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A FREE BOUNDARY CHARACTERIZATION OF MEASURE-VALUED SOLUTIONS FOR FORWARD-BACKWARD DIFFUSION

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Abstract. We consider a nonlinear one-dimensional scalar equation of diffusion type in which, depending on the gradient of the solution, the diffusion coefficient may be positive or negative. We compare two concepts of Young measure solutions which are based on different methods to construct approximate solutions, the SP-solutions (singular perturbation) and the EM-solution (energy minimization). We show that the SP-solution can recover classical solutions where the EM-solution fails to do so, and that EM-solutions are more stable under perturbations of the initial values. We characterize the EM-solution with a free boundary problem and determine its long-time behavior.

1. Introduction

Equations with forward-backward diffusion were studied for many reasons in the last years, among the applications are image enhancement processes and population dynamics. Both lead to equations of the form

$$\partial_t u = \nabla \cdot q\,(\nabla u) + f(u) \text{ in } (0,T) \times \Omega, \tag{1.1}$$

with a non-monotone function q. Regarding biological models with backward character we refer to [1, 6, 9]. In the image enhancement process suggested by Perona and Malik in [10], one studies $q(\nabla u) = \bar{q}(|\nabla u|)\nabla u$, with a diffusion function $\bar{q}(\cdot)$ that changes its sign, which allows for the effect of edge

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sharpening. In this work we study backward diffusion without reaction term, i.e.,

$$\partial_t u = \nabla \cdot q(\nabla u), \tag{1.2}$$

we restrict to the one-dimensional case, $q = \Phi'$, and we assume that Φ has quadratic growth at infinity. We note that the latter assumption is not satisfied in the application of the Perona-Malik equation. The backward diffusion is due to the non-convexity of Φ.

Despite its importance in applications, the theory of forward-backward diffusion is not yet fully developed. For studies from the theoretical point of view we refer to [2, 4, 5, 6, 7, 8, 9, 11] and the references therein. In general, one cannot expect the uniqueness of solutions as it was demonstrated for $\Omega = [0, 1]$ in [4, 5] for a piecewise linear heat flux $\bar{q}(s) \cdot s$, decreasing on the interval $[a, b] \subset \Omega$, and homogeneous Neumann boundary conditions. For the case of the Perona-Malik evolution equation it was shown that a continuum of weak solutions exists for finite time.

Furthermore, for general initial data we cannot expect the existence of classical solutions as was shown with an example in [8]. A positive result is the following maximum principle [7]: Suppose that $\bar{q}(s)s \geq 0$ for all $s \in \mathbb{R}$ and that u is a Lipschitz continuous weak solution to the Perona-Malik evolution equation equipped with either no-flux or homogeneous Dirichlet boundary conditions. Then, for every $p \in [2, \infty]$, there holds

$$||u(t, \cdot)||_{L^p(\Omega)} \leq ||u_0(\cdot)||_{L^p(\Omega)}.$$

A solution concept which is capable to provide existence and uniqueness is that of Young measure solutions, which we will describe in some detail below. Two different methods were employed to show the existence of a Young measure solution. Slemrod used in [11] a regularized version of equation (1.2) to construct a sequence of approximate solutions, and defines a generalized solution as the limit of such sequences. He justifies this method as being the physical relevant approximation with a vanishing viscosity. In the following, we refer to this solution as the SP-solution (for 'singular perturbation', see Definition 1). Another concept was introduced by Demoulini in [2]. She considered a time discretization and an implicit scheme in which, in each time step, an energy is minimized. The limit of such approximate solutions is again a Young measure solution and we refer to it as the EM-solution (for 'energy minimization', see Definition 1).

Our main contributions are as follows. We first note that the two concept provide different solutions in general (Observation A). This is a consequence of the following two facts. 1) The SP-solution coincides with the classical solution if no backward diffusion is encountered. 2) The EM-solution is characterized by a free boundary problem in which all gradients of non-optimal energy are avoided (Theorem 1). The solutions are different because the region with backward diffusion (s_-, s_+) is strictly contained in the set of non-optimal gradients, (z_-, z_+).

The characterization of the EM-solution with a free boundary problem also allows to characterize the long-time behavior of solutions. We furthermore note that, compared with the SP-solution approach, the EM-solution concept is more stable. Indeed, if we consider initial data for which the two solutions do not coincide, we find small perturbations of the initial values such that the corresponding SP-solutions are close to the EM-solution. In Theorem 3 we construct a sequence of initial values u_0^δ converging to the initial values u_0 uniformly for $\delta \to 0$, with the property that the corresponding SP-solutions u^δ fail to approximate the SP-solution to u_0 and, instead, converge to the EM-solution to u_0.

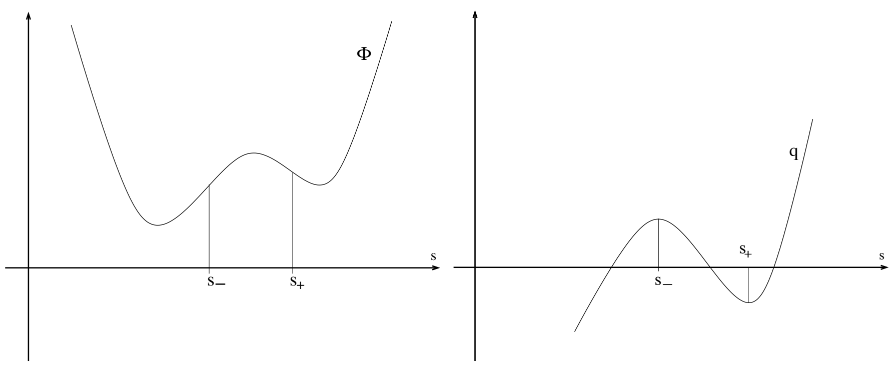

FIGURE 1. Possible shape of the functions $\Phi(s)$ and $q(s)$. Note that Φ may have only one local minimum in which case q changes sign only once.

We next want to fix the notations for this paper, the boundary conditions and the constitutive functions. We consider a potential $\Phi : \mathbb{R} \to \mathbb{R}$ with at least one local minimum and two turning points $s_\pm$, and $q = \nabla\Phi = \Phi'$ with two local extrema, one maximum at s_- and one minimum at s_+. We study (1.2) on space-time cylinders $\Omega_T := (0,T) \times \Omega$ with initial condition $u(0) = u_0$ and homogeneous Dirichlet boundary conditions. We are only considering the one-dimensional case and may therefore write (1.2) also as

$$\partial_t u = \partial_x q(\partial_x u) = q'(\partial_x u)\partial_x^2 u.$$

In particular, the equation exhibits backward diffusion for $s_- < \partial_x u < s_+$.

We consider the spatial domain $x \in \Omega := (0, L)$, a time-interval $(0, T)$, and functions $u : \Omega_T \to \mathbb{R}$ living on $\Omega_T := \Omega \times (0, T)$. The Dirichlet boundary condition thus reads, $u(0, \cdot) = 0 = u(L, \cdot)$ on $(0, T)$. On the initial values u_0 we always assume $u_0(0) = 0 = u_0(L)$, for the construction of SP-solutions we demand additionally $u_0 \in C^2([0, L])$ and $\Delta u_0(0) = 0 = \Delta u_0(L)$. On Φ and $q = \Phi'$ we always assume $q \in C^{1,1}(\mathbb{R})$ and

$$s_- < s_+ \text{ are the unique local extrema of } q, \tag{1.3}$$

$$\lim_{s\to\pm\infty} \frac{q(s)}{(1+|s|)} \text{ exist, and } \lim_{s\to\pm\infty} \frac{\Phi(s)}{(1+|s|^2)} \text{ exist and are positive.} \tag{1.4}$$

We will exploit the Maxwell construction. The convex envelope Φ^{**} of the potential Φ coincides with Φ outside some interval (z_-, z_+), in the interval $[z_-, z_+]$ the function Φ^{**} is affine. We have thus defined points $z_- < s_- < s_+ < z_+$, and can use $q^{**} = (\Phi^{**})'$ and the constant slope $q^* = q^{**}|_{[z_-,z_+]}$ in the following:

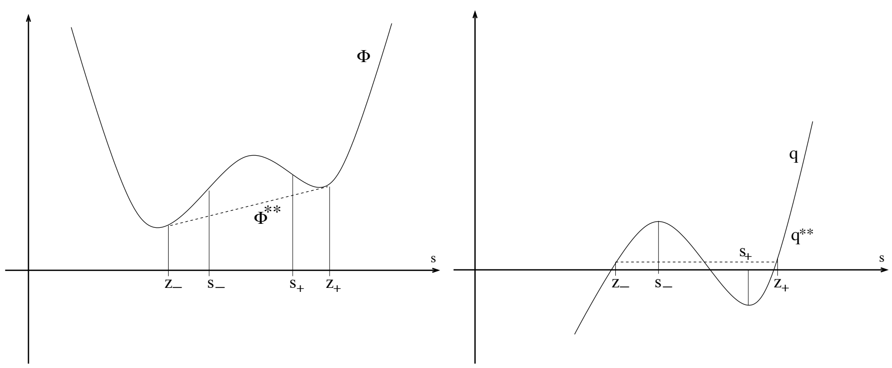

FIGURE 2. Left: $\Phi(s)$ and its convexification $\Phi^{**}(s)$. Right: corresponding function q and q^{**}. Marked is the interval (z_-, s_-) where $q \neq q^{**}$ although the equation exhibits forward diffusion.

