Advances in Differential Equations Volume 13, Numbers 7-8 (2008), 681–732

AN EXPLICIT FINITE DIFFERENCE SCHEME FOR THE CAMASSA-HOLM EQUATION

Giuseppe Maria Coclite
Dipartimento di Matematica, Università degli Studi di Bari
Via E. Orabona 4, 70125 Bari, Italy

Kenneth H. Karlsen
Centre of Mathematics for Applications, University of Oslo
P.O. Box 1053, Blindern, N–0316 Oslo, Norway
and
Department of Scientific Computing, Simula Research Laboratory
P.O. Box 134, N–1325 Lysaker, Norway

Nils Henrik Risebro
Centre of Mathematics for Applications, University of Oslo
P.O. Box 1053, Blindern, N–0316 Oslo, Norway

(Submitted by: J.A. Goldstein)

Abstract. We put forward and analyze an explicit finite difference scheme for the Camassa-Holm shallow water equation that can handle general H^1 initial data and thus peakon-antipeakon interactions. Assuming a specified condition restricting the time step in terms of the spatial discretization parameter, we prove that the difference scheme converges strongly in H^1 towards a dissipative weak solution of the Camassa-Holm equation.

1. Introduction

In this paper, we present and analyze an explicit finite difference scheme for the Camassa-Holm partial differential equation [7]

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u = 2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u, \quad (t,x) \in (0,T) \times \mathbb{R}, \tag{1.1}$$

which we augment with an initial condition:

$$u|_{t=0} = u_0, \qquad u_0 \in H^1(\mathbb{R}), u_0 \neq 0. \tag{1.2}$$

Accepted for publication: April 2008.
AMS Subject Classifications: 35G25, 35L05, 65M06, 65M12.
This work was supported by the Research Council of Norway through the project WaveMaker and an Outstanding Young Investigators Award of K.H. Karlsen.

Rewriting equation (1.1) as

$$(1-\partial_{xx}^2)\,[\partial_t u + u\partial_x u] + \partial_x\big(u^2 + \tfrac{1}{2}(\partial_x u)^2\big) = 0,$$

we see that (for smooth solutions) (1.1) is equivalent to the elliptic-hyperbolic system

$$\partial_t u + u\partial_x u + \partial_x P = 0, \qquad -\partial_{xx}^2 P + P = u^2 + \tfrac{1}{2}(\partial_x u)^2. \tag{1.3}$$

Recalling that $e^{-|x|}/2$ is the Green's function of the operator $1-\partial_{xx}^2$, (1.3) can be written as

$$\partial_t u + \partial_x F(u, \partial_x u) = 0, \quad F(u,\partial_x u) = \tfrac{1}{2}\left[u^2 + e^{-|x|} \star \left(u^2 + \tfrac{1}{2}(\partial_x u)^2\right)\right], \tag{1.4}$$

which can be viewed as a conservation law with nonlocal flux function. In this paper, the relevant formulation of the Camassa-Holm equation (1.1) is the one provided by the hyperbolic-elliptic system (1.3) or (1.4).

The Camassa-Holm equation can be viewed as a model for the propagation of unidirectional shallow water waves [7, 32]; it is a member of the class of weakly non-linear and weakly dispersive shallow water models, a class which already contains the Korteweg-de Vries (KdV) and Benjamin-Bona-Mahony (BBM) equations. In another interpretation, the Camassa-Holm equation models finite length, small-amplitude radial deformation waves in cylindrical compressible hyperelastic rods [21]. It arises also in the context of differential geometry as an equation for geodesics of the H^1-metric on the diffeomorphism group, see for example [17, 18, 30, 36]. The Camassa-Holm equation possesses several striking properties such as an infinite number of conserved integrals, a bi-Hamiltonian structure, and complete integrability [2, 7, 19, 14, 26]. Moreover, it enjoys an infinite number of non-smooth solitary wave solutions, called peakons, which are weak solutions of (1.4).

From a mathematical point of view, the Camassa-Holm equation has by now become rather well studied. While it is impossible to give a complete overview of the mathematical literature, we shall here mention a few typical results, starting with the local(-in-time) existence results in [15, 34, 37] and those using Besov spaces in [23, 22]. It is well known that global solutions do not exist and wave-breaking occurs [7]. Wave-breaking means that the solution itself stays bounded while the spatial derivative becomes unbounded in finite time.

In view of what we have said so far (peakon solutions/wave-breaking), it is clear that a theory based on weak solutions is essential. In the literature there are a number of results on (dissipative and conservative) weak solutions of the Camassa-Holm equation, see [3, 4, 5, 16, 11, 20, 29, 39, 40] and the references cited therein. In this paper, we are interested specifically in the class of *dissipative weak solutions* studied by Xin and Zhang [39, 40]. Their results show, among other things, that there exists a global dissipative weak solution of (1.1)-(1.2) for any H^1 initial data u_0 (peakon-antipeakon interactions are covered). These solutions are global in the sense that they are defined past the blow-up time (wave-breaking). More precisely, suppose $u_0 \in H^1(\mathbb{R})$. Then there exists a global weak (distributional) solution $u \in L^\infty(0,T;H^1(\mathbb{R}))$ of (1.1) satisfying the following properties: $t \mapsto \|u(t,\cdot)\|_{H^1(\mathbb{R})}$ is non-increasing; $\partial_x u \in L^p_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$, $p < 3$;

$$\partial_x u(t,x) \leq \frac{2}{t} + C\,\|u_0\|_{H^1(\mathbb{R})}, \quad \text{for} \quad t > 0, \tag{1.5}$$

for some positive constant C. This last item presumably singles out a unique weak solution. As an example of how this may work we consider the "peakon-antipeakon" solution given by

$$u(t,x) = \tanh(t-1)\big(e^{-|x-y(t-1)|} - e^{-|x+y(t-1)|}\big), \quad y(t) = \log\big(\cosh(t)\big). \tag{1.6}$$

This formula represents a peakon ($e^{-|x+y|}$) colliding with an antipeakon ($-e^{-|x-y|}$) at $x = 0$ and $t = 1$. Note that $u(1,x) = 0$. How this solution is extended to $t > 1$ depends on which solution concept we adopt. If we use the formula (1.6) also for $t > 1$ we get the conservative solution for which $\|u(t,\cdot)\|_{H^1(\mathbb{R})}$ is constant for almost all t. We can also extend the solution by defining $u(t,x) = 0$ for $t > 1$. Obviously, the "entropy condition" (1.5) will only be satisfied for this dissipative solution.

Let us now turn to the topic of the present paper, which is the design and analysis of numerical schemes. The first numerical results for the Camassa-Holm equation are presented in [8] using a pseudo-spectral scheme. Numerical simulations with pesudo-spectral schemes are also reported in [25, 31]. Numerical schemes based on multipeakons (thereby exploiting the Hamiltonian structure of the Camassa-Holm equation) are examined in [6, 9, 10]. In [28], the authors prove that the multipeakon algorithm from [9, 10] converges to the solution of the Camassa-Holm equation as the number of peakons tends to infinity. This convergence result applies to the specific situation where the initial function $u_0 \in H^1$ is such that $(1-\partial^2_{xx})u_0$ is a positive measure. For the same class of initial data, in [27] the authors prove that a semi-discrete finite difference scheme based on the variable $m := (1-\partial^2_{xx})u$ converges strongly in H^1 to the weak solution identified in [16, 20]. In [33], the authors establish error estimates for a spectral projection scheme for smooth solutions. In a different direction, an adaptive high-resolution finite volume scheme is developed and used in [1]. The local discontinuous Galerkin method is adapted to the Camassa-Holm equation in [41]. Although this work does not provide a rigorous convergence result for general (non-smooth) solutions, they show that the discrete total energy is non-increasing in time, thereby suggesting that the approximate solutions are of dissipative nature. Besides, they establish an error estimate for smooth solutions. Finally, multi-symplectic schemes possessing good conservative properties are suggested and demonstrated in the recent work [13].

It seems rather difficult to construct numerical schemes for which one can prove the convergence to a (non-smooth) solution of the Camassa-Holm equation. This statement is particularly accurate in the case of general H^1 initial data and peakon-antipeakon interactions. Indeed, in this context we are only aware of the recent work [12] in which we prove convergence of a tailored semi-discrete difference scheme to a dissipative weak solution. Before we can outline this scheme, let us discretize the spatial domain $\mathbb{R}$ by specifying the mesh points $x_j = j\Delta x$, $x_{j+1/2} = (j + 1/2)\Delta x$, $j = 0, \pm 1, \pm 2, \dots$, where $\Delta x > 0$ is the length between two consecutive mesh points (the spatial discretization parameter). Let D_-, D, and D_+ denote the corresponding backward, central, and forward difference operators, respectively. The scheme proposed in [12], which is based on the formulation (1.3), reads

$$\begin{aligned} &\frac{d}{dt}u_{j+1/2} + \big(u_{j+1/2} \vee 0\big)D_-u_{j+1/2} + \big(u_{j+1/2} \wedge 0\big)D_+u_{j+1/2} + D_+P_j = 0, \\ &- D_-D_+P_j + P_j = \big(u_{j+1/2} \vee 0\big)^2 + \big(u_{j-1/2} \wedge 0\big)^2 + \frac{1}{2}\big(D_-u_{j+1/2}\big)^2, \end{aligned} \tag{1.7}$$

where

$$u_{j+1/2}(t) \approx u(t, x_{j+1/2}), \quad P_j(t) \approx P(t, x_j), \quad \text{for } t \geq 0 \text{ and } j \in \mathbb{Z}.$$

If we interpret the Camassa-Holm equation (1.4) as a "perturbation" of the inviscid Burgers equation, then the u-part of (1.7) might not come across as a reasonable (upwind) difference scheme. On the other hand, as pointed out in [12], the key point is that with (1.7) the quantity $q_j := D_-u_{j+1/2}$ satisfies a difference scheme which contains proper upwinding of the transport term in the equation for $q := \partial_x u$, which reads $\partial_t q + u\partial_x q + \frac{q^2}{2} + P - u^2 = 0$. Consequently, as is proved in [12], the scheme (1.7) satisfies a total energy inequality, in which only the q-part of the total energy is dissipated (not the u-part, which is after all continuous). This is the essential starting point for the entire convergence analysis in [12].

The "semi-discrete" equation in (1.7) constitutes an infinite system of ordinary differential equations which must be solved by some numerical method. The main purpose of the present paper is to show that a fully discrete version of the scheme used in [12] produces a convergent sequence of approximate solutions, and that the limit is a dissipative weak solution to (1.1). The fully discrete version that is analyzed in this paper is based on replacing the time derivative in (1.7) by a forward difference; i.e.,

$$u'_{j+1/2}(t) \to D^t_+ u^n_{j+1/2} := \frac{u^{n+1}_{j+1/2} - u^n_{j+1/2}}{\Delta t},$$

and evaluating the rest of (1.7) at $t^n := n\Delta t$. Now, $u^n_{j+1/2}$ should approximate the exact solution u at the point $(t^n, x_{j+1/2})$. This gives the fully discrete scheme

$$\begin{aligned} &D^t_+ u^n_{j+1/2} + \big(u^n_{j+1/2} \vee 0\big) D_- u^n_{j+1/2} + \big(u^n_{j+1/2} \wedge 0\big) D_+ u^n_{j+1/2} + D_+ P^n_j = 0,\\ &- D_- D_+ P^n_j + P^n_j = \big(u^n_{j-1/2} \vee 0\big)^2 + \big(u^n_{j+1/2} \wedge 0\big)^2 + \frac{1}{2}\big(D_- u^n_{j+1/2}\big)^2, \end{aligned} \tag{1.8}$$

where P^n_j approximates $P(t^n, x_j)$. As in [12] this is a difference scheme which is tailored so that it gives an upwind scheme for the equation satisfied by $q := \partial_x u$.

The main aim of this paper is prove that the fully discrete (explicit) scheme (1.8) converges to a dissipative weak solution of the Camassa-Holm equation. The starting point of the analysis is a total energy estimate, showing that the H^1 norm of the approximate solutions is (almost) non-increasing in time. To this end, we must assume that

$$\Delta t = \mathcal{O}(\Delta x^2 \log(1 + \Delta x^\theta)), \tag{1.9}$$

for some $\theta > 0$ as $\Delta x \to 0$. This is a very severe condition, and it may seem that when using this method in practice one should use very small time steps. However, this is not a Neumann type stability criterion, and we do not have blow up if it is violated. Indeed, practical experiments indicate stability and convergence if $\Delta t = \mathcal{O}(\Delta x)$.

By appropriately extending the difference solution (1.8) to a function $u_{\Delta x}(t, x)$ defined at all points (t, x) in the domain, we prove under condition (1.9) that $\{u_{\Delta x}\}_{\Delta x > 0}$ converges strongly in H^1 to a dissipative weak solution of the Camassa-Holm equation (1.1)-(1.2). Regarding the proof, we adapt the "renormalization" approach used in [12] for the semi-discrete scheme, but there are several essential deviations and many parts of the convergence proof are substantially more involved and/or different. These differences are mainly due to the fact that the semi-discrete scheme, when viewed as a fully discrete scheme with "infinitely small time steps," has a large and stabilizing numerical viscosity. Regarding the fully discrete (explicit) scheme (1.8), to account for this lack of numerical viscosity the convergence analysis relies heavily on the CFL condition (1.9), and is different with reference to the differences between the semi-discrete and fully discrete schemes. Let us here point out just one aspect, namely that the H^1 norm of the fully discrete approximation is not entirely non-increasing but can grow slightly with a growth factor that, however, tends to zero as $\Delta x \to 0$. Compared to the semi-implicit case [12], the proof is notably more complicated and involves working with a version of the scheme (1.8) in which the quadratic terms have been suitably truncated.

The paper is organized as follows. In Section 2, we introduce some notation to improve the readability and recall a few mathematical results relevant to the convergence analysis. The finite difference scheme and its convergence theorem are stated in Section 3. The convergence theorem is a consequence of the results proved in Sections 4-8. Finally, we present a numerical example in Section 9.

Throughout this paper, we use C to denote a generic constant; the actual value of C may change from one line to the next in a calculation. We also use the notation $a_i \lesssim b_i$ to mean that $a_i \leq C b_i$ for some positive constant C which is independent of i.

2. Preliminaries

In what follows, Δx and Δt denote two small positive numbers. Unless otherwise stated, the indices j and n will run over $\mathbb{Z}$ and $0, \dots N$, respectively, where $N\Delta t = T$ for a fixed final time $T > 0$. For such indices we set $x_j = j\Delta x$, $x_{j+1/2} = (j+1/2)\Delta x$, $t^n = n\Delta t$, and introduce the grid cells

$$I_j = [x_{j-1/2}, x_{j+1/2}), \quad I^n = [t^n, t^{n+1}), \quad \text{and} \quad I_j^n = I_j \times I^n.$$

The following notation will be used frequently:

$$a \vee 0 = \max\{a, 0\} = \frac{a + |a|}{2}, \qquad a \wedge 0 = \min\{a, 0\} = \frac{a - |a|}{2}.$$

For $n \in \{0, \dots, N\}$, let $v^n = \{v_j^n\}_{j\in\mathbb{Z}}$ denote an arbitrary sequence, where n refers to "time" and j to "space." We will frequently employ the following finite difference operators:

$$D_+ v_j^n := \frac{v_{j+1}^n - v_j^n}{\Delta x}, \qquad D_- v_j^n := \frac{v_j^n - v_{j-1}^n}{\Delta x},$$

$$Dv_j^n := \frac{D_+v_j^n + D_-v_j^n}{2} = \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x}, \quad D_+^t v_j^n = \frac{v_j^{n+1} - v_j^n}{\Delta t}.$$

We also use the notation

$$\|v^n\|_{\ell^p} := \Big(\Delta x \sum_{j\in\mathbb{Z}} |v_j^n|^p\Big)^{\frac{1}{p}},\ 1 \le p < \infty, \qquad \|v^n\|_{\ell^\infty} := \sup_j |v_j^n|,$$

$$\|v^n\|_{h^1} := \Big(\Delta x \sum_{j\in\mathbb{Z}} \Big[\big(v_j^n\big)^2 + (D_-v_j^n)^2\Big]\Big)^{\frac{1}{2}}.$$

Occasionally, we also use the "space-time" ℓ^p norms of $v = \{v^n\}_{n=0}^N = \{v_j^n\}_{j,n}$:

$$\|v\|_{\ell^p} := \Big(\Delta t \sum_{n=0}^N \|v^n\|_{\ell^p}^p\Big)^{1/p}.$$

Note that, if $v \in \ell^p$, $p < \infty$, then $\lim_{j\to\pm\infty} v_j = 0$.

Let $\{v_j\}_{j\in\mathbb{Z}}$ and $\{w_j\}_{j\in\mathbb{Z}}$ denote two arbitrary (spatial) sequences. Suppose $\big\|\{v_j\}_j\big\|_{h^1} < \infty$. Then the following discrete Sobolev inequality holds:

$$\|\{v_j\}_j\|_{\ell^\infty} \le \frac{1}{\sqrt{2}} \|\{v_j\}_j\|_{h^1}\,. \tag{2.1}$$

The discrete product rule takes the form

$$D_\pm(v_jw_j) = v_jD_\pm w_j + D_\pm v_j w_{j\pm1}. \tag{2.2}$$

Moreover, the discrete chain rule states

$$D_\pm f(v_j) = f'(v_j)D_\pm v_j \pm \frac{\Delta x}{2} f''(\xi_j^\pm)(D_\pm v_j)^2, \qquad f \in C^2, \tag{2.3}$$

for some number $\xi_j^\pm$ between $v_{j\pm1}$ and v_j.

We continue to collect some handy results for later use, starting with a discrete Gronwall inequality.

Lemma 2.1. *Assume that $c^k \ge 0$ and $f^k \ge 0$ for all $k = 0, \dots, N$, and that the sequence $\{u^n\}_{n=0}^N$ satisfies the difference inequality*

$$D_+^t u^n + f^n \le c^n u^n, \qquad n = 0, \dots, N-1. \tag{2.4}$$

If $u^n \ge 0$ for all $n = 0, \dots, N$, then

$$u^N + \exp\Big(\Delta t \sum_{n=0}^{N-1} c^n\Big)\Delta t \sum_{n=0}^{N-1} \exp\Big(-\Delta t \sum_{k=0}^{n} c^k\Big) f^n \le \exp\Big(\Delta t \sum_{n=0}^{N-1} c^n\Big) u^0.$$

Proof. Set $R^n = \exp\big(-\Delta t \sum_{k=0}^{n-1} c^k\big)$. Then we have

$$D_+^t R^n = \frac{1}{\Delta t}\big(R^{n+1} - R^n\big) = \exp\Big(-\Delta t \sum_{k=0}^{n} c^k\Big) \frac{1}{\Delta t}\Big(1 - \exp\Big(\Delta t c^n\Big)\Big)$$
$$\leq R^{n+1} \frac{1}{\Delta t}\big(1 - (1 + c^n \Delta t)\big) = -c^n R^{n+1}.$$

Hence, multiplying (2.4) by R^{n+1} we arrive at

$$D_+^t\big(R^n u^n\big) = D_+^t u^n R^{n+1} + D_+^t R^n u^n \leq -f^n R^{n+1}.$$

Multiplying this by Δt and summing over n, we see that the lemma holds. □

The next lemma contains estimates for the solution of a discrete version of the differential equation $P - \partial_{xx}^2 P = f$.

