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COUNTABLE BRANCHING OF SIMILARITY SOLUTIONS OF HIGHER-ORDER POROUS MEDIUM TYPE EQUATIONS

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Abstract. A countable set of self-similar solutions of the fourth-order porous medium equation (the PME−4),

$$u_t = -(|u|^n u)_{xxxx} \quad \text{in } \mathbf{R} \times \mathbf{R}_+ \quad (n > 0), \tag{0.1}$$

is described. The similarity solutions under consideration have the standard form

$$u_l(x,t) = t^{-\alpha_l} f_l(y), \quad y = x/t^{\beta_l}, \quad \beta_l = \tfrac{1-\alpha_l n}{2}, \quad l = 0,1,2,\dots,$$

where α_l and $f_l(y)$ are eigenvalues and eigenfunctions of a nonlinear eigenvalue problem (actually, for a linear pencil of ordinary differential operators) written as

$$\mathbf{B}_n(f) \equiv -(|f|^n f)^{(4)} + \tfrac{1-\alpha n}{4}\, y\, f' + \alpha f = 0 \quad \text{in} \quad \mathbf{R}, \quad f \neq 0, \;\; f \in C_0(\mathbf{R}).$$

First four nonlinear eigenfunctions were obtained by Bernis and McLeod in the 1980s. In order to identify the full countable set of eigenfunctions $\{f_l\}$, we check their appearance at the branching point $n = 0$ from the eigenfunctions $\{\psi_l\}$ of the non self-adjoint operator $\mathbf{B}_0 = -D_y^4 + \frac{1}{4}\, y D_y + \frac{1}{4}\, I$, with the point spectrum $\sigma(\mathbf{B}_0) = \{\lambda_l = -\frac{l}{4},\, l \geq 0\}$, where I is the identity. These eigenfunctions give the solutions

$$u_l(x,t) = t^{-\frac{N}{4} + \lambda_l} \psi_l(y), \quad y = x/t^{\frac{1}{4}}, \quad l = 0,1,2,\dots,$$

of the linear bi-harmonic equation $u_t = -u_{xxxx}$, which is (0.1) for $n = 0$. The results extend to the PME−4 posed in $\mathbf{R}^N$. A similar classification is performed for the PME−6

$$u_t = (|u|^n u)_{xxxxxx} \quad \text{in } \mathbf{R} \times \mathbf{R}_+ \quad (u_t = u_{xxxxxx} \text{ for } n = 0). \tag{0.2}$$

The general methodology of the study of (0.1) and (0.2) is associated with that developed for the classic second-order PME $u_t = (|u|^n u)_{xx}$ in $\mathbf{R} \times \mathbf{R}_+$, which systematic study was began by the discovery of the first ZKB source-type solution by Zel'dovich, Kompaneetz, and Barenblatt in 1950-52.

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1. Introduction: from classic PME to higher-order diffusion equations

In this paper, we study a countable set of similarity solutions of the *fourth-order porous medium equation* (the PME−4):

$$\boxed{u_t = -\big(|u|^n u\big)_{xxxx} \quad \text{in} \quad \mathbf{R} \times \mathbf{R}_+ \quad (n > 0).} \tag{1.1}$$

First four similarity patterns as weak compactly supported solutions of (1.1) corresponding to initial data given by derivatives of Dirac's delta, $\delta(x)$, $\delta'(x)$, $\delta''(x)$, and $\delta'''(x)$ (the ODE is then integrated once), were obtained by Bernis and McLeod [6, 8] in 1988–1991. We plan to justify existence of a *countable set* of the corresponding nonlinear eigenfunctions.

Before discussing this main nonlinear *higher-order diffusion model* and explaining the mathematical methodology of our analysis, it is necessary and is natural to concentrate initially on the classic second-order *porous medium equation* (PME), which modern evolution theory of quasilinear degenerate parabolic equations appeared from.

The PME burst into nonlinear parabolic theory in the 1950s and continuously played a vital part for fifty years afterwards. We will touch and briefly discuss several well and not that well-known aspects of PME theory, and even related asymptotic properties of the linear *heat equation* when necessary. Eventually, this will lead us to principal complicated and often open problems of the twenty first century concerning $2m$th-order quasilinear degenerate parabolic equations.

1.1. Filtration theory, classic PME, and beginning of amazing history: Leibenzon, Richard's, Muskat, and Zhukovskii's legacy.

The classic second-order PME is the quadratic evolution PDE of parabolic type

$$u_t = \Delta(u^2) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+ \quad (u \geq 0), \tag{1.2}$$

where Δ is the Laplace operator in $\mathbf{R}^N$. It is a single representative of the family of similar equations with parameter $n > 0$,

$$u_t = \Delta(|u|^n u) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \tag{1.3}$$

so that setting $n = 1$ yields (1.2) in the class of nonnegative solutions $u = u(x,t) \geq 0$. Notice that, for $n = 1$, precisely (1.3) yields another PDE

$$u_t = \Delta(|u|u) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \tag{1.4}$$

which coincides with the classical one (1.2) for nonnegative solutions and remains parabolic on solutions of changing sign. Equations (1.3) and (1.4) are called *signed PMEs*.

Parabolic PDE models such as (1.2), (1.3), and others in filtration theory of liquids and gases in porous media were first derived and used by Leibenzon[1] in the 1920s and 1930s [39], by Richard's (1931) [46], and Muskat (1937) [42]. On the other hand, for a long time, the PME type nonlinear operators were also attributed to combustion theory, where different types of nonlinear parabolic equations served as reaction-diffusion models; see the fundamental monograph [55].

In fact, modern filtration theory goes back to the beginning of the twentieth century, to works by N.Ye. Zhukovskii[2], 1847–1921. In his published posthumously paper [59], Zhukovskii performed a modification of Kirchoff's method in jet flows theory to solve seepage problems with free surfaces. Zhukovskii's contribution to the "theory of ground waters" is explained in Kochina's paper [36]; see also [5]. It is worth mentioning here that Kochina's earlier paper [45] in 1948 was the first one where a systematic approach to similarity solutions of nonlinear parabolic equations $u_t = (k(u)u_x)_x$ was stressed upon.

Indeed, Zhukovskii is much better known by his fundamental research in various general and applied areas of aerodynamics (Zhukovskii's airfoil, theorem, condition, force), hydrodynamics of ideal fluids (formula, hypothesis, function, method, force, condition[3], etc.; see [35, Ch. 6]), mechanisms and machines theory (Zhukovskii lever), etc.; for a full account of Zhukovskii's applied works, see Stepanov's jubelee paper [50].

It seems that it is not often widely enough remembered that earlier Zhukovskii's approach to the "strength of motion" was one of key forerunners of Lyapunov's stability theory[4]. His doctoral thesis defended in the Moscow University in 1882 was precisely called "On the Strength of Motion" (see [57]), where he developed linear theory of stability for conservative dynamical systems. In fact, Zhukovskii's original rigidity (sometimes translated as "firmness") concept is intermediate between the concepts of orbital stability and the Lyapunov's one; see recent developments, extensions, and further

[1] One of Zhukovskii's former students.

[2] Or Joukowski, according to his first publications in French.

[3] A most full name is "Chaplygin–Zhukovskii–Kutta condition".

[4] The basis of the "first Lyapunov method" of stability theory appeared in Lyapunov's "hydrodynamic" paper "On steady screw motion of a solid body in a liquid" that was published a few years later in 1888 in "Proceedings of Kharkov Mathematical Society". Lyapunov's outstanding doctoral thesis "A General Theory of the Stability of Motion" was defended in the Moscow University in 1892, with Zhukovskii being one of the Opponents (the Referees).

references in [14, 15]. Concerning another Zhukovskii's contribution to ODE theory, on his non-oscillation test of 1892 [58], see [24, p. 19].

1.2. **The heat equation is the limit case of the PME.** Indeed, the PME (1.3) is a difficult nonlinear parabolic PDE. It is crucial that it is *degenerate* at points where $u = 0$, i.e., at the *singularity level* $\{u = 0\}$. This results in a number of new key properties of solutions including *finite propagation*, which was discovered in 1950 as a paradoxical feature of filtration and heat conduction in nonlinear media. Before, we digress for a moment from the PME and turn our attention to the following its key feature:

An interesting and principal feature of the PME (1.3) is that, at $n = 0$, it becomes the linear *heat equation* (the HE) being the canonical parabolic equation,

$$u_t = \Delta u \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+. \tag{1.5}$$

The Cauchy problem for (1.5) with bounded initial data,

$$u(x,0) = u_0(x) \quad \text{in} \quad \mathbf{R}^N, \tag{1.6}$$

has the unique global solution given by the Poisson formula

$$u(x,t) = b(x-\cdot,t) * u_0(\cdot) \equiv (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, u_0(y)\,\mathrm{d}y, \tag{1.7}$$

where $b(x,t)$ denotes the fundamental solution of the parabolic operator $D_t - \Delta$,

$$b(x,t) = t^{-\frac{N}{2}} F\big(\tfrac{x}{\sqrt{t}}\big) \equiv t^{-\frac{N}{2}} (4\pi)^{-\frac{N}{2}}\ \mathrm{e}^{-\frac{|x|^2}{4t}}, \quad b(x,0) = \delta(x). \tag{1.8}$$

Here, $\delta(x)$ is Dirac's delta and, as usual, this initial condition is understood in the sense of bounded measures in $\mathbf{R}^N$.

Substituting (1.8) into (1.5) yields the following problem for the rescaled Gaussian kernel $F(y)$:

$$\mathbf{B}_0 F \equiv \Delta F + \frac{1}{2}\, y \cdot \nabla F + \frac{N}{2}\, F = 0 \quad \text{in} \quad \mathbf{R}^N, \quad \int F = 1, \tag{1.9}$$

which admits the unique solution $F(y)$ given in (1.8). Note that $\mathbf{B}_0$ appeared in (1.9) is also the classic linear *Hermite operator* of mathematical physics. Its spectral properties have been known since the nineteenth century[5]. It is symmetric in $C_0^\infty(\mathbf{R}^N)$ in the metric of the weighted space $L^2_\rho(\mathbf{R}^N)$, since

$$\mathbf{B}_0 = \frac{1}{\rho}\, \nabla \cdot (\rho \nabla) + \frac{N}{2}\, I, \quad \text{where} \quad \rho(y) = \mathrm{e}^{\frac{1}{4}\,|y|^2}, \tag{1.10}$$

[5] For instance, its spectrum and eigenfunctions in 1D were derived by Sturm in 1836 [51]; see Sturm's original computations in [24, pp. 3-4].

and admits Friedrichs' self-adjoint extension; see [9]. Moreover, $\mathbf{B}_0$ has the discrete spectrum (here $\gamma = (\gamma_1, ..., \gamma_N)$, $|\gamma| = \gamma_1 + ... + \gamma_N$, stands for a multiindex)

$$\mathbf{B}_0\psi = \lambda\psi, \;\; \psi \in L^2_\rho \Longrightarrow \sigma(\mathbf{B}_0) = \big\{\lambda_\gamma = -\frac{|\gamma|}{2}, \; l = |\gamma| = 0, 1, 2, ...\big\}. \quad (1.11)$$

Eigenvalues have finite multiplicity and the corresponding eigenfunctions are given by separable *Hermite polynomials* $H_\gamma(y) = H_{\gamma_1}(y_1)...H_{\gamma_N}(y_N)$ in $\mathbf{R}^N$, which are obtained via differentiating the Gaussian kernel in (1.8) (see precise statements and properties in [9, p. 48])

$$\psi_\gamma(y) = c_\gamma D^\gamma F(y) \equiv c_\gamma H_\gamma(y) F(y), \quad c_\gamma = (2^{|\gamma|}\gamma!)^{-\frac{1}{2}} \quad \big(D^\gamma = D^{\gamma_1}_{y_1}...D^{\gamma_N}_{y_N}\big). \quad (1.12)$$

The eigenfunction set is complete and closed in $L^2_\rho(\mathbf{R}^N)$. These are classical facts from linear self-adjoint operator theory [9].

The PME (1.3) can be considered as a *nonlinear extension* of the classic heat equation (1.5), which is obtained from the former in the limit $n \to 0$. This limit has been carefully studied in the literature (see references below on PME theory). We then say that, in the sense of convergence of wide classes of solutions, these equations can be continuously (homotopically) deformed to each other with respect to the parameter n, i.e., formally,

$$\boxed{(\text{PME}) \to (\text{HE}) \quad \text{as} \quad n \to 0^+.} \quad (1.13)$$

In other words, the family of equations (1.3) with $n \in [0, 1]$ represents a *homotopic path* connecting the linear heat equation (1.5) with the quadratic PME (1.4). These PDEs are also said to belong to same homotopic class (if they can be continuously deformed, in the sense of their solutions, to each other). These notations around homotopic concepts are introduced for convenience and are not directly associated with classic homotopy approaches in nonlinear operator theory; see [38, Ch. 2].