1.1. **Young measure solutions.** Since $q'(\partial_x u)$ can be negative, equation (1.2) does not admit classical solutions in general. Slemrod constructed in [11] Young measure solutions as follows. He considers the finite viscosity approximation

$$\partial_t u^\varepsilon = \partial_x[q(\partial_x u^\varepsilon)] - \varepsilon^2 \partial_x^4 u^\varepsilon, \tag{1.5}$$

which is a semilinear parabolic equation. For all initial values u_0 as above, this equation has, locally in time, a solution u^ε, in the one-dimensional case the solution can be extended globally in time. The limit

$$u(t,x) := \lim_{\varepsilon \to 0} u^\varepsilon(t,x) \text{ a.e. in } \Omega_T \tag{1.6}$$

together with the Young measure $\nu_{t,x}$ generated by $\partial_x u^\varepsilon$ can be shown to be a Young measure solution, the SP-solution. We next give the definitions for Young measure solutions. We write $\mathrm{Prob}(\mathbb{R})$ for the space of all probability measures on $\mathbb{R}$. A measure $\mu \in \mathrm{Prob}(\mathbb{R})$ can be identified with a linear functional $\mu \in C_0(\mathbb{R},\mathbb{R})'$. For a function $g \in C_0(\mathbb{R},\mathbb{R})$ we write the application as

$$\langle \mu, g \rangle := \int_{\mathbb{R}} g(\lambda)\, d\mu(\lambda).$$

For technical reasons we introduce the set

$$\mathcal{E}_0^2(\mathbb{R}) := \left\{ \varphi \in C(\mathbb{R}) : \lim_{|a|\to\infty} \frac{\varphi(a)}{(1+|a|^2)} \text{ exists} \right\}.$$

A measure $\mu \in \mathrm{Prob}(\mathbb{R})$, which is additionally contained in the dual $\mathcal{E}_0^2(\mathbb{R})'$ (for instance because it has compact support), can be applied in particular to $\mathrm{id}_{\mathbb{R}}$, q, and $\mathrm{id}_{\mathbb{R}} \cdot q$.

Definition 1. *A pair (u,ν) is called a Young measure solution (YM-solution) to equation (1.2) if $u \in H^1(\Omega_T)$ and $\nu : \Omega_T \to \mathrm{Prob}(\mathbb{R}) \cap \mathcal{E}_0^2(\mathbb{R})'$ satisfy*

$$\partial_t u = \nabla \cdot \bar{q} \quad \textit{in the sense of distributions with} \tag{1.7}$$

$$\bar{q}(t,x) = \langle \nu_{t,x}, q \rangle, \tag{1.8}$$

$$\nabla u(t,x) = \langle \nu_{t,x}, \mathrm{id}_{\mathbb{R}} \rangle \quad \textit{for a.e. } (t,x) \in \Omega_T. \tag{1.9}$$

Furthermore, we demand that u satisfies the initial and boundary conditions in the sense of traces.

(1) *A YM-solution (u,ν) is called an SP-solution to equation (1.2) if for a sequence $\varepsilon \to 0$ and the corresponding sequence of solutions u_ε to equation (1.5) the function u is the limit a.e. of u_ε and ν is the Young measure generated by $\partial_x u_\varepsilon$.*

(2) *A YM-solution (u,ν) is called an EM-solution to equation (1.2), if it satisfies*

$$\langle \nu_{t,x}, q^{**} \rangle = \langle \nu_{t,x}, q \rangle \qquad \textit{for a.e. } (t,x) \in \Omega_T, \tag{1.10}$$

$$\langle \nu_{t,x}, \mathrm{id}_{\mathbb{R}} \cdot q^{**} \rangle = \langle \nu_{t,x}, \mathrm{id}_{\mathbb{R}} \rangle \cdot \langle \nu_{t,x}, q^{**} \rangle \qquad \textit{for a.e. } (t,x) \in \Omega_T. \tag{1.11}$$

Demoulini calls (1.10) and (1.11) the independence properties. They are satisfied e.g. if $\nu_{t,x}$ is, at every point (t,x), either a Dirac mass of the form $\nu_{t,x} = \delta_s$ with $s \notin [z_-, z_+]$, or of the form $\nu_{t,x} = \lambda\delta_{z_-} + (1-\lambda)\delta_{z_+}$ with $\lambda \in [0,1]$. The interpretation of such a measure is that either u is a function with derivative s, or u with slope $\lambda z_- + (1-\lambda)z_+$ represents the limit of functions with fine oscillations between the derivatives z_- and z_+.

1.2. **The results of S. Demoulini and M. Slemrod.** The existence of an SP-solution is established under quite general assumptions in [11], which are formulated for general space dimension.

(S1) $s \cdot q(s) \geq 0$, and $q = \nabla\Phi : \mathbb{R}^N \to \mathbb{R}^N$ for $\Phi \in C^1(\mathbb{R}^N)$,
(S2) q is a continuous mapping $\mathbb{R}^N \to \mathbb{R}^N$ satisfying

$$|q(s)| \leq c_1(1 + |s|^\gamma),\ 1 \leq \gamma < 2$$
$$c_2|s|^2 - c_3 \leq \Phi(s) \text{ for all } s \in \mathbb{R},$$

where $c_1,\ c_2,\ c_3$ are positive constants,
(S3) $\{s \in \mathbb{R}^N \mid s \cdot q(s) = 0\} \subset \{s \in \mathbb{R}^N \mid |s| < \rho_0\}$ for some $\rho_0 > 0$.

We note that our assumptions on q imply (S2) and (S3), and that we do not assume (S1). The main result of [11] is the existence of an SP-solution in one space dimension under assumptions (S1)-(S3).

Besides the existence, Slemrod proves some results on the asymptotic behavior of the SP-solutions, for example that for the homogeneous Dirichlet problem the ω-limit set for the initial data u_0 is the singleton $\{0\}$, and that the SP-Solution converges weakly to 0 as $t \to \infty$ in $H_0^1(\Omega)$, if $ker\{s \cdot q(s)\} := \{s \in \mathbb{R}^N \mid s \cdot q(s) = 0\} \subseteq (-\infty, 0]$ or $ker(q) \subseteq [0, \infty)$.

Demoulini's assumptions on q in [2] are more restrictive as regards the asymptotic behavior of q and Φ; the assumptions coincide with (1.4) in the one-dimensional case.

In [2] the existence of an EM-solutions is shown. Furthermore, it is shown that the u-part of the EM-solution is unique. The proof consists in making rigorous the following calculation. For two solutions $(\bar{u}, \bar{\nu})$ with $\bar{q} = \langle \bar{\nu}, q \rangle$, and $(\tilde{u}, \tilde{\nu})$ with $\tilde{q} = \langle \tilde{\nu}, q \rangle$ we multiply $\partial_t(\bar{u} - \tilde{u}) = \nabla \cdot (\bar{q} - \tilde{q})$ with $\bar{u} - \tilde{u}$. On the right hand side appears

$$\int_{\Omega_T} (\bar{q} - \tilde{q}) \cdot \nabla(\bar{u} - \tilde{u})$$
$$= \int_{\Omega_T} \int_{\mathbb{R}^2} q(\lambda_1)\lambda_2 \, d\bar{\nu}_{t,x}(\lambda_1) d\bar{\nu}_{t,x}(\lambda_2) - \int_{\Omega_T} \int_{\mathbb{R}^2} q(\lambda_1)\lambda_2 \, d\bar{\nu}_{t,x}(\lambda_1) d\tilde{\nu}_{t,x}(\lambda_2)$$

$$- \int_{\Omega_T} \int_{\mathbb{R}^2} q(\lambda_2)\lambda_1 \, d\bar{\nu}_{t,x}(\lambda_1) d\tilde{\nu}_{t,x}(\lambda_2) + \int_{\Omega_T} \int_{\mathbb{R}^2} q(\lambda_1)\lambda_2 \, d\bar{\nu}_{t,x}(\lambda_1) d\tilde{\nu}_{t,x}(\lambda_2)$$
$$= \int_{\Omega_T} \int_{\mathbb{R}^2} [q^{**}(\lambda_1) - q^{**}(\lambda_2)] \cdot (\lambda_1 - \lambda_2) \, d\bar{\nu}_{t,x}(\lambda_1) d\tilde{\nu}_{t,x}(\lambda_2) \geq 0,$$

where we use first (1.10) and then (1.11) in the second equality. The inequality in the last line is due to the convexity of Φ^{**}. The calculation implies that

$$- \int_0^T \int_\Omega \partial_t(\bar{u} - \tilde{u}) \cdot (\bar{u} - \tilde{u}) \, dx \, dt \geq 0,$$

and thus,

$$||\bar{u}(T, \cdot) - \tilde{u}(T, \cdot)||_{L^2(\Omega)} \leq ||\bar{u}_0(\cdot) - \tilde{u}_0(\cdot)||_{L^2(\Omega)}$$

for all $T > 0$, and in particular the uniqueness of solutions.

As we have seen, the uniqueness follows essentially from equations (1.10) and (1.11), which have not to be fulfilled by SP-solutions. In fact, no uniqueness result is known for the SP-solution. Demoulini also proves results concerning the long time behaviour of solutions. She shows that the ω-limit set of every EM-solution is a singleton $\{z\}$, and that the EM-solution converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ towards the unique function $z \in H_0^1(\Omega)$ as $t \to \infty$, where the equilibrium solution z is a measure valued solution of the steady-state problem in $H^{-1}(\Omega \times \mathbb{R}^+)$.

With this work we explicitly calculate the nontrivial singleton z in the one-dimensional case under general assumptions on u_0.

2. Main results

Our first aim is to demonstrate that the SP-solution may in fact differ from the EM-solution. One difference of the two concepts lies in the method how gradients in the intervals (z_-, s_-) and (s_+, z_+) are treated. For such gradients the equation has forward character, since q' is positive. Consequently, for the SP-solution we approximate a parabolic problem with singular perturbations and will recover the classical solution of the parabolic problem. In contrast, a solution satisfying (1.10) and (1.11) must avoid gradients in (z_-, z_+). The idea is that the function is replaced by a fine mixture of gradients in $\{z_\pm\}$, and the evolution equation is no longer the original parabolic problem.

Observation A. *Assume $z_- < 0 < s_-$ and consider initial values $u_0 \in H^4(\Omega)$ on $\Omega = (0, L)$ satisfying $u_0 = \Delta u_0 = 0$ on $\partial\Omega$ and $|\partial_x u_0| < m := min\{|z_-|, |s_-|\}$. Then, for any $T > 0$, there exists a classical solution u_c*

to equation (1.2) *on* $[0,T)$. *There exists* $T_0 > 0$ *such that the SP-solution coincides with the classical solution* u_c *on the time interval* $[0,T_0)$. *Instead, the EM-solution is given by the constant function*

$$u(t,x) = u_0(x) \quad \forall t \in [0,T], x \in [0,L].$$

Proof. The results concerning the classical solution and the SP-solution regard classical solution concepts for forward parabolic equations and are shown in the Appendix.

