Advances in Differential Equations Volume 13, Numbers 7-8 (2008), 733–752

EXPONENTIAL DECAY OF TIMOSHENKO SYSTEMS WITH INDEFINITE MEMORY DISSIPATION

Jaime E. Muñoz Rivera and Hugo D. Fernández Sare
Department of Research and Development
National Laboratory for Scientific Computation
Rua Getulio Vargas 333
Quitandinha CEP 25651-070. Petrópolis, Rio de Janeiro – Brazil

(Submitted by: Viorel Barbu)

Abstract. We study the asymptotic behavior of Timoshenko systems with memory, where the memory is given by a non-dissipative kernel and is acting only on one equation of the system. We show that the exponential stability depends on conditions regarding the decay rate of the kernel and a nice relationship between the coefficients of the system. Moreover, with full-memory effect in the system, we will show exponential stability in the general case.

1. Introduction

In this paper, we consider linear Timoshenko systems with memory effect acting only in one equation of the system; that is,

$$\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0 \qquad \text{in} \quad (0,L)\times(0,\infty), \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + g*[b\psi_{xx} - k(\varphi_x+\psi)] = 0 \quad \text{in} \quad (0,L)\times(0,\infty), \tag{1.2}$$

where $\rho_1, \rho_2, k,\ b$ are positive constants and $g*f$ is the convolution term defined by

$$g*f := \int_0^t g(t-s)f(s)ds.$$

The functions φ and ψ describe the transverse displacement of the beam and the rotation angle of a filament of the beam, respectively. In this paper, we consider the following boundary condition

$$\varphi(0,t)=\varphi(L,t)=\psi_x(0,t)=\psi_x(L,t) \quad = \quad 0, \quad t\geq 0, \tag{1.3}$$

Accepted for publication: April 2008.
AMS Subject Classifications: 35B40, 74D99, 47D06, 45D05.
Supported by a CNPq-DLR grant.

but our results are also valid for several boundary conditions, for example, Dirichlet boundary conditions on φ and ψ. The initial conditions are given by

$$\varphi(\cdot,0)=\varphi_0,\quad \varphi_t(\cdot,0)=\varphi_1,\quad \psi(\cdot,0)=\psi_0,\quad \psi_t(\cdot,0)=\psi_1 \quad \text{in} \quad (0,L). \tag{1.4}$$

Here, we will study the asymptotic behavior of the solution of system (1.1)-(1.2) for non-dissipative kernels $g(s)$, where we define as dissipative kernel those with decreasing positive functions $g(s)$; these are the hypotheses considered in almost every system with memory.

We will mention some known results about stability of the system (1.1)-(1.2). Note that, when $g \equiv 0$, system (1.1)-(1.2) is a hyperbolic conservative system, therefore there is no decay. On the other hand, when the memory kernel is of dissipative type, it is proved in [1] that the solution of the system has exponential stability if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}; \tag{1.5}$$

that is, when the wave speeds associated to (1.1)-(1.2) are equal. A similar result also holds when thermal dissipation is introduced in equation (1.2), see [6]. When the memory term in (1.2) is replaced by a frictional damping, that is, replaced by the function $\overline{b}(x)\psi_t, \overline{b}>0$, Soufyne [10] and M. Rivera, R. Racke [6], proved the exponential stability of the linearized system if and only if (1.5) holds. In a recent work [4], the authors investigated the stability of the system replacing the memory term in (1.2) by a history term, $\int_0^\infty g(t-s)\psi_{xx}(s,x)ds$, using also (1.5).

One important point to note here is the following: in papers involving memory or history effect, as in [1, 4], the authors considered kernels g with dissipative properties, that is, $g(t)>0$, $g'(t)<0$, and also $g''(t)>0$. The main result of this work is to remove these dissipative conditions and instead to consider the positivity of the kernel at $t=0$; that is,

$$g \in W^{2,1}(\mathbb{R}) \cap C^2(\mathbb{R}^+) \tag{1.6}$$

$$g_0 = g(0) > 0 \tag{1.7}$$

$$|g(t)| \le C_g e^{-\gamma t},\quad |g'(t)| \le C_g' e^{-\gamma t},\quad |g''(t)| \le C_g'' e^{-\gamma t},\quad \forall t \ge 0, \tag{1.8}$$

with γ, C_g positive constants, and show that the system (1.1)-(1.2) is exponentially stable if and only if (1.5) holds. Note that, in particular, (1.6)-(1.8) implies that the system is not necessarily of dissipative type. In other words, we will show the exponential decay of the solution of system (1.1)-(1.2) when g, g' and g'' may change sign. In particular, our result is valid for kernels which vanish in $[a, \infty[$, for some $a > 0$. Our methodology will be based on fixed point arguments, similar to that used in [5].

This paper is divided into 5 sections. In Section 2, we establish the main tools and procedures used in this paper. In Section 3, we study the Timoshenko system with frictional damping as a starting point to obtain the exponential stability of the non-dissipative system (1.1)-(1.2), which is proved in Section 4. Finally, in Section 5, we consider the fully dissipative Timoshenko system. That is, we introduce the memory effect in both equations of the system. In this case, the restriction on the coefficients of the system (1.5) is not used; that is, to show the exponential decay it will be only necessary to use the hypotheses on the memory kernels.

2. Notation and semigroup formulation

In this section we will establish the main tools that we will use in the next sections. Moreover, we explain the methodology used to obtain the exponential stability of the system (1.1)-(1.2).

We define $r(t)$ as the resolvent kernel of $g(t)$; that is, let $r(t)$ be the solution of the following Volterra equation:

$$r(t) - (g * r)(t) = g(t). \tag{2.1}$$

Then, in the general case, we have the following.

Lemma 2.1. *Suppose that h, g are continuous functions satisfying the following conditions*

$$|h(t)| \leq C_h e^{-\gamma t} \qquad \text{and} \qquad |g(t)| \leq C_g e^{-\gamma t}, \qquad \forall t > 0,$$

with $\gamma, C_h, C_g > 0$ such that $C_g < \gamma$. Then, the solution $r(t)$ of the Volterra equation

$$r(t) = h(t) + (g * r)(t), \tag{2.2}$$

satisfies

$$|r(t)| \leq \frac{C_h(\gamma - \gamma_r)}{\gamma - \gamma_r - C_g} e^{-\gamma_r t}, \quad \forall t > 0,$$

for some $\gamma_r > 0$ such that $C_g < \gamma - \gamma_r$.

Proof. See [5]. □

Remark 2.2. Using the previous Lemma 2.1 we can obtain that the resolvent kernel $r(t)$, defined by (2.1), decays exponentially as time goes to infinity. Moreover, we can obtain that $r'(t), r''(t)$ also decays exponentially. In fact, differentiating (2.2) we get

$$r'(t) = w(t) + (g * r')(t),$$

where $w(t) := g'(t) + r_0 g(t)$, and $r_0 = r(0) = g(0) > 0$, by hypotheses (1.7). Hence, using Lemma 2.1,

$$|r'(t)| \leq \frac{C_w(\gamma - \gamma_{r'})}{\gamma - \gamma_{r'} - C_g} e^{-\gamma_{r'} t}, \quad \forall t > 0,$$

where $\gamma_{r'} > 0$ such that $C_g < \gamma - \gamma_{r'}$. Similarly, differentiating (2.2) twice we get

$$r''(t) = \tilde{w}(t) + (g * r'')(t),$$

where $\tilde{w}(t) := g''(t) + r_0 g'(t) + r_0 g(t)$. Using Lemma 2.1 once more we arrive at

$$|r''(t)| \leq \frac{C_{\tilde{w}}(\gamma - \gamma_{r''})}{\gamma - \gamma_{r''} - C_g} e^{-\gamma_{r''} t}, \quad \forall t > 0,$$

where $\gamma_{r''} > 0$ such that $C_g < \gamma - \gamma_{r''}$.