Lemma 2.2. *Let $\{f_j\}_{j\in\mathbb{Z}}$ be a sequence in $\ell^1 \cap \ell^2$, and denote by $\{P_j\}_{j\in\mathbb{Z}}$ the solution to the difference equation*

$$P_j - D_- D_+ P_j = f_j, \qquad j \in \mathbb{Z}. \tag{2.5}$$

Introducing the notation

$$h = \Big(1 + 2\frac{1 - e^{-\kappa}}{(\Delta x)^2}\Big)^{-1}, \quad \kappa = \ln\Big(1 + \frac{\Delta x^2}{2} + \frac{\Delta x}{2}\sqrt{4 + \Delta x^2}\Big),$$

the solution $\{P_j\}_{j\in\mathbb{Z}}$ takes the form

$$P_j = h \sum_{i\in\mathbb{Z}} e^{-\kappa|j-i|} f_i, \qquad j \in \mathbb{Z}. \tag{2.6}$$

Moreover, the following estimates hold:

$$\|\{P_j\}_j\|_{\ell^\infty}, \|\{P_j\}_j\|_{\ell^1} \leq C \|\{f_j\}\|_{\ell^1}, \tag{2.7}$$
$$\|\{D_+ P_j\}_j\|_{\ell^\infty}, \|\{D_+ P_j\}_j\|_{\ell^1} \leq C \|\{f_j\}_j\|_{\ell^1}, \tag{2.8}$$
$$\|\{P_j\}_j\|_{h^1} \leq C \|\{f_j\}_j\|_{\ell^2}, \tag{2.9}$$

where $C > 0$ is a constant independent of Δx.

Proof. To verify the solution formula (2.6), we define p_i by $p_i = ce^{-\sigma|i|}$, for some constants c and σ yet to be found. We shall choose these so that

$$(I - D_+ D_-)p_i = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

If we find that this holds with $\sigma = \kappa$ and $c = h$ then (2.6) holds. We observe that for $i \neq 0$

$$D_+D_-p_i = ce^{-\sigma|i|}2\frac{\cosh(\sigma)-1}{\Delta x^2}.$$

Hence, σ must satisfy

$$\sigma = \cosh^{-1}\left(1+\frac{\Delta x^2}{2}\right) = \kappa.$$

For $i = 0$ we find that

$$p_0 - D_+D_-p_0 = c\Big(1 - \frac{2}{\Delta x^2}\big(e^{-\kappa}-1\big)\Big).$$

If this is to be equal to 1 then $c = h$.

For later use, one should observe that

$$h = \frac{\Delta x}{2} + \mathcal{O}(\Delta x^2), \quad \frac{|e^{\kappa}-1|}{\Delta x} = 1 + \mathcal{O}(\Delta x), \quad \frac{|e^{-\kappa}-1|}{\Delta x} = 1 + \mathcal{O}(\Delta x). \quad (2.10)$$

For any $j \in \mathbb{Z}$, we have $|P_j| \lesssim \|\{f_j\}\|_{\ell^1}$. Furthermore,

$$\|\{P_j\}\|_{\ell^1} \leq 2h \sum_i \Big[\Delta x \sum_j e^{-\kappa|j-i|}\Big] |f_i| \lesssim \|\{f_i\}\|_{\ell^1}.$$

Hence, we have proved (2.7).

From (2.6),

$$\begin{aligned} D_+P_j = \frac{P_{j+1}-P_j}{\Delta x} &= h\sum_i \frac{e^{-\kappa|i-j-1|} - e^{-\kappa|i-j|}}{\Delta x} f_i \\ &= h\sum_{i=j}^{\infty} \frac{e^{-\kappa(i-j-1)} - e^{-\kappa(i-j)}}{\Delta x} f_i + h\sum_{i=-\infty}^{j-1} \frac{e^{\kappa(i-j-1)} - e^{\kappa(i-j)}}{\Delta x} f_i \\ &= h\sum_{i=j}^{\infty} e^{-\kappa(i-j)}\frac{e^{\kappa}-1}{\Delta x} f_i + h\sum_{i=-\infty}^{j-1} e^{\kappa(i-j)}\frac{e^{-\kappa}-1}{\Delta x} f_i. \end{aligned}$$

Using (2.10) we acquire from this the following two estimates:

$$|D_+P_j| \lesssim h\sum_i e^{-\kappa|i-j|}|f_i| \lesssim \|\{f_i\}\|_{\ell^1},$$

$$\|\{D_+P_j\}_j\|_{\ell^1} \lesssim h\Delta x \sum_{j,i} e^{-\kappa|i-j|}|f_i| \lesssim \|\{f_i\}\|_{\ell^1}.$$

Therefore, (2.8) holds.

It remains to prove (2.9). To this end, we multiply the equation (2.5) by $\Delta x P_j$ and perform a summation by parts to discover

$$\|\{P_j\}_j\|_{h^1}^2 = \Delta x \sum_j P_j f_j \leq \frac{1}{2} \|\{P_j\}_j\|_{\ell^2}^2 + \frac{1}{2} \|\{f_j\}_j\|_{\ell^2}^2 ,$$

from which (2.9) follows. □

We shall routinely use some well-known results related to weak convergence, which we collect in a lemma (for proofs, see, e.g., [24]). Throughout the paper we use overbars to denote weak limits.

Lemma 2.3. *Let O be a bounded open subset of $\mathbb{R}^M$, with $M \geq 1$. Let $\{v_n\}_{n\geq 1}$ be a sequence of measurable functions on O for which*

$$\sup_{n\geq 1} \int_O \Phi(|v_n(y)|)\, dy < \infty,$$

for some given continuous function $\Phi : [0,\infty) \to [0,\infty)$. Then along a subsequence as $n \to \infty$, $g(v_n) \rightharpoonup \overline{g(v)}$ in $L^1(O)$ for all continuous functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|v|\to\infty} \frac{|g(v)|}{\Phi(|v|)} = 0.$$

Let $g\colon \mathbb{R} \to (-\infty,\infty]$ be a lower semicontinuous convex function and $\{v_n\}_{n\geq 1}$ a sequence of measurable functions on O, for which

$$v_n \rightharpoonup v \text{ in } L^1(O),\ g(v_n) \in L^1(O) \text{ for each } n,\ g(v_n) \rightharpoonup \overline{g(v)} \text{ in } L^1(O).$$

Then $g(v) \leq \overline{g(v)}$ almost everywhere on O. Moreover, $g(v) \in L^1(O)$ and

$$\int_O g(v)\, dy \leq \liminf_{n\to\infty} \int_O g(v_n)\, dy.$$

If, in addition, g is strictly convex on an open interval $(a,b) \subset \mathbb{R}$ and $g(v) = \overline{g(v)}$ almost everywhere on O, then, passing to a subsequence if necessary,

$$v_n(y) \to v(y) \text{ for a.e. } y \in \{y \in O : v(y) \in (a,b)\}.$$

Let X be a Banach space and denote by $X^\star$ its dual. The space $X^\star$ equipped with the weak-$\star$ topology is denoted by $X^\star_{\text{weak}}$, while X equipped with the weak topology is denoted by X_{weak}. By the Banach-Alaoglu theorem, a bounded ball in $X^\star$ is $\sigma(X^\star, X)$-compact. If X is separable, then the weak-$\star$ topology is metrizable on bounded sets in $X^\star$, and thus one can consider the metric space $C\big([0,T]; X^\star_{\text{weak}}\big)$ of functions $v : [0,T] \to X^\star$ that are continuous with respect to the weak topology. We have $v_n \to v$ in

$C\big([0,T];X^\star_{\text{weak}}\big)$ if $\langle v_n(t),\phi\rangle_{X^\star,X} \to \langle v(t),\phi\rangle_{X^\star,X}$ uniformly with respect to t, for any $\phi \in X$. The following lemma is a consequence of the Arzelà-Ascoli theorem:

Lemma 2.4. *Let X be a separable Banach space, and suppose $v_n\colon [0,T] \to X^\star$, $n = 1,2,\dots,$ is a sequence of measurable functions such that*

$$\|v_n\|_{L^\infty([0,T];X^\star)} \le C,$$

for some constant C independent of n. Suppose the sequence

$$[0,T] \ni t \mapsto \langle v_n(t),\Phi\rangle_{X^\star,X}, \quad n = 1,2,\dots,$$

is equi-continuous for every Φ that belongs to a dense subset of X. Then v_n belongs to $C\big([0,T];X^\star_{\text{weak}}\big)$ for every $n = 1,2,\dots,$ and there exists a function $v \in C\big([0,T];X^\star_{\text{weak}}\big)$ such that along a subsequence as $n \to \infty$

$$v_n \to v \text{ in } C\big([0,T];X^\star_{\text{weak}}\big).$$

3. Explicit scheme and main result

In this section, we present the fully discrete (explicit) difference scheme for the Cammassa-Holm equation (1.3), which generates sequences $\{u^n_{j+1/2}\}$ and $\{P^n_j\}$ for $(n,j) \in \{0,\dots,N\} \times \mathbb{Z}$. We let $\{u^n_{j+1/2}\}$ solve the explicit difference equation

$$D^t_+ u^n_{j+1/2} + \big(u^n_{j+1/2} \vee 0\big) D_- u^n_{j+1/2} + \big(u^n_{j+1/2} \wedge 0\big) D_+ u^n_{j+1/2} + D_+ P^n_j = 0, \tag{3.1}$$

where the initial values are specified as follows:

$$u^0_{j+1/2} = u_0(x_{j+1/2}). \tag{3.2}$$

Given $\{u^n_{j+1/2}\}$, we determine $\{P^n_j\}$ by solving

$$-D_- D_+ P^n_j + P^n_j = \big(u^n_{j+1/2} \vee 0\big)^2 + \big(u^n_{j-1/2} \wedge 0\big)^2 + \frac{1}{2}\big(D_- u^n_{j+1/2}\big)^2, \tag{3.3}$$

which is a linear system of equations that can be solved as outlined in Lemma 2.2.

Next, let us derive the difference scheme satisfied by

$$q^n_j = D_- u^n_{j+1/2}. \tag{3.4}$$

This will be done by applying the difference operator D_- to the u-equation (3.1). To this end, we apply the discrete product rule to find

$$D_- \Big[(u^n_{j+1/2} \vee 0) D_- u^n_{j+1/2}\Big] = (u^n_{j-1/2} \vee 0) D_- q_j + D_-(u^n_{j+1/2} \vee 0) q^n_j,$$

and

$$D_- \left[(u^n_{j+1/2} \wedge 0) D_+ u^n_{j+1/2} \right] = (u^n_{j+1/2} \wedge 0) D_+ q_j + D_- (u^n_{j+1/2} \wedge 0) q_j,$$

so that

$$\begin{aligned} &D_- \left[(u^n_{j+1/2} \vee 0) D_- u^n_{j+1/2} + (u^n_{j+1/2} \wedge 0) D_+ u^n_{j+1/2} \right] \\ &\qquad = (u^n_{j-1/2} \vee 0) D_- q^n_j + (u^n_{j+1/2} \wedge 0) D_+ q^n_j + (q^n_j)^2. \end{aligned} \tag{3.5}$$

The P-equation (3.3) rephrased in terms of q reads

$$-D_- D_+ P^n_j + P^n_j = \left(u^n_{j+1/2} \vee 0\right)^2 + \left(u^n_{j-1/2} \wedge 0\right)^2 + \frac{1}{2}\left(q^n_j\right)^2. \tag{3.6}$$

Employing (3.5) and (3.6) when applying D_- to the u-equation in (3.1) yields

$$\begin{aligned} &D^t_+ q^n_j + \left(u^n_{j-1/2} \vee 0\right) D_- q^n_j + \left(u^n_{j+1/2} \wedge 0\right) D_+ q^n_j \\ &\qquad + \frac{\left(q^n_j\right)^2}{2} + P^n_j - \left(u^n_{j+1/2} \vee 0\right)^2 - \left(u^n_{j-1/2} \wedge 0\right)^2 = 0. \end{aligned} \tag{3.7}$$

Regarding the initial values, in view of (3.4) and (3.2), we observe that

$$q^0_j = \frac{1}{\Delta x} \int_{I_j} \partial_x u_0(x)\, dx, \qquad j \in \mathbb{Z}. \tag{3.8}$$

Since the variable $q = \partial_x u$ can be discontinuous, (3.7) represents a natural upwind discretization of the equation for q, $\partial_t q + u \partial_x q + \frac{q^2}{2} - u^2 + P = 0$.

The main result of this paper is the convergence of the scheme to a dissipative weak solution of (1.1)-(1.2), which is defined in the following sense [39, 40]:

Definition 3.1. *Fix a final time $T > 0$. We call a function $u\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ a weak solution of the Cauchy problem for* (1.1)-(1.2) *on $[0,T] \times \mathbb{R}$ if*

(**D.1**) $u \in C([0,T] \times \mathbb{R}) \cap L^\infty(0,T; H^1(\mathbb{R}))$.

(**D.2**) *For all s and t in $[0,T]$, with $s \le t$, we have* $\|u(t,\cdot)\|_{H^1(\mathbb{R})} \le \|u(s,\cdot)\|_{H^1(\mathbb{R})}$.

(**D.3**) *u satisfies* (1.3) *in the sense of distributions on $(0,T) \times \mathbb{R}$.*

(**D.4**) *$u(0,x) = u_0(x)$ for every $x \in \mathbb{R}$.*

(**D.5**) *If, in addition, there exists a positive constant K such that*

$$u_x(t,x) \le \frac{2}{t} + K \|u_0\|^2_{H^1(\mathbb{R})}, \qquad (t,x) \in (0,T] \times \mathbb{R},$$

then we call u a dissipative weak solution of the Cauchy problem (1.1)-(1.2).

In addition to $\partial_x u \in L^\infty(0,T;L^2(\mathbb{R}))$, cf. (**D.2**), the dissipative weak solutions u that we construct in this paper will possess an improved integrability property, namely $\partial_x u \in L^p_{\mathrm{loc}}((0,T)\times\mathbb{R})$ for $p<3$; i.e.,

$$\int_0^T \int_a^b |\partial_x u|^p \, dx\, dt \le C(a,b,T,p), \quad \forall a,b\in\mathbb{R},\ a<b.$$

To state our main convergence result and also for later use, we need to introduce some functions (interpolations of the difference approximations) that are defined at all points (t,x) in the domain. We begin by defining the functions

$$\begin{cases} q_j(t) = q_j^n + (t-t^n)D_+^t q_j^n, \\ u_{j+1/2}(t) = u_{j+1/2}^n + (t-t^n)D_+^t u_{j+1/2}^n \end{cases} \quad \text{for } t\in I^n.$$

With the aid of these we define

$$q_{\Delta x}(t,x) = q_j(t), \qquad (t,x)\in I_j^n, \tag{3.9}$$

and

$$u_{\Delta x}(t,x) = u_{j-1/2}(t) + \big(x - x_{j-1/2}\big) q_j(t), \qquad \text{for } (t,x)\in I_j^n, \tag{3.10}$$

for $j\in\mathbb{Z}$, $n=0,\dots,N-1$. Note that $t\mapsto u_{\Delta x}(t,x)$ is a continuous function, since $u_{j-1/2}(t)$ and $q_j(t)$ are continuous. Regarding the continuity in x we have that

$$\begin{aligned} \lim_{x\uparrow x_{j+1/2}} u_{\Delta x}(t,x) &= u_{j-1/2}(t) + \big(x_{x+1/2} - x_{j-1/2}\big) q_j(t) \\ &= u_{j-1/2}^n + \big(t-t^n\big) D_+^t u_{j-1/2}^n + \Delta x\big(q_j^n + (t-t^n) D_+^t q_j^n\big) \\ &= u_{j-1/2}^n + \big(t-t^n\big) D_+^t u_{j-1/2}^n + \big(u_{j+1/2}^n - u_{j-1/2}^n\big) \\ &\qquad + \big(t-t^n\big) D_+^t \big(u_{j+1/2}^n - u_{j-1/2}^n\big) = u_{j+1/2}(t), \end{aligned}$$

and therefore $u_{\Delta x}$ is continuous, and furthermore $\partial_x u_{\Delta x} = q_{\Delta x}$ almost everywhere. Observe also that, due to (3.8), there holds $q_{\Delta x}(0,x)\to\partial_x u_0$ in $L^2(\mathbb{R})$ as $\Delta x\to 0$. Similarly to $u_{\Delta x}$, we define a function $P_{\Delta x}$ by bilinear interpolation. First, let

$$P_j(t) = P_j^n + (t-t^n) D_+^t P_j^n, \qquad t\in I^n,$$

and then define

$$P_{\Delta x}(t,x) = P_j(t) + (x-x_j) D_+ P_j(t), \qquad (t,x)\in I_j^n, \tag{3.11}$$

for $j\in\mathbb{Z}$, $n=0,\dots,N-1$.

We are now in a position to state our main result.

Theorem 3.1. *Suppose* (1.2) *holds. Let* $\{u_{\Delta x}\}_{\Delta x>0}$ *be a sequence defined by* (3.10) *and* (3.1)-(3.4). *Then, along a subsequence as* $\Delta x \downarrow 0$,

$$u_{\Delta x} \to u \text{ in } H^1_{\mathrm{loc}}((0,T)\times\mathbb{R}),$$

where u *is a dissipative weak solution of the Cauchy problem* (1.1)-(1.2).

This theorem is a consequence of the results stated and proved in Sections 4-8.

4. Total energy estimate and some consequences

The purpose of this section is to establish a discrete total energy estimate for the difference scheme (3.1)-(3.4).

Lemma 4.1. *Assume that* Δx *and* Δt *are related through the CFL type condition*

$$\Delta t < \frac{\log(1+\Delta x^{\theta})\,\Delta x^2}{C\,\|u_0\|^2_{H^1(\mathbb{R})}\,(1+\Delta x^2)}, \tag{4.1}$$

where C *is a constant (to be detailed in the proof of the lemma) that is independent of* Δx *and* u_0 *and* $\theta > 0$. *Then, for any* $N_0 \in \{0,\dots,N\}$, *and for all sufficiently small* Δx,

$$\left\|u^{N_0}\right\|^2_{h^1} + \Delta x^2\Delta t\sum_{n=0}^{N_0-1}\sum_{j\in\mathbb{Z}}|u^n_{j+1/2}|\big(D_-D_+u^n_{j+1/2}\big)^2 \le e^{t^N\Delta x^{\theta}}\left\|u^0\right\|^2_{h^1}. \tag{4.2}$$

Proof. For the proof of (4.2), we shall need to introduce an auxiliary difference scheme. To this end, we start by defining the cut-off function

$$f^M(u) = \begin{cases} -M, & u < -M,\\ u, & u \in [-M,M],\\ +M & u > M,\end{cases}$$

where $M > 0$ is a fixed constant (to be determined later on). Now, let $\{\tilde u^n_{j+1/2}\}$ and $\{\tilde P^n_j\}$ solve the following system of difference equations:

$$\begin{aligned} D^t_+\tilde u^n_{j+1/2} &+ \big(f^M(\tilde u^n_{j+1/2})\vee 0\big)D_-\tilde u^n_{j+1/2}\\ &+ \big(f^M(\tilde u^n_{j+1/2})\wedge 0\big)D_+\tilde u^n_{j+1/2} + D_+\tilde P^n_j = 0,\end{aligned} \tag{4.3}$$

for $n = 0,\dots,N-1$ and $j\in\mathbb{Z}$, and

$$\begin{aligned} -D_-D_+\tilde P^n_j + \tilde P^n_j &= \big(f^M(\tilde u^n_{j+1/2})\vee 0\big)(\tilde u^n_{j+1/2}\vee 0)\\ &+ \big(f^M(\tilde u^n_{j-1/2})\wedge 0\big)(\tilde u^n_{j-1/2}\wedge 0) + \frac{1}{2}D_-\big(f^M(\tilde u^n_{j+1/2})\big)D_-\tilde u^n_{j+1/2},\end{aligned}$$

for $n = 0, \dots, N$ and $j \in \mathbb{Z}$. Regarding the initial data, we set $\tilde{u}_j^0 = u_j^0$ for $j \in \mathbb{Z}$.