Concerning the second part of the observation we have to show that the constant function u_0 is an EM-solution i.e., that (1.9)-(1.13) are satisfied for $u(t,x) = u_0(x)$. We find $\sigma : (0,L) \to [0,1]$ as the fraction of points where gradient z_- is used, and can then define ν.

$$\sigma(x)z_- + (1-\sigma(x))z_+ = \partial_x u_0(x),\ \nu_{t,x} := \sigma(x)\delta_{z_-} + (1-\sigma(x))\delta_{z_+}.$$

The definition of ν is made in order to match (1.11). Equation (1.10) holds by definition for

$$\bar{q}(t,x) = \langle \nu_{t,x}, q\rangle = \sigma(x)q(z_-) + (1-\sigma(x))q(z_+) = q^*.$$

We find that $\bar{q}$ is independent of x. Since u is independent of t, there holds $\partial_x u = 0 = \partial_x \bar{q}$, which is (1.9). We conclude by calculation the independence properties for $p = q^{**}$,

$$\begin{aligned}
\langle \nu_{t,x}, p\rangle &= \sigma p(z_-) + (1-\sigma)p(z_+) \ = \ \sigma q(z_-) + (1-\sigma)q(z_+) = \langle \nu_{t,x}, q\rangle,\\
\langle \nu_{t,x}, id_{\mathbb{R}} \cdot p\rangle &= \sigma z_- p(z_-) + (1-\sigma)z_+ p(z_+) = \sigma z_- q^* + (1-\sigma))z_+ q^*,\\
\langle \nu_{t,x}, id_{\mathbb{R}}\rangle &= \sigma z_- + (1-\sigma)z_+,\\
\langle \nu_{t,x}, p\rangle &= \sigma p(z_-) + (1-\sigma)p(z_+) = q^*.
\end{aligned}$$

Therefore, (1.12) and (1.13) are satisfied and $u(t,\cdot) = u_0(\cdot)$ is the EM-solution. □

We remark that the assumption $|\partial_x u_0| < m := min\{|z_-|, |s_-|\}$ is only needed for the statement about the EM-solutions. In fact, the assertion on the EM-solution in Observation A is a special case of the following result.

Theorem 1 (Characterization of the EM-solution)**.** *Let the initial values* $u_0 \in C^2(\bar{\Omega})$ *satisfy homogeneous Dirichlet boundary conditions and*

$$\begin{aligned}
\partial_x u_0 &\geq z_+ \ \textit{in}\ [0,x_1),\\
\partial_x u_0 &\in (z_-, z_+) \ \textit{in}\ (x_1,x_2),\\
\partial_x u_0 &\leq z_- \ \textit{in}\ (x_2,L].
\end{aligned}$$

Then the EM-solution is locally given by the unique solution of the following free boundary problem (P).

Problem (P) : *Find* $u \in C(\bar{\Omega} \times [0,T], \mathbb{R}) \cap L^2((0,T), H^1(\Omega,\mathbb{R}))$ *with* $u(t,0)=0=u(t,L)$*, and monotone* $X_1, X_2 \in C([0,T],[0,L]) \cap W^{1,1}((0,T),\mathbb{R})$*,* X_1 *non-decreasing and* X_2 *non-increasing, such that*

$$X_1(0) = x_1, \ \ X_2(0) = x_2, \tag{2.1}$$

$$\partial_t u(t,\cdot) = 0 \ \text{ in } (X_1(t), X_2(t)), \tag{2.2}$$

$$\partial_t u(t,\cdot) = \partial_x[q(\partial_x u)] \ \text{ in } (0,L) \setminus (X_1(t), X_2(t)), \tag{2.3}$$

$$\partial_x u(t, X_1(t)-\varepsilon) \to z_+, \partial_x u(t, X_2(t)+\varepsilon) \to z_-, \quad \text{for } \varepsilon \searrow 0. \tag{2.4}$$

We will construct classical solutions to these equations. Uniqueness holds on the larger class of weak solutions since it is a consequence of the uniqueness result for EM-solutions. To write (2.3) (2.4) in a weak form one demands (written for the left interval)

$$-\int_0^T \int_0^{X_1(t)} u \cdot \partial_t \Phi - \int_0^T u_0(X_1(t))\Phi(t, X_1(t))\, \partial_t X_1(t)\ dt$$
$$= -\int_0^T \int_0^{X_1(t)} q(\partial_x u)\, \partial_x \Phi + \int_0^T q^* \,\Phi(t, X_1(t)) \quad \forall \Phi \in C_0^\infty((0,T)\times(0,\infty)).$$

In fact, the boundary condition (2.4) should be understood as the condition of a continuous flux function. The jump of the effective flus q^{**} vanishes,

$$[q^{**}(\partial_x u(t,.))] = 0 \text{ in } X_{1,2}(t).$$

The characterization of EM-solutions in Theorem 1 allows to study the long-time behavior of solutions.

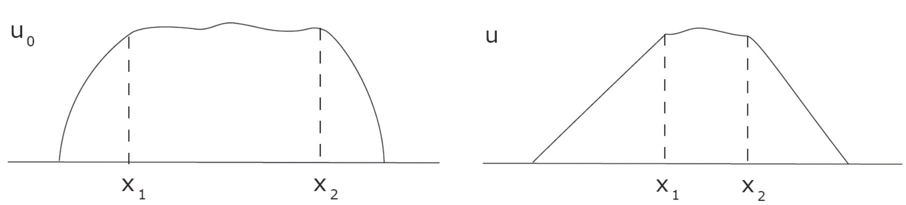

FIGURE 3. Evolution of the free boundary according to Theorem 2.

Theorem 2 (Asymptotic behavior of the EM-solution)**.** *Let the initial values* $u_0 \in C^2(\bar{\Omega})$ *be as in Theorem 1 with* $z_- < 0 < z_+$*. Then we can characterize the long time behavior of EM-solutions.*

(1) *If* $u_0(\xi) < \min\{z_+\xi, z_-(\xi - L)\}$ *for some* $\xi \in (0, L)$, *then the EM-solution* u *exists for all times and is characterized for all times by problem* (P). *For all* $x \in [0, L]$ *we find*

$$u(x,t) \to \min\{u_0(x), z_+x, z_-(x - L)\} \text{ for } t \to \infty.$$

(2) *If* $u_0(\xi) \leq \min\{z_+\xi, z_-(\xi - L)\}$ *for all* $\xi \in (0, L)$, *and* $\partial_x^2 u_0 \leq 0$ *on* $(0, x_1) \cup (x_2, L)$, *then the EM-solution is characterized by problem* (P) *on an interval* $(0, T)$ *for some* $T \in (0, \infty]$, *and by the free boundary problem* (P') *of page 20) on* (T, ∞). *For all* $x \in [0, L]$ *there holds*

$$u(x,t) \to \min\{z_+x, z_-(x - L)\} \text{ for } t \to \infty.$$

We refer to Remark 1 on page 22 for comments on the case $\mathrm{sgn}(z_-) = \mathrm{sgn}(z_+)$.

The next theorem shows an instability property in the SP-solution concept. To be precise, we show that an arbitrary small perturbation of the initial values may lead to a very different solution.

Theorem 3 (Discontinuous dependence of the SP-solution). *We study the case* $z_- < 0 < z_+$ *and initial values* $u_0 \in C^1(\bar{\Omega})$ *with* $z_- < \partial_x u_0 < z_+$ *satisfying the homogeneous Dirichlet condition. We use the constant continuation* $u_{EM}(x,t) = u_0(x)$, *which is the EM-solution to the problem by Observation A. There exists a sequence of perturbations* $u_0^\delta \in C^1(\bar{\Omega})$ *of the initial values,*

$$\|u_0^\delta - u_0\|_{L^\infty} \to 0 \text{ for } \delta \to 0,$$

such that every corresponding SP-solution $u_{SP}^\delta : \Omega \times [0, T] \to \mathbb{R}$ *satisfies*

$$u_{SP}^\delta(t, \cdot) \nrightarrow u_{EM} \text{ in } L^\infty(\Omega) \text{ for } \delta \to 0. \tag{2.5}$$

The theorem has a direct geometric interpretation. The initial values have gradients that are not optimal in the following sense: we find piecewise affine functions with the same global shape that use only gradients, that are energetically favored (compare figure 2). The modified initial values are in the vicinity of a minimum of the energy landscape, and the corresponding time-dependent solutions can not escape from the minimum.

Remarks on the higher dimensional case. Observation A and Theorem 3 concern the case of initial values with small gradients and carry over to the higher dimensional case. We recall that in the n-dimensional case one considers $\Phi : \mathbb{R}^n \to \mathbb{R}$ with $q := \nabla\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and the equation $\partial_t u(x) = \nabla \cdot [q(\nabla u(x))]$ (see [2] for precise assumptions on Φ and existence

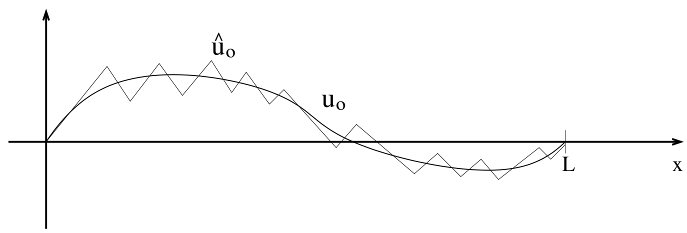

FIGURE 4. Illustration of Theorem 3 and its proof: Initial data u_0 and an L^∞ approximation $\hat{u}_0$.

and uniqueness results for the EM-solution). As in the one-dimensional case, we denote by Φ^{**} the convexification of Φ and define the convex set

$$D := \{v \in \mathbb{R}^n \mid \Phi(v) \neq \Phi^{**}(v)\}\,.$$

Observation A in n dimensions reads as follows. Denote by

$$F := \left\{v \in \mathbb{R}^n \mid D^2\Phi(v) \text{ positive definite}\right\}$$

the collection of gradients that generate forward diffusion. Note that the complement of F is contained in D, but, in general, the set $F \cap D$ is non-empty. Let us assume that the ball $B_r(0) \in \mathbb{R}^n$ with $r > 0$ is contained in $F \cap D$ and that the initial values $u_0 \in H^4(\Omega)$ and Δu_0 satisfy homogeneous Dirichlet conditions and that $|\nabla u_0(x)| < r$ for almost all $x \in \Omega$. Then, the EM-solution is the constant function $u_{EM}(t,x) = u_0(x)$. Instead, the SP-solution, if it exists, coincides with the classical solution of the forward problem.

