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CONVERGENCE OF SOLUTIONS OF A SEMILINEAR PARABOLIC EQUATION TO SELFSIMILAR SOLUTIONS OF THE LINEAR HEAT EQUATION

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Abstract. We find the rate of convergence of solutions of a semilinear parabolic equation to selfsimilar solutions of the linear heat equation. In particular, we show that the rate is not affected by the nonlinearity for some range of parameters, while in a complementary range the rate depends explicitly on the nonlinearity.

1. Introduction

This paper is a continuation of our research on a quantitative description of the behavior of global solutions of the Cauchy problem for a semilinear parabolic equation

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 1$, $p > (N+2)/N$, and u_0 is a positive continuous function.

In our previous papers, given a specific decay rate of u_0 as $|x| \to \infty$, we determined the exact rate of the grow-up of solutions [1, 2, 5], the rate of convergence to regular steady states [6, 10] and to the singular steady state [4], the rate of the decay to the trivial solution [3, 7], and the rate of the convergence to selfsimilar solutions [8]. In particular, the aim of [3] was to show that under certain conditions on p and u_0, which will be stated more

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precisely later, the solution of (1.1) approaches zero with the same order as the solution of the linear heat equation

$$U_t = \Delta U, \qquad x \in \mathbb{R}^N, \quad t > 0, \tag{1.2}$$

with the same initial condition $U(x,0) = u_0(x)$. In order to state this result more precisely, we introduce the following conditions that assure the global existence of solutions:

(A1) $(N-2)p > N$ and

$$0 \le u_0(x) \le \phi_\infty(|x|), \qquad |x| > 0,$$

where

$$\phi_\infty(|x|) := L|x|^{-m}, \qquad |x| > 0,$$

with

$$m := \frac{2}{p-1}, \quad L := \{m(N-2-m)\}^{\frac{1}{p-1}}$$

a singular steady state of (1.1).

(A2) $p > (N+2)/N$ and

$$0 \le u_0(x) \le \phi_\alpha(|x|), \qquad x \in \mathbb{R}^N,$$

for some $\alpha > 0$, where ϕ_α is a positive solution of the initial-value problem

$$\begin{cases} \phi_{\rho\rho} + \frac{N-1}{\rho}\phi_\rho + \frac{\rho}{2}\phi_\rho + \frac{m}{2}\phi + |\phi|^{p-1}\phi = 0, & \rho > 0, \\ \phi(0) = \alpha, \qquad \phi_\rho(0) = 0, \end{cases} \tag{1.3}$$

satisfying

$$\phi_\alpha(\rho) = L_\alpha \rho^{-m} + o(\rho^{-m}) \quad \text{as } \rho \to \infty \tag{1.4}$$

with some constant $L_\alpha > 0$.

It is known [9] that (1.3) has a unique global solution ϕ_α, and it satisfies (1.4) with some constant L_α depending on α. If $\phi_\alpha(\rho)$ is positive for all $\rho > 0$, then

$$u_\alpha(x,t) := (t+1)^{-\frac{1}{p-1}}\phi_\alpha\big((t+1)^{-\frac{1}{2}}|x|\big), \qquad x \in \mathbb{R}^N,\ t \ge 0, \tag{1.5}$$

solves (1.1) with $u_0(x) = \phi_\alpha(|x|)$. We note that $\phi_\infty(\rho) = L\rho^{-m}$ satisfies the equation in (1.3).

In our previous paper [3], we showed that if (A1) or (A2) holds and if u_0 satisfies

$$c_1(|x|+1)^{-l} \le u_0(x) \le c_2(|x|+1)^{-l}, \quad x \in \mathbb{R}^N, \tag{1.6}$$

with some $l > m$ and positive constants c_1, c_2, then the solution of (1.1) exists globally in time and there are constants $C_1, C_2 > 0$ and $t_1 > 1$ such that

$$C_1 g_l(t) \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C_2 g_l(t), \qquad t \ge t_1, \tag{1.7}$$

where

$$g_l(t) := \begin{cases} t^{-\frac{l}{2}} & \text{if } m < l < N, \\ t^{-\frac{N}{2}} \ln t & \text{if } l = N, \\ t^{-\frac{N}{2}} & \text{if } l > N. \end{cases} \tag{1.8}$$

In other words, the solution of (1.1) decays to zero at the same rate as the solution of the linear heat equation (1.2) with the same initial value u_0. This result is an extension of an earlier work by Lee and Ni [13], in which they obtained the same result by assuming that c_1 and c_2 are sufficiently small. See also [11, 12, 14, 15] for other works on the convergence to the trivial solution.

In this paper, assuming more about the behavior of u_0 as $|x| \to \infty$, we show that the solution of (1.1) converges to a selfsimilar solution of the linear heat equation. Let $w = w_l(\rho)$ be the unique solution of

$$\begin{cases} w_{\rho\rho} + \frac{N-1}{\rho} w_\rho + \frac{\rho}{2} w_\rho + \frac{l}{2} w = 0, \qquad \rho > 0, \\ w(0) = 1, \quad w_\rho(0) = 0. \end{cases} \tag{1.9}$$

By direct computation, we can show that

$$U = W_l(x,t) := (t+1)^{-\frac{l}{2}} w_l\big((t+1)^{-\frac{1}{2}}|x|\big)$$

is a solution of (1.2) with the initial value $W_l(x,0) = w_l(|x|)$. We call such a solution a (forward) selfsimilar solution of the linear heat equation.

