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ENTIRE SOLUTIONS AND GLOBAL BIFURCATIONS FOR A BIHARMONIC EQUATION WITH SINGULAR NON-LINEARITY IN $\mathbb{R}^3$

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Abstract. We study the structure of solutions of the boundary-value problem

$$\Delta^2 u = \frac{\lambda}{(1-u)^2} \ \text{ in } B, \quad u = \Delta u = 0 \ \text{ on } \partial B, \tag{0.1}$$

where Δ^2 is the biharmonic operator and $B \subset \mathbb{R}^3$ is the unit ball. We show that there are infinitely many turning points of the branch of the radial solutions of (0.1). The structure of solutions depends on the classification of the radial solutions of the equation

$$-\Delta^2 u = u^{-2} \quad \text{in } \mathbb{R}^3. \tag{0.2}$$

This is in sharp contrast with the corresponding result in $\mathbb{R}^2$.

1. Introduction

In this paper we investigate existence, uniqueness, asymptotic behavior, and further qualitative properties of radial solutions of the biharmonic equation

$$-\Delta^2 u = u^{-2} \quad \text{in } \mathbb{R}^3. \tag{1.1}$$

The motivation for studying (1.1) is to understand the structure of solutions of the Navier boundary-value problem

$$-\Delta^2 u = \lambda (L+u)^{-2} \ \text{ in } \Omega, \quad u = \Delta u = 0 \ \text{ on } \partial\Omega, \tag{1.2}$$

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where $L > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. The physical dimension should be $N = 2$ or 3. Problem (1.2) is a special case (with $T = 0$, $D = 1$) of the problem

$$T\Delta u - D\Delta^2 u = \lambda(L+u)^{-2} \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega, \tag{1.3}$$

where $T \geq 0$, $D \geq 0$ and $L > 0$. Problem (1.3) models the deflection of charged plates in electrostatic actuators (Lin and Yang [22]). Here, $\lambda = aV^2$ where V is the electric voltage and a is constant. Associated with (1.3) is the following energy functional

$$E(u) = \int_\Omega \left\{ \frac{T}{2}|\nabla u|^2 + \frac{D}{2}|\Delta u|^2 - \frac{\lambda}{L+u} \right\}, \tag{1.4}$$

where

$$P = \int_\Omega \frac{T}{2}|\nabla u|^2 dx$$

is the stretching energy,

$$Q = \int_\Omega \frac{D}{2}|\Delta u|^2 dx$$

corresponds to the bending energy, and

$$W = -\int_\Omega \frac{\lambda}{L+u(x)} dx$$

is the electric potential energy.

Lin and Yang ([22]) considered two kinds of boundary conditions: pinned boundary condition

$$u = \Delta u = 0 \text{ on } \partial\Omega,$$

and clamped boundary condition

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

For the pinned boundary condition, they found that there exists $0 < \lambda_c < \infty$ such that, for $\lambda \in (0, \lambda_c)$, (1.3) has a maximal regular solution u_λ, which can be obtained from an iterative scheme. (By a regular solution u_λ of (1.3), we mean that $u_\lambda \in C^4(\Omega) \cap C^3(\overline{\Omega})$ satisfies (1.3).) For $\lambda > \lambda_c$, (1.3) does not have any regular solution. Moreover, if λ', $\lambda'' \in (0, \lambda_c)$ and $\lambda' < \lambda''$, then the corresponding maximal solutions $u_{\lambda'}$ and $u_{\lambda''}$ satisfy

$$u_{\lambda'} > u_{\lambda''} \quad \text{in } \Omega.$$

Physically, this is a natural relation because a higher supply of voltage results in a greater elastic deformation or deflection.

The number λ_c, which determines the pull-in voltage, is called the pull-in threshold. It is known from [22] that, for $\lambda \in (0, \lambda_c)$, $\min_\Omega(L + u_\lambda) > 0$. Let $\Sigma_\lambda = \{x \in \Omega : \ L + u_\lambda(x) = 0\}$ be the singular set of (1.3). An interesting question is to study the limit of u_λ as $\lambda \nearrow \lambda_c$. The monotonicity of u_λ with respect to λ implies that there is a well-defined function U so that

$$U(x) = \lim_{\lambda \to \lambda_c^-} u_\lambda(x); \quad -L \le U(x) < 0, \quad x \in \Omega.$$

However, $U(x)$ may touch down to $-L$ and cease to be a regular solution to (1.3). (By [22], $U \in W^{2,2}_{loc}(\Omega)$.) For the one-dimensional case, Lin and Yang showed that U is a regular solution; that is, the set $\Sigma_{\lambda_c} = \emptyset$.

In our previous paper ([13]), we showed that *for $N = 2$ or 3, U is a regular solution*. Moreover, we also showed that there is a *unique solution for* (1.3) *at $\lambda = \lambda_c$*. For two-dimensional convex domains, we also established the existence of a *second solution for every* $\lambda \in (0, \lambda_c)$. This shows that, at least in two-dimensional domains, problem (1.3) behaves *subcritically*. (Numerical computations as well as asymptotic behavior as $D \to 0$ are done in [23].)

In this paper, we initiate the study of (1.2) on three dimensional domains. Note that our main results in this paper are still true for (1.3). We concentrate on the case of the unit ball. We shall establish the following results: *for λ small, the maximal solution is unique. There exists $\lambda_* < \lambda_c$ such that the solution branch has infinitely many turning points for λ near λ_**. This shows that problem (1.2) behaves *supercritically* in $\mathbb{R}^3$. This is somehow surprising. We remark that, when $N = 3$, the formal critical exponent for Δ^2 is $\frac{N+4}{N-4} = -7$. One expects that u^{-3} should be less critical than u^{-7} and behave *subcritically*.

We remark that problem (1.2) can find applications in thin film problems, see [1, 2, 3, 18, 19, 20, 21]. When $D = 0$, Problem (1.3) can also find applications in MEMS devices, see [6, 7, 9, 10, 11, 17, 28, 29, 25]. The qualitative behavior of solutions has been studied in [14, 15, 16] and [24].

By a change $v = -u$, we see that v satisfies

$$\Delta^2 v = \frac{\lambda}{(L - v)^2}, \ \ 0 < v < L \ \text{ in } \Omega, \quad v = \Delta v = 0 \ \text{ on } \partial\Omega. \tag{1.5}$$

For simplicity, we assume that $L = 1$. We shall consider throughout the paper the following problem

$$\Delta^2 v = \frac{\lambda}{(1 - v)^2}, \ \ 0 < v < 1 \ \text{ in } \Omega, \quad v = \Delta v = 0 \ \text{ on } \partial\Omega. \tag{1.6}$$

We first study the properties of entire radial solutions of (1.1). We seek solutions u of (1.1) which only depend on $|x|$. Due to the homogeneity, (1.1) is invariant under a suitable rescaling. This means that existence of a solution immediately implies the existence of infinitely many solutions, each one of them being characterized by its value at the origin. To ensure smoothness of the solution, one needs to require that $u'(0) = u'''(0) = 0$. We see that solutions of (1.1) can be determined only by fixing a priori also the value of $u''(0)$. In this paper, the proofs are performed with a shooting method which uses as a free parameter the "shooting concavity," namely the initial second derivative $u''(0)$.

Hence, we consider the following initial-value problem

$$\begin{aligned} &\Delta^2 u = -u^{-2}, \;\; u = u(r), \text{ in } \mathbb{R}^3 \\ &u(0) = 1, \;\; u'(0) = u'''(0) = 0, \;\; u''(0) = \gamma > 0. \end{aligned} \tag{1.7}$$

Our first theorem is on the classification of entire solutions to (1.7):

Theorem 1.1. *There exists a unique $\gamma^* \in (0, \infty)$ such that, for $\gamma \in (0, \gamma^*)$, there is a unique $R_\gamma \in (0, \infty)$ such that $\Delta u_\gamma(R_\gamma) = 0$ and $(\Delta u_\gamma)'(r) < 0$ for $r \in (0, R_\gamma)$. The function R_γ is continuous and increasing with respect to γ and $R_\gamma \to \infty$ as $\gamma \to \gamma^*$. For $\gamma > \gamma^*$, there exists $C := C(\gamma) > 0$ such that $(\Delta u_\gamma)'(r) < 0$ for $r > 0$, $\Delta u_\gamma(r) \to C$ as $r \to \infty$ and u_γ has the growth Cr^2 near ∞. For $\gamma = \gamma^*$, we have $(\Delta u_{\gamma^*})'(r) < 0$ for $r > 0$, $\Delta u_{\gamma^*}(r) \to 0$ as $r \to \infty$. Thus, $\Delta u_{\gamma^*}(r) > 0$ for $r \in (0, \infty)$ and $u'_{\gamma^*}(r) > 0$ for $r \in (0, \infty)$.*

To show the difference between dimension two and dimension three, we prove the following theorem in dimension two:

Theorem 1.2. *Consider the following problem*

$$\begin{cases} \Delta^2 u = -u^{-2}, u = u(r), \quad \textit{in } \mathbb{R}^2, \\ u(0) = 1, u'(0) = u'''(0) = 0, \Delta u(0) = \gamma. \end{cases} \tag{1.8}$$

For any $\gamma \in (0, \infty)$, there is a unique $R_\gamma \in (0, \infty)$ such that $\Delta u_\gamma(R_\gamma) = 0$ and $(\Delta u_\gamma)'(r) < 0$ for $r \in (0, R_\gamma)$. The function $\gamma \mapsto R_\gamma$ is increasing and $R_\gamma \to \infty$ as $\gamma \to \infty$. Moreover, $\Delta u_\gamma(r) \to \infty$, $u_\gamma(r) \to \infty$ for $r \in (0, \infty)$ as $\gamma \to \infty$.

It is easy to know that the equation in (1.1) has a singular solution

$$U_0(r) = \left(\frac{56}{3^4}\right)^{-1/3} r^{4/3}. \tag{1.9}$$

Theorem 1.1 implies that there exists a unique entire solution to (1.7). Our second theorem is on the qualitative properties of this entire solution u_{γ^*}.

Theorem 1.3. *Let* $u_{\gamma^*}(r)$ *be the entire solution to* (1.7) (*given by Theorem* 1.1). *Then*

$$\lim_{r\to\infty} r^{-\frac{4}{3}} u_{\gamma^*}(r) = \left(\frac{56}{3^4}\right)^{-1/3}, \tag{1.10}$$

and $u_{\gamma^*}(r) - U_0(r)$ *has infinitely many intersections.*

Finally, we consider the structure of radial solutions of (1.2) with

$$\Omega = B = \{x \in \mathbb{R}^3 : \ |x| < 1\}.$$

Namely, we study existence and the property of non-minimal radially symmetric solutions of the problem

$$\Delta^2 u = \lambda(1-u)^{-2}, \ \ 0 \le u < 1 \ \text{ in } B, \quad u = \Delta u = 0 \ \text{ on } \partial B. \tag{1.11}$$

To state the results, we put

$$\mathcal{C}_r^\lambda = \{u \in C^4(B) \cap C^3(\overline{B}) : \ u = u(|x|) \ \text{ solves (1.11)}\}$$
$$\mathcal{C}_r = \cup_{\lambda>0}\{\lambda\} \times \mathcal{C}^\lambda.$$

Theorem 1.4. *There are no secondary bifurcation points on* $\mathcal{C}_r$ *and* $\mathcal{C}_r$ *is homeomorphic to* $\mathbb{R}$ *with the end points* $(0,0)$ *and* $(\lambda_*, 1-|x|^{4/3})$, *where*

$$\lambda_* = \frac{56}{81}.$$

Moreover, $\mathcal{C}_r$ *bends infinitely many times with respect to* λ *around* λ_* *and the Morse index of the solutions approach* $+\infty$ *for* λ *near* λ_*.

We remark that the techniques in this paper have been extended in [12] to give a complete characterization of entire radial solutions to

$$\Delta^2 u = u^p, u > 0 \ \text{ in } \mathbb{R}^n, \tag{1.12}$$

where $n \ge 5$ and $p > \frac{n+4}{n-4}$. This problem has been studied recently by Gazzola and Grunau [8].

The organization of this paper is as follows: In Section 2, we prove Theorem 1.1 and in Section 3, we prove Theorem 1.2. In Section 4, we study the properties of entire solutions and prove Theorem 1.3. Section 5 is devoted to the proof of Theorem 1.4.

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2. The case of $N = 2$: Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Since we are only interested in the radial solutions, by a shooting method, keeping $u(0)$ fixed, say $u(0) = 1$, we look for solutions u of the initial-value problem over $[0, \infty)$:

$$\begin{aligned} &u^{(4)}(r) + \tfrac{2}{r}u'''(r) - \tfrac{1}{r^2}u''(r) + \tfrac{1}{r^3}u'(r) = -u^{-2}(r) \\ &u(0) = 1, \ \ u'(0) = u'''(0) = 0, \ \ u''(0) = \gamma > 0, \end{aligned} \tag{2.1}$$

which is the radial version of equation (1.8). By standard ODE theory, we see that, for each $\gamma > 0$, (2.1) has a unique solution $u_\gamma(r)$ for r near 0.

If $u = u(r)$ is a radial positive solution of (2.1), then

$$u_a := au(a^{-\frac{3}{4}}r) \ \ (a > 0)$$

is a radial positive solution of the equation in (2.1) such that $u_a(0) = a$.

We apply a shooting method with initial second derivative as parameter. We remark that $2u''(0) = \Delta u$ and that by l'Hospital's rule

$$(\Delta u)'(0) = u'''(0) + \lim_{r\to 0} \frac{ru''(r) - u'(r)}{r^2} = \frac{3}{2}u'''(0).$$

This means that the initial conditions in (2.1) also read as

$$u(0) = 1, \ \ u'(0) = (\Delta u)'(0) = 0, \quad \Delta u(0) = 2\gamma > 0. \tag{2.2}$$

For all $\gamma > 0$, (2.1)-(2.2) admit a unique local smooth solution u_γ defined on some right neighborhood of $r = 0$. Let

$$R_\gamma = \begin{cases} +\infty & \text{if } u_\gamma(r)(\Delta u_\gamma)(r) > 0, \ \ \forall r > 0 \\ \min\{r > 0;\ u_\gamma(r)(\Delta u_\gamma)(r) = 0\} & \text{otherwise.} \end{cases}$$

From now on we understand that u_γ is continued on $[0, R_\gamma)$. Let

$$I^+ = \{\gamma > 0 : \ \ R_\gamma < \infty, \ \ u_\gamma(R_\gamma) = \infty\},$$

$$I^- = \{\gamma > 0 : \ \ R_\gamma < \infty, \ \ (\Delta u_\gamma)(R_\gamma) = 0\}.$$

We first prove the following statement:

Lemma 2.1. *Assume $N = 2$. Then $I^- = (0, \infty)$ and $R_\gamma \to \infty$ as $\gamma \to \infty$.*

To prove this lemma, we need a comparison principle, which has been observed by McKenna-Reichel [24] and which will turn out to be useful also in the proof of Lemma 2.1.

Lemma 2.2. (*Comparison Principle*). *Assume that* $f:(0,\infty)\to(0,\infty)$ *is locally Lipschitz and strictly increasing. Let* $u,\ v\in C^4([0,R))$ *be such that*

$$\begin{cases} \forall r\in[0,R):\ \Delta^2 v(r)-f(v(r))\geq\Delta^2 u(r)-f(u(r)) \\ v(0)\geq u(0),\ \ v'(0)=u'(0)=0, \\ \Delta v(0)\geq\Delta u(0),\ \ (\Delta v)'(0)=(\Delta u)'(0)=0. \end{cases} \tag{2.3}$$

Then we have for all $r\in[0,R)$:

$$v(r)\geq u(r),\ \ v'(r)\geq u'(r),\ \ \Delta v(r)\geq\Delta u(r),\ \ (\Delta v)'(r)\geq(\Delta u)'(r). \tag{2.4}$$

Moreover,

(*i*) *the initial point 0 can be replaced by any initial point* $\rho>0$ *if all four initial data are weakly ordered;*

(*ii*) *a strict inequality in one of the initial data at* $\rho\geq 0$ *or in the differential inequality on* (ρ,R) *implies a strict ordering of* v, v', Δv, $(\Delta v)'$ *and* u, u', Δu, $(\Delta u)'$ *on* (ρ,R).

Proof of Lemma 2.1. We first show that $I^+=\emptyset$. Supposing the contrary, there is $0<\gamma_0<\infty$ and $R_{\gamma_0}<\infty$ such that $\lim_{r\to R_{\gamma_0}^-}u_{\gamma_0}(r)=\infty$. Noticing that u_{γ_0} satisfies the equation

$$(r(\Delta u_{\gamma_0})'(r))'=-ru_{\gamma_0}^{-2}(r)\leq 0,$$

we see that $(\Delta u_{\gamma_0})'(r)\leq 0$ for $r\in(0,R_{\gamma_0})$. The fact that $\Delta u_{\gamma_0}(0)=2\gamma_0<\infty$ implies that $(\Delta u_{\gamma_0})(r)\leq 2\gamma_0$ for $r\in[0,R_{\gamma_0}]$. Thus,

$$\Delta(u_{\gamma_0}-2\gamma_0\phi)\leq 0\ \text{ on } \overline{B_{R_{\gamma_0}}}, \tag{2.5}$$

where $B_{R_{\gamma_0}}$ is the ball with center at 0 and radius of R_{γ_0} and ϕ is the unique solution of the problem

$$-\Delta\phi=1\ \text{ in } B_{R_{\gamma_0}},\quad \phi=0\ \text{ on } \partial B_{R_{\gamma_0}}.$$

(2.5) and the maximum principle then implies that u_{γ_0} can not be ∞ on $\partial B_{R_{\gamma_0}}$. Thus, $I^+=\emptyset$.

Now we show that $I^-\neq\emptyset$. Considering the problem

$$\Delta^2 v=\lambda(1-v)^{-2}\ \text{ in } B,\quad v=\Delta v=0\ \text{ on } \partial B, \tag{2.6}$$

where B is the unit ball of $\mathbb{R}^2$, we see from [22] that there is $0<\lambda_c<\infty$ such that, for $\lambda\in(0,\lambda_c]$, (2.6) has a minimal positive solution $v_\lambda\in C^4(B)$ satisfying $0<v_\lambda<1$. The minimality of v_λ implies that $v_\lambda(x)=v_\lambda(r)$. Defining $w_\lambda=1-v_\lambda$, we see that w_λ satisfies the problem

$$-\Delta^2 w_\lambda=\lambda w_\lambda^{-2}\ \text{ in } B,\quad w_\lambda=1,\ \ \Delta w_\lambda=0\ \text{ on } \partial B.$$

Setting $\xi_\lambda = \min_B w_\lambda$, $y = \lambda^{1/4}\xi_\lambda^{-3/4} r$, and $\tilde{w}_\lambda = w_\lambda(r)/\xi_\lambda$, we see that $\tilde{w}_\lambda$ with $\tilde{w}_\lambda(0) = \min_B \tilde{w}_\lambda = 1$ satisfies the problem

$$-\Delta_y^2 \tilde{w}_\lambda = \tilde{w}_\lambda^{-2} \text{ in } B_\lambda, \quad \tilde{w}_\lambda = \frac{1}{\xi_\lambda}, \ \ \Delta_y \tilde{w}_\lambda = 0 \text{ on } \partial B_\lambda,$$

where $B_\lambda = \{y \in \mathbb{R}^2 : \ |y| < \lambda^{1/4}\xi_\lambda^{-3/4}\}$. Denote $2\gamma_\lambda = (\Delta\tilde{w}_\lambda)(0)$. We see that $\gamma_\lambda \in I^-$. Moreover,

$$R_{\gamma_\lambda} = \lambda^{1/4}\xi_\lambda^{-3/4}.$$

Now, we use Lemma 2.2 to show that R_γ is an increasing function of γ. For any $\gamma_1, \gamma_2 \in I^-$ and $\gamma_1 > \gamma_2$, by Lemma 2.2, we see that $u_{\gamma_1}(r) > u_{\gamma_2}(r)$ and $\Delta u_{\gamma_1}(r) > \Delta u_{\gamma_2}(r)$ for $r \in (0, \min\{R_{\gamma_1}, R_{\gamma_2}\}]$. This clearly implies that $R_{\gamma_1} > R_{\gamma_2}$. The continuity of R_γ on γ can be obtained by standard ODE theory.

Now we claim that

$$\sup\{\gamma \in I^-\} = \infty. \tag{2.7}$$

Suppose $\sup\{\gamma \in I^-\} = \gamma^* < \infty$. We show that

$$\lim_{\gamma \to \gamma^*} R_\gamma = \infty. \tag{2.8}$$

If (2.8) does not hold, we see that $R_\lambda \le R^* < \infty$ for all $\gamma \in I^-$. Considering the problem

$$\Delta^2 v = \lambda(1-v)^{-2} \text{ in } B_{R^*}, \quad v = \Delta v = 0 \text{ on } \partial B_{R^*}, \tag{2.9}$$

we see from [22] that there exists $\lambda^{**} > 0$ depending on R^* such that, for $\lambda \in (0, \lambda^{**})$, (2.9) has a minimal solution $v_\lambda \in C^4(B_{R^*})$. By arguments similar to those in the proof of $I^- \neq \emptyset$, we can obtain $\tilde{w}^*$ with $\min_{B_\lambda} \tilde{w}^* = 1$ and $\tilde{w}^*$ satisfies the equation

$$-\Delta^2 \tilde{w}^* = [\tilde{w}^*]^{-2} \text{ in } B_\lambda, \ \ \tilde{w}^* = \Delta\tilde{w}^* = 0 \text{ on } \partial B_\lambda,$$

where

$$B_\lambda = \{y \in \mathbb{R}^2 : \ |y| < R^*[\lambda \min_{B_{R^*}} (1-v_\lambda)^{-3}]^{1/4}\}.$$

It is known from [13] that $\lambda \min_{B_{R^*}} (1-v_\lambda)^{-3} \to \infty$ as $\lambda \to 0$. Thus, $R^*[\lambda \min_{B_{R^*}} (1-v_\lambda)^{-3}]^{1/4} > R^*$ for λ sufficiently small. Denoting $2\gamma^{**} = (\Delta\tilde{w}^*)(0)$, we see that $R_{\gamma^{**}} > R^*$. This contradicts the fact that R_γ is an increasing function of γ. The monotonicity of Δu and the facts that $\gamma^* < \infty$, $R_{\gamma^*} = +\infty$ imply that (1.1) has a solution $u(r)$ for $r \in (0, \infty)$

with $0 \le \Delta u(r) \le 2\gamma^*$ for $r \in [0,\infty)$. On the other hand, we see from the equation of u that

$$-(\Delta u)'(r) = \frac{1}{r}\int_0^r \xi u^{-2}(\xi)d\xi \ge \frac{C}{r} \quad \text{for } r \text{ large},$$

and this implies that

$$\Delta u(r_0) - \Delta u(r) \ge C\ln\frac{r}{r_0} \quad \text{for } r > r_0,$$

where $r_0 > 1$ is a large number. This contradicts the fact that $0 \le \Delta u(r) \le 2\gamma^*$ for $r \in [0,\infty)$. Thus, claim (2.7) holds.

Proof of Theorem 1.2. The proof of the first part of this theorem can be obtained from Lemma 2.1. We only need to show the last part of this theorem.

Suppose that there is $R^* > 0$ such that $\Delta u_\gamma(R^*) < \infty$ as $\gamma \to \infty$. Since $-\Delta(\Delta u_\gamma) \le 1$ in B_{R^*} and $\Delta u_\gamma < \infty$ on ∂B_{R^*}, the standard *a priori* estimate implies that $\Delta u_\gamma < \infty$ in B_{R^*} as $\gamma \to \infty$. This contradicts the fact that $\Delta u_\gamma(0) = 2\gamma \to \infty$. Thus, $\Delta u_\gamma(r) \to \infty$ for $r \in [0,\infty)$ as $\gamma \to \infty$. This also implies that $u_\gamma(r) \to \infty$ for $r \in (0,\infty)$ as $\gamma \to \infty$. This completes the proof. □

3. The case of $N = 3$: Proof of Theorem 1.1

In this section, we consider the case of $N = 3$. As in Section 2, (1.7) is equivalent to the following initial-value problem over $[0,\infty)$:

$$\begin{aligned} &u^{(4)}(r) + \tfrac{4}{r}u'''(r) = -u^{-2}(r), \quad r \in [0,\infty) \\ &u(0) = 1, \quad u'(0) = u'''(0) = 0, \quad u''(0) = \gamma > 0. \end{aligned} \tag{3.1}$$

By standard ODE theory, we see that, for each $\gamma > 0$, (3.1) admits a unique local smooth solution u_γ defined on some right neighborhood of $r = 0$. Let R_γ, I^+, I^- be defined as in Section 2.

By arguments similar to those in the proof of Lemma 2.1, we find that the set

$$I^- = \{\gamma \in (0,\infty) : \ R_\gamma < \infty, \ \ (\Delta u_\gamma)(R_\gamma) = 0\} \ne \emptyset.$$

Define $\gamma^* = \sup I^-$. We will show that $\gamma^* < \infty$. Indeed, for $\epsilon > 0$ sufficiently small (e.g. $\epsilon < 2/3$) and $b > 0$ sufficiently large, it follows from Lemma 3.5 of [24] that the function $v_\epsilon(r) = (1 + b^2r^2)^{1-\frac{\epsilon}{2}}$ satisfies

$$\Delta^2 v + v^{-2} \le 0 \ \text{ on } (0,\infty).$$

Now we construct a subsolution to the equation with the growth $O(r^2)$ in (3.1). Let $V(r) = 1 + r^2 + v_\epsilon(r)$. We see that

$$\Delta^2 V + V^{-2} \le \Delta^2 v_\epsilon + v_\epsilon^{-2} \le 0 \text{ on } (0, \infty).$$

We easily see that $\Delta V(r) > 0$ for $r \in (0, \infty)$ and $\Delta V(r) \to 6$ as $r \to \infty$. Setting $\tilde{\gamma} = V''(0)$, we see that the solution $u_{\tilde{\gamma}} \ge V$ and $\Delta u_{\tilde{\gamma}} \ge \Delta V$ on $(0, \infty)$. On the other hand, the function $\overline{V}(r) = Ar^2$ $(A > 0)$ is a supersolution to the equation in (3.1), thus by choosing A sufficiently large and applying Lemma 2.2, we see that $u_{\tilde{\gamma}} \le \overline{V}$ on $(0, \infty)$. Thus, $u_{\tilde{\gamma}}$ is a solution of (3.1) with growth $O(r^2)$ near ∞. The comparison principle implies that $\gamma^* < \tilde{\gamma}$. We easily know that $\Delta u_{\gamma^*}(r) \to 0$ as $r \to \infty$.

Now, we show that u_{γ^*} is the unique solution of (3.1) with $\Delta u(r) \to 0$ as $r \to \infty$. Supposing the contrary, there are $\gamma^{**} > \gamma^*$ such that $\Delta u_{\gamma^{**}}(r) \to 0$, $\Delta u_{\gamma^*}(r) \to 0$ as $r \to \infty$. Then, it follows from the comparison principle that

$$u_{\gamma^{**}} > u_{\gamma^*} \text{ on } (0, \infty).$$

But, it follows from the equations of u_{γ^*} and $u_{\gamma^{**}}$ that

$$3\gamma^* = \int_0^\infty \frac{1}{r^2} \int_0^r \frac{\xi^2}{u^2_{\gamma^*}(\xi)} d\xi dr, \tag{3.2}$$

and

$$3\gamma^{**} = \int_0^\infty \frac{1}{r^2} \int_0^r \frac{\xi^2}{u^2_{\gamma^{**}}(\xi)} d\xi dr. \tag{3.3}$$

Clearly, (3.2) and (3.3) imply a contradiction. This implies that u_{γ^*} is the unique solution of (3.1) satisfying $\Delta u(r) \to 0$ as $r \to \infty$. Since $\Delta u_{\gamma^*}(r) > 0$ for $r \in (0, \infty)$, we see that $(r^2 u'_{\gamma^*}(r))' > 0$ for $r \in (0, \infty)$. Integrating it on $(0, r)$ and noting that $u'_{\gamma^*}(0) = 0$, we see that $u'_{\gamma^*}(r) > 0$ for $r \in (0, \infty)$. This completes the proof. □

4. Properties of Entire Solutions: Proof of Theorem 1.3

Let u_{γ^*} be given by Theorem 1.1. We prove Theorem 1.3 in this section. In fact, we prove the following theorem which gives the asymptotic behavior of u_{γ^*} :

Theorem 4.1. *The following linearized problem*

$$\Delta^2 \psi = \frac{2}{u^3_{\gamma^*}} \psi, \quad \psi = \psi(r), \quad \psi'(0) = \psi'''(0) = 0, \quad \Delta\psi \to 0 \quad \text{as } r \to \infty \tag{4.1}$$

admits the only solution $\psi = c(\frac{4}{3}u_{\gamma^*} - ru'_{\gamma^*})$*, for some constant* c*. As a consequence, we have for* r *large*

$$u_{\gamma^*}(r) = \left(\frac{56}{81}\right)^{-1/3} r^{4/3} + M_1 r^{1/2}\cos(\beta\ln r) + M_2 r^{1/2}\sin(\beta\ln r) + O(r^{\frac{4}{3}+\nu_1}), \tag{4.2}$$

where $\nu_1 = -\frac{5+\sqrt{45+4\sqrt{193}}}{6}$ *and* $\beta = \frac{\sqrt{4\sqrt{193}-45}}{6}$.

It is easy to see that Theorem 1.1 follows from Theorem 4.1.

We first show (1.10). We use some ideas from [8]. To this end, we use the Emden-Fowler transformation:

$$u_{\gamma^*}(r) = r^{\frac{4}{3}}v(t), \quad t = \ln r \ (r > 0). \tag{4.3}$$

Therefore, after the change of (4.3), the equation in (3.1) may be rewritten as

$$v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = -v^{-2}(t), \ \ t \in \mathbb{R}, \tag{4.4}$$

where $K_0 = -\frac{56}{81}$, $K_1 = -\frac{50}{27}$, $K_2 = \frac{15}{9}$, $K_3 = \frac{10}{3}$. This implies that the entire solution of (3.1) corresponds to a solution of (4.4). For $\gamma > \gamma^*$, the solution u_γ has a growth $O(r^2)$; this corresponds to $v(t) \to \infty$ as $t \to \infty$. Thus, we show that $u_{\gamma^*}(r)$ corresponds to the solution v of (4.4) satisfying $\lim_{t\to\infty} v(t) = (-K_0)^{-1/3}$.

Note that (4.4) admits the constant solution $v_s = (-K_0)^{-1/3}$, which, by (4.3), corresponds to the singular solution $U_0(r) = (-K_0)^{-1/3}r^{4/3}$ of (3.1).

We now write (4.4) as a system in $\mathbb{R}^4$. By (4.3) we have

$$u'_{\gamma^*}(r) = 0 \ \ \Leftrightarrow \ \ v'(t) = -\tfrac{4}{3}v(t).$$

This fact suggests that we define

$$w_1(t) = v(t), \qquad w_2(t) = v'(t) + \tfrac{4}{3}v(t),$$

$$w_3(t) = v''(t) + \tfrac{4}{3}v'(t), \qquad w_4(t) = v'''(t) + \tfrac{4}{3}v''(t),$$

so that (4.4) becomes

$$\begin{cases} w_1'(t) = -\frac{4}{3}w_1(t) + w_2(t), \ \ w_2'(t) = w_3(t), \ \ w_3'(t) = w_4(t) \\ w_4'(t) = C_2 w_2(t) + C_3 w_3(t) + C_4 w_4(t) - w_1^{-2}(t), \end{cases} \tag{4.5}$$

where

$$C_m = -\Sigma_{i=m-1}^4 \frac{4^{i+1-m}K_i}{(-1)^{i+1-m}3^{i+1-m}},$$

for $m = 1, 2, 3, 4$ with $K_4 = 1$. This gives first that $C_1 = 0$ so that the term $C_1 w_1(t)$ does not appear in the last equation of (4.5). Moreover, we have the explicit formulae:

$$C_2 = -\tfrac{3}{4}K_0, \ \ C_3 = 1, \ \ C_4 = -2.$$

System (4.5) has one stationary point (corresponding to $\mathbf{w}_s$)

$$P\Big((-K_0)^{-1/3}, \ \tfrac{4}{3}(-K_0)^{-1/3}, 0, 0\Big).$$

Around this "singular point" P the linearized matrix of the system (4.5) is given by

$$M_P = \begin{pmatrix} -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2K_0 & C_2 & C_3 & C_4 \end{pmatrix}. \tag{4.6}$$

The corresponding characteristic polynomial is

$$\nu \mapsto \nu^4 + K_3\nu^3 + K_2\nu^2 + K_1\nu + 3K_0,$$

and the eigenvalues are given by

$$\nu_1 = -\frac{5 + \sqrt{45 + 4\sqrt{193}}}{6}, \quad \nu_2 = -\frac{5 - \sqrt{45 + 4\sqrt{193}}}{6},$$
$$\nu_3 = -\frac{5 + \sqrt{45 - 4\sqrt{193}}}{6}, \quad \nu_4 = -\frac{5 - \sqrt{45 - 4\sqrt{193}}}{6}.$$

It is clear that

$$\nu_1 < 0 < \nu_2, \quad \nu_3, \nu_4 \notin \mathbb{R}, \ \ \Re\nu_3 = \Re\nu_4 = -\frac{5}{6} < 0.$$

This means that P has a three-dimensional stable manifold and a one-dimensional unstable manifold.

Let u be the unique entire solution of (3.1) with $\Delta u(r) \to 0$ as $r \to \infty$. Let v be defined according to (4.3) so that it solves (4.4), and $\mathbf{w}(t) = (w_1(t), w_2(t), w_3(t), w_4(t))$ be the vector solution of the corresponding first-order system (4.5). Then, we see from $\Delta u(r) \to 0$ as $r \to \infty$, that

$$e^{-\frac{2}{3}t}\Big[v''(t) + \frac{11}{3}v'(t) + \frac{28}{9}v(t)\Big] \to 0 \ \text{ as } t \to \infty. \tag{4.7}$$

Proposition 4.2. *We have*

$$\lim_{t\to\infty} \mathbf{w}(t) = P.$$

In particular, the trajectory $\mathbf{w}$ *is on the stable manifold of* P.

To prove this proposition, we first prove some useful lemmas.

Lemma 4.3. *Let v be the global solution and assume $L \in [0, \infty]$ such that*

$$\lim_{t\to\infty} v(t) = L.$$

Then $L = (-K_0)^{-1/3}$.

Proof. We first exclude the case $L = +\infty$. By (4.7), we see that

$$v''(t) + \frac{11}{3}v'(t) + \frac{28}{9}v(t) := g(t) = o(e^{\frac{2}{3}t}) \text{ as } t \to \infty.$$

Thus, the standard ODE theory implies that

$$\begin{aligned} v(t) &= B_1 e^{-\frac{7}{3}t} + B_2 e^{-\frac{4}{3}t} + \int_T^t \Big(e^{-\frac{7}{3}(t-s)} - e^{-\frac{4}{3}(t-s)}\Big) g(s)ds \\ &\le B_3 e^{-\frac{4}{3}t} + B_4 e^{-\frac{4}{3}t} \int_T^t e^{\frac{4}{3}s} g(s)ds = o(e^{\frac{2}{3}t}) \text{ as } t \to \infty, \end{aligned}$$

where $T > 0$ is sufficiently large. On the other hand, since $v(t) \to +\infty$ as $t \to \infty$, we see from (3.1) that

$$v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = o(1) \text{ as } t \to \infty. \quad (4.8)$$

The corresponding characteristic polynomial is

$$\rho^4 + \frac{10}{3}\rho^3 + \frac{15}{3^2}\rho^2 - \frac{50}{3^3}\rho - \frac{56}{3^4} = (\rho - \frac{2}{3})\Big(\rho^3 + \frac{12}{3}\rho^2 + \frac{39}{3^2}\rho + \frac{28}{3^3}\Big),$$

and the unique positive eigenvalue is $\rho = \frac{2}{3}$. Therefore,

$$e^{-\frac{2}{3}t} v(t) \to c, \; c > 0 \text{ as } t \to \infty.$$

This contradicts the fact obtained above that $v(t) = o(e^{\frac{2}{3}t})$.

If $L \neq (-K_0)^{-1/3}$, then $-v^{-2}(t) - K_0 v(t) \to \alpha \neq 0$ and for $\epsilon > 0$ sufficiently small there exists $T > 0$ such that

$$\alpha - \epsilon \le v^{-4}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) \le \alpha + \epsilon \quad \forall t \ge T. \quad (4.9)$$

Take $\epsilon < |\alpha|$ so that $\alpha - \epsilon$ and $\alpha + \epsilon$ have the same sign and let

$$\delta := \sup_{t \ge T} |v(t) - v(T)| < \infty.$$

Integrating (4.9) over $[T, t]$ for any $t \ge T$ yields

$$\begin{aligned} (\alpha - \epsilon)(t - T) + C - |K_1|\delta &\le v'''(t) + K_3 v''(t) + K_2 v'(t) \\ &\le (\alpha + \epsilon)(t - T) + C + |K_1|\delta, \quad \forall t \ge T, \end{aligned}$$

where $C = C(T)$ is a constant containing all the terms $v(T)$, $v'(T)$, $v''(T)$ and $v'''(T)$. Repeating this procedure twice gives

$$\frac{\alpha-\epsilon}{6}(t-T)^3+O(t^2)\le v'(t)\le \frac{\alpha+\epsilon}{6}(t-T)^3+O(t^2) \text{ as } t\to\infty.$$

This contradicts the assumption that v admits a finite limit as $t\to\infty$. This completes the proof. □

If v is eventually monotone, then Lemma 4.3 implies that (1.10) holds. So, we need to consider the case that v oscillates infinitely many times near $t=\infty$; i.e., v has an unbounded sequence of consecutive local maxima and minima. In the sequel we always restrict to these kinds of solutions without explicit mention.

We define the energy function

$$E(t)=\frac{1}{v(t)}-\frac{K_0}{2}v^2(t)-\frac{K_2}{2}(v'(t))^2+\frac{1}{2}(v''(t))^2. \tag{4.10}$$

We prove first that on consecutive extrema of v, the energy is decreasing. For the proof of the following lemma, the sign of the coefficients K_1, K_3 in front of the odd-order derivatives in equation (4.4) is absolutely crucial.

Lemma 4.4. *Assume that $t_0 < t_1$ and that $v'(t_0)=v'(t_1)=0$. Then*

$$E(t_0)\ge E(t_1).$$

If v is not constant, then the inequality is strict.

Proof. From the equation (4.4) we find

$$\begin{aligned}
E'(t) &= -v^{-2}(t)v'(t)-K_0v(t)v'(t)-K_2v'(t)v''(t)+v''v''' \\
&= (-v^{-2}-K_0v-K_2v'')v'+v''v''' \\
&= (v^{(4)}(t)+K_3v'''+K_1v')v'(t)+v''v'''.
\end{aligned}$$

Integrating by parts, this yields

$$\begin{aligned}
E(t_1)-E(t_0) &= \int_{t_0}^{t_1}E'(s)ds=-\int_{t_0}^{t_1}v'''(s)v''(s)ds-K_3\int_{t_0}^{t_1}|v''(s)|^2ds \\
&\quad +K_1\int_{t_0}^{t_1}|v'(s)|^2ds+\int_{t_0}^{t_1}v'''(s)v''(s)ds \\
&= -K_3\int_{t_0}^{t_1}|v''(s)|^2ds+K_1\int_{t_0}^{t_1}|v'(s)|^2ds\le 0,
\end{aligned}$$

since $K_3>0$ and $K_1<0$. If v is not a constant, the inequality is strict. □

Lemma 4.5. *There are* $0 < \theta_1 < \theta_2$ *such that*

$$\theta_1 \le v(t) \le \theta_2 \ \text{ for } t \text{ sufficiently large.} \tag{4.11}$$

Proof. Letting $\{t_k\}_{k\in\mathbb{N}}$ denote the sequence of consecutive positive critical points of v, we see that there are $\theta_1, \theta_2 > 0$ such that $\theta_1 \le v(t_k) \le \theta_2$ for all k. Supposing the contrary, we can find a subsequence (still denoted by $\{t_k\}$) such that $v(t_k) \to 0$ or $v(t_k) \to \infty$ as $k \to \infty$. We only consider the first case; the second case is similar. By Lemma 4.4, we see that

$$E(t_1) \ge E(t_k) \ \text{ for any large } k. \tag{4.12}$$

Since $v(t_k) \to 0$ as $t \to \infty$, we easily see that $E(t_k) \to \infty$ as $k \to \infty$, this contradicts (4.12). This completes the proof. □

Lemma 4.6. *For* $T > 0$ *sufficiently large,*

$$\int_T^\infty |v'(s)|^2 ds + \int_T^\infty |v''(s)|^2 ds < \infty.$$

Proof. We take the same sequence $\{t_k\}_{k\in\mathbb{N}}$ as in the proof of Lemma 4.5. We assume that $T > t_1$. Then for any k

$$-K_3 \int_{t_1}^{t_k} |v''(s)|^2 ds + K_1 \int_{t_1}^{t_k} |v'(s)|^2 ds = E(t_k) - E(t_1) \ge -E(t_1) > -\infty.$$

The statement follows by letting $k \to \infty$ and using again the fact that $K_3 > 0$ and $K_1 < 0$. □

Lemma 4.7.

$$\int_T^\infty |v'''(s)|^2 ds < \infty.$$

Proof. Since $u'_{\gamma^*}(r) > 0$ for $r \in (0, \infty)$, we see that $v'(t) + \frac{4}{3}v(t) > 0$ for $t \in (-\infty, \infty)$ and thus

$$-v'(t) < \frac{4}{3}v(t) \quad \text{for } t \in (-\infty, \infty). \tag{4.13}$$

We choose $\{t_k\}_{k\in\mathbb{N}}$ as in the previous lemmas. Now, we can choose another monotonically increasing diverging sequence $\{\tau_k\}_{k\in\mathbb{N}}$ of flex points of v such that v is decreasing there. We choose

$$\tau_k > T, \ \ \tau_k \nearrow \infty, \quad v'(\tau_k) \le 0, \quad v''(\tau_k) = 0.$$

It follows from (4.13) and Lemma 4.5 that $-v'(\tau_k) < \frac{4}{3}v(\tau_k)$ and thus $|v'(\tau_k)| \le 2\theta_2$ for all k. We multiply the equation (4.4) by v'' and integrate over (T, τ_k) to obtain

$$\int_T^{\tau_k} \Big(v^{(4)}(s) + K_3 v'''(s) + K_2 v''(s) + K_1 v'(s) + K_0 v(s)\Big) v''(s) ds \tag{4.14}$$
$$= -\int_T^{\tau_k} v^{-2}(s) v''(s) ds.$$

We show that all the lower-order terms remain bounded, when $k \to \infty$. We see that

$$\Big| \int_T^{\tau_k} v^{-2}(s) v''(s) ds \Big| = \Big| [v^{-2} v']_T^{\tau_k} - 2\int_T^{\tau_k} v^{-3}(s) |v'(s)|^2 ds \Big| \le C, \tag{4.15}$$

by Lemmas 4.5 and 4.6. With the same argument, one also obtains

$$\Big| \int_T^{\tau_k} v(s) v''(s) ds \Big| \le C. \tag{4.16}$$

The Hölder inequality and Lemma 4.6 imply

$$\Big| \int_T^{\tau_k} v'(s) v''(s) ds \Big| \le C. \tag{4.17}$$

By our choice of τ_k (recall that $v''(\tau_k) = 0$), we obtain

$$\Big| \int_T^{\tau_k} v'''(s) v''(s) ds \Big| = \frac{1}{2} |v''(T)|^2 \le C. \tag{4.18}$$

Finally, integrating by parts, we find from (4.14)-(4.18) that

$$\int_T^{\tau_k} (v'''(s))^2 ds \le \Big| \int_T^{\tau_k} v^{(4)}(s) v''(s) ds \Big| + |v'''(T) v''(T)| \le C. \tag{4.19}$$

Letting $k \to \infty$, we obtain our conclusion. □

Lemma 4.8.

$$\int_T^{\infty} |v^{(4)}(s)|^2 ds < \infty.$$

Proof. In view of Lemmas 4.5-4.7 we may find a sequence $\{s_k\}$ such that

$$\lim_{k\to\infty} s_k = \infty, \quad v(s_k) = O(1), \quad \lim_{k\to\infty} v'(s_k) = \lim_{k\to\infty} v''(s_k) = \lim_{k\to\infty} v'''(s_k) = 0.$$

We multiply the equation (4.4) by $v^{(4)}$ and integrate over $[T, s_k)$ to obtain

$$\int_T^{s_k} (v^{(4)}(s))^2 ds$$
$$= \int_T^{s_k} (-v^{-2}(s) - K_0 v(s) - K_1 v'(s) - K_2 v''(s) - K_3 v'''(s)) v^{(4)}(s) ds. \tag{4.20}$$

By using Lemmas 4.5-4.7 and arguing as in the previous proofs we obtain

$$\int_T^{s_k} v^{(4)}(s)v'''(s)ds = \Big[\frac{1}{2}|v'''(s)|^2\Big]_T^{s_k} = O(1);$$

$$\int_T^{s_k} v^{(4)}(s)v''(s)ds = O(1) - \int_T^{s_k} |v'''(s)|^2 ds = O(1);$$

$$\int_T^{s_k} v^{(4)}(s)v'(s)ds = O(1) - \int_T^{s_k} v'''(s)v''(s)ds = O(1);$$

$$\begin{aligned}\int_T^{s_k} v^{(4)}(s)v(s)ds &= O(1) - \int_T^{s_k} v'''(s)v'(s)ds \\ &= O(1) + \int_T^{s_k} |v''(s)|^2 ds = O(1);\end{aligned}$$

$$\begin{aligned}\int_T^{s_k} v^{(4)}v^{-2}(s)ds &= O(1) + 2\int_T^{s_k} v^{-3}v'''(s)v'(s)ds \\ &\le O(1) + C\Big(\int_T^{s_k} |v'''(s)|^2 ds\Big)^{1/2}\Big(|v'(s)|^2 ds\Big)^{1/2} \le O(1).\end{aligned}$$

Inserting all these estimates into (4.20), the claim follows. □

Lemma 4.9.

$$\int_T^{\infty} v^2(s)(v^{-3}(s) + K_0)^2 ds < \infty.$$

Proof. From the equation (4.4), we conclude

$$(v^{(4)}(s) + K_3 v'''(s) + K_2 v''(s) + K_1 v'(s))^2 = v^2(s)(v^{-3}(s) + K_0)^2.$$

The statement follows now immediately from Lemmas 4.5-4.8. □

The proof of Proposition 4.2 and (1.10) will be completed by showing the following:

Lemma 4.10. *Let* $\mathbf{w} = (w_1, w_2, w_3, w_4)$ *be as in Proposition* 4.2. *We assume further that* $v = w_1$ *has an unbounded sequence of consecutive local maxima and minima near* $t = \infty$. *Then it follows that*

$$\lim_{t\to\infty} \mathbf{w}(t) = P. \tag{4.21}$$

In particular, $\lim_{t\to\infty} v(t) = (-K_0)^{-1/3}$.

Proof. We first show that the limit of $v'(t)$ as $t \to \infty$ exists. Define

$$h(t) := \int_T^t v'(\xi)v''(\xi)d\xi \quad \text{for } t > T.$$

We easily see that the limit of $h(t)$ as $t \to \infty$ exists. Indeed, for any large t_1, t_2 with $T < t_1 < t_2$, we see from Lemma 4.6 that

$$|h(t_2)-h(t_1)| \le \Big(\int_{t_1}^{t_2} (v'(\xi))^2 d\xi\Big)^{1/2} \Big(\int_{t_1}^{t_2} |v''(\xi)|^2 d\xi\Big)^{1/2} \to 0 \ \text{ as } t_1, t_2 \to \infty.$$

Thus, $\lim_{t\to\infty} h(t)$ exists and this implies $\lim_{t\to\infty} |v'(t)|$ exists. Lemma 4.6 implies that $\lim_{t\to\infty} v'(t) = 0$. Thus, we can obtain that $\lim_{t\to\infty} v''(t) = 0$, $\lim_{t\to\infty} v'''(t) = 0$ and $\lim_{t\to\infty} v^{(4)}(t) = 0$. It is easily seen from the equation (4.4) that

$$\lim_{t\to\infty} (v^{-2}(t) + K_0 v(t)) = 0.$$

This implies that

$$\lim_{t\to\infty} v(t) = (-K_0)^{-1/3}.$$

This completes the proof. □

Finally we complete the proof of Theorem 4.1.

Proof of Theorem 4.1. To prove the first part of this theorem, we just need to show that there is no solution to (4.1) with $\psi(0) = 0$, $\Delta\psi(0) = 1$. In fact, if there is a solution to (4.1) with $\psi(0) = 0$, $\Delta\psi(0) = 1$, then we claim that $\psi(r)$ can't have zeroes in $(0, +\infty)$. In fact, if $\psi(r) > 0$ for $r \in (0, R)$, $\psi(R) = 0$, then $\Delta(\Delta\psi) > 0$ in $(0, R)$, and hence, $(\Delta\psi)'(r) > 0$ and $(\Delta\psi)(r) \ge (\Delta\psi)(0) = 1$ for $r \in (0, R)$, which then implies that $\psi \ge \frac{1}{4}r^2$ for $r \in (0, R)$, a contradiction of the fact that $\psi(R) = 0$. Note that $\psi(r) > 0$ for $r > 0$ and so $\Delta(\Delta\psi) > 0$. Hence $\Delta\psi \ge 1$ for $r > 0$. This is a contradiction of our assumption.

Thus, $\psi = c(\frac{4}{3}u_{\gamma^*} - r u'_{\gamma^*})$, for some constant c.

Using the Emden-Fowler transformation (4.3) and letting

$$v(t) = (-K_0)^{-1/3} + h(t),$$

we see that $h(t)$ satisfies

$$h^{(4)}(t) + K_3 h'''(t) + K_2 h''(t) + K_1 h'(t) + 3K_0 h(t) + O(h^2) = 0, \quad t > 1. \tag{4.22}$$

Therefore, in the leading order, we can write

$$h(t) = M_1 e^{-\frac{5}{6}t} \cos\beta t + M_2 e^{-\frac{5}{6}t} \sin\beta t + M_3 e^{\nu_1 t} + o(e^{\nu_1 t}), \tag{4.23}$$

where $\beta = \frac{\sqrt{4\sqrt{193}-45}}{6}$, since $\mathbf{w}$ is on the stable manifold of the singular point P. This then implies that, as $r \to +\infty$,

$$k(r) = M_1 r^{1/2}\cos(\beta \ln r) + M_2 r^{1/2}\sin(\beta \ln r) + M_3 r^{\frac{4}{3}+\nu_1} + o(r^{4/3+\nu_1}), \tag{4.24}$$

where $k(r) = r^{4/3}h(t) := u_{\gamma^*}(r) - U_0(r)$, $t = \ln r$.

We now show that $M_1^2 + M_2^2 \neq 0$. Suppose now that $M_1 = M_2 = 0$. Then we have

$$k(r) \sim r^{-1-\kappa} \text{ as } r \to +\infty, \tag{4.25}$$

where $\kappa = -\frac{7}{3} - \nu_1 > 0$. Furthermore, $k(r)$ has no zeroes for r large. We show that this is impossible. In fact, it is easy to see that k must change sign in $(0, +\infty)$. Otherwise, we assume $k > 0$. Then using the behavior of k near ∞ and integrating the equation $\Delta^2 k = -\frac{u_{\gamma^*}^{-2} - U_0^{-2}}{u_{\gamma^*} - U_0} k$ over $\mathbb{R}^3$, we see that

$$\int_0^\infty \frac{r^2(u_{\gamma^*}^{-2} - U_0^{-2})}{u_{\gamma^*} - U_0} k(r)dr = 0,$$

which contradicts the fact that $k > 0$. (The integral exists because of (4.25).)

Supposing $k(r)$ has exactly j zeroes in $(0, +\infty)$ (recalling that k has no zeroes when r is large) and $k(r) \sim r^{-1-\kappa}$ as $r \to \infty$, we easily see that $r^2 k'(r)$ has j zeroes. On the other hand, since the function $\eta(r) := r^2 k'(r)$ satisfies $\eta(0) = 0$ and $\eta(r) \to 0$ as $r \to \infty$, we see that $\eta'(r)$ has $j+1$ zeroes. Thus, $\Delta k(r) = \frac{1}{r^2}\eta'(r)$ has at least $j+1$ zeroes. A similar idea implies that $r^2(\Delta k)'(r)$ has at least j zeroes and $(r^2(\Delta k)'(r))'$ has at least $j+1$ zeroes. Therefore, $\Delta^2 k = \frac{1}{r^2}(r^2(\Delta k)'(r))'$ has at least $j+1$ zeroes. This contradicts our assumption that k has j zeroes, since $\Delta^2 k = -\frac{u_{\gamma^*}^{-2} - U_0^{-2}}{u_{\gamma^*} - U_0} k > 0$. This proves our claim and completes the proof of Theorem 4.1. □

5. Structure of radial solutions of (1.11): Proof of Theorem 1.4

In this section, we study the structure of radial solutions of (1.11) and prove Theorem 1.4. Note that (1.11) is reduced to

$$\begin{cases} u^{(4)}(r) + \frac{4}{r}u'''(r) = \frac{\lambda}{(1-u(r))^2} \ \text{ for } r \in (0,1) \\ 0 \le u(r) < 1 \\ u(1) = 0, \ \ u''(1) + 2u'(1) = 0, \ \ u'(0) = u'''(0) = 0, \end{cases} \tag{5.1}$$

where $u = u(r)$ for $r = |x|$. We apply the phase plane analysis as in [26], [27], but our case is more complicated since the operator in our equation is 4th order.

Next, we introduce the initial-value problem

$$\begin{cases} u^{(4)}(r) + \frac{4}{r}u'''(r) = \frac{\lambda}{(1-u(r))^2} \text{ for } r \in (0,1) \\ u(0) = A \in (0,1), \;\; u'(0) = u'''(0) = 0. \end{cases} \tag{5.2}$$

Make the changes:

$$v(y) = \frac{1-u(r)}{1-A}, \quad y = \lambda^{1/4}(1-A)^{-3/4}r.$$

Then (5.2) is reduced to

$$\begin{cases} v^{(4)}(y) + \frac{4}{y}v'''(y) = -v^{-2}(y) \text{ for } y \in (0, \lambda^{1/4}(1-A)^{-3/4}) \\ 0 < v \leq \frac{1}{1-A}, \qquad v(0) = 1, \;\; v'(0) = v'''(0) = 0. \end{cases} \tag{5.3}$$

Setting $\theta = \lambda^{1/4}(1-A)^{-3/4}$, we see that the solution $v(y)$ of (5.3) depends on θ; we denote it by v_θ. Moreover, $v_\theta(\theta) = \frac{1}{1-A}$, $(\Delta_y v_\theta)(\theta) = 0$. We claim that $v_\theta(y) \to u_{\gamma^*}(y)$ for all $y \in (0,\infty)$ as $\theta \to \infty$. This can be seen from Theorem 1.1. Note that, for each θ, there is a unique γ_θ such that $(v_\theta)''(0) = \gamma_\theta$. We easily see that $\gamma_\theta \to \gamma^*$ as $\theta \to \infty$, where γ^* is defined in Theorem 1.1. The standard ODE theory implies that our claim holds.

We apply the Emden-Fowler transformation:

$$z_\tau(t) = y^{-\frac{4}{3}}v_\theta(y), \quad t = \ln y,$$

where $\tau = \ln\theta$. Then (5.3) changes to

$$\begin{cases} z_\tau^{(4)}(t) + K_3 z_\tau'''(t) + K_2 z_\tau''(t) + K_1 z_\tau'(t) + K_0 z_\tau(t) = -z_\tau^{-2} \\ 0 < z_\tau(t) < \frac{1}{1-A}e^{-\frac{4}{3}t} \\ \lim_{t\to-\infty} e^{\frac{4}{3}t}z_\tau(t) = 1, \; \lim_{t\to-\infty} e^{\frac{4}{3}t}z_\tau'(t) = -\dfrac{4}{3}, \\ \lim_{t\to-\infty} e^{\frac{4}{3}t}z_\tau''(t) = \dfrac{16}{9}, \end{cases} \tag{5.4}$$

for $t \in (-\infty, \tau)$. Through the above transformation, the boundary conditions $u(1) = \Delta u(1) = 0$ correspond to

$$z_\tau(\tau) = \lambda^{-1/3}, \;\; (z_\tau)''(\tau) + \frac{11}{3}(z_\tau)'(\tau) + \frac{28}{9}z_\tau(\tau) = 0.$$

In other words, for any $\tau \in \mathbb{R}$, (λ_τ, u_τ) defined by

$$\begin{cases} u_\tau(r) = 1 - \frac{z_\tau(\tau+\ln r)}{z_\tau(\tau)} r^{\frac{4}{3}}, \quad \lambda_\tau = \frac{1}{z_\tau^3(\tau)}, \quad A_\tau = 1 - \frac{1}{e^{\frac{4}{3}\tau} z_\tau(\tau)} \\ (z_\tau)''(\tau) + \frac{11}{3}(z_\tau)'(\tau) + \frac{28}{9} z_\tau(\tau) = 0 \end{cases} \tag{5.5}$$

satisfies (5.1), and conversely, every solution of (5.1) is written in the form of (5.5). Hence, $\mathcal{C}_r$ is homeomorphic to $\mathbb{R}$. Since $v_\theta(y) \to u_{\gamma^*}(y)$ for all $y \in (0, \infty)$ as $\theta \to \infty$, we easily see that $z_\tau(t) \to Z(t)$ for all $t \in (-\infty, \infty)$ as $\tau \to \infty$ with $Z(t) := y^{-4/3} u_{\gamma^*}(y)$ is a solution of the problem

$$\begin{cases} Z^{(4)}(t) + K_3 Z'''(t) + K_2 Z''(t) + K_1 Z'(t) + K_0 Z(t) = -Z^{-2} \text{ for } t \in \mathbb{R} \\ \lim_{t \to -\infty} e^{\frac{4}{3}t} Z(t) = 1, \ \lim_{t \to -\infty} e^{\frac{4}{3}t} Z'(t) = -\dfrac{4}{3}, \ \lim_{t \to -\infty} e^{\frac{4}{3}t} Z''(t) = \dfrac{16}{9}. \end{cases}$$

Note that $\tau \to \infty$ as $\theta \to \infty$. It is clear that $Z(t) \equiv v(t)$ and $v(t)$ is given in (4.3). The singular point $\mathbf{w} = P$ corresponds to $(\lambda, u) = (\lambda_*, 1 - |x|^{\frac{4}{3}})$ since $z_\tau(\tau) \to (-K_0)^{-1/3}$ as $\tau \to \infty$, where $\lambda_* = -K_0$.

To prove that $\mathcal{C}_r$ bends infinitely many times with respect to λ around λ^*, we only need to show that P is a spiral attractor. Since $\mathbf{w}$ is on the stable manifold of the singular point P, we see that all trajectories of system (4.5) are eventually tangential to the space

$$S := \{s_1 \mathbf{x}_1 + s_2 \mathbf{x}_2 + b\mathbf{y} : \ s_1, s_2, b \in \mathbb{R}\}.$$

Here, $\mathbf{x}_1 \pm i\mathbf{x}_2$ denotes eigenvectors of the matrix M_P defined in (4.6) corresponding to the complex eigenvalues ν_3, ν_4. $\mathbf{y}$ denotes the eigenvector of the matrix M_P defined in (4.6) corresponding to the real eigenvalue ν_1. But by Theorem 4.1, we have

$$v(t) = (-K_0)^{-1/3} + M_1 e^{-\frac{5}{6}t} \cos \beta t + M_2 e^{-\frac{5}{6}t} \sin \beta t + M_3 e^{\nu_1 t} + o(e^{\nu_1 t}), \tag{5.6}$$

where $M_1^2 + M_2^2 \neq 0$. Thus P is a spiral attractor. This shows that $\mathcal{C}_r$ must bend infinitely many times with respect to λ around λ_*.

Next, we show that the secondary bifurcation point of $\mathcal{C}_r$ does not occur, which is the content of the following lemma.

Lemma 5.1. *For any $\kappa \in (0, 1)$, there is at most one $\tilde{\lambda} := \tilde{\lambda}(\kappa) \in (0, \lambda_c]$ with $(\tilde{\lambda}, u_{\tilde{\lambda}}) \in \mathcal{C}_r$ and $u_{\tilde{\lambda}}(0) = \kappa$.*

Proof. Suppose there are λ_1, $\lambda_2 \in (0, \lambda_c]$ with $\lambda_1 \neq \lambda_2$, say $\lambda_1 > \lambda_2$ and $(\lambda_1, u_{\lambda_1})$, $(\lambda_2, u_{\lambda_2}) \in \mathcal{C}_r$ such that $u_{\lambda_1}(0) = u_{\lambda_2}(0) = \kappa$. If we set $u_1 \equiv u_{\lambda_1}$,

$u_2 \equiv u_{\lambda_2}$ and $z_j = 1 - u_j(r)$ for $j = 1, 2$, then

$$-\Delta^2 z_j = \lambda_j z_j^{-2}, z_j(0) = 1 - \kappa, z_j'(0) = z_j'''(0) = 0, z_j(1) = 1, (\Delta z_j)(1) = 0. \tag{5.7}$$

Let $\tilde{z}_j(y) = \frac{z_j((1-\kappa)^{3/4}\lambda_j^{-1/4}y)}{1-\kappa}$. We see that $\tilde{z}_j$ $(j = 1, 2)$ satisfies

$$\Delta_y^2 v_j = -v_j^{-2}, v_j(0) = 1,\ v_j'(0) = v_j'''(0) = 0, v_j(\tau_j) = \frac{1}{1-\kappa}, (\Delta_y v_j)(\tau_j) = 0, \tag{5.8}$$

where $\tau_j = \lambda_j^{1/4}(1-\kappa)^{-3/4}$. Since $\lambda_1 > \lambda_2$, we see that $\tau_1 > \tau_2$. Supposing $(v_1)_{yy}(0) > (v_2)_{yy}(0)$, by the comparison principle (see Lemma 2.2), we see that $v_1(y) > v_2(y)$ for $y \in (0, \tau_2]$. This contradicts the fact that $v_1(\tau_2) < v_2(\tau_2) = \frac{1}{1-\kappa}$. Supposing $(v_1)_{yy}(0) < (v_2)_{yy}(0)$, by the comparison principle again, we see that $(\Delta v_1)(\tau_2) < (\Delta v_2)(\tau_2) = 0$, but this contradicts the fact that $(\Delta v_1)(\tau_2) > (\Delta v_1)(\tau_1) = 0$. Thus, $(v_1)_{yy}(0) = (v_2)_{yy}(0)$ and thus, $v_1 \equiv v_2$. Therefore, $\lambda_1 = \lambda_2$. This is a contradiction and completes the proof. □

Finally, we analyze the Morse index of the solutions. Given $u \in \mathcal{C}_r^\lambda$, the linearized eigenvalue problem is defined as follows:

$$\Delta^2 \varphi = \frac{2\lambda}{(1-u)^3}\varphi + \mu\varphi \quad \text{in } B, \quad \varphi = \Delta\varphi = 0 \ \text{ on } \partial B. \tag{5.9}$$

Then, the number of its negative eigenvalues, denoted by $i_R = i_R(\lambda, u)$, is called the radial Morse index. Equivalently, one can define i_R to be the maximum dimension of the space $\mathcal{V}$ in $H^2(B) \cap H_0^1(B)$ such that the following quadratic form

$$Q[\phi] = \int_B [|\Delta\phi|^2 - \frac{2\lambda}{(1-u)^3}\phi^2] \tag{5.10}$$

is negative.

We have:

Theorem 5.2. *Under the assumption of Theorem* 1.4, $i_R = i_R(\lambda, u) \to \infty$ *as* $(\lambda, u) \to (\lambda_*, 1 - |x|^{4/3})$.

Proof. Each $(\lambda, u) \in \mathcal{C}_r$ can be parametrized by $\tau \in \mathbb{R}$: $(\lambda, u) = (\lambda(\tau), u(\tau))$. We denote by $\mu_{i,\lambda(\tau)}(u(\tau))$ the i-th eigenvalue of the linearized eigenvalue problem with radially symmetric eigenfunction; i.e.,

$$\Delta^2 \phi = \frac{2\lambda}{(1-u)^3}\phi + \mu\phi \ \ r \in (0, 1), \tag{5.11}$$

$$\phi(1) = \phi''(1) + 2\phi'(1) = 0,\ \ \phi'(0) = \phi'''(0) = 0.$$

Each $\mu_{i,\lambda(\tau)}(u(\tau))$ is simple. If $(\lambda(\tau), u(\tau))$ is on the turning point of $\mathcal{C}_r$, then there is $i \geq 1$ such that $\mu_{i,\lambda(\tau)}(u(\tau)) = 0$ by the implicit function theorem. If $\mu_{i,\lambda(\tau)}(u(\tau)) = 0$ holds for some $i \geq 1$ with $(\lambda(\tau), u(\tau)) \in \mathcal{C}_r$ not on the turning point, then it is actually the secondary bifurcation point of $\mathcal{C}_r$. We will show in the next lemma that this case does not occur. Therefore, $(\lambda(\tau), u(\tau))$ is on the turning point of $\mathcal{C}_r$ if and only of (5.11) has the eigenvalue 0. Note that the lemma below also implies that the curve $\mathcal{C}_r$ has no intersection. Theorem 1.1 implies that the functions $\mu_{i,\lambda(\tau)}(u(\tau))$ are continuous, piecewise analytic, and have only isolated zeroes. We will show that, for any positive integer i, $\mu_{i,\lambda(\tau)}(u(\tau)) < 0$ for large τ. This means that, for any $\zeta > 0$, the operator

$$\Delta^2 - \frac{2\lambda(\tau)}{(1-u(\tau))^3} I, \tag{5.12}$$

on $(0,1)$ with the Navier boundary conditions has at least ζ negative eigenvalues for τ large. Hence, we see that there is a sequence $\{\tau_j\}$ with $\tau_j \to \infty$ as $j \to \infty$ such that the number of negative eigenvalues of (5.12) changes at τ_j. (Recall that $\mu_{i,\lambda(-\infty)}(u(-\infty)) = \mu_i(\Delta^2) \to +\infty$ as $i \to \infty$.) Each $(\lambda(\tau_j), u(\tau_j))$ must be a turning point. Otherwise, the solution curve near $(\lambda(\tau_j), u(\tau_j))$ can be parametrized by λ and the critical groups of these solutions must be locally independent of λ by homotopy invariance of the critical groups (where critical groups are defined in Chang [4]). By the formula for the critical groups at a non-degenerate point (see [4], page 33), this implies that the number of negative eigenvalues of the linearization must be constant in a deleted neighborhood of $(\lambda(\tau_j), u(\tau_j))$ which contradicts our choice of τ_j. (There is a minor technical point here. We need to work in the space

$$\mathcal{H}_0(B) = \{h \in H^2(B) \cap H_0^1(B) : \ h(x) = h(|x|) \in H^2, \ \ h(1) = 0\}.$$

We choose $\|u(\tau_j)\|_\infty < \eta < 1$ and then smoothly truncate the function $\frac{1}{(1-s)^2}$ such that it equals $\frac{1}{(1-\eta)^2}$ for $1 > s > \eta$ so the equation makes sense on $\mathcal{H}_0(B)$. Note that the truncation will not affect the solutions close to $(u(\tau_j), \lambda(\tau_j))$ in $\mathcal{H}_0(B) \times \mathbb{R}$.) We also see that each $(u(\tau_j), \lambda(\tau_j))$ is a turning point. (This argument has been used by Dancer [5].)

To prove our claim on $\mu_{i,\lambda(\tau)}(u(\tau))$ for large τ, we need to consider positive solutions (λ_j, u_j) of (1.11) such that $\lambda_j \to \lambda_*$ and $\|u_j\|_\infty \to 1$ as $j \to \infty$. Thus, we see that there is τ_j with $\tau_j \to \infty$ such that $\lambda(\tau_j) = \lambda_j$ and $u(\tau_j) =$

u_j. We use a blowing up argument. If we define $\epsilon_j = 1 - \|u_j\|_\infty$ and

$$U_j(y) = \frac{1 - u_j(\epsilon_j^{3/4}\lambda_j^{-1/4}y)}{\epsilon_j}, \quad y \in B_j = \{y : \ \epsilon_j^{3/4}\lambda_j^{-1/4}y \in (0,1)\},$$

then $U_j(0) = \min_{B_j} U_j = 1$. A rather standard limiting argument shows that a subsequence of U_j converges uniformly to the unique positive solution u_{γ^*} of (1.1) with $u(0) = 1$. Moreover, $\lim_{y\to\infty} y^{-\frac{4}{3}} u_{\gamma^*}(y) = (-K_0)^{-1/3}$.

By Theorem 4.1, we see that the solution q of

$$k^{(4)}(y) + \frac{4}{y}k'''(y) = \frac{2}{u_{\gamma^*}^3(y)}k(y), \quad k(0) = 1, \quad k'(0) = k'''(0) = 0, \quad k''(0) = 1 \tag{5.13}$$

has infinitely many positive zeroes. (We still do not know that relation between the zeroes of $q(y)$ and the Morse index. In the case of second-order equations, we know that there is a relation between the zeroes of $q(y)$ and the Morse index.)

We are now in the position to complete the proof of this theorem. We consider the equation

$$h^{(4)}(r) + \frac{4}{r}h'''(r) = \frac{-2K_0}{r^4}h(r). \tag{5.14}$$

Making the transformations:

$$\phi(t) = r^{-\frac{1}{2}}h(r), \quad t = \ln r,$$

we see from the equation (5.14) that $\phi(t)$ satisfies the equation

$$\phi^{(4)}(t) - \frac{5}{2}\phi''(t) + \Big(\frac{9}{16} - \frac{112}{81}\Big)\phi = 0. \tag{5.15}$$

Now, we show that there exists $0 < \ell < \infty$ such that the problem

$$\begin{cases} \phi^{(4)}(t) - \frac{5}{2}\phi''(t) + \Big(\frac{9}{16} - \frac{112}{81}\Big)\phi = 0 \quad \text{in } (0,\ell), \\ \phi(0) = \phi'(0) = 0, \phi(\ell) = \phi'(\ell) = 0 \end{cases} \tag{5.16}$$

has a solution $\phi_\ell(t)$. Since the eigenvalues of the equation in (5.16) are β_i $(i = 1,2,3,4)$ with

$$\beta_{1,2}^2 = \frac{5}{4} + \frac{\sqrt{193}}{9}, \qquad \beta_{3,4}^2 = \frac{5}{4} - \frac{\sqrt{193}}{9},$$

the solution $\phi(t)$ of the equation (5.16) can be written in the form

$$\phi(t) = A_1 \cos\beta_1 t + A_2 \sin\beta_1 t + A_3 \cosh\beta_2 t + A_4 \sinh\beta_2 t, \tag{5.17}$$

where

$$\beta_1 = \sqrt{\frac{\sqrt{193}}{9} - \frac{5}{4}}, \quad \beta_2 = \sqrt{\frac{\sqrt{193}}{9} + \frac{5}{4}}.$$

Substituting the boundary conditions into (5.17), we see that

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta_1 & 0 & \beta_2 \\ \cos\beta_1\ell & \sin\beta_1\ell & \cosh\beta_2\ell & \sinh\beta_2\ell \\ -\beta_1\sin\beta_1\ell & \beta_1\cos\beta_1\ell & \beta_2\sinh\beta_2\ell & \beta_2\cosh\beta_2\ell \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.18}$$

To obtain a non-trivial (A_1, A_2, A_3, A_4), we need that

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta_1 & 0 & \beta_2 \\ \cos\beta_1\ell & \sin\beta_1\ell & \cosh\beta_2\ell & \sinh\beta_2\ell \\ -\beta_1\sin\beta_1\ell & \beta_1\cos\beta_1\ell & \beta_2\sinh\beta_2\ell & \beta_2\cosh\beta_2\ell \end{vmatrix} = 0; \tag{5.19}$$

i.e.,

$$\begin{aligned} \rho(s) := {} & \beta_1\beta_2(\cosh\beta_2\ell - \cos\beta_1\ell)^2 \\ & -[\beta_1\beta_2\sinh^2\beta_2\ell + (\beta_1^2 - \beta_2^2)\sinh\beta_2\ell\sin\beta_1\ell - \beta_1\beta_2\sin^2\beta_1\ell] = 0. \end{aligned}$$

A simple calculation implies that

$$\rho(s) = 2\beta_1\beta_2 - 2\beta_1\beta_2\cosh\beta_2\ell\cos\beta_1\ell + (\beta_2^2 - \beta_1^2)\sinh\beta_2\ell\sin\beta_1\ell. \tag{5.20}$$

It is clear that

$$\rho(0) = 0, \quad \rho(\frac{2n\pi}{\beta_1}) < 0, \quad \rho(\frac{(2n+1)\pi}{\beta_1}) > 0,$$

where $n \in \mathbb{N}^+$. Thus, we can find $\ell_0 \in \left(\frac{2n\pi}{\beta_1}, \frac{(2n+1)\pi}{\beta_1}\right)$ such that $\rho(\ell_0) = 0$ for n large. This implies that (5.16) has a solution $\phi_{\ell_0}(t)$ for $t \in (0, \ell_0)$. This implies that there is $h_{\ell_0}(r)$ for $r \in (R, e^{\ell_0}R)$ which satisfies the problem

$$\begin{aligned} & h^{(4)}(r) + \tfrac{4}{r}h'''(r) = \tfrac{-2K_0}{r^4}h(r) \quad \text{in } (R, e^{\ell_0}R), \\ & h(R) = h'(R) = 0, \quad h(e^{\ell_0}R) = h'(e^{\ell_0}R) = 0. \end{aligned} \tag{5.21}$$

Extending $h_{\ell_0}(r)$ by 0 outside the interval $(R, e^{\ell_0}R)$, we see that $h_{\ell_0} \in W^{2,2}(\mathbb{R}^3)$. Similar arguments imply that there are infinitely many intervals $J_1, J_2, \ldots, J_k, \ldots$ such that $J_k \cap J_l = \emptyset$ for $k \neq l$, $|J_k| = e^{\ell_0} - 1$, $J_1 = (1, e^{\ell_0})$ such that (5.21) with the similar boundary conditions has a solution h_k on J_k.

If $M > 0$ and σ is small and negative, we see by continuous dependence that there are M intervals $I_1, I_2, \dots, I_M$, such that $I_k \cap I_l = \emptyset$ for $k \neq l$ such that, for each k, the problem

$$m^{(4)}(r) + \frac{4}{r}m'''(r) = \Big[\frac{2}{u_{\gamma^*}^3(r)} + \sigma\Big]m(r) \ \text{ in } I_k, \tag{5.22}$$

with the Dirichlet boundary conditions on two end points of I_k has a solution m_k. Let m_j be the solution of (5.22) and be zero otherwise. Then, $m_j \in W^{2,2}(\mathbb{R}^3)$, m_i are orthogonal if $i \neq j$ (in the product of $(h,k) = \int_{\mathbb{R}^3} \Delta h \Delta k dx$) and by multiplying (5.22) by m_j and integrating between these intervals we see that

$$Q(m) = \int_{\mathbb{R}^3} \Big[\frac{1}{2}|\Delta m|^2 - \frac{2}{u_{\gamma^*}^3}m^2\Big]$$

is strictly negative at each m_i. Hence, the span of m_i is an M-dimensional subspace of $C_0^\infty(\mathbb{R}^3)$ such that $Q(m) < \tilde{\mu} < 0$ if m is in the unit sphere of E, where E is the span of m_j in $W^{2,2}(\mathbb{R}^3)$. Since m_j has compact support it follows easily that there is an M-dimensional subspace of $\mathcal{H}_0(B_j)$ such that

$$\int_{B_j} \Big[|(\Delta m)(z)|^2 - \frac{2(1-\|u(\tau_j)\|_\infty)^3}{(1-u(\tau_j))^3(\rho_j z)}m^2(z)\Big]dz < 0,$$

where $\rho_j = (1-\|u(\tau_j)\|_\infty)^{3/4}[\lambda(\tau_j)]^{-1/4}$ for large τ_j if m is in the unit sphere in E. (Note that B_j, which is B rescaled, has the property that each function in E is supported in B_j for large j.)

Hence, returning to the original scaling (using the transformation $x = \rho_j z$) we see that there is an M-dimensional subspace E_j of $H^2(B) \cap H_0^1(B)$ such that

$$\int_B \Big[|\Delta m(x)|^2 - \frac{2\lambda(\tau)}{(1-u(\tau)(x))^3}m^2(x)\Big]dx < 0,$$

for m in the unit sphere of E_j and τ large. By the variational characterization of eigenvalues, this implies that $\mu_{i,\lambda(\tau)}(u(\tau)) < 0$ for $1 \le i \le M$ if τ is large. Since M is arbitrary, this proves our claim and completes the proof of this theorem. $\square$

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