Concerning Theorem 3 in n dimensions we assume that the initial values $u_0 \in C^1(\bar{\Omega})$ satisfy homogeneous Dirichlet conditions and $\nabla u_0(x) \in D$ for all $x \in \Omega$. Then, the EM-solution to these initial values is again the constant function $u_{EM}(t,x) = u_0(x)$. On the other hand, there exists a sequence $u_0^\delta \to u_0$ in L^∞ as in Theorem 3 satisfying (2.5) for every corresponding sequence of SP-solutions.

The proofs for the described generalizations are straightforward adaptations of the one-dimensional case. The existence of the classical solution and the convergence of the ε-solutions are as in the appendix with the exception that one has to assume the existence of the SP-solution as this result is not available in higher space dimensions. The calculation for the generalization of Theorem 3 is identical to that in the one-dimensional case starting on page 23. One has to approximate u_0 by a function u_0^δ with gradient almost everywhere in ∂D.

A challenging problem is to generalize Theorem 1 to n space dimensions, i.e., to characterize the EM-solution with a free boundary problem.

3. PROOFS

Proof of Theorem 1. The first part of the proof consists in checking that every solution (X_1, X_2, u) of the free boundary problem (2.1)-(2.4) provides an EM-solution. Indeed, we may set $\Sigma_t := (X_1(t), X_2(t))$ and

$$\nu_{t,x} := \begin{cases} \sigma\,\delta_{z_-} + (1-\sigma)\,\delta_{z_+} & \text{for } x \in \Sigma_t, \partial_x u(t,x) = \sigma z_- + (1-\sigma) z_+, \\ \delta_{\partial_x u(t,x)} & \text{else.} \end{cases}$$

Then, with

$$\bar{q}(t,x) := \langle \nu_{t,x}, q \rangle = \begin{cases} q(z_-) & \text{for } x \in \Sigma_t, \\ q(\partial_x u(t,x)) & \text{else,} \end{cases}$$

one checks that (1.7)-(1.11) are satisfied.

The second part of the proof consists in verifying that the free boundary problem has a solution, at least locally in time. This is shown in Lemma 2, assertion 1. Note that the uniqueness result for Young measure solutions yields also the uniqueness of solutions to the free boundary value problem. □

Our next aim is to prove the existence result for the free boundary problem which will be done with methods similar to those developed in [3]. To this end we first note that the problem can be treated with the same methods on $(0, X_1(t))$ and on $(X_2(t), L)$, we thus restrict ourselves to the existence proof on a single interval $(0, X(t))$. We write the problem with $x_0 = x_1$, $z = z_+$, $a(t,x) = q'(\partial_x u(t,x))$, and $\tilde{u}_0(x) := u_0(x)$. We search for strong solutions, i.e., for (u, X) satisfying

$$(P_u) \begin{cases} X(\cdot) \in C^0([0,T], \mathbb{R}), 0 \le \partial_t X \in L^1((0,T)), \quad X(0) = x_0, \\ u(t,\cdot) \in H^2((0,X(t)), \mathbb{R}) \text{ bounded, uniformly in } t, \quad u(0,\cdot) = u_0, \\ \partial_t u = a \partial_x^2 u \text{ on } \bigcup_{t \in (0,T)} \{t\} \times (0, X(t)), \\ u(t,0) = 0, \quad u(t, X(t)) = \tilde{u}_0(X(t)), \quad \partial_x u(t, X(t)) = z \quad \forall t. \end{cases}$$

The solvability of this system provides the solvability of the free boundary problem (P). We will construct solutions (u, X) of the above system by integrating solutions (v, X) of the subsequent system, which is formally obtained by introducing $v := \partial_x u - z$, and by setting $\tilde{v}_0 := v_0 := \partial_x u_0 - z$.

A critical quantity in the problem is the boundary datum $\tilde{v}_0$. While the initial values must be non-negative on $(0, x_0)$ in order to apply a maximum principle, we wish to have the datum $\tilde{v}_0$, which determines the evolution of the free boundary, strictly negative in an interval on $(x_0, x_0 + \Delta x)$. We therefore distinguish the two functions. Thus we can approximate the initial datum $\tilde{v}_0$ with a sequence of functions $\tilde{v}_0^\varepsilon$ satisfying $\tilde{v}_0^\varepsilon < -\varepsilon$. We refer to the following problem for (v, X) as problem (P_v^ε).

$$(P_v^\varepsilon)\begin{cases} X(\cdot) \in C^0([0,T],\mathbb{R}), 0 \leq \partial_t X \in L^1((0,T)), \\ X(0) = x_0, \\ v(t,\cdot) \in H^1((0,X(t)),\mathbb{R}) \text{ bounded, uniformly in } t, \\ v(0,\cdot) = v_0, \\ \partial_t v = \partial_x(a\partial_x v) \text{ on } \bigcup_{t\in(0,T)} \{t\} \times (0, X(t)), \\ \partial_x v(t,0) = 0 \quad \forall t, \\ v(t,X(t)) = 0 \quad \forall t, \\ \partial_x v(t,X(t)) = \dfrac{1}{a(t,X(t))} \tilde{v}_0^\varepsilon(X(t))\, \partial_t X(t) \quad \forall t. \end{cases}$$

The last equation is understood in the weak sense as the natural Neumann condition. To be precise, we demand that

$$\partial_t \int_0^{X(t)} v(t,x)\varphi(x)\,dx = -\int_0^{X(t)} a(t,x)\partial_x v(t,x)\partial_x\varphi(x)\,dx + \partial_t X(t)\tilde{v}_0^\varepsilon(X(t))\varphi(X(t)) \tag{3.1}$$

in the distributional sense on $(0,T)$ for all $\varphi \in C_0^1([0,\infty),\mathbb{R})$.

In order to make clear the connection between the two problems, let us consider a solution v^ε of (P_v^ε). We can define a function

$$u^\varepsilon(t,x) := \int_0^x [v^\varepsilon(t,s) + z]\,ds$$

and calculate

$$\begin{aligned} u^\varepsilon(0,x) &= \int_0^x [v_0(t,\cdot) + z] = u_0(x), \quad u^\varepsilon(t,0) = 0, \\ \partial_t u^\varepsilon(t,x) &= \int_0^x \partial_t v^\varepsilon(t,\cdot) = a(t,x)\partial_x v^\varepsilon(t,x) = a(t,x)\partial_x^2 u^\varepsilon(t,x), \\ \partial_x u^\varepsilon(t,X(t)) &= v^\varepsilon(t,X(t)) + z = z. \end{aligned}$$

For the remaining equality we formally differentiate the values at the free boundary with respect to time,

$$\begin{aligned}\frac{d}{dt}u^\varepsilon(t,X(t)) &= \frac{d}{dt}[zX(t)] + \frac{d}{dt}\int_0^{X(t)} v^\varepsilon(t,s)\,ds \\ &= [z + v^\varepsilon(t,X(t))]\,\partial_t X(t) + \int_0^{X(t)} \partial_t v^\varepsilon(t,s)\,ds \\ &= z\,\partial_t X(t) + a(t,X(t))\partial_x v^\varepsilon(t,X(t)) = [z + \tilde{v}_0^\varepsilon(t,X(t))]\,\partial_t X(t).\end{aligned}$$

An integration over time yields the remaining u-equation, $u^\varepsilon(t,X(t)) = \tilde{u}_0(X(t))$, where $\tilde{u}_0^\varepsilon$ is the primitive of $\tilde{v}_0^\varepsilon$. We conclude that every limit of u^ε for $\varepsilon \to 0$ is a solution of (P_u). The main task is therefore to show that (P_v^ε) has a solution. The principal idea in the next two lemmas is to solve (P_v^ε) for strictly negative $\tilde{v}_0^\varepsilon$ on a time interval, which does not depend on the supremum of $\tilde{v}_0^\varepsilon$. This allows to take the limit $\varepsilon \to 0$.

Lemma 1 (Solution of (P_v^ε))**.** *For points $0 < x_0 < x_1 < \infty$ let initial values $v_0 \in H^1((0,x_0),\mathbb{R})$ be non-negative with $v_0(x_0) = 0$, and the free boundary datum $\tilde{v}_0^\varepsilon = \tilde{v}_0 \in H^1([x_0,x_1),\mathbb{R})$ be negative.*

(1) *Given coefficients $a \in L^\infty((0,T),H^1((0,x_1))) \cap H^1((0,T),L^2((0,x_1)))$ with $a \geq a_{\min} > 0$, we define the number*

$$\mu_0 := \frac{1}{a_{\min}} + \|\partial_t a\|_{L^2([0,T]\times(0,x_1))} + \frac{1}{x_0} + \frac{1}{x_1 - x_0} + \frac{1}{\int_{x_0}^{x_1} |\tilde{v}_0|}.$$

There exists $T > 0$ depending only on μ_0, such that problem (P_v^ε) has a solution (v,X) on $(0,T)$. The solution satisfies an estimate

$$\begin{aligned}\sup_t \|v\|^2_{H^1((0,X(t))} + \int_0^T \|v\|^2_{H^2((0,X(t))}\,dt \\ + \int_0^T |\partial_x v(t,X(t))|^2\,\partial_t X(t)\,dt \leq C(\mu_0, \|v_0\|_{H^1}).\end{aligned} \tag{3.2}$$

(2) *There exists $T > 0$ depending only on x_0, x_1, and $\|v_0\|_{H^1}$, and a solution of (P_v^ε) on $(0,T)$ with coefficients $a(t,x) = q'(v(t,x))$.*

Proof. *Ad 1.* We define the positive number

$$\begin{aligned}\varepsilon := \min\{ &\inf_{x\in[x_0,x_1]} (-\tilde{v}_0(x)), \\ &1/(\|a\|_{L^\infty H^1} + \|v_0\|_{H^1((0,x_0))} + \|\tilde{v}_0\|_{C^0((0,x_0))} + \mu_0)\}\,.\end{aligned} \tag{3.3}$$

Now we prove assertion 1 in two steps: In Step 1 we solve the problem on an interval $(0,\tau_\varepsilon)$ which depends on ε. In Step 2 we continue the solution to a time-interval $(0,T)$ which depends on μ_0, but is independent of ε.