If we define $\tilde{q}_j^n := D_- \tilde{u}_{j+1/2}^n$, then it is straightforward to see that $\{\tilde{q}_j^n\}$ satisfies the difference equation

$$\begin{aligned} D_+^t \tilde{q}_j^n &+ \big(f^M(\tilde{u}_{j-1/2}^n) \vee 0\big) D_- \tilde{q}_j^n + \big(f^M(\tilde{u}_{j+1/2}^n) \wedge 0\big) D_+ \tilde{q}_j^n \\ &+ \frac{1}{2} D_- \big(f^M(\tilde{u}_{j+1/2}^n)\big) \tilde{q}_j^n - \big(f^M(\tilde{u}_{j+1/2}^n) \vee 0\big)(\tilde{u}_{j+1/2}^n \vee 0) \\ &- \big(f^M(\tilde{u}_{j-1/2}^n) \wedge 0\big)(\tilde{u}_{j-1/2}^n \wedge 0) + \tilde{P}_j^n = 0. \end{aligned} \tag{4.4}$$

Multiplying (4.3) by $\tilde{u}_{j+1/2}^n$ we find that

$$\begin{aligned} &\tilde{u}_{j+1/2}^n D_+^t \tilde{u}_{j+1/2}^n + \big(f^M(\tilde{u}_{j+1/2}^n) \vee 0\big)(\tilde{u}_{j+1/2}^n \vee 0)\tilde{q}_j^n \\ &\quad + \big(f^M(\tilde{u}_{j+1/2}^n) \wedge 0\big)(\tilde{u}_{j+1/2}^n \wedge 0)\tilde{q}_{j+1}^n + D_+ \tilde{P}_j^n \tilde{u}_{j+1/2}^n = 0, \end{aligned} \tag{4.5}$$

while multiplying (4.4) by $\tilde{q}_j^n$ gives us

$$\begin{aligned} \tilde{q}_j^n D_+^t \tilde{q}_j^n &+ \big(f^M(\tilde{u}_{j-1/2}^n) \vee 0\big)(D_- \tilde{q}_j^n)\tilde{q}_j^n + \big(f^M(\tilde{u}_{j+1/2}^n) \wedge 0\big)(D_+ \tilde{q}_j^n)\tilde{q}_j^n \\ &+ \frac{1}{2} D_-(f^M(\tilde{u}_{j+1/2}^n))(\tilde{q}_j^n)^2 - \big(f^M(\tilde{u}_{j+1/2}^n) \vee 0\big)(\tilde{u}_{j+1/2}^n \vee 0)\tilde{q}_j^n \\ &- \big(f^M(\tilde{u}_{j-1/2}^n) \wedge 0\big)(\tilde{u}_{j-1/2}^n \wedge 0)\tilde{q}_j^n + \tilde{P}_j^n \tilde{q}_j^n = 0. \end{aligned} \tag{4.6}$$

Adding (4.5) and (4.6), multiplying the result by Δx, and summing over j yields

$$\Delta x \sum_{j \in \mathbb{Z}} \Big(\tilde{u}_{j+1/2}^n D_+^t \tilde{u}_{j+1/2}^n + \tilde{q}_j^n D_+^t \tilde{q}_j^n \Big) + \mathrm{I} + \mathrm{II} + \mathrm{III} = 0,$$

where

$$\begin{aligned} \mathrm{I} &= \Delta x \sum_{j \in \mathbb{Z}} \Big(f^M(\tilde{u}_{j-1/2}^n) \vee 0 \Big)(D_- \tilde{q}_j^n)\tilde{q}_j^n \\ &\quad + \Delta x \sum_{j \in \mathbb{Z}} \Big(f^M(\tilde{u}_{j+1/2}^n) \wedge 0 \Big)(D_+ \tilde{q}_j^n)\tilde{q}_j^n + \frac{\Delta x}{2} \sum_{j \in \mathbb{Z}} D_- \Big(f^M(u_{j+1/2}) \Big)(\tilde{q}_j^n)^2, \\ \mathrm{II} &= \Delta x \sum_{j \in \mathbb{Z}} \Big(f^M(\tilde{u}_{j+1/2}^n) \vee 0 \Big)(\tilde{u}_{j+1/2}^n \vee 0)\tilde{q}_j^n \\ &\qquad + \Delta x \sum_{j \in \mathbb{Z}} \Big(f^M(\tilde{u}_{j+1/2}^n) \wedge 0 \Big)(\tilde{u}_{j+1/2}^n \wedge 0)\tilde{q}_{j+1}^n \\ &\qquad - \Delta x \sum_{j \in \mathbb{Z}} \Big(f^M(\tilde{u}_{j+1/2}^n) \vee 0 \Big)(\tilde{u}_{j+1/2}^n \vee 0)\tilde{q}_j^n \end{aligned}$$

$$- \Delta x \sum_{j\in\mathbb{Z}} \Big(f^M(\tilde{u}^n_{j-1/2}) \wedge 0\Big)(\tilde{u}^n_{j-1/2} \wedge 0)\tilde{q}^n_j \equiv 0$$

(by shifting indices),

$$\mathrm{III} = \Delta x \sum_{j\in\mathbb{Z}} D_+ \tilde{P}^n_j \tilde{u}^n_{j+1/2} + \Delta x \sum_{j\in\mathbb{Z}} \tilde{P}^n_j \tilde{q}^n_j \equiv 0$$

(by summation by parts, cf. (3.4)).

Let us now deal with term I. The discrete chain rule (2.3) tells us that

$$(D_\pm \tilde{q}^n_j)\tilde{q}^n_j = D_\pm\Big(\frac{(\tilde{q}^n_j)^2}{2}\Big) \mp \frac{\Delta x}{2}(D_\pm \tilde{q}^n_j)^2.$$

Hence,

$$\begin{aligned} \mathrm{I} = \Delta x \sum_{j\in\mathbb{Z}} \Big(f^M(\tilde{u}^n_{j-1/2}) \vee 0\Big)\Big[D_-\Big(\frac{(\tilde{q}^n_j)^2}{2}\Big) + \frac{\Delta x}{2}(D_- \tilde{q}^n_j)^2\Big] \\ + \Delta x \sum_{j\in\mathbb{Z}} \Big(f^M(\tilde{u}^n_{j+1/2}) \wedge 0\Big)\Big[D_+\Big(\frac{(\tilde{q}^n_j)^2}{2}\Big) - \frac{\Delta x}{2}(D_+ \tilde{q}^n_j)^2\Big] \\ + \frac{\Delta x}{2} \sum_{j\in\mathbb{Z}} D_-\Big(f^M(\tilde{u}^n_{j+1/2})\Big)(\tilde{q}^n_j)^2 = \mathrm{I}_1 + \mathrm{I}_2, \end{aligned}$$

where

$$\begin{aligned} \mathrm{I}_1 = \Delta x \sum_{j\in\mathbb{Z}} \Big(f^M(\tilde{u}^n_{j-1/2}) \vee 0\Big) D_-\Big(\frac{(\tilde{q}^n_j)^2}{2}\Big) \\ + \Delta x \sum_{j\in\mathbb{Z}} \Big(f^M(\tilde{u}^n_{j+1/2}) \wedge 0\Big) D_+\Big(\frac{(\tilde{q}^n_j)^2}{2}\Big) \\ + \frac{\Delta x}{2} \sum_{j\in\mathbb{Z}} D_-\Big(f^M\Big(\tilde{u}^n_{j+1/2}\Big)\Big)(\tilde{q}^n_j)^2, \end{aligned}$$

$$\begin{aligned} \mathrm{I}_2 &= \frac{\Delta x^2}{2} \sum_{j\in\mathbb{Z}} \Big[\Big(f^M\Big(\tilde{u}^n_{j-1/2}\Big) \vee 0\Big)(D_- \tilde{q}^n_j)^2 - \Big(f^M\Big(\tilde{u}^n_{j+1/2}\Big) \wedge 0\Big)(D_+ \tilde{q}^n_j)^2\Big] \\ &= \frac{\Delta x^2}{2} \sum_{j\in\mathbb{Z}} \Big[\Big(f^M\Big(\tilde{u}^n_{j+1/2}\Big) \vee 0\Big)(D_+ \tilde{q}^n_j)^2 - \Big(f^M\Big(\tilde{u}^n_{j+1/2}\Big) \wedge 0\Big)(D_+ \tilde{q}^n_j)^2\Big] \\ &= \frac{\Delta x^2}{2} \sum_{j\in\mathbb{Z}} |f^M\Big(\tilde{u}^n_{j+1/2}\Big)| \Big(D_+ \tilde{q}^n_j\Big)^2 \geq 0. \end{aligned}$$

To handle the I_1-term, we use the discrete product rule (2.2):

$$
\begin{aligned}
&D_-\Big[\Big(f^M(\tilde{u}^n_{j+1/2})\vee 0\Big)\frac{(\tilde{q}^n_j)^2}{2}\Big]\\
&\quad=\Big(f^M(\tilde{u}^n_{j-1/2})\vee 0\Big)D_-\Big(\frac{(\tilde{q}^n_j)^2}{2}\Big)+D_-\Big(f^M(\tilde{u}^n_{j+1/2})\vee 0\Big)\frac{(\tilde{q}^n_j)^2}{2},\\
&D_+\Big[\Big(f^M(\tilde{u}^n_{j-1/2})\wedge 0\Big)\frac{(\tilde{q}^n_j)^2}{2}\Big]\\
&\quad=\Big(f^M(\tilde{u}^n_{j+1/2})\wedge 0\Big)D_+\Big(\frac{(\tilde{q}^n_j)^2}{2}\Big)+D_+\Big(f^M(\tilde{u}^n_{j-1/2})\wedge 0\Big)\frac{(\tilde{q}^n_j)^2}{2}\\
&\quad=\Big(f^M(\tilde{u}^n_{j+1/2})\wedge 0\Big)D_+\Big(\frac{(\tilde{q}^n_j)^2}{2}\Big)+D_-\Big(f^M(\tilde{u}^n_{j+1/2})\wedge 0\Big)\frac{(\tilde{q}^n_j)^2}{2}.
\end{aligned}
$$

Using this we find that

$$
\begin{aligned}
\mathrm{I}_1 &= \Delta x\sum_{j\in\mathbb{Z}}D_-\Big[\Big(f^M(\tilde{u}^n_{j+1/2})\vee 0\Big)\frac{(\tilde{q}^n_j)^2}{2}\Big]-\Delta x\sum_{j\in\mathbb{Z}}D_-\Big(f^M(\tilde{u}^n_{j+1/2})\vee 0\Big)\frac{(\tilde{q}^n_j)^2}{2}\\
&\quad+\Delta x\sum_{j\in\mathbb{Z}}D_+\Big[\Big(f^M(\tilde{u}^n_{j-1/2})\wedge 0\Big)\frac{(\tilde{q}^n_j)^2}{2}\Big]\\
&\quad-\Delta x\sum_{j\in\mathbb{Z}}D_-\Big(f^M(\tilde{u}^n_{j+1/2})\wedge 0\Big)\frac{(\tilde{q}^n_j)^2}{2}+\frac{\Delta x}{2}\sum_{j\in\mathbb{Z}}D_-\Big(f^M(\tilde{u}^n_{j+1/2})\Big)(\tilde{q}^n_j)^2\\
&=-\Delta x\sum_{j\in\mathbb{Z}}D_-\Big(f^M(\tilde{u}^n_{j+1/2})\Big)\frac{(\tilde{q}^n_j)^2}{2}+\frac{\Delta x}{2}\sum_{j\in\mathbb{Z}}D_-\Big(f^M(\tilde{u}^n_{j+1/2})\Big)(\tilde{q}^n_j)^2=0.
\end{aligned}
$$

Summarizing our findings so far:

$$
\Delta x\sum_{j\in\mathbb{Z}}\Big(\tilde{u}^n_{j+1/2}D^t_+\tilde{u}^n_{j+1/2}+\tilde{q}^n_jD^t_+\tilde{q}^n_j\Big)+\frac{\Delta x^2}{2}\sum_{j\in\mathbb{Z}}\Big|f^M\Big(\tilde{u}^n_{j+1/2}\Big)\Big|\Big(D_+\tilde{q}^n_j\Big)^2=0. \tag{4.7}
$$

Next, by (2.3),

$$
\begin{aligned}
\Delta x\sum_{j\in\mathbb{Z}}\Big(\tilde{u}^n_{j+1/2}D^t_+\tilde{u}^n_{j+1/2}+q_jD^t_+\tilde{q}^n_j\Big)&=D^t_+\Big[\frac{\Delta x}{2}\sum_{j\in\mathbb{Z}}\Big((\tilde{u}^n_{j+1/2})^2+(\tilde{q}^n_j)^2\Big)\Big]\\
&\quad-\frac{1}{2}\Delta t\Delta x\sum_{j\in\mathbb{Z}}\Big((D^t_+\tilde{u}^n_{j+1/2})^2+(D^t_+\tilde{q}^n_j)^2\Big).
\end{aligned} \tag{4.8}
$$

Hence, we must now estimate

$$\Delta t\Delta x\sum_{j\in\mathbb{Z}}\left(\left(D_+^t\tilde{u}_{j+1/2}^n\right)^2+\left(D_+^t\tilde{q}_j^n\right)^2\right).$$

Using (4.3), (4.4), and the basic inequality

$$\left(\sum_{\ell=1}^{l}a_l\right)^2\le 2^{l-1}\sum_{\ell=1}^{l}(a_\ell)^2,$$

which holds for any sequence $\{a_\ell\}_{\ell=1}^l$ of positive real numbers, there is a positive constant c_1 that does not depend on Δx such that

$$\begin{aligned}
&\Delta t\Delta x\sum_{j\in\mathbb{Z}}\left(\left(D_+^t\tilde{u}_{j+1/2}^n\right)^2+\left(D_+^t\tilde{q}_j^n\right)^2\right)\\
&\le c_1\Delta t\Delta x\sum_{j\in\mathbb{Z}}\Big[\left(f^M\left(\tilde{u}_{j+1/2}^n\right)\vee 0\right)^2\left(\tilde{q}_j^n\right)^2+\left(f^M\left(\tilde{u}_{j+1/2}^n\right)\wedge 0\right)^2\left(\tilde{q}_{j+1}^n\right)^2\\
&+\left(D_+\tilde{P}_j^n\right)^2+\left(f^M\left(\tilde{u}_{j-1/2}^n\right)\vee 0\right)^2\left(D_-\tilde{q}_j^n\right)^2+\left(f^M\left(\tilde{u}_{j+1/2}^n\right)\wedge 0\right)^2\left(D_+\tilde{q}_j^n\right)^2\\
&+\left(D_-\left(f^M\left(\tilde{u}_{j+1/2}^n\right)\right)\right)^2\left(\tilde{q}_j^n\right)^2+\left(f^M\left(\tilde{u}_{j+1/2}^n\right)\vee 0\right)^2\left(\tilde{u}_{j+1/2}^n\vee 0\right)^2\\
&+\left(f^M\left(\tilde{u}_{j-1/2}^n\right)\wedge 0\right)^2\left(\tilde{u}_{j-1/2}^n\wedge 0\right)^2+\left(\tilde{P}_j^n\right)^2\Big]\\
&\qquad\le c_1\Delta t\Big(J_1+J_2+J_3\Big),
\end{aligned}$$

where

$$\begin{aligned}
J_1&=\Delta x\sum_{j\in\mathbb{Z}}\Big[\left(f^M(\tilde{u}_{j+1/2}^n)\vee 0\right)^2(\tilde{q}_j^n)^2+\left(f^M(\tilde{u}_{j+1/2}^n)\wedge 0\right)^2(\tilde{q}_{j+1}^n)^2\\
&\quad+\left(f^M(\tilde{u}_{j+1/2}^n)\vee 0\right)^2\left(\tilde{u}_{j+1/2}^n\vee 0\right)^2+\left(f^M(\tilde{u}_{j-1/2}^n)\wedge 0\right)^2\left(\tilde{u}_{j-1/2}^n\wedge 0\right)^2\Big],\\
J_2&=\Delta x\sum_{j\in\mathbb{Z}}\Big[\left(f^M(\tilde{u}_{j-1/2}^n)\vee 0\right)^2(D_-\tilde{q}_j^n)^2\\
&\qquad\quad+\left(f^M(u_{j+1/2})\wedge 0\right)^2(D_+\tilde{q}_j^n)^2+\left(D_-\left(f^M(\tilde{u}_{j+1/2}^n)\right)\right)^2(\tilde{q}_j^n)^2\Big],\\
J_3&=\Delta x\sum_{j\in\mathbb{Z}}\Big[(D_+\tilde{P}_j^n)^2+(\tilde{P}_j^n)^2\Big].
\end{aligned}$$

Since $|f^M(u)| \leq M$, the following bounds hold:

$$J_1 \leq 2M^2 \Delta x \sum_{j\in\mathbb{Z}} \Big[(\tilde{u}^n_{j+1/2})^2 + (\tilde{q}^n_j)^2 \Big] = 2M^2 \left\| \tilde{u}^n \right\|_{h^1}, \tag{4.9}$$

and

$$\begin{aligned} J_2 &= \frac{\Delta x}{\Delta x^2} \sum_{j\in\mathbb{Z}} \Big[\Big(f^M(\tilde{u}^n_{j-1/2}) \vee 0 \Big)^2 (D_- \tilde{q}^n_j \Delta x)^2 \\ &\quad + \Big(f^M(\tilde{u}^n_{j+1/2}) \wedge 0 \Big)^2 (D_+ \tilde{q}^n_j \Delta x)^2 + \Big(D_- \Big(f^M(\tilde{u}^n_{j+1/2}) \Big) \Delta x \Big)^2 (\tilde{q}^n_j)^2 \Big] \\ &\leq c_2 M^2 \frac{\Delta x}{\Delta x^2} \sum_{j\in\mathbb{Z}} (q_j)^2 \leq c_2 \frac{M^2}{\Delta x^2} \left\| \tilde{u}^n \right\|_{h^1}, \end{aligned} \tag{4.10}$$

for some constant $c_2 > 0$ independent of Δx.