The following properties of solutions of (1.9) are known (see the proof of Lemma 3.1 in [1]):

(i) If $0 < l < N$, then $w_l > 0$ for all $\rho > 0$. Moreover, there exist positive constants c_l^-, c_l^+ such that

$$c_l^-(\rho+1)^{-l} \le w_l(\rho) \le c_l^+(\rho+1)^{-l}, \qquad \rho \ge 0. \tag{1.10}$$

(ii) If $l = N$, then w_l is written explicitly as $w_l(\rho) = e^{-\rho^2/4}$.

(iii) If $l > N$, then w_l changes its sign.

Our aim is to show that for $l \in (m, N)$, if u_0 is close to kw_l with some positive constant k for $|x| \simeq \infty$, then the solution u of (1.1) approaches $W_l(x,t)$ as $t \to \infty$.

Our first result is stated as follows.

Theorem 1.1. *Assume that* (A1) *or* (A2) *holds. If*

$$|x|^l\{u_0(x) - kw_l(x)\} \to 0 \quad \text{as } |x| \to \infty \tag{1.11}$$

for some $l \in (m, N)$ *and* $k > 0$, *then*

$$(t+1)^{\frac{l}{2}} \|u(\cdot, t) - kW_l(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty. \tag{1.12}$$

Remark. We may replace (1.11) by

$$u_0(x) = kc_l|x|^{-l} + o(|x|^{-l}) \quad \text{as } |x| \to \infty.$$

The above theorem applies to the case when it is only known that $u_0(x)$ behaves like $kw_l(|x|)$ for large $|x|$, regardless of the actual difference between $u_0(x)$ and $kw_l(|x|)$, but does not provide any precise rate of convergence. If the behavior of u_0 up to a second-order term is specified, then we can determine the exact convergence rate as follows.

Theorem 1.2. *Assume that* (A1) *or* (A2) *holds. Assume further that* u_0 *satisfies*

$$|u_0(x) - kw_l(|x|)| \le \kappa(|x|+1)^{-\lambda}, \qquad x \in \mathbb{R}^N,$$

for some $l \in (m, N)$, $\lambda > l$ *and positive constants* k *and* κ.

(i) *If* $l < \lambda < \min\{pl-2, N\}$, *then for any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$\|u(\cdot, t) - kW_l(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \le C_\varepsilon (t+1)^{-\frac{\lambda}{2}+\varepsilon}$$

for all $t \ge 0$.

(ii) *If* $m < l \le (N+2)/p$ *and* $\lambda \ge pl-2$, *then for any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$\|u(\cdot, t) - kW_l(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \le C_\varepsilon (t+1)^{-\frac{pl-2}{2}+\varepsilon}$$

for all $t \ge 0$.

(iii) *If* $(N+2)/p < l < N$ *and* $\lambda \ge N$, *then for any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$\|u(\cdot, t) - kW_l(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \le C_\varepsilon (t+1)^{-\frac{N}{2}+\varepsilon}$$

for all $t \ge 0$.

The following lower estimates show that the convergence rate given in Theorem 1.2 is sharp (up to ε).

Theorem 1.3. *Let* $p > (N+2)/N$, $l \in (m, N)$ *and let* u *be a global solution of* (1.1).

(i) *If*

$$u_0(x) \ge kw_l(|x|) + \kappa(|x|+1)^{-\lambda}, \qquad x \in \mathbb{R}^N,$$

for some $\lambda \in (l, N)$, $k > 0$ *and* $\kappa > 0$, *then there exists* $c > 0$ *such that*

$$\|u(\cdot,t) - kW_l(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \geq c(t+1)^{-\frac{\lambda}{2}}$$

for all $t \geq 0$.

(ii) *If*

$$u_0(x) > kw_l(|x|), \qquad x \in \mathbb{R}^N,$$

for some $k > 0$, *then there exists* $c > 0$ *such that*

$$\|u(\cdot,t) - kW_l(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \geq c(t+1)^{-\frac{pl-2}{2}}$$

for all $t \geq 0$.

(iii) *If*

$$u_0(x) > kw_l(|x|), \qquad x \in \mathbb{R}^N,$$

for some $k > 0$, *then for any* $\varepsilon > 0$ *there exists* $c_\varepsilon > 0$ *such that*

$$\|u(\cdot,t) - kW_l(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \geq c_\varepsilon(t+1)^{-\frac{N}{2}-\varepsilon}$$

for all $t \geq 0$.

Figure 1 illustrates the above results in the $l-\lambda$ space when $p > (N+2)/N$ is fixed. The shaded triangle shows the region where $l \in (m, N)$ and $l < \lambda < \min\{pl-2, N\}$.

This paper is organized as follows. In Section 2, we derive an upper bound on solutions by assuming the radial symmetry of solutions. Because of the source term u^p, the upper estimate is the most technical part. We apply the bootstrap argument to obtain an optimal estimate. By combining this with a lower bound, we prove Theorems 1.1 and 1.2 in Section 3. Section 4 is devoted to the proof of Theorem 1.3.

2. Upper bound on radially symmetric solutions

In this section we assume throughout that u is radially symmetric with respect to the origin, and accordingly write $u_0 = u_0(r)$ and $u = u(r,t)$ instead of $u_0 = u_0(x)$ and $u = u(x,t)$, respectively. Then (1.1) can be written as

$$\begin{cases} u_t = u_{rr} + \frac{N-1}{r}u_r + u^p, & r > 0, \quad t > 0, \\ u(r,0) = u_0(r), & r \geq 0. \end{cases} \tag{2.1}$$

The goal of this section is to prove the following result concerning an upper bound on solutions.

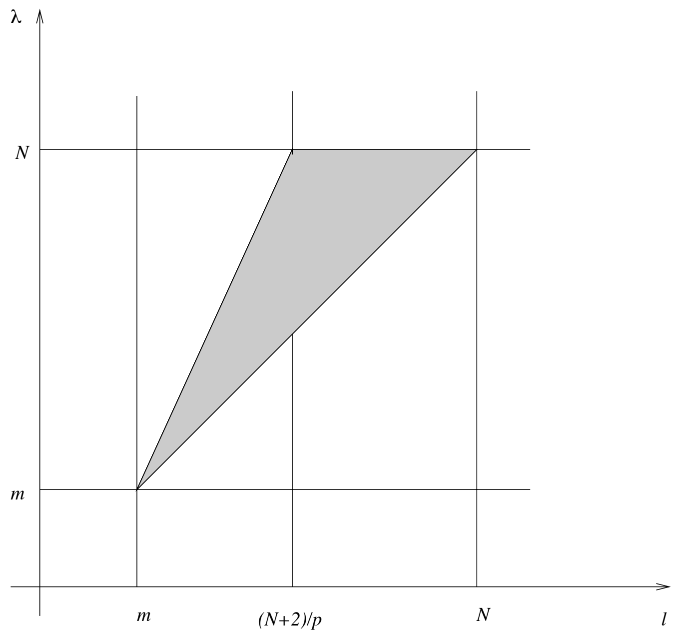

Figure 1.

Proposition 2.1. *Assume that* (A1) *or* (A2) *holds. Assume further that* u_0 *satisfies*

$$u_0(r) \le k w_l(r) + \kappa (r+1)^{-\lambda}, \qquad r \ge 0,$$

for some $l \in (m, N)$*,* $\lambda > l$ *and positive constants* k *and* κ*.*

(i) *If* $l < \lambda < \min\{pl-2, N\}$*, then for any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$u(r,t) \le (t+1)^{-\frac{l}{2}} k w_l\big((t+1)^{-\frac{1}{2}} r\big) + C_\varepsilon (t+1)^{-\frac{\lambda}{2}+\varepsilon}, \qquad r \ge 0,$$

for all $t \ge 0$*.*

(ii) *If* $m < l < (N+2)/p$ *and* $\lambda \ge pl-2$*, then for any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$u(r,t) \le (t+1)^{-\frac{l}{2}} k w_l\big((t+1)^{-\frac{1}{2}} r\big) + C_\varepsilon (t+1)^{-\frac{pl-2}{2}+\varepsilon}, \qquad r \ge 0,$$

for all $t \ge 0$*.*

(iii) *If* $(N+2)/p \le l < N$ *and* $\lambda \ge N$*, then for any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$u(r,t) \le (t+1)^{-\frac{l}{2}} k w_l\big((t+1)^{-\frac{1}{2}} r\big) + C_\varepsilon (t+1)^{-\frac{N}{2}+\varepsilon}, \qquad r \ge 0,$$

for all $t \geq 0$.

We note that under the hypotheses of Proposition 2.1, the initial value satisfies

$$u_0(r) \leq \hat{k} r^{-l}, \qquad r > 1, \tag{2.2}$$

for some positive constant $\hat{k}$.

The proof of Proposition 2.1 consists of several steps. First of all, we introduce the transformation defined by

$$v(\rho, s) := (t+1)^{\frac{l}{2}} u(r,t), \qquad \rho := (t+1)^{-\frac{1}{2}} r, \quad s := \ln(t+1). \tag{2.3}$$

Then v satisfies

$$v_s = v_{\rho\rho} + \frac{N-1}{\rho} v_\rho + \frac{\rho}{2} v_\rho + \frac{l}{2} v + e^{-\frac{l-m}{m} s} v^p, \quad \rho > 0,\ s > 0, \tag{2.4}$$

with $v(\rho, 0) = u_0(\rho)$. In particular, for $l = m$,

$$v_S(\rho, s) := (t+1)^{\frac{m}{2}} u(r,t)$$

satisfies the autonomous equation

$$(v_S)_s = (v_S)_{\rho\rho} + \frac{N-1}{\rho}(v_S)_\rho + \frac{\rho}{2}(v_S)_\rho + \frac{m}{2} v_S + v_S^p, \quad \rho > 0,\ s > 0. \tag{2.5}$$

It was one objective of our previous paper [3] to prove that if u_0 satisfies certain decay assumptions at $r \simeq \infty$, then v_S converges to zero as $s \to \infty$. More precisely, the next result follows from [3].

Lemma 2.2. *Suppose that the hypotheses of Proposition* 2.1 *are satisfied. Then there exist positive constants* c_0, c_1 *and* $l_0 > m$ *such that*

$$v_S(\rho, s) \leq c_0 (1+\rho)^{-l_0}, \qquad \rho \geq 0,\ s \geq 0, \tag{2.6}$$

and

$$v_S(\rho, s) \leq c_1 e^{-\frac{l-m}{2} s}, \qquad \rho \geq 0,\ s \geq 0. \tag{2.7}$$

Proof. If we assume (A1) then (2.6) follows from [3, Lemma 3.1] and (2.7) from [3, Theorems 1.3 and 1.2]. Under the assumption (A2), the proof of [3, Theorem 1.5] implies that (2.6) holds and [3, Theorems 1.5 and 1.2] yield (2.7). Since (2.7) is only an upper bound, it is sufficient to assume the second inequality in (1.8) in [3]. □

The next assertion can be interpreted as a slight improvement of the result obtained from (2.6) and (2.7).

Lemma 2.3. *Suppose that the hypotheses of Proposition* 2.1 *are satisfied. Suppose further that there exist* $K_1 > 0, 0 \le \theta \le (l-m)/2$ *and* $\lambda \in (m, l]$ *such that*

$$v_S(\rho, s) \le K_1 e^{-\theta s} \rho^{-\lambda}, \qquad \rho > 1,\ s \ge 0. \tag{2.8}$$

Then for any $\vartheta \ge \theta$ *and* $\mu \in [\lambda, l]$ *satisfying*

$$\mu > m + 2\vartheta, \tag{2.9}$$

$$\mu < p\lambda, \tag{2.10}$$

$$\vartheta \le \frac{p(l-m)}{2} - \frac{(l-m-2\theta)\mu}{2\lambda}, \tag{2.11}$$

there exists $K_2 > 0$ *such that*

$$v_S(\rho, s) \le K_2 e^{-\vartheta s} \rho^{-\mu}, \qquad \rho > 1,\ s \ge 0. \tag{2.12}$$

Proof. Since $m < \lambda \le \mu \le l < N$, the solution w_μ of (1.9) is positive for all $\rho \ge 0$ and satisfies (1.10) with l replaced by μ. With a large constant $A > 0$ to be specified below, we define

$$\bar{v}(\rho, s) := Ae^{-\vartheta s} w_\mu(\rho), \qquad \rho \ge 0,\ s \ge 0.$$

Then (1.10) and the fact that $\mu \le l$ imply that, if $A > \frac{\hat{k}}{2^{-\mu} c_\mu^-}$, then

$$\bar{v}(\rho, 0) \ge Ac_\mu^-(1+\rho)^{-\mu} \ge A2^{-\mu} c_\mu^- \rho^{-\mu} > \hat{k}\rho^{-l}, \qquad \rho > 1.$$

Hence, by (2.2), $\bar{v}$ lies above v_S initially if A is sufficiently large.

By direct computation using (1.9), $\bar{v}$ satisfies

$$\begin{aligned} I &:= \bar{v}_s - \bar{v}_{\rho\rho} - \tfrac{N-1}{\rho}\bar{v}_\rho - \tfrac{\rho}{2}\bar{v}_\rho - \tfrac{m}{2}\bar{v} - v_S^p \\ &= -A\, e^{-\vartheta s}\Big\{(w_\mu)_{\rho\rho} + \tfrac{N-1}{\rho}(w_\mu)_\rho + \tfrac{\rho}{2}(w_\mu)_\rho + \tfrac{m+2\vartheta}{2} w_\mu\Big\} - v_S^p \\ &= \tfrac{\mu-m-2\vartheta}{2}\, A\, e^{-\vartheta s} w_\mu - v_S^p, \qquad \rho > 0,\ s > 0. \end{aligned} \tag{2.13}$$

For $\rho > 1$ we can use (2.8) and (1.10) to estimate

$$I \ge \tfrac{\mu-m-2\vartheta}{2}\, A2^{-\mu} c_\mu^- e^{-\vartheta s} \rho^{-\mu} - K_1^p\, e^{-p\theta s} \rho^{-p\lambda},$$

where $\mu - m - 2\vartheta > 0$ by (2.9). Therefore I is nonnegative if $\rho > 1$ and

$$\rho \ge \left\{ \frac{K_1^p}{(\mu - m - 2\vartheta)A2^{-\mu-1}c_\mu^-} \right\}^{\frac{1}{p\lambda-\mu}} e^{\frac{\vartheta - p\theta}{p\lambda-\mu}\, s}, \tag{2.14}$$

where we have used the fact that $p\lambda - \mu > 0$ due to (2.10).

For small ρ, we estimate v_S by using (2.7) rather than (2.8). Accordingly, it follows from (2.13) that

$$I \geq \tfrac{\mu-m-2\vartheta}{2} A\, e^{-\vartheta s} w_\mu - c_1^p\, e^{-\frac{p(l-m)}{2}\, s}. \tag{2.15}$$

Hence if $\rho > 1$ but (2.14) is violated, then

$$\begin{aligned} I &\geq (\mu - m - 2\vartheta) A 2^{-\mu-1} c_\mu^-\, e^{-\vartheta s} \rho^{-\mu} - c_1^p\, e^{-\frac{p(l-m)}{2}\, s} \\ &\geq (\mu - m - 2\vartheta) A 2^{-\mu-1} c_\mu^-\, e^{-\vartheta s} \left\{ \frac{(\mu - m - 2\vartheta) A 2^{-\mu-1} c_\mu^-}{K_1^p} \right\}^{\frac{\mu}{p\lambda-\mu}} e^{-\frac{(\vartheta - p\theta)\mu}{p\lambda-\mu}\, s} \\ &\qquad - c_1^p\, e^{-\frac{p(l-m)}{2}\, s} \\ &\geq (\mu - m - 2\vartheta) A 2^{-\mu-1} c_\mu^- \left\{ \frac{(\mu - m - 2\vartheta) A 2^{-\mu-1} c_\mu^-}{K_1^p} \right\}^{\frac{\mu}{p\lambda-\mu}} e^{-\frac{p(\lambda\vartheta - \mu\theta)}{p\lambda-\mu}\, s} \\ &\qquad - c_1^p\, e^{-\frac{p(l-m)}{2}\, s}. \end{aligned}$$

Thus, $I \geq 0$ for such ρ if A is large enough, provided that

$$\frac{p(\lambda\vartheta - \mu\theta)}{p\lambda - \mu} \leq \frac{p(l-m)}{2}.$$

This, however, is guaranteed by the hypothesis (2.11).

Finally, for $\rho \leq 1$, (2.11) and the assumption $\theta \leq (l-m)/2$ imply that $\vartheta \leq p(l-m)/2$. Therefore, from (2.15) and the strict positivity of w_μ for $\rho \leq 1$, we see that I is nonnegative also for such ρ if A is large enough.

Altogether, we have proved that $\bar{v}$ is a supersolution of (2.5). Hence, by recalling the initial ordering, we conclude $v_S \leq \bar{v}$ for all $\rho \geq 0, s \geq 0$ by the comparison principle. In view of (1.10), this implies (2.12) with $K_2 := c_\mu^+ A$. □

We now repeatedly apply the above lemma to obtain the following.

Lemma 2.4. *Suppose that the hypotheses of Proposition 2.1 are satisfied. Then for any $\nu > 0$ there exists $c_\nu > 0$ such that*

$$v_S(\rho, s) \leq c_\nu\, e^{-\frac{l-m-\nu}{2}\, s} \rho^{-l}, \qquad \rho > 1,\ s \geq 0. \tag{2.16}$$

Proof. By Lemma 2.2, there exist $l_0 \in (m, l)$ and $c_0 > 0$ such that

$$v_S(\rho, s) \leq c_0\, e^{-\alpha_0 s} \rho^{-l_0}, \qquad \rho > 1,\ s \geq 0,$$

for $\alpha_0 := 0 < (l-m)/2$. Our plan is to apply Lemma 2.3 finitely many times so as to arrive at (2.16).

For this purpose, we may assume that $\nu < l_0 - m$. We take $n \in \mathbb{N}$ large enough such that

$$\gamma := \left(\frac{l}{l_0}\right)^{\frac{1}{n}} \in (1, p).$$

We now define sequences $(l_j)_{j\in\mathbb{N}}$ and $(\alpha_j)_{j\in\mathbb{N}}$ by

$$l_j := \begin{cases} \gamma^j l_0, & 1 \le j \le n, \\ l, & j > n, \end{cases}$$

and, recursively,

$$\alpha_j := \min\left\{\frac{l_j - m - \nu}{2}, \frac{p(l-m)}{2} - \frac{(l - m - 2\alpha_{j-1})l_j}{2l_{j-1}}\right\}, \qquad j \ge 1.$$

We first observe that for j with $\alpha_j < (l_j - m - \nu)/2$, since $\alpha_{j-1} < (l-m)/2$ and $l_j \le \gamma l_{j-1}$, we have

$$\begin{aligned} \alpha_j &= \frac{p(l-m)}{2} - \frac{(l - m - 2\alpha_{j-1})l_j}{2l_{j-1}} \\ &\ge \frac{p(l-m)}{2} - \frac{(l - m - 2\alpha_{j-1})\gamma}{2} \qquad (2.17) \\ &= \frac{(p-\gamma)(l-m)}{2} + \gamma\alpha_{j-1} \\ &> \gamma\alpha_{j-1}. \qquad (2.18) \end{aligned}$$

On the other hand, if $\alpha_j = (l_j - m - \nu)/2$ then trivially $\alpha_j \ge (l_{j-1} - m - \nu)/2 \ge \alpha_{j-1}$ (note that this is true also if $j - 1 = 0$). Thus the sequence $(\alpha_j)_{j\in\mathbb{N}}$ is nondecreasing. Now, by (2.17), there exists $j_0 \ge n$ such that $\alpha_j = (l - m - \nu)/2$ for all $j \ge j_0$, for otherwise we would have $\alpha_j \ge \gamma^{j-n}\alpha_n \ge \gamma^{j-n}\alpha_1$. Since $\alpha_1 > 0$ by (2.17), this would mean that $\alpha_j \to \infty$ as $j \to \infty$ and thereby contradict the fact that $\alpha_j \le (l - m - \nu)/2$ for all j by definition.

In order to apply Lemma 2.3 with $\theta := \alpha_{j-1} < (l-m)/2, \lambda := l_{j-1} \in (m, l], \vartheta := \alpha_j \ge \alpha_{j-1}$ and $\mu := l_j \in [l_{j-1}, l]$, $j = 1, \ldots, j_0$, we only need to check that (2.9)-(2.11) are satisfied, while (2.10) follows from $l_j \le \gamma l_{j-1}$ for all j and the fact that $\gamma < p$, (2.11) is a direct consequence of the definition of α_j. Finally, (2.9) follows from

$$\alpha_j \le \frac{l_j - m - \nu}{2} < \frac{l_j - m}{2}.$$

Consequently, for $j = j_0$ we obtain the estimate

$$v_S(\rho,s) \le ce^{-\alpha_{j_0} s}\rho^{-l_{j_0}} = ce^{-\frac{l-m-\nu}{2}s}\rho^{-l}, \qquad \rho > 1,\ s \ge 0,$$

with some $c > 0$. This completes the proof. □

Lemma 2.5. *Suppose that the hypotheses of Proposition* 2.1 *are satisfied. Suppose further that*

$$u_0(r) \le kw_l(r) + \kappa r^{-\lambda}, \qquad r > 0, \tag{2.19}$$

for some $l \in (m,N)$, $\lambda \in (l,N)$, $\lambda \le pl-2$ *and positive constants* k *and* κ. *Then for any* $\delta > 0$ *there exists* $c_\delta > 0$ *such that the solution* v *of* (2.4) *satisfies*

$$v(\rho,s) \le kw_l(\rho) + c_\delta\, e^{-\frac{\lambda-l-\delta}{2}s}(1+\rho)^{-\lambda}, \qquad \rho \ge 0,\ s \ge 0. \tag{2.20}$$

Proof. Set

$$\nu := \frac{2l}{\lambda}\left(\frac{l-m}{m} - \frac{\lambda-l-\delta}{2}\right).$$

We note that ν is positive since our assumption $\lambda \le pl-2$ implies $(l-m)/m \ge (\lambda-l)/2$. Our goal is to compare v with $\bar{v}$ defined by

$$\bar{v}(\rho,s) := kw_l(\rho) + Be^{-\frac{\lambda-l-\delta}{2}s}w_\lambda(\rho), \qquad \rho \ge 0,\ s \ge 0,$$

with appropriately large $B > 0$. By (1.10), we have

$$\bar{v}(\rho,0) \ge kw_l(\rho) + B2^{-\lambda}c_\lambda^-\rho^{-\lambda}, \qquad \rho > 1.$$

Hence, by (2.19), $\bar{v}(\rho,0) \ge v(\rho,0) = u_0(\rho)$ holds if

$$B \ge \frac{2^\lambda}{c_\lambda^-}\max\{\kappa, \|u_0\|_{L^\infty(\mathbb{R}^N)}\}.$$

Using (1.9), we see that $\bar{v}$ satisfies

$$\begin{aligned}
J \quad := \quad & \bar{v}_s - \bar{v}_{\rho\rho} - \tfrac{N-1}{\rho}\bar{v}_\rho - \tfrac{\rho}{2}\bar{v}_\rho - \tfrac{l}{2}\bar{v} - e^{-\frac{l-m}{m}s}v^p \\
= \quad & -\tfrac{\lambda-l-\delta}{2}\,B\,e^{-\frac{\lambda-l-\delta}{2}s}w_\lambda \\
& -k\left((w_l)_{\rho\rho} + \tfrac{N-1}{\rho}(w_l)_\rho + \tfrac{\rho}{2}(w_l)_\rho + \tfrac{l}{2}w_l\right) \\
& -Be^{-\frac{\lambda-l-\delta}{2}s}\left((w_\lambda)_{\rho\rho} + \tfrac{N-1}{\rho}(w_\lambda)_\rho + \tfrac{\rho}{2}(w_\lambda)_\rho + \tfrac{l}{2}w_\lambda\right) - e^{-\frac{l-m}{m}s}v^p \\
= \quad & \tfrac{\delta}{2}Be^{-\frac{\lambda-l-\delta}{2}s}w_\lambda - e^{-\frac{l-m}{m}s}v^p.
\end{aligned} \tag{2.21}$$

Hence, by (1.10) and Lemma 2.4, which says that $v \equiv e^{\frac{l-m}{2}s} v_S$ satisfies $v \le c\, e^{\frac{\nu}{2}s} \rho^{-l}$ with some $c > 0$, we obtain

$$J \ge 2^{-\lambda-1} c_\lambda^- \delta B e^{-\frac{\lambda-l-\delta}{2}s} \rho^{-\lambda} - c^p\, e^{\left(\frac{p\nu}{2} - \frac{l-m}{m}\right)s} \rho^{-pl}, \qquad \rho > 1,\ s \ge 0.$$

Thus, $J \ge 0$ if additionally

$$\rho \ge \left(\frac{c^p}{2^{-\lambda-1} c_\lambda^- \delta B} \right)^{\frac{1}{pl-\lambda}} e^{\frac{1}{pl-\lambda}\left(\frac{p\nu}{2} - \frac{l-m}{m} + \frac{\lambda-l-\delta}{2}\right)s}, \tag{2.22}$$

because $\lambda \le pl - 2 < pl$.

On the other hand, we see from Lemma 2.2 that $v = e^{\frac{l-m}{2}s} v_S \le c_1$ for $\rho \ge 0$ and $s \ge 0$, and hence the right-hand side of (2.21) can alternatively be estimated as

$$J \ge \tfrac{\delta}{2} B e^{-\frac{\lambda-l-\delta}{2}s} w_\lambda - c_1^p\, e^{-\frac{l-m}{m}s}, \qquad \rho \ge 0,\ s \ge 0.$$

Hence, by (1.10),

$$J \ge 2^{-\lambda-1} \delta B e^{-\frac{\lambda-l-\delta}{2}s} - c_1^p\, e^{-\frac{l-m}{m}s} \ge 0, \qquad \rho \le 1$$

if $B \ge \frac{2^{\lambda+1} c_1^p}{\delta}$, where we have used $(l-m)/m \ge (\lambda - l)/2$ again.

However, if $\rho > 1$ but (2.22) is false, then

$$\begin{aligned} J &\ge 2^{-\lambda-1} c_\lambda^- \delta B\, e^{-\frac{\lambda-l-\delta}{2}s} \rho^{-\lambda} - c_1^p\, e^{-\frac{l-m}{m}s} \\ &\ge 2^{-\lambda-1} c_\lambda^- \delta B \left(\frac{2^{-\lambda-1} c_\lambda^- \delta B}{c^p} \right)^{\frac{\lambda}{pl-\lambda}} e^{-\frac{\lambda}{pl-\lambda}\left(\frac{p\nu}{2} - \frac{l-m}{m} + \frac{\lambda-l-\delta}{2}\right)s} e^{-\frac{\lambda-l-\delta}{2}s} \\ &\quad - c_1^p\, e^{-\frac{l-m}{m}s}. \end{aligned} \tag{2.23}$$

Here an elementary calculation involving the definition of ν shows that

$$\begin{aligned} &\frac{\lambda}{pl-\lambda} \left(\frac{p\nu}{2} - \frac{l-m}{m} + \frac{\lambda-l-\delta}{2} \right) - \frac{\lambda-l-\delta}{2} \\ = &\frac{\lambda}{pl-\lambda} \left(\frac{pl}{\lambda} \left(\frac{l-m}{m} - \frac{\lambda-l-\delta}{2} \right) - \frac{l-m}{m} + \frac{\lambda-l-\delta}{2} \right) - \frac{\lambda-l-\delta}{2} \\ = &\frac{(pl-\lambda)\frac{l-m}{m} - (pl-\lambda)\frac{\lambda-l-\delta}{2}}{pl-\lambda} - \frac{\lambda-l-\delta}{2} = \frac{l-m}{m}. \end{aligned}$$

Therefore (2.23) yields that $J \ge 0$ for such ρ if B is suitably large.

Applying the comparison principle, we conclude that there exists $B > 0$ such that

$$v(\rho, s) \le k w_l(\rho) + B e^{-\frac{\lambda-l-\delta}{2}s} w_\lambda(\rho), \qquad \rho \ge 0,\ s \ge 0,$$

which immediately gives (2.20) in view of (1.10). □

Now we are in a position to prove Proposition 2.1.

Proof of Proposition 2.1. The assertion (i) is a direct consequence of Lemma 2.5 and (2.3). Consider next the assertion (ii). Setting $\lambda = pl - 2 < N$ and $\delta = 2\varepsilon$ in (2.20) of Lemma 2.5, we have

$$v(\rho, s) \le kw_l(\rho) + c_{2\varepsilon}\, e^{(-\frac{\lambda-l}{2}+\varepsilon)\, s}(1+\rho)^{-\lambda}.$$

By (2.3), this implies (ii). Similarly, setting $\lambda = N - \varepsilon$ and $\delta = \varepsilon$ with $\varepsilon < N - l$, we can show (iii). □

3. Proofs of Theorems 1.1 and 1.2

We first give a lower bound of the solution of (2.1).

Lemma 3.1. *Let $p > (N+2)/N$, and let u be a global solution of* (2.1)*. If*

$$u_0(r) \ge kw_l(r) - \kappa(r+1)^{-\lambda}, \qquad r \ge 0, \tag{3.1}$$

for some $l \in (m, N)$, $\lambda \in (l, N)$, and positive constants k and κ, then there exists a constant $c > 0$ such that

$$u(r,t) \ge (t+1)^{-\frac{l}{2}} kw_l\big((t+1)^{-\frac{1}{2}} r\big) - c(t+1)^{-\frac{\lambda}{2}}, \qquad r \ge 0,\ t \ge 0. \tag{3.2}$$

Proof. Define

$$\underline{v}(\rho, s) := kw_l(\rho) - Be^{-\beta s} w_\lambda(\rho), \qquad \rho \ge 0,\ s \ge 0,$$

with $\beta = (\lambda - l)/2$. Since $\lambda \in (l, N)$, it follows from (1.10) and (3.1) that

$$v(\rho, 0) = u_0(\rho) \ge kw_l(\rho) - Bw_\lambda(\rho), \qquad \rho \ge 0,$$

if $B > 0$ is sufficiently large. Moreover, by using (1.9), we compute

$$\begin{aligned}
&\underline{v}_s - \underline{v}_{\rho\rho} - \tfrac{N-1}{\rho}\underline{v}_\rho - \tfrac{\rho}{2}\underline{v}_\rho - \tfrac{l}{2}\underline{v} - e^{-\frac{l-m}{m}s}\underline{v}^p \\
&\quad \le \beta Be^{-\beta s} w_\lambda - k\Big\{(w_l)_{\rho\rho} + \tfrac{N-1}{\rho}(w_l)_\rho + \tfrac{\rho}{2}(w_l)_\rho + \tfrac{l}{2}w_l\Big\} \\
&\qquad + Be^{-\beta s}\Big\{(w_\lambda)_{\rho\rho} + \tfrac{N-1}{\rho}(w_\lambda)_\rho + \tfrac{\rho}{2}(w_\lambda)_\rho + \tfrac{l}{2}w_\lambda\Big\} \\
&\quad = \big(\beta - \tfrac{\lambda-l}{2}\big)\, Be^{-\beta s} w_\lambda = 0.
\end{aligned}$$

Hence $\underline{v}$ is a subsolution of (2.4). Then, by the comparison principle, we have

$$\begin{aligned}
v(\rho, s) &\ge kw_l(\rho) - Be^{-\beta s} w_\lambda(\rho) \\
&\ge kw_l(\rho) - c_\lambda^+ Be^{-\beta s}, \qquad \rho \ge 0,\ s \ge 0.
\end{aligned}$$

By (2.3), this is equivalent to (3.2) with $c := c_\lambda^- B$. □

Now, let us complete the proofs of Theorems 1.1 and 1.2. We first prove Theorem 1.2.

Proof of Theorem 1.2. Let $u_0^+(r)$ and $u_0^-(r)$ be radially symmetric functions that satisfy the hypotheses of Theorem 1.2, and denote by $u^+(r,t)$ and $u^-(r,t)$ the solutions of (2.1) with initial data $u_0^+(r)$ and $u_0^-(r)$, respectively.

First, when $l < \lambda < \min\{pl-2, N\}$, we take $u_0^+(r)$ and $u_0^-(r)$ such that

$$u_0(x) \le u_0^+(|x|) \le kw_l(|x|) + \kappa(|x|+1)^{-\lambda}, \qquad x \in \mathbb{R}^N, \tag{3.3}$$

and

$$kw_l(|x|) - \kappa(|x|+1)^{-\lambda} \le u_0^-(|x|) \le u_0(x), \qquad x \in \mathbb{R}^N. \tag{3.4}$$

Then, by Proposition 2.1 (i) and Lemma 3.1, we have

$$u^+(r,t) \le (t+1)^{-\frac{l}{2}} kw_l\big((t+1)^{-\frac{1}{2}} r\big) + C_\varepsilon (t+1)^{-\frac{\lambda}{2}+\varepsilon}, \qquad r \ge 0,\ t \ge 0,$$

and

$$u^-(r,t) \ge (t+1)^{-\frac{l}{2}} kw_l\big((t+1)^{-\frac{1}{2}} r\big) - c(t+1)^{-\frac{\lambda}{2}} \qquad r \ge 0,\ t \ge 0.$$

By the comparison principle, these inequalities imply (i).

Next, when $m < l \le (N+2)/p$ and $\lambda \ge pl-2$, we take $u_0^+(r)$ as in (3.3) and take $u_0^-(r)$ such that

$$kw_l(|x|) - \kappa(|x|+1)^{-(pl-2-2\varepsilon)} \le u_0^-(|x|) \le u_0(x), \qquad x \in \mathbb{R}^N,$$

instead of (3.4). Then, by Proposition 2.1 (ii) and Lemma 3.1 with $\lambda = pl-2-2\varepsilon$, we have

$$u^+(r,t) \le (t+1)^{-\frac{l}{2}} kw_l\big((t+1)^{-\frac{1}{2}} r\big) + C_\varepsilon (t+1)^{-\frac{pl-2}{2}+\varepsilon}, \qquad r \ge 0,\ t \ge 0,$$

and

$$u^-(r,t) \ge (t+1)^{-\frac{l}{2}} kw_l\big((t+1)^{-\frac{1}{2}} r\big) - c(t+1)^{-\frac{pl-2}{2}+\varepsilon} \qquad r \ge 0,\ t \ge 0.$$

By the comparison principle, these inequalities imply (ii).

Similarly, when $(N+2)/p < l < N$ and $\lambda \ge N$, we obtain (iii) by applying Proposition 2.1 (iii) and Lemma 3.1 with $\lambda = N - 2\varepsilon$. □

Next, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. We first consider the radially symmetric case. Given $\varepsilon > 0$, we take $\sigma > 0$ such that $\sigma c_l^+ \le \varepsilon/2$, where c_l^+ is as in (1.10). Then we take ρ_0 large such that

$$\rho^l u_0(\rho) \le \Big(k + \frac{\sigma}{2}\Big) c_l, \qquad \rho \ge \rho_0 \tag{3.5}$$

and

$$\rho^l w_l(\rho) \geq \frac{k+\frac{\sigma}{2}}{k+\sigma} \cdot c_l, \qquad \rho \geq \rho_0. \tag{3.6}$$

Fix any λ with $l < \lambda < \min\{pl-2, N\}$, and set

$$\kappa := \max_{\rho \leq \rho_0} \rho^\lambda u_0(\rho).$$

Then for $\rho \leq \rho_0$ we have

$$u_0(\rho) \leq \kappa \rho^{-\lambda},$$

whereas for $\rho > \rho_0$ we use (3.5) and (3.6) to show that

$$u_0(\rho) \leq (k + \tfrac{\sigma}{2}) c_l \rho^{-l} \leq (k+\sigma) w_l(\rho),$$

so that

$$u_0(\rho) \leq (k+\sigma) w_l(\rho) + \kappa \rho^{-\lambda}$$

holds for all $\rho > 0$. Now, by Lemma 2.5, there exist positive constants α and c such that the function v given by (2.3) satisfies

$$v(\rho, s) \leq (k+\sigma) w_l(\rho) + c e^{-\alpha s}, \qquad \rho \geq 0, \ s \geq 0.$$

Recalling the definition of σ and setting

$$s_\varepsilon := \frac{1}{\alpha} \ln \frac{2c}{\varepsilon},$$

we obtain

$$\begin{aligned} v(\rho, s) - k w_l(\rho) &\leq \sigma w_l(\rho) + c\, e^{-\alpha s} \\ &\leq \sigma c_l^+ + c\, e^{-\alpha s_\varepsilon} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \qquad \rho \geq 0, \ s \geq s_\varepsilon. \end{aligned}$$

By (2.3), this implies

$$(t+1)^{\frac{l}{2}} u(r,t) - k w_l\big((t+1)^{-\frac{1}{2}} r\big) \leq \varepsilon, \qquad r \geq 0, \ t \geq e^{s_\varepsilon} - 1.$$

Similarly, we can show

$$(t+1)^{\frac{l}{2}} u(r,t) - k w_l\big((t+1)^{-\frac{1}{2}} r\big) \geq -\varepsilon, \qquad r \geq 0, \ t \geq e^{s_\varepsilon} - 1.$$

Suppose here that u_0 is not necessarily radially symmetric. We take radially symmetric functions $u_0^+(r)$ and $u_0^-(r)$ that satisfy the hypotheses of Theorem 1.1 and

$$u_0^-(|x|) \leq u_0(x) \leq u_0^+(|x|), \qquad x \in \mathbb{R}^N.$$

We denote by $u^+(r,t)$ and $u^-(r,t)$ the solutions of (2.1) