\{\frac{C_p(\overline{B_r(x_0)}\setminus\Omega, B_{2r}(x_0))}{r^{n-p_0}}\Big\}^{\frac{1}{p_0-1}}\frac{dr}{r}\Big\} \le \frac{1}{4}. \tag{5.74}$$

Now we take ε from the condition

$$\gamma(t_0)\Big(\frac{\varepsilon}{R_0^n}\Big)^{\frac{1}{(1+\frac{\lambda}{2})(p_0-1)}} = \frac{1}{4}. \tag{5.75}$$

This guarantees the validity of the inequality

$$\bar{l} \leq \frac{1}{2}. \tag{5.76}$$

Using Theorem 5.2 and condition (5.1), we have for $l > \bar{l}$

$$\int_0 \frac{\Gamma_p(\overline{Q_r \cap \{u < l\}})}{r^{n+1}}\, dr = \infty. \tag{5.77}$$

This inequality and (5.76) imply

$$\liminf_{(x,t)\to(x_0,t_0)} u(x,t) \leq \tfrac{3}{4}, \quad (x,t) \in \Omega_T, \tag{5.78}$$

which proves that the point (x_0, t_0) is irregular. Theorem 2.1 is proved. □

References

[1] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal, **164** (2002), 213–259.

[2] E. Acerbi, G. Mingione, and Seregin G. A., *Regularity results for parabolic systems related to a class of non-Newtonian fluids*, Annales de l'Institut Henri Poincaré, Analyse Non Linéaire, **21** (2004), 25–60.

[3] E. Acerbi and N. Fusco, *A transmission problem in the Calculus of variations*, Cal. Var., **2** (1994), 1–16.

[4] Y. A. Alkhutov, *The Harnak inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition*, Differ. Uravn., **33**(12) (1997), 1651–1660.

[5] Y. A. Alkhutov and O. V. Krasheninnikava, *Continuity at boundary point of solutions of quasilinear elliptic equations with a non-standard growth condition*, Izvestija RAN, Ser. Math., **68**(6) (2004), 2–60.

[6] E. Di Benedetto, *Degenerate Parabolic Equations*, Springer–Verlag, New York, 1993.

[7] V. Chiado Piat and A. Coscia, *Hölder continuity of minimizers of functionals with variable growth exponent*, Manuscripta Math., **93**, (1997), 283–299.

[8] X. L. Fan and D. Zhao, *On the generalized Orlicz-Sobolev space* $W^{1,p(x)}(\Omega)$, J. Gansu Educ. College, **12**(1), (1998), 1–6.

[9] X. L. Fan and Q. H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlin. Anal., **52**, (2003), 1843–1852.

[10] X. L. Fan and D. Zhao, *A class of De Giorgi type and Hölder continuity*, Nonlinear Anal. T.M.A., **36** (1999), 295–318.

[11] R. Gariepy and W. P. Ziemer, *A regularity condition at the boundary for solutions of quasilinear elliiptic equations*, Arch. Rational Mech. Anal., **67** (1977), 25–39.

[12] T. Kilpelainen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math., **172** (1994), 137–161.

[13] O. A. Ladyzhenskaya and N. N. Ural'tseva, Linear and quasilinear elliptic equations, Nauka, Moscow, 1964, English transl., Academic Press, New York–London, 1968.

[14] J.-L. Lions, Quelques method de resolution des problems aux limites non lineaires, Guathies–Villlars, Paris, 1969.

[15] Meng Xu, Chen Ya Zhe, *Hölder Continuity of Weak Solutions for Parabolic Equations with Nonstandard Growth Conditions*, Acta Mathmatica Sinica, Engl. Series, **22**(3) (2006), 793–806.

[16] M. Ružhička, Electrorheological fluids: modeling and mathematical theory, Lecture Notes in Math., 1748, (2000).

[17] I. I. Skrypnik, *Regularity of a boundary point for degenerate quasilinear parabolic equations with measurable coefficients*, Ukrain. Math. Zh., **52**(11) (2000), 1550–1565.

[18] I. I. Skrypnik,*A necessaty condition for the regularity of a boundary point for degenerate parabolic equations with measutable coefficients*, Trudy Inst. Prikl. Math. Mech. Ukr. NAN, **8**, (2003), 147–167.

[19] I. I. Skrypnik,*Regularity of a boundaty point for singular parabolic equations with measurable coefficients*, Ukrain. Math. Zh., **56**(6) (2004), 818–836.

[20] I. I. Skrypnik,*On the Necessity of the Wiener Condition for Nonlinear Singular Paraboic Equations*, *Ukrain. Math. Bulletin*, **1**(3) (2004), 377–406.

[21] I. V. Skrypnik, *A necessary condition for the regularity of a boundary point for quasilinear parabolic equation*, Mat. Sbornik, **183**(7), (1992), 3–22.

[22] I. V. Skrypnik, Methods for Analysis of Nonlinear Elliptic Boundary Value Problems, Nauka, Moscow, 1990, English transl. Translations of Math. Monographs, 1994.

[23] N. Wiener, *Certain notions in potential theory*, J. Math. Phys., **3** (1924), 24–51.

[24] V. V. Zhikov, *On Lavrentiev's phenomen*, Russian J. Math. Phys, **3**:2 (1994), 249–269.

[25] V. V. Zhikov, *On some variational problems*, Russian J. Math. Phys, **5**(1) (1996), 105–116.

[26] V. V. Zhikov, *Meyer's type estimates for the solution of a nonlinear Stokes system*, Differ. Uravn., **33**(1) (1997), 107–114.

[27] W. P. Ziemer, *Behavior at the boundary of solutions of quasilinear parabolic equations*, J. Diff. Equat., **35**(3), (1980), 291–305.

[28] W. P. Ziemer, *Interior and boundary continuity of weak solutions of degenerate parabolic equations*, TAMS, **271**(2), (1982), 733–747.