To estimate J_3 we use Lemma 2.2, specifically (2.9), which implies that

$$\begin{aligned} J_3 &\leq c_3 \Delta x \sum_{j\in\mathbb{Z}} \Big[\Big(f^M(\tilde{u}^n_{j+1/2}) \vee 0 \Big)^2 \Big(\tilde{u}^n_{j+1/2} \vee 0 \Big)^2 \\ &\quad + \Big(f^M(\tilde{u}^n_{j-1/2}) \wedge 0 \Big)^2 \Big(\tilde{u}^n_{j-1/2} \wedge 0 \Big)^2 + \Big(D_- \Big(f^M(\tilde{u}^n_{j+1/2}) \Big) \Big)^2 (\tilde{q}^n_j)^2 \Big] \\ &\leq 2c_3 M^2 \Delta x \sum_{i\in\mathbb{Z}} (\tilde{u}^n_{j+1/2})^2 + c_3 M^2 \frac{\Delta x}{\Delta x^2} \sum_{i\in\mathbb{Z}} (\tilde{q}^n_j)^2 \\ &\leq c_3 M^2 \Big(1 + \frac{1}{\Delta x^2} \Big) \left\| \tilde{u}^n \right\|_{h^1}, \end{aligned} \tag{4.11}$$

for some constant $c_3 > 0$ independent of Δx.

Blending (4.9), (4.10), and (4.11) we derive the bound

$$\Delta t \Delta x \sum_{j\in\mathbb{Z}} \Big((D^t_+ \tilde{u}^n_{j+1/2})^2 + (D^t_+ \tilde{q}^n_j)^2 \Big) \leq C M^2 \Big(\Delta t + \frac{\Delta t}{\Delta x^2} \Big) \left\| \tilde{u}^n \right\|_{h^1}, \tag{4.12}$$

where the constant C is independent of Δx.

Combining (4.7), (4.8), and (4.12), it follows that $\{\tilde{u}^n_{j+1/2}\}$ obeys the following discrete energy estimate:

$$D^t_+ \left\| \tilde{u}^n \right\|^2_{h^1} + \underbrace{\frac{\Delta x^2}{2} \sum_{j\in\mathbb{Z}} \left| f^M(\tilde{u}^n_{j+1/2}) \right| \Big(D_+ \tilde{q}^n_j \Big)^2}_{=: Z^n} \leq \underbrace{C M^2 \Big(\Delta t + \frac{\Delta t}{\Delta x^2} \Big)}_{=: \omega} \left\| \tilde{u}^n \right\|^2_{h^1}.$$

By the discrete Gronwall inequality, cf. Lemma 2.1,

$$\left\|\tilde{u}^{N_0}\right\|_{h^1}^2 + \sum_{n=0}^{N_0-1} e^{CM^2\omega(t^{N_0-1}-t^n)} Z^n \leq e^{CM^2\omega t^{N_0-1}} \left\|u^0\right\|_{h^1}^2.$$

Choosing $M = \|u_0\|_{H^1(\mathbb{R})}$ and recalling the CFL type condition (4.1), we deduce

$$\begin{aligned} e^{CM^2\omega t^N} &= e^{C\|u_0\|_{H^1(\mathbb{R})}^2 \Delta t^2((1+\Delta x^2)/\Delta x^2)N} \\ &\leq (1+\Delta x^\theta)^{N\Delta t} \leq e^{t^N \Delta x^\theta} \leq 2 \quad \text{if} \quad \Delta x^\theta \leq \frac{\log 2}{T}. \end{aligned}$$

Therefore, in particular

$$\|\tilde{u}^n\|_{h^1} \leq \sqrt{2} \left\|u^0\right\|_{h^1} \quad \text{for } n = 0, \dots, N_0.$$

By the discrete Sobolev inequality (2.1), we find that

$$\|\tilde{u}^n\|_{\ell^\infty} \leq \frac{1}{\sqrt{2}} \|\tilde{u}^n\|_{h^1} \leq \left\|u^0\right\|_{h^1} \leq M \quad \text{for } n = 0, \dots, N_0.$$

This means that $\tilde{u}^n$ will "never notice" f^M, since

$$f^M(\tilde{u}_{j+1/2}^n) = \tilde{u}_{j+1/2}^n \quad \text{for } j \in \mathbb{Z} \text{ and } n = 0, \dots, N_0.$$

Therefore,

$$\tilde{u}_{j+1/2}^n = u_{j+1/2}^n \quad \text{and} \quad \tilde{P}_j^n = P_j^n \quad \text{for } j \in \mathbb{Z} \text{ and } n = 0, \dots, N_0.$$

Finally, (4.2) follows by noting that

$$e^{CM^2\omega(t^{N_0-1}-t^n)} \geq 1 \quad \text{for } n = 0, \dots, N_0 - 1. \qquad \square$$

We conclude this section by stating some immediate consequences of (4.2).

Lemma 4.2. *For $n = 0, \dots, N$,*

$$\|P^n\|_{\ell^\infty}, \|P^n\|_{\ell^1} \leq C \|u_0\|_{H^1(\mathbb{R})}^2,$$
$$\|D_+P^n\|_{\ell^\infty}, \|D_+P^n\|_{\ell^1} \leq C \|u_0\|_{H^1(\mathbb{R})}^2,$$

where $C > 0$ is a constant independent of Δx.

Proof. This follows immediately from Lemma 2.2, noting that in this case

$$f_j = \left(u_{j+1/2}^n \vee 0\right)^2 + \left(u_{j-1/2}^n \wedge 0\right)^2 + \frac{1}{2}\left(D_- u_{j+1/2}^n\right)^2,$$

and thus

$$\|f_j\|_{\ell^1} \leq \|u^n\|_{h^1}^2 \leq (1+\Delta x)^2 \left\|u^0\right\|_{H^1(\mathbb{R})}^2. \qquad \square$$

5. One-sided sup-norm estimate

Lemma 5.1. *Assume that Δt satisfies the CFL type condition* (4.1) *and that Δx is sufficiently small. For $n = 0, \dots, N$ and $j \in \mathbb{Z}$, we then have*

$$q_j^n \le \frac{2}{t^n} + C \left\| u^0 \right\|_{h^1}, \tag{5.1}$$

where $C > 0$ is a finite constant.

Proof. We can write the difference equation for $\{q_j^n\}$, see (3.7), as

$$q_j^{n+1} = q_j^n(1 - \lambda a - \lambda b) + q_{j-1}^n \lambda a + q_{j+1}^n \lambda b - \Delta t \frac{q_j^{n\,2}}{2} + \Delta t\big((u_{j-1/2}^n \vee 0)^2 + (u_{j+1/2}^n \wedge 0)^2 - P_j^n\big),$$

where

$$a = \lambda(u_{j-1/2}^n \vee 0), \quad b = -\lambda(u_{j+1/2}^n \wedge 0), \quad \lambda = \Delta t/\Delta x.$$

Now, we have uniform bounds on $\|u^n\|_{\ell^\infty}$ and $\|P^n\|_{\ell^\infty}$ and thus

$$q_j^{n+1} \le q_j^n \Big(1 - \lambda a - \lambda b\Big) + q_{j-1}^n \lambda a + q_{j+1}^n \lambda b - \Delta t \frac{q_j^{n\,2}}{2} + \Delta t L, \tag{5.2}$$

for some finite constant $L \lesssim \left\| u^0 \right\|_{h^1}^2$.

Set $\bar{q}_j^n = \max\{q_j^n, q_{j-1}^n, q_{j+1}^n\}$. We claim that

$$q_j^{n+1} \le \bar{q}_j^n - \Delta t \frac{(\bar{q}_j^n)^2}{2} + \Delta t L, \tag{5.3}$$

if Δt is chosen sufficiently small.

First, we choose Δt so small that $\lambda(a+b) < 1/2$. Then, if $\bar{q}_j^n = q_j^n$ the claim follows immediately from (5.2).

Next, assume $\bar{q}_j^n = q_{j-1}^n$. Then note that

$$\begin{aligned}(q_j^n)^2 &= (q_{j-1}^n)^2 + (q_j^n)^2 - (q_{j-1}^n)^2 \\ &= (q_{j-1}^n)^2 + \frac{q_j^n + q_{j-1}^n}{2} \Delta x D_- q_j^n = (q_{j-1}^n)^2 + \frac{u_{j+1/2}^n - u_{j-3/2}^n}{2} D_- q_j^n.\end{aligned}$$

Since $D_- q_j^n < 0$, we find that

$$(q_j^n)^2 \le (q_{j-1}^n)^2 - \|u^n\|_{\ell^\infty} D_- q_j^n.$$

Using this we can rephrase (5.2) as

$$q_j^{n+1} \le q_j^n \Big(1 - \lambda(a + \|u^n\|_{\ell^\infty} + b)\Big) + q_{j-1}^n \lambda(a + \|u^n\|_{\ell^\infty}) + q_{j+1}^n \lambda b$$

$$- \Delta t \frac{(q_{j-1}^n)^2}{2} + \Delta t L$$
$$\leq \bar{q}_j^n - \Delta t \frac{(\bar{q}_j^n)^2}{2} + \Delta t L =: F(\bar{q}_j^n),$$

if $\lambda \left\| u^n \right\|_{\ell^\infty} < \frac{1}{2}$. The proof of (5.3) if $\bar{q}_j^n = q_{j+1}^n$ is similar.

Note that $F'(q) = 1 - \Delta t q$, and thus F is increasing for $q < 1/\Delta t$. Furthermore, by the CFL type condition (4.1), $\Delta t = \mathcal{O}(\Delta x^3)$, and by the bounds on $\left\| u^n \right\|_{\ell^\infty}$,

$$q_j^n \leq |q_j^n| \leq \mathcal{O}\Big(\frac{1}{\Delta x}\Big) = \mathcal{O}\Big(\frac{1}{\Delta t^{1/3}}\Big) \leq \frac{1}{\Delta t},$$

for sufficiently small Δt. Therefore, setting[1] $M^n = \max_j q_j^n$, from (5.3) we get

$$M^{n+1} \leq F(M^n).$$

Now, set $Z^n = M^n - \sqrt{2L}$. Then

$$\begin{aligned} Z^{n+1} &\leq F(Z^n + \sqrt{2L}) - \sqrt{2L} \\ &= Z^n + \sqrt{2L} - \frac{\Delta t}{2}(Z^n + \sqrt{2L})^2 + L\Delta t - \sqrt{2L} \\ &= Z^n(1 - \Delta t \sqrt{2L}) - \frac{\Delta t}{2}(Z^n)^2 \leq Z^n - \frac{\Delta t}{2}(Z^n)^2. \end{aligned}$$

Now, clearly if $Z^n \leq 0$, then $Z^{n+1} \leq 0$. Hence, if $Z^0 \leq 0$, then $Z^n \leq 0$ for all $n > 0$. If $Z^n > 0$, by [38, page 271],

$$Z^n \leq \frac{2}{t^n + 1/Z^0} \leq \frac{2}{t^n}.$$

This finishes the proof. □

6. Higher integrability estimate

We begin this section by deriving a "renormalized form" of the finite difference scheme for q_j; so let f be a non-linear function (renormalization) of appropriate regularity and growth. Multiplying (3.7) by $f'(q_j)$ and using the discrete chain rule, which in the present context reads

$$\begin{aligned} f'(q_j^n) D_\pm q_j^n &= D_\pm f(q_j^n) \mp \frac{\Delta x}{2} f''(q_{j\pm 1/2}^n)(D_\pm q_j^n)^2, \\ f'(q_j^n) D_+^t q_j^n &= D_+^t f(q_j^n) - \frac{\Delta t}{2} f''(q_j^{n+1/2})(D_+^t q_j^n)^2, \end{aligned}$$

[1]This maximum exists since $q^n \in \ell^2$.

where $q^n_{j\pm1/2}$ is a number between q_j and $q_{j\pm1}$, and $q_j^{n+1/2}$ is a number between q_j^n and q_j^{n+1}. Multiplying the scheme (3.7) by $f'(q_j^n)$ we obtain

$$\begin{aligned}D^t_+ f(q^n_j) &+ (u^n_{j-1/2}\vee 0)D_- f(q^n_j) + (u^n_{j+1/2}\wedge 0)D_+ f(q^n_j) \\ &+ \frac{(q^n_j)^2}{2} f'(q^n_j) + \Big[P^n_j - (u^n_{j+1/2}\vee 0)^2 - (u^n_{j-1/2}\wedge 0)^2\Big] f'(q^n_j) \\ &+ I_{\Delta x, f'', j} = \frac{\Delta t}{2} f''(q^{n+1/2}_j)(D^t_+ q^n_j)^2,\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}I_{\Delta x, f'', j} := \frac{\Delta x}{2}\Big\{&(u^n_{j-1/2}\vee 0) f''(q^n_{j-1/2})(D_- q^n_j)^2 \\ &- (u^n_{j+1/2}\wedge 0) f''(q^n_{j+1/2})(D_+ q^n_j)^2\Big\}.\end{aligned}$$

Let us now write (6.1) in divergence form. To this end, observe that the discrete product rule (2.2) implies the following relations:

$$\begin{aligned}D_-\Big[(u^n_{j+1/2}\vee 0) f(q^n_j)\Big] &= (u^n_{j-1/2}\vee 0)D_- f(q^n_j) + D_-(u^n_{j+1/2}\vee 0) f(q^n_j), \\ D_+\Big[(u^n_{j-1/2}\wedge 0) f(q^n_j)\Big] &= (u^n_{j+1/2}\wedge 0)D_+ f(q^n_j) + D_+(u^n_{j-1/2}\wedge 0) f(q^n_j) \\ &= (u^n_{j+1/2}\wedge 0)D_+ f(q^n_j) + D_-(u^n_{j+1/2}\wedge 0) f(q^n_j),\end{aligned}$$

and therefore, using that $q^n_j = D_- u^n_{j+1/2}$,

$$\begin{aligned}&(u^n_{j-1/2}\vee 0)D_- f(q^n_j) + (u^n_{j+1/2}\wedge 0)D_+ f(q^n_j) \\ &\quad = D_-\Big[(u^n_{j+1/2}\vee 0) f(q^n_j)\Big] + D_+\Big[(u^n_{j-1/2}\wedge 0) f(q^n_j)\Big] - q^n_j f(q^n_j).\end{aligned}$$

Hence, we end up with the following divergence-form variant of the renormalized difference scheme (6.1):

$$\begin{aligned}D^t_+ f(q^n_j) &+ D_-\Big[(u^n_{j+1/2}\vee 0) f(q^n_j)\Big] + D_+\Big[(u^n_{j-1/2}\wedge 0) f(q^n_j)\Big] \\ &+ \frac{(q^n_j)^2}{2} f'(q^n_j) - q^n_j f(q^n_j) + \Big[P^n_j - (u^n_{j+1/2}\vee 0)^2 - (u^n_{j-1/2}\wedge 0)^2\Big] f'(q^n_j) \\ &+ I_{\Delta x, f'', j} = \frac{\Delta t}{2} f''(q^{n+1/2}_j)(D^t_+ q^n_j)^2.\end{aligned} \tag{6.2}$$

To ensure that the limit of $(\partial_x u_{\Delta x})^2$, cf. (4.2), is non-singular (i.e., not a measure), we shall need the following higher integrability estimate:

Lemma 6.1. *Let $q_{\Delta x}$ be defined by* (3.9)*, and assume that the CFL type condition* (4.1) *holds. Then, for all finite numbers a, b, α with $a < b$ and $\alpha \in (0, \theta)$,*

$$\int_0^T \int_a^b |q_{\Delta x}|^{2+\alpha}\, dx\, dt \leq C, \tag{6.3}$$

for some constant $C = C(a, b, T, \alpha)$ that is independent of Δx.

Proof. Define $\eta_\varepsilon(q) = \sqrt{\varepsilon^2 + q^2} - \varepsilon$. Note that $\eta_\varepsilon(q) \approx |q|$ for small ε and

$$-1 < \eta_\varepsilon'(q) = \frac{q}{\sqrt{\varepsilon^2 + q^2}} < 1 \quad \text{and} \quad 0 < \eta_\varepsilon''(q) = \frac{\varepsilon^2}{(\varepsilon^2 + q^2)^{3/2}} \leq \frac{1}{\varepsilon}.$$

In (6.2), we then specify $f(q) = \eta_\varepsilon(q)(1 + q^2)^{\alpha/2}$. One can easily check that

$$\begin{aligned} f'(q) &= \eta_\varepsilon'(q)(1 + q^2)^{\alpha/2} + \alpha\eta_\varepsilon(q)q(1 + q^2)^{\alpha/2-1}, \\ f''(q) &= \eta_\varepsilon''(q)(1 + q^2)^{\alpha/2} + 2\alpha\eta_\varepsilon'(q)q(1 + q^2)^{\alpha/2-1} \\ &\qquad + \alpha\eta_\varepsilon(q)\Big((1 + q^2)^{\alpha/2-1} + 2(\frac{\alpha}{2} - 1)(1 + q^2)^{\alpha/2-2}\Big), \end{aligned}$$

so that in particular $f''(q) \geq 0$ and

$$f''(q) = \eta_\varepsilon''(q)(1 + q^2)^{\alpha/2} + \text{bounded terms}. \tag{6.4}$$

Next, set

$$\begin{aligned} H(q) &:= \frac{q^2}{2} f'(q) - q f(q) \\ &= \frac{q^2}{2}\Big(\eta_\varepsilon'(q)(1 + q^2)^{\alpha/2} + \alpha\eta_\varepsilon(q)(1 + q^2)^{\alpha/2-1}q\Big) - q\eta_\varepsilon(q)(1 + q^2)^{\alpha/2} \\ &= q\eta_\varepsilon(q)(1 + q^2)^{\alpha/2}\Big[\frac{q\eta_\varepsilon'(q)}{2\eta_\varepsilon(q)} + \frac{\alpha q^2}{2(1 + q^2)} - 1\Big] \\ &= q\eta_\varepsilon(q)(1 + q^2)^{\alpha/2}\frac{1}{2}\Big[\frac{\alpha q^2}{1 + q^2} + \frac{\varepsilon}{\sqrt{\varepsilon^2 + q^2}} - 1\Big] =: H_\varepsilon(q)h_\varepsilon(q), \end{aligned}$$

with

$$H_\varepsilon(q) = q\eta_\varepsilon(q)\Big(1 + q^2\Big)^{\alpha/2}, \qquad h_\varepsilon(q) = \frac{1}{2}\Big[\frac{\alpha q^2}{1 + q^2} + \frac{\varepsilon}{\sqrt{\varepsilon^2 + q^2}} - 1\Big].$$

Now, note that

$$\lim_{q \to -\infty} h_\varepsilon(q) \leq \lim_{q \to -\infty} h_1(q) = \frac{\alpha - 1}{2} < 0.$$

Hence, for $\alpha, \varepsilon < 1$, we can find a constant $K > 0$ such that

$$h_\varepsilon(q) < \frac{\alpha - 1}{4} \quad \text{for all } q < -K. \tag{6.5}$$

Let us continue by defining the sets

$$\mathcal{N}^n = \{j \in \mathbb{Z} : q_j^n < -K\}, \quad \mathcal{C}^n = \{j \in \mathbb{Z} : -K \le q_j^n \le 0\},$$
$$\text{and} \quad \mathcal{P}^n = \{j \in \mathbb{Z} : q_j^n > 0\},$$

where K is defined in (6.5). Moreover, let $0 \le \chi(x) \le 1$ be a smooth cutoff function satisfying

$$\chi(x) = \begin{cases} 0, & x < a - 1, \\ 1, & x \in [a, b], \\ 0, & x > b + 1. \end{cases}$$

We multiply (6.2) by $\chi(x_j)\Delta t \Delta x$ and sum over $(n, j) \in \{0, \dots, N-1\} \times \mathbb{Z}$ to get

$$\Delta t \Delta x \sum_n \sum_{j \in \mathcal{N}^n} h_\varepsilon\left(q_j^n\right) H_\varepsilon\left(q_j^n\right) \chi(x_j)$$
$$\le -\Delta t \Delta x \sum_n \sum_{j \in \mathcal{C}^n} h_\varepsilon\left(q_j^n\right) H_\varepsilon\left(q_j^n\right) \chi\left(x_j\right) \tag{6.6}$$
$$- \Delta t \Delta x \sum_n \sum_{j \in \mathcal{P}^n} h_\varepsilon\left(q_j^n\right) H_\varepsilon\left(q_j^n\right) \chi\left(x_j\right) \tag{6.7}$$
$$\begin{aligned} -\Delta t \Delta x \sum_{j,n} \Big[& D_-\left(\left(u_{j+1/2}^n \vee 0\right) f\left(q_j^n\right)\right) \chi(x_j) \\ & D_+\left(\left(u_{j-1/2}^n \wedge 0\right) f\left(q_j^n\right)\right) \chi(x_j)\Big] \end{aligned} \tag{6.8}$$
$$- \Delta t \Delta x \sum_{n,j} A_j^n \chi\left(x_j\right) \tag{6.9}$$
$$+ \Delta x \sum_j \left(f\left(q_j^0\right) - f\left(q_j^N\right)\right) \chi(x_j) \tag{6.10}$$
$$+ \Delta x \Delta t \sum_{n,j} \frac{\Delta t}{2} f''\left(q_j^{n+1/2}\right) \chi(x_j) \left(D_+^t q_j^n\right)^2, \tag{6.11}$$

where

$$A_j^n = P_j^n - (u_{j+1/2}^n \vee 0)^2 - (u_{j-1/2}^n \wedge 0)^2. \tag{6.12}$$

Now, for $j \in \mathcal{N}^n$ we have that

$$\frac{1-\alpha}{4}|q_j^n|\eta_\varepsilon(q_j^n)\big(1+(q_j^n)^2\big)^{\alpha/2} \le h_\varepsilon(q_j^n)H_\varepsilon(q_j^n).$$

Therefore,

$$\begin{aligned}\frac{1-\alpha}{4}\Delta t\Delta x\sum_n\sum_{j\in\mathcal{N}^n}|q_j^n|\eta_\varepsilon(q_j^n)\big(1+(q_j^n)^2\big)^{\alpha/2}\chi(x_j)\\ \le |(6.6)|+|(6.7)|+|(6.8)|+|(6.9)|+|(6.10)|+|(6.11)|.\end{aligned}$$

We shall now find bounds on all the terms on the right-hand side; in what follows, we let C denote a generic constant independent of Δx, ε, and α.

We start with (6.11). By (3.7)

$$\begin{aligned}(D_+^t q_j^n)^2 \le C\Big[&\Big((u_{j-1/2}^n\vee 0)D_-q_j^n\Big)^2+\Big((u_{j+1/2}^n\wedge 0)D_+q_j^n\Big)^2\\ &+(P_j^n)^2+(u_{j+1/2}^n\vee 0)^4+(u_{j-1/2}^n\wedge 0)^4+(q_j^n)^4\Big].\end{aligned}$$

To bound the "integrals" of these terms we must use the CFL type condition (4.1), which implies that $\Delta t = \mathcal{O}(\Delta x^{2+\theta})$. First,

$$\begin{aligned}&\Delta x\Delta t\sum_j\Big((u_{j-1/2}^n\vee 0)D_-q_j^n\Big)^2\\ &\quad\le C\Delta x^{1+\theta}\sum_j(\Delta x D_-q_j^n)^2 \le C\Delta x^{1+\theta}\sum_j(q_j^n)^2 \le C\Delta x^\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Delta t\Delta x\sum_{n,j}\Delta t\Big[\Big((u_{j-1/2}^n\vee 0)D_-q_j^n\Big)^2+\Big((u_{j+1/2}^n\wedge 0)D_+q_j^n\Big)^2\Big]\chi(x_j)\\ &\quad\le C\Delta x^\theta T\to 0 \quad \text{as } \Delta x\to 0.\end{aligned}$$

We also find that

$$\begin{aligned}\Delta x\sum_j&\Big[(P_j^n)^2+(u_{j+1/2}^n\vee 0)^4+(u_{j-1/2}^n\wedge 0)^4\Big]\\ &\le C\Delta x\sum_j\Big[|P_j^n|+(u_{j+1/2}^n\vee 0)^2+(u_{j-1/2}^n\wedge 0)^2\Big]\\ &\le C\Delta x\sum_j\Big[|P_j^n|+(u_{j+1/2}^n)^2\Big]\le C,\end{aligned}$$

since $u_{\Delta x}$ and P_j^n are uniformly bounded. Thus

$$\Delta t\Delta x\sum_{n,j}\Delta t\left[(P_j^n)^2+(u_{j+1/2}^n\vee 0)^4+(u_{j-1/2}^n\wedge 0)^4\right]\chi(x_j)\to 0,$$

as $\Delta x\to 0$. Additionally,

$$\begin{aligned}
&\Delta x\sum_j\Delta t(q_j^n)^4\le C\Delta x\sum_j\Delta x^\theta(\Delta x q_j^n)^2(q_j^n)^2\\
&\le C\Delta x^{1+\theta}\sum_j\left[(u_{j+1/2}^n)^2+(u_{j-1/2}^n)^2\right](q_j^n)^2\le C\Delta x^{1+\theta}\sum_j(q_j^n)^2\le C\Delta x^\theta.
\end{aligned}$$

Therefore,

$$\Delta x\Delta t\sum_j\Delta t(q_j^n)^4\chi(x_j)\le CT\Delta x^\theta\to 0\quad\text{as }\Delta x\to 0.$$

Now, we have established that

$$\Delta x\Delta t\sum_{n,j}\Delta t(D_+^t q_j^n)^2\chi(x_j)=\mathcal{O}(\Delta x^\theta)\quad\text{as }\Delta x\to 0. \tag{6.13}$$

Recalling (6.4) this implies that

$$|(6.11)|\le C\Delta x\Delta t\sum_{n,j}\frac{\Delta t}{\varepsilon}\left(1+(q_j^{n+1/2})^2\right)^{\alpha/2}(D_+^t q_j^n)^2\chi(x_j)+\mathcal{O}(\Delta x^\theta).$$

When we established (6.13) we always had a "Δx^θ to spare," which we can use now. With $\beta=\theta-\alpha>0$, we get

$$\begin{aligned}
&\Delta t\Delta x\sum_{n,j}\frac{\Delta t}{\varepsilon}\left(1+(q_j^{n+1/2})^2\right)^{\alpha/2}(D_+^t q_j^n)^2\\
&\qquad\le C\Delta x\Delta t\sum_{n,j}\frac{\Delta x^\beta}{\varepsilon}\Delta x^\alpha\left(1+(q_j^{n+1/2})^2\right)^{\alpha/2}\Delta x^2(D_+^t q_j^n)^2\\
&\qquad\le C\Delta x\Delta t\sum_{n,j}\frac{\Delta x^\beta}{\varepsilon}\left(\Delta x^2+(\Delta x q_j^{n+1/2})^2\right)^{\alpha/2}\Delta x^2(D_+^t q_j^n)^2\\
&\qquad\le C\frac{\Delta x^\beta}{\varepsilon}\Delta x\Delta t\sum_{n,j}\Delta x^2(D_+^t q_j^n)^2\le C\frac{\Delta x^\beta}{\varepsilon},
\end{aligned}$$

since

$$\left(\Delta x^2+(\Delta x q_j^{n+1/2})^2\right)^{\alpha/2}\le\left(\Delta x^2+4(\|u^n\|_{\ell^\infty})^2\right)^{\alpha/2}\le C.$$

Now, choosing $\varepsilon = \Delta x^{\beta}$, we finally conclude that $|(6.11)|$ is bounded.

Next, we turn to (6.6). For $-K \le q \le 0$, we have that $|h_\varepsilon(q)H_\varepsilon(q)| \le C$, where C is independent of ε. Therefore,

$$|(6.6)| \le CT(b-a+2).$$

To estimate $|(6.7)|$, observe that

$$|(6.7)| \le C\Delta t\Delta x \sum_n \sum_{j\in\mathcal{P}^n} \left(1+|q_j^n|^{2+\alpha}\right)\chi(x_j).$$

Now, equipped with (4.2) and (5.1), it is possible to bound the right-hand side by a "Δx independent" constant exactly as was done in [12].

Regarding (6.8), observe that

$$\begin{aligned}
&\left|\Delta t\Delta x \sum_{j,n} D_-\left[(u_{j+1/2}^n \vee 0) f(q_j^n)\right]\chi(j\Delta x)\right| \\
&= \left|\Delta t\Delta x \sum_{j,n} (u_{j+1/2}^n \vee 0) D_+\chi(j\Delta x) f(q_j^n)\right| \\
&\le C\Delta t\Delta x \sum_{j,n} |u_{j+1/2}^n||D_+\chi(j\Delta x)|(1+|q_j^n|^{1+\alpha}) \\
&\le CT\Big(\sup_n \|u^n\|_{\ell^\infty} \|\{D_+\chi(x_j)\}_j\|_{\ell^1} \\
&\qquad\qquad + \sup_n \left\|\{|q_j^n|^{1+\alpha}\}_j\right\|_{\ell^{\frac{2}{1+\alpha}}} \|\{D_+\chi(j\Delta x)\}_j\|_{\ell^{\frac{2}{1-\alpha}}}\Big) \\
&= cT\Big(\sup_n \|u^n\|_{\ell^\infty} \left\|\{D_+\chi(x_j)\}_j\right\|_{\ell^1} + \sup_n \|q^n\|_{\ell^2}^{1+\alpha} \|\{D_+\chi(j\Delta x)\}_j\|_{\ell^{\frac{2}{1-\alpha}}}\Big).
\end{aligned}$$

Therefore, $|(6.8)|$ is also bounded independently of Δx.

Next, we focus on (6.9). Remembering that $|f'(q)| \le C(1+|q|^{\alpha})$, we find

$$\begin{aligned}
|(6.9)| &\le \Delta t\Delta x \sum_{n,j} \left(\|P^n\|_{\ell^\infty} + 2\|u^n\|_{\ell^\infty}^2\right)|f'(q_j^n)|\chi(x_j) \\
&\le C\Delta t\Delta x \sum_{n,j} \left(\|P^n\|_{\ell^\infty} + 2\|u^n\|_{\ell^\infty}^2\right)\left(1+|q_j^n|^{\alpha}\right)\chi(x_j) \\
&\le CT\Big(\sup_n \|P^n\|_{\ell^\infty} + 2\sup_n \|u^n\|_{\ell^\infty}^2\Big) \\
&\qquad \times \left(\|\{\chi(x_j)\}_j\|_{\ell^1} + \sup_n \left\|\{|q_j^n|^{\alpha}\}_j\right\|_{\ell^{\frac{2}{\alpha}}} \|\{\chi(j\Delta x)\}_j\|_{\ell^{\frac{2}{2-\alpha}}}\right) \\
&\le CT\Big(\sup_n \|P^n\|_{\ell^\infty} + 2\sup_n \|u^n\|_{\ell^\infty}^2\Big)
\end{aligned}$$

$$\times \Big(\|\{\chi(j\Delta x)\}_j\|_{\ell^1} + \sup_n \|q^n\|_{\ell^2}^{\alpha} \|\{\chi(j\Delta x)\}_j\|_{\ell^{\frac{2}{2-\alpha}}} \Big).$$

Finally, keeping in mind that $f \geq 0$, we treat (6.10) as follows:

$$\begin{aligned}
|(6.10)| &\leq \Delta x \sum_j \big(f(q_j^N) + f(q_j^0)\big)\chi(x_j) \\
&\leq C\Delta x \sum_j \big(2 + |q_j^N|^{\alpha} + |q_j^0|^{\alpha}\big)\chi(x_j) \\
&\leq \|\{\chi(x_j)\}_j\|_{\ell^1} \\
&\quad + \Big(\big\|\{|q_j^N|^{1+\alpha}\}_j\big\|_{\ell^{\frac{2}{1+\alpha}}} + \big\|\{|q_j^0|^{1+\alpha}\}_j\big\|_{\ell^{\frac{2}{1+\alpha}}} \Big) \|\{\chi(x_j)\}_j\|_{\ell^{\frac{2}{1-\alpha}}} \\
&= \|\{\chi(x_j)\}_j\|_{\ell^1} + \Big(\big\|q^N\big\|_{\ell^2}^{1+\alpha} + \big\|q^0\big\|_{\ell^2}^{1+\alpha} \Big) \|\{\chi(x_j)\}_j\|_{\ell^{\frac{2}{1-\alpha}}}.
\end{aligned}$$

Summarizing, we have established

$$\Delta t \Delta x \sum_n \sum_j |q_j^n| \eta_\varepsilon(q_j^n)(1 + (q_j^n)^2)^{\alpha/2} \leq C.$$

The statement of the lemma follows by noting that

$$|q|^{2+\alpha} \leq |q|\eta_\varepsilon(q)(1+q^2)^{\alpha/2} + |q|^{1+\alpha},$$

and using, in combination with (4.2), the bound

$$\Delta t \Delta x \sum_{j,n} |q_j^n|^{1+\alpha} \chi(x_j) \leq CT \sup_n \|q^n\|_{\ell^2}^{1+\alpha} \|\{\chi(x_j)\}_j\|_{\ell^{\frac{2}{1-\alpha}}}. \qquad \square$$

7. Basic convergence results

The purpose of this section is to present some straightforward consequences of the a priori estimates established in the foregoing sections. More precisely, we prove that the two families $\{u_{\Delta x}\}_{\Delta x>0}$ and $\{P_{\Delta x}\}_{\Delta x>0}$, cf. (3.10) and (3.11), have strongly converging subsequences, starting with the former.

Lemma 7.1. *There exists a limit function $u \in L^\infty(0,T;H^1(\mathbb{R})) \cap C([0,T] \times \mathbb{R})$, such that along a subsequence as $\Delta x \to 0$*

$$u_{\Delta x} \overset{\star}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H^1(\mathbb{R})), \tag{7.1}$$

$$u_{\Delta x} \to u \quad \text{uniformly in } [a,b] \times [0,T], \text{ for any } a < b. \tag{7.2}$$

Additionally,

$$t \mapsto \|u(t,\cdot)\|_{H^1(\mathbb{R})} \text{ is non-increasing, and} \tag{7.3}$$

$$\lim_{t\to 0} u(t,x) = u_0(x), \qquad x \in \mathbb{R}. \tag{7.4}$$

Proof. First we note that, for $t \in I^n$,

$$\begin{aligned}
&\int_{\mathbb{R}} (u_{\Delta x}(t,x))^2\, dx \\
&\quad = \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \Big(\frac{1}{\Delta x}(x_{j+1/2}-x)u_{j-1/2}(t) + (x-x_{j-1/2})u_{j+1/2}(t)\Big)^2 dx \\
&\quad \le \frac{\Delta x}{2} \sum_j \Big((u_{j-1/2}(t))^2 + (u_{j+1/2}(t))^2\Big) \\
&\quad \le \frac{1}{\Delta t}\Big((t-t^n)\Delta x \sum_j (u^{n+1}_{j+1/2})^2 + (t^{n+1}-t)\Delta x \sum_j (u^n_{j+1/2})^2\Big),
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} \Big(\partial_x u_{\Delta x}(t,x)\Big)^2 dx &= \Delta x \sum_j (q_j(t))^2 \\
&\le \frac{1}{\Delta t}\Big((t-t^n)\Delta x \sum_j (q_j^{n+1})^2 + (t^{n+1}-t)\Delta x \sum_j (q_j^n)^2\Big).
\end{aligned}$$

Hence,

$$\|u_{\Delta x}(t,\cdot)\|^2_{H^1(\mathbb{R})} \le \frac{1}{\Delta t}\Big((t-t^n)\left\|u^{n+1}\right\|^2_{H^1(\mathbb{R})} + (t^{n+1}-t)\left\|u^n\right\|^2_{H^1(\mathbb{R})}\Big).$$

Let $s \in I^m$ with $m \le n$. Then, using (4.2),

$$\begin{aligned}
\|u_{\Delta x}(t,\cdot)\|^2_{H^1(\mathbb{R})} \le& \frac{1}{\Delta t}\Big((t-t^n)\left\|u^{n+1}\right\|^2_{H^1(\mathbb{R})} + (t^{n+1}-t)\left\|u^n\right\|^2_{H^1(\mathbb{R})}\Big) \\
\le& e^{\Delta t(n-m)\Delta x^\theta}\|u_{\Delta x}(s,\cdot)\|^2_{H^1(\mathbb{R})} + \frac{e^{\Delta t(n-m)\Delta x^\theta}}{\Delta t}\Big[(t-t^n) \\
&\qquad \times \Big(\left\|u^{m+1}\right\|^2_{H^1(\mathbb{R})} - \|u_{\Delta x}(s,\cdot)\|^2_{H^1(\mathbb{R})}\Big) \\
&\qquad + (t^{n+1}-t)\Big(\|u^m\|^2_{H^1(\mathbb{R})} - \|u_{\Delta x}(s,\cdot)\|^2_{H^1(\mathbb{R})}\Big)\Big] \\
\le& e^{\Delta t(n-m)\Delta x^\theta}\|u_{\Delta x}(s,\cdot)\|^2_{H^1(\mathbb{R})} \\
&\quad + e^{\Delta t(n-m)\Delta x^\theta}\left|\left\|u^{m+1}\right\|^2_{H^1(\mathbb{R})} - \|u^m\|^2_{H^1(\mathbb{R})}\right| \\
\le& e^{\Delta t(n-m)\Delta x^\theta}\|u_{\Delta x}(s,\cdot)\|^2_{H^1(\mathbb{R})} \\
&\quad + e^{\Delta t(n-m)\Delta x^\theta}(e^{\Delta t\Delta x^\theta}-1)\|u^m\|^2_{H^1(\mathbb{R})}.
\end{aligned}$$

This implies (7.1) and (7.3).

Next, we prove that $\{u_{\Delta x}\}_{\Delta x>0}$ is uniformly bounded in $W^{1,2+\alpha}((0,T)\times(a,b))$. We can assume that $a-1=x_{j_a}$ and $b+1=x_{j_b}$ for some integers j_a and j_b. Since $q\mapsto |q|^{2+\alpha}$ is convex,

$$\begin{aligned}\int_0^T\int_a^b |\partial_x u_{\Delta x}|^{2+\alpha}\,dx \\ \leq \sum_n \Delta x\sum_j\left(\left(t^{n+1}-t\right)|q_j^n|^{2+\alpha}+\left(t-t^n\right)|q_j^n|^{2+\alpha}\right)\leq C,\end{aligned}\tag{7.5}$$

for some constant $C=C(\alpha,a,b,u_0)$, where we have also used (6.3).

Now, set $\sigma=(x-x_{j-1/2})/\Delta x$. Then, for $x\in I_j$, we have

$$\begin{aligned}|\partial_t u_{\Delta x}(t,x)| &= |(1-\sigma)D_+^t u_{j-1/2}^n+\sigma D_+^t u_{j+1/2}^n| \\ &\leq (1-\sigma)|D_+^t u_{j-1/2}^n|+\sigma|D_+^t u_{j+1/2}^n|.\end{aligned}$$

Furthermore, by the uniform bounds on $D_+P_j^n$ and $u_{\Delta x}$,

$$|D_+^t u_{j+1/2}^n|\leq C(1+|q_j^n|+|q_{j-1}^n|).$$

Using this,

$$\int_0^T\int_a^b |\partial_t u_{\Delta x}|^{2+\alpha}\,dx\leq C\Big(1+\Delta t\sum_{n=0}^N\Delta x\sum_{j_a}^{j_b}|q_j^n|^{2+\alpha}\Big)\leq C.$$

Now, $\{u_{\Delta x}\}_{\Delta x>0}\subset W^{1,2+\alpha}\subset\subset C^{0,\ell}$ on $(0,T)\times(a,b)$ with $\ell=1-2/(2+\alpha)$. Therefore, along a subsequence, $u_{\Delta x}\to u$ uniformly in $(0,T)\times(a,b)$ as $\Delta x\to 0$.

Let us show that the limit satisfies the initial condition (7.4). Fix $\bar{x}\in\mathbb{R}$ and let $t\in(0,1)$. We have $\bar{x}\in I_j$ for some j and $u_{\Delta x}(x_{j-1/2},0)=u_0(x_{j-1/2})$, so that

$$\begin{aligned}|u_{\Delta x}(0,\bar{x})-u_0(\bar{x})| &\leq |u_{\Delta x}(0,\bar{x})-u_{\Delta x}(0,x_{j-1/2})|+|u_0(x_{j-1/2})-u_0(\bar{x})| \\ &\leq C(\bar{x}-x_{j-1/2})^{\ell}.\end{aligned}$$

Consequently,

$$\begin{aligned}|u(t,\bar{x})-u_0(\bar{x})| &\leq |u(t,\bar{x})-u_{\Delta x}(t,\bar{x})| \\ &\qquad +|u_{\Delta x}(t,\bar{x})-u_{\Delta x}(0,\bar{x})|+|u_{\Delta x}(0,\bar{x})-u_0(\bar{x})| \\ &\leq |u(t,\bar{x})-u_{\Delta x}(t,\bar{x})|+Ct^{\ell}+C\Delta x^{\ell}.\end{aligned}$$

Now, we can let $\Delta x\to 0$ and then $t\to 0$ to conclude that $u(\bar{x},0)=u_0(\bar{x})$. This concludes the proof of the lemma. □

Lemma 7.2. *There exists a limit function*

$$P \in L^\infty(0,T;W^{1,\infty}(\mathbb{R})) \cap L^\infty(0,T;W^{1,1}(\mathbb{R}))$$

such that along a subsequence as $\Delta x \to 0$

$$P_{\Delta x} \to P \text{ in } L^p_{\mathrm{loc}}((0,T)\times\mathbb{R}), \quad 1 \le p < \infty. \tag{7.6}$$

Proof. By the bounds on P_j^n in Lemma 4.2, we see that $P_{\Delta x}$ is bounded in L^∞ uniformly in Δx. Next, we show that $\{\partial_t P_{\Delta x}\}_{\Delta x>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$. For $t \in [t^n, t^{n+1})$ and $x \in I_{j+1/2}$,

$$\partial_t P_{\Delta x}(t,x) = D_+^t P_j^n + (x - x_j) D_+^t D_+ P_j^n = (1-\sigma) D_+^t P_j^n + \sigma D_+^t P_{j+1}^n,$$

where $\sigma = (x - x_j)/\Delta x$. Write $D_+^t P_j^n = X_j^n + Y_j^n$, where X_j^n and Y_j^n solve

$$X_j^n - D_- D_+ X_j^n = D_+^t \Big((u_{j-1/2}^n \wedge 0)^2 + (u_{j+1/2}^n \vee 0)^2 \Big),$$

$$Y_j^n - D_- D_+ Y_j^n = \frac{1}{2} D_+^t \big((q_j^n)^2 \big).$$

Then $\|X^n\|_{\ell^1}$ is bounded by the ℓ^1 norm of the corresponding right-hand side above. By the discrete chain rule

$$D_+^t \big(u_{j-1/2}^n \wedge 0\big)^2 \le 2\big(u_{j-1/2}^n \wedge 0\big) D_+^t u_{j-1/2}^n + \Delta t \big(D_+^t u_{j-1/2}^n\big)^2.$$

Estimating the first term here,

$$\begin{aligned}
&\big(u_{j-1/2}^n \wedge 0\big) D_+^t u_{j-1/2}^n \\
&\qquad = -\big(u_{j-1/2}^n \wedge 0\big) \Big[\big(u_{j-1/2}^n \vee 0\big) q_{j-1}^n + \big(u_{j-1/2}^n \wedge 0\big) q_j^n + D_+ P_j^n \Big] \\
&\qquad = -\big(u_{j-1/2}^n \wedge 0\big) \Big[\big(u_{j-1/2}^n \wedge 0\big) q_j^n + D_+ P_j^n \Big].
\end{aligned}$$

This means that

$$\big\| \big(u^n \wedge 0\big) D_+^t u^n \big\|_{\ell^1} \le C \Big[\|u^n\|_{\ell^2} \|q^n\|_{\ell^2} + \|D_+ P^n\|_{\ell^1} \Big] \le C.$$

Similarly, we have

$$\begin{aligned}
&\big(D_+^t u_{j+1/2}^n\big)^2 \\
&\qquad \le C \Big[\big(u_{j+1/2}^n \vee 0\big)^2 + (q_j^n)^2 + \big(u_{j+1/2}^n \wedge 0\big)^2 + \big(q_{j+1}^n\big)^2 + \big(D_+ P_j^n\big)^2 \Big] \\
&\qquad \le C \Big[\big(u_{j+1/2}^n\big)^2 + \big(q_j^n\big)^2 + \big(q_{j+1}^n\big)^2 + |D_+ P_j^n| \Big],
\end{aligned}$$

and thus $\big\|(D_+^t u^n)^2\big\|_{\ell^1} \le C$. We have shown that

$$\|X^n\|_{\ell^1} \le \Big\| \Big\{ D_+^t \big[(u_{j-1/2}^n \wedge 0)^2 + (u_{j+1/2}^n \vee 0)^2 \big] \Big\}_j \Big\|_{\ell^1} \le C. \tag{7.7}$$

Next, using (6.2) with $f(q) = q^2/2$ we have that

$$\begin{aligned}\frac{1}{2}D_+^t(q_j^n)^2 = \underbrace{-D_-\left[\left(u_{j+1/2}^n \vee 0\right)\left(q_j^n\right)^2\right] + D_+\left[\left(u_{j-1/2}^n \wedge 0\right)\left(q_j^n\right)^2\right]}_{a_j^n} \\ + \underbrace{A_j^n q_j^n - I_{\Delta x, f'', j} + \Delta t\left(D_+^t q_j^n\right)^2}_{b_j^n},\end{aligned} \tag{7.8}$$

where A_j^n is defined in (6.12) and

$$I_{\Delta x, f'', j} = \Delta x \left\{\left(u_{j-1/2}^n \vee 0\right)\left(D_- q_j^n\right)^2 - \left(u_{j+1/2}^n \wedge 0\right)\left(D_+ q_j^n\right)^2\right\} \geq 0.$$

We write $Y_j^n = Y_j^{a,n} + Y_j^{b,n}$ where

$$Y_j^{a,n} = \left(I - D_- D_+\right)^{-1} a_j^n \quad \text{and} \quad Y_j^{b,n} = \left(I - D_- D_+\right)^{-1} b_j^n.$$

Now, $\left\|Y^{b,n}\right\|_{\ell^1} \leq \|b^n\|_{\ell^1}$, and therefore we compute

$$\begin{aligned}\|A^n q^n\|_{\ell^1} &= \Delta x \sum_j |A_j^n q_j^n| \leq C \|q^n\|_{\ell^2} \left(\Delta x \sum_j \left[\left(P_j^n\right)^2 + \left(u_{j+1/2}^n\right)^2\right]\right)^{1/2} \\ &\leq C \|q^n\|_{\ell^2} \left(\Delta x \sum_j \left[|P_j^n| + \left(u_{j+1/2}^n\right)^2\right]\right)^{1/2} \leq C.\end{aligned}$$

By (6.13), the L^1 norm of $\Delta t(D_+^t q_j^n)^2$ is of the same order as Δx^θ. Then, summing (7.8) over n and j, we arrive at

$$\Delta t \Delta x \sum_{n,j} I_{\Delta x, f'', j} \leq CT + \Delta x \sum_j \left[\left(q_j^N\right)^2 + \left(q_j^0\right)^2\right] + \mathcal{O}\left(\Delta x^\theta\right) \leq C.$$

This means that

$$\Delta t \Delta x \sum_{n,j} |Y_j^{b,n}| \leq C. \tag{7.9}$$

Now, let

$$L_j = h \sum_i e^{-\kappa|i-j|} D_\pm K_i = -h \sum_i D_\mp\left(e^{-\kappa|i-j|}\right) K_i,$$

for some sequence $\{K_j\}_j \in \ell^1$. Since

$$|D_\pm e^{-\kappa|i-j|}| \leq C e^{-\kappa|i-j|},$$

we get

$$\|L\|_{\ell^1} = \Delta x \sum_j |L_j| \le \Delta x h C \sum_{i,j} e^{-\kappa|i-j|} |K_i| \le C \|K\|_{\ell^1}.$$

Using this,

$$\|Y^{a,n}\|_{\ell^1} \le C \|q^n\|_{\ell^2}^2.$$

Combining this with (7.9) and (7.7) we see that

$$\Delta t \Delta x \sum_{n,j} |D_+^t P_j^n| \le C,$$

and therefore

$$\int_0^T \int_{\mathbb{R}} |\partial_t P_{\Delta x}|\, dx\, dt \le C.$$

Hence, $\{P_{\Delta x}\}_{\Delta x>0}$ is bounded in $W^{1,1}((0,T)\times\mathbb{R})$. Combining this with the L^∞ estimates found in Lemma 4.2 yields the existence of a convergent subsequence as claimed in (7.6). □

8. Strong convergence result

We now show that the sequence $\{q_{\Delta x}\}_{\Delta x>0}$, cf. (3.9), has a strongly converging subsequence. This result is a key point of the convergence analysis.

Lemma 8.1. *Fix $1 \le p < 3$ and $1 \le r < 1 + \frac{\theta}{2}$. Then there exist two functions $q \in L^p_{\mathrm{loc}}((0,T)\times\mathbb{R})$, $\overline{q^2} \in L^r_{\mathrm{loc}}((0,T)\times\mathbb{R})$ such that, for a subsequence as $\Delta x \to 0$,*

$$q_{\Delta x} \overset{\star}{\rightharpoonup} q \text{ in } L^\infty(0,T;L^2(\mathbb{R})), \qquad q_{\Delta x} \rightharpoonup q \text{ in } L^p_{\mathrm{loc}}((0,T)\times\mathbb{R}), \tag{8.1}$$

$$q_{\Delta x}^2 \rightharpoonup \overline{q^2} \text{ in } L^r_{\mathrm{loc}}((0,T)\times\mathbb{R}), \tag{8.2}$$

for all $a,b \in \mathbb{R}$, $a<b$. Moreover,

$$q^2(t,x) \le \overline{q^2}(t,x) \text{ for a.e. } (t,x) \in (0,T)\times\mathbb{R}, \tag{8.3}$$

and

$$\partial_x u = q \quad \text{in the sense of distributions on } (0,T)\times\mathbb{R}. \tag{8.4}$$

Finally, there is a positive constant C such that

$$q(t,x) \le \frac{2}{t} + C\|u_0\|_{H^1(\mathbb{R})}, \qquad t \in (0,T),\ x \in \mathbb{R}. \tag{8.5}$$

Proof. Claims (8.1), (8.2) are direct consequences of Lemmas 4.1 and 6.1. Claim (8.3) is true thanks to (8.2) and the convexity of $g = q^2$, cf. Lemma 2.3, while (8.4) is a consequence of the definitions of $q_{\Delta x}$ and $u_{\Delta x}$, cf. (3.9) and (3.10).

We conclude by proving (8.5). Fix $t > 0$, and let Δt be so small that $t \in I^n$ with $n > 0$. Note that $n \to \infty$ as $\Delta t \to 0$, and t^n and t^{n+1} both tend to t. From the definition of $q_{\Delta x}$ and (5.1) we have that

$$\begin{aligned} q_{\Delta x}(t,x) &= q_j^n + (t - t^n) D_+^t q_j^n = \frac{t^{n+1} - t}{\Delta t} q_j^n + \frac{t - t^n}{\Delta t} q_j^{n+1} \\ &\le \frac{t^{n+1} - t}{\Delta t}\big(\frac{2}{t^n} + \hat{C}\big) + \frac{t - t^n}{\Delta t}\big(\frac{2}{t^n} + \hat{C}\big) = \frac{2}{t} + \hat{C} + 2 f_{\Delta t}(t), \end{aligned} \tag{8.6}$$

where $\hat{C} := C \left\| u_0 \right\|_{H^1(\mathbb{R})}$ and for every $t \in [t^n, t^{n+1})$, $x \in I_j$, with

$$f_{\Delta t}(t) = \frac{t - t^n}{\Delta t} \frac{1}{t^{n+1}} + \frac{t^{n+1} - t}{\Delta t} \frac{1}{t^n} - \frac{1}{t}.$$

Observe that

$$f'_{\Delta t}(t) = \frac{1}{\Delta t}\Big(\frac{1}{t^{n+1}} - \frac{1}{t^n}\Big) + \frac{1}{t^2} = -\frac{1}{t^n t^{n+1}} + \frac{1}{t^2},$$

so $f'_{\Delta t}(t) = 0$ if and only if $t = \sqrt{t^n t^{n+1}} \in (t_n, t_{n+1})$, and in particular

$$\begin{aligned} \sup_{t \in [t^n, t^{n+1})} f_{\Delta t}(t) &= f_{\Delta t}(\sqrt{t^n t^{n+1}}) = \frac{\left(\sqrt{t^{n+1}} - \sqrt{t^n}\right)^2}{t^n t^{n+1}} \\ &\le \Big(\frac{\Delta t}{2\sqrt{t^n}}\Big)^2 \frac{1}{t^n t^{n+1}} \le \frac{\Delta t^2}{8 (t^n)^2 t^{n+1}} \to 0. \end{aligned}$$

Therefore, (8.5) follows from (8.1) and (8.6). □

In view of the weak convergences stated in (8.1), we have that for any function $f \in C^1(\mathbb{R})$ with f' bounded

$$\begin{aligned} &f(q_{\Delta x}) \overset{\star}{\rightharpoonup} \overline{f(q)} \quad \text{in } L^\infty(0,T; L^2(\mathbb{R})), \\ &f(q_{\Delta x}) \rightharpoonup \overline{f(q)} \text{ in } L^p_{\text{loc}}((0,T) \times \mathbb{R}),\ 1 \le p < 3, \end{aligned} \tag{8.7}$$

where the same subsequence of $\Delta x \to 0$ applies to any f from the specified class.

In what follows, we let $\overline{q f(q)}$ and $\overline{f'(q) q^2}$ denote the weak limits of the families $q_{\Delta x} f(q_{\Delta x})$ and $f'(q_{\Delta x}) q_{\Delta x}^2$, respectively, in $L^r_{\text{loc}}((0,T) \times \mathbb{R})$, for each $1 \le r < \frac{3}{2}$.

Lemma 8.2. *For any convex function $f \in C^1(\mathbb{R})$ with f' bounded we have that*

$$\iint_{(0,T)\times\mathbb{R}} \Big(\overline{f(q)}\partial_t\varphi + u\overline{f(q)}\partial_x\varphi\Big)\,dx\,dt + \int_{\mathbb{R}} f(q_0(x))\varphi(0,x)\,dx \geq \iint_{(0,T)\times\mathbb{R}} \Big(\frac{1}{2}\overline{f'(q)q^2} - \overline{qf(q)} + \Big(P-u^2\Big)\overline{f'(q)}\Big)\varphi\,dx\,dt,$$

for any nonnegative $\varphi \in C_c^\infty([0,T)\times\mathbb{R})$.

Proof. Set

$$\varphi_j(t) = \frac{1}{\Delta x}\int_{I_j}\varphi(t,x)\,dx, \quad \varphi_j^n = \frac{1}{\Delta x\Delta t}\iint_{I_j^n}\varphi(t,x)\,dx\,dt. \tag{8.8}$$

We multiply (6.2) by $\Delta x\Delta t\varphi_j^n$, sum over n, j, and take into account the convexity of f. After partial summations, the final result reads

$$E_0 + E_1 + E_2 + E_3 + E_4 + E_5 \geq 0, \tag{8.9}$$

where

$$\begin{aligned}
E_0 =& \Delta x\sum_j f(q_j^0)\varphi_j^0,\\
E_1 =& \Delta t\Delta x\sum_{n,j} f(q_j^n)D_t^-\varphi_j^n,\\
E_2 =& \Delta t\Delta x\sum_{n,j}\Big[\big(u_{j+1/2}^n\vee 0\big)f(q_j^n)D_+\varphi_j^n + \big(u_{j-1/2}^n\wedge 0\big)f(q_j^n)D_-\varphi_j^n\Big],\\
E_3 =& -\Delta t\Delta x\sum_{n,j}\Big[\frac{(q_j^n)^2}{2}f'(q_j^n) - q_j^n f(q_j^n)\Big]\varphi_j^n,\\
E_4 =& -\Delta t\Delta x\sum_{n,j} A_j^n f'(q_j^n)\varphi_j^n,\\
E_5 =& \Delta t^2\Delta x\sum_{n,j} f''\big(q_j^{n+1/2}\big)\big(D_+^t q_j^n\big)^2\varphi_j^n.
\end{aligned}$$

By (3.8),

$$\|q_0 - q_{\Delta x,0}\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } \Delta x \to 0,$$

where $q_0 := \partial_x u_0$ and $q_{\Delta x,0} := \partial_x u_{\Delta x}|_{t=0}$, so that

$$\Delta x\sum_j f(q_j^0)\varphi_j^0 \to \int_{\mathbb{R}} f(q_0(x))\varphi(0,x)\,dx \quad \text{as } \Delta x \to 0.$$

We split E_1 into three parts:

$$E_1 = E_{1,1} + E_{1,2} + E_{1,3}, \tag{8.10}$$

where

$$\begin{aligned} E_{1,1} &= \iint_{(0,T)\times\mathbb{R}} f(q_{\Delta x})\partial_t\varphi \, dx \, dt, \\ E_{1,2} &= \sum_{n,j} \iint_{I_j^n} \Big(f(q_j^n) - f(q_{\Delta x}) \Big) D_t^- \varphi_j^n \, dx \, dt, \\ E_{1,3} &= \sum_{n,j} \iint_{I_j^n} f(q_{\Delta x}) \Big(D_t^- \varphi_j^n - \partial_t\varphi \Big) \, dt. \end{aligned}$$

Due to (8.7)

$$E_{1,1} \to \iint_{(0,T)\times\mathbb{R}} \overline{f(q)}\partial_t\varphi \, dx \, dt. \tag{8.11}$$