On the other hand, in order to give a semigroup formulation of the problem, we introduce the Hilbert space

$$\mathcal{H} = H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L_*^2(0,L), \tag{2.3}$$

with

$$L_*^2(0,L) := \Big\{ v \in L^2(0,L) : \int_0^L v\, dx = 0 \Big\},$$
$$H_*^1(0,L) := \Big\{ v \in H^1(0,L) : \int_0^L v\, dx = 0 \Big\},$$

and norm given by

$$\begin{aligned} ||U||_{\mathcal{H}}^2 &= ||(u^1, u^2, u^3, u^4)'||_{\mathcal{H}}^2 \\ &= \rho_1 ||u^2||_{L^2}^2 + \rho_2 ||u^4||_{L^2}^2 + b||u_x^3||_{L^2}^2 + k||u_x^1 + u^3||_{L^2}^2. \end{aligned} \tag{2.4}$$

Additionally, let $\mathcal{K}_{C,\varepsilon}$ and $\mathcal{M}_{C,\varepsilon}$ be the spaces of exponential decay where the solutions will be defined, given by

$$\mathcal{K}_{C,\varepsilon} = \Big\{ \phi \in L^\infty(\mathbb{R}^+; H_*^1(0,L)) : \sup_{t \in \mathbb{R}^+} \big[e^{t\varepsilon} ||\phi_x||_{L^2} \big] \leq C \Big\} \tag{2.5}$$

$$\begin{aligned} \mathcal{M}_{C,\varepsilon} = \Big\{ &(\phi^1, \phi^2) \in L^\infty(\mathbb{R}^+; H_0^1 \times H_*^1(0,L)) : \\ &\sup_{t \in \mathbb{R}^+} \big[e^{t\varepsilon} (||\phi_x^2||_{L^2} + ||\phi_x^1 + \phi^2||_{L^2}) \big] \leq C \Big\}, \end{aligned} \tag{2.6}$$

where C, ε are positive constants to be fixed last.

In order to obtain the exponential stability for the system (1.1)-(1.2), we use the resolvent equation to transform the original system into another for which we obtain, using fixed point arguments, that the solution decays exponentially to zero as time goes to infinity. Using this result we will show that the solution of the original system also decays exponentially to zero.

In fact, denoting F by $F := \rho_2\psi_{tt}$ and using (1.2) we have

$$F = [b\psi_{xx} - k(\varphi_x + \psi)] - g * [b\psi_{xx} - k(\varphi_x + \psi)]\,. \tag{2.7}$$

Then, using the resolvent kernel identity (2.1), we rewrite (2.7) as

$$b\psi_{xx} - k(\varphi_x + \psi) = F + r * F.$$

Therefore, denoting $\beta := \rho_2 r_0 > 0$, the system (1.1)-(1.2) is equivalent to

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{2.8}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\psi_t = P, \tag{2.9}$$

where $P = \rho_2 r\psi_1 + \rho_2 r'\psi_0 - \rho_2 r_0'\psi - \rho_2(r'' * \psi)$. Note that systems (2.8)-(2.9) and (1.1)-(1.2) are equivalent. Therefore, in order to show that the solutions of the system (1.1)-(1.2) are exponentially stable, we will show that the solutions of (2.8)-(2.9) decay exponentially to zero. To do this, first we define an application $\mathcal{T}$, that for any $f \in \mathcal{K}_{C,\varepsilon}$ associates the solution ψ of the system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{2.10}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\psi_t = P_f, \tag{2.11}$$

where $P_f = \rho_2 r\psi_1 + \rho_2 r'\psi_0 - \rho_2 r_0' f - \rho_2(r'' * f)$; that is, $\mathcal{T}(f) = \psi$.

We will prove that $\mathcal{T}$ has a fixed point in $\mathcal{K}_{C,\varepsilon}$; that is, the solution of system (2.10)-(2.11) decays exponentially to zero. Then, using the fact that $r(t)$ and $r'(t)$ are exponentially decreasing, we conclude that the solution of the non-homogeneous system (2.8)-(2.9) decays exponentially to zero, which means that the solution of system (1.1)-(1.2) also decays exponentially to zero.

Our starting point will be the homogeneous system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \tag{2.12}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\psi_t = 0. \tag{2.13}$$

Using semigroup theory and Prüss' results [8], we will show that the system is exponentially stable if and only if (1.5) holds. Then, using the fixed point theorem, we will prove that this decay rate also holds for the system (2.8)-(2.9).

3. Homogeneous system

In this section we study the system (2.12)-(2.13). Our objective is to obtain an explicit rate of decay, depending on the constants of the system: $\rho_1, \rho_2, b, k, \beta$. For this purpose we rewrite the system (2.12)-(2.13) as an evolution equation for $U = (\varphi, \varphi_t, \tilde{\psi}, \tilde{\psi}_t)' \equiv (u^1, u^2, u^3, u^4)'$. Then, U formally satisfies

$$U_t = \mathcal{A}U, \quad U(0) = U_0,$$

where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1)'$ and $\mathcal{A}$ is the differential operator

$$\mathcal{A} = \begin{pmatrix} 0 & I(\cdot) & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2(\cdot) & 0 & \frac{k}{\rho_1}\partial_x(\cdot) & 0 \\ 0 & 0 & 0 & I(\cdot) \\ -\frac{k}{\rho_2}\partial_x(\cdot) & 0 & \Big(-\frac{k}{\rho_2}I + \frac{b}{\rho_2}\partial_x^2\Big)(\cdot) & -r_0 I(\cdot) \end{pmatrix} \tag{3.1}$$

with domain

$$D(\mathcal{A}) \;:=\; \Big\{U = (u^1, u^2, u^3, u^4)' \in \mathcal{H} : u^1 \in H^2(0,L),\ u^2 \in H_0^1(0,L),$$
$$u^3 \in H^2(0,L),\ u_x^3 \in H_0^1(0,L),\ u^4 \in H_0^1(0,L)\Big\},$$

where the Hilbert space $\mathcal{H}$ is given by (2.3). It is not difficult to show that $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$.

