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# REACTION-DIFFUSION PROBLEMS WITH NON-FREDHOLM OPERATORS

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**Abstract.** The paper is devoted to the study of a multi-dimensional semi-linear elliptic system of equations in an unbounded cylinder with a linear dependence of the components of the non-linearity vector. Problems of this type describe reaction-diffusion waves with the Lewis number different from 1. Due to this property of non-linearity, the corresponding operator does not possess the Fredholm property. Therefore the usual solvability conditions and the conventional methods of non-linear analysis cannot be directly applied.

We reduce the elliptic problem to an integro-differential system of equations and show how to apply the implicit function theorem to it. It allows us to prove existence of waves for the Lewis number different from 1 and sufficiently close to it. Next we prove the Fredholm property of integro-differential operators, their properness, and construct the topological degree. The latter is used to study bifurcations of solutions.

## 1. Introduction

The aim of this paper is to study reaction-diffusion systems of the form

$$-\Delta\theta + \alpha(y)\frac{\partial\theta}{\partial x} - \kappa(\theta,\psi) = 0 \tag{1.1}$$

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$$-\Lambda\Delta\psi + \alpha(y)\frac{\partial\psi}{\partial x} + \kappa(\theta,\psi) = 0, \tag{1.2}$$

in an unbounded cylinder $\Omega = \{(x,y) : -\infty < x < +\infty, \quad y \in \omega\}$, where $\omega$ is an open regular bounded subset of $\mathbb{R}^d$ with $d = 1, 2$. This system is supplemented with the boundary conditions

$$\frac{\partial\theta}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0 \ \text{ on } \partial\Omega, \tag{1.3}$$

and the following conditions at infinity:

$$\theta(-\infty,y) = 0, \ \ \psi(-\infty,y) = 1, \ \ \theta(+\infty,y) = 1, \ \ \psi(+\infty,y) = 0. \tag{1.4}$$

We assume that $\kappa$ is a $C^2$ function on $\mathbb{R}^2$ that satisfies

$$\kappa(0,1) = \kappa(1,0) = 0, \tag{1.5}$$

$$-\frac{\partial\kappa}{\partial\theta}(0,1) + \frac{\partial\kappa}{\partial\psi}(0,1) > 0, \ \ -\frac{\partial\kappa}{\partial\theta}(1,0) + \frac{\partial\kappa}{\partial\psi}(1,0) > 0. \tag{1.6}$$

In view of these conditions the function $\kappa(\theta, 1-\theta)$ vanishes at $\theta = 0$ and $\theta = 1$ and possesses negative derivatives at these points. Also we assume that the function $\alpha$ satisfies

$$\alpha \text{ is continuous on } \bar{\omega} \text{ and } \bar{\alpha} = \int_\omega \alpha(y)dy > 0. \tag{1.7}$$

We will also consider systems of the form

$$-\Delta\theta + \beta(c,y)\frac{\partial\theta}{\partial x} - \kappa(\theta,\psi) = 0 \tag{1.8}$$

$$-\Lambda\Delta\psi + \beta(c,y)\frac{\partial\psi}{\partial x} + \kappa(\theta,\psi) = 0, \tag{1.9}$$

where now the function $\beta(c,y)$ depends on some *unknown* parameter $c$. Typically, $\beta(c,y)$ takes the form $c + \gamma(y)$ or $c\gamma(y)$ where $\gamma(y)$ is some given non-negative function. In the first case, $c$ corresponds to the speed of a travelling wave propagating with a given velocity field $\gamma(y)$ parallel to the axis of the cylinder. In the second case, $c$ allows us to adjust the velocity field so that there exist solutions to the system.

The specific property of such systems is that the non-linear terms cancel by adding the equations ((1.1) and (1.2) or (1.8) and (1.9)). Consequently the underlying elliptic operator does not satisfy the Fredholm property. Indeed by linearizing (1.1)-(1.3) we easily see that the limiting system obtained by taking the limit $x \to \pm\infty$, that reads

$$-\Delta\theta + \alpha(y)\frac{\partial\theta}{\partial x} - a_\pm\theta - b_\pm\psi = 0,$$

$$-\Lambda\Delta\psi + \alpha(y)\frac{\partial\psi}{\partial x} + a_{\pm}\theta + b_{\pm}\psi = 0,$$
$$\frac{\partial\theta}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0 \ \text{ on } \ \partial\Omega,$$

where

$$a_+ = \kappa'_\theta(1,0),\ a_- = \kappa'_\theta(0,1),\ b_+ = \kappa'_\psi(1,0),\ b_- = \kappa'_\psi(0,1),$$

have non-zero solutions. Indeed, any constant vector $(\theta, \psi)$ satisfying the equality $a_{\pm}\theta + b_{\pm}\psi = 0$ is a solution. Therefore, the corresponding operator does not satisfy the Fredholm property, neither in Hölder nor in Sobolev spaces (see [28]). Consequently, the solvability conditions for the linear operators can not be directly applied here, and strictly speaking we can not apply asymptotic expansions, bifurcation analysis, or other methods, like the implicit function theorem or topological degree to study these non-linear problems.

In this work we will develop an approach that allows us to overcome this major difficulty. The key point will be the reduction of the differential operator without the Fredholm property to an integro-differential operator that possesses it. Our method applies to problem (1.1)-(1.4) but can be extended to more general systems where the components of the non-linearity vector are linearly dependent. These results will be published elsewhere.

Let us briefly sketch the construction of the integro-differential operator that is detailed in Section 2. We first introduce the function

$$h = \theta + \psi - 1. \tag{1.10}$$

By adding equations (1.1) and (1.2) we see that it satisfies the linear equation

$$-\Delta h + \alpha(y)\frac{\partial h}{\partial x} = (\Lambda - 1)\Delta\psi, \tag{1.11}$$

together with the boundary condition

$$\frac{\partial h}{\partial\nu} = 0 \ \text{ on } \partial\Omega, \quad h(\pm\infty, y) = 0. \tag{1.12}$$

We will check that Problem (1.11), (1.12) possesses a unique solution $h$. This will provide $h$ as a function of $\psi$. The corresponding operator $h = \mathcal{H}(\psi)$ turns out to be bounded and differentiable in appropriate spaces. This will allow the reduction of system (1.1)-(1.3) into the integro-differential equation

$$-\Lambda\Delta\psi + \alpha(y)\frac{\partial\psi}{\partial x} + \kappa(\mathcal{H}(\psi) + 1 - \psi, \psi) = 0, \tag{1.13}$$

$$\frac{\partial\psi}{\partial\nu} = 0 \ \text{ on } \partial\Omega. \tag{1.14}$$

The resolution of this problem will provide $\psi$ while the unknown $\theta$ will be given by $\theta = \mathcal{H}(\psi) + 1 - \psi$.

Let us introduce the linearized operator associated to (1.13) at some point $\psi$ that reads

$$Lv = -\Lambda\Delta v + \alpha(y)\frac{\partial v}{\partial x} + \kappa'_\theta(\theta,\psi)(\mathcal{H}'(\psi)v - v) + \kappa'_\psi(\theta,\psi)v. \tag{1.15}$$

In Section 2.2, the integro-differential formulation is first used to derive existence results for problem (1.8)-(1.9) for $\Lambda$ close to one. Note that for $\Lambda = 1$ the system reduces to a scalar equation since $\psi \equiv 1-\theta$. The existence of multidimensional waves described by such scalar equations is studied in [3, 4, 9, 10, 12, 27, 29]. Few results have been previously derived for systems. Monotone systems of equations are considered in [29, 30, 31]. Problem (1.8)-(1.9) with $\Lambda \neq 1$ and close to 1 is studied in [7, 8] with a completely different and less general method. The existence of solutions is first proved in a bounded rectangle. Next a priori estimates independent of the rectangle allow the proof of the existence of solutions in the unbounded strip. Also, parabolic problems with the Lewis number different from one are studied in [13, 14, 16, 17, 18, 19].

Coming back to system (1.8), (1.9), we assume that

$$\begin{aligned}&\beta(c,y) \text{ is continuous on } \mathbb{R}\times\bar{\omega} \text{ and}\\ &\text{has a derivative with respect to } c, \text{ denoted by } \beta'(c,y)\\ &\text{that is continuous and non-negative on } \mathbb{R}\times\bar{\omega}.\end{aligned} \tag{1.16}$$

It will be convenient to set

$$\overline{\beta}(c) = \int_\omega \beta(c,y)dy. \tag{1.17}$$

As already mentioned existence results for the scalar equation ($\Lambda = 1$) are well known. We aim to investigate the existence of solutions of the system in the neighborhood of these scalar solutions for $\Lambda$ close to 1. More precisely the following result holds.

**Theorem 1.1.** *Suppose that for $\Lambda = 1$ problem* (1.8), (1.9), (1.3), (1.4) *has a solution* $(\theta_0, 1-\theta_0, c_0)$ *with* $\bar{\beta}(c_0) > 0$. *Then, under conditions* (1.5), (1.6), (1.16), *for all $\Lambda$ sufficiently close to* 1, *problem* (1.8), (1.9), (1.3), (1.4) *has a solution* $(\theta,\psi,c)$.

The proof of this theorem is based on the integro-differential formulation that is given here by (1.13), (1.14) with $\alpha(y) = \beta(c,y)$. The linearized operator at some point $\psi$ is given by (1.15). Now, if $\Lambda = 1$, we have $\psi \equiv 1-\theta$ so that $\mathcal{H}(\psi) = \mathcal{H}'(\psi) \equiv 0$. In that particular case the operator $L$ is the usual differential operator with well-known properties. These properties will allow us to apply the implicit function theorem and to derive the above existence result.

The result in Theorem 1.1 has been mentioned in the 1990s in lecture [2] but to our knowledge was not published. Since we need it for this and for subsequent works we prove it independently.

In the general case, the above perturbation argument requires us to take into account properties of the operator $\mathcal{H}'$. One of the main results of this work establishes the Fredholm property of the linearized operator. We need the additional condition on the non-linearity

$$\frac{\partial\kappa}{\partial\theta}(1,0) = \frac{\partial\kappa}{\partial\theta}(0,1) = 0. \tag{1.18}$$

We will discuss it in more detail in Section 3.1 and will see that it is not restrictive for combustion problems.

**Theorem 1.2.** *Let $\kappa$ and $\alpha$ be given and satisfy* (1.5)–(1.7), (1.18). *Let also $\theta$, $\psi$ be two functions in $H^2(\Omega)$ satisfying the boundary conditions* (1.3). *Then, the operator $L$ defined in* (1.15) *and considered as acting from $H^2(\Omega)$ with boundary condition* (1.14) *into $L^2(\Omega)$ is Fredholm with the zero index.*

An application of this result is illustrated with an existence theorem (Theorem 3.3). If in the previous case we prove existence of solutions close to a multi-dimensional solution with $\Lambda = 1$, in this case we consider any $\Lambda$ and prove existence of multi-dimensional solutions close to a one-dimensional solution under the assumption that the function $\alpha(y)$ is close to a constant. This difference appears to be rather essential. In the first case the linearized operator does not contain the operator $\mathcal{H}$ because it is identically zero for $\Lambda = 1$. In the second case the linearized operator contains $\mathcal{H}$, and we need to prove its Fredholm property to obtain solvability conditions.

The remaining part of the paper is related to the topological degree. There are several degree constructions for Fredholm and proper operators with the zero index (see [21], [28] and the references therein). Here we use the degree constructed in [28], which is well adapted for elliptic problems in unbounded domains. We recall that properness is understood here in the sense that the intersection of the inverse image of a compact set with any bounded closed set is compact.

An essential point of the construction is the use of weighted spaces because otherwise elliptic problems in unbounded domains are not necessarily proper [28]. We study the Fredholm property of our operators in weighted spaces in Section 4, and prove their properness in Section 5. In Section 6 we define the topological degree.

The last section of the paper is devoted to some applications of the topological degree to bifurcations of solutions. The classical bifurcation result affirms that a bifurcation occurs when a simple eigenvalue of the linearized operator crosses the origin. We cannot apply this result for the original differential operators because they do not satisfy the Fredholm property and the degree is not defined for them. We have defined the degree for the corresponding integro-differential operators but we now need to verify that bifurcation points remain the same. This means that if a real eigenvalue crosses the origin for the differential operator, then it is also the same for the integro-differential operator. This is checked in Section 7 and we also prove the bifurcation result. In the framework of combustion theory this bifurcation results in the appearance of cellular flames.

## 2. The integro-differential problem and application

2.1. **The integro-differential formulation.** As mentioned in the Introduction we will re-write problem (1.1)-(1.4) as an integro-differential equation with respect to $\psi$. For that purpose, we first need to investigate problem (1.11), (1.12).

It will be useful to consider this linear problem with a more general right-hand side lying in the space

$$V = \Big\{ f \in L^2(\Omega)^{d+1} : \nabla . f \in L^2(\Omega) \ \text{ and } \ f.\nu = 0 \text{ on } \partial\Omega \Big\}. \tag{2.1}$$

It is well known (see for example [25]) that $V$ equipped with the norm

$$\|f\|_V^2 = \|f\|_{L^2(\Omega)}^2 + \|\nabla . f\|_{L^2(\Omega)}^2$$

is a Banach space. Also we introduce the open set

$$\mathcal{V} = \Big\{ \alpha \in \mathcal{C}^0(\bar{\omega}) : \ \bar{\alpha} = \int_\omega \alpha(y) dy > 0 \Big\}, \tag{2.2}$$

equipped with the supremum norm.

Next assume that $f \in V$ and $\alpha \in \mathcal{V}$ are given. We consider the equation

$$-\Delta g + \alpha(y) \frac{\partial g}{\partial x} = \nabla . f \tag{2.3}$$

supplemented with the boundary condition

$$\frac{\partial g}{\partial \nu} = 0 \ \text{ on } \partial\Omega \tag{2.4}$$

and the limits at infinity

$$\lim_{x \to \pm\infty} g(x,y) = 0, \ \text{ uniformly with respect to } y \in \omega. \tag{2.5}$$

Let us set

$$E = \Big\{ v \in H^2(\Omega) : \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \partial\Omega \Big\}. \tag{2.6}$$

The existence of a solution is guaranteed by the following:

**Lemma 2.1.** *For all $f \in V$ and $\alpha \in \mathcal{V}$, problem* (2.3)-(2.5) *possesses a unique solution $g$ in $E$.*

**Proof.** The difficulty lies in the unboundness of the open set $\Omega$. We consider the following problem posed in a bounded open set $\Omega_a = (-a, a) \times \omega$, $a > 0$,

$$-\Delta g + \alpha(y) \frac{\partial g}{\partial x} = \nabla . f \ \text{ in } \Omega_a, \tag{2.7}$$

$$\frac{\partial g}{\partial \nu} = 0 \ \text{ on } (-a, a) \times \partial\omega, \tag{2.8}$$

$$-\frac{\partial g}{\partial x}(-a, y) + \alpha(y) g(-a, y) = f_1(-a, y), \ \ g(a, y) = 0, \ \text{ for } y \in \omega. \tag{2.9}$$

Here $f = (f_1, f_2)$ with $f_1(x, y) \in \mathbb{R}$ and $f_2(x, y) \in \mathbb{R}^d$. Note that $f_1(-a, y)$ is well defined since $f \in V$. This linear problem possesses a unique solution $g_a$. Next, computations similar to those in Lemma 3.2 below allow us to obtain estimates for $g_a$ independent of $a > 0$. Then taking the limit as $a \to +\infty$ provides a solution of (2.3)-(2.5). The details are left to the reader. □

The preceding lemma allows us to consider the operator $\mathcal{G} : V \times \mathcal{V} \to E$, defined by the resolution of problem (2.3)-(2.5). We now aim to derive some estimates for this operator.