Due to the boundedness of f',

$$|f(q_j^n) - f(q_{\Delta x})| \le c_1(t - t^n)|D_+^t q_j^n|, \tag{8.12}$$

for each $(t, x) \in [t^n, t^{n+1}) \times I_j$, where $c_1 > 0$ is a finite constant. In view of (6.13),

$$\begin{aligned} |E_{1,2}| &\le c_1 \Delta x \sum_{n,j} |D_t^- \varphi_j^n||D_+^t q_j^n| \int_{t^n}^{t^{n+1}} (t - t^n) \, dt \qquad (8.13) \\ &\le c_1 \Delta x \Delta t^2 \sum_{n,j} |D_t^- \varphi_j^n||D_+^t q_j^n| \le c_1 \Delta t \left\| D_t^- \varphi \right\|_{\ell^2} \left\| D_+^t q \right\|_{\ell^2} \to 0, \end{aligned}$$

as $\Delta x \to 0$. Finally, since

$$|D_t^- \varphi - \partial_t\varphi| \le c_2 \Delta x,$$

for some constant $c_2 > 0$, we have that

$$|E_{1,3}| \le c_2 \Delta x \iint_{\mathrm{supp}(\varphi)} |f(q_{\Delta x})| \, dx \, dt \to 0 \quad \text{as } \Delta x \to 0. \tag{8.14}$$

Clearly, (8.10), (8.11), (8.13), and (8.14) imply

$$E_1 \to \iint_{(0,T)\times\mathbb{R}} \overline{f(q)}\partial_t\varphi \, dx \, dt \quad \text{as } \Delta x \to 0. \tag{8.15}$$

Next, we split E_2 into four parts:

$$E_2 = E_{2,1} + E_{2,2} + E_{2,3} + E_{2.4}, \tag{8.16}$$

where

$$
\begin{aligned}
E_{2,1} =& \iint_{(0,T)\times\mathbb{R}} u_{\Delta x} f(q_{\Delta x}) \partial_x \varphi \, dx \, dt, \\
E_{2,2} =& \sum_{n,j} \iint_{I_j^n} \Big[\Big((u^n_{j+1/2} \vee 0) - (u_{\Delta x} \vee 0) \Big) f(q_j^n) D_+ \varphi_j^n \\
&\qquad + \Big((u^n_{j-1/2} \wedge 0) - (u_{\Delta x} \wedge 0) \Big) f(q_j^n) D_- \varphi_j^n \Big] \, dx \, dt, \\
E_{2,3} =& \sum_{n,j} \iint_{I_j^n} \Big[(u_{\Delta x} \vee 0) \Big(f(q_j^n) - f(q_{\Delta x}) \Big) D_+ \varphi_j^n \\
&\qquad + (u_{\Delta x} \wedge 0) \Big(f(q_j^n) - f(q_{\Delta x}) \Big) D_- \varphi_j^n \Big] \, dx \, dt, \\
E_{2,4} =& \sum_{n,j} \iint_{I_j^n} \Big[\Big(u_{\Delta x} \vee 0 \Big) f(q_{\Delta x}) \Big(D_+ \varphi_j^n - \partial_x \varphi \Big) \\
&\qquad + (u_{\Delta x} \wedge 0) f(q_{\Delta x}) \Big(D_- \varphi_j^n - \partial_x \varphi \Big) \Big] \, dx \, dt.
\end{aligned}
$$

Due to (8.7),

$$
E_{2,1} \to \iint_{(0,T)\times\mathbb{R}} \overline{q f(q)} \partial_x \varphi \, dx \, dt. \tag{8.17}
$$

Using the definition of $u_{\Delta x}$,

$$
\begin{aligned}
|u^n_{j-1/2} - u_{\Delta x}| &\le \Delta x \Big(|q_j^n| + |q_j^{n+1}| \Big), \\
|u^n_{j-1/2} - u_{\Delta x}| &\le \Delta x q_j^n \Big(|q_j^n| + |q_j^{n+1}| \Big) + |u^n_{j+1/2} - u^n_{j-1/2}|,
\end{aligned}
$$

so, by Lemmas 4.1 and 7.1,

$$
\begin{aligned}
|E_{2,2}| \le& \Delta x^2 \Delta t \sum_{n,j} \Big(|q_j^n| + |q_j^{n+1}| \Big) |f(q_j^n)| \Big(|D_+ \varphi_j^n| + |D_- \varphi_j^n| \Big) \qquad (8.18) \\
&\qquad + \Delta x \Delta t \sum_{n,j} |u^n_{j+1/2} - u^n_{j-1/2}| |f(q_j^n)| |D_+ \varphi_j^n| \\
\le& 2\Delta x \, \|D_+ \varphi\|_{\ell^\infty} \, \|q\|_{\ell^2} \, \|f(q)\|_{\ell^2} \qquad (8.19) \\
&\qquad + \|f(q)\|_{\ell^2} \left\| \Big\{ \Big(u^n_{j+1/2} - u^n_{j-1/2} \Big) D_+ \varphi_j^n \Big\}_{n,j} \right\|_{\ell^2} \to 0,
\end{aligned}
$$

as $\Delta x \to 0$. Using (8.12) and (6.13), we deduce

$$\begin{aligned} |E_{2,3}| &\le c_1 \Delta x \sum_{n,j} \Big(|D_-\varphi_j^n| + |D_+\varphi_j^n|\Big) \\ &\qquad\qquad \times \Big(|u_{j+1/2}^n| + |u_{j-1/2}^n|\Big) \int_{t^n}^{t^{n+1}} (t-t^n)\,dt \\ &= c_1 \Delta x \Delta t^2 \sum_{n,j} \Big(|D_-\varphi_j^n| + |D_+\varphi_j^n|\Big)\Big(|u_{j+1/2}^n| + |u_{j-1/2}^n|\Big) \\ &\le 4c_1 \Delta t \, \|D_+\varphi\|_{\ell^2} \, \|u\|_{\ell^2} \to 0, \end{aligned} \tag{8.20}$$

as $\Delta x \to 0$. Finally, since $|D_\pm \varphi - \partial_t \varphi| \le c_3 \Delta x$, for some constant $c_3 > 0$, we obtain

$$|E_{2,4}| \le c_3 \Delta x \iint_{\mathrm{supp}(\varphi)} |q_{\Delta x}||f(q_{\Delta x})|\,dt\,dx \to 0 \quad \text{as } \Delta x \to 0. \tag{8.21}$$

Note that (8.16), (8.17), (8.18), (8.20), and (8.21) imply

$$E_2 \to \iint_{(0,T)\times\mathbb{R}} \overline{qf(q)}\,\partial_x \varphi\,dx\,dt \quad \text{as } \Delta x \to 0. \tag{8.22}$$

We split E_3 into three parts:

$$E_3 = E_{3,1} + E_{3,2} + E_{3,3}, \tag{8.23}$$

where

$$\begin{aligned} E_{3,1} &= -\iint_{(0,T)\times\mathbb{R}} \Big[\frac{(q_{\Delta x})^2}{2} f'(q_{\Delta x}) - q_{\Delta x} f(q_{\Delta x})\Big]\varphi\,dx\,dt, \\ E_{3,2} &= -\sum_{n,j} \iint_{I_j^n} \Big[\frac{(q_j^n)^2}{2} f'(q_j^n) - q_j^n f(q_j^n) \\ &\qquad\qquad - \frac{(q_{\Delta x})^2}{2} f'(q_{\Delta x}) + q_{\Delta x} f(q_{\Delta x})\Big]\varphi_j^n\,dx\,dt, \\ E_{3,3} &= -\sum_{n,j} \iint_{I_j^n} \Big[\frac{(q_{\Delta x})^2}{2} f'(q_{\Delta x}) - q_{\Delta x} f(q_{\Delta x})\Big]\big(\varphi_j^n - \varphi\big)\,dx\,dt. \end{aligned}$$

Due to (8.7),

$$E_{3,1} \to -\iint_{(0,T)\times\mathbb{R}} \Big(\frac{\overline{q^2 f'(q)}}{2} - \overline{qf(q)}\Big)\varphi\,dx\,dt \quad \text{as } \Delta x \to 0. \tag{8.24}$$

Using the boundedness of f', we can estimate as follows:

$$\begin{aligned}
&|\frac{(q_j^n)^2}{2}f'(q_j^n) - q_j^n f(q_j^n) - \frac{(q_{\Delta x})^2}{2}f'(q_{\Delta x}) + q_{\Delta x}f(q_{\Delta x})|\\
&\leq \frac{|(q_j^n)^2 - q_{\Delta x}^2|}{2}|f'(q_j^n)| + \frac{(q_{\Delta x})^2}{2}|f'(q_j^n) - f'(q_{\Delta x})|\\
&\leq \frac{|q_j^n - q_{\Delta x}|(|q_j^n| + |q_{\Delta x}|)}{2}\left\|f'\right\|_{L^\infty} + \frac{(q_{\Delta x})^2}{2}\left\|f''\right\|_{L^\infty}|q_j^n - q_{\Delta x}|\\
&\leq \frac{(t-t^n)|D_+^t q_j^n|(|q_j^n| + |q_{\Delta x}|)}{2}\left\|f'\right\|_{L^\infty} + \frac{(q_{\Delta x})^2}{2}\left\|f''\right\|_{L^\infty}(t-t^n)|D_+^t q_j^n|.
\end{aligned}$$

Hence, taking into account (6.13) and (6.3),

$$\begin{aligned}
|E_{3,2}| &\leq \Delta x \sum_{n,j}\int_{I^n}\Big[\frac{(t-t^n)|D_+^t q_j^n|(|q_j^n| + |q_{\Delta x}|)}{2}\left\|f'\right\|_{L^\infty} \\
&\qquad\qquad + \frac{(q_{\Delta x})^2}{2}\left\|f''\right\|_{L^\infty}(t-t^n)|D_+^t q_j^n|\Big]\varphi_j^n\,dt \\
&\leq \Delta x \Delta t^2 \sum_{n,j}\Big[\frac{|D_+^t q_j^n|(|q_j^n| + |q_{\Delta x}|)}{2}\left\|f'\right\|_{L^\infty} \\
&\qquad\qquad + \frac{(q_{\Delta x})^2}{2}\left\|f''\right\|_{L^\infty}|D_+^t q_j^n|\Big]\varphi_j^n \\
&\leq \Delta t\Big[\frac{\left\|D_+^t q\right\|_{\ell^2}(\left\|q^n\right\|_{\ell^2} + \left\|q_{\Delta x}\right\|_{L^2(\mathbb{R})})}{2}\left\|f'\right\|_{L^\infty} \\
&\qquad\qquad + \mathcal{O}\Big(\frac{1}{\Delta x}\Big)\left\|u\right\|_{\ell^\infty}\left\|q\right\|_{\ell^2}\left\|D_+^t q^n\right\|_{\ell^2}\Big]\left\|\varphi\right\|_{L^\infty} \\
&\leq \Delta t\Big[\mathcal{O}\Big(\sqrt{\frac{\Delta x^\theta}{\Delta t}} + \sqrt{\frac{\Delta x^\theta}{\Delta t}}\frac{1}{\Delta x}\Big)\Big] = \mathcal{O}\Big(\Delta x^\theta\Big) \to 0 \quad \text{as } \Delta x \to 0.
\end{aligned} \tag{8.25}$$

Since $|\varphi_j^n - \varphi| = \mathcal{O}(\Delta x)$,

$$|E_{3,3}| \leq \mathcal{O}(\Delta x)\iint_{\mathrm{supp}\,(\varphi)}|\frac{(q_{\Delta x})^2}{2}f'(q_{\Delta x}) - q_{\Delta x}f(q_{\Delta x})|\,dx\,dt \to 0, \tag{8.26}$$

as $\Delta x \to 0$. We have that (8.23), (8.24), (8.25), and (8.26) imply

$$E_3 \to \iint_{(0,T)\times\mathbb{R}} \overline{qf(q)}\partial_x\varphi\,dx\,dt \quad \text{as } \Delta x \to 0. \tag{8.27}$$

We split the term E_4 into three parts: $E_4 = E_{4,1} + E_{4,2} + E_{4,3}$, where

$$\begin{aligned} E_{4,1} &= -\iint_{(0,T)\times\mathbb{R}} A_{\Delta x} f'(q_{\Delta x})\varphi\, dx\, dt, \\ E_{4,2} &= -\sum_{n,j} \iint_{I_j^n} \Big(A_j^n f'(q_j^n) - A_{\Delta x} f'(q_{\Delta x})\Big)\varphi_j^n\, dx\, dt, \\ E_{4,3} &= -\sum_{n,j} \iint_{I_j^n} A_{\Delta x} f'(q_{\Delta x})\Big(\varphi_j^n - \varphi\Big)\, dx\, dt, \end{aligned}$$

where $A_{\Delta x} = P_{\Delta x} - (u_{\Delta x})^2$. Lemmas 7.1 and 7.2, cf. also (8.7), imply that

$$E_{4,1} \to -\iint_{(0,T)\times\mathbb{R}} (P - u^2)\overline{f'(q)}\varphi\, dx\, dt \quad \text{as } \Delta x \to 0.$$

Continuing, it is not hard to see that $|E_{4,2}| = \mathcal{O}(\Delta x) \to 0$ as $\Delta x \to 0$. Moreover, since $\varphi_j^n - \varphi = \mathcal{O}(\Delta x)$, $E_{4,3} \to 0$ as $\Delta x \to 0$. Summarizing,

$$E_4 \to -\iint_{(0,T)\times\mathbb{R}} (P - u^2)\overline{f'(q)}\varphi\, dx\, dt \quad \text{as } \Delta x \to 0. \tag{8.28}$$

Finally, regarding E_5, due to (6.13), we conclude that as $\Delta x \to 0$

$$|E_5| \le \|\varphi\|_{L^\infty((0,T)\times\mathbb{R})} \Delta t^2 \Delta x \sum_{n,j} f''(q_j^{n+1/2})(D_+^t q_j^n)^2 \varphi_j^n \to 0. \tag{8.29}$$

The lemma now follows from (8.9), (8.15), (8.22), (8.27), (8.28), and (8.29). □

We know that $\{(q_{\Delta x})^2\}_{\Delta x>0}$ is bounded in $L^\infty(0,T;L^1(\mathbb{R})) \cap L^r_{\text{loc}}((0,T)\times\mathbb{R})$, for any $1 \le r < 1 + \theta/2$. Additionally, using (6.2) with $f(q) = \frac{q^2}{2}$, we can show that the mapping $t \mapsto \int_{\mathbb{R}} (q_{\Delta x})^2 \varphi\, dx$ is equi-continuous on $[0,T]$, for every $\varphi \in C_c^\infty(\mathbb{R})$. Hence, in view of Lemma 2.4,

$$\int_{\mathbb{R}} (q_{\Delta x})^2 \varphi\, dx \to \int_{\mathbb{R}} \overline{q^2}\varphi\, dx \quad \text{uniformly on } [0,T], \tag{8.30}$$

and

$$t \mapsto \int_{\mathbb{R}} \overline{q^2}\varphi\, dx \quad \text{is continuous on } [0,T]. \tag{8.31}$$

The statements (8.30) and (8.31) hold with $q_{\Delta x}^2$, $\overline{q^2}$ replaced respectively by $f(q_{\Delta x})$, $\overline{f(q_{\Delta x})}$, for any convex function $f \in C^1(\mathbb{R})$ with f' bounded.

Lemma 8.3. *Let q and $\overline{q^2}$ be the weak limits identified in Lemma 8.1. Then*

$$\iint_{(0,T)\times\mathbb{R}} \big(q\partial_t\varphi + uq\partial_x\varphi\big)\,dx\,dt + \int_{\mathbb{R}} q_0(x)\varphi(0,x)\,dx \tag{8.32}$$
$$= \iint_{(0,T)\times\mathbb{R}} \Big(-\frac{1}{2}\overline{q^2} + \big(P-u^2\big)\Big)\varphi\,dx\,dt, \quad \forall\varphi\in C_c^\infty((0,T)\times\mathbb{R}).$$

Proof. Starting from (6.2) with $f(q)=q$, we argue as in the proof of Lemma 8.2 to conclude the validity of (8.32). □

The next lemma tells us that the weak limits in Lemma 8.1 satisfy the initial data in an appropriate sense.

Lemma 8.4. *Let q and $\overline{q^2}$ be the weak limits identified in Lemma 8.1. Then*

$$\lim_{t\to 0}\int_{\mathbb{R}} q^2(t,x)\,dx = \int_{\mathbb{R}} (\partial_x u_0)^2\,dx, \tag{8.33}$$
$$\lim_{t\to 0}\int_{\mathbb{R}} \overline{q^2}(t,x)\,dx = \int_{\mathbb{R}} (\partial_x u_0)^2\,dx.$$

Proof. The proof is similar to that in [12]. □

We can now wrap up the proof of the strong convergence of $\{q_{\Delta x}\}_{\Delta x>0}$.

Lemma 8.5. *Let q and $\overline{q^2}$ be the weak limits identified in Lemma 8.1. Then*

$$\overline{q^2}(t,x) = q^2(t,x) \text{ for a.e. } (t,x)\in(0,T)\times\mathbb{R}. \tag{8.34}$$

Consequently, as $\Delta x\to 0$ (along a subsequence if necessary)

$$q_{\Delta x}\to q \text{ in } L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}) \text{ and a.e. in } (0,T)\times\mathbb{R}. \tag{8.35}$$

Proof. By Lemma 8.2,

$$\partial_t\overline{f(q)} + \partial_x\big(u\overline{f(q)}\big) \le \overline{qf(q)} - \frac{1}{2}\overline{f'(q)q^2} + \big(u^2-P\big)\overline{f'(q)}, \tag{8.36}$$

in the sense of distributions on $(0,T)\times\mathbb{R}$, for any convex function $f\in C^1(\mathbb{R})$ with f' bounded. Moreover, by Lemma 8.3,

$$\partial_t q + \partial_x(uq) = \frac{1}{2}\overline{q^2} + u^2 - P, \tag{8.37}$$

in the sense of distributions on $(0,T)\times\mathbb{R}$. Equipped with (8.36), (8.37), (8.33), and (8.5), we can argue exactly as in Xin and Zhang [39] to arrive at (8.34). In view of Lemma 2.3, the claim (8.35) follows immediately from (8.34) and (7.5). □

We now prove that the limit u satisfies (**D.3**).