Concerning the rate of decay, in [7] the authors proved that the semigroup $S(t)$ on $\mathcal{H}$, associated to system (2.12)-(2.13), is exponentially stable, but no information is given about explicit rates of decay. Here, we will use similar arguments to obtain an explicit rate of decay, which depends on the constants of the system. To prove this, by well-known characterizations of the exponential stability of contraction semigroups (see [2, 3, 9]) it is sufficient and necessary to prove that there exists $\mu > 0$ such that

$$\{\lambda \in \mathbb{C} : Re(\lambda) \geq -\mu\} \subset \varrho(\mathcal{A}) \qquad \text{(resolvent set)}, \tag{3.2}$$

and

$$\exists M > 0 : ||(\lambda I - \mathcal{A})^{-1}|| \leq M \quad \text{for all} \quad Re(\lambda) \geq -\mu. \tag{3.3}$$

As mentioned previously, we have that the system (2.12)-(2.13) is exponentially stable, see [7]. Then there exists a constant $\mu_0 > 0$ satisfying conditions (3.2)-(3.3). We define

$$\gamma_0 \;:=\; \frac{r_0}{r_0 K_1 + K_2 + K_3 + 1} > 0, \tag{3.4}$$

with

$$K_1 := 16\rho_2\Big(\frac{1}{\rho_1}+\frac{1}{b}\Big) > 0, \qquad K_2 := 8\Big(1+\frac{\beta^2}{2\rho_2 k}\Big) > 0, \tag{3.5}$$

$$K_3 := \Big(1+\frac{4kL^2}{b}\Big)\Big(2+\frac{\beta^2}{b\rho_2}+\frac{2\rho_1 L^2}{\rho_2}\Big(1+\frac{4kL^2}{b}\Big)\Big) > 0. \tag{3.6}$$

Note that γ_0 depends only on the constants of the system. We will prove that γ_0 is an explicit rate of exponential decay for system (2.12)-(2.13); that is, we will prove that γ_0 satisfies conditions (3.2)-(3.3).

In fact, if $\gamma_0 \le \mu_0$ the result is obvious, thus it is sufficient to consider the case $\gamma_0 > \mu_0$. That is, it is sufficient to prove (3.2)-(3.3) for the set

$$\Gamma := \Big\{\lambda \in \mathbb{C} : -\gamma_0 \le \mathrm{Re}(\lambda) < -\mu_0\Big\}. \tag{3.7}$$

Note that the resolvent equation is given by

$$\lambda u^1 - u^2 = f^1 \tag{3.8}$$

$$\lambda\rho_1 u^2 - k(u^1_x + u^3)_x = \rho_1 f^2 \tag{3.9}$$

$$\lambda u^3 - u^4 = f^3 \tag{3.10}$$

$$\lambda\rho_2 u^4 - b u^3_{xx} + k(u^1_x + u^3) + \beta u^4 = \rho_2 f^4. \tag{3.11}$$

We will get inequality (3.3) using the following lemmas.

Lemma 3.1. *Let us suppose that conditions* (1.6)-(1.8) *holds. Then, for any* $F \in \mathcal{H}$, *there exists a positive* $C > 0$ *such that*

$$(Re\lambda)\|U\|^2_{\mathcal{H}} + \beta\int_0^L |u^4|\,dx \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Proof. Multiplying (3.9) by u^2 in $L^2(0,L)$ we get

$$\lambda\rho_1\int_0^L |u^2|^2\,dx + k\int_0^L (u^1_x+u^3)\overline{u^2_x}\,dx = \rho_1\int_0^L f^2\overline{u^2}\,dx,$$

and, using equation (3.8),

$$\lambda\rho_1\int_0^L |u^2|^2\,dx + \overline{\lambda}k\int_0^L (u^1_x+u^3)\overline{u^1_x}\,dx = \rho_1\int_0^L f^2\overline{u^2}\,dx + k\int_0^L (u^1_x+u^3)\overline{f^1_x}\,dx. \tag{3.12}$$

On the other hand, multiplying equation (3.11) by $\overline{u^4}$ and integrating over $[0,L]$, we get

$$\lambda\rho_2\int_0^L |u^4|^2\,dx + \underbrace{b\int_0^L u_x^3\overline{u_x^4}\,dx}_{:=I_1} + \underbrace{k\int_0^L (u_x^1+u^3)\overline{u^4}\,dx}_{:=I_2} + \beta\int_0^L |u^4|^2\,dx$$
$$= \rho_2\int_0^L f^4\overline{u^4}\,dx.$$

Substituting u^4 given by (3.10) into I_1 and I_2, we get

$$\lambda\rho_2\int_0^L |u^4|^2\,dx + b\overline{\lambda}\int_0^L |u_x^3|^2\,dx + k\overline{\lambda}\int_0^L (u_x^1+u^3)\overline{u^3}\,dx + \beta\int_0^L |u^4|\,dx$$
$$= \rho_2\int_0^L f^4\overline{u^4}\,dx + b\int_0^L u_x^3\overline{f^3}\,dx + k\int_0^L (u_x^1+u^3)\overline{f^3}\,dx. \tag{3.13}$$

Adding (3.12) and (3.13) and taking the real part our conclusion follows. □

To estimate u^3 we introduce the following multiplier

$$-w_{xx} = u_x^3 \quad , \qquad w(0) = w(L) = 0.$$

Note that the solution w can be written as

$$w(x) = -\int_0^x u^3(y)dy + \frac{x}{L}\int_0^L u^3(y)\,dx \equiv G(u^3)(x).$$

Using this multiplier we have the next result.

Lemma 3.2. *With the same hypotheses as in Lemma* 3.1*, there exists* $C>0$ *such that*

$$b\int_0^L |u_x^3|^2\,dx \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \rho_2\int_0^L |u^4|^2\,dx$$
$$+ \beta\int_0^L |u^4||u^3|\,dx + \rho_1\int_0^L |u^2||G(u^4)|\,dx.$$

Proof. Multiplying (3.11) by $\overline{u^3}$ yields

$$\underbrace{\lambda\rho_2\int_0^L u^4\overline{u^3}\,dx}_{:=I_4} + b\int_0^L |u_x^3|^2\,dx + k\int_0^L u_x^1\overline{u^3}\,dx + k\int_0^L |u^3|^2\,dx + \beta\int_0^L u^4\overline{u^3}\,dx$$
$$= \rho_2\int_0^L f^4\overline{u^3}\,dx,$$

and substituting u^3 given by (3.10) into I_4 gives us

$$b\int_0^L |u_x^3|^2\,dx + k\int_0^L |u^3|^2\,dx + k\int_0^L u_x^1\overline{u^3}\,dx \tag{3.14}$$
$$= -\frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L |u^4|^2\,dx - \beta\int_0^L u^4\overline{u^3}\,dx + \rho_2\int_0^L f^4\overline{u^3}\,dx - \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L u^4\overline{f^3}\,dx.$$

On the other hand, multiplying (3.9) by $\overline{w}$ we have

$$k\int_0^L u_x^1\overline{w_x}\,dx - k\int_0^L |w_x|^2\,dx \tag{3.15}$$
$$= -\frac{\lambda\rho_1}{\overline{\lambda}}\int_0^L u^2\left[\overline{G(u^4)} + \overline{G(f^3)}\right]\,dx + \rho_1\int_0^L f^2\overline{w}\,dx.$$

Since

$$\int_0^L u_x^1\overline{w_x}\,dx = -\int_0^L u_x^1\overline{u^3}\,dx,$$

we conclude from (3.14)-(3.15) that

$$b\int_0^L |u_x^3|^2\,dx + k\int_0^L |u^3|^2\,dx - k\int_0^L |w_x|^2\,dx$$
$$= -\frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L |u^4|^2\,dx - \frac{\lambda\rho_1}{\overline{\lambda}}\int_0^L u^2\overline{G(u^4)}\,dx - \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L u^2\overline{G(f^3)}\,dx$$
$$-\beta\int_0^L u^4\overline{u^3}\,dx + \rho_1\int_0^L f^2\overline{w}\,dx + \rho_2\int_0^L f^4\overline{u^3}\,dx - \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L u^4\overline{f^3}\,dx.$$

Finally, since

$$\int_0^L |w_x|^2\,dx \le \int_0^L |u^3|^2\,dx,$$

by taking the real part in the above equation our conclusion follows. □

The following lemma is used to estimate the term $||u_x^1 + u^3||_{L^2}^2$.