Actually, the remarkable limit (1.13), though being known for a long time, was not somehow essentially used in existence, uniqueness, or asymptotic theory of PME-type equations. Mathematical study of PMEs embracing a number of fundamental ideas and discoveries that founded the basement of more general theory of nonlinear degenerate PDEs was systematically developed since the 1950s; see e.g., monographs [13, 22, 53, 32] for history, key references, and main results.

1.3. **ZKB and BZ similarity solutions of the PME in the 1950s: first nonlinear eigenfunctions.** ZKB, SOURCE-TYPE SOLUTION AND FIRST n-BRANCH. Let us return to the cornerstone of PME theory: the discovery by

Zel'dovich, Kompaneetz, and Barenblatt in 1950–1952 [56, 2] the *source-type* or, nowadays, *ZKB-solution* of (1.3), which is again denoted by $b(x,t)$,

$$u_0(x,t) \equiv b(x,t) = t^{-\alpha} f_0(y), \quad y = x/t^{\beta}, \tag{1.14}$$

where $\beta = \frac{1-\alpha n}{2}$, $\alpha = \alpha_0 = \frac{N}{2+nN}$. The rescaled kernel $f_0(y)$ is a solution of the following *nonlinear elliptic eigenvalue problem* (in what follows, we will always deal with continuous compactly supported weak solutions $f \in C_0(\mathbf{R}^N)$):

$$\mathbf{B}_n(f) \equiv \Delta(|f|^n f) + \frac{1-\alpha n}{2} y \cdot \nabla f + \alpha f = 0 \ \text{ in } \ \mathbf{R}^N, \quad f \neq 0, \ \ f \in C_0(\mathbf{R}^N). \tag{1.15}$$

Indeed, for the first eigenvalue $\alpha = \alpha_0$ as in (1.14), this gives the unique (up to scaling) explicit solution

$$f_0(y) = \big[A_0\big(a_0^2 - |y|^2\big)_+\big]^{\frac{1}{n}}, \tag{1.16}$$

with the constant $A_0 = \frac{n}{2(n+1)(2+nN)}$, where $(\cdot)_+$ denotes the positive part $\max\{(\cdot), 0\}$. The constant $a_0 > 0$ is arbitrary and characterizes the preserved total mass of the solution. If we want $u_0(x,t) = b(x,t)$ to initially take Dirac's delta as in (1.8), i.e.,

$$\int f_0 = 1,$$

the following holds (see e.g., [48, p. 21]):

$$1 = \int_{\mathbf{R}^N} f_0(y) \mathrm{d}y \equiv N\omega_N \int_0^{a_0} z^{N-1} \big[A_0(a_0^2 - z^2)\big]^{\frac{1}{n}} \mathrm{d}z \tag{1.17}$$
$$\Longrightarrow a_0^{\frac{2}{n}+N}(n) = \pi^{-\frac{N}{2}} A_0^{-\frac{1}{n}} \frac{\Gamma\big(\frac{n+1}{n} + \frac{N}{2}\big)}{\Gamma\big(\frac{n+1}{n}\big)},$$

where Γ is Euler's Gamma function, and $\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ denotes the volume of the unit ball in $\mathbf{R}^N$. We should remember that the explicit integration of (1.15) to get (1.16) is associated with the *first conservation law* for the PDE (1.3): for initial data $u_0 \in L^1(\mathbf{R}^N)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u(x,t)\,\mathrm{d}x = 0 \Longrightarrow \int u(x,t)\,\mathrm{d}x = \int u_0(x)\,\mathrm{d}x \quad \text{for} \quad t \geq 0. \tag{1.18}$$

Returning to the rescaled fundamental profile (1.16), it follows that, unlike (1.8) for the heat equation, $b(x,t)$ is compactly supported in x for any $t > 0$. This is a striking property of *finite propagation* for the quasilinear degenerate parabolic equation (1.3). At the free-boundary (interface), where $|y| = a_0$, the profile $f_0(y)$ has finite regularity, and $f_0^n(y)$ is Lipschitz continuous.

Comparing formulae (1.8) and (1.16), it seems that these fundamental solutions of the HE and the PME correspond to entirely different functional settings. Nevertheless, (1.13) takes place and this striking continuity with respect to the exponent $n \geq 0$ is not difficult to detect by passing to the limit as $n \to 0^+$ in (1.16). Using that, in (1.17), the ratio of Gamma functions satisfies as $n \to 0$,

$$\frac{\Gamma\big(\frac{n+1}{n} + \frac{N}{2}\big)}{\Gamma\big(\frac{n+1}{n}\big)} = \big(\frac{n+1}{n}\big)^{\frac{N}{2}} + \cdots ,$$

one can see that the nonlinear kernel (1.16) converges to the linear Gaussian one (1.8), and, uniformly in y,

$$f_0(y) = \big[A_0\big(a_0^2(n) - |y|^2\big)_+\big]^{\frac{1}{n}} \to \big(4\pi\big)^{-\frac{N}{2}} \, \mathrm{e}^{-\frac{|y|^2}{4}} \quad \text{as} \quad n \to 0^+. \tag{1.19}$$

This implies a remarkable conclusion that the first *nonlinear eigenfunction* $f_0(y)$ of the *nonlinear eigenvalues problem* (1.15) with the first eigenvalue

$$\alpha_0 = \tfrac{N}{2+nN} \tag{1.20}$$

is obtained as $n \to 0^+$ from the rescaled Gaussian kernel $F(y)$ in (1.8), which is the first eigenfunction with $\lambda_0 = 0$ of the linear eigenvalue problem in (1.11). Here, we observe a continuous curve of fundamental rescaled kernels called the *n-branch* in nonlinear operator theory [38, p. 374], which is originated from the linear counterpart at the *branching point* $n = 0$. Thus, for the PME, there exists

$$\text{A continuous } n\text{-branch of eigenfunctions } f_0 = f_0(y; n) \text{ for all } n \geq 0. \tag{1.21}$$

Of course, this is just a straightforward consequence of some simple manipulations with explicitly given linear and nonlinear kernels, but we should remember it for further use in more difficult problems, where the mathematics is not easy and explicit.

Dipole BZ solution: the second explicit n-branch. It is convenient to continue our description of further similarity solutions for the PME in one-dimension,

$$u_t = \big(|u|^n u\big)_{xx} \quad \text{in} \quad \mathbf{R} \times \mathbf{R}_+. \tag{1.22}$$

It admits explicit *Barenblatt-Zel'dovich* (BZ) *dipole solution* constructed in 1957 [3],

$$u_1(x,t) = t^{-\frac{1}{n+1}} f_1(y), \quad y = x/t^{\frac{1}{2(n+1)}}, \tag{1.23}$$

where $f_1(y)$ is the second nonlinear eigenfunction of the problem (1.15) in $\mathbf{R}$ with the eigenvalue

$$\alpha_1 = \tfrac{1}{n+1} \quad (\alpha = 2\beta). \tag{1.24}$$

Then the ODE (1.15) is also explicitly integrated to give

$$f_1(y) = |y|^{\frac{1}{n+1}} \big[B_0 \big(b_0^{\frac{n+2}{n+1}} - |y|^{\frac{n+2}{n+1}} \big)_+ \big]^{\frac{1}{n}} \operatorname{sign} y. \tag{1.25}$$

The constant $b_0 = b_0(n) > 0$ can be chosen so that, in the sense of distributions,

$$u_1(x, 0) = \delta'(x), \tag{1.26}$$

and this justifies the term *dipole* applied to the solution (1.23). In addition, this is associated with the *second conservation law* of the first moment of solutions of the PME (1.22): for data $xu_0(x) \in L^1(\mathbf{R})$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int xu(x,t)\mathrm{d}x = 0 \Longrightarrow \int xu(x,t)\mathrm{d}x = \int xu_0(x)\mathrm{d}x \quad \text{for} \quad t \ge 0, \tag{1.27}$$

where, for the odd function as in (1.23), (1.25), $\int xu_0(x) > 0$.

In $\mathbf{R}^N$, i.e., for the PME (1.3), the conservation law (1.27) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int x_1 u(x,t)\mathrm{d}x = 0 \Longrightarrow \int x_1 u(x,t)\,\mathrm{d}x = \int x_1 u_0(x)\mathrm{d}x \quad \text{for} \quad t \ge 0, \tag{1.28}$$

which deals with the x_1-moment of the solution. Taking the corresponding similarity solution in the same form (1.14), i.e.,

$$u_1(x,t) = t^{-\alpha} f_1(y), \quad y = x/t^{\beta}, \quad \beta = \tfrac{1-\alpha n}{2}, \tag{1.29}$$

we find from (1.28) that

$$\int x_1 u_1(x,t)\,\mathrm{d}x = t^{-\alpha + \beta(N+1)} \int y_1 f_1(y)\,\mathrm{d}y,$$

so that this gives the second nonlinear eigenvalue

$$-\alpha + \beta(N+1) = 0 \text{ or } \alpha = \tfrac{1-\alpha n}{2}(N+1) \Longrightarrow \alpha_1 = \tfrac{N+1}{2+n(N+1)}. \tag{1.30}$$

The corresponding elliptic equation (1.15) with the eigenvalue (1.30) is known to admit a solution $f_1(y)$ [34], which is still a rare example of non-radially symmetric nonlinear eigenfunctions of the present eigenvalue problem. Thus, (1.29), (1.30) represent

$$\text{The second } n\text{-branch of dipole eigenfunctions } f_1(y;n) \text{ for all } n \ge 0, \tag{1.31}$$

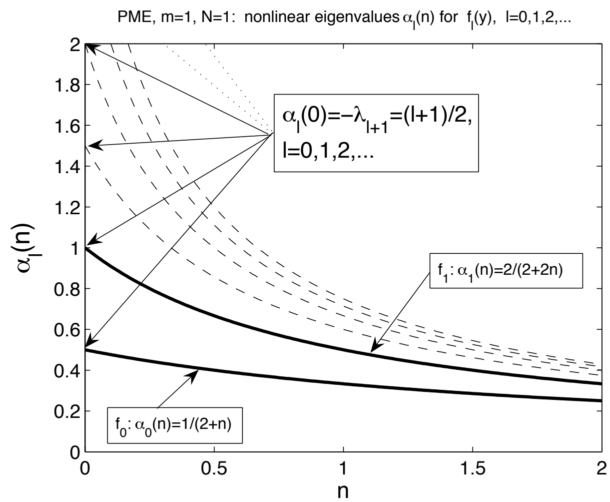


FIGURE 1. n-branches of eigenfunctions of (1.15) for $N = 1$.

which at $n = 0$ is originated from the eigenfunction (1.12) with the multiindex $\gamma^1 = (1, 0, ..., 0)$, i.e.,

$$\psi_{\gamma^1}(y) = \tfrac{1}{2\sqrt{2}}\,(4\pi)^{-\frac{N}{2}}\,y_1\,\mathrm{e}^{-\frac{|y|^2}{4}}. \tag{1.32}$$

In Figure 1, we show two explicit n-branches (1.20) and (1.24) for $N = 1$ corresponding to eigenfunctions f_0 and f_1. Other branches denoted by dashed lines are not explicit but definitely exist as the results of [33] and [27] suggest; see below.

1.4. **On a countable set of nonlinear eigenfunctions.** The explicit nonlinear eigenfunctions (1.16) and (1.25) of the PME in 1D are special ones among many non-explicit others. It has been proved that the third function $f_2(y)$ exists and has anomalous exponents (see more details and applications in [7]), i.e., the corresponding eigenvalue $\alpha = \alpha_2$ in (1.15) corresponds to *self-similarity of the second kind.* This term, unlike the standard dimensional (via invariants of a known group) self-similar solutions of the first kind, was introduced by Zel'dovich in 1956 [54].

It turns out that the PME in 1D or in the radial geometry in $\mathbf{R}^N$ admits a countable subset of different similarity solutions

$$u_l(x,t) = t^{-\alpha_l} f_l(x/t^{\beta_l}), \quad \beta_l = \tfrac{1-\alpha_l n}{2}, \quad l = 0, 1, 2, 3, \ldots . \tag{1.33}$$

The proof in [33] uses a kind of phase-plane analysis for a second-order ODE for $f(y)$ (it can be reduced to a first-order equation), where each new pattern f_l appears as the result of occurring an extra rotation of the vector field by changing the parameter α from α_l to α_{l+1}. As a consequence, which is also guaranteed by the Maximum Principle for second-order ODEs, the set of nonlinear eigenfunctions of the problem (1.15)

$$\Phi = \{f_l,\ l \geq 0\} \quad (N = 1) \tag{1.34}$$

is ordered by Sturm's zero set property:

$$\text{Each } f_l(y) \text{ has precisely } l+1 \text{ local extrema and } l \text{ sign changes.} \tag{1.35}$$

Infinite countable and even uncountable sets of similarity profiles are typical in blow-up problems and were detected earlier [28] in reaction-diffusion problem with the gradient diffusivity

$$u_t = \nabla \cdot (|\nabla u|^n \nabla u) + u^p \quad (n > 0,\ p > n+1); \tag{1.36}$$

see [11] for further related references. Notice that the pure *p-Laplacian equation*, with $p = n+2$, occurring in the theory of turbulent diffusion and non-Newtonian, dilatable, or pseudo-plastic fluids (see details in [44]),

$$u_t = \nabla \cdot (|\nabla u|^n \nabla u) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \tag{1.37}$$

also has interesting history of similarity analysis and admits an analogous classification of nonlinear eigenfunctions as the PME (1.3), which our study is concentrated upon.

1.5. **On construction of non-radial nonlinear eigenfunctions via MP.** As we have mentioned, [33] established existence of an infinite countable set of *radially symmetric* nonlinear eigenfunction of the problem (1.15). On the other hand, the homotopy concept (1.13) suggests that there exist many non-radial eigenfunctions, since this is true for $n = 0$, where all the linear eigenfunctions are given explicitly by (1.12). The corresponding branching phenomena at $n = 0$ for (1.15) and related possible geometries of nonlinear eigenfunctions for small $n > 0$ are discussed in [27, Append. A], which we are going to use in a brief account of these ideas.

Without loss of generality, we fix $N = 2$, so that $y = (y_1, y_2)$. We introduce the polar coordinates (r, σ). As in [27, § A.4], in order to construct the pattern $f_l(y)$ for any $l \geq 1$, we consider equation (1.15) in the unbounded sector

$$\Psi_l = \left\{0 < \sigma < \tfrac{\pi}{l}\right\} \tag{1.38}$$

with the Dirichlet boundary conditions on the lateral boundary. If a nontrivial solution $f_l(y) \geq 0$ in Ψ_l, $f_l(y) = 0$ for all $|y| \gg 1$, exists for some eigenvalue $\alpha = \alpha_l > 0$, this allows to reflect $f_l \mapsto -f_l$ relative to all lateral boundaries (rays) to get a weak solution in $\mathbf{R}^2$, as Figure 6 in [27] explains. This construction gives an eigenfunction $f_l(y)$ that has l "sign changes" (a characteristic of Sturm–Morse index of the solution) along those rays.

Concerning existence of the necessary eigenvalue $\alpha_l > 0$, for the second-order elliptic equation (1.15), we can use the Maximal Principle arguments, where we consider the corresponding parabolic flow

$$v_\tau = \mathbf{B}_n(v) \quad \text{in} \quad \Psi_l \times \mathbf{R}_+, \tag{1.39}$$

with some small compactly supported initial data $v_0(y) \geq 0$. In view of the Maximum Principle and concepts of upper and lower solutions for second-order parabolic equations, equation (1.39) represents a partially gradient flow at least for monotone in time solutions satisfying $v_\tau \geq 0$ or ≤ 0 (this happens if initial data represent a lower or upper weak solution). This class is enough to study stabilization to equilibria as $\tau \to +\infty$; see such arguments in [32, p. 116].

We next use the standard argument of continuation in parameter $\alpha \in \big(0, \frac{1}{n}\big)$:

(i) For $\alpha = \frac{1}{n}$, we obtain the well-known gradient flow (cf. [1])

$$v_\tau = \Delta(|f|^n f) + \tfrac{1}{n}\, f, \tag{1.40}$$

which guarantees stabilization to the unique *positive unbounded* equilibrium,

$$v(y,\tau) \to \bar f(y) > 0 \quad \text{as} \quad \tau \to +\infty \tag{1.41}$$

uniformly on compact subsets from Ψ_l, where

$$\bar f(y) \to +\infty \quad \text{as} \quad y \to \infty. \tag{1.42}$$

(ii) On the other hand, for $\alpha = 0$, the corresponding right-hand side of (1.39)

$$v_t = \Delta(|v|^n v) + \tfrac{1}{2}\, y \cdot \nabla v \tag{1.43}$$

is composed from two negative operators (check multiplying by v in L^2), so that

$$v(y,\tau) \to 0 \quad \text{as} \quad \tau \to +\infty. \tag{1.44}$$

It follows from typical continuous dependence on parameters of sufficiently regular parabolic flows, in between the unbounded flow satisfying (1.41), (1.42) for $\alpha = \frac{1}{n}$ and the trivial one (1.44) for $\alpha = 0$, there must exist the case for some $\alpha_l \in \big(0, \frac{1}{n}\big)$ such that $v(y,\tau)$ stabilizes to a nontrivial stationary compactly supported solution $f_l(y)$. Of course, to complete this analysis, extra work and additional estimates of flows are necessary, but anyway it is well understood that these arguments based on the MP and related gradient properties of such parabolic flows allow one to complete this study of order-preserving cases. In this connection, we refer to the pioneering results in [43], where some crucial solutions of parabolic equations in the supercritical Sobolev range were first detected by analogous arguments.

Note that sign changing solution of (1.15) in the sector (1.38), being extended by anti-symmetric reflections to the whole plane $\mathbf{R}^2$, will give further nonlinear eigenfunctions, which exhibit a discrete symmetry relative to the angle σ, and, possibly, no further symmetries. We do not know how to classify all such eigenfunctions. Recall that all of them have a linear counterpart at $n = 0$ among linear eigenfunctions from set (1.12), and this leads to a difficult *algebraic branching system*, [27, § A.1].

1.6. **Open problems for the PME:**
Nonlinear eigenfunctions and their evolution completeness. Some fundamental problems for the second-order PME (1.3) remain open still. We stress our attention to the following two associated with the nonlinear eigenvalue problem (1.15), which we have discussed already.

Conjecture 1.1. *For any $n > 0$, up to invariant scalings and orthogonal transformations,* (1.15) *for any $n > 0$ has not more than a countable set of nonlinear eigenfunctions*

$$\Phi = \{f_\gamma(y),\ |\gamma| = l \ge 0\}, \tag{1.45}$$

with the monotone increasing sequence of eigenvalues $\{\alpha_l\}$ satisfying

$$\alpha_l \uparrow \tfrac{1}{n} \quad \text{as} \quad l \to \infty. \tag{1.46}$$

Let us mention again that this result is known to hold in 1D and in the radial geometry in $\mathbf{R}^N$, and was proved by Hulshof [33]. As we have seen above, some (but not all) nonlinear eigenfunctions can be determined by using discrete symmetries of equation (1.15) by using the MP ingredients of parabolic theory; [27].

It is curious that in the linear case $n = 0$, when (1.15) is sufficiently "continuously" deformed as $n \to 0$ into the linear eigenvalue problem (1.11) for the self-adjoint operator (1.10), Conjecture 1.1 is obviously true. Even the limit (1.46) is included since, obviously, setting $n = 0$ in (1.15) yields

$$\alpha_\gamma(0) = -\lambda_\gamma + \tfrac{N}{2} = \tfrac{l+N}{2} \to +\infty \quad \text{as} \quad l = |\gamma| \to \infty. \tag{1.47}$$

The next open problem concerning *evolution completeness* (see precise formulation in [23, 27]) of the eigenfunction set (1.45) is also trivial in the linear case; see explanations immediately after the statement.

Conjecture 1.2. *For any fixed $n > 0$ and any initial data $u_0 \in C_0(\mathbf{R}^N)$, $u_0 \ne 0$, there exists a finite $l \ge 0$ and a nonlinear eigenfunction f_γ with*

$|\gamma| = l$ *from* (1.45) *such that after necessary scalings and orthogonal transformations, the solution* $u(x,t)$ *of the Cauchy problem for the PME* (1.3), (1.6) *satisfies*

$$u(x,t) = t^{-\alpha_l}\big[f_\gamma(x/t^{\beta_l}) + o(1)\big] \quad \textit{as} \quad t \to \infty \quad \big(\beta_l = \tfrac{1-\alpha_l n}{2}\big) \tag{1.48}$$

uniformly on compact subsets in $\mathbf{R}^N$.

The result for $n = 0$ in L^2_ρ is obvious in view of the closure of the eigenfunctions (1.12). Indeed, performing first the fundamental scaling in the heat equation (1.5),

$$u(x,t) = (1+t)^{-\frac{N}{2}} v(y,\tau), \quad y = x/(1+t)^{\frac{1}{2}}, \quad \tau = \ln(1+t), \tag{1.49}$$

we obtain the problem with the linear operator (1.9),

$$v_\tau = \mathbf{B}_0 v \quad \text{for} \quad \tau > 0, \quad v_0 = u_0. \tag{1.50}$$

Therefore, taking arbitrary initial data

$$v_0(y) = \sum_{(\gamma)} c_\gamma \psi_\gamma(y), \tag{1.51}$$

we obtain the unique solution via converging in the mean series

$$v(y,\tau) = \sum_{(\gamma)} c_\gamma \mathrm{e}^{\lambda_\gamma \tau} \psi_\gamma(y). \tag{1.52}$$

This gives all the possible asymptotic patterns that are available in the problem (1.50),

$$v_\gamma(y,\tau) \sim \mathrm{e}^{\lambda_\gamma \tau} \psi_\gamma(y), \tag{1.53}$$

where $l = |\gamma| \ge 0$ is arbitrary and always *finite*. Indeed, $l = \infty$ in (1.53) would mean by (1.51) that $v(y,\tau) \equiv 0$.

In other words, in typical linear problems, the evolution completeness follows from the completeness-closure of eigenfunction subsets, which is a classic problem of linear operator theory, [9].

For nonlinear PDEs, evolution completeness of eigenfunction subsets becomes an independent important problem, while standard completeness and closure notions often make no sense, as having the linear origin. There are some special examples, including PME-type equations, where the evolution completeness of nonlinear eigenfunctions is rigorously proved; see [23].

It follows that the homotopic connection (1.13) could give an extra flavour to Conjectures 1.1 and 1.2, bearing in mind that both open problems are true for $n = 0$. We mean that existence of a countable set of nonlinear eigenfunctions for $n > 0$ can be connected with the same fact at $n = 0$ if continuous n-branches (cf. (1.21)) of eigenfunctions are available. This branching phenomenon was discussed in [27, Append. A], where questions of evolution completeness for the PME are also addressed in some particular geometries.

Meantime, we proceed to more difficult equation of the PME type, where these ideas of n-branching of nonlinear eigenfunctions and their homotopic connections with linear PDEs and eigenvalue problems become not only crucial, but seems the only available at hands.

2. Fourth-order PME-type equation in 1D: Preliminaries and formulation of similarity solutions

2.1. **PME-4.** We now begin to deal with the PME−4 (1.1). It is key that for $n = 0$, this reduces to the classic *bi-harmonic equation*

$$u_t = -u_{xxxx}, \tag{2.1}$$

properties of which we will describe in Section 3.

We consider for (1.1) the Cauchy problem with bounded compactly supported data

$$u(x,0) = u_0(x) \quad \text{in} \quad \mathbf{R}. \tag{2.2}$$

The PDE (1.1) contains a fully divergent, monotone, and potential differential operator. Moreover, (1.1) is a gradient system and admits strong estimates via multiplication by $(|u|^n u)_t$ in L^2. The PME operator is potential in the metric of H^{-2} and is also monotone, so local existence of a unique continuous solution follows from classic theory of monotone operators; see [40, Ch. 2]. Finite propagation in the PDE (1.1) is proved by energy estimates via Saint–Venant's principle; see [49] and a survey in [30].

It is curious that though the principal fact on the oscillatory behaviour of solutions close to interfaces of (1.1) was rigorously established (together with existence and uniqueness of the ZKB source-type and some other similarity solutions; see below) twenty years ago, [6, 8], a detailed generic structure of such oscillations for (1.1) was not under scrutiny until now. We address this curious behaviour in Section 4.

2.2. **Blow-up similarity solutions:**
Problem setting and preliminaries. Similar to (1.14), the parabolic PDE (1.1) possesses the following similarity solutions:

$$u_S(x,t) \equiv b(x,t) = t^{-\alpha} f(y), \quad y = x/t^{\beta}, \quad \text{where } \beta = \tfrac{1-\alpha n}{4}, \tag{2.3}$$

where the similarity kernel $f(y)$ satisfies the *nonlinear eigenvalue problem*

$$\mathbf{B}_n(f) \equiv -(|f|^n f)^{(4)} + \tfrac{1-\alpha n}{4}\, y\, f' + \alpha f = 0 \quad \text{in} \quad \mathbf{R}, \quad f \neq 0, \;\; f \in C_0(\mathbf{R}). \tag{2.4}$$

Collecting all terms with the eigenvalue α on the right-hand side yields

$$\mathbf{B}_n^1(f) \equiv -(|f|^n f)^{(4)} + \tfrac{1}{4}\, y\, f' = \alpha\big(\tfrac{n}{4}\, yf' - f\big) \equiv \alpha \mathcal{L}_n f. \qquad (2.5)$$

Formally, (2.5) can be treated as an eigenvalue problem for a *linear pencil* of two, nonlinear $\mathbf{B}_n^1$ and linear $\mathcal{L}_n$, ordinary differential operators. Notice that both operators are neither divergent (symmetric) nor variational. In a whole, the eigenvalue problem (2.5) does not admit a standard variational formulation. For $n = 0$, (2.4) becomes a standard linear eigenvalue problem, but for a non self-adjoint operator; see details below.