Step 1. Solution on a time-interval $(0,\tau_\varepsilon)$. We can transform the system into new coordinates $\xi = x/X(t)$ such that we have to solve for the unknown function $V : [0,T]\times[0,1] \to \mathbb{R}$, $V(t,\xi) = v(t,\xi X(t))$. The V-equations now read

$$\begin{aligned}&\partial_t V(t,\xi) = X(t)^{-1}\partial_\xi\left[\frac{a(t,\xi X(t))}{X(t)}\,\partial_\xi V(t,\xi)\right] + \xi\frac{\partial_t X(t)}{X(t)}\,\partial_\xi V(t,\xi),\\ &\partial_\xi V(t,0) = 0, \quad V(t,1) = 0, \quad V(0,\xi) = v_0(\xi x_0).\end{aligned} \tag{3.4}$$

Given a guess for $X(\cdot) \in H^1((0,\tau),\mathbb{R})$ problem (3.4) is a standard parabolic problem and we can solve for V on a short time interval depending on ε. We then improve our guess for X by inverting the last equation of (P_v^ε), i.e. we integrate $X_{new}(0) = x_0$ and

$$\partial_t X_{new}(t) = \frac{1}{X(t)}\partial_\xi V(t,1)\,a(t,X(t))\,[\tilde v_0(X(t))]^{-1}. \tag{3.5}$$

It remains to specify function spaces for this iteration.

Claim. There exists $\bar q > 2$ (independent of ε), $\tau_\varepsilon > 0$, and a solution X, V on $(0,\tau_\varepsilon)$ with

$$X \in W^{1,\bar q}([0,\tau_\varepsilon],\mathbb{R}), \quad V \in L^2([0,\tau_\varepsilon],H^2((0,1))) \cap L^\infty([0,\tau_\varepsilon],H^1((0,1))).$$

Proof of the Claim. We use the interpolation inequality which states that the embedding

$$j : L^2([0,\tau_\varepsilon],H^2) \cap L^\infty([0,\tau_\varepsilon],H^1) \to L^q([0,\tau_\varepsilon],H^{7/4})$$

is well-defined and continuous for some $q > 2$ (in fact for $q \le 8/3$). We fix such q and set $\bar q := (2+q)/2$.

To see that the iteration defined by (3.4) and (3.5) is indeed well defined in the above spaces, it suffices to multiply (3.4) with $\partial_\xi^2 V$ and to integrate. Since a and X are strictly positive on a small time interval, we find an a priori estimate for V in the spaces above. Here, we exploit that the terms without a determined sign (coming from the derivative of a and from the last term) can be estimated by

$$C\int_0^\tau\int_0^1 |\partial_x a|\,|\partial_\xi V|\,|\partial_\xi^2 V| \le C(\tau)^{(q-2)/2q}\|a\|_{L^\infty H^1}\,\|\partial_\xi V\|_{L^q L^\infty}\,\|V\|_{L^2H^2},$$

$$C\int_0^\tau\int_0^1 \partial_t X\ |\partial_\xi V|\ |\partial_\xi^2 V| \le C(\tau)^{(\bar q-2)/2\bar q}\|\partial_t X\|_{L^{\bar q}}\ \|\partial_\xi V\|_{L^\infty L^2}\ \|V\|_{L^2H^2}.$$

For small τ, these expressions are small compared to the norms of V and X of the claim, since q and $\bar q$ are larger than 2.

By similar calculations one verifies that the above iteration is contractive. Let us consider two different guesses for X, say $\bar X$ and $\tilde X$, that are mapped to two different solutions $\bar V$ and $\tilde V$ of (3.4). These satisfy

$$\|\bar V-\tilde V\|_{L^q([0,\tau_\varepsilon],H^{7/4})} \le C\|\bar X-\tilde X\|_{W^{1,\bar q}([0,\tau_\varepsilon],\mathbb{R})}$$

on a small time interval. The corresponding free boundary position functions $\bar X_{new}$ and $\tilde X_{new}$ then satisfy

$$\begin{aligned}\|\bar X_{new}-\tilde X_{new}\|_{W^{1,\bar q}([0,\tau_\varepsilon],\mathbb{R})} &\le C\ \|\partial_t(\bar X_{new}-\tilde X_{new})\|_{L^{\bar q}}\\ &\le C\ (\tau_\varepsilon)^{(q-2)/q(2+q)}\|\partial_t(\bar X_{new}-\tilde X_{new})\|_{L^q}\\ &\le C\ (\tau_\varepsilon)^{(q-2)/q(2+q)}\cdot\frac{1}{\varepsilon^2}\|\bar V-\tilde V\|_{L^q([0,\tau_\varepsilon],H^{7/4})},\end{aligned}$$

where we used Hölders inequality with

$$\frac{1}{q}+\frac{1}{q'}=\frac{1}{\bar q}\ \text{for}\ q'=\frac{q(2+q)}{q-2}.$$

This provides the contractiveness of the iteration map for small τ_ε. We have thus verified the claim.

In order to continue the solution in Step 2 to a larger time interval, we note that V remains non-negative by the maximum principle. As a consequence, the time derivative of X in (3.5) remains non-negative.

Step 2. Continuation of the solution to a fixed time-interval. The solution can be continued to a fixed time interval $[0,T]$ if the number ε of (3.3) remains positive with a lower bound. To show this, it suffices to verify that $\|v^\varepsilon(t)\|_{H^1}$ remains bounded, and to exclude $X(t)\to x_1$ on $(0,T)$.

We derive the a priori estimate for v^ε in the original variables on the time-dependent interval $(0,X)=(0,X^\varepsilon)$.

$$\begin{aligned}&\frac{d}{dt}\int_0^{X(t)}\frac12 a|\partial_x v|^2(t)\\ &=\frac12 a|\partial_x v(t,X(t))|^2\,\partial_t X(t)+\int_0^{X(t)} a\partial_x v\cdot\partial_x\partial_t v+\int_0^{X(t)}\frac12\partial_t a|\partial_x v|^2\\ &=\frac12 a|\partial_x v(X(t))|^2\,\partial_t X(t)+a(t,X(t))\partial_x v(t,X(t))\cdot\partial_t v(X(t))\end{aligned}$$

$$-\int_0^{X(t)} |\partial_x(a\partial_x v)|^2 + \int_0^{X(t)} \frac{1}{2}\partial_t a|\partial_x v|^2$$

In order to evaluate the second term we must determine $\partial_t v(t, X(t))$. To this end we differentiate $v(t, X(t)) = 0$ with respect to time and find

$$\partial_t v(t, X(t)) = -\partial_x v(t, X(t))\, \partial_t X(t).$$

The second term equals the first up to a factor $1/2$ and we find

$$\begin{aligned}\frac{d}{dt}\int_0^{X(t)} \frac{a}{2}|\partial_x v|^2 + \frac{a}{2}|\partial_x v(X(t))|^2\, \partial_t X(t) + \int_0^{X(t)} |\partial_x(a\partial_x v)|^2 \\ = \int_0^{X(t)} \frac{1}{2}\partial_t a|\partial_x v|^2.\end{aligned}$$

With $q > 2$ as in Step 1, c_0 depending on the lower bound of a, and with $C_0 := \|v_0\|_{H^1}^2$, we can calculate

$$\begin{aligned}c_0(\|v\|_{L^\infty H^1}^2 + \|v\|_{L^2H^2}^2) &\le C_0 + \int_0^T \int_0^{X(t)} \frac{1}{2}\partial_t a(t,x)|\partial_x v(t,x)|^2\, dx\, dt \\ &\le C_0 + \int_0^T \|\partial_x v(t,.)\|_{L^\infty} \int_0^{X(t)} |\partial_t a(t,x)|\; |\partial_x v(t,x)|\, dx\, dt \\ &\le C_0 + T^\delta \|\partial_x v\|_{L^qL^\infty} \cdot \|\partial_t a\|_{L^2L^2} \cdot \|v\|_{L^\infty H^1} \\ &\le C_0 + C(\mu_0)T^\delta(\|v\|_{L^\infty H^1}^2 + \|v\|_{L^2H^2}^2),\end{aligned}$$

for some $\delta > 0$ by $q > 2$. On a small time interval with length depending on μ_0 and $\|v_0\|_{H^1}$, we find the estimate (3.2), and, in particular, the $L^\infty H^1$-estimate for v which allows to continue the solution beyond time τ_ε.

The fact that $x_1 - X(t)$ remains bounded from below follows from estimate (3.2) together with the last identity in (P_v). We find that $\tilde{v}_0^\varepsilon(X(t))\partial_t X(t)$ is bounded in $L^3((0,T))$, thus, since the integral of $\tilde{v}_0^\varepsilon$ over (x_0, x_1) is bounded from below, we find a positive lower bound for the time span that X needs to reach the value x_1. In particular, we can continue the solution up to a time $T > 0$, which may depend on μ_0, but not on ε. This shows assertion 1.

Ad 2. In order to solve the nonlinear problem with coefficients $a = q'(v)$ we study an iteration map $a \mapsto u \mapsto a^{new}$. With $a_{\min} := \inf q'(v_0)$ we chose $R_1, R_2 > 0$ (specified below), and consider the set

$$\begin{aligned}A_T := \big\{a \in H^1((0,T)\times(0,x_1)) \cap L^\infty H^1 : \\ \inf a \ge a_{\min}/2, \|\partial_t a\|_{L^2} \le R_1, \|a\|_{L^\infty H^1} \le R_2\big\}.\end{aligned} \tag{3.6}$$

We remark that the set A_T is closed and that functions $a \in A$ are continuous. From assertion 1 we know that, given $a \in A_T$, we can solve problem (P_v) with coefficients a on an interval $(0, T_0)$, where T_0 depends only on $a_{\min}$ and R_1. We can then choose $T = T_0(a_{\min}, R_1)$.