Lemma 3.3. *Under the same hypotheses of Lemma* 3.1 *and, in addition,*

$$\frac{\rho_1}{k} = \frac{\rho_2}{b},$$

then there exists positive constants C such that

$$k\int_0^L |u_x^1 + u^3|^2\,dx \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \rho_2\int_0^L |u^4|^2\,dx$$
$$+\beta\int_0^L |u^4||u_x^1 + u^3|\,dx + 4\rho_2|Re(\lambda)|\int_0^L |u_x^3||u^2|\,dx,$$

where $\lambda \in \Gamma$ is given by (3.7).

Proof. Multiplying (3.11) by $(u_x^1 + u^3)$ in $L^2(0,L)$ gives us

$$\lambda\rho_2 \int_0^L u^4\overline{(u_x^1+u^3)}\,dx + \underbrace{b\int_0^L u_x^3\overline{(u_x^1+u^3)_x}\,dx}_{:=I_5} + k\int_0^L |u_x^1+u^3|\,dx$$

$$+\,\beta\int_0^L u^4\overline{(u_x^1+u^3)}\,dx = \rho_2\int_0^L f^4\overline{(u_x^1+u^3)}\,dx.$$

Substituting $(u_x^1+u^3)_x$ given by (3.9) into I_5, we get

$$\underbrace{\lambda\rho_2\int_0^L u^4\overline{u_x^1}\,dx}_{:=I_6} + \underbrace{\lambda\rho_2\int_0^L u^4\overline{u^3}\,dx}_{:=I_7} + \frac{\overline{\lambda}b\rho_1}{k}\int_0^L u_x^3\overline{u^2}\,dx + k\int_0^L |u_x^1+u^3|^2\,dx$$

$$+\,\beta\int_0^L u^4\overline{(u_x^1+u^3)}\,dx - \frac{b\rho_1}{k}\int_0^L u_x^3\overline{f^2}\,dx = \rho_2\int_0^L f^4\overline{(u_x^1+u^3)}\,dx. \tag{3.16}$$

Substituting u^1 given by (3.8) and u^4 given by (3.10) into I_6, we get

$$I_6 = -\frac{\lambda^2\rho_2}{\overline{\lambda}}\int_0^L u_x^3\overline{u^2}\,dx + \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L u^4\overline{f_x^1}\,dx - \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L f^3\overline{u_x^2}\,dx. \tag{3.17}$$

Using (3.10) in I_7 we get

$$I_7 = \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L |u^4|^2\,dx + \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L u^4\overline{f^3}\,dx. \tag{3.18}$$

Then, replacing (3.17)-(3.18) in (3.16) we obtain

$$\begin{aligned} k\int_0^L |u_x^1+u^3|^2\,dx &= \left(\frac{\lambda^2\rho_2}{\overline{\lambda}} - \frac{\rho_1 b\overline{\lambda}}{k}\right)\int_0^L u_x^3\overline{u^2}\,dx - \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L |u^4|^2\,dx \\ &- \beta\int_0^L u^4\overline{(u_x^1+u^3)}\,dx + \frac{\rho_1 b}{k}\int_0^L u_x^3\overline{f^2}\,dx + \rho_2\int_0^L f^4\overline{(u_x^1+u^3)}\,dx \\ &- \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L f_x^3\overline{u^2}\,dx - \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L u^4\overline{f^3}\,dx - \frac{\lambda\rho_2}{\overline{\lambda}}\int_0^L u^4\overline{f_x^1}\,dx. \end{aligned} \tag{3.19}$$

Let $\lambda = \gamma + i\xi$, then using hypotheses (1.5) and equation (3.8), we can deduce that

$$\left|\left(\frac{\lambda^2\rho_2}{\overline{\lambda}} - \frac{b\rho_1\overline{\lambda}}{k}\right)\int_0^L u^3\overline{u_x^2}dx\right| = \left|\frac{\rho_2}{\overline{\lambda}}\left(\lambda^2 - \overline{\lambda}^2\right)\int_0^L u^3\overline{u_x^2}dx\right|$$

$$\leq 4\rho_2 \frac{|\gamma\xi|}{\sqrt{\gamma^2+\xi^2}} \int_0^L |u_x^3||u^2|dx \leq 4\rho_2|\gamma| \int_0^L |u_x^3||u^2|dx$$
$$\leq 4\rho_2|Re(\lambda)| \int_0^L |u_x^3||u^2|dx. \tag{3.20}$$

Then, using (3.20) in (3.19) our result follows. □

Lemma 3.4. *With the notation above there exists $C > 0$ such that*

$$\rho_1 \int_0^L |u^2|^2\, dx \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{3k}{2}\int_0^L |u_x^1+u^3|^2\, dx + 2kL^2 \int_0^L |u_x^3|^2\, dx.$$

Proof. Multiplying (3.9) by u^1 in $L^2(0,L)$ we get

$$\underbrace{\lambda\rho_1 \int_0^L u^2\overline{u^1}\, dx}_{:=I_8} + k\int_0^L (u_x^1+u^3)\overline{u_x^1}\, dx = \rho_1 \int_0^L f^2\overline{u^1}\, dx.$$

Then, substituting u^1 given by (3.8) in I_8, and using Poincaré's inequality, our conclusion follows. □

Now, we are in position to show the main result of this section.

Theorem 3.5. *Let us assume hypotheses* (1.6)-(1.8), *assume that the initial data* (1.4) *satisfies* $(\varphi_0, \varphi_1, \psi_0, \psi_1)' \in \mathcal{H}$ *given by* (2.3) *and suppose that*

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}.$$

Then, the energy associated to the system (2.12)-(2.13) *with boundary conditions* (1.3) *decays exponentially to zero; that is, there exist, positive constants M, α independent of the initial data, such that*

$$E(t) \leq ME(0)e^{-\alpha t}, \qquad \forall t \geq 0.$$

Proof. We will prove conditions (3.2) and (3.3), see [8]. In fact, let $U = (u^1, u^2, u^3, u^4)'$ and $F = (f^1, f^2, f^3, f^4)'$ satisfying (3.8)-(3.11), then, from Lemma 3.1, we get

$$(Re\lambda)\|U\|_{\mathcal{H}}^2 + \beta||u^4||^2 \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.21}$$

Similarly, from Lemma 3.4, we have

$$2\rho_1||u^2||_{L^2}^2 \leq C||F||_{\mathcal{H}}||U||_{\mathcal{H}} + 3k||u_x^1+u^3||_{L^2}^2 + 4kL^2||u_x^3||_{L^2}^2. \tag{3.22}$$