Lemma 8.6. *For any* $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} u\varphi_t + (\frac{u^2}{2}+P)\varphi_x\,dx\,dt = 0,\\ \int_0^T\int_{\mathbb{R}} P\big(\varphi-\varphi_{xx}\big)\,dx\,dt = \int_0^T\int_{\mathbb{R}}\big(u^2+\frac{1}{2}\big(\partial_x u\big)^2\big)\varphi\,dx\,dt.\end{aligned} \tag{8.38}$$

Proof. It is not difficult to establish the equation for P, since we have already established that $\partial_x u_{\Delta x}\to\partial_x u$ in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R})$, cf. (8.35). Indeed, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} P_{\Delta x}\big(\varphi-\varphi_{xx}\big)\,dx\,dt &= \sum_{n,j}\iint_{I^n_{j-1/2}} P_{\Delta x}\big(\varphi-\varphi_{xx}\big)\,dx\,dt\\ &= \sum_{n,j}P_j^n\iint_{I^n_{j-1/2}}\varphi-\varphi_{xx}\,dx\,dt\\ &\quad+\underbrace{\sum_{n,j}\iint_{I^n_{j-1/2}}\Big(P_j^n-P_{\Delta x}\Big)\Big(\varphi-\varphi_{xx}\Big)\,dx\,dt}_{w_1}\\ &= \Delta t\Delta x\sum_{n,j}P_j^n\Big(\varphi_j^n-D_-D_+\varphi_j^n\Big)\\ &\quad+w_1+\underbrace{\sum_{n,j}P_j^n\iint_{I^n_{j-1/2}}\Big(\Big(\varphi-\varphi_j^n\Big)+\Big(\varphi_{xx}-D_-D_+\varphi_j^n\Big)\Big)\,dx\,dt}_{w_2}.\end{aligned}$$

Since $|P_{\Delta x}(t,x)-P_j^n|\le C\Delta x$ for $(t,x)\in I^n_{j-1/2}$, $w_1\to 0$ as $\Delta x\to 0$. Similarly, since $\varphi-\varphi_{xx}$ is close to $\varphi_j^n-D_-D_+\varphi_j^n$ in $I^n_{j-1/2}$ and P_j^n is bounded, we conclude that $w_2\to 0$ as $\Delta x\to 0$. Set

$$\begin{aligned}f_j^n &= \big(u^n_{j-1/2}\wedge 0\big)^2+\big(u^n_{j+1/2}\vee 0\big)^2+\frac{1}{2}\big(q_j^n\big)^2,\\ f_{\Delta x} &= \big(u_{\Delta x}\big)^2+\frac{1}{2}\big(\partial_x u_{\Delta x}\big)^2.\end{aligned}$$

Using the scheme for P_j^n, cf. (3.3),

$$\Delta t\Delta x\sum_{n,j}P_j^n\Big(\varphi_j^n-D_-D_+\varphi_j^n\Big)=\Delta t\Delta x\sum_{n,j}f_j^n\varphi_j^n=\int_0^T\int_{\mathbb{R}} f_{\Delta x}\varphi\,dx\,dt$$

$$+\underbrace{\sum_{n,j}\iint_{I^n_{j-1/2}}\Big(f^n_j - f_{\Delta x}\Big)\varphi\,dx\,dt}_{w_3} + \sum_{n,j} f^n_j \underbrace{\iint_{I^n_{j-1/2}}\Big(\varphi - \varphi^n_j\Big)\,dx\,dt}_{=0}.$$

By the definition of $q_{\Delta x}$ we have that $q_{\Delta x} = q^n_j + (t - t^n)D^t_+ q^n_j$ for $t \in I^n$. Hence

$$(q_{\Delta x})^2 - \big(q^n_j\big)^2 = 2(t-t^n)q^n_j D^t_+ q^n_j + \big(t-t^n\big)^2\big(D^t_+ q^n_j\big)^2 \quad \text{in } I^n_{j-1/2},$$

so that, assuming $\operatorname{supp}(\varphi) \subset [x_{j_a}, x_{j_b}]$ for some integers j_a and j_b,

$$\begin{aligned}
&\sum_{n,j}\iint_{I^n_{j-1/2}} |\big(q^n_j\big)^2 - \big(q_{\Delta x}\big)^2||\varphi|\,dx\,dt \\
&\quad\le C\Delta x \sum_n \sum_{j_a}^{j_b} \int_{t^n}^{t^{n+1}} \Big(2(t-t^n)q^n_j D^t_+ q^n_j + \big(t-t^n\big)^2\big(D^t_+ q^n_j\big)^2\Big)\,dt \\
&\quad= C\Delta t\Delta x \sum_n \sum_{j=j_a}^{j_b} \Big(\Delta t q^n_j D^t_+ q^n_j + \big(\Delta t D^t_+ q^n_j\big)^2\Big) \\
&\quad= \mathcal{O}\Big(\sqrt{\Delta t\Delta x^\theta} + \Delta t\Delta x^\theta\Big).
\end{aligned}$$

By the Hölder continuity of $u_{\Delta x}$, recalling that $u_{\Delta x} \in C^{0,\ell}$ with $\ell = 1 - 2/(2+\alpha)$, we find

$$|\big(u^n_{j-1/2} \wedge 0\big)^2 + \big(u^n_{j+1/2} \vee 0\big)^2 - \big(u_{\Delta x}\big)^2| = \mathcal{O}(\Delta x^\ell + \Delta t^\ell),$$

and therefore $w_3 \to 0$ as $\Delta x \to 0$. Hence, using (7.2), (8.4), and (8.35),

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}} P\big(\varphi - \varphi_{xx}\big)\,dx\,dt &= \lim_{\Delta x\to 0}\int_0^T\int_{\mathbb{R}} P_{\Delta x}\big(\varphi - \varphi_{xx}\big)\,dx\,dt \\
&= \lim_{\Delta x\downarrow 0}\int_0^T\int_{\mathbb{R}} \Big(\big(u_{\Delta x}\big)^2 + \frac{1}{2}\big(\partial_x u_{\Delta x}\big)^2\Big)\varphi\,dx\,dt \\
&= \int_0^T\int_{\mathbb{R}} \Big(u^2 + \frac{1}{2}\big(\partial_x u\big)^2\Big)\varphi\,dx\,dt.
\end{aligned}$$

This means that the second equation in (8.38) holds.

To establish the first equality in (8.38), we derive a divergence-form version of the scheme (3.1). To this end, introduce the functions $f_\vee(u) = \frac{1}{2}(u \vee 0)^2$ and $f_\wedge(u) = \frac{1}{2}(u \wedge 0)^2$. Observe that $f_\vee$ and $f_\wedge$ are piecewise C^2, and the absolute value of the second derivatives are bounded by 1. By the discrete chain rule,

$$\big(u^n_{j+1/2} \vee 0\big) D_- u^n_{j+1/2} = D_- f_\vee(u^n_{j+1/2}) + \mathcal{O}\Big(\Delta x \big(D_- u^n_{j+1/2}\big)^2\Big),$$

and

$$\big(u^n_{j+1/2} \wedge 0\big) D_+ u^n_{j+1/2} = D_+ f_\wedge(u^n_{j+1/2}) + \mathcal{O}\Big(\Delta x \big(D_+ u^n_{j+1/2}\big)^2\Big).$$

Consequently, we can replace (3.1) by

$$\begin{aligned} D^t_+ u^n_{j+1/2} + D_- f_\vee(u^n_{j+1/2}) + D_+ f_\wedge(u^n_{j+1/2}) + D_+ P^n_j \\ = \mathcal{O}\Big(\Delta x \Big\{ \big(D_- u^n_{j+1/2}\big)^2 + \big(D_+ u^n_{j+1/2}\big)^2 \Big\}\Big). \end{aligned} \tag{8.39}$$

Observe that

$$D = \frac{D_- + D_+}{2}, \quad \Delta x D_- D_+ = D_+ - D_-, \quad f_\vee + f_\wedge = \frac{u^2}{2}. \tag{8.40}$$

Using these identities, we can restate (8.39) as

$$\begin{aligned} D^t_+ u^n_{j+1/2} &+ D_-\Big[\frac{(u^n_{j+1/2})^2}{4} + \frac{1}{2}\Big(f_\vee(u^n_{j+1/2}) - f_\wedge(u^n_{j+1/2})\Big)\Big] \\ &+ D_+\Big[\frac{(u^n_{j+1/2})^2}{4} + \frac{1}{2}\Big(f_\wedge(u^n_{j+1/2}) - f_\vee(u^n_{j+1/2})\Big)\Big] + D_+ P^n_j \\ &= \mathcal{O}\Big(\Delta x \Big\{ \big(D_- u^n_{j+1/2}\big)^2 + \big(D_+ u^n_{j+1/2}\big)^2 \Big\}\Big). \end{aligned} \tag{8.41}$$

Using (8.40),

$$\begin{aligned} D_-\Big(f_\vee(u^n_{j+1/2}) - f_\wedge(u^n_{j+1/2})\Big) + D_+\Big(f_\wedge(u^n_{j+1/2}) - f_\vee(u^n_{j+1/2})\Big) \\ = \Delta x D_- D_+ f_\wedge(u^n_{j+1/2}) - \Delta x D_- D_+ f_\vee(u^n_{j+1/2}), \end{aligned}$$

equation (8.41) becomes

$$\begin{aligned} D^t_+ u^n_{j+1/2} &+ D\Big(\frac{(u^n_{j+1/2})^2}{2}\Big) + D_+ P^n_j \\ &= \mathcal{O}\Big(\Delta x \Big\{ (D_- u^n_{j+1/2})^2 + (D_+ u^n_{j+1/2})^2 \Big\}\Big) \\ &\quad + \Delta x \left\{ D_- D_+ f_\vee(u^n_{j+1/2}) - D_- D_+ f_\wedge(u^n_{j+1/2}) \right\}. \end{aligned} \tag{8.42}$$

Now, fix $\varphi \in C_c^2((0,T)\times\mathbb{R})$ and define φ_j^n as before, cf. (8.8). Multiplying (8.42) by $\varphi_j^n \Delta t\Delta x$ and performing partial summations gives

$$\underbrace{\Delta t\Delta x\sum_{n,j} u_{j+1/2}^n D_+^t\varphi_j^n}_{E_1} + \underbrace{\Delta t\Delta x\sum_{n,j}\frac{\left(u_{j+1/2}^n\right)^2}{2}D\varphi_j^n}_{E_2} + \underbrace{\Delta t\Delta x\sum_{n,j}P_j^n D_-\varphi_j^n}_{E_3} = \mathcal{O}(\Delta x),$$

by using (4.2). We have $|u_{\Delta x} - u_{j-1/2}^n| \le C\Big(\Delta x|q_j^n| + \Delta t|D_+^t u_{j+1/2}^n|\Big)$. Using this and (7.2), we compute as follows:

$$\begin{aligned}
E_1 &= \int_0^T\int_{\mathbb{R}} u_{\Delta x}\varphi_t\,dx\,dt + \sum_{n,j}\iint_{I_j^n}\left(u_{j+1/2}^n - u_{\Delta x}\right)\varphi_t\,dx\,dt \\
&\qquad + \sum_{n,j} u_{j+1/2}^n \iint_{I_j^n}\left(D_+^t\varphi_j^n - \varphi_t\right)dx\,dt \\
&= \int_0^T\int_{\mathbb{R}} u_{\Delta x}\varphi_t\,dx\,dt + \mathcal{O}(\Delta x) \to \int_0^T\int_{\mathbb{R}} u\varphi_t\,dx\,dt \quad \text{as } \Delta x\to 0.
\end{aligned}$$

In the same way, equipped with (7.2) and (7.6), we can show that

$$E_2 \to \int_0^T\int_{\mathbb{R}}\frac{u^2}{2}\varphi_x\,dx\,dt, \quad E_3 \to \int_0^T\int_{\mathbb{R}} P\varphi_x\,dx\,dt \quad \text{as } \Delta x\to 0,$$

thus proving the first equality in (8.38). This concludes the proof of the lemma. □

9. Numerical examples

We have tried the difference method presented here on several examples, and in doing this found that the first-order method analyzed in this paper exhibits very slow convergence, and thus requires a very small mesh size Δx to compute reasonable solutions. This is not surprising and appears to be the case with other schemes in the literature as well. Therefore we have implemented a second-order extension of the method. This second-order extension is based on the conservative version of the scheme

$$D_+^t u_{j+1/2}^n + D_-\left[\left(u_{j+1/2}^n\vee 0\right)u_{j+1/2}^n + \left(u_{j+1/2}^n\wedge 0\right)u_{j+3/2}^n + P_{j+1}^n\right]$$

$$= D_-\big(u^n_{j+1/2} \vee 0\big)u^n_{j-1/2} + D_-\big(u^n_{j+1/2} \wedge 0\big)u^n_{j+1/2}, \quad (9.1)$$

which can be viewed as a balance equation with a flux across $x = x_{j+1/2}$ given by

$$F^n_{j+1/2} = \big(u^n_{j+1/2} \vee 0\big)u^n_{j+1/2} + \big(u^n_{j+1/2} \wedge 0\big)u^n_{j+3/2} + P^n_{j+1}.$$

Taking this viewpoint, we *define* the second-order finite volume scheme by

$$D^t_+ u^n_j + D_- F^{n+1/2}_{j+1/2} = u^n_j D_-\big(u^{n+1/2}_{j+1/2}\big). \quad (9.2)$$

Here, $u^{n+1/2}_{j+1/2}$ is a first-order approximation of the value at the point $x = x_{j+1/2}$, $t = t^n + \Delta t/2$. This approximation is found by setting

$$u^n_{j+1/2} = \tfrac{1}{2}\big(u^n_j + u^n_{j+1}\big),$$

and then using the scheme (3.1) for half a time step (i.e., $\Delta t/2$). This scheme is a formally second-order accurate finite volume approximation, and this simple adaptation produces significantly more accurate approximations.

In Figure 1, we show the approximations calculated by the first-order scheme (3.1) and the second-order scheme (9.2) for the single peakon example. In this case the exact solution reads $u(x,t) = e^{-|x-t|}$. Figure 1 shows the solutions calculated using 2^9 equally spaced grid points in the interval $[-10, 30]$ for $t = 20$. We see that the second-order method is much more accurate than the first-order method. In passing, we note that we have not

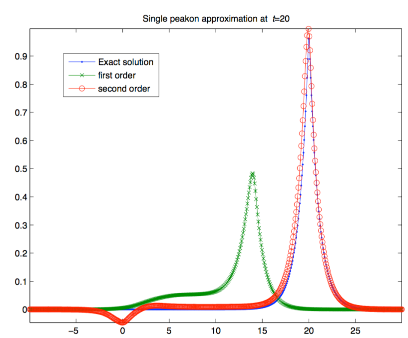

FIGURE 1. Approximations using $\Delta x = 40/2^9$, at $t = 20$ to the single peakon.

used the strict CFL-condition (4.1), but the more natural condition

$$\Delta t \leq \max_j\{u^n_j\}\Delta x.$$

This holds for the second-order scheme as well.

In order to investigate the convergence properties of the two methods, we computed errors in L^1 for the two schemes. Table 1 shows the computed L^1 errors in the case of a single peakon under mesh refinement. In this context, the L^1 error is defined as

$$L^1 \text{ error} = \Delta x \sum_i |u_{\Delta x}(x_i,t) - u(x_i,t)|,$$

where u is the exact solution. We used $t = 20$ and $\Delta x = 40/2^k$ for $k = 5, \dots, 12$. As expected, and as reported in [12], the first-order method converges very slowly. One other notable feature of Table 1 is that the

k	5	6	7	8	9	10	11	12	13
1^{st}	2.92	3.23	3.41	3.53	3.57	3.51	3.32	3.01	2.64
2^{nd}	5.36	5.17	3.29	1.27	0.60	0.36	0.21	0.13	0.09

TABLE 1. L^1 errors for the single peakon case, at $t = 20$, for $x \in [-10, 30]$, $\Delta x = 40/2^k$, $k = 5, \dots, 13$

second-order method seems to converge at a rate slightly less than 1.

The two-peakon solution is considerably more complicated than the single peakon, and this is also a much harder challenge computationally, see e.g., [1] and [33]. We use the two-peakon solution given by

$$u(x,t) = m_1(t)e^{-|x-x_1(t)|} + m_2(t)e^{-|x-x_2(t)|}, \tag{9.3}$$

with

$$x_1(t) = \log\left(\frac{18e^{t-10}}{e^{(t-10)/2}+6}\right), \quad x_2(t) = \log\left(40e^{t-10} + 60e^{(t-10)/2}\right),$$
$$m_1(t) = \frac{e^{(t-10)/2}+6}{2e^{(t-10)/2}+3}, \quad m_2(t) = \frac{e^{(t-10)/2}+\frac{2}{3}}{e^{(t-10)/2}+3}.$$

These formulas were taken from [35]. Figure 2 shows a contour plot of the approximate solutions found by using the first- and second-order methods, and $\Delta x = 40/2^{10}$ for $x \in [-15, 25]$ and $t \in [0, 25]$. We see that the interaction between the two peakons is poorly represented by the first-order method. Both the location as well as the magnitude of the peaks are far from the correct value. This is also illustrated in Figure 3 where we show the approximations using $\Delta x = 40/2^8$ at $t = 25$. We have also calculated errors for the two-peakon case. Indeed for $\Delta x \geq 40/2^{12}$, the first-order method did not seem to converge, and in order to give meaningful answers, this method

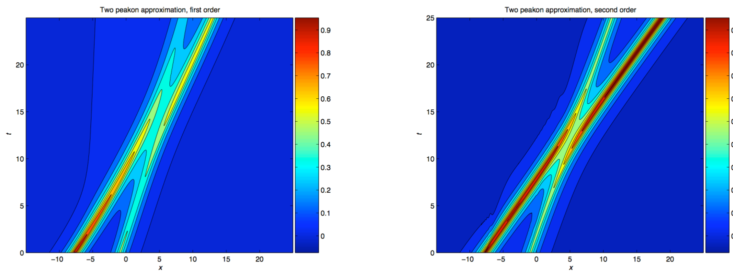

FIGURE 2. Approximations to (9.3) using $\Delta x = 40/2^{10}$. Left: first-order method(3.1). Right : second-order method (9.2).

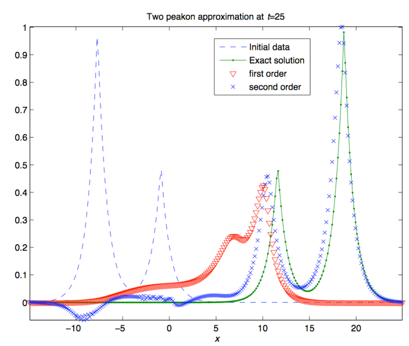

FIGURE 3. The approximations to (9.3) at $t = 25$ and $\Delta x = 40/2^8$.

demands very fine discretizations. These results are reported in Table 2. For $\Delta x > 40/2^8$ none of the methods gave satisfactory results. In our fi-

k	8	9	10	11	12	13
1^{st}	4.56	3.64	3.97	4.18	4.05	3.70
2^{nd}	1.88	1.04	0.63	0.38	0.22	0.16

TABLE 2. L^1 errors for the approximation to (9.2), $t = 25$, $x \in [-15, 25]$, $\Delta x = 40/2^k$, $k = 8, \dots, 13$.

nal example we choose initial data corresponding to a peakon-antipeakon collision:

$$u_0(x) = -\tanh(6)\big(e^{-|x+y(6)|} - e^{-|x-y(6)|}\big), \tag{9.4}$$

where $y(t) = \log(\cosh(t))$. In this case, we have a "peakon anti-peakon collision" at $t = 6$. In Figure 4, we exhibit the approximations generated by the first-order (left) and the second-order method for $t \in [0, 10]$ and $\Delta x = 24/2^{12}$. It is clear that the first-order scheme generates the dissipative solution, and for t larger than the collision time, the first-order approximation vanishes. Regarding the second-order approximation, it seems to continue as a peakon moving to the right, and an anti-peakon moving to the left. The magnitudes and speeds of these features are however far from the conservative solution, and we have indicated the conservative solution in the right-hand figure.

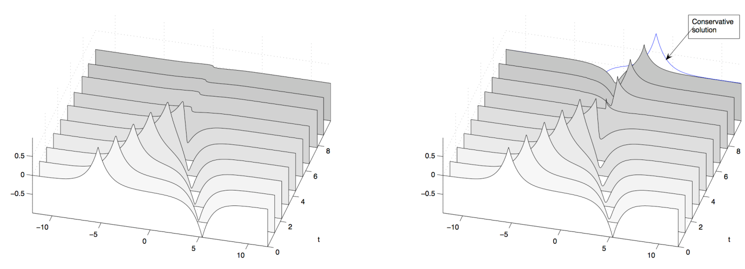

FIGURE 4. The numerical solutions to the initial-value problem (9.4). Left: first-order method, right: second-order version.

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