Note that (2.4) is invariant under the group of scaling transformations

$$f \mapsto \varepsilon^{\frac{4}{n}} f, \quad y \mapsto \varepsilon\, y \quad (\varepsilon > 0), \qquad (2.6)$$

so that, for a unique representation of necessary solutions, one needs an additional normalization. As in [8], for even solutions $f_l(y)$ with even $l = 0, 2, ...,$ it is convenient to impose the following normalization and the symmetry conditions at the origin $y = 0$:

$$f(0) = 1, \quad \text{and} \quad f'(0) = 0, \;\; f'''(0) = 0 \quad (\text{if } \; f(0) \neq 0), \qquad (2.7)$$

while for odd solutions with $l = 1, 3, ...,$ we set

$$f'(0) = 1, \quad \text{and} \quad f(0) = 0, \;\; f''(0) = 0. \qquad (2.8)$$

Note that the nonlinear eigenvalue problem (2.5) is formulated for a degenerate nonlinear fourth-order operator, and hence is much more difficult than that for the 1D second-order one (1.15). The later one reduces to a first-order ODE that can be studied on the phase-plane, [33]; see also another example in [48, p. 136]. For (2.5), any simple geometric approach is not possible, and the shooting problem is always at least 3D; cf. (4.8). For the PME-6 (7.1), the parametric space becomes 4D. Therefore, we cannot in principle rely on geometric ideas coming from the second-order case, which also obeys the Maximum Principle and various comparison-barrier features that fail for higher-order operators. In fact, we need to develop a new unified approach to such nonlinear eigenvalue problems that, in general, can be applied to any PME$-2m$ such as

$$u_t = (-1)^{m+1} D_x^{2m}(|u|^n u) \quad \text{in} \quad \mathbf{R} \times \mathbf{R}_+ \quad (m = 2, 3, 4, ...). \qquad (2.9)$$

3. Fundamental solution and spectral properties of $\mathbf{B}_0$ for $n = 0$

We begin with description of some auxiliary properties that occur in the case $n = 0$ where (2.5) leads to linear differential operators and a standard eigenvalue problem. These properties will be taken into account for the sake of extensions for sufficiently small $n > 0$.

Consider the bi-harmonic equation (2.1). The corresponding *fundamental solution* of the operator $D_t + D_y^4$ has the similarity form

$$b(x,t) = t^{-\frac{1}{4}} F(y), \quad y = x/t^{\frac{1}{4}}, \tag{3.1}$$

where the rescaled kernel F is the unique solution of the ODE

$$\mathbf{B}_0 F \equiv -F^{(4)} + \frac{1}{4}\, yF' + \frac{1}{4} F = 0 \quad \text{in } \mathbf{R}, \quad \text{with} \quad \int F = 1. \tag{3.2}$$

On integration once, we obtain a third-order equation,

$$-F''' + \tfrac{1}{4}\, yF = 0 \quad \text{in } \mathbf{R}. \tag{3.3}$$

The kernel $F = F(|y|)$ is radial, has exponential decay, oscillates as $|y| \to \infty$, and, for some positive constant D and $d = 3 \cdot 2^{-11/3}$ (see [17, p. 46]),

$$|F(y)| \le D\, \mathrm{e}^{-d|y|^{4/3}} \quad \text{in } \mathbf{R}. \tag{3.4}$$

It is key that setting $n = 0$ in (2.4) yields the standard eigenvalue problem for $\mathbf{B}_0$,

$$\mathbf{B}_0 \psi = \lambda \psi, \quad \text{where} \quad \lambda = \tfrac{1}{4} - \alpha. \tag{3.5}$$

The necessary spectral properties of the linear non self-adjoint operator $\mathbf{B}_0$ and the corresponding adjoint operator $\mathbf{B}_0^*$ are of importance in the asymptotic analysis and are explained in [16] for similar $2m$th-order operators (see also [19, § 4]). In particular, $\mathbf{B}_0$ is naturally defined in a weighted space $L^2_\rho(\mathbf{R})$, with the domain $H^4_\rho(\mathbf{R})$, where $\rho(y) = \mathrm{e}^{a|y|^{4/3}}$, where $a \in (0, 2d)$ is a constant, and has the discrete (point) spectrum

$$\sigma(\mathbf{B}_0) = \big\{\lambda_l = -\tfrac{l}{4},\ l = 0, 1, 2, ...\big\}. \tag{3.6}$$

The corresponding eigenfunctions are normalized derivatives of the rescaled kernel,

$$\psi_l(y) = \frac{(-1)^l}{\sqrt{l!}} F^{(l)}(y), \quad l = 0, 1, 2, ... \tag{3.7}$$

The adjoint operator

$$\mathbf{B}_0^* = -D_y^4 - \tfrac{1}{4}\, y D_y \tag{3.8}$$

has the same spectrum (3.6) and polynomial eigenfunctions

$$\psi_l^*(y) = \frac{1}{\sqrt{l!}} \sum_{j=0}^{\lfloor \frac{l}{4} \rfloor} \frac{1}{j!} D_y^{4j} y^l, \quad l = 0, 1, 2, ... , \tag{3.9}$$

which form a complete subset in $L^2_{\rho^*}(\mathbf{R})$, where $\rho^* = \frac{1}{\rho}$. In particular,

$$\psi_0^* = 1,\ \psi_1^* = y,\ \psi_2^* = \tfrac{1}{\sqrt{2}}\, y^2,\ \psi_3^* = \tfrac{1}{\sqrt{6}}\, y^3,\ \psi_4^* = \tfrac{1}{\sqrt{24}}\, (y^4 + 24), \tag{3.10}$$

etc. As $\mathbf{B}_0$, the adjoint operator $\mathbf{B}_0^*$ has the compact resolvent $(\mathbf{B}_0^* - \lambda I)^{-1}$. It is not difficult to see that, integrating by parts, the eigenfunctions (3.7) are orthonormal to polynomial eigenfunctions $\{\psi_l^*\}$ of the adjoint operator $\mathbf{B}_0^*$, so

$$\langle \psi_l, \psi_k^* \rangle = \delta_{lk}, \tag{3.11}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard (being dual here) scalar product in $L^2(\mathbf{R})$.

4. LOCAL PROPERTIES OF OSCILLATORY SIMILARITY PATTERNS NEAR INTERFACES

In this section, we describe the generic oscillatory behaviour of solutions of (2.4) close to finite interfaces. According to pioneering results in [6, 8], ODEs like (2.4) admit compactly supported solutions of changing sign near finite interfaces. Let $f(y)$ vanish at the interface $y \to y_0 > 0$, so that $f(y) \equiv 0$ for $y > y_0$. Then making for convenience the reflection $y \mapsto y_0 - y$, with $y > 0$ small enough, and keeping for $y \approx 0^+$ the leading first two terms in (2.4), after integration once we obtain an exponentially small perturbation of the following third-order equation:

$$(|f|^n f)''' = -\beta y_0 f \quad (\beta > 0).$$

We scale out the positive constant βy_0 to get the ODE

$$(|f|^n f)''' = -f \quad \text{for} \quad y > 0, \quad f(0) = 0. \tag{4.1}$$

Next, it is convenient to use the natural change

$$F = |f|^n f \quad \Longrightarrow \quad F''' = -\big|F\big|^{-\frac{n}{n+1}} F. \tag{4.2}$$

We next describe oscillatory solution of changing sign of the ODE (4.2), with zeros concentrating at the given interface point $y = 0^+$. Let us mention again that oscillatory properties of solutions are a common feature of higher-order degenerate ODEs. We refer to first results in [6, 8], to [20, 21] (thin film equations), and to [31, Ch. 3-5], where further examples can be found.

To this end, by the scaling invariance of (4.2), we look for its solutions of the form

$$F(y) = y^\mu \varphi(s), \quad s = \ln y, \quad \text{where } \mu = \tfrac{3(n+1)}{n} > 3 \text{ for } n > 0, \tag{4.3}$$

where $\varphi(s)$ is called the *oscillatory component*. Substituting (4.3) into (4.1) yields the following third-order equation for $\varphi(s)$:

$$P_3(\varphi) = -|\varphi|^{-\frac{n}{n+1}} \varphi, \tag{4.4}$$

where P_k denote linear differential polynomials obtained by a simple recursion procedure (see [31, p. 140]), so that

$$P_1(\varphi) = \varphi' + \mu\varphi, \quad P_2(\varphi) = \varphi'' + (2\mu - 1)\varphi' + \mu(\mu - 1)\varphi,$$
$$P_3(\varphi) = \varphi''' + 3(\mu - 1)\varphi'' + (3\mu^2 - 6\mu + 2)\varphi' + \mu(\mu - 1)(\mu - 2)\varphi.$$

According to (4.3), we are interested in uniformly bounded global solutions $\varphi(s)$ that are well defined as $s = \ln y \to -\infty$, i.e., as $y \to 0^+$. The best candidates for such global orbits of (4.4) are periodic solutions $\varphi_*(s)$ that are defined for all $s \in \mathbf{R}$. Indeed, they can describe suitable (and, possibly, generic) connections with the interface at $s = -\infty$.

Proposition 4.1. *For any $n > 0$, (4.4) has a periodic solution $\varphi_*(s)$ of changing sign.*

Proof. Existence of a nontrivial sign-changing periodic solution $\varphi_*(s)$ of (4.4) is proved by shooting argument as explained in [20, p. 292]; local transversal zeros of solutions $\varphi(s)$ are discussed on p. 291 therein, which gives local continuation of solutions through weakly singular points at $\varphi = 0$. The crucial part is played by the fact that equation (4.4) does not admit unbounded orbits as $s \to +\infty$. Indeed, linearizing the semilinear operator (4.4) at infinity following [38, p. 54] yields the linear operator and the polynomial

$$\begin{aligned} \mathbf{P}_n &= D_s^3 + 3(\mu - 1)D_s^2 + (3\mu^2 - 6\mu + 2)D_s + \mu(\mu - 1)(\mu - 2)I, \\ P_n(\lambda) &= \lambda^3 + 3(\mu - 1)\lambda^2 + (3\mu^2 - 6\mu + 2)\lambda + \mu(\mu - 1)(\mu - 2). \end{aligned} \tag{4.5}$$

One can see that

$$P_n(\lambda) = P_\infty(\lambda + \mu - 3), \quad \text{where} \quad P_\infty(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda + 3). \tag{4.6}$$

By (4.5) and (4.6), there exist three negative roots of the characteristic equation,

$$P_n(\lambda) = 0 \quad \Longrightarrow \quad \lambda_1 = 2 - \mu, \quad \lambda_2 = 1 - \mu, \quad \lambda_3 = -\mu, \tag{4.7}$$

so there are no growing without bounds orbits and infinity cannot attract orbits. This gives a 3D set of bounded orbits that is enough to generate a periodic motion inside and is crucial to complete the proof of (i) as in [20, § 7.1]. □

Two problems remain open:
(i) uniqueness of the periodic solution $\varphi_*(s)$, and
(ii) stability (hyperbolicity) of $\varphi_*(s)$ as $s \to +\infty$.

Numerically, we have obtained positive answers to both questions. In particular, (i) and (ii) imply that there exists a unique (up to translation) periodic bounded connection with $s = -\infty$, where the interface is situated.

The convergence to the unique stable periodic solution of (4.4) is shown in Figure 2 for various $n > 0$. Different curves therein correspond to different Cauchy data $\varphi(0)$, $\varphi'(0)$, $\varphi''(0)$ prescribed at $s = 0$. For n smaller than $\frac{3}{4}$, the oscillatory component gets extremely small, so an extra scaling is necessary, which is explained in [20, § 7.3]. A more accurate passage to the limit $n \to 0$ in (4.4) is done there in Section 7.6 and in Appendix B. This explains the continuous deformation as $n \to 0$ of oscillatory structures in (4.3) to linear ones in the exponential tail of the fundamental kernel $F(y)$ given by (3.2).

In (d), we also present the periodic solution for $n = +\infty$ where (4.4) takes a simpler form (see an algebraic construction of the unique periodic solution in [20, § 7.4])

$$P_3(\varphi) = -\operatorname{sign}\varphi.$$

Finally, given the periodic $\varphi_*(s)$ of (4.4), as a natural way to approach the interface point $y_0 = 0$ according to (4.3), we have that the ODE (4.1) generates at the singularity set $\{f = 0\}$

$$\begin{aligned}&\text{A 3D local asymptotic bundle with parameters } y_0,\\&\text{phase shift in } s \mapsto s + s_0,\end{aligned} \tag{4.8}$$

and the parameter $\varepsilon > 0$ of the scaling group in (2.6). Notice that this scaling invariance has been lost in the approximate ODE (4.1).

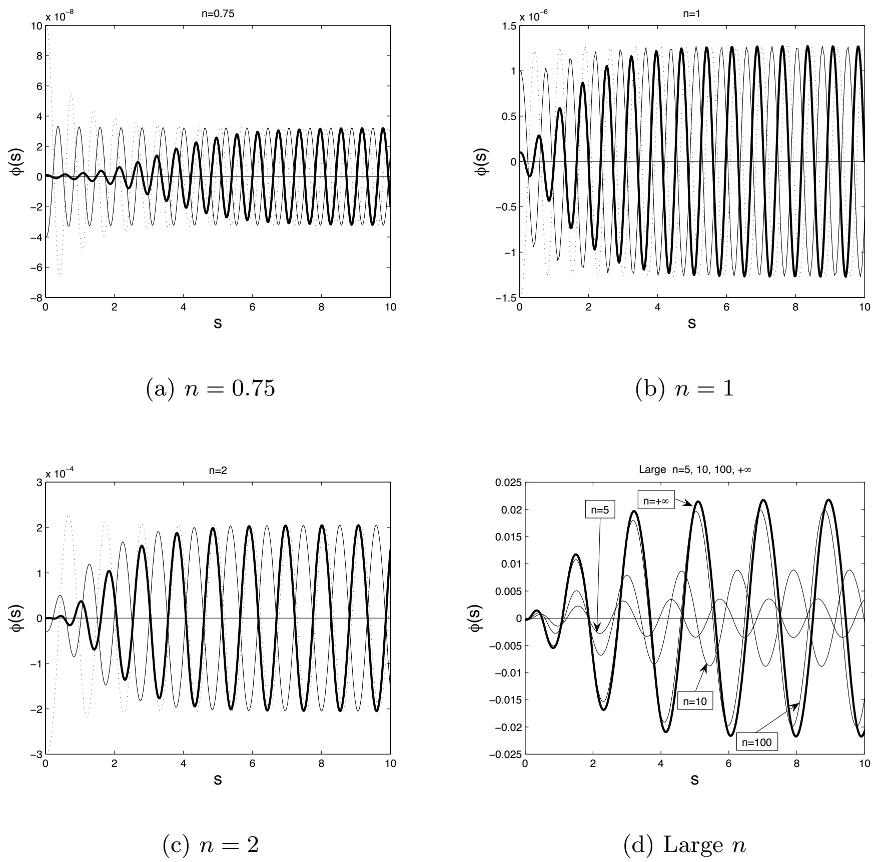
5. General description of nonlinear eigenfunctions for PME-4

We return to the nonlinear eigenvalue problem (2.4).

5.1. **First observation: the set nonlinear eigenfunctions is expected to be countable.** First of all, it is crucial that, taking into account the local result (4.8) and bearing in mind the *three* boundary conditions in (2.7) or (2.8), we may expect that, up to scaling invariance (2.6),

$$\text{There exists not more than a countable set } \{f_k\} \text{ of solutions.} \tag{5.1}$$

In other words, the 3D bundle in (4.8) is well designed to satisfy also three boundary conditions. This speculations assume a certain "analyticity" hypothesis concerning the dependence on parameters in the degenerate ODE (2.4), which is plausible but not easy to prove. Actually, this means that, relative to the parameter $n > 0$, we can expect at most a countable set of n-branches of similarity profiles. To begin with, this is true for the linear case $n = 0$, where there exists a countable set of linearly independent eigenfunctions (3.7).



(a) $n = 0.75$ (b) $n = 1$

(c) $n = 2$ (d) Large n

FIGURE 2. Convergence to a stable periodic orbit of the ODE (4.2) for various $n > 0$.

5.2. **First four n-branches: eigenvalues are given explicitly.** Unlike the PME case, the problem (2.4) does not admit any explicit representation of nonlinear eigenfunctions $f_l(y)$. But first four eigenvalues α_l can be obtained from the original PDE (1.1) using the following conservation laws reflecting highly divergent structure of the operator: for $u_0 \in C_0(\mathbf{R})$ and $l = 0, 1, 2, 3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int x^l u(x,t)\,\mathrm{d}x = 0 \Longrightarrow \int x^l u(x,t)\,\mathrm{d}x = \int x^l u_0(x)\,\mathrm{d}x \quad \text{for} \quad t \ge 0. \tag{5.2}$$

For the similarity solution (2.3) this yields

$$\int x^l u_S(x,t)\,\mathrm{d}x = t^{-\alpha+(l+1)\beta} \int y^l f(y)\,\mathrm{d}y, \tag{5.3}$$

and therefore

$$-\alpha + (l+1)\,\tfrac{1-\alpha n}{4} = 0 \Longrightarrow \alpha_l(n) = \tfrac{l+1}{4+(l+1)n} \;\text{ for }\; l = 0,1,2,3. \tag{5.4}$$

The corresponding to (5.4) nonlinear eigenfunctions $f_l(y)$ of (2.4) were constructed by Bernis and McLeod [6, 8], where the authors also proved uniqueness of such solutions of (2.4) up to scaling. It is also proved that solutions are compactly supported and are oscillatory near interfaces, i.e., have infinitely many sign changes; see more details in Section 4.

Concerning the corresponding nonlinear eigenfunctions, we reduce (2.4) to the semilinear problem,

$$F = |f|^n f \Longrightarrow -F^{(4)} + \beta(1-\mu)|F|^{-\mu}F'y + \alpha|F|^{-\mu}F = 0,\; \mu = \frac{n}{n+1}. \tag{5.5}$$

We next present numerical results concerning existence and multiplicity of solutions for the problem (5.5) and stress some principal properties and difficulties. In Figure 3 constructed by `MatLab`, we show the first basic symmetric pattern for (5.5) that is again called the $F_0(y)$ for $n = 0$, 0.5, 1, and 2. For completeness, we also include the case of the negative

$$n = -0.5, \tag{5.6}$$

which is not studied here. Nevertheless, the common geometries of patterns in Figure 3 for $n > 0$ and $n < 0$ suggest that several properties, branching, and evolution completeness setting remain similar for $n \in (-1, 0)$ (excluding finite propagation which is then infinite). We also observe that the profiles $F(y)$ are also oscillatory with larger amplitudes at the "interface" at $y = +\infty$.

In Figure 4, we show next four nonlinear eigenfunctions from the family

$$\Phi = \{F_l,\, l = 0,1,2,...\}$$

of (5.5) for the same values of n.

Remark on approximate Sturm property. Actually, if we forget for a moment about the complicated oscillatory structure of solutions near interfaces, where infinite numbers of extrema and zeros occur, the dominant geometry of profiles in Figure 4 approximately obeys Sturm's classic zero set property (1.35), which is true rigorously for the second-order ODEs only. Namely, we see from Figures 4 and 3 that each eigenfunctions

$$F_l(y) \;\text{ has precisely } l+1 \text{ "dominant" local extrema,} \tag{5.7}$$

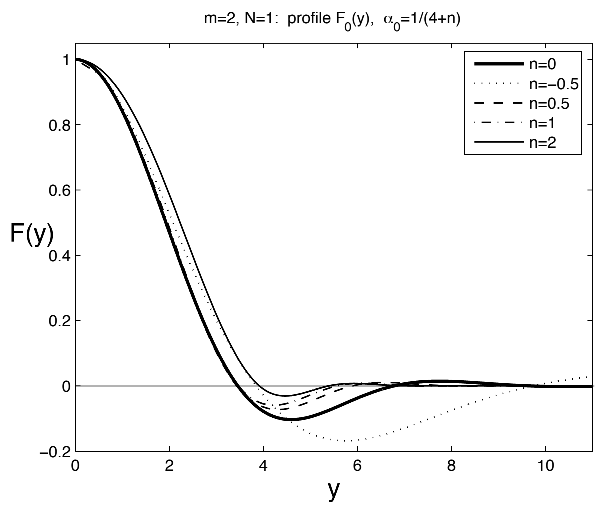


FIGURE 3. The first solution $F_0(y)$ of (5.5) for various n.

and, as a consequence, l "non-tail, transversal" zeros (this excludes infinite number of zeros from the oscillatory tail with the behaviour (4.3)); see [29, § 5] and [26, § 4] for some further details. We will present an extra explanation of this approximate Sturm property in the next subsection.

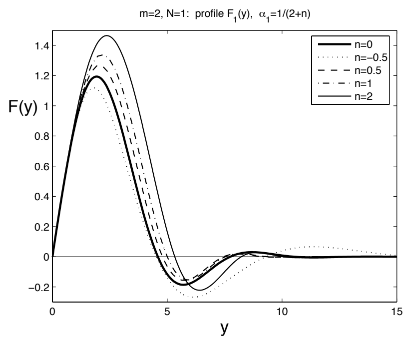
As we know, the eigenfunctions F_0, F_1, F_2, and F_3 have the explicit eigenvalues (5.4), while the F_4 in (d) has not, and this makes computations of it (and further higher-order) much more difficult. Our numerics show that, surprisingly, for $n = -0.5$, 0.5, and 1,

$$\alpha_4(n) \approx \tfrac{5}{4+5n} \quad \big(\text{e.g.} \quad \alpha_4(1) \approx \tfrac{5}{9} = 0.55...\big), \tag{5.8}$$

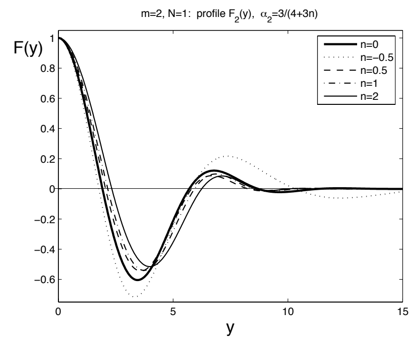
i.e., this eigenvalue is still close to that in (5.4) with $l = 4$ for sufficiently small n. Rigorously, (5.8) is not allowed since the corresponding conservation law is nonexistent. Further eigenfunctions are more difficult to detect numerically. For $n = 2$, (5.8) is not applied, and numerically we estimate

$$\alpha_4(2) = 0.365... \quad \big((5.8) \text{ yields} \quad \tfrac{5}{14} = 0.357...\big).$$

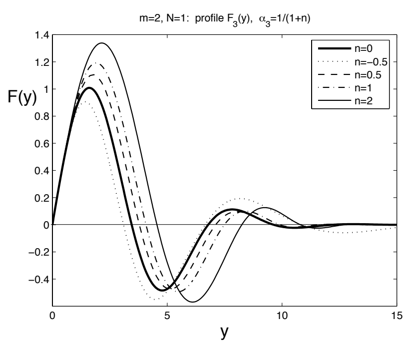
Further eigenfunctions are more difficult to calculate. For instance, in Figure 5, we show the even profiles F_6 and F_{10} for $n = 0$ $\big(\alpha_6(0) = \frac{7}{4} = 1.75$, $\alpha_{10}(0) = \frac{11}{4} = 2.75\big)$ and $n = 1$ ($\alpha_6(1) = 0.70...$, $\alpha_{10}(1) = 0.76...$).



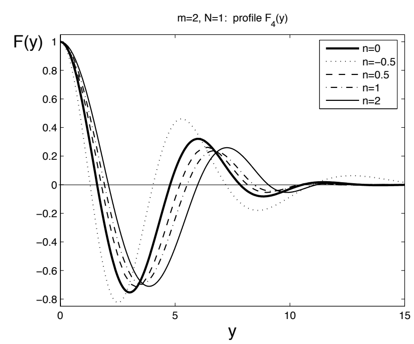
(a) $F_1(y)$



(b) $F_2(y)$



(c) $F_3(y)$



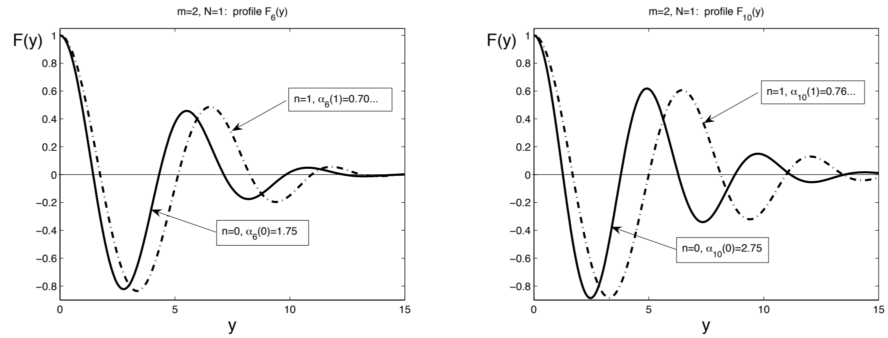
(d) $F_4(y)$

FIGURE 4. The next four nonlinear eigenfunctions of the ODE (5.5).

In Figure 6, we show first explicit n-branches given in (5.4). Other branches are not given explicitly. We next show how to estimate their behaviour via branching at the *branching point* $n = 0$ from eigenfunctions of the corresponding linear eigenvalue problem.