On the other hand, we can deduce that,

$$\beta\int_0^L |u^4||u_x^1+u^3|\, dx \leq \frac{\beta^2}{2k}\int_0^L |u^4|^2\, dx + \frac{k}{2}\int_0^L |u_x^1+u^3|^2\, dx, \tag{3.23}$$

and

$$4\rho_2|Re(\lambda)|\int_0^L|u_x^3||u^2|\;dx \le 2\rho_2|Re(\lambda)|\Big(\frac{1}{\rho_1}+\frac{1}{b}\Big)\|U\|_{\mathcal{H}}^2. \tag{3.24}$$

Since $\lambda \in \Gamma$ is given by (3.7), it follows that $Re(\lambda) < 0$. Therefore, using (3.23)-(3.24) in Lemma 3.3, we have

$$\begin{aligned} 4k||u_x^1+u^3||^2 \le C_2\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + 8\Big(\rho_2+\frac{\beta^2}{2k}\Big)||u^4||^2 \\ -16\rho_2 Re(\lambda)\Big(\frac{1}{\rho_1}+\frac{1}{b}\Big)\|U\|_{\mathcal{H}}^2. \end{aligned} \tag{3.25}$$

Then, adding (3.22) and (3.25), there exists $C_3 > 0$ such that

$$\begin{aligned} &2\rho_1||u^2||_{L^2}^2 + k||u_x^1+u^3||_{L^2}^2 \\ &\le C_3||F||_{\mathcal{H}}||U||_{\mathcal{H}} + K_2\rho_2||u^4||_{L^2}^2 - K_1 Re(\lambda)||U||_{\mathcal{H}}^2 + 4kL^2||u_x^3||_{L^2}^2. \end{aligned} \tag{3.26}$$

Moreover, from Lemma 3.2 and using the definition of K_3 given by (3.6), it follows that

$$\Big(1+\frac{4kL^2}{b}\Big)b||u_x^3||_{L^2}^2 \le C_4||F||_{\mathcal{H}}||U||_{\mathcal{H}} + K_3\rho_2||u^4||_{L^2}^2 + \rho_1||u^2||_{L^2}^2. \tag{3.27}$$

Then, adding (3.26) and (3.27), we obtain

$$\begin{aligned} &\rho_1||u^2||_{L^2}^2 + k||u_x^1+u^3||_{L^2}^2 + b||u_x^3||_{L^2}^2 \\ &\le C_5||F||_{\mathcal{H}}||U||_{\mathcal{H}} + (K_2+K_3)\rho_2||u^4||_{L^2}^2 - K_1 Re(\lambda)||U||_{\mathcal{H}}^2. \end{aligned} \tag{3.28}$$

Finally, from (3.21) and using the definition of $\beta(= r_0\rho_2)$, we can conclude that

$$\Big(\frac{K_2+K_3+1}{r_0}\Big)Re(\lambda)\|U\|_{\mathcal{H}}^2 + \Big(K_2+K_3+1\Big)\rho_2||u^4||_{L^2}^2 \le C_k\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{3.29}$$

therefore, adding (3.28) and (3.29), we have that

$$\Big(\frac{K_2+K_3+1}{r_0}\Big)(Re\lambda)\|U\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^2 \le C_k\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} - Re(\lambda)K_1\|U\|_{\mathcal{H}}^2,$$

whence we get

$$\Big[Re(\lambda) + \frac{r_0}{r_0K_1+K_2+K_3+1}\Big]\|U\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}.$$

This implies that

$$||(\lambda I - \mathcal{A})^{-1}|| \le M \quad \text{for all} \quad Re(\lambda) > -\gamma_0,$$

where γ_0 is defined by (3.4), and it is a rate of exponential decay for the solution of the system (2.12)-(2.13); that is,

$$||U||_{\mathcal{H}} \quad \leq \quad C||U_0||_{\mathcal{H}} e^{-\gamma_0 t}, \quad \forall t \geq 0. \tag{3.30}$$

Therefore, our conclusion follows. □

Remark 3.6. When (1.5) does not hold, the semigroup associated to the system (2.12)-(2.13) is not exponentially stable, see [7].

4. Exponential decay

We consider now the system given by (2.10)-(2.11); that is, we consider the following system:

$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x \quad = \quad 0, \tag{4.1}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\psi_t \quad = \quad R, \tag{4.2}$$

where $R = -\rho_2 r_0' f - \rho_2 (r'' * f)$ and $f \in \mathcal{K}_{C,\epsilon}$. Our goal in this section is to show that the system (4.1)-(4.2) has exponentially stable solutions and consequently, using the properties of $r(t)$ and $r'(t)$, show that the solutions of the system (1.1)-(1.2) are also exponentially stable.

First, we rewrite the system (4.1)-(4.2) as evolution equation for $U = (\varphi, \varphi_t, \psi, \psi_t)'$ and

$$\mathcal{B}(t) = (0, 0, 0, r(t)\psi_1(x) - r_0' f + r'(t)\psi_0(x) - (r'' * f))'. \tag{4.3}$$

Then, the system (4.1)-(4.2) can be rewritten as

$$U_t(t) = \mathcal{A}U(t) + \mathcal{B}(t), \quad U(0) = U_0,$$

where the operator $\mathcal{A}$ is defined as in the previous section. The mild solution

$$U(t) := (u_1(t), u_2(t), u_3(t), u_4(t))'$$

of the system (4.1)-(4.2) is given by

$$U(t) = S(t)U(0) + \int_0^t S(t-s)\mathcal{B}(s)ds. \tag{4.4}$$

We now introduce a function $\mathcal{T}$, defined as, for all t, $f \in \mathcal{K}_{C,\varepsilon}$, $\mathcal{T}(f) = u_3(t)$, where $u_3(t)$ is the third component of the solution of (4.1)-(4.2), by semigroup theory.

Recalling the constants $\gamma, C_h, C_g, \gamma_r, \gamma_{r'}$ as used in Lemma 2.1, the following lemma holds.

Lemma 4.1. *Assume hypotheses* (1.6)-(1.8). *Let* $f \in \mathcal{K}_{C,\varepsilon}$ *with* $0 < \varepsilon < \min\{\gamma_0, \gamma_{r''}\}$, *where* $\mathcal{K}_{C,\varepsilon}$ *is given by* (2.5). *If*

$$\tau := \sqrt{\frac{\rho_2}{b}}\Big(\frac{2L|r_0'|}{\gamma_0} + \frac{2LC_h(\gamma-\gamma_{r''})}{(\gamma-\gamma_{r''}-C_g)\gamma_{r''}\gamma_0}\Big) < 1,$$

then $\mathcal{T}(\mathcal{K}_{C,\varepsilon}) \subset \mathcal{K}_{C,\varepsilon}$ *is verified for* C *large enough.*

Proof. From (4.4), we have

$$\begin{aligned} ||U(t)||_{\mathcal{H}} &\leq ||S(t)U(0)||_{\mathcal{H}} + \int_0^t ||S(t-s)\mathcal{B}(s)||_{\mathcal{H}} ds \\ &\leq ||U(0)||_{\mathcal{H}} e^{-\gamma_0 t} + \int_0^t e^{-\gamma_0(t-s)} ||\mathcal{B}(s)||_{\mathcal{H}} ds, \end{aligned}$$

where $\mathcal{B}(t)$ is given by (4.3) and satisfies

$$||\mathcal{B}(s)||_{\mathcal{H}} \quad = \quad \sqrt{\rho_2}|| - \underbrace{r_0' f(s)}_{:=I_1} - \underbrace{(r'' * f)(s)}_{:=I_2} ||_{L^2}. \tag{4.5}$$