We now define an iteration map $Q : A_T \to A_T$, $a \mapsto a^{new}$ as follows. Given coefficients $a \in A_T$ we solve problem (P_v) with (v, X). We then insert into q' and set $a^{new}(t,x) = q'(v(t,x))$ for $x < X(t)$ and $a^{new}(x) = q'(z)$ for $x \geq X(t)$. Choosing R_1 and R_2 large (depending on $a_{\min}$, $\|v_0\|_{H^1}$, and the function q), and then $T = T_0(a_{\min}, R_1)$, the iteration map Q is well defined into A_T by the estimates (3.2) on solutions v. For small T we can achieve that $a^{new} = Qa$ satisfies all inequalities of the definition of (3.6) strictly. With this choice of T, let now (v, X) be any solution of the equation with $a(t,x) = q'(v(t,x))$ on an interval $(0,\tau)$ with $\tau \leq T$. Then, by continuity, the coefficients $a = Qa$ are in the interior of the set A_τ. We conclude that a local existence result implies an existence result on $(0,T)$, since the coefficients can never leave the set A_T.

In order to construct local solutions v for coefficients $a = q'(v)$ we consider the above mapping Q on a set A_T with $T = \tau_\varepsilon > 0$ possibly depending on ε. We claim that the above mapping Q is contractive. We use the transformed quantity V and exploit that (V, X) solves (3.4) and (3.5) with $X_{new} = X$. Let a_1 and a_2 be two different coefficient fields with solutions (V_1, X_1) and (V_2, X_2). Then $V := V_1 - V_2$ solves

$$\begin{aligned}
\partial_t V(t,\xi) &= X_1(t)^{-1} \partial_\xi \Big[\frac{a_1(t, \xi X_1(t))}{X_1(t)} \, \partial_\xi V(t,\xi) \Big] \\
&\quad + X_1(t)^{-1} \partial_\xi \Big[\frac{a_1(t, \xi X_1(t)) - a_2(t, \xi X_2(t))}{X_1(t)} \, \partial_\xi V_2(t,\xi) \Big] \\
&\quad + \frac{1}{X_1(t)} \partial_\xi \Big[\frac{a_2(t, \xi X_2(t))}{X_1(t)} \, \partial_\xi V_2(t,\xi) \Big] - \frac{1}{X_2(t)} \partial_\xi \Big[\frac{a_2(t, \xi X_2(t))}{X_2(t)} \, \partial_\xi V_2(t,\xi) \Big] \\
&\quad + \xi \frac{\partial_t X_1(t)}{X_1(t)} \, \partial_\xi V(t,\xi) + \xi \Big(\frac{\partial_t X_2(t)}{X_2(t)} - \frac{\partial_t X_1(t)}{X_1(t)} \Big) \partial_\xi V_2(t,\xi).
\end{aligned}$$

Again, multiplication of this equation with $\partial_\xi^2 V$ and exploiting strict negativity of $\tilde{v}_0$ in (3.5) yields the contractiveness of Q on a $\tilde{v}_0$-dependent time interval $(0, \tau_\varepsilon)$. Since we can continue solutions until the ε-independent time T, this concludes the existence proof. □

Lemma 2 (Solution of (P_u))**.** *We consider* $0 < x_0 < x_1$ *and initial values* $u_0 \in H^2([0, x_1], \mathbb{R})$ *with* $\partial_x u_0 \geq z$ *on* $(0, x_0)$, $\partial_x u_0(x_0) = z$, *and* $\partial_x u_0 < z$ *on* $(x_0, x_1]$. *With* $\tilde{u}_0 = u_0$ *there holds*

(1) *For positive* x_0, $x_1 - x_0$, $\int_{x_0}^{x_1} |\partial_x u_0 - z|$, *and* $\|u_0\|_{H^2((x_0,x_1))}$, *there exists* $T > 0$ *and a solution of problem* (P_u) *with* $a(t,x) = q'(\partial_x u(t,x))$ *on the time interval* $(0,T)$.

(2) *If* u_0 *is of class* $C^2([0,L],\mathbb{R})$ *and* T *is chosen maximal, then* $X(T) = x_1$ *or* $T = \infty$.

Concavity remains preserved: For initial data with $\partial_x^2 u_0 \le 0$ *on* $(0,x_0)$, *the solutions satisfy* $\partial_x^2 u(t,.) \le 0$ *on* $(0,X(t))$ *for all* $t \in (0,T)$.

Proof. Ad 1. We set $v_0 = \partial_x u_0 - z$ and $\tilde v_0 = \partial_x u_0 - z - \varepsilon$ and find a solution $(v^\varepsilon, X^\varepsilon)$ of (P_v^ε) with Lemma 1, assertion 2, where the family of solutions is defined on an ε-independent interval $(0,T)$. Integrating v^ε over x yields an approximate solution $u^\varepsilon(t,x)$ of problem (P_u). We send $\varepsilon \to 0$ and take an $L^\infty H^2$ weak-$*$ limit u of u^ε. The norms of u allow to take limits in all equalities of (P_u). In particular, the convergence $v^\varepsilon \to v$ is uniform by the bound of (3.2) and the v-equation, and we can take strong limits in the coefficients, $a(v^\varepsilon) \to a(v)$.

Ad 2. For $C^2(\bar\Omega)$-initial data u_0, the quantity

$$w^\varepsilon = q'(v^\varepsilon)\partial_x v^\varepsilon = q'(v^\varepsilon)\partial_x^2 u^\varepsilon = \partial_t u^\varepsilon$$

is initially bounded. We want to exploit a maximum principle for w^ε to find global solutions of the ε-problems. The limit $\varepsilon \to 0$ then provides global solutions to the original problem.

In order to apply the maximum principle, we note that the ε-solutions are classical, and that $w^\varepsilon = q'(v^\varepsilon)\partial_x v^\varepsilon$ satisfies with the positive coefficients $a = q'(v^\varepsilon)$

$$\partial_t w^\varepsilon = \partial_t \partial_x [q(v^\varepsilon)] = \partial_x [q'(v^\varepsilon)\partial_t v^\varepsilon] = \partial_x [q'(v^\varepsilon)\partial_x w^\varepsilon].$$

As boundary conditions, we have $w^\varepsilon(t,0) = 0$ and $w^\varepsilon(t,X(t)) \le 0$. We conclude that w is bounded from above by the maximum of its initial values, and that concavity remains preserved. Furthermore, differentiating the last equation of (P_u) with respect to time we find

$$0 = \frac{d}{dt}[\partial_x u(t,X(t)^-)] = \partial_x w(t,X(t)^-) + \frac{1}{a} w(t,X(t)^-)\partial_t X.$$

We conclude that no negative minimum of w^ε can appear at the right boundary $X(t)$ and hence that w^ε remains additionally bounded from below. For the rigorous derivation of the maximum principles on domains $\bigcup_t \{t\} \times (0, X^\varepsilon(t))$ one considers the transformed equation (3.4).

Note that relations such as $v^\varepsilon \geq z$ remain as well satisfied by the maximum principle for v^ε. By assertion 1, we can extend the solution globally in time as long as $X(t)$ does not tend to a boundary point. □

The original problem (P) decouples into two problems of form (P_u), as long as the two points $X_1(t)$ and $X_2(t)$ do not meet. It remains to study the case that the two points meet after finite time, $\lim_{t\nearrow T} X_1(t) = x_1 = x_2 = \lim_{t\nearrow T} X_2(t)$. In this case, the EM-solution is described for $t > T$ by the following modified free boundary problem (P').

Find $u \in C([0,T] \times [0,L], \mathbb{R})$ and $X \in C([0,T],[0,L])$, such that $u(0,.) = u_0$, $u(t,0) = u(t,L) = 0$ for all t, and

$$X(0) = x_0, \tag{3.7}$$

$$\partial_t u(t,\cdot) = \partial_x[q(\partial_x u)] \text{ in } (0,L) \setminus \{X(t)\}, \tag{3.8}$$

$$\partial_x u(t, X(t) \pm \varepsilon) \to z_\mp \text{ for } \varepsilon \searrow 0. \tag{3.9}$$

The subsequent proposition collects results for (P'). They are analogous to the results for (P_u).

Proposition 1 (Problem (P'))**.** *For initial values $u_0 \in C^0(\bar{\Omega})$ such that, for some $x_0 \in (0,L)$, we have $u_0|_{[0,x_0]} \in C^2([0,x_0])$, $u_0|_{[x_0,L]} \in C^2([x_0,L])$, and*

$$\partial_x u_0 > z_+ \text{ in } [0,x_0),$$
$$\partial_x u_0(x_0 \pm \varepsilon) \to z_\mp \text{ for } \varepsilon \searrow 0,$$
$$\partial_x u_0 < z_- \text{ in } (x_0, L],$$

there exists $T > 0$ and a solution to problem (P') on $[0,T]$. The (P') solution is the EM-solution of (1.2).

In the case $z_- < 0 < z_+$ and for initial values with $\partial_x^2 u_0 \leq 0$ on $(0,L) \setminus \{x_0\}$, the solution exists globally in time and there exists $x_\infty \in (0,L)$ such that $X(t) \to x_\infty$ for $t \to \infty$.

Proof. We only sketch the proof of this proposition which is similar to the proof of Theorem 1, but simpler, since an ε-regularization is not necessary. One studies again the spatial derivative $v = \partial_x u$ which formally solves the following system of equations.

$$X(0) = x_0,$$

$$\partial_t v = \partial_x[a\partial_x v] \text{ in } \bigcup_t \{t\} \times ((0,L) \setminus \{X(t)\}),$$

$$v(0,\cdot) = \partial_x u_0, \quad \partial_x v(t,0) = \partial_x v(t,1) = 0 \quad \forall t,$$

$$v(t, X(t)^{\pm}) = z_{\mp},$$
$$(z_+ - z_-)\partial_t X(t) = a(t, X(t))(\partial_x v(t, X(t)^+) - \partial_x v(t, X(t)^-).$$

Here, the last line follows by differentiating the jump condition $u(X(t)^+) = u(X(t)^-)$ with respect to time and exploiting the differential equation for u. The above problem can now be treated similarly to problem (P_v), the ε-regularization is not necessary here since the derivatives of u are strictly different on both sides of $X(t)$. The main steps in the proof are as follows.

1. One derives a short time existence result with the help of an iteration. Given an evolution for $X(t)$ one solves the v-equations with Dirichlet conditions at $X(t)$. Given this solution v one improves $X(t)$ by integrating the last equation.

2. One considers the iteration $a \mapsto (v, X) \mapsto a^{new}$, where (v, X) is the solution of step 1 and the new approximation of a is given by $a^{new} := q'(v)$. The iteration yields a solution u of the original system (P') with $a := q'(\partial_x u)$ on a short time interval.

3. We claim that the maximum principle provides uniform bounds for the non-positive function $\partial_x^2 u$. Once this is shown, one can continue the solution globally in time.

Indeed, let us again study the variable $w := a\partial_x v = \partial_t u$. The last equation for v provides

$$w(t, X(t)^+) - w(t, X(t)^-) = (z_+ - z_-)\partial_t X(t),$$

whereas the time derivative of the jump condition for u implies that for the limits from both sides (indicated by $X(t)^+$ for right limits and by $X(t)^-$ for left limits) there holds

$$0 = \frac{d}{dt}\partial_x u(t, X(t)^{\pm}) = \partial_x w(t, X(t)^{\pm}) + \frac{1}{a} w(t, X(t)^{\pm})\partial_t X.$$

Let us assume that at time t we have $\partial_t X(t) > 0$. Then $w(t, X(t)^+) > w(t, X(t)^-)$ and a maximum of w necessarily appears in $X(t)^+$. But there we find,

$$\partial_x w(t, X(t)^+) = -\frac{1}{a} w(t, X(t)^+)\partial_t X > 0,$$

hence a maximum $w = 0$ cannot appear at $x = X(t)$. Let us consider negative minima of w. They can only appear on the right of $X(t)$, but, again,

$$\partial_x w(t, X(t)^-) = -\frac{1}{a} w(t, X(t)^-)\partial_t X > 0,$$

hence a negative minimum cannot appear at $x = X(t)$. The same calculations for the case $\partial_t X(t) < 0$ provide that w remains non-positive and bounded for all times.

4. One excludes $X(t) \to 0$ and $X(t) \to L$ for $t \to \infty$ with the help of the original system (P'). The calculation that u is the EM-solution is identical to the calculation for Theorem 1. □

Remark 1. Theorem 2 and the above proposition are formulated for the case $z_- < 0 < z_+$ for notational convenience. Let us assume that $0 < z_- < z_+$. In this case the two points $X_1(t)$ and $X_2(t)$ move towards each other and they necessarily meet after finite time T_0. The evolution is described by (P') on an interval (T_0, T_1), where $T_1 < \infty$ is a time instance when $X(t)$ hits 0. On the interval (T_1, ∞) the evolution is described by a forward parabolic equation with Dirichlet conditions. Therefore, the solution u eventually converges to the trivial solution $u_\infty \equiv 0$. In the case $z_- < z_+ < 0$, the analogous result holds with the point 0 replaced by L. We omit the calculations that verify these facts that can be conceived with geometric intuition.

Proof of Theorem 2. Lemma 2 provides a maximal interval of existence $(0, T)$ and a solution to problem (P). The functions X_1 and X_2 are monotonically non-decreasing and monotonically non-increasing, respectively. There holds either $T = \infty$ or $\lim_{t \nearrow T} X_1(t) = \lim_{t \nearrow T} X_2(t)$ by Lemma 2, claim 2.

In the case $T = \infty$ the EM-solution is given by (P) and we find

$$\lim_{t \to \infty} X_1(t) = x_1 \leq x_2 = \lim_{t \to \infty} X_2(t).$$

In this case the limiting function is given by $\min\{u_0(x), z_+ x, z_- (x - L)\}$. To prove this fact one has to verify the technical result that every solution of a forward parabolic problem on an interval $(0, X_1(t))$ with $X_1(t) \nearrow x_1$ converges to the solution of the elliptic problem on $(0, x_1)$ for $t \to \infty$.

In the case $T < \infty$, the EM-solution is given on $(0, T)$ by problem (P) and on (T, ∞) by problem (P') (compare Proposition 1). Multiplication of (P') with $\partial_t u$ and integration by parts yields

$$\begin{aligned} \int_0^L |\partial_t u|^2 &= -\int_0^L q(\partial_x u) \partial_t \partial_x u \\ &\qquad + q(\partial_x u(X(t)^-)) \partial_t u(X(t)^-) - q(\partial_x u(X(t)^+)) \partial_t u(X(t)^+) \\ &= -\int_0^L \partial_t \Phi(\partial_x u) + q(z_+) \partial_t u(X(t)^-) - q(z_-) \partial_t u(X(t)^+). \end{aligned}$$

The values $q(z_-)$ and $q(z_+)$ coincide with q^* by definition of $z_\pm$. For the time derivative of u we remark that (assuming differentiability)

$$\partial_t u(X(t)^-) + \partial_x u(X(t)^-)\partial_t X(t) = \partial_t u(X(t)^+) + \partial_x u(X(t)^+)\partial_t X(t),$$

where we know the values $\partial_x u(X(t)^\pm) = z_\mp$. Inserting above and integration over a time interval (T, τ) we find

$$\int_T^\tau \int_0^L |\partial_t u|^2 + \int_0^L \Phi(\partial_x u)\Big|_T^\tau = q^* \int_T^\tau [z_+ - z_-]\partial_t X(t)\, dt.$$

The right hand side is bounded independent of τ and we find the convergence of u to a stationary solution of (P'). The only stationary solution of (P') is given by $\min\{z_+ x, z_-(x-L)\}$. □

Proof of Theorem 3. The proof is based on an energy estimate for solutions of the singularly perturbed problems. Testing (1.5) with $\partial_t u^{\varepsilon,\delta}$ yields

$$\begin{aligned}\int_\Omega |\partial_t u^{\varepsilon,\delta}|^2\, dx &= \int_\Omega q(\partial_x u^{\varepsilon,\delta})\partial_t\partial_x u^{\varepsilon,\delta}\, dx - \varepsilon^2 \int_\Omega \partial_x^2 u^{\varepsilon,\delta}\partial_t\partial_x^2 u^{\varepsilon,\delta}\, dx \\ &= \partial_t \int_\Omega \Phi(\partial_x u^{\varepsilon,\delta})\, dx - \varepsilon^2 \partial_t \int_\Omega \frac{1}{2}|\partial_x^2 u^{\varepsilon,\delta}|^2\, dx,\end{aligned}$$

and therefore

$$\int_0^T \int_\Omega |\partial_t u^{\varepsilon,\delta}|^2\, dx\, dt + \left[\varepsilon^2 \int_\Omega \frac{1}{2}|\partial_x^2 u^{\varepsilon,\delta}|^2\, dx\right]_0^T + \left[\int_\Omega \Phi(\partial_x u^{\varepsilon,\delta})\, dx\right]_0^T = 0 \tag{3.10}$$

We set $\phi_- = \Phi(z_-)$ and $\phi_+ = \Phi(z_+)$. In the case $\phi_- = \phi_+$ the claim follows immediately from (3.10). It suffices to approximate u_0 with a piecewise affine function $\hat{u}_0$ with slope either z_+ or z_-, and to chose as u_0^δ a smoothed (on scale δ^2) version of $\hat{u}_0$. We thus achieve that

$$\varepsilon^2 \int_\Omega \frac{1}{2}|\partial_x^2 u_0^\delta|^2\, dx + \int_\Omega \Phi(\partial_x u_0^\delta)\, dx = L \cdot \phi_- + \rho(\delta, \varepsilon)$$

for some function $\rho(\delta, \varepsilon) \to 0$ with $\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \rho(\delta,\varepsilon) = 0$. Since $\Phi \geq \phi_-$, estimate (3.10) together with the weak lower semicontinuity of norms implies $\partial_t u_{SP}^\delta \to 0$ in $L^2(\Omega \times (0,T))$.

In the general case, not assuming $\phi_- = \phi_+$, we chose $\lambda \in (0,1)$ with $\lambda z_- + (1-\lambda)z_+ = 0$. Jensen's Lemma implies that

$$\inf_{\substack{u\in C^1(\Omega,\mathbb{R})\\ u|_{\partial\Omega}=0}} \frac{1}{|\Omega|}\int_\Omega \Phi(\partial_x u)dx \quad \geq \quad \inf_{\substack{u\in C^1(\Omega,\mathbb{R})\\ u|_{\partial\Omega}=0}} \frac{1}{|\Omega|}\int_\Omega \Phi^{**}(\partial_x u)dx$$

$$\geq \inf_{\substack{u\in C^1(\Omega,\mathbb{R})\\ u|_{\partial\Omega}=0}} \Phi^{**}\left(\frac{1}{|\Omega|}\int_\Omega \partial_x u dx\right) \geq \Phi^{**}(0).$$

Therefore, we chose now initial data u_0^δ which are smoothed versions of affine functions with slopes $z_\pm$. We can achieve that

$$\int_\Omega \Phi(\partial_x u_0^\delta)dx \leq O(\delta) + (\lambda\phi_- + (1-\lambda)\phi_+)\cdot L = O(\delta) + L\cdot\Phi^{**}(0).$$

For such initial data we conclude from (3.10) that

$$\begin{aligned}
&\int_0^T\int_\Omega |\partial_t u^{\varepsilon,\delta}|^2\,dx\,dt + \varepsilon^2\int_\Omega \frac{1}{2}|\partial_x^2 u^{\varepsilon,\delta}(T,\cdot)|^2 dx + L\cdot\Phi^{**}(0)\\
&\leq \varepsilon^2\int_\Omega \frac{1}{2}|\partial_x^2 u_0^\delta|^2\,dx + \int_\Omega \Phi(\partial_x u_0^\delta)\,dx\\
&= \varepsilon^2\int_\Omega \frac{1}{2}|\partial_x^2 u_0^\delta|^2\,dx + L\cdot\Phi^{**}(0) + O(\delta).
\end{aligned}$$

As a consequence, the SP-solutions to these initial values all have small energy and, in particular,

$$\int_{\Omega_T} |\partial_t u_{SP}^\delta|^2 = O(\delta).$$

This shows that, in the limit $\delta\to 0$, the solutions u_{SP}^δ approximate a stationary solution, i.e., the EM-solution. □

APPENDIX A. APPENDIX

In this appendix we prove those assertions of Observation A that regard the forward regime in the evolution equation. The proof of existence uses techniques similar to those in [7].