6. COUNTABLE BRANCHING OF EIGENFUNCTIONS AT $n = 0$: PME-4 IN $\mathbf{R}^N$

We study the behaviour of nonlinear eigenfunction curves appeared at the branching point $n = 0$ from linear eigenfunctions. We then will be able to explain existence of a countable set of similarity solutions of the PME−4



(a) $F_6(y)$ (b) $F_{10}(y)$

FIGURE 5. The eigenfunctions F_6 and F_{10} of (2.4) for $n = 0$ and $n = 1$.

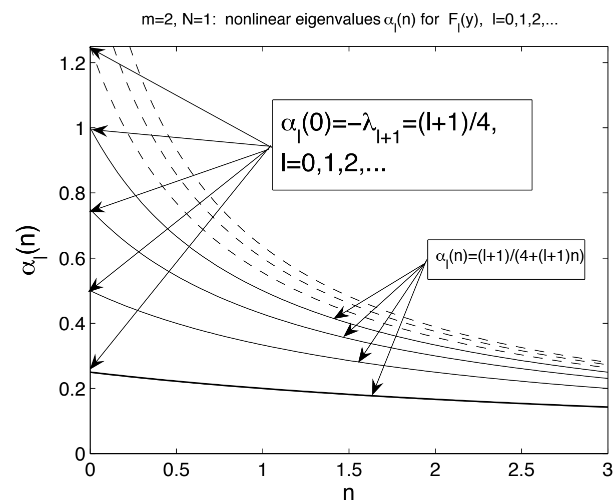


FIGURE 6. n-branches of eigenfunctions of (2.4).

(1.1). We then refer to classic branching theory in the case of nonlinearities of finite regularity; see [52, § 27], [38, Ch. 8], and [12, § 28].

Without any extra difficulties, this analysis will be performed in the maximal generality for the N-dimensional PME−4

$$\boxed{u_t = -\Delta^2(|u|^n u) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+.} \tag{6.1}$$

Then the similarity solutions have the same form (1.14), where f is an eigenfunction of the following linear pencil of elliptic operators (cf. (2.5) for

$N = 1$):

$$\mathbf{B}_n(f) \equiv -\Delta^2(|f|^n f) + \tfrac{1-\alpha n}{4}\, y \cdot \nabla f + \alpha f = 0 \;\; \text{in} \;\; \mathbf{R}^N, \; f \ne 0, \; f \in C_0(\mathbf{R}^N). \tag{6.2}$$

6.1. **Linear eigenvalue problem in $\mathbf{R}^N$ for $n = 0$.** In the linear case $n = 0$, we have again the *bi-harmonic equation*,

$$u_t = -\Delta^2 u \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+. \tag{6.3}$$

The first similarity pattern $F_0(y)$ is indeed unique (up to mass scaling), and is the rescaled kernel of the fundamental solution of (6.3) given by

$$\begin{gathered} b(x,t) = t^{-\frac{N}{4}} F(y), \quad y = x/t^{\frac14}, \quad \text{where} \\ \mathbf{B}_0 F \equiv -\Delta^2 F + \tfrac14\, y \cdot \nabla F + \tfrac{N}{4}\, F = 0, \quad \textstyle\int F = 1; \end{gathered} \tag{6.4}$$

see [19, § 4]. The functional setting for $\mathbf{B}_0$ and the adjoint operator $\mathbf{B}_0^*$, admitting bi-orthonormal system of eigenfunctions, is explained in [16] in detail. In particular, these operators are naturally defined in the weighted spaces $L^2_\rho(\mathbf{R}^N)$ and $L^2_{\rho^*}(\mathbf{R}^N)$ with the exponential weights

$$\rho(y) = \mathrm{e}^{a|y|^{4/3}}, \quad \rho^*(y) = \frac{1}{\rho(y)}, \tag{6.5}$$

where $a > 0$ is a sufficiently small constant. As usual, the corresponding domains of bounded operators $\mathbf{B}_0$ and $\mathbf{B}_0^*$ are Hilbert (Sobolev) spaces $H^{2m}_\rho(\mathbf{R}^N)$ and $H^{2m}_{\rho^*}(\mathbf{R}^N)$.

Thus, there exists a countable set of eigenfunctions $\{\psi_\gamma,\, l = |\gamma| \ge 0\}$ of the corresponding rescaled operator $\mathbf{B}_0$ with the discrete spectrum [16]

$$\sigma(\mathbf{B}_0) = \big\{ -\tfrac{l}{4}, \;\; l = 0, 1, 2, ... \big\}. \tag{6.6}$$

The eigenfunctions are derivatives of the rescaled kernel F (cf. (1.12) for $m = 1$),

$$\psi_\gamma(y) = \frac{(-1)^{|\gamma|}}{\sqrt{\gamma!}}\, D^\gamma F(y). \tag{6.7}$$

The adjoint orthonormal basis $\{\psi_\gamma^*\}$ consists of finite polynomials, [16].

6.2. **Branching of eigenfunctions at $n = 0$.** Thus, for $n > 0$ small enough, we consider the nonlinear eigenvalue problem (6.2) as a perturbation of the linear one for the operator $\mathbf{B}_0$ in (6.4). To this end, using that

$$\beta = \tfrac14 - \tfrac{\alpha}{4}\, n \tag{6.8}$$

and other elementary manipulations, we write the equation in (6.2) as

$$\mathbf{B}_0 f = g(f,n) \equiv \Delta^2[(|f|^n - 1)f] + n\tfrac{\alpha}{4}\, y \cdot \nabla f - \big(\alpha - \tfrac{N}{4}\big) f. \tag{6.9}$$

Next, since $\mathbf{B}_0$ is known to have compact resolvent in $L^2_\rho(\mathbf{R}^N)$, we form the strictly negative operator $\mathbf{B}_0 - I$ and, instead of (6.9), consider the equivalent integral equation

$$f = \mathbf{A}(f, n) \equiv (\mathbf{B}_0 - I)^{-1}(g(f, n) - f), \tag{6.10}$$

where the nonlinear operator on the right-hand side is treated as a compact Hammerstein operator in L^p_ρ-spaces ($p \geq 2$), as classic theory suggests [38, § 17]. Therefore, performing below necessary computations, we bear in mind using bifurcation-branching theory for compact integral operators as in (6.10). We do not discuss here specific aspects of compact integral operators in weighted L^p-spaces; see [16] for extra details. Note that, in a suitable metric, $\mathbf{B}_0$ is a sectorial operator, [25, § 5.1]. This puts the asymptotic analysis into the standard framework of invariant manifold theory, [41].

Notice that (6.9) and (6.10) contain two parameters, n and α, so we deal with bifurcation (branching) problem for

$$\mu = (n, \alpha)^T \in \mathbf{R}^2. \tag{6.11}$$

For instance, as immediately follows from (6.9) and (6.4), the first n-branch is supposed to appear from the rescaled kernel F at the branching point

$$\mu_0 = \big(0, \tfrac{N}{4}\big)^T.$$

For derivation of branching equations, we use general branching theory in Banach spaces; see Vainberg–Trenogin [52, Ch. 7]. Beforehand, we need to discuss a few typical difficulties associated with the above eigenvalue problem.

It follows that the crucial part is played by the nonlinearity

$$Q(n, f) = (|f|^n - 1)f \text{ in a neighbourhood of the point } P_0 = \{f = 0, n = 0^+\}. \tag{6.12}$$

We conveniently put $Q \equiv 0$ for $n \leq 0$, so that the function is defined in a whole neighbourhood of $n = 0$ and $f = F$, with $F \neq 0$ to be determined as an eigenfunction of the linear equation. Notice that the derivative $Q'_f(n, f)$ *is not* continuous at P_0, since

$$Q'_f(n, f) = [(n+1)|f|^n - 1]\big|_{f=0} = -1 \neq Q'_f(n, 0)|_{n=0} = 0. \tag{6.13}$$

One can see that along the curves

$$f(n) = \mathrm{e}^{-\frac{k}{n}} \to 0 \quad \text{as} \quad n \to 0^+ \quad (k > 0),$$

the derivative (6.13) have various limits $\mathrm{e}^{-k} - 1$. Therefore, the standard bifurcation and branching results that assume continuous differentiability in a neighbourhood of both $n = 0$ and in $f = F$ (see e.g., [12, p. 380] and

[52, § 27]) do not apply. Index-degree (rotation of vector fields; see Theorem 56.4 in [38, p. 373]) theory assumes less regularity on nonlinear terms but anyway demands differentiability at $f = F$ for all $n \approx 0$.

In view of our difficulties with the regularity (and also with compactness of operators involved), we will rely on Theorem 28.1 in [12, p. 381] that is formulated in the linearized setting where differentiability is replaced by the control of higher-order nonlinear terms as in (6.12) in a neighborhood. As usual, the key principle of branching is that it occurs if the corresponding eigenvalue has odd multiplicity that can be easily checked in some cases (the even multiplicity case needs an additional treatment, which is also a routine procedure not to be focused here); see further comments below.

Notice that the condition on the nonlinearity (6.12) is rather tricky to check that it satisfies the estimate (b) in Theorem 28.1 in [12, p. 381] for the variable x in $f = F + x$. Namely, setting

$$G(n, x) = Q(n, F + x)$$

we have to have that, in L^2_ρ,

$$|G(n, x) - G(n, \bar{x})| \le \varphi(r)|x - \bar{x}| \tag{6.14}$$

for all $n \approx 0$, $x, \bar{x} \in B_r(0)$, and with a real-valued function satisfying $\varphi(r) \to 0$ as $r \to 0$. It turns out that this estimate is true provided that $F(y)$ has zeros that are "transversal" a.e.; this property will be justified below for known linear eigenfunctions $F(y)$. Besides this, certain efforts are necessary to adapt the cited approach in [12] to the case of the bifurcation parameter (6.11), which we do not do here and, in addition to previous books, refer to modern general bifurcation theory, [10, Ch. 8], where further references can be found.

Thus, application of known bifurcation theory to equation with the leading nonlinearities such as (6.12) that is not continuously differentiable in a necessary neighbourhood is not entirely straightforward and demands extra difficult and technical analysis.

6.3. **Asymptotic expansion of branches for small $n > 0$.** It follows from (6.10) that branching is possible under the following non-trivial kernel assumption: for $n = 0$,

$$\alpha - \tfrac{N}{4} = -\lambda_l = \tfrac{l}{4} \quad \Longrightarrow \quad \alpha_l(0) = \tfrac{N+l}{4}, \quad l \ge 0. \tag{6.15}$$

This gives an approximation of the countable sequence of critical exponents $\{\alpha_l(n), \beta_l(n)\}$ (to be determined) of the similarity patterns (2.3) of the PME-4 for small $n > 0$. In particular, it follows from Theorem 29.1 [12, p. 401] (compactness is necessary; see also [38, Th. 56.4]) that n-branching does occur if λ_l has odd multiplicity. Moreover, under this assumption, there exists a continuous n-branch of such solutions [38, Th. 56.6].

We next perform a further study of the behaviour of nonlinear eigenfunctions close to the *branching point* $n = 0$, and derive a Lyapunov-Schmidt type equation as a solvability criterion under some weaker regularity assumptions.

To this end, we use in (6.9) the expansion

$$|f|^n f = f + nf \ln |f| + o(n) \quad \text{as} \quad n \to 0^+, \tag{6.16}$$

which, as can be easily seen, is true uniformly on bounded intervals in f. In addition, this is understood in the weak sense (for a wide class of sufficiently good functions $f(y)$) associated with the equivalent integral equation; see Proposition 6.1 below. Substituting expansions (6.8) and (6.16) into (6.2) yields, still formally,

$$\mathbf{B}_n(f) \equiv \mathbf{B}_0 f + \big(\alpha - \tfrac{N}{4}\big) f + n\mathcal{L}(f) + o(n) = 0, \tag{6.17}$$

with the perturbation operator

$$\mathcal{L}(f) = -\big[\Delta^2(f \ln |f|) + \tfrac{\alpha}{4}\, y \cdot \nabla f\big]. \tag{6.18}$$

We next describe the behaviour of solutions for small $n > 0$ and apply the classical Lyapunov-Schmidt method [38, Ch. 8] to equation (6.17). Recall that in this linearized setting, we naturally arrive at the functional framework that is suitable for the linear operator $\mathbf{B}_0$, i.e., it is $L^2_\rho(\mathbf{R}^N)$, with the domain $H^4_\rho(\mathbf{R}^N)$, etc., and a similar setting for the adjoint operator $\mathbf{B}_0^*$; see above and further details in [16].

Therefore, for $n = 0$, we have to study continuation (we also call this branching meaning appearance of an n-branch) of a nonlinear eigenfunction from the linear one. Therefore, we will perform linearization about f being a certain nontrivial finite linear combination of eigenfunctions from the given eigenspace with a fixed $\lambda_l = -\frac{l}{4}$, i.e.,

$$f = \phi_l = \sum_{|\beta|=l} C_\beta \psi_\beta \quad (\neq 0). \tag{6.19}$$

These eigenfunctions are derivatives (6.7) of the analytic radially symmetric rescaled kernel $F = F(|y|)$ of the fundamental solution (6.4). Therefore, the nodal (zero) set of such f in (6.19) is well understood and consists of a countable set of isolated sufficiently smooth hypersurfaces which are concentrated as $y \to \infty$, where

$$\phi_l(y) \to 0 \quad \text{as} \quad y \to \infty \text{ uniformly and exponentially fast.} \tag{6.20}$$

Moreover, (6.7) clearly shows that all zeros are transversal (in the usual sense) a.e. in $\mathbf{R}^N$ that is necessary for checking the key hypothesis on the nonlinearity (6.14). It is not difficult to check that, for such functions (6.19),

$$\mathcal{L}(\phi_l) \in L^2_\rho(\mathbf{R}^N). \tag{6.21}$$

This is important for the asymptotic expansion used below.

Returning to the crucial limit (6.16), as a key illustration and for convenience, we state the following:

Proposition 6.1. *For a function f given by* (6.19)*, in the sense of distributions and in the weak sense in* $L^\infty(\mathbf{R}^N)$

$$\frac{1}{n}\, f(|f|^n - 1) \rightharpoonup f \ln|f| \quad as \quad n \to 0^+. \tag{6.22}$$

According to (6.22), analyzing the integral equation for f, we can use the fact that, for any function $\phi \in \mathcal{L}$, $\phi \in L^1(\mathbf{R}^N)$ (or $\phi \in C_0(\mathbf{R}^N)$)

$$\int_{\mathbf{R}^N} f(y)(|f(y)|^n - 1)\phi(y)\,\mathrm{d}y = n\Big[\int_{\mathbf{R}^N} f(y)\ln|f(y)|\phi(y)\,\mathrm{d}y + o(1)\Big]. \tag{6.23}$$

By [19, Lemma 4.1], the kernel of the linearized operator

$$E_0 = \ker(\mathbf{B}_0 - \lambda_l I) = \mathrm{Span}\,\{\psi_\beta,\ |\beta| = l\}$$

is non-trivial and finite-dimensional. Hence, denoting by E_1 the complementary (orthogonal to E_0) invariant subspace, we set

$$f = \phi_l + V_1, \quad \phi_l \in E_0, \quad \text{and} \quad V_1 = \sum_{|\beta|>l} c_\beta \psi_\beta \in E_1. \tag{6.24}$$

According to the known spectral properties of operator $\mathbf{B}_0$, we define P_0 and P_1, $P_0 + P_1 = I$, to be projections onto E_0 and E_1 respectively. We also introduce a perturbation of the parameter α by setting

$$\alpha_l(n) = \alpha_l(0) + \delta, \quad \text{with some} \quad \delta = \delta(n). \tag{6.25}$$

The perturbation δ is estimated from the orthogonality condition by substituting into (6.17) and multiplying by ψ^*_β. This gives

$$\delta(n) = c_l n + o(n), \tag{6.26}$$

where the constant c_l is obtained from the system

$$\boxed{\langle \mathcal{L}(\phi_l), \psi^*_\beta \rangle = c_l, \quad |\beta| = l.} \tag{6.27}$$

Since according to (6.24), ϕ_l is given by (6.19), (6.27) is an *algebraic system* for unknowns $\{C_\beta\}$ and c_l that represents the branching *Lyapunov–Schmidt equation*; see [52, § 23]. It can be solved, for instance, for $N = 1$ (then eigenvalues are always simple), in the radial geometry in $\mathbf{R}^N$ for even $l = 0, 2, 4, ...$, and in some other cases (including those where the dimension of the kernel is odd; even dimensions need additional treatment); see some other details in [27, App. A].

Global bifurcation results on continuous branches of solutions originated at $n = 0$ are already given in Krasnosel'skii [37, p. 196] (the first Russian edition was published in 1956). Concerning further results and extensions, see references in [12, Ch. 10] (especially, see [12, p. 401] for typical global continuation of bifurcation branches), and also [38, § 56.4] and [10, Ch. 9] for further results and references concerning global bifurcation theory.

Thus, the total number of n-branches that are originated at the branching point $n = 0$ for a fixed eigenvalue λ_l depends on the total number of solutions of (6.27); see more details on one-to-one correspondence between actual solutions and solutions of branching equations in [52, p. 329, 363].

Finally, plunging into (6.17) $f = \phi_l + V_1$, with

$$V_1 = nY + o(n), \tag{6.28}$$

we obtain, passing to the limit $n \to 0^+$, the following equation for Y:

$$\mathbf{B}_0 Y = -c_l \phi_l + \mathcal{L}(\phi_l), \tag{6.29}$$

which, by Fredholm's theory, in view of the orthogonality, admits a unique solution $Y \in E_1$.

In general, the above analysis shows that, up to solvability of the nonlinear algebraic systems, the PME-4 (6.2) admits a countable set of different similarity solutions (2.3) at least for small $n > 0$, where the parameters $\alpha_l(n)$ are given by

$$\alpha_l(n) = \tfrac{N+l}{4} + c_l n + o(n) \quad \text{as} \quad n \to 0^+, \quad l = 0, 1, 2, \ldots . \tag{6.30}$$

At $n = 0$, these solutions are originated from suitable eigenfunctions of the linear operator in (6.4). The global extensions of these n-branches of similarity solutions for larger $n > 0$ represent a difficult open problem, to be treated numerically later on.

6.4. **Remark on approximate Sturm property, continued.** Thus, we take $N = 1$, so that at $n = 0$ we have branching from the eigenfunctions of the operator $\mathbf{B}_0$ given in (3.2), i.e.,

$$\mathbf{B}_0 \psi \equiv -\psi^{(4)} + \tfrac{1}{4}\, y\psi' + \tfrac{1}{4}\, \psi = \lambda_l \psi, \quad \psi \in L^2_\rho(\mathbf{R}) \quad \big(\lambda_l = -\tfrac{l}{4}\big). \tag{6.31}$$

This is a fourth-order ordinary differential operator with unbounded non-autonomous coefficients, and Sturm zero set property is not valid for the eigenvalue problem (6.31) either. Nevertheless, $\mathbf{B}_0$ can be viewed as a perturbation via the first-order operator $\frac{1}{4}\,yD_y + \frac{1}{4}\,I$ of the standard negative symmetric (with self-adjoint extensions) one in L^2,

$$\mathbf{B} = -D_y^4 \equiv -(-D_y^2)(-D_y^2), \tag{6.32}$$

which is an iteration of the positive operator $-D_y^2$ with the Maximum Principle. Therefore, according to Elias [18] (see also applications to some nonlinear higher-order eigenvalue problems in [47, 4]), for standard symmetric settings, eigenvalues of (6.32) rigorously obeys Sturm's zero and local extrema properties. Eventually, this Sturm's geometric ordering, now in an "approximate" sense, remains valid for the perturbed linear operator (6.31), and finally is inherited by the nonlinear eigenvalue problem (2.4). The questions of Sturm's properties and index for variational higher-order nonlinear operators are discussed in [29, § 7] in greater detail.

6.5. **On global continuation of n-branches: open problem.** Global continuation of branches of nonlinear eigenfunctions of (6.2) appearing at the branching point $n = 0$ is an intriguing open problem. In general, for the integral equation (6.10) with compact Hammerstein operators in weighted L^2-spaces, it is known since Rabinowitz's study (1971) that branches are infinitely extensible and can end up at further bifurcation points; see [12, § 29] for information stated in the framework of bifurcation analysis and [38, § 56.4]. Therefore, such n-extensions of the branches are possible at any $n = n_0 > 0$ provided that the linearized operator has proper spectral properties. For oscillatory profiles $f(y)$ of changing sigh, with a complicated nodal set, this is difficult to check in general but, in principle, can be done in simpler 1D or radial geometries, assuming that the oscillatory structure near interfaces is known in detail. Nevertheless, for the non-variational eigenvalue problem (6.2) with non-divergent and non-monotone operators, a rigorous treatment of global behaviour of branches is difficult and remains open.

Bearing in mind the expected evolution completeness of nonlinear eigenfunctions (see the next subsection) and the known completeness-closure of eigenfunctions for $n = 0$, we state the following:

Conjecture 4.2. (i) *All the n-branches of nonlinear eigenfunctions of* (6.2) *that are originated at the branching point $n = 0$ are globally extensible for all $n > 0$.*

(ii) *For $N = 1$, all the n-branches appeared at $n = 0$ do not have turning* (*saddle-node*) *bifurcations at any $n > 0$.*

6.6. **Evolution completeness for the PME-4: open problems.** Firstly, for the nonlinear eigenvalue problem (6.2), we arrive at the same Conjecture 1.1 formulated in Section 1 for the second-order case.

Secondly, we recognize that the problem of evolution completeness of the nonlinear eigenfunctions, even in the 1D case $N = 1$, is expected to be extremely difficult. Recall that it has not been completely solved for the standard PME (1.22) obeying the Maximum Principle (see Conjecture 1.2). In this connection, we state the following:

Open Problem 4.3. *Evolution completeness in* $C_0(\mathbf{R}^N)$ *of nonlinear eigenfunctions of* (6.2) *for small* $n > 0$ *can be associated with the completeness-closure in* $L^2_\rho(\mathbf{R}^N)$ *of the eigenfunctions* (6.7) *of the linear operator* $\mathbf{B}_0$ *given in* (6.4) *obtained at the branching point* $n = 0$.

7. PME-6: countable set of similarity solutions and nonlinear eigenfunctions

7.1. **The model: degenerate sixth-order parabolic equation.** Consider the *sixth-order porous medium equation* (PME−6):

$$u_t = \big(|u|^n u\big)_{xxxxxx} \quad \text{in} \quad \mathbf{R} \times \mathbf{R}_+ \quad (n > 0). \tag{7.1}$$

For $n = 0$, we obtain the *tri-harmonic equation*

$$u_t = u_{xxxxxx}, \tag{7.2}$$

with the known fundamental solution and spectral properties of the corresponding operators $\mathbf{B}_0$ and $\mathbf{B}_0^*$; see below.

Existence of a unique weak solution of the CP for (7.1) with compactly supported data (2.2) follows from classic parabolic theory, [40, Ch. 2]. For finite propagation, see again [49] and references in [30].

7.2. **Blow-up similarity solutions: problem setting.** The PDE (7.1) possesses the similarity solutions

$$u_S(x,t) \equiv b(x,t) = t^{-\alpha} f(y), \quad y = x/t^{\beta}, \quad \text{where } \beta = \tfrac{1-\alpha n}{6}, \tag{7.3}$$

and $f(y)$ is an eigenfunction of the linear pencil,

$$\mathbf{B}_n(f) \equiv (|f|^n f)^{(6)} + \tfrac{1-\alpha n}{6}\, y\, f' + \alpha f = 0 \quad \text{in} \quad \mathbf{R}, \quad f \neq 0, \;\; f \in C_0(\mathbf{R}). \tag{7.4}$$

Bearing in mind the invariant group (2.6) with the exponent $\frac{4}{n} \mapsto \frac{6}{n}$, we perform the following normalization: for $l = 0, 2, 4, \dots$,

$$f(0) = 1, \quad \text{and} \quad f'(0) = f'''(0) = f^{(5)}(0) = 0, \tag{7.5}$$

and for $l = 1, 3, 5, ...$,

$$f'(0) = 1, \quad \text{and} \quad f(0) = f''(0) = f^{(4)}(0) = 0. \tag{7.6}$$

7.3. **Oscillatory properties near interface.** Keeping leading two terms in (7.4) and integrating yields

$$(|f|^n f)^{(5)} = -\beta y_0 f \quad (\beta > 0).$$

Reflecting relative the interface, $y_0 - y \mapsto y > 0$, and scaling out the positive constant βy_0 leads to

$$(|f|^n f)^{(5)} = f \quad \text{for} \quad y > 0, \quad f(0) = 0, \tag{7.7}$$

and finally,

$$F = |f|^n f \quad \Longrightarrow \quad F^{(5)} = \big|F\big|^{-\frac{n}{n+1}} F. \tag{7.8}$$

The oscillatory component φ is now introduced by

$$F(y) = y^\mu \varphi(s), \quad s = \ln y, \quad \text{where } \mu = \tfrac{5(n+1)}{n} > 5 \text{ for all } n > 0, \tag{7.9}$$

and $\varphi(s)$ solves the following fifth-order ODE:

$$\begin{gathered} P_5(\varphi) \equiv \varphi^{(5)} + 5(\mu-2)\varphi^{(4)} + 5(2\mu^2 - 8\mu + 7)\varphi''' \\ + 5(\mu-2)(2\mu^2 - 8\mu + 5)\varphi'' + (5\mu^4 - 40\mu^3 + 105\mu^2 - 100\mu + 24)\varphi' \\ + \mu(\mu-1)(\mu-2)(\mu-3)(\mu-4)\varphi = |\varphi|^{-\frac{n}{n+1}}\varphi. \end{gathered} \tag{7.10}$$

It is remarkable that the same ODE for $\varphi(s)$ occurs for the *sixth-order thin film equation* (TFE-6) that was studied in [31, pp. 139-148] and in [21, § 12]. The ODE (7.10) admits an unstable (as $s \to +\infty$) periodic solution for all $n > 0$. Taking into account its stable manifold towards the interface as $s = -\infty$ together with the group of scalings similar to (2.6) yields, instead of (4.8), that the ODE (7.7) generates at the singularity set $\{f = 0\}$

$$\text{a 4D local asymptotic bundle.} \tag{7.11}$$

7.4. **On countable set of nonlinear eigenfunctions.** Again taking into account the local result (7.11), the four boundary conditions in (7.5) or (7.6) lead us to the conclusion (5.1). Hence, we can expect at most a countable set of n-branches of similarity profiles, which is true for the linear case $n = 0$; see below.