**Lemma 2.2.** *The operator $\mathcal{G}$ given by the resolution of* (2.3)-(2.5) *satisfies the estimates*

$$\|\mathcal{G}(f, \alpha)\|_{H^1} \leq M(\alpha) \|f\|_{L^2} \ \ \textit{for } f \in V, \ \alpha \in \mathcal{V}, \tag{2.10}$$

$$\|\mathcal{G}(f, \alpha)\|_{H^2} \leq M(\alpha) \|f\|_{V} \ \ \textit{for } f \in V, \ \alpha \in \mathcal{V}, \tag{2.11}$$

*where $M$ denotes a function depending only on $\alpha$ and that is locally bounded on $\mathcal{V}$.*

**Proof.** Let $f = (f_1, f_2) \in V$ and $\alpha \in \mathcal{V}$ be given functions. We introduce an orthonormal basis of $L^2(\omega)$ consisting of the eigenfunctions of the operator $-\Delta$ in $L^2(\omega)$ together with the homogeneous Neumann condition

$$\begin{aligned} &-\Delta \phi_n = \lambda_n \phi_n \ \text{ in } \omega, \ \frac{\partial \phi_n}{\partial \nu} = 0 \text{ on } \partial\omega, \\ &0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ..., \ \ \lambda_n \to +\infty \ \text{ for } n \to +\infty. \end{aligned} \tag{2.12}$$

For $n \geq 0$, we set

$$\gamma_n = (g, \phi_n)_{L^2(\omega)}. \tag{2.13}$$

We have

$$\int_\omega \Delta_y g g dy = \sum_{n=0}^{+\infty} (\Delta_y g, \phi_n)_{L^2(\omega)} (g, \phi_n)_{L^2(\omega)} = -\sum_{n=1}^{+\infty} \lambda_n |\gamma_n|^2. \tag{2.14}$$

Next, multiplying equation (2.3) by $g$ and integrating over $\omega$, we obtain for almost every $x \in \mathbb{R}$

$$-\int_\omega \frac{\partial^2 g}{\partial x^2} g dy + \sum_{n=1}^{+\infty} \lambda_n |\gamma_n|^2 + \int_\omega \alpha(y) \frac{\partial g}{\partial x} g dy = \int_\omega \nabla . f g dy. \tag{2.15}$$

Then, integrating with respect to $x \in \mathbb{R}$ implies that

$$\iint_\Omega (\frac{\partial g}{\partial x})^2 dxdy + \sum_{n=1}^{+\infty} \lambda_n \|\gamma_n\|^2_{L^2(\mathbb{R})} + \iint_\Omega \alpha(y) \frac{\partial g}{\partial x} g dxdy = \iint_\Omega \nabla . f g dxdy. \tag{2.16}$$

Therefore, since $f \in V$,

$$\lambda_1 \sum_{n=1}^{+\infty} \|\gamma_n\|^2_{L^2(\mathbb{R})} \leq \iint_\Omega \nabla . f g dxdy = -\iint_\Omega f \nabla g dxdy. \tag{2.17}$$

Next, multiplying equation (2.3) by $g$ and integrating over $\Omega$, we see that

$$\iint_\Omega |\nabla g|^2 dxdy \leq \iint_\Omega |f|^2 dxdy. \tag{2.18}$$

Therefore, we infer from (2.16)-(2.18) that

$$\sum_{n=1}^{+\infty} \|\gamma_n\|^2_{L^2(\mathbb{R})} \leq \frac{1}{\lambda_1} \|f\|^2_{L^2(\Omega)}. \tag{2.19}$$

Thus, setting $\tilde{g} = g - \gamma_0$, we obtain the following estimate:

$$\|\tilde{g}\|^2_{L^2} \leq \frac{1}{\lambda_1} \|f\|^2_{L^2(\Omega)}. \tag{2.20}$$

Now we need to estimate the norm of $\gamma_0$. Denoting by $'$ the derivative with respect to $x$, it is easy to check the equation

$$-\gamma_0'' + \bar{\alpha} \gamma_0' = u_1' - j_0', \tag{2.21}$$

where we have set

$$u_1 = (f_1, \phi_0)_{L^2(\omega)}, \; j_0 = (\alpha \tilde{g}, \phi_0)_{L^2(\omega)}. \tag{2.22}$$

Applying the Fourier transform to equation (2.21), we obtain

$$\hat{\gamma_0}(\xi) = \frac{i(\hat{u_1}(\xi) - \hat{j_0}(\xi))}{\xi + i\bar{\alpha}}. \tag{2.23}$$

Then we have

$$\|\gamma_0\|^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} \frac{|\hat{u_1}(\xi) - \hat{j_0}(\xi)|^2}{\xi^2 + \bar{\alpha}^2} d\xi \leq \frac{1}{\bar{\alpha}^2} \int_{\mathbb{R}} |\hat{u_1}(\xi) - \hat{j_0}(\xi)|^2 d\xi. \tag{2.24}$$

Recalling the definition (2.22) and (2.20), we see that

$$\begin{aligned} \|\gamma_0\|_{L^2(\mathbb{R})} &\leq \frac{1}{\bar{\alpha}}(\|f_1\|_{L^2(\Omega)} + \|\alpha\|_\infty \|\tilde{g}\|_{L^2(\Omega)}) \\ &\leq \frac{1}{\bar{\alpha}}(\|f\|_{L^2(\Omega)} + \frac{\|\alpha\|_\infty}{\sqrt{\lambda_1}} \|f\|_{L^2(\Omega)}). \end{aligned} \tag{2.25}$$

Combining (2.25) and (2.20) we conclude

$$\|g\|_{L^2(\Omega)} \leq N(\alpha) \|f\|_{L^2(\Omega)}, \tag{2.26}$$

where the function $N$ is defined by the equality

$$N(\alpha) = \frac{1}{\bar{\alpha}}(1 + \frac{\|\alpha\|_\infty}{\sqrt{\lambda_1}}) + \frac{1}{\sqrt{\lambda_1}}. \tag{2.27}$$

Finally, we multiply equation (2.3) by $\Delta g$ and integrate over $\Omega$. This implies that

$$\iint_\Omega |\Delta g|^2 dxdy = \iint_\Omega (\nabla . f - \alpha(y) g_x)(\Delta g) dxdy, \tag{2.28}$$

and thanks to (2.18),

$$\|\Delta g\|_{L^2} \leq (1 + \|\alpha\|_\infty) \|f\|_V. \tag{2.29}$$

To conclude, estimates (2.18), (2.26) and (2.29) imply (2.10) and (2.11). □

In the following, we derive some regularity properties of the operator $\mathcal{G}$.

**Lemma 2.3.** *The operator $\mathcal{G}$ is continuous from $V \times \mathcal{V}$ into $E$.*

**Proof.** Let $(f_1, \alpha_1)$ and $(f_2, \alpha_2)$ be in $V \times \mathcal{V}$. Then, setting

$$\delta = \mathcal{G}(f_1, \alpha_1) - \mathcal{G}(f_2, \alpha_2),$$

we have the equation

$$-\Delta\delta + \alpha_1(y)\delta_x = \nabla.(f_1 - f_2) + (\alpha_2(y) - \alpha_1(y)) \frac{\partial}{\partial x} \mathcal{G}(f_2, \alpha_2) = \nabla . k_1,$$

where

$$k_1 = f_1 - f_2 + (\alpha_2(y) - \alpha_1(y))(\mathcal{G}(f_2, \alpha_2), 0) \in V.$$

Then estimate (2.11) yields that

$$\begin{aligned}\|\delta\|_{H^2} &\le M(\alpha_1)\|k_1\|_V \\ &\le M(\alpha_1)\big(\|f_1-f_2\|_V+\|\alpha_2-\alpha_1\|_\infty\|\mathcal{G}(f_2,\alpha_2)\|_{H^1}\big) \\ &\le M(\alpha_1)\big(\|f_1-f_2\|_V+\|\alpha_2-\alpha_1\|_\infty M(\alpha_2)\|f_2\|_{L^2}\big),\end{aligned} \tag{2.30}$$

which provides the continuity of the operator $\mathcal{G}$. □

**Lemma 2.4.** *The operator $\mathcal{G}$ is of class $\mathcal{C}^1$ from $V\times\mathcal{V}$ into $E$. The partial Fréchet derivatives of the operator $\mathcal{G}$ are given by the equalities*

$$\mathcal{G}'_f(f_0,\alpha_0).f=\mathcal{G}(f,\alpha_0), \tag{2.31}$$

$$\mathcal{G}'_\alpha(f_0,\alpha_0)\alpha=-\mathcal{G}\big((\alpha\mathcal{G}(f_0,\alpha_0),0),\alpha_0\big). \tag{2.32}$$

**Proof.** First of all we note that $\mathcal{G}(f,\alpha)$ depends linearly on $f$, so that its derivative with respect to $f$ is given by (2.31). Next let $(f_0,\alpha_0)\in V\times\mathcal{V}$ and let $\alpha$ be such that $\alpha_0+\alpha\in\mathcal{V}$. We set

$$g_0=\mathcal{G}(f_0,\alpha_0),\quad g=\mathcal{G}(f_0,\alpha_0+\alpha),\quad \tilde{g}=g-g_0. \tag{2.33}$$

Clearly, $\tilde{g}$ satisfies

$$-\Delta\tilde{g}+\alpha_0(y)\frac{\partial\tilde{g}}{\partial x}=-\alpha(y)\frac{\partial g}{\partial x}, \tag{2.34}$$

and (2.10) yields that

$$\|\tilde{g}\|_{H^1}\le M(\alpha_0)\|g\|_{L^2}\|\alpha\|_\infty. \tag{2.35}$$

Next we consider a solution $\bar{g}$ of

$$\begin{aligned}&-\Delta\bar{g}+\alpha_0(y)\frac{\partial\bar{g}}{\partial x}=-\alpha(y)\frac{\partial g_0}{\partial x},\\ &\frac{\partial\bar{g}}{\partial\nu}=0\ \text{ on }\partial\Omega,\quad \bar{g}(\pm\infty,y)=0\ \text{ for } y\in\omega.\end{aligned} \tag{2.36}$$

Then

$$-\Delta(\tilde{g}-\bar{g})+\alpha_0(y)\frac{\partial(\tilde{g}-\bar{g})}{\partial x}=-\alpha(y)\frac{\partial\tilde{g}}{\partial x}, \tag{2.37}$$

and in view of (2.11) we have

$$\|\tilde{g}-\bar{g}\|_{H^2}\le M(\alpha_0)\|\tilde{g}\|_{H^1}\|\alpha\|_\infty\le M(\alpha_0)^2\|g\|_{L^2}\|\alpha\|_\infty^2. \tag{2.38}$$

This shows that $\mathcal{G}$ is differentiable with respect to $\alpha$ and $\mathcal{G}'_\alpha(f_0,\alpha_0).\alpha=\bar{g}$.

The proof of the continuity of the derivatives is left to the reader. □

We are now able to introduce the integro-differential formulation of problem (1.1)-(1.4). We first take care of the non-homogeneous boundary conditions by introducing some fixed $C^\infty$ function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) = 1 \quad \text{if } x < -1 \text{ and } \phi(x) = 0 \text{ if } x > 1. \tag{2.39}$$

We set

$$v = \psi - \phi \ \text{ and } \ u = \theta + \phi - 1. \tag{2.40}$$

The functions $u$ and $v$ satisfy homogeneous conditions at infinity and will be our new unknowns.

Now, if $(\theta, \psi) = (u+1-\phi, v+\phi)$ is a solution of (1.1)-(1.4), then $h = u+v$ satisfies

$$\begin{aligned} &-\Delta h + \alpha(y)\frac{\partial h}{\partial x} = (\Lambda - 1)\Delta(v+\phi), \\ &\frac{\partial h}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \quad h(\pm\infty, y) = 0 \ \text{ for } y \in \omega. \end{aligned} \tag{2.41}$$

Recalling the definitions (2.6) and (2.1) of the spaces $E$ and $V$, for any $v \in E$, we have that $\nabla(v+\phi) \in V$. Therefore, according to Lemma 2.1, problem (2.41) possesses a unique solution $h$ and we can set $h = \mathcal{H}(v)$. In view of Lemma 2.4, the operator $\mathcal{H}$ is continuously differentiable as acting from $E$ into $E$.

Consequently, for a solution $(u+1-\phi, v+\phi)$ of (1.1)-(1.4), we have $u+v = \mathcal{H}(v)$ so that $u = \mathcal{H}(v) - v$. This leads us to substitute into problem (1.1)-(1.4) the following integro-differential equation for the scalar unknown $v$ :

$$\begin{aligned} &-\Lambda\Delta(v+\phi) + \alpha(y)\frac{\partial(v+\phi)}{\partial x} + \kappa(h - v + 1 - \phi, v + \phi) = 0, \\ &\frac{\partial v}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \quad v(\pm\infty, y) = 0 \ \text{ for } y \in \omega. \end{aligned} \tag{2.42}$$

Clearly, once $v$ is determined, the other unknown $u$ is given by $u = \mathcal{H}(v) - v$.

For later use let us introduce the operator corresponding to (2.42). It reads

$$\mathcal{A}(v) = -\Lambda\Delta(v+\phi) + \alpha(y)\frac{\partial(v+\phi)}{\partial x} + \kappa\big(\mathcal{H}(v) - v + 1 - \phi, v + \phi\big). \tag{2.43}$$

This operator maps $E$ into $F = L^2(\Omega)$ and is continuously differentiable.

2.2. **Existence of travelling waves for $\Lambda$ close to one.** Here we aim to apply the above approach to investigate the existence of solutions of problem (1.8), (1.9) together with the conditions (1.3), (1.4).

Recall that the quantity $\beta(c,y)$ depends on the unknown real parameter $c$ and satisfies (1.16). In view of condition (1.7) it is natural to introduce

$$\mathcal{O} = \left\{ c \in \mathbb{R} : \overline{\beta}(c) = \int_\omega \beta(c,y)dy > 0 \right\} \tag{2.44}$$

that is an open subset of $\mathbb{R}$.

As already mentioned, for $\Lambda = 1$ the system reduces to a scalar equation ($\theta \equiv 1-\psi$) for which existence results are well known. We will consider the general system for $\Lambda$ close to 1 in the neighborhood of these scalar solutions.

**Theorem 2.5.** *Assume that* (1.5), (1.6), (1.16) *hold. Moreover suppose that problem* (1.8), (1.9), (1.3), (1.4) *for* $\Lambda = 1$ *has a solution* $(1-\psi_0, \psi_0, c_0)$ *with* $c_0 \in \mathcal{O}$. *Then there exists* $\epsilon > 0$ *such that for all* $\Lambda$ *such that* $|\Lambda - 1| < \epsilon$ *there exists a solution* $(\theta_\Lambda, \psi_\Lambda, c_\Lambda) \in C^1_{loc}(\bar\Omega) \times C^1_{loc}(\bar\Omega) \times \mathcal{O}$ *of the problem*

$$-\Delta\theta_\Lambda + \beta(c_\Lambda, y)\frac{\partial\theta_\Lambda}{\partial x} = \kappa(\theta_\Lambda, \psi_\Lambda) \ \ \text{in } \Omega, \tag{2.45}$$

$$-\Lambda\Delta\psi_\Lambda + \beta(c_\Lambda, y)\frac{\partial\psi_\Lambda}{\partial x} = -\kappa(\theta_\Lambda, \psi_\Lambda) \ \ \text{in } \Omega, \tag{2.46}$$

$$\frac{\partial\theta_\Lambda}{\partial\nu} = \frac{\partial\psi_\Lambda}{\partial\nu} = 0 \ \ \text{on } \partial\Omega, \tag{2.47}$$

$$\begin{aligned} &\lim_{x\to-\infty} \theta_\Lambda(x,y) = 0, \quad \lim_{x\to+\infty} \theta_\Lambda(x,y) = 1, \\ &\lim_{x\to-\infty} \psi_\Lambda(x,y) = 1, \quad \lim_{x\to+\infty} \psi_\Lambda(x,y) = 0. \end{aligned} \tag{2.48}$$

*The limits* (2.48) *are uniform with respect to* $y \in \omega$. *In addition, the solution* $(\theta_\Lambda, \psi_\Lambda, c_\Lambda)$ *depends continuously on* $\Lambda$ *in the topology of* $C^1_{loc}(\bar\Omega) \times C^1_{loc}(\bar\Omega) \times \mathcal{O}$.

**Remark 2.6.** The above framework in particular allows $\beta(c,y)$ to take the form $\beta(c,y) = c + \gamma(y)$ as well as $\beta(c,y) = c\gamma(y)$ with $\gamma(y)$ satisfying the analog of (1.7). For $\beta(c,y) = c\gamma(y)$ conditions on the scalar equation that guarantee that $c_0 \in \mathcal{O}$ are well known. For example this holds true if $\int_0^1 \kappa(s, 1-s)ds = 0$ (see [4, 9, 10, 12, 27, 29]).

**Remark 2.7.** The specific form of the function $\kappa(\theta,\psi)$ that corresponds to chemical kinetics with the first-order reaction is

$$\kappa(\theta,\psi) = \kappa_0(\theta)\psi. \tag{2.49}$$

In combustion theory, the function $\kappa_0(\theta)$ is usually taken in the form of the Arrhenius exponential, which can be approximated by the usual exponential

$$\kappa_0(\theta) = ke^{Z(\theta-1)}, \tag{2.50}$$

where $Z$ is the Zeldovich number and $k$ is some constant. As is well known, there is a so-called cold boundary difficulty. This means that the function $\kappa_0(\theta)$ is everywhere positive and, consequently, problem (1.1)-(1.3) can not have bounded solutions in $\Omega$. If the Zeldovich number is sufficiently large, which is the case for combustion fronts, then $\kappa_0(\theta)$ is very small for low temperatures. With a very good accuracy it can be replaced by a cut-off function identically equal to zero if $\theta \leq \theta^*$ for some given temperature $\theta^*$.

The cut-off procedure, though widely used, has its own disadvantage from the point of view of properties of the corresponding operators. Since the function is identically zero on some interval of temperature including its value at infinity, the essential spectrum of the linearized operator contains zero even in the case where the non-linearity vector is not linearly dependent. This problem can be easily removed by the introduction of weighted spaces with a small exponential weight that moves the essential spectrum to the left half-plane. To avoid this technical complication we will assume that the function $\kappa(\theta, \psi)$ is not identically zero for small $\theta$ but is zero only at $\theta = 0, \psi = 1$ (the exact condition is given by (1.6)). This assumption simplifies the presentation, and is not essential from the point of view of physical applications.

Let us prove Theorem 2.5. As above, we take care of the non-homogeneous conditions at infinity by considering the function $\phi$ given by (2.39) and introducing the new unknowns $u = \theta + \phi - 1$ and $v = \psi - \phi$.

Problem (1.8), (1.9) is similar to (1.1), (1.2) except for the presence of the additional unknown $c$. Also the dependence with respect to $\Lambda$ plays an important role in the sequel. This leads us to consider $X = E \times \mathcal{O} \times (0, +\infty)$ where $\mathcal{O}$ is given by (2.44). For any $(v, c, \Lambda) \in X$, we can solve the problem

$$\begin{aligned} &-\Delta h + \beta(c, y)\frac{\partial h}{\partial x} = (\Lambda - 1)\Delta(v + \phi), \\ &\frac{\partial h}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad h(\pm\infty, y) = 0 \text{ for } y \in \omega, \end{aligned} \tag{2.51}$$

since this problem takes the form (2.3) for an appropriate choice of $(\alpha, f)$. Consequently (2.51) possesses a unique solution $h$ and we can write $h = \mathcal{H}(v, c, \Lambda)$. In view of Lemma 2.4 the operator $\mathcal{H}$ is continuously differentiable from $X$ into $E$.

The reduction of problem (1.8), (1.9), (1.3), (1.4) for the unknown

$$(\theta, \psi, c) = (u + 1 - \phi, v + \phi, c)$$

to an integro-differential equation for $(v, c)$ is as described in the end of Section 2.1. Here $h = u + v$ satisfies (2.51). Therefore we will look for $(v, c)$

such that

$$\begin{aligned} &-\Lambda\Delta(v+\phi)+\beta(c,y)\frac{\partial(v+\phi)}{\partial x}+\kappa(\mathcal{H}(v,c,\Lambda)-v+1-\phi,v+\phi)=0,\\ &\frac{\partial v}{\partial \nu}=0 \text{ on } \partial\Omega, \quad v(\pm\infty,y)=0 \text{ for } y\in\omega. \end{aligned} \tag{2.52}$$

Then, assuming that $(v,c)$ is determined, the last unknown will be given by $u=\mathcal{H}(v,c,\Lambda)-v$.

Following (2.52) it is convenient to introduce the function $\tilde{\kappa}:\mathbb{R}^2\to\mathbb{R}$ defined by

$$\tilde{\kappa}(h,\psi)=\kappa(h+1-\psi,\psi) \tag{2.53}$$

and the operator $\mathcal{B}:X\to F=L^2(\Omega)$ given by

$$\mathcal{B}(v,c,\Lambda)=-\Lambda\Delta(v+\phi)+\beta(c,y)\frac{\partial(v+\phi)}{\partial x}+\tilde{\kappa}(\mathcal{H}(v,c,\Lambda),v+\phi). \tag{2.54}$$

With this notation solutions of problem (1.8), (1.9), (1.3), (1.4) are associated to zeros of $\mathcal{B}$.

The operator $\mathcal{B}$ is differentiable as proved in the following.

**Lemma 2.8.** *The operator $\mathcal{B}$ acting from $X$ into $F$ is continuously differentiable.*

**Proof.** We first check the continuity of $\mathcal{B}$. Clearly the first two terms in (2.54) depend continuously on their arguments so that we only need to verify the continuity of the mapping

$$(v,c,\Lambda)\to\tilde{\kappa}(\mathcal{H}(v,c,\Lambda),v+\phi).$$

Recall that the operator $\mathcal{H}$ is continuous from $X$ into $E$. Consequently, the desired continuity follows from the embedding $H^2\hookrightarrow L^\infty$ since the function $\tilde{\kappa}$ is locally Lipschitz continuous.

The differentiability of $\mathcal{B}$ can easily be derived. The proof is left to the reader. □

**Proof of Theorem 2.5.** Let us assume that $(1-\psi_0,\psi_0,c_0)$ is a solution of (1.8), (1.9), (1.3), (1.4) for $\Lambda=1$ such that $c_0\in\mathcal{O}$. Let us introduce $v_0=\psi_0-\phi$. Since $\Lambda=1$ the solution of (2.51) is equal to zero, so that $\mathcal{H}(v_0,c_0,1)=0$. Also $(v_0,c_0)$ satisfies (2.52) (with $\Lambda=1$), that may be rewritten, thanks to definition (2.54), as

$$\mathcal{B}(v_0,c_0,1)=0. \tag{2.55}$$

We aim to show the existence of zeros of $\mathcal{B}$ in the vicinity of $(v_0,c_0,1)$ by applying the implicit function theorem. For that purpose we need to investigate the operator $\mathcal{B}'_{v,c}(v_0, c_0, 1)$. As already mentioned, we have $\mathcal{H}(v, c, 1) \equiv 0$. Therefore the derivative at $(v_0, c_0, 1)$ with respect to $(v, c)$ reads

$$\mathcal{B}'_{v,c}(v_0, c_0, 1).(w, d) = -\Delta w + \beta(c_0, y)\frac{\partial w}{\partial x} + \beta'(c_0, y)d\frac{\partial(v_0 + \phi)}{\partial x} + \Psi(v_0)w,$$

where we have set

$$\Psi(v_0) = \frac{\partial \tilde{\kappa}}{\partial \psi}(0, v_0 + \phi). \tag{2.56}$$

We now study the equation

$$\mathcal{B}'_{v,c}(v_0, c_0, 1).(w, d) = f \ \text{ for } f \in L^2(\Omega). \tag{2.57}$$

It can be written in the form

$$Lw = f - d\beta'(c_0, y)\frac{\partial(v_0 + \phi)}{\partial x}, \tag{2.58}$$

where

$$Lw = -\Delta w + \beta(c_0, y)\frac{\partial w}{\partial x} + \Psi(v_0)w. \tag{2.59}$$

Let us check that (2.58) is solvable in $E$ for a unique suitable value of $d$. Recalling definition (2.53), assumption (1.6) guarantees that

$$\frac{\partial \tilde{\kappa}}{\partial \psi}(0, 0)0 \ \text{ and } \ \frac{\partial \tilde{\kappa}}{\partial \psi}(0, 1) > 0. \tag{2.60}$$

Therefore, $\Psi(v_0)$ has positive limits as $x \to \pm\infty$. Consequently the essential spectrum of the operator $L$ lies in the right half-plane. Thus it is a Fredholm operator with zero index. Moreover, it possesses zero as an algebraically simple eigenvalue; the subspace $kerL$ is spanned by $w_0 = \frac{\partial \psi_0}{\partial x}$, that is a negative function (see [4]). Therefore, the equation

$$Lw = g \tag{2.61}$$

is solvable in $E$ if and only if $g$ is orthogonal in $L^2(\Omega)$ to the eigenfunction $w^*$ corresponding to the zero eigenvalue of the formally adjoint operator

$$L^* w^* = -\Delta w^* - \beta(c_0, y)\frac{\partial w^*}{\partial x} + \Psi(v_0)w^*$$

(see [1]). The eigenfunction $w^*$ has also a constant sign, positive for instance (see [30]).

Thus, the solvability condition applied to equation (2.58) reads

$$\iint_\Omega (f - d\beta'(c_0, y)\frac{\partial(v_0 + \phi)}{\partial x})w^* dxdy = 0.$$

Therefore, for any $f \in L^2(\Omega)$ we can choose $d$ such that it is satisfied provided that

$$\iint_\Omega \beta'(c_0, y)\frac{\partial(v_0+\phi)}{\partial x} w^* dxdy \neq 0. \tag{2.62}$$

Now recall that $w^*$ is positive and $\frac{\partial(v_0+\phi)}{\partial x}$ is negative. Therefore, thanks to (1.16) we conclude that condition (2.62) holds, so that, for any $f \in L^2(\Omega)$, problem (2.57) possesses a solution $(w, d) \in E \times \mathbb{R}$.

The operator $\mathcal{B}'_{v,c}(v_0, c_0, 1)$ maps $E \times \mathbb{R}$ onto $F$ but is not invertible since, for any right-hand side, equation (2.57) has a one-dimensional family of solutions $(w_1 + \tau w_0, d)$ where $\tau \in \mathbb{R}$, $(w_1, d)$ is a solution of problem (2.57), and $w_0$ is generating $kerL$. However, equation (2.57) possesses a unique solution in $E_0 \times \mathbb{R}$ with

$$E_0 = (kerL)^\perp = \Big\{ w \in E : \iint_\Omega w w_0 \, dxdy = 0 \Big\}. \tag{2.63}$$

Let us consider the operator $\mathcal{B}$ as acting from the sub-manifold of $X$ defined by $M = (v_0+E_0)\times\mathcal{O}\times(0, +\infty)$ into $F$. The operator $\mathcal{B} : M \to F$ is continuously differentiable and its derivative with respect to $(v, c)$ at $(v_0, c_0, 1) \in M$ is a linear isomorphism from the tangent space $E_0 \times \mathbb{R}$ into $F$. This operator satisfies the hypothesis of the implicit function theorem in the neighborhood of $(v_0, c_0, 1)$, so that the equation $\mathcal{B}(v, c, \Lambda) = 0$ possesses a solution $(v, c)$ for all values of the parameter $\Lambda$ sufficiently close to 1.

Finally, for such values of $\Lambda$, let us define the functions

$$\psi = v + \phi \text{ and } \theta = 1 - \psi + \mathcal{H}(v, c, \Lambda). \tag{2.64}$$

Then $(\theta, \psi, c)$ is a solution of problem (2.45)-(2.48). The implicit function theorem also provides the continuous dependence of this solution with respect to the parameter $\Lambda$. □

## 3. Fredholm property and applications

In this section, we prove the Fredholm property for the operator introduced in (2.43) (Section 3.1). Next, this property is used to investigate some existence results for problem (1.8), (1.9) (Section 3.2).

3.1. **Fredholm property.** Let $\Lambda > 0$ and $\alpha$ satisfying (1.7) be given. Also we are given $\kappa$ a $C^2$ function from $\mathbb{R}^2$ into $\mathbb{R}$ satisfying (1.5), (1.6), (1.18) and a function $\phi$ as in (2.39). Recall that we defined the function $\tilde{\kappa}$ by (2.53).

We consider the operator given by (2.43); that is,

$$\mathcal{A}(v) = -\Lambda\Delta(v+\phi) + \alpha(y)\frac{\partial(v+\phi)}{\partial x} + \tilde{\kappa}\big(\mathcal{H}(v), v+\phi\big),$$

where $\mathcal{H}(v)$ is the solution of (2.41). As already mentioned this operator maps $E$ into $F$ and is continuously differentiable. The operator linearized about $v_0 \in E$ takes the form

$$Aw = -\Lambda\Delta w + \alpha(y)\frac{\partial w}{\partial x} + a_0(x,y)w + b_0(x,y)H(w), \tag{3.1}$$

where $H = H(w)$ is defined by the resolution of

$$\begin{aligned} &-\Delta H + \alpha(y)\frac{\partial H}{\partial x} = (\Lambda - 1)\Delta w, \\ &\frac{\partial H}{\partial \nu} = 0 \ \text{ on } \partial\Omega \ \text{ and } H(\pm\infty, y) = 0. \end{aligned} \tag{3.2}$$

Here we have set

$$a_0(x,y) = \frac{\partial\tilde{\kappa}}{\partial\psi}(h_0,\psi_0), \ \ b_0(x,y) = \frac{\partial\tilde{\kappa}}{\partial h}(h_0,\psi_0), \tag{3.3}$$

where $\tilde{\kappa}$ is given by (2.53), $\psi_0 = v_0 + \phi$ and $h_0$ is the solution of (2.41) for $v = v_0$. Here, due to (2.60), the function $a_0$ satisfies

$$a_0 \ \text{ is bounded and } \lim_{x\to\pm\infty} a_0(x,y) = a^{\pm} > 0, \tag{3.4}$$

while, due to (1.18), the function $b_0$ is such that

$$b_0 \ \text{ is bounded and } \lim_{x\to\pm\infty} b_0(x,y) = 0, \tag{3.5}$$

where the limits hold uniformly with respect to $y \in \omega$.

We can now state the main result of this section.

**Proposition 3.1.** *Let $\Lambda > 0$ and $\alpha$ satisfying* (1.7) *be given. Let also $\phi$ be a function as in* (2.39) *and let $\mathcal{A}$ be given by* (2.43). *Then, under conditions* (1.5), (1.6) *and* (1.18), *for any $l \geq 0$, the operator $\mathcal{A} + lI$ acting from $E$ into $F$ is Fredholm with zero index.*

**Remark 3.2.** As in the previous section we consider the example specific for combustion theory. In view of (2.49)-(2.50), it reads

$$\kappa(\theta,\psi) = ke^{Z(\theta-1)}\psi = ke^{ZH}e^{-Z\psi}\psi.$$

We approximate the function $e^{-Z\psi}\psi$ by a function $\kappa_0(\psi)$ such that

$$\kappa_0(0) = \kappa_0(1) = 0, \ \ \kappa_0'(0) < 0, \ \kappa_0'(1) < 0,$$

and set

$$\kappa(\theta,\psi) = ke^{Z(\theta+\psi-1)}\kappa_0(\psi). \tag{3.6}$$

Then this modified function $\kappa$ satisfies (1.18).

**Proof.** In order to study the Fredholm property for the non-linear operator $\mathcal{A}$, it is sufficient to study the Fredholm property for the linearized operator at any point $v_0 \in E$, that is given by (3.1). Let us denote by $B$ and $C$ the operators from $E$ into $F$ defined by

$$Bw = -\Lambda \Delta w + \alpha(y)\frac{\partial w}{\partial x} + a_0(x,y)w, \tag{3.7}$$

$$Cv = b_0(x,y)H(w). \tag{3.8}$$

The Fredholm property of elliptic problems in unbounded domains is studied in a number of works (see [28], [22] and the references therein). In particular it is known that, due to property (3.4), for any $l \geq 0$, the operator $B + lI$ is Fredholm with zero index (see [32]). Now, since $A = B + C$, the proof of Proposition 3.1 consists in checking that the operator $C$ is compact.

From estimate (2.11) it follows that the operator $H$ defined by solving problem (3.2) is bounded from $E$ into $E$. Consequently, in order to prove the compactness of the operator $C$, it is sufficient to prove that the operator $J$ from $H^1(\Omega)$ into $L^2(\Omega)$, given by

$$Jw = b_0(x,y)w, \tag{3.9}$$

is compact. This last property follows from (3.5). Indeed let $(w_n)$ be a bounded sequence in $H^1(\Omega)$. Let us show that the sequence $Jw_n$ is relatively compact in $F$. From the compact embedding $H^1 \hookrightarrow L^2$ in bounded domains, we obtain the fact that there exist a subsequence still denoted by $(w_n)$ and a function $w_0 \in F$ such that $(w_n)$ tends to $w_0$ in $L^2_{loc}(\Omega)$. Next, since the function $b_0$ converges uniformly to 0 at infinity, for any $\epsilon > 0$ there exists $R(\epsilon) > 0$ such that

$$|b_0(x,y)| < \epsilon, \quad \text{for all } |x| > R(\epsilon) \text{ and all } y \in \omega. \tag{3.10}$$

Therefore, for any $\epsilon > 0$, we have

$$\begin{aligned} &\iint_\Omega |b_0(x,y)(w_n - w_0)|^2 dxdy \qquad (3.11)\\ &\leq \|b_0\|^2_\infty \iint_{|x|<R(\epsilon), y\in\omega} |w_n - w_0|^2 dxdy + \epsilon^2 \iint_{|x|>R(\epsilon), y\in\omega} |w_n - w_0|^2 dxdy. \end{aligned}$$

Since the sequence $(w_n)$ is bounded in $L^2(\Omega)$ and converges to $w_0$ in $L^2_{loc}(\Omega)$, we obtain the fact that there is some constant $C$ such that for any $\epsilon > 0$

$$\overline{\lim_{n\to+\infty}} \iint_\Omega |b_0(x,y)(w_n - w_0)|^2 dxdy \leq \epsilon^2 C. \tag{3.12}$$

Therefore $Jw_n = b_0 w_n$ tends to $b_0 w_0$ in $L^2(\Omega)$ so that the operator $J$ is compact from $H^1(\Omega)$ into $F$. This completes the proof of Proposition 3.1. □

3.2. **Application of the Fredholm property.** In this section we come back to existence results of travelling waves. We will consider problem (1.8), (1.9) in the particular case where $\beta(c, y)$ takes the form $\beta(c, y) = c\alpha(y)$; other cases will be considered elsewhere. Therefore we investigate the system

$$-\Delta\theta + c\alpha(y)\frac{\partial\theta}{\partial x} - \kappa(\theta, \psi) = 0, \tag{3.13}$$

$$-\Lambda\Delta\psi + c\alpha(y)\frac{\partial\psi}{\partial x} + \kappa(\theta, \psi) = 0, \tag{3.14}$$

together with conditions (1.3), (1.4). Here the unknowns are $\theta$, $\psi$ and $c$.

For $\alpha(y) \equiv 1$, the existence of one-dimensional solutions of problem (3.13), (3.14) has been extensively studied [5, 6, 11, 20, 33]. Here we aim to prove the existence of multi-dimensional solutions of (3.13), (3.14) for $\alpha = \alpha(y)$ close to the constant function 1. For that purpose we need to introduce an assumption on the zero eigenvalue of problem (3.13), (3.14) for $\alpha(y) \equiv 1$ linearized about the one-dimensional solution $(\theta_0(x), \psi_0(x), c_0)$. More precisely, we suppose that the zero eigenvalue for this problem is simple; that is, there exists a unique, up to a constant factor, eigenfunction $(z_0, w_0)$ corresponding to the zero eigenvalue, and the problem

$$\begin{aligned} &-\Delta z + c_0\frac{\partial z}{\partial x} - \kappa'_\theta(\theta_0, \psi_0)z - \kappa'_\psi(\theta_0, \psi_0)w = z_0, \\ &-\Lambda\Delta w + c_0\frac{\partial w}{\partial x} + \kappa'_\theta(\theta_0, \psi_0)z + \kappa'_\psi(\theta_0, \psi_0)w = w_0, \\ &\frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \quad z(\pm\infty, y) = w(\pm\infty, y) = 0, \end{aligned} \tag{3.15}$$

does not have bounded solutions.

Note that the derivative $(\theta_0'(x), \psi_0'(x))$ of the solution is an eigenfunction corresponding to the zero eigenvalue. The simplicity assumption implies that we do not deal with bifurcation points here.

Under this additional condition, we can state the following result:

**Theorem 3.3.** *Suppose that conditions* (1.5), (1.6) *and* (1.18) *hold. Let* $\Lambda > 0$ *be given. Assume that* $(\theta_0, \psi_0, c_0)$ *is a one-dimensional solution of* (3.13), (3.14), (1.3), (1.4) *for* $\alpha \equiv 1$ *with* $c_0 > 0$. *Assume in addition that* 0 *is a simple eigenvalue of problem* (3.13)-(3.14) *linearized about* $(\theta_0, \psi_0)$. *Then there exists* $\epsilon > 0$ *such that problem* (3.13)-(3.14) *together with* (1.3) *and* (1.4) *has a solution* $(\theta, \psi, c)$ *for any* $\alpha(y)$ *such that* $\|\alpha - 1\|_\infty < \epsilon$.

The proof of this theorem relies on the integro-differential operator corresponding to equations (3.13), (3.14). As usual we introduce the unknowns $u = \theta + \phi - 1, v = \psi - \phi$ where $\phi$ is a given function satisfying (2.39). In the sequel the function $\alpha$ will be allowed to vary. This leads us to consider the operator $\mathcal{F} : E \times (0, +\infty) \times \mathcal{V} \to L^2(\Omega)$ defined by

$$\mathcal{F}(v, c, \alpha) = -\Lambda\Delta(v + \phi) + c\alpha(y)\frac{\partial(v + \phi)}{\partial x} + \Upsilon(v, c, \alpha), \tag{3.16}$$

where $\mathcal{V}$ is given by (2.2). Here, we have set

$$\Upsilon(v, c, \alpha) = \tilde{\kappa}\big(h, v + \phi\big), \tag{3.17}$$

where $h = h(v, c, \alpha)$ satisfies

$$\begin{aligned} &-\Delta h + c\alpha(y)\frac{\partial h}{\partial x} = (\Lambda - 1)\Delta(v + \phi), \\ &\frac{\partial h}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \quad h(\pm\infty, y) = 0. \end{aligned} \tag{3.18}$$

The solutions of problem (3.13)-(3.14),(1.3)-(1.4) are associated to the zeros of $\mathcal{F}$.

**Proof of Theorem 3.3.** Let $(\theta_0, \psi_0, c_0)$ denote a one-dimensional solution of (3.13), (3.14), (1.3), (1.4) for $\alpha \equiv 1$ with $c_0 > 0$. We set $u_0 = \theta_0 - 1 + \phi$ and $v_0 = \psi_0 - \phi$. In terms of the integro-differential formulation, we have

$$\mathcal{F}(v_0, c_0, 1) = 0. \tag{3.19}$$

Our existence result will be obtained by applying the implicit function theorem. The operator $\mathcal{F}$ is of class $C^1$. We need to study the operator $(\mathcal{F})'_{(v,c)}(v_0, c_0, 1)$. The expression for this operator is given by

$$(\mathcal{F})'_{(v,c)}(v_0, c_0, 1).(w, d) = Aw + d\big((v_0 + \phi)' + \Upsilon'_c(v_0, c_0, 1)\big). \tag{3.20}$$

Here the operator $A$ defined by

$$Aw = -\Lambda\Delta w + c_0\frac{\partial w}{\partial x} + \Upsilon'_v(v_0, c_0, 1).w \tag{3.21}$$

is the one studied in Section 3.1 (for $\alpha(y) \equiv c_0$). Proposition 3.1 yields that it is Fredholm with the zero index. Consequently the solvability condition for the equation

$$(\mathcal{F})'_{(v,c)}(v_0, c_0, 1).(w, d) = f, \ \text{ with } f \in L^2(\Omega), \tag{3.22}$$

reads

$$\iint_\Omega \Big(d\big((v_0 + \phi)' + \Upsilon'_c(v_0, c_0, 1)\big) - f\Big)w^* dxdy = 0, \tag{3.23}$$

where $w^*$ is an eigenfunction corresponding to the zero eigenvalue of the adjoint operator $A^*$. We note that there exists a unique solvability condition because the operator $A$ has the zero index and its kernel has dimension 1 (see the proof of Lemma 3.4 below). The operator $A^*$ is defined on $L^2(\Omega)$ which determines the specific form of the solvability condition (3.23). This condition is satisfied for some $d$ if

$$\iint_\Omega \big((v_0+\phi)' + \Upsilon_c'(v_0,c_0,1)\big)w^* dxdy \neq 0. \tag{3.24}$$

This inequality in its turn is satisfied because otherwise the equation

$$Aw = (v_0+\phi)' + \Upsilon_c'(v_0,c_0,1)$$

has a non-zero solution. Coming back to definition (3.20) this implies that

$$(\mathcal{F})'_{(v,c)}(v_0,c_0,1).(w,-1) = 0.$$

This is in contradiction with Lemma 3.4 below whose proof is postponed.

Finally, as in the proof of Theorem 2.5, we conclude that the operator $(\mathcal{F})'_{(v,c)}(v_0,c_0,1)$ is invertible on a convenient subspace so that the implicit function theorem can be applied. The details are left to the reader. □

It remains to state and prove the following lemma:

**Lemma 3.4.** *Suppose that* 0 *is a simple eigenvalue of problem* (3.13)-(3.14) *linearized about the one-dimensional solution* $(\theta_0,\psi_0)$ *for some* $\Lambda>0$. *Then the kernel of* $(\mathcal{F})'_{(v,c)}(v_0,c_0,1)$ *is a one-dimensional space spanned by* $(\psi_0',0)$.

**Proof of Lemma 3.4.** We set $h_0=\theta_0+\psi_0-1$. Recalling definition (3.3) we easily see that the problem $(\mathcal{F})'_{(v,c)}(v_0,c_0,1).(w,d)=0$ is equivalent to the following system for $w$, $d$ and $H$ :

$$\begin{aligned} &-\Lambda\Delta w + c_0\frac{\partial w}{\partial x} + a_0 w + b_0 H = -d\psi_0',\\ &-\Delta H + c_0\frac{\partial H}{\partial x} = (\Lambda-1)\Delta w - dh_0', \end{aligned} \tag{3.25}$$

supplemented with the homogeneous Neumann boundary condition and homogeneous limits at infinity.

Assume that $(w,H,d)$ is a solution of (3.25). Setting $z=H-w$, we readily check that we have

$$\begin{aligned} &-\Lambda\Delta w + c_0\frac{\partial w}{\partial x} + (a_0+b_0)w + b_0 z = -d\psi_0',\\ &-\Delta z + c_0\frac{\partial z}{\partial x} - (a_0+b_0)w - b_0 z = -d\theta_0'. \end{aligned} \tag{3.26}$$

Recalling the definition (2.53), we obtain the fact that the functions $z$ and $w$ are such that

$$\begin{aligned} &-\Delta z + c_0 \frac{\partial z}{\partial x} - \kappa'_\theta(\theta_0,\psi_0) z - \kappa'_\psi(\theta_0,\psi_0) w = -d\theta'_0, \\ &-\Lambda \Delta w + c_0 \frac{\partial w}{\partial x} + \kappa'_\theta(\theta_0,\psi_0) z + \kappa'_\psi(\theta_0,\psi_0) w = -d\psi'_0. \end{aligned} \tag{3.27}$$

Now $(\theta'_0, \psi'_0)$ is an eigenfunction corresponding to the zero eigenvalue for problem (3.13), (3.14). Since this eigenvalue is assumed to be simple, system (3.27) can not have solutions if $d \neq 0$. Consequently $d = 0$ and $(z, w)$ belongs to the space spanned by $(\theta'_0, \psi'_0)$. □

## 4. FREDHOLM PROPERTY IN WEIGHTED SPACES

In this section we aim to derive the Fredholm property in weighted spaces for the operator $\mathcal{A}$ introduced in (2.43). Such a property will be necessary in order to define the topological degree.

In the sequel for the sake of simplicity we assume that the function $\alpha$ does not depend on the transversal variable $y$; that is,

$$\alpha(y) = \alpha \in (0, +\infty), \quad \forall y \in \omega. \tag{4.1}$$

However, our arguments can be extended to more general cases.

The function $\alpha$ being a constant, Problem (2.3)-(2.5) reads

$$-\Delta g + \alpha \frac{\partial g}{\partial x} = \nabla . f \tag{4.2}$$

$$\frac{\partial g}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \ g(\pm\infty, y) = 0. \tag{4.3}$$

We first investigate this problem in weighted spaces (Section 4.1). We then derive some Fredholm properties for the operator $\mathcal{A}$ in these spaces (Section 4.2).

### 4.1. The operator $\mathcal{G}$ in weighted spaces.

Let us introduce the weight function $\mu(x) = 1 + x^2$ and the spaces

$$V_\mu = \big\{ f \in V : \mu f \in V \big\}, \quad E_\mu = \big\{ v \in E : \mu v \in E \big\}. \tag{4.4}$$

According to Lemma 2.1, for $f \in V_\mu \subset V$, Problem (4.2)-(4.3) possesses a unique solution $g = \mathcal{G}(f)$. We aim to show the following:

**Proposition 4.1.** *Let $\alpha > 0$ be given. The operator $\mathcal{G}$ given by the resolution of (4.2)-(4.3) maps $V_\mu$ into $E_\mu$. Furthermore we have*

$$\|\mathcal{G}(f)\|_{E_\mu} \le C \|f\|_{V_\mu}, \quad \text{for any } f \in V_\mu, \tag{4.5}$$

*where $C$ is some constant.*

**Proof.** Let $f = (f_1, f_2) \in V_\mu$. As in the proof of Lemma 2.2, we introduce $\gamma_0 = (g, \phi_0)_{L^2(\omega)}$ and $\tilde{g} = g - \gamma_0$, where $\phi_0$ is given by (2.12). We also set

$$\delta_0 = \mu\gamma_0, \quad \tilde{\delta} = \mu\tilde{g}, \quad \delta = \mu g = \delta_0 + \tilde{\delta}. \tag{4.6}$$

Here, since $\alpha$ is a constant, the equation for $\gamma_0$ reads (see (2.21))

$$-\gamma_0' + \alpha\gamma_0 = \phi, \tag{4.7}$$

where $\phi = (f_1, \phi_0)_{L^2(\omega)}$. Multiplying equation (4.7) by $\mu^2\gamma_0$ and integrating over $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}} \delta_0^2\big(\alpha + \frac{\mu'}{\mu}\big)dx = \int_{\mathbb{R}} (\mu\phi)\delta_0 dx \le \frac{\alpha}{2}\|\delta_0\|_{L^2}^2 + \frac{1}{2\alpha}\|\mu\phi\|_{L^2}^2. \tag{4.8}$$

Hence,

$$\int_{\mathbb{R}} \delta_0^2\big(\alpha + \frac{2\mu'}{\mu}\big)dx \le \frac{1}{\alpha}\|\mu\phi\|_{L^2}^2. \tag{4.9}$$

Since $\frac{\mu'}{\mu}$ goes to 0 as $x \to \pm\infty$, there exists $a > 0$ such that, for all $|x| > a$, $\alpha + \frac{\mu'(x)}{\mu(x)} > \alpha/2$. Therefore, (4.9) yields that

$$\frac{\alpha}{2}\int_{|x|>a} \delta_0^2 dx \le \frac{1}{\alpha}\|\mu\phi\|_{L^2}^2 + \int_{|x|<a} \delta_0^2\Big|\alpha + \frac{2\mu'}{\mu}\Big|dx \le \frac{1}{\alpha}\|\mu\phi\|_{L^2}^2 + C_a\|\gamma_0\|_{L^2}^2, \tag{4.10}$$

where $C_a$ is some positive constant depending on $a$. Thanks to estimate (2.25) for $\|\gamma_0\|_{L^2}$, (4.10) implies

$$\|\delta_0\|_{L^2} \le C\|\mu\phi\|_{L^2} \le C\|f\|_{V_\mu}. \tag{4.11}$$

Next, note that in view of (4.2) $\delta$ satisfies the equation

$$-\Delta\delta + \Big(\alpha + 2\frac{\mu'}{\mu}\Big)\frac{\partial\delta}{\partial x} + k\delta = \mu\nabla.f, \tag{4.12}$$

where we have set

$$k = -\alpha\frac{\mu'}{\mu} - 2\Big(\frac{\mu'}{\mu}\Big)^2 + \frac{\mu''}{\mu}. \tag{4.13}$$

Multiplying equation (4.12) by $\delta$ and integrating over $\Omega$, we obtain

$$\iint_\Omega \big|\frac{\partial\delta}{\partial x}\big|^2 dxdy - \iint_\Omega \Delta_y\delta\delta dxdy = \iint_\Omega \mu\nabla.f\delta dxdy + \iint_\Omega \delta^2(-k + (\frac{\mu'}{\mu})')dxdy. \tag{4.14}$$

Due to the definition of $\delta$, we have that

$$-\iint_\Omega \Delta_y\delta\delta dxdy \ge \lambda_1\|\tilde{\delta}\|_{L^2}^2, \tag{4.15}$$

$\tilde{\delta}$ given by (4.6). Also the function $|-k+(\frac{\mu'}{\mu})'|$ goes to 0 as $x \to \pm\infty$; hence there exists $b > 0$ such that

$$|-k(x)+\Big(\frac{\mu'}{\mu}\Big)'(x)| < \frac{\lambda_1}{4}, \quad \text{for all } |x| > b. \tag{4.16}$$

Next we have

$$\iint_{|x|<b, y\in\omega} \Big|-k+\Big(\frac{\mu'}{\mu}\Big)'\Big|\delta^2 dxdy \le k_b \iint_\Omega |\mathcal{G}(f)|^2 dxdy, \tag{4.17}$$

and using (2.10), we obtain that

$$\iint_{|x|<b, y\in\omega} \Big|-k+\Big(\frac{\mu'}{\mu}\Big)'\Big|\delta^2 dxdy \le k_b M(\alpha)^2 \|f\|_{L^2}^2 \le k_b M(\alpha)^2 \|\mu f\|_{L^2}^2. \tag{4.18}$$

In view of (4.15), (4.16) and (4.18) we infer from (4.14) that

$$\iint_\Omega \Big|\frac{\partial\delta}{\partial x}\Big|^2 dxdy + \lambda_1 \|\tilde{\delta}\|_{L^2}^2 \le \|\mu\nabla.f\|_{L^2}\|\delta\|_{L^2} + \frac{\lambda_1}{4}\|\delta\|_{L^2}^2 + k_b M(\alpha)^2 \|\mu f\|_{L^2}^2. \tag{4.19}$$

This inequality yields the desired bound on $\|\tilde{\delta}\|_{L^2}$ since $\delta = \delta_0 + \tilde{\delta}$ and we have the estimate (4.11) of $\|\delta_0\|_{L^2}$.

Next the left-hand side of (4.14) is also equal to $\|\nabla\delta\|_{L^2}^2$. Hence, (4.14) implies

$$\|\nabla\delta\|_{L^2}^2 \le C(\|\mu\nabla.f\|_{L^2}^2 + \|\delta\|_{L^2}^2). \tag{4.20}$$

Finally, equation (4.12) combined together with the estimate for $\delta$ in $H^1(\Omega)$ implies

$$\|\delta\|_{H^2} \le C\|\mu f\|_V. \tag{4.21}$$

This concludes the proof of Proposition 4.1. □

4.2. **Fredholm property of the operator $\mathcal{A} : E_\mu \to F_\mu$.** In this section we prove the Fredholm property for the operator $\mathcal{A}$ in weighted spaces. Under condition (4.1), this operator takes the form

$$\mathcal{A}(v) = -\Lambda\Delta(v+\phi) + \alpha\frac{\partial(v+\phi)}{\partial x} + \tilde{\kappa}\big(\mathcal{H}(v), v+\phi\big), \tag{4.22}$$

where $h = \mathcal{H}(v)$ is given by the resolution of the following:

$$\begin{aligned} &-\Delta h + \alpha\frac{\partial h}{\partial x} = (\Lambda - 1)\Delta(v+\phi), \\ &\frac{\partial h}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \quad h(\pm\infty, y) = 0 \ \ y \in \omega. \end{aligned} \tag{4.23}$$

It is easy to check that the operator $\mathcal{A}$ is bounded as acting from $E_\mu$ into $F_\mu$. Indeed for the first two terms in the right-hand side of (4.22), this property follows from the boundedness of the functions $\mu'/\mu$ and $\mu''/\mu$ while for the last term it follows from Proposition 4.1.

**Proposition 4.2.** *Let $\Lambda > 0$ and $\alpha > 0$ be given. Also let $\kappa$ be a $C^2$ function satisfying* (1.5), (1.6), (1.18) *and $\phi$ a function as in* (2.39). *Then, the operator $\mathcal{A} : E_\mu \to F_\mu$ given by* (4.22) *is Fredholm with zero index.*

**Proof.** The proof of the proposition is based on the following lemma (see [28] for the proof).

**Lemma 4.3.** *Suppose that the operator $L : E \to F$ is normally solvable and has a finite-dimensional kernel. Furthermore assume that the operator defined by*

$$Kw = \mu Lw - L(\mu w) : E_\mu \to F \tag{4.24}$$

*is compact. Then the operator $L : E_\mu \to F_\mu$ is normally solvable and has a finite-dimensional kernel.*

Let $v_0 \in E_\mu$ and $A$ be the derivative of $\mathcal{A}$ at $v_0$ given by (see (3.1))

$$Aw = -\Lambda\Delta w + \alpha\frac{\partial w}{\partial x} + a_0(x,y)w + b_0(x,y)H(w), \tag{4.25}$$

where

$$\begin{aligned} & -\Delta H + \alpha\frac{\partial H}{\partial x} = (\Lambda - 1)\Delta w, \\ & \frac{\partial H}{\partial \nu} = 0 \text{ on } \partial\Omega \text{ and } H(\pm\infty, y) = 0. \end{aligned} \tag{4.26}$$

In order to apply Lemma 4.3, we introduce the operator $K : E_\mu \to L^2(\Omega)$ defined as follows:

$$\begin{aligned} K(w) &= A(\mu w) - \mu A(w) \qquad\qquad (4.27)\\ &= -\Lambda(\Delta(\mu w) - \mu\Delta w) + \alpha(\frac{\partial(\mu w)}{\partial x} - \mu\frac{\partial w}{\partial x}) + b_0(x,y)(H(\mu w) - \mu H(w)). \end{aligned}$$

We aim to show the following:

**Lemma 4.4.** *The operator $K$ is compact from $E_\mu$ into $F$.*

Before proving this lemma, let us complete the proof of Proposition 4.2. In view of Proposition 3.1 and Lemma 4.4, we see that the operator $A : E_\mu \to F_\mu$ is normally solvable and has a finite-dimensional kernel. Next, for $\Lambda = 1$, the operator $A : E_\mu \to F_\mu$ is Fredholm with zero index. Since the index does not change during a homotopy in the class of normally solvable operators with a finite-dimensional kernel, then the operator $A : E_\mu \to F_\mu$ is Fredholm with zero index. □

It remains to prove Lemma 4.4.

**Proof of Lemma 4.4.** In expression (4.27) for $K(w)$, the first terms clearly define a compact operator. Therefore, since $b_0(x,y)$ is bounded (see (3.5)) we only need to show that the operator

$$K_1 w = H(\mu w) - \mu H(w) \tag{4.28}$$

is compact.

Let $(w^n)_{n\geq 0}$ be a bounded sequence in $E_\mu$. It possesses a subsequence, still denoted by $(w^n)$, that is convergent in $C^1_{loc}(\Omega)$ and $H^1(\Omega)$ strongly and in $E_\mu$ weakly towards some $w^0 \in E_\mu$. Since the operator $K_1$ is linear, we can suppose that $w^0 = 0$. Then the above convergence in $H^1(\Omega)$ also implies that $H(w^n) \to 0$ in $L^2(\Omega)$ (see estimate (2.10)).

Let us show that $K_1(w^n) \to 0$ as $n \to +\infty$ in $L^2(\Omega)$ strongly. We first study the behavior of the quantity

$$k_0^n = (K_1(w^n), \phi_0)_{L^2(\omega)} = h_0(\mu w^n) - \mu h_0(w^n),$$

where $\phi_0$ is given by (2.12) and we have set $h_0(w) = (H(w), \phi_0)_{L^2(\omega)}$. Similarly to (4.7), the functions $h_0(\mu u^n)$ and $h_0(u^n)$ satisfy the equations

$$\begin{aligned} -h_0(\mu w^n)' + \alpha h_0(\mu w^n) &= \lambda(\frac{\partial(\mu u^n)}{\partial x}, \phi_0)_{L^2(\omega)}, \\ -h_0(w^n)' + \alpha h_0(w^n) &= \lambda(\frac{\partial w^n}{\partial x}, \phi_0)_{L^2(\omega)}, \end{aligned} \tag{4.29}$$

where $\lambda = \Lambda - 1$, so that we have, for the function $k_0^n$,

$$-(k_0^n)' + \alpha k_0^n = \frac{\mu'}{\mu}\big(\lambda\mu(w^n, \phi_0)_{L^2(\omega)} + \mu h_0(w^n)\big). \tag{4.30}$$

Then, thanks to estimate (2.25), we have

$$\|k_0^n\|^2_{L^2(\mathbb{R})} \leq \frac{1}{\alpha^2}\|\frac{\mu'}{\mu}\big(\lambda\mu(w^n, \phi_0)_{L^2(\omega)} + \mu h_0(w^n)\big)\|^2_{L^2(\mathbb{R})}. \tag{4.31}$$

Hence, for all $R > 0$, we can find $C_R > 0$ such that

$$\begin{aligned} \|k_0^n\|^2_{L^2(\mathbb{R})} \leq & \frac{1}{\alpha^2}\int_{|x|>R} |\frac{\mu'}{\mu}|^2 |\lambda\mu(w^n, \phi_0)_{L^2(\omega)} + \mu h_0(w^n)|^2 dx \\ & + C_R \int_{|x|<R} |\lambda\mu(w^n, \phi_0)_{L^2(\omega)} + \mu h_0(w^n)|^2 dx. \end{aligned} \tag{4.32}$$

Due to the boundedness of the sequence $(w^n)_n$ in $E_\mu$ and to estimate (4.5), the quantity $\mu h_0(w^n)$ is bounded in $L^2(\mathbb{R})$ so that $\lambda\mu(w^n, \phi_0)_{L^2(\omega)} + \mu h_0(w^n)$ is also bounded in $L^2(\mathbb{R})$. Moreover since the sequences $H(w^n)$ and $w^n$ go to zero in $L^2(\Omega)$ strongly, $\lambda\mu(w^n,\phi_0)_{L^2(\omega)}+\mu h_0(w^n)$ goes to zero in $L^2_{loc}(\mathbb{R})$. Therefore there exists some constant $C$ independent of $R$ such that

$$\overline{\lim_{n\to+\infty}}\|k_0^n\|^2_{L^2(\mathbb{R})}\leq C\sup_{|x|>R}|\frac{\mu'}{\mu}|^2(x),\quad \text{for all } R>0. \tag{4.33}$$

Recalling the form of the weight function, $\mu(x)=1+x^2$, we obtain that the right-hand side of this inequality tends to zero as $R\to+\infty$. Hence we have proved that

$$\lim_{n\to+\infty}\|k_0^n\|_{L^2(\mathbb{R})}=0. \tag{4.34}$$

It remains to show that $\tilde{K_1^n}=K_1(w^n)-k_0^n$ tends to zero in $L^2$. First note that $K_1(w^n)$ satisfies the equation

$$\begin{aligned}-\Delta K_1(w^n)+\alpha\frac{\partial K_1(w^n)}{\partial x}&=-2\lambda\frac{\mu'}{\mu}(\mu\frac{\partial w^n}{\partial x})-\lambda\frac{\mu''}{\mu}(\mu w^n)\\&-(\alpha\frac{\mu'}{\mu}-\frac{\mu''}{\mu})(\mu H(w^n))+2\frac{\mu'}{\mu}(\mu\frac{\partial H(w^n)}{\partial x}).\end{aligned} \tag{4.35}$$

Let us denote by $f^n$ the right-hand side of equation (4.35). Due to the assumptions and (4.5), the quantities $\mu\frac{\partial w^n}{\partial x}$, $\mu w^n$, $\mu H(w^n)$ and $\mu\frac{\partial H(w^n)}{\partial x}$ are bounded in $L^2(\Omega)$ and tend to zero in $L^2_{loc}(\Omega)$ while

$$\frac{\mu'(x)}{\mu(x)},\ \ \frac{\mu''(x)}{\mu(x)}\ \to 0\ \text{ as } x\to\pm\infty. \tag{4.36}$$

Therefore, the sequence $f^n$ tends to 0 in $L^2(\Omega)$.

Then, multiplying equation (4.35) by $K_1(w^n)$ and integrating over $\Omega$, we obtain that

$$\iint_\Omega|\frac{\partial K_1(w^n)}{\partial x}|^2dxdy-\iint_\Omega\Delta_yK_1(w^n)K_1(w^n)dxdy=\iint_\Omega f^nK_1(w^n)dxdy. \tag{4.37}$$

Recalling now that

$$-\iint_\Omega\Delta_yK_1(w^n)K_1(w^n)dxdy\geq\lambda_1\|\tilde{K_1^n}\|^2_{L^2(\Omega)}, \tag{4.38}$$

we conclude that

$$\lambda_1\|\tilde{K_1^n}\|^2_{L^2(\Omega)}\leq\iint_\Omega|f^n||\tilde{K_1^n}+k_0^n|dxdy. \tag{4.39}$$

The last inequality combined with the behavior of the sequences $k_0^n$ and $f^n$ as $n\to+\infty$ shows that the sequence $\|\tilde{K_1^n}\|_{L^2(\Omega)}$ tends to zero as $n\to+\infty$. Lemma 4.4 is proved. □

## 5. PROPERNESS IN THE WEIGHTED SPACES

In this section we will consider the real number $\lambda = \Lambda - 1$ as a homotopy parameter and we will explicitly write down the dependence with respect to this parameter. The operator $\mathcal{A}_\lambda(u)$ is defined for $(u, \lambda) \in E_\mu \times I$ where $I$ is a compact subinterval of $(-1, +\infty)$.

We will show that the non-linear operator $\mathcal{A}_\lambda$ from $E_\mu \times I$ into $F_\mu$ is proper. We recall that properness is understood here in the sense that the intersection of the inverse image of a compact set with any bounded closed set is compact.

**Proposition 5.1.** *Let $\alpha > 0$. Also let $\kappa$ be a $C^2$ function satisfying* (1.5), (1.6), (1.18) *and $\phi$ a function as in* (2.39). *Then, the operator $\mathcal{A}_\lambda(v)$ given by* (4.22) *with $\Lambda = \lambda + 1$ is proper with respect to $(v, \lambda) \in E_\mu \times I$.*

In order to prove Proposition 5.1, we introduce a weaker topology in $E_\mu$, denoted by $\rightharpoonup$ and defined by the following convergence: $u_n \rightharpoonup u_0$ means that $u_n \to u_0$ in $H^1(\Omega)$ and in $C^0_{loc}(\Omega)$. We note that the embedding $(E_\mu, \to) \hookrightarrow (E_\mu, \rightharpoonup)$ is compact.

The proof of the properness is based on the following lemma (see [28]):

**Lemma 5.2.** *Suppose that $D$ is a bounded subset of $E_\mu$, the operator $\mathcal{A}_\lambda(v)$ is closed, and for any $v_0 \in D$ and $\lambda_0 \in I$ there exists a bounded operator $S(v_0, \lambda_0) : E_\mu \to F_\mu$, which has a closed range, a finite-dimensional kernel and is continuous in the operator norm with respect to the parameter $\lambda_0$, such that for any sequence $\{v_n\}$, $v_n \in D$, $v_n \rightharpoonup v_0 \in D$ and $\lambda_n \to \lambda_0$ we have*

$$\|\mathcal{A}_{\lambda_0}(v_0) - \mathcal{A}_{\lambda_n}(v_n) - S(v_0, \lambda_0)(v_0 - v_n)\|_{F_\mu} \to 0. \tag{5.1}$$

*Then $\mathcal{A}_\lambda(v)$ is a proper operator with respect to both variables $v$ and $\lambda$.*

**Proof of Proposition 5.1.** Let us now check that the assumptions of Lemma 5.2 hold true. Let $v_0 \in E_\mu$ and $\lambda_0 \in I$. We set $\psi_0 = v_0 + \phi$. For simplicity in this section we will use the following notation: for $w \in H^2_{loc}(\Omega)$ with $\nabla w \in V$ we denote by $H_0(w)$ the solution of the equation

$$-\Delta H_0 + \alpha \frac{\partial H_0}{\partial x} = \Delta w, \tag{5.2}$$

together with the homogeneous Neumann conditions at the boundary and the homogeneous limits at infinity. With this notation, the solution of (4.23) reads

$$\mathcal{H}(v, \lambda) = \lambda H_0(v + \phi) = \lambda H_0(\psi), \quad \text{with } \psi = v + \phi, \tag{5.3}$$

and

$$\frac{\partial \mathcal{H}}{\partial v}(v_0,\lambda_0).v = \lambda H_0(v), \quad \frac{\partial \mathcal{H}}{\partial \lambda}(v_0,\lambda_0).\lambda = \lambda H_0(\psi). \tag{5.4}$$

For $v \in E_\mu$ and $\lambda \in I$, we have

$$\mathcal{A}_{\lambda_0}(v_0) - \mathcal{A}_\lambda(v) = S(v_0,\lambda_0)(v_0 - v) + \epsilon_0(v,\lambda), \tag{5.5}$$

where

$$\begin{aligned} S(v_0,\lambda_0)z = -(1+\lambda_0)\Delta z + \alpha\frac{\partial z}{\partial x} + \lambda_0 \frac{\partial \tilde{\kappa}}{\partial h}(\lambda_0 H_0(\psi_0),\psi_0)H_0(z) \\ + \frac{\partial \tilde{\kappa}}{\partial \psi}(\lambda_0 H_0(\psi_0),\psi_0)z, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \epsilon_0(v,\lambda) = -(\lambda_0-\lambda)\Delta\psi + \frac{\partial \tilde{\kappa}}{\partial h}(\lambda_0 H_0(\psi_0),\psi_0)(\lambda_0-\lambda)H_0(\psi) \\ + \int_0^1 (1-s)X(v,\lambda)^* D^2\tilde{\kappa}(\lambda_0 H_0(\psi_0) \\ + s(\lambda H_0(\psi) - \lambda_0 H_0(\psi_0)), \psi_0 + s(v-v_0))X(v,\lambda)ds, \end{aligned} \tag{5.7}$$

and $X(v,\lambda)$ is the vector

$$X(v,\lambda) = \begin{pmatrix} (\lambda_0-\lambda)H_0(\psi) + \lambda_0 H_0(v_0 - v) \\ v_0 - v \end{pmatrix}. \tag{5.8}$$

Note that, thanks to Proposition 4.2, $S(v_0,\lambda_0)$ is a Fredholm operator, so that it satisfies the assumptions of Lemma 5.2.

Next let $U_p = (v_p,\lambda_p) \in E_\mu \times I$, $p \in \mathbb{N}$, be such that

$$\begin{cases} \|\mu v_p\|_{H^2} \le 1 \ \text{ for } p \in \mathbb{N}, \ \text{ and } \ v_p \rightharpoonup v_0 \ \text{ as } p \to +\infty \\ \lambda_p \to \lambda_0 \ \text{ as } p \to +\infty. \end{cases} \tag{5.9}$$

We aim to show that

$$\mu\epsilon_0(v_p,\lambda_p) \to 0, \ \text{ in } L^2(\Omega). \tag{5.10}$$

In view of (5.5), this will guarantee that $\mathcal{A}_\lambda(v)$ satisfies (5.1) and conclude the proof of Proposition 5.1 by applying Lemma 5.2.

We have

$$\begin{aligned} \epsilon_0(v_p,\lambda_p) = -(\lambda_0-\lambda_p)\Delta\psi_p + \frac{\partial \tilde{\kappa}}{\partial h}(\lambda_0 H_0(\psi_0),\psi_0)(\lambda_0-\lambda_p)H_0(\psi_p) \\ + \int_0^1 (1-s)X_p^* D^2\tilde{\kappa}_p X_p ds, \end{aligned} \tag{5.11}$$

where $\psi_p = v_p + \phi$, $X_p = X(v_p,\lambda_p)$ and

$$D^2\tilde{\kappa}_p = D^2\tilde{\kappa}(\lambda_0 H_0(\psi_0) + s(\lambda_p H_0(\psi_p) - \lambda_0 H_0(\psi_0)), \psi_0 + s(v_p - v_0)).$$

Due to (5.9) the first term in the right-hand side of (5.11) goes to zero in $L^2_\mu(\Omega)$ as $p \to +\infty$. Next, (5.9) together with Proposition 4.1 enable us to say that $H_0(\psi_p)$ is bounded in $E_\mu$. Consequently, the second term in (5.11) goes to zero in $L^2_\mu(\Omega)$ as $p \to +\infty$. Also $D^2\tilde{\kappa}_p$ is bounded in $L^\infty(\Omega)$. Therefore, it is sufficient to check that $\mu X(v_p, \lambda_p)^* X(v_p, \lambda_p)$ goes to zero in $L^2(\Omega)$. In view of definition (5.8), we easily see that it is sufficient to prove that

$$\begin{aligned} &\mu(v_0 - v_p)^2 \to 0, \ \text{ in } L^2(\Omega), \\ &\mu H_0(v_0 - v_p)^2 \to 0, \ \text{ in } L^2(\Omega). \end{aligned} \tag{5.12}$$

Due to condition (5.9) the sequences $\mu(v_0 - v_p)$ and $\mu H_0(v_0 - v_p)$ are bounded in $L^\infty(\Omega)$ while the quantities $(v_0 - v_p)$ and $H_0(v_0 - v_p)$ tend to zero in $H^1(\Omega)$. Therefore, we have

$$\begin{aligned} &\iint_\Omega \mu^2 (v_0 - v_p)^4 dxdy = \iint_\Omega \mu^4 \frac{(v_0 - v_p)^4}{\mu^2} dxdy \qquad (5.13) \\ &\leq C_R \iint_{|x|<R, y\in\omega} (v_0 - v_p)^4 dxdy + \frac{1}{\mu(R)^2} \iint_{|x|>R, y\in\omega} \mu^4 (v_0 - v_p)^4 dxdy, \\ &\qquad\qquad \text{for all } R > 0 \\ &\leq C_R \iint_{|x|<R, y\in\omega} (v_0 - v_p)^4 dxdy \\ &\qquad + \frac{1}{\mu(R)^2} \|\mu(v_0 - v_p)\|^2_\infty \iint_{|x|>R, y\in\omega} \mu^2 (v_0 - v_p)^2 dxdy, \end{aligned}$$

where $R > 0$ and $C_R$ is some constant depending on $R$. Since $\|\mu(v_0 - v_p)\|_\infty$ and $\|\mu(v_0 - v_p)\|_{L^2}$ are bounded, and $(v_0 - v_p) \to 0$ in $L^2(\Omega)$ as $p \to +\infty$, we obtain that, for all $R > 0$,

$$\overline{\lim_{p\to+\infty}} \iint_\Omega \mu^2 (v_0 - v_p)^4 dxdy \leq \frac{C}{\mu(R)^2}. \tag{5.14}$$

Now taking the limit $R \to +\infty$, we conclude that

$$\mu(v_0 - v_p)^2 \to 0 \ \text{ in } L^2(\Omega). \tag{5.15}$$

The proof of the second limit in (5.12) is similar since $\|\mu H_0(v_0 - v_p)\|_\infty$ and $\|\mu H_0(v_0 - v_p)\|_{L^2}$ are bounded, and $H_0(v_0 - v_p) \to 0$ in $L^2(\Omega)$. $\square$

## 6. Topological degree

There are several degree constructions for Fredholm and proper operators (see [21], [28] and the references therein). We will use here the degree constructed in [28], which is well adapted for elliptic problems in unbounded domains.

Let us first recall some results on the topological degree constructed in [28]. Let $E_0$ and $E_1$ be two Banach spaces with $E_0 \subset E_1$, the injection of $E_0$ into $E_1$ being continuous. Let $G \subset E_0$ be an open bounded set. We consider the following classes of operators:

Class $\Phi$ is a class of linear bounded operators $A : E_0 \to E_1$ such that the operator $A + kI : E_0 \to E_1$ is Fredholm with zero index for all $k \geq 0$ and there exists $k_0 = k_0(A)$ such that the operators $A + kI : E_0 \to E_1$ have inverse that are uniformly bounded for all $k \geq k_0$.

Class $F$ is a class of operators $f \in C^1(G, E_1)$ that are proper on closed bounded sets such that for any $x \in G$ the Fréchet derivative $f'(x)$ belongs to $\Phi$.

Finally, we introduce the following class of homotopies. Class $H$ is a class of proper operators $f(x,t) \in C^1(G \times [0,1], E_1)$ such that, for any $t \in [0,1]$, $f(.,t)$ belongs to class $F$. Then, it is shown in [28] that the topological degree can be constructed for the classes $F$ and $H$.

These results will be used to prove the following theorem.