Note that

$$I_1 \leq 2CL\sqrt{\rho_2}|r_0'|e^{-\varepsilon s}, \qquad I_2 \leq \frac{2CLC_h(\gamma-\gamma_{r''})}{(\gamma-\gamma_{r''}-C_g)(\gamma_{r''}-\varepsilon)} e^{-\varepsilon s}.$$

Using the above inequalities, conditions (1.8), and Lemma 2.1, we get

$$||\mathcal{B}(s)||_{\mathcal{H}} \leq 2CL\sqrt{\rho_2}|r_0'|e^{-\varepsilon s} + \frac{2CLC_h(\gamma-\gamma_{r''})}{(\gamma-\gamma_{r''}-C_g)(\gamma_{r''}-\varepsilon)} e^{-\varepsilon s}, \tag{4.6}$$

provided that $0 < \varepsilon < \gamma_{r''}$ and $C_g < \min\{\gamma - \gamma_r, \gamma - \gamma_{r''}\}$. Then we have

$$\begin{aligned} ||U(t)||_{\mathcal{H}} \leq \Big[& ||U(0)||_{\mathcal{H}} \\ & + \sqrt{\rho_2} C \Big(\frac{2L|r_0'|}{\gamma_0 - \varepsilon} + \frac{2LC_h(\gamma-\gamma_{r''})}{(\gamma-\gamma_{r''}-C_g)(\gamma_{r''}-\varepsilon)(\gamma_0-\varepsilon)}\Big)\Big] e^{-\varepsilon t}, \end{aligned} \tag{4.7}$$

provided that $0 < \varepsilon < \gamma_0$. Since

$$||U||_{\mathcal{H}}^2 = \rho_1 ||u^2||_{L^2}^2 + \rho_2 ||u^4||_{L^2}^2 + b||u_x^3||_{L^2}^2 + k||u_x^1 + u^3||_{L^2}^2 \geq b||u_x^3||_{L^2}^2,$$

then

$$||U||_{\mathcal{H}} \geq \sqrt{b}||u_x^3||_{L^2}. \tag{4.8}$$

Using (4.8) in (4.7) we find that

$$||u_x^3||_{L^2} \leq \sqrt{\frac{\rho_2}{b}} \Big[||U(0)||_{\mathcal{H}}$$

$$+ C\Big(\frac{2L|r_0'|}{\gamma_0 - \varepsilon} + \frac{2LC_h(\gamma - \gamma_{r''})}{(\gamma - \gamma_{r''} - C_g)(\gamma_{r''} - \varepsilon)(\gamma_0 - \varepsilon)}\Big)\Big] e^{-\varepsilon t}, \qquad (4.9)$$

but, by hypothesis, there exists $0 < \varepsilon < \min\{\gamma_0, \gamma_r, \gamma_{r''}\}$ such that

$$\sqrt{\frac{\rho_2}{b}}\Big(\frac{2L|r_0'|}{\gamma_0 - \varepsilon} + \frac{2LC_h(\gamma - \gamma_{r''})}{(\gamma - \gamma_{r''} - C_g)(\gamma_{r''} - \varepsilon)(\gamma_0 - \varepsilon)}\Big) := (1-\delta) < 1.$$

Then, choosing C large enough such that $\sqrt{\frac{\rho_2}{b}}||U(0)||_{\mathcal{H}} < \delta C$, we have from (4.9) that

$$||u_x^3||_{L^2} \quad \leq \quad Ce^{-\varepsilon t}. \qquad (4.10)$$

This implies that $\mathcal{T}(f) = u_3(t) \in \mathcal{K}_{C,\varepsilon}$, which completes the result. □

We will prove now that $\mathcal{T}$ is a contraction.

Lemma 4.2. *Assume the same hypotheses as in Lemma 4.1; then $\mathcal{T}$ is a contraction.*

Proof. We consider the following systems

$$\begin{cases} \rho_1\varphi^1_{tt} - k(\varphi^1_x + \psi^1)_x = 0 \\ \rho_2\psi^1_{tt} - b\psi^1_{xx} + k(\varphi^1_x + \psi^1) + \beta\psi^1_t \\ \qquad = \rho_2 r(t)\psi_1(x) - \rho_2 r_0' f^1 + \rho_2 r'(t)\psi_0(x) - \rho_2(r'' * f^1) \\ \varphi^1(0) = \varphi_0, \quad \varphi^1_t(0) = \varphi_1, \quad \psi^1(0) = \psi_0, \quad \psi^1_t(0) = \psi_1, \end{cases}$$

and

$$\begin{cases} \rho_1\varphi^2_{tt} - k(\varphi^2_x + \psi^2)_x = 0 \\ \rho_2\psi^2_{tt} - b\psi^2_{xx} + k(\varphi^2_x + \psi^2) + \beta\psi^2_t \\ \qquad = \rho_2 r(t)\psi_1(x) - \rho_2 r_0' f^2 + \rho_2 r'(t)\psi_0(x) - \rho_2(r'' * f^2) \\ \varphi^2(0) = \varphi_0, \quad \varphi^2_t(0) = \varphi_1, \quad \psi^2(0) = \psi_0, \quad \psi^2_t(0) = \psi_1. \end{cases}$$

Setting $\nu = \varphi^1 - \varphi^2$, $\eta = \psi^1 - \psi^2$ and $\phi = f^1 - f^2$, we find

$$\begin{cases} \rho_1\nu_{tt} - k(\nu_x + \eta)_x = 0 \\ \rho_2\eta_{tt} - b\eta_{xx} + k(\nu_x + \eta) + \beta\eta_t = -\rho_2 r_0'\phi - \rho_2(r'' * \phi) \\ \nu(0) = 0, \quad \nu_t(0) = 0, \quad \eta(0) = 0, \quad \eta_t(0) = 0. \end{cases}$$

Denoting by $V := (\nu, \nu_t, \eta, \eta_t)'$ and $\mathcal{B}_\phi := (0, 0, 0, -r_0'\phi - r'' * \phi)'$, we can write V in terms of the semigroup in the following way:

$$V(t) = \int_0^t S(t-s)\mathcal{B}_\phi(s)ds.$$

Since $S(t)$ is exponentially stable (see Theorem 3.5), we have that

$$||V(t)||_{\mathcal{H}} \leq \int_0^t e^{-\gamma_0(t-s)}||\mathcal{B}_\phi(s)||_{\mathcal{H}}ds.$$

We can estimate

$$\begin{aligned}||\mathcal{B}_\phi(s)||_{\mathcal{H}} &\leq \sqrt{\rho_2}\left[|r_0'|||\phi(s)||_{L^2} + ||(r''*\phi)(s)||_{L^2}\right]\\ &\leq 2L\sqrt{\rho_2}|r_0'|||\phi_x(s)||_{L^2} + 2L\sqrt{\rho_2}\int_0^s|r''(s-\tau)|.||\phi_x(\tau)||_{L^2}d\tau.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}||V(t)||_{\mathcal{H}} &\leq 2L\sqrt{\rho_2}|r_0'|.||\phi_x(s)||_{L^\infty(0,T;L^2)}\int_0^t e^{-\gamma_0(t-s)}ds\\ &+ 2L\sqrt{\rho_2}||\phi_x(\tau)||_{L^\infty(0,T;L^2)}\int_0^t e^{-\gamma_0(t-s)}ds\int_0^t|r''(\tau)|d\tau\\ &\leq 2L\sqrt{\rho_2}|r_0'|.||\phi_x(s)||_{L^\infty(0,T;L^2)}\frac{1}{\gamma_0}\\ &+ 2L\sqrt{\rho_2}||\phi_x(\tau)||_{L^\infty(0,T;L^2)}\frac{1}{\gamma_0}\int_0^t|r''(\tau)|d\tau.\end{aligned}$$

On the other hand, by (4.8), we can also deduce that

$$||V(t)||_{\mathcal{H}} \geq \sqrt{b}||u_x^3||_{L^2}.$$

From the above two inequalities we get

$$\begin{aligned}||u_x^3||_{L^2} &\leq \sqrt{\frac{\rho_2}{b}}\frac{2L|r_0'|}{\gamma_0}||\phi_x(s)||_{L^\infty(0,T;L^2)}\\ &+ 2L\sqrt{\frac{\rho_2}{b}}||\phi_x(\tau)||_{L^\infty(0,T;L^2)}\frac{1}{\gamma_0}\int_0^\infty|r''(\tau)|d\tau,\end{aligned}$$

and using Lemma 2.1 we find

$$||u_x^3||_{L^2} \quad \leq \quad \sqrt{\frac{\rho_2}{b}}\Big(\frac{2L|r_0'|}{\gamma_0} + \frac{2LC_h(\gamma-\gamma_{r''})}{(\gamma-\gamma_{r''}-C_g)\gamma_{r''}\gamma_0}\Big)||u_x^3||_{L^2},$$

which implies, using Lemma 4.1, that $\mathcal{T}$ is a contraction. □

Using these previous lemmas, now we will show the exponential stability of the system (1.1)-(1.2), using fixed point arguments.

Theorem 4.3. *Assume the same hypotheses as in Lemma* 4.1. *Then the solutions of the system* (1.1)–(1.2) *with boundary conditions* (1.3) *decays exponentially.*

Proof. Since the operator $\mathcal{T}$ is a contraction on $L^\infty(\mathbb{R}^+; H^1_*(0,L))$ and is invariant over $\mathcal{K}_{C,\varepsilon}$, there exists a fixed point, say f, in $\mathcal{K}_{C,\varepsilon}$; that is,

$$\mathcal{T}(f) = f.$$

Note that, by definition $\mathcal{T}(f) = u_3(t)$ is the third component of the solution $U(t)$ of (4.1)-(4.2); that is,

$$U(t) = (u_1(t), u_2(t), u_3(t), u_4(t)) := (\varphi, \varphi_t, \psi, \psi_t).$$

Therefore, the system (4.1)-(4.2) can be written as

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{4.11}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\psi_t = -\rho_2 r_0'\psi - \rho_2(r'' * \psi). \tag{4.12}$$

Then we have that $\psi \in \mathcal{K}_{C,\varepsilon}$ implies that the right-hand side of the system (4.11)-(4.12) decays exponentially, thus the solution (φ, ψ) decays exponentially. On the other hand, using the fact that the kernels $r(t)$ and $r'(t)$ are exponentially decreasing, we obtain that the solutions of the system

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x &= 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\psi_t &= \rho_2 r\psi_1 - \rho_2 r_0'\psi + \rho_2 r'\psi_0 - \rho_2(r'' * \psi) \end{aligned}$$

are exponentially stable. Since the above system is equivalent to the system (1.1)-(1.2), our conclusion follows. □

APPLICATION

For example, our result can be applied to study the asymptotic behavior of the solution related to the problem (1.1)-(1.2) when $L = 1$ and $\rho_1 = \rho_2 = k = b = 1$. Let us consider the relaxation function $g(t) = c\cos(\beta t)e^{-\gamma t}$, with $c, \gamma \in \mathbb{R}^+, \beta \in \mathbb{R}$. In that case,

$$C_g = c, \quad g_0 = c, \quad \gamma_0 = \frac{c}{9c^2 + 32c + 69}, \quad \beta = 3c.$$

Since

$$\begin{aligned} &g'(t) = -\ c\ [\beta\sin(\beta t) + \gamma\cos(\beta t)]\ e^{-\gamma t}, \quad g_0' = -c\gamma, \quad r_0' = (c-\gamma)c, \\ &g''(t) = c[(\gamma^2 - \beta^2)\cos(\beta t) + 2\beta\gamma\sin(\beta t)]e^{-\gamma t}, \end{aligned}$$

which can be written as

$$g'(t) = -c\sqrt{\gamma^2 + \beta^2}\sin(\beta t + \theta_1)e^{-\gamma t}, \qquad \theta_1 = \arccos\frac{\beta}{\sqrt{\beta^2+\gamma^2}}$$

$$g''(t) = -c(\gamma^2 + \beta^2)\sin(\beta t + \theta_2)e^{-\gamma t}, \qquad \theta_2 = \arccos\frac{2\beta\gamma}{\beta^2+\gamma^2},$$

and recalling Lemma 2.1, the function $h(t)$ can be written as

$$\begin{aligned} h(t) &= -c(\gamma^2+\beta^2)\sin(\beta t+\theta_2)e^{-\gamma t} - c^2\sqrt{\gamma^2+\beta^2}\sin(\beta t+\theta_1)e^{-\gamma t} \\ &\quad +(c-\gamma)c^2\cos(\beta t)e^{-\gamma t}. \end{aligned}$$

Thus, $C_h = c(\gamma^2+\beta^2)+c^2\sqrt{\gamma^2+\beta^2}+|c-\gamma|c^2$. Therefore, setting $\gamma = 2c$, we have $C_h \le (13+\sqrt{13}+1)c^3$. Our task is to find c and γ such that

$$\tau = \Big(\frac{2|r_0'|}{\gamma_0} + \frac{2C_h(\gamma-\gamma_{r''})}{(\gamma-\gamma_{r''}-C_g)\gamma_{r''}\gamma_0}\Big) < 1;$$

that is,

$$\tau = \Big(2c(9c^2+32c+69) + \frac{2C_h(\gamma-\gamma_{r''})(9c^2+32c+69)}{(\gamma-\gamma_{r''}-C_g)\gamma_{r''}c}\Big) < 1.$$

Let us take $\gamma_r = \gamma_{r''} = c/2$. We look for c such that

$$G(c) := [2+3(13+\sqrt{13}+1)](9c^2+32c+69)c < 1.$$

Choosing for example $c \le 9.10^{-4}$, we have that $G(c) < 1$. Then the relaxation function can be chosen as $g(t) = c\cos(3ct)e^{-2ct}$.

5. The full–memory effect

In this section we consider the system

$$\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + f*[k(\varphi_x+\psi)_x] = 0 \quad \text{in} \quad (0,L)\times\mathbb{R}^+, \tag{5.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + g*[b\psi_{xx} - k(\varphi_x+\psi)] = 0 \quad \text{in} \quad (0,L)\times\mathbb{R}^+, \tag{5.2}$$

where f has the same hypothesis as g in (1.6)-(1.8). So, in this case, we will show the exponential decay of the solution for non dissipative kernels f and g, for any positive constants ρ_1, ρ_2, b and k. To do this we follow the same approach as was used in the previous section.

Denoting by r_1 and r_2 the resolvent kernels of f and g respectively, we get that system (5.1)-(5.2) is equivalent to

$$\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + \rho_1 v_0\varphi_t - \rho_1 v\varphi_1 + \rho_1 v_0'\varphi - \rho_1 v'\varphi_0 + \rho_1(v''*\varphi) = 0, \tag{5.3}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + \beta\psi_t - \rho_2 r\psi_1 + \rho_2 r_0'\psi - \rho_2 r'\psi_0 + \rho_2(r''*\psi) = 0. \tag{5.4}$$

As in the previous case, our starting point is the study of the homogeneous system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \rho_1 v_0 \varphi_t = 0, \tag{5.5}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\psi_t = 0. \tag{5.6}$$

For this system, using the same arguments as in Theorem 3.5, we have the next result about the exponential stability.

Theorem 5.1. *Let us assume that f and g satisfy the hypotheses* (1.6)-(1.8) *and suppose that the initial data satisfies $(\varphi_0, \varphi_1, \psi_0, \psi_1)' \in \mathcal{H}$ given by* (2.3). *Then, the exponential decay to solutions of the system* (5.5)-(5.6) *holds for any $\rho_1, \rho_2, k, b, v_0, \beta$ positive constants.*

Remark 5.2. In other words, Theorem (5.1) affirms that there exist positive constants M, μ_0 independent of the initial data such that

$$E(t) \le ME(0)e^{-\mu_0 t}, \qquad \forall t \ge 0,$$

where $E(t)$ is the energy associated to the system (5.5)-(5.6). Here, applying the same arguments used in Theorem (3.5) we can obtain a specific μ_0 depending on the constants of the system; that is, we can obtain $\mu_0 = \frac{R}{K_4+1} > 0$ with $R := \min\{r_0, v_0\} > 0$ and

$$K_4 := \max\left\{2 + 4\rho_1 L^2 v_0^2\Big(\frac{1}{k} + \frac{8L^2}{b}\Big), 2 + \frac{8\rho_2 L^2 r_0^2}{b}\right\} > 0.$$

Now, using fixed point arguments, we show the exponential stability of the equivalent system (5.3)-(5.4). We consider the following system

$$\rho_1\tilde{\varphi}_{tt} - k(\tilde{\varphi}_x + \tilde{\psi})_x + \rho_1 v_0 \tilde{\varphi}_t = -\rho_1 v_0' f^1 - \rho_1(v'' * f^1), \tag{5.7}$$

$$\rho_2\tilde{\psi}_{tt} - b\tilde{\psi}_{xx} + b(\tilde{\varphi}_x + \tilde{\psi}) + r_0\rho_2\tilde{\psi}_t = -\rho_2 r_0' f^2 - \rho_2(r'' * f^2), \tag{5.8}$$

where f^1, f^2 are functions which will be chosen later. Setting $U(t) = (u_1(t), u_2(t), u_3(t), u_4(t))'$ and

$$\mathcal{B}_1(t) \;=\; (0, -v_0' f^1 - (v'' * f^1), -r_0' f^2 - (r'' * f^2))', \tag{5.9}$$

where the operator $\mathcal{A}$ is defined by (3.1). The mild solution of system (5.7)-(5.8) can be written as

$$U(t) = S(t)U(0) + \int_0^t S(t-s)\mathcal{B}_1(s)ds. \tag{5.10}$$

We introduce now a function $\mathcal{T}_1$ defined as

$$\forall (f^1, f^2) \in \mathcal{M}_{C,\varepsilon}, \quad \mathcal{T}_1(f^1, f^2) = (u_1(t), u_3(t)),$$

where $U(t) = (u_1(t), u_2(t), u_3(t), u_4(t))'$ is given by (5.10), and $\mathcal{M}_{C,\varepsilon}$ is defined by (2.6).

Then, as in Lemma 4.1, we can show that $\mathcal{T}_1(\mathcal{M}_{C,\varepsilon}) \subset \mathcal{M}_{C,\varepsilon}$, and consequently $\mathcal{T}_1$ is a contraction over $L^\infty(\mathbb{R}^+; H_0^1 \times H_*^1(0,L))$; that is,

$$||\mathcal{T}_1(f^1, f^2)||_{L^\infty(\mathbb{R}^+;H_0^1\times H_*^1(0,L))} \leq \tau||(f^1, f^2)||_{L^\infty(\mathbb{R}^+;H_0^1\times H_*^1(0,L))},$$

where the contraction constant $\tau > 0$ depends only on the exponential decay rates of the kernels f, g. Consequently, using the exponential decay of $v(t)$, $v'(t)$, $r(t)$, $r'(t)$ and using fixed point arguments (see Theorem 4.3), we have the next theorem.

Theorem 5.3. *Let us assume that f and g satisfy the hypotheses* (1.6)-(1.8). *Then for any positive constants ρ_1, ρ_2, b and k the solutions of system* (5.1)-(5.2) *with boundary conditions* (1.3) *decay exponentially.*

Remark 5.4. In particular, the arguments used to obtain exponential stability in Sections 4 and 5 will be valid for kernels $h(t)$ satisfying (1.6), (1.7) and

$$|h(t)| \leq C\epsilon e^{-\epsilon t}, \quad |h'(t)| \leq C\epsilon^2 e^{-\epsilon t}, \quad |h''(t)| \leq C\epsilon^3 e^{-\epsilon t},$$

where $C > 0$ and $\epsilon > 0$ small enough.

REFERENCES

[1] F. Ammar Khodja Ammar, A. Benabdallah, J.E. Muñoz Rivera, and R. Racke, *Energy decay for Timoshenko systems of memory type,* J. Differential Equations, 194 (2003), 82–115.

[2] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert space,* Trans. Amer. Math. Soc., 236 (1978), 385–394.

[3] Z. Liu and S. Zheng, "Semigroups Associated with Dissipative Systems," π Research Notes Math. 398, Chapman&Hall/CRC, Boca Raton (1999).

[4] J. Muñoz Rivera and H. Fernández Sare, *Stability from Timoshenko systems with past history,* J. Mathematical Analysis and Applications, 339 (2008), 482–502.

[5] J. Muñoz Rivera and M. Grazia Naso, *On the decay of the energy for systems with memory and indifinite dissipation,* Asymptotic Analysis, 49 (2006), 189–204.

[6] J. Muñoz Rivera and R. Racke, *Mildly dissipative nonlinear Timoshenko system–global existence and expoential stability,* J. Mathematical Analysis and Applications, 276 (2002), 248–278.

[7] Muñoz Rivera, J.E., Racke, R.: *Global stability for damped Timoshenko systems,* Discrete and Continuous Dynamical Systems, 9 (2003), 1625–1639.

[8] Prüss, J.: *On the spectrum of C_0-semigroups,* Trans. Amer. Math. Soc., 284 (1984), 847-857.

[9] Renardy, M,: *On the type of certain C_0-semigroups,* Commun. in Partial Differential Equations, 18 (1993), 1299–1307.

[10] Soufyane, A.: *Stabilisation de la poutre de Timoshenko,* C. R. Acad. Sci. Paris, Sér. I, 328 (1999), 731–734.