Proof of the first part of Observation A. *1. Existence of a classical solution u_c.* Since Ω is bounded and $|\partial_x u_0| < s_-$, there exists $0 < s_1 < s_-$ such that $|\partial_x u_0| \leq s_1 < s_-$. We can modify the flux-function q outside of the interval $[z_-, s_1]$ to obtain a function $\tilde{q}\in C^1(\mathbb{R},\mathbb{R})$ with $\tilde{q}' \geq c > 0$. The modified problem

$$\partial_t\tilde{u} = \partial_x\tilde{q}(\partial_x\tilde{u}) \tag{A.1}$$

with Dirichlet conditions is a forward parabolic equation and has a classical solution $\tilde{u}\in C^\alpha(\overline{\Omega_T})$ for every $T>0$ with $\partial_x^2\tilde{u}\in C^\alpha$ bounded. The solution

$\tilde{u}$ and its derivative $\partial_x \tilde{u}$ satisfy the weak maximum principle, hence

$$\sup_{\Omega_T} |\partial_x \tilde{u}(t,x)| \leq \sup_{\Omega} |\partial_x u_0| \leq s_1,$$

and consequently $u_c = \tilde{u}$ is also a classical solution of the q-problem (1.2).

2. Uniqueness of weak solutions. We claim that every weak solution $v \in L^2(0,T;H^1(\Omega))$ of the modified problem (A.1) satisfies $v = u_c$. Indeed, $u = u_c$ is also a weak solution of problem (A.1) and we can calculate (both equalities must be interpreted in the weak sense)

$$\partial_t u - \partial_t v = \partial_x[\tilde{q}(\partial_x u)] - \partial_x[\tilde{q}(\partial_x v)],$$

$$\partial_t \int_\Omega \frac{1}{2}|u-v|^2 = -\int_\Omega [\tilde{q}(\partial_x u) - \tilde{q}(\partial_x v)] \cdot (\partial_x u - \partial_x v) \leq 0$$

by monotonicity of $\tilde{q}$. Hence $u = v$.

3. The SP-solution coincides with the classical solution. Lemma A below asserts that, on a small time interval $(0,T)$, the solutions u^ε of the problems

$$\partial_t u^\varepsilon = \partial_x \tilde{q}(\partial_x u^\varepsilon) - \varepsilon^2 \Delta^2 u^\varepsilon \tag{A.2}$$

remain uniformly close to the initial values in $H^2(\Omega)$. In particular, u^ε coincides with the SP-approximations that are defined with q instead of $\tilde{q}$. We consider a sequence $\varepsilon_k \to 0$ such that $u^{\varepsilon_k} \to u$ almost everywhere, where u is the SP-solution. Since $u^{\varepsilon_k} \in L^\infty((0,T),H^2)$ are uniformly bounded, upon selecting a further subsequence, we may assume that $u^{\varepsilon_k} \to u$ weakly-$*$ in $L^\infty((0,T),H^2)$ and, by compactness, strongly in $L^2((0,T),H^1(\Omega))$. We conclude that u is a weak solution of (A.1), since, for $\tilde{q}$ with linear growth, we can take limits $\varepsilon = \varepsilon_k \to 0$ in

$$\int_0^T \int_\Omega \partial_t u^\varepsilon \cdot \varphi = -\int_0^T \int_\Omega \tilde{q}(\partial_x u^\varepsilon) \cdot \partial_x \varphi - \int_0^T \int_\Omega \varepsilon^2 \Delta u^\varepsilon \cdot \Delta \varphi.$$

By step 2, the weak limit u coincides with the classical solution u_c. □

Lemma A. *Suppose $u_0 \in H^4(\Omega)$ satisfies $u_0(\cdot) = 0$ and $\Delta u_0(\cdot) = 0$ on $\partial\Omega$. Let u^ε be the solution of the initial boundary value problem*

$$\begin{aligned} &\partial_t u^\varepsilon = \partial_x \left[q(\partial_x u^\varepsilon)\right] - \varepsilon^2 \Delta^2 u^\varepsilon && \text{in } \Omega_T \\ &u^\varepsilon = 0 \text{ and } \Delta u^\varepsilon = 0 && \text{on } \partial\Omega_T \\ &u^\varepsilon = u_0 && \text{in } \Omega \times \{t=0\}, \end{aligned}$$

where $q : \mathbb{R} \to \mathbb{R}$ satisfies $q' \geq q_0 > 0$ with q' and q'' bounded. Then, for all $\delta > 0$ there exists a time $T > 0$ independent of ε such that

$$||u^\varepsilon - u_0||_{L^\infty(0,T;H^2(\Omega))} \leq \delta. \tag{A.3}$$

Proof of Lemma A. The function $v^\varepsilon := u^\varepsilon - u_0$ solves the problem

$$\begin{aligned}
\partial_t v^\varepsilon &= \partial_x \left[q(\partial_x u_0 + \partial_x v^\varepsilon)\right] - \varepsilon^2 \Delta^2 u_0 - \varepsilon^2 \Delta^2 v^\varepsilon \\
&= q'(\partial_x u_0 + \partial_x v^\varepsilon)\left(\partial_x^2 u_0 + \partial_x^2 v^\varepsilon\right) - \varepsilon^2 \Delta^2 u_0 - \varepsilon^2 \Delta^2 v^\varepsilon \text{ in } \Omega_T, \\
v^\varepsilon(0,\cdot) &= 0 \text{ in } \Omega, \\
\Delta v^\varepsilon &= 0 \text{ on } \partial\Omega \times \{t > 0\}.
\end{aligned}$$

We set $w^\varepsilon = \partial_x v^\varepsilon$ and note that w^ε solves

$$\partial_t w^\varepsilon = \partial_x^2 [q(\partial_x u_0 + w^\varepsilon)] - \varepsilon^2 \Delta^2 u_0 - \varepsilon^2 \Delta^2 w^\varepsilon$$

with Neumann boundary conditions. We can therefore calculate

$$\begin{aligned}
&\partial_t \int_\Omega \frac{1}{2} |\partial_x w^\varepsilon|^2 dx = -\int_\Omega \partial_x^2 w^\varepsilon \cdot \partial_t w^\varepsilon \\
&= -\int_\Omega q'(\partial_x u_0 + w^\varepsilon) \partial_x^3 u_0 \partial_x^2 w^\varepsilon dx - \int_\Omega q'(\partial_x u_0 + w^\varepsilon) |\partial_x^2 w^\varepsilon|^2 dx \\
&\quad - \int_\Omega q''(\partial_x u_0 + w^\varepsilon) \left(\partial_x^2 u_0 + \partial_x w^\varepsilon\right)^2 \partial_x^2 w^\varepsilon dx \\
&\quad - \int_\Omega \varepsilon^2 \left(\Delta^2 u_0 \partial_x^2 w^\varepsilon + |\partial_x^3 w^\varepsilon|^2\right) dx.
\end{aligned}$$

By uniform positivity of q', boundedness of $q', q'' \in L^\infty$, and boundedness of $u_0 \in H^4$, an integration over time yields the estimate

$$\begin{aligned}
&A := \|w^\varepsilon\|^2_{L^\infty(0,T;H^1(\Omega))} + \|w^\varepsilon\|^2_{L^2(0,T;H^2(\Omega))} + \varepsilon^2 \|w^\varepsilon\|^2_{L^2(0,T;H^3(\Omega))} \\
&\leq C \left\{ \int_0^T \|w^\varepsilon(t)\|_{H^2(\Omega)} \, dt + \int_0^T \|(\partial_x w^\varepsilon(t))^2\|_{L^2(\Omega)} \cdot \|w^\varepsilon(t)\|_{H^2(\Omega)} \, dt \right\} \\
&\leq C \left\{ \sqrt{T} (\|w^\varepsilon(t)\|^2_{L^2(0,T;H^2(\Omega))} + 1) \right. \\
&\quad \left. + \int_0^T \|\partial_x w^\varepsilon(t)\|_{L^\infty(\Omega)} \|\partial_x w^\varepsilon(t)\|_{L^2(\Omega)} \|w^\varepsilon(t)\|_{H^2(\Omega)} \, dt \right\}.
\end{aligned}$$

For small $T > 0$ the first term can be absorbed in the right hand side and we find $A \leq C(\sqrt{T} + A^{3/2})$. Choosing a smaller value of T if necessary, we find that non-negative solutions A to this inequality satisfy, for some $a_- < a_+ \in \mathbb{R}$, $a_- > 0$ small, that either $A \leq a_-$ or $A \geq a_+$. Here, the constants are independent of ε.

For $T = 0$ we have that

$$||w^\varepsilon(0,\cdot)||^2_{L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} = 0,$$

and, furthermore, the mapping

$$t \mapsto ||w^\varepsilon(t,\cdot)||_{H^1(\Omega)}$$

is continuous for all $\varepsilon > 0$. We conclude that $A \leq a_-$ remains valid for all times, hence A remains small. Concerning the continuity of A it suffices to exploit that for $w^\varepsilon \in L^2((0,T),H^3)$ with $\partial_t w^\varepsilon \in L^2((0,T),H^{-1})$ the function is continuos into $H^1(\Omega)$. Even though the bounds are not uniform in ε this time, we can conclude the continuity of the H^1-norm. □

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