First six eigenvalues α_l of (7.4) are the PDE (7.1) using the conservation laws (5.2), which now are valid for $l = 0, 1, 2, 3, 4, 5$. This yields

$$-\alpha + (l+1)\,\frac{1 - \alpha n}{6} = 0 \quad \Longrightarrow \quad \alpha_l = \frac{l+1}{6 + (l+1)n} \text{ for } l = 0, 1, 2, 3, 4, 5. \tag{7.12}$$

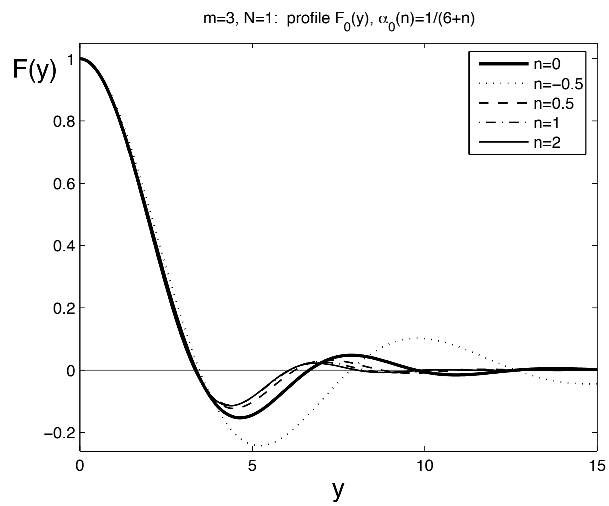


Figure 7. The first eigenfunction $F_0(y)$ of (7.13) for various n.

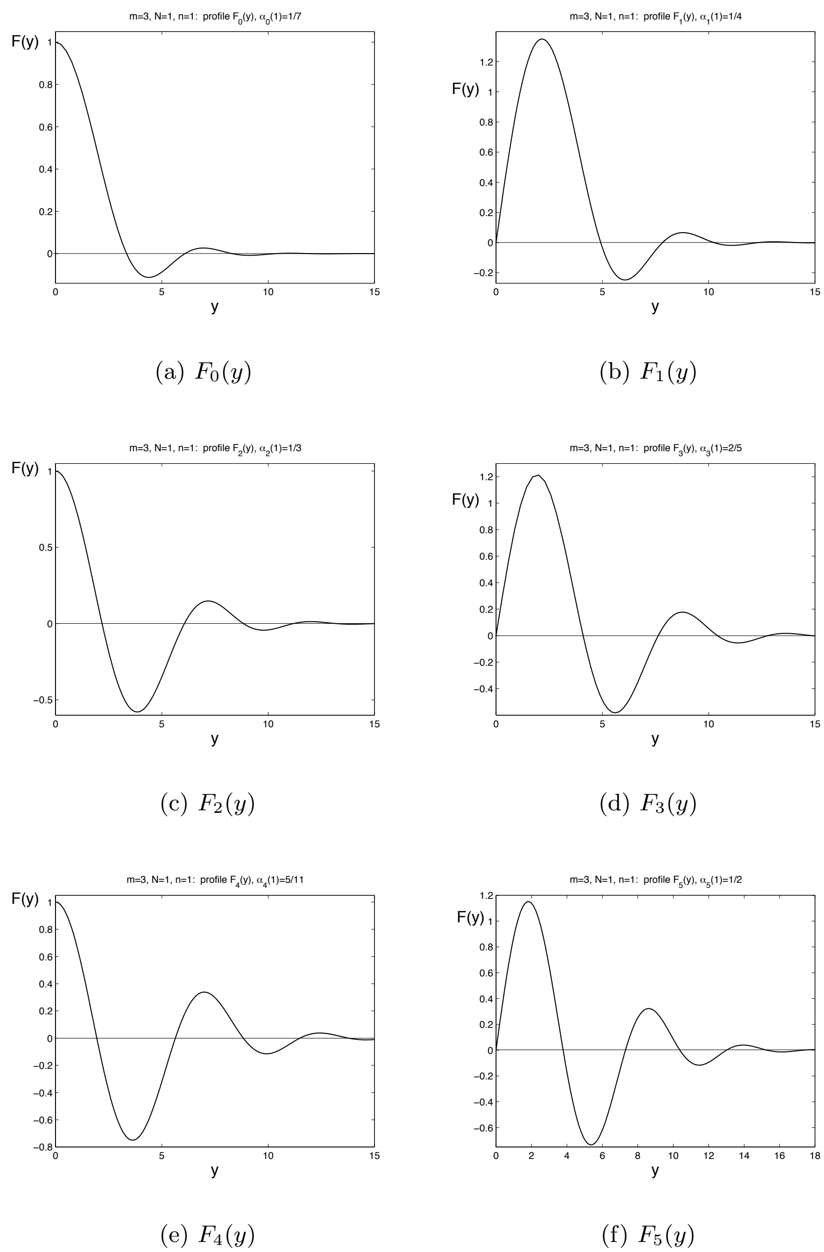
As usual, we reduce (7.4) to the semilinear problem,

$$F = |f|^n f \implies F^{(6)} + \beta(1-\mu)|F|^{-\mu}F'y + \alpha|F|^{-\mu}F = 0 \quad \big(\mu = \tfrac{n}{n+1}\big). \tag{7.13}$$

Existence and uniqueness of solutions of (7.13) for eigenvalues (7.12), for which the ODE can be integrated once, remain an open problem. Note that some properties of possible solutions are known. For instance, if a solution exists, it is compactly supported. This is proved by Bernis' energy estimates as in [8, § 7], or by more advanced PDE approaches as in [49]. Here we do not concentrate on mathematics concerning the nonlinear eigenvalue problem (7.13) and just present numerical results clearly showing existence and uniqueness of various branches of eigenvalues.

In Figure 7, the first eigenfunction F_0 of (7.13) is shown for $n = 0$, 0.5, 1, and 2, including the negative $n = -0.5$, as a special example for posing similarity settings for the "sixth-order fast diffusion equation" with $n \in (-1, 0)$.

In Figure 8, we show the first six nonlinear eigenfunctions from the family $\Phi = \{F_l,\, l = 0, 1, 2, ...\}$ of (7.13) for $n = 1$. We again observe that these eigenfunctions perfectly obey the approximate Sturm property and also reflect some typical aspects of application Lusternik–Schnirel'man category theory of calculus of variation; see [29, § 5]. Recall that (7.13) cannot be reduced to a variational eigenvalue problem.

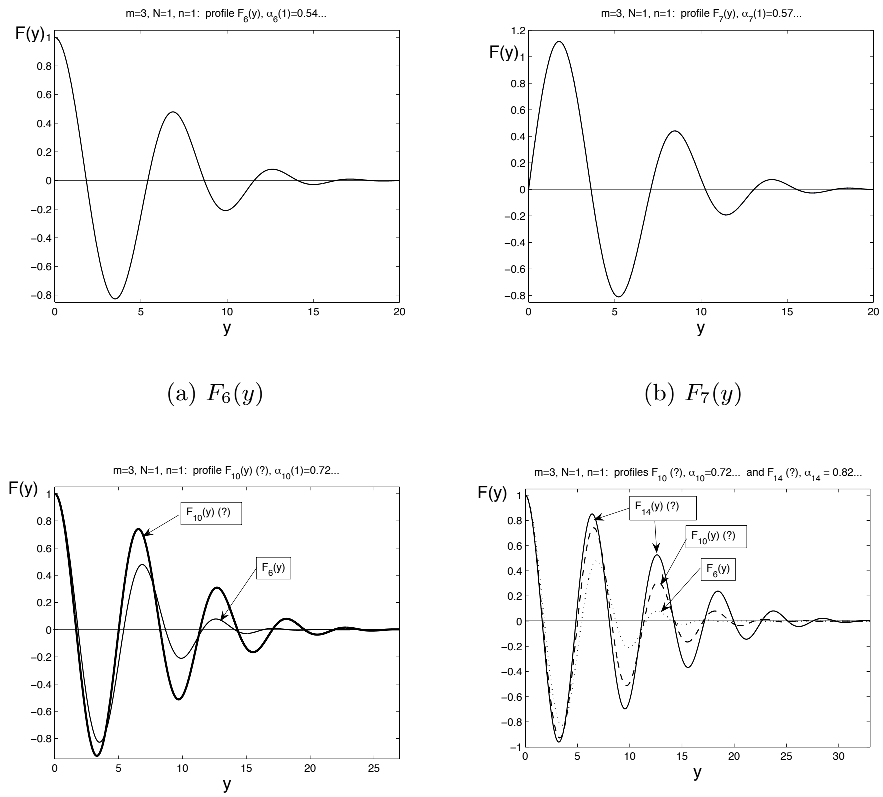


(a) $F_0(y)$ (b) $F_1(y)$

(c) $F_2(y)$ (d) $F_3(y)$

(e) $F_4(y)$ (f) $F_5(y)$

Figure 8. First six nonlinear eigenfunctions of (7.13) for $n = 1$.



(a) $F_6(y)$ (b) $F_7(y)$

(c) $F_{10}(y)$ (d) $F_{14}(y)$

Figure 9. Higher-order nonlinear eigenfunctions of (7.13) for $n = 1$.

Extra patterns are shown in Figure 9. Numerical construction of patterns F_l for $l \geq 8$ becomes rather difficult, and we are not sure that eigenvalues α_l for F_{10}, and F_{14} are constructed with reasonable accuracy (this is reflected by question marks in (c) and (d)).

The general behaviour of n-branches given in (7.12) is similar to that in Figure 6.

7.5. **On n-branches of nonlinear eigenfunctions.** This analysis is similar to that in Section 6. We just need to present the corresponding spectra properties.

Thus, consider the PME−6

$$u_t = \Delta^3(|u|^n u) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+, \tag{7.14}$$

for which the similarity solutions have the same form (7.3), where f are eigenfunctions of the linear pencil of linear and nonlinear elliptic operators,

$$\mathbf{B}_n(f) \equiv \Delta^3(|f|^n f) + \frac{1-\alpha n}{6}\, y \cdot \nabla f + \alpha f = 0 \quad \text{in} \quad \mathbf{R}^N, \;\; f \ne 0, \;\; f \in C_0(\mathbf{R}^N). \tag{7.15}$$

For $n = 0$, we get the *tri-harmonic equation*,

$$u_t = -\Delta^2 u \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+. \tag{7.16}$$

The first eigenfunction $F_0(y)$ is the rescaled kernel of the fundamental solution

$$\begin{gathered} b(x,t) = t^{-\frac{N}{6}} F(y), \quad y = x/t^{\frac{1}{6}}, \quad \text{where} \\ \mathbf{B}_0 F \equiv \Delta^3 F + \tfrac{1}{6}\, y \cdot \nabla F + \tfrac{N}{6}\, F = 0 \quad \text{in} \quad \mathbf{R}^N, \quad \textstyle\int F = 1. \end{gathered} \tag{7.17}$$

There exists a countable set of eigenfunctions $\{\psi_\gamma,\, l = |\gamma| \ge 0\}$ of the corresponding rescaled operator $\mathbf{B}_0$ with the discrete spectrum [16]

$$\sigma(\mathbf{B}_0) = \big\{ -\tfrac{l}{6},\;\; l = 0,1,2,... \big\}. \tag{7.18}$$

The eigenfunctions are normalized derivatives of the rescaled kernel F, so (6.7) holds and the adjoint orthonormal basis $\{\psi^*_\gamma\}$ consists of finite polynomials, [16].

Applications of branching theory [12, § 28] to (7.15) is similar to that in Section 6. It is easy to translate the basic computations for small $n > 0$ in Section 6 to the sixth-order nonlinear eigenvalue problem (7.15).

Finally, we note that the approach of extensions from the branching point $n = 0$ applies to any PME$-2m$

$$u_t = (-1)^{m+1}\Delta^m(|u|^n u) \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}_+; \quad m = 2,3,4,\ldots\,.$$

Spectral theory for the corresponding $2m$th-order linear bounded operators $\mathbf{B}_0$ and $\mathbf{B}_0^*$ is available in [16]; principles of formation of the branching equation remain the same.

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