**Theorem 6.1.** *Let $\Lambda > 0$ and $\alpha > 0$ be given. Also let $\kappa$ be a $C^2$ function satisfying* (1.5), (1.6), (1.18) *and $\phi$ a function as in* (2.39). *Then, we can define the topological degree for the operator $\mathcal{A} : E_\mu \to F_\mu$ defined by* (4.22). *In addition, the topological degree is determined by the orientation of the derivative operator $A$.*

**Proof.** In order to prove this theorem, we will apply the results recalled above. Here we set $E_0 = E_\mu$, $E_1 = F_\mu$ and $G$ is any open ball in $E_0$. Let us show that $\mathcal{A}$ belongs to the class $F$. In view of Proposition 5.1, it remains to show that the derivative operator $A$ is in the class $\Phi$.

We first note that the operator $A+kI$ considered as acting in the weighted spaces is Fredholm with zero index. Indeed by Proposition 4.2 this holds true for $A$; next it can be easily extended to $A + kI$ thanks to Lemma 4.3.

Now consider the equation

$$(A + kI)w = 0. \tag{6.1}$$

We aim to check that (6.1) possesses a unique solution if $k$ is sufficiently large.

Recall that $A$ is given by (4.25). Therefore, (6.1) can be rewritten as

$$(L_1 + kI)w = -b_0(x,y)H(w), \tag{6.2}$$

where

$$L_1 w = -\Lambda \Delta w + \alpha \frac{\partial w}{\partial x} + a_0(x,y)w. \tag{6.3}$$

Here $L_1$ is the usual reaction-diffusion operator. It is well known that $L_1$ is a sectorial operator ([15]), so that any solution of (6.2) satisfies

$$\|w\|_{L^2} \le C\|b_0\|_\infty \frac{\|H(w)\|_{L^2}}{k}. \tag{6.4}$$

Next we claim that there exists some constant $C$ such that, for $u \in E$,

$$\|H(w)\|_{L^2} \le C\|w\|_{L^2}. \tag{6.5}$$

Indeed consider

$$H_1(w) = H(w) + (\Lambda - 1)w. \tag{6.6}$$

In view of (4.26) it satisfies the equation

$$-\Delta H_1(w) + \alpha \frac{\partial H_1(w)}{\partial x} = (\Lambda - 1)\alpha \frac{\partial w}{\partial x}. \tag{6.7}$$

Consequently, the estimate (2.10) yields that

$$\|H_1(w)\|_{H^1} \le C|\Lambda - 1|\alpha\|w\|_{L^2}, \tag{6.8}$$

and (6.8) together with (6.6) provide (6.5).

Now combining (6.4) and (6.5) we see that

$$\|w\|_{L^2} \le C_0 \frac{\|w\|_{L^2}}{k}. \tag{6.9}$$

This inequality implies that $w = 0$ for $k > k_0 = C_0$.

Since the operator $A + kI$ is Fredholm with zero index, we obtain that the operator $A + kI$ is inversible for all $k > k_0$. The uniform bound with respect to sufficiently large $k$ follows from the sectorial property for the operator $A$ proved in the lemma below. Theorem 6.1 is proved. □

**Lemma 6.2.** *The operator $A$ is sectorial with arbitrarily small semi-angle.*

**Proof.** We consider the operator $A$ given by (4.25) as acting in $\widetilde{E} = E + iE$. For $w \in \widetilde{E}$ we have

$$(Aw, \bar{w}) = \Lambda \iint_\Omega |\nabla w|^2 dxdy + \iint_\Omega \alpha \frac{\partial w}{\partial x} \bar{w} dxdy + \iint_\Omega (a_0(x,y)w + b_0(x,y)H(w))\bar{w} dxdy. \tag{6.10}$$

Thanks to estimate (6.5), we obtain

$$Re(Aw, \bar{w}) \geq \Lambda \iint_{\Omega} |\nabla w|^2 dxdy - \alpha \|\nabla w\|_{L^2} \|w\|_{L^2} - M\|w\|_{L^2}^2, \tag{6.11}$$

where $M$ is some positive constant, while

$$|Im(Aw, \bar{w})| \leq \alpha \|\nabla w\|_{L^2} \|w\|_{L^2} + M\|w\|_{L^2}^2. \tag{6.12}$$

Then, for any given $\gamma > 0$, we have

$$\begin{aligned} & Re(Aw, \bar{w}) - \gamma |Im(Aw, \bar{w})| \qquad (6.13) \\ & \geq \Lambda \iint_{\Omega} |\nabla w|^2 dxdy - (1+\gamma)\alpha \|\nabla w\|_{L^2} \|w\|_{L^2} - (1+\gamma)M\|w\|_{L^2}^2 \\ & \geq (\Lambda - \epsilon(1+\gamma)\alpha)\|\nabla w\|_{L^2}^2 - \left( \frac{(1+\gamma)\alpha}{4\epsilon} + (1+\gamma)M \right) \|w\|_{L^2}^2, \end{aligned}$$

where $\epsilon > 0$ is arbitrary. If $\epsilon$ is chosen small enough so that $\Lambda - \epsilon(1+\gamma)\alpha > 0$, we have

$$Re(Aw, \bar{w}) - \gamma |Im(Aw, \bar{w})| \geq -\beta \|w\|_{L^2}^2, \tag{6.14}$$

for some positive number $\beta$. From this last estimate we conclude

$$|Im(Aw, \bar{w})| \leq \frac{1}{\gamma} Re((A+\beta)w, \bar{w}). \tag{6.15}$$

Consequently $A$ is a sectorial operator with the semi-angle $\arctan(\frac{1}{\gamma})$ for all $\gamma > 0$. This proves Lemma 6.2. □

## 7. Bifurcations

There are a number of works where Fredholm theory and topological degree are used to study bifurcations of solutions for elliptic problems in unbounded domains (see [33, 23, 26] and the references therein).

In this section we will apply topological degree to study bifurcations of solutions for the problem (3.13)-(3.14) with $\alpha(y) \equiv 1$. We recall that the essential spectrum of this problem passes through the origin and the general bifurcation theory does not apply directly in this case.

We consider the following problem

$$-\Delta\theta + c\frac{\partial \theta}{\partial x} - \kappa(\theta, \psi, \tau) = 0 \tag{7.1}$$

$$-\Lambda(\tau)\Delta\psi + c\frac{\partial \psi}{\partial x} + \kappa(\theta, \psi, \tau) = 0, \tag{7.2}$$

together with

$$\theta(-\infty, y) = 0, \quad \psi(-\infty, y) = 1, \quad \theta(+\infty, y) = 1, \quad \psi(+\infty, y) = 0, \tag{7.3}$$

$$\frac{\partial\theta}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{7.4}$$

We assume that the dependence with respect to $\tau$ is of class $\mathcal{C}^1$.

In the sequel, we will assume that, for all $\tau$, problem (7.1)-(7.4) possesses a one-dimensional solution denoted by $(\theta_\tau, \psi_\tau, c_\tau)$.

Let us introduce the integro-differential operator corresponding to this problem. As usual we consider the unknowns $u = \theta + \phi - 1, v = \psi - \phi$ where $\phi$ is a given function satisfying (2.39). The integro-differential equation takes the form

$$-\Lambda(\tau)\Delta(v+\phi) + c\frac{\partial(v+\phi)}{\partial x} + \kappa\big(h - v + 1 - \phi, v + \phi, \tau\big) = 0, \tag{7.5}$$

where $h = h(v, c, \tau)$ satisfies

$$\begin{aligned} &-\Delta h + c\frac{\partial h}{\partial x} = (\Lambda(\tau) - 1)\Delta(v+\phi), \\ &\frac{\partial h}{\partial\nu} = 0 \text{ on } \partial\Omega, \quad h(\pm\infty, y) = 0. \end{aligned} \tag{7.6}$$

We first aim to compare spectral properties of problems (7.1)-(7.4) and problem (7.5). Hereafter we consider the system (7.1)-(7.4) linearized about $(\theta_\tau, \psi_\tau)$. The eigenvalue problem for that linear operator reads as follows: find $\lambda^{(1)}$, $z^{(1)}$ and $w^{(1)}$ such that

$$-\Delta z^{(1)} + c_\tau\frac{\partial z^{(1)}}{\partial x} - \kappa'_\theta z^{(1)} - \kappa'_\psi w^{(1)} = \lambda^{(1)} z^{(1)}, \tag{7.7}$$

$$-\Lambda(\tau)\Delta w^{(1)} + c_\tau\frac{\partial w^{(1)}}{\partial x} + \kappa'_\theta z^{(1)} + \kappa'_\psi w^{(1)} = \lambda^{(1)} w^{(1)}, \tag{7.8}$$

where we have set

$$\kappa'_\theta = \frac{\partial\kappa}{\partial\theta}(\theta_\tau(x), \psi_\tau(x), \tau), \quad \kappa'_\psi = \frac{\partial\kappa}{\partial\psi}(\theta_\tau(x), \psi_\tau(x), \tau). \tag{7.9}$$

Consider now the eigenvalue problem for equation (7.5) in the neigbourhood of the one-dimensional solution $(v_\tau, c_\tau)$ with $v_\tau = \psi_\tau - \phi$. It reads: find $\lambda^{(2)}$ and $w^{(2)}$ such that

$$-\Lambda(\tau)\Delta w^{(2)} + c_\tau\frac{\partial w^{(2)}}{\partial x} + \kappa'_\theta(H^{(2)} - w^{(2)}) + \kappa'_\psi w^{(2)} = \lambda^{(2)} w^{(2)}, \tag{7.10}$$

where $H^{(2)}$ is defined by the resolution of

$$\begin{aligned} &-\Delta H^{(2)} + c_\tau\frac{\partial H^{(2)}}{\partial x} = (\Lambda(\tau) - 1)\Delta w^{(2)}, \\ &\frac{\partial H^{(2)}}{\partial\nu} = 0 \text{ on } \partial\Omega \text{ and } H^{(2)}(\pm\infty, y) = 0. \end{aligned} \tag{7.11}$$

We aim to compare some spectral properties of problems (7.7)-(7.8) and (7.10). For that purpose we will suppose that both eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ are differentiable with respect to the parameter $\tau$.

**Lemma 7.1.** *Suppose that for $\tau = \tau_0$, $\lambda^{(1)}(\tau_0) = 0$ is a simple eigenvalue of problem* (7.7), (7.8) *with corresponding non-trivial eigenvector $(z^{(1)}, w^{(1)})$. Furthermore, assume that*

$$\frac{d\lambda^{(1)}}{d\tau}(\tau_0) \neq 0. \tag{7.12}$$

*Then $w^{(1)}$ is an eigenvector for problem* (7.10) *associated to the eigenvalue $\lambda^{(2)}(\tau_0)$ where*

$$\lambda^{(2)}(\tau_0) = 0 \;\; and \;\; \frac{d\lambda^{(2)}}{d\tau}(\tau_0) \neq 0. \tag{7.13}$$

**Proof.** Let $(z^{(1)}, w^{(1)})$ be an eigenvector for problem (7.7), (7.8) associated to the eigenvalue $\lambda^{(1)}$ such that (7.12) holds true. Setting $H^{(1)} = z^{(1)} + w^{(1)}$, we readily see that

$$-\Lambda(\tau)\Delta w^{(1)} + c_\tau \frac{\partial w^{(1)}}{\partial x} + \kappa'_\theta (H^{(1)} - w^{(1)}) + \kappa'_\psi w^{(1)} = \lambda^{(1)} w^{(1)}, \tag{7.14}$$

$$-\Delta H^{(1)} + c_\tau \frac{\partial H^{(1)}}{\partial x} - \lambda^{(1)} H^{(1)} = (\Lambda(\tau) - 1)\Delta w^{(1)}. \tag{7.15}$$

Now at $\tau = \tau_0$, we have $\lambda^{(1)}(\tau_0) = 0$ . Consequently at $\tau = \tau_0$ (7.14)-(7.15) is similar to (7.10)-(3.2). This shows that (7.10) also possesses the zero eigenvalue $\lambda^{(2)}(\tau_0) = 0$ associated to $w^{(2)} = w^{(1)}$ and $H^{(2)} = H^{(1)}$.

It remains to check the second condition in (7.13). Let us argue by contradiction and assume that

$$\frac{d\lambda^{(2)}}{d\tau}(\tau_0) = 0. \tag{7.16}$$

Differentiating both equations (7.15) and (3.2) with respect to the parameter $\tau$, we obtain

$$\begin{aligned} -\Delta \frac{\partial H^{(1)}}{\partial \tau} + c'_\tau \frac{\partial H^{(1)}}{\partial x} + c_\tau \frac{\partial}{\partial x}\frac{\partial H^{(1)}}{\partial \tau} - \frac{d\lambda^{(1)}}{d\tau} H^{(1)} - \lambda^{(1)} \frac{\partial H^{(1)}}{\partial \tau} \\ = \Lambda'(\tau)\Delta w^{(1)} + (\Lambda(\tau) - 1)\Delta \frac{\partial w^{(1)}}{\partial \tau}, \end{aligned} \tag{7.17}$$

$$-\Delta \frac{\partial H^{(2)}}{\partial \tau} + c'_\tau \frac{\partial H^{(2)}}{\partial x} + c_\tau \frac{\partial}{\partial x}\frac{\partial H^{(2)}}{\partial \tau} = \Lambda'(\tau)\Delta w^{(2)} + (\Lambda(\tau) - 1)\Delta \frac{\partial w^{(2)}}{\partial \tau}. \tag{7.18}$$

Let us set

$$\delta_H = \frac{\partial H^{(1)}}{\partial \tau} - \frac{\partial H^{(2)}}{\partial \tau}, \quad \delta_w = \frac{\partial w^{(1)}}{\partial \tau} - \frac{\partial w^{(2)}}{\partial \tau}, \quad \delta_z = \delta_H - \delta_w. \tag{7.19}$$

Subtracting (7.17) from (7.18), we obtain for $\tau = \tau_0$

$$-\Delta \delta_H + c_{\tau_0} \frac{\partial \delta_H}{\partial x} - \frac{d\lambda^{(1)}}{d\tau}(\tau_0) H^{(1)} = (\Lambda(\tau_0) - 1)\Delta \delta_w. \tag{7.20}$$

This equation can also be written in terms of the functions $\delta_w$ and $\delta_z$ as

$$-\Lambda(\tau_0)\Delta \delta_w - \Delta \delta_z + c_{\tau_0} \frac{\partial(\delta_w + \delta_z)}{\partial x} - \frac{d\lambda^{(1)}}{d\tau}(\tau_0) H^{(1)} = 0. \tag{7.21}$$

Next we differentiate equations (7.14) and (7.10) with respect to $\tau$. For $i = 1, 2$, this provides

$$\begin{aligned}
&- \Lambda'(\tau)\Delta w^{(i)} + c'_\tau \frac{\partial w^{(i)}}{\partial x} + \kappa''_{\theta\tau}(H^{(i)} - w^{(i)}) + \kappa''_{\psi\tau} w^{(i)} \\
&- \Lambda(\tau)\Delta \frac{\partial w^{(i)}}{\partial \tau} + c_\tau \frac{\partial}{\partial x}\frac{\partial w^{(i)}}{\partial \tau} + \kappa'_\theta\Big(\frac{\partial H^{(i)}}{\partial \tau} - \frac{\partial w^{(i)}}{\partial \tau}\Big) + \kappa'_\psi \frac{\partial w^{(i)}}{\partial \tau} \\
&\qquad = \frac{d\lambda^{(i)}}{d\tau} w^{(i)} + \lambda^{(i)} \frac{\partial w^{(i)}}{\partial \tau}.
\end{aligned} \tag{7.22}$$

Now, at $\tau = \tau_0$, (7.22) yields

$$-\Lambda(\tau_0)\Delta \delta_w + c_{\tau_0} \frac{\partial \delta_w}{\partial x} + \kappa'_\theta(\delta_H - \delta_w) + \kappa'_\psi \delta_w = \frac{d\lambda^{(1)}}{d\tau}(\tau_0) w^{(1)}, \tag{7.23}$$

and with $\delta_z = \delta_H - \delta_w$,

$$-\Lambda(\tau_0)\Delta \delta_w + c_{\tau_0} \frac{\partial \delta_w}{\partial x} + \kappa'_\theta \delta_z + \kappa'_\psi \delta_w = \frac{d\lambda^{(1)}(\tau_0)}{d\tau} w^{(1)}. \tag{7.24}$$

Subtracting (7.24) from (7.21), we see that

$$-\Delta \delta_z + c_{\tau_0} \frac{\partial \delta_z}{\partial x} - \kappa'_\theta(\tau_0, x)\delta_z - \kappa'_\psi(\tau_0, x)\delta_w = \frac{d\lambda^{(1)}}{d\tau}(\tau_0) z^{(1)}, \tag{7.25}$$

where $z^{(1)} = H^{(1)} - w^{(1)}$. The linear operator in (7.24), (7.25) is the same as in (7.7), (7.8). Therefore, the simplicity of the zero eigenvalue together with (7.12) implies that (7.24), (7.25) cannot have solutions. This leads to a contradiction and concludes the proof. □

This lemma and the topological degree constructed above allow us to prove that if a simple eigenvalue of the linearized problem crosses the origin, then a bifurcation occurs and some other solutions appear in a neighbourhood of the one-dimensional solution.

Recall that, if $\Lambda = 1$, system (7.1), (7.2) can be reduced to a single equation. The principle eigenvalue of the corresponding linearized operator is simple, and bifurcation phenomena can not occur.

As is known from formal asymptotic expansions, for some $\Lambda = \Lambda(\tau_0)$ sufficiently large, a real eigenvalue crosses the origin resulting in the appearance of two-dimensional solutions called in combustion theory cellular flames [24]. The next theorem proves that $\tau = \tau_0$ is indeed a bifurcation point.

**Theorem 7.2.** *Assume that conditions* (1.5), (1.6) *and* (1.18) *hold. Suppose that for* $\tau \neq \tau_0$ *the zero eigenvalue of problem* (7.7), (7.8) *is simple, for* $\tau = \tau_0$ *it has multiplicity* 2, *and there exists an eigenvalue* $\lambda(\tau)$ *such that*

$$\lambda(\tau_0) = 0, \quad \frac{d\lambda(\tau_0)}{d\tau} \neq 0. \tag{7.26}$$

*Then* $\tau = \tau_0$ *is a bifurcation point.*

**Proof.** Consider the operator

$$\mathcal{F}(v, c, \tau) = -\Lambda(\tau)\Delta(v + \phi) + c\frac{\partial(v + \phi)}{\partial x} + \kappa(h - v + 1 - \phi, v + \phi, \tau) \tag{7.27}$$

corresponding to equation (7.5). The triplet $(v, c, \tau)$ will vary in $E_\mu \times (0, +\infty) \times I$, where $E_\mu$ is the weighted space introduced in Section 4.1 and $I$ is the interval described by the parameter $\tau$.

According to our assumptions for each $\tau$ there exists a one-dimensional solution $(\theta_\tau, \psi_\tau, c_\tau)$ of problem (7.1)-(7.4). Therefore, the equation

$$\mathcal{F}(v, c, \tau) = 0 \tag{7.28}$$

has a solution $(v_\tau, c_\tau, \tau)$, where $v_\tau = \psi_\tau - \phi$. When a simple eigenvalue of the operator linearized about this solution passes through the origin, we can expect that a bifurcation will occur and other solution will appear. This is a conventional result based on the application of topological degree. However, in our case we cannot apply it directly. Indeed, the solution $(\theta_\tau, \psi_\tau, c_\tau)$ of problem (7.1)-(7.4) is invariant with respect to translation in space. Consequently the operator $A_\tau w = \mathcal{F}'_v(v_\tau, c_\tau, \tau).w$ linearized about $v_\tau$ with respect to $v$ for $c = c_\tau$ fixed has the zero eigenvalue with the corresponding eigenfunction $\psi'_\tau(x)$.

Because of the existence of the family of solutions and of the zero eigenvalue we can not apply topological degree directly. This situation is specific for travelling waves. A method to define topological degree in this case is developed in [33] (see also [32] and [26]). Instead of the unknown constant

$c$ we introduce a functional $c(v)$. This allows us to get rid of the invariance of solutions with respect to translation and of the corresponding zero eigenvalue of the linearized operator.

Here we will develop another approach. We will construct a subspace which removes the simple zero eigenvalue related to the invariance of the one-dimensional solution with respect to translation in space. We will also require an additional condition on the subspace that will play an important role.

*Construction of the subspace.* Consider the function $f_{\tau_0} = \mathcal{F}'_c(v_{\tau_0}, c_{\tau_0}, \tau_0)$. We can find a function $g_{\tau_0}$ such that

$$(g_{\tau_0}, \psi'_{\tau_0}) \neq 0 \text{ and } (g_{\tau_0}, f_{\tau_0}) \neq 0 \tag{7.29}$$

where ( , ) denotes the inner product in $L^2(\Omega)$. Indeed, let us first check that $f_{\tau_0} \not\equiv 0$. We have

$$f_{\tau_0}(x) = \psi'_{\tau_0}(x) + (1-\Lambda)\kappa'_\theta(\theta_{\tau_0}, \psi_{\tau_0}, \tau_0) \int_x^{+\infty} \int_t^{+\infty} e^{c_{\tau_0}(x-y)} \psi'_{\tau_0}(y) dy dt.$$

From assumption (1.18), we easily obtain that

$$f_{\tau_0}(x) \sim \psi'_{\tau_0}(x) \text{ as } x \to +\infty.$$

Next the construction of $g_{\tau_0}$ relies on the quantity $(\psi'_{\tau_0}, f_{\tau_0})$. If $(\psi'_{\tau_0}, f_{\tau_0}) \neq 0$, we set $g_{\tau_0} = \psi'_{\tau_0}$ and if $(\psi'_{\tau_0}, f_{\tau_0}) = 0$ we set $g_{\tau_0} = \psi'_{\tau_0} + f_{\tau_0}$.

Consider the subspace $E_{0,\mu}$ of functions $v \in E_\mu$ such that

$$\iint_\Omega g_{\tau_0} v dx dy = 0. \tag{7.30}$$

Thanks to the first condition in (7.29), there is no invariance with respect to the translation in $E_{0,\mu}$ and the corresponding zero eigenvalue is removed. Moreover, the operator $A_\tau$ considered as acting from $E_{0,\mu}$ does not have a zero eigenvalue for $\tau$ close to $\tau_0$. Indeed, since $(g_{\tau_0}, \psi'_{\tau_0}) \neq 0$ and $(g_{\tau_0}, \psi'_\tau) \to (g_{\tau_0}, \psi'_{\tau_0})$ as $\tau \to \tau_0$, we obtain that $\psi'_\tau \notin E_{0,\mu}$ and the zero eigenvalue corresponding to the translation invariance is removed for $\tau$ close to $\tau_0$ (up to the restriction of the interval $I$ we suppose that this property holds for all $\tau \in I$).

Denote by $\mathcal{F}_0$ the restriction of the operator $\mathcal{F}$ to $E_{0,\mu} \times (0, +\infty) \times I$, and $A_{0,\tau}$ the restriction of the operator $A_\tau$ to $E_{0,\mu}$. Since the operator $A_\tau$ is normally solvable with a finite-dimensional kernel, the operator $A_{0,\tau}$ also satisfies these properties. Similarly, since the operator $\mathcal{F}^\tau := \mathcal{F}(.,.,\tau)$ is proper, the operator $\mathcal{F}_0^\tau := \mathcal{F}_0(.,.,\tau)$ is also proper. In both cases we take an intersection of a set of solutions with a closed subspace. Therefore the dimension of the kernel of the linear operator remains bounded, the image is closed, and the inverse image of a compact set with respect to the non-linear operator remains compact.

Consider next the operator $(\mathcal{F})'_{(v,c)} = (\mathcal{F})'_{(v,c)}(v_\tau, c_\tau, \tau)$ obtained as a linearization of $\mathcal{F}$ with respect to both $v$ and $c$:

$$(\mathcal{F})'_{(v,c)}.(w,d) = \mathcal{F}'_v(v_\tau, c_\tau, \tau).w + \mathcal{F}'_c(v_\tau, c_\tau, \tau)d = A_\tau w + f_\tau d.$$

To use the degree construction similar to that described in Section 6, let us show that the operator $(\mathcal{F})'_{(v,c)}.(w,d) + kw$ is invertible as acting from $E_{0,\mu} \times \mathbb{R}$ into $F_\mu$ for $k$ large enough and for $\tau$ close to $\tau_0$. We will first check that its kernel is empty. Indeed the equation

$$(\mathcal{F})'_{(v,c)}.(w,d) + kw = 0 \tag{7.31}$$

can be rewritten

$$A_\tau w + kw = -df_\tau. \tag{7.32}$$

In view of Lemma 7.3 below and equation (7.33) for $k$ large enough and for $\tau$ close to $\tau_0$ we have $w = -dw^1(\tau,k)$ and $w^1(\tau,k) \notin E_{0,\mu}$. Therefore the condition $w \in E_{0,\mu}$ implies that $d = 0$ and $w = 0$.

We easily complete the invertibility property. The details are left to the reader.

Thus, by adapting the degree contraction recalled in Section 6, we can define topological degree for the operator $\mathcal{F}_0^\tau$ as well as for the homotopy $\mathcal{F}_0$. It has the eigenvalue $\lambda(\tau)$. For $\tau = \tau_0$ it is an algebraically simple zero eigenvalue.

We can now apply the standard arguments to show that $\tau_0$ is a bifurcation point. Indeed since for $\tau \neq \tau_0$ the operator $\mathcal{F}_0^\tau$ does not have the zero eigenvalue, the index of the one-dimensional solution $(v_\tau, c_\tau, \tau)$ can be computed as $(-1)^\nu$, where $\nu$ is the number of negative eigenvalues of the linearized operator together with their multiplicities. Since $\lambda(\tau)$ crosses the origin for $\tau = \tau_0$, then the quantity $(-1)^\nu$ changes. On the other hand, the degree is a homotopy invariant and does not change with a change of the parameter. Therefore there are other solutions for the equation $\mathcal{F}(v,c,\tau) = 0$ in a neighborhood of the one-dimensional solution for $\tau = \tau_0$. Therefore, there are also other solutions of system (7.1)-(7.4) in the neighborhood of the one-dimensional solution. The theorem is proved. □

It remains to state and prove the following result:

**Lemma 7.3.** *For $k$ large enough and $\tau$ sufficiently close to $\tau_0$, the unique solution $w^1 = w^1(\tau,k)$ in $E_\mu$ of the equation*

$$(A_\tau + kI)w^1 = f_\tau \tag{7.33}$$

*satisfies* $(w^1(\tau, k), g_{\tau_0}) \neq 0$, *so that* $w^1(\tau, k) \notin E_{0,\mu}$.

**Proof of Lemma 7.3.** The proof of Theorem 6.1 implies that $A_\tau + kI$ is invertible on $E_\mu$ for $k$ large enough. Therefore, (7.33) possesses a unique solution denoted by $w^1 = w^1(\tau, k)$. The main step of the proof consists in deriving that $kw^1(\tau, k)$ tends towards $f_{\tau_0}$ weakly in $L^2(\Omega)$ as $k \to +\infty$ and $\tau \to \tau_0$.

First we note that the sectoriality of the operator $A_\tau$ implies that

$$\|w^1(\tau, k)\|_{L^2} \leq \frac{\|f_\tau\|_{L^2}}{k}. \tag{7.34}$$

Therefore, $w^1(\tau, k)$ tends to zero strongly in $L^2(\Omega)$ as $k \to +\infty$ and $k\|w^1(\tau, k)\|_{L^2}$ is bounded independently of both $k$ and $\tau$.

Next the linear problem (7.33) reads

$$-\Lambda(\tau)\Delta w^1(\tau, k) + c_\tau \frac{\partial w^1(\tau, k)}{\partial x} + kw^1(\tau, k) = f(\tau, k), \tag{7.35}$$

where the vector $f(\tau, k)$ is defined by

$$f(\tau, k) = f_\tau - \kappa'_\theta(H^1(\tau, k) - w^1(\tau, k)) + \kappa'_\psi w^1(\tau, k). \tag{7.36}$$

Here $H^1(\tau, k) = H(w^1)$ is given by the resolution of (3.2) with $w = w^1(\tau, k)$.

It follows from estimates (7.34) and (6.5) that the function $f(\tau, k)$ is bounded in $L^2(\Omega)$ independently of $k$ and $\tau$ and tends to $f_{\tau_0}$ strongly in $L^2(\Omega)$ as $k \to +\infty$ and $\tau \to \tau_0$.

Next, multiplying equation (7.35) by $w^1(\tau, k)$ and integrating over $\Omega$, we obtain

$$\begin{aligned} &\Lambda(\tau) \iint_\Omega |\nabla w^1(\tau, k)|^2 dxdy + k \iint_\Omega w^1(\tau, k)^2 dxdy \\ &= \iint_\Omega f(\tau, k) w^1(\tau, k) dxdy \leq \frac{k}{2}\|w^1(\tau, k)\|^2_{L^2} + \frac{1}{2k}\|f(\tau, k)\|^2_{L^2}. \end{aligned} \tag{7.37}$$

From this estimate and (7.34) it follows that $w^1(\tau, k)$ tends to zero strongly in $H^1(\Omega)$. By virtue of equation (7.35), we easily obtain an estimate for $\|\Delta w^1(\tau, k)\|_{L^2}$ independent of $k$ and $\tau$, so that $w^1(\tau, k)$ is bounded in $H^2(\Omega)$. In addition, any subsequence of $w^1(\tau, k)$ weakly convergent in $H^2(\Omega)$ tends to zero because of (7.34). Therefore $w^1(\tau, k)$ tends to zero weakly in $H^2(\Omega)$ as $k \to +\infty$ and $\tau \to \tau_0$, so that $\Delta w^1(\tau, k) \rightharpoonup 0$ weakly in $L^2(\Omega)$.

From (7.35) it follows that $kw^1(\tau, k)$ tends to $f_{\tau_0}$ weakly in $L^2(\Omega)$. Thus $(kw^1(\tau, k), g_{\tau_0})$ tends to $(f_{\tau_0}, g_{\tau_0})$ as $k \to +\infty$ and $\tau \to \tau_0$. Since by (7.29) we have $(f_{\tau_0}, g_{\tau_0}) \neq 0$, this concludes the proof of Lemma 7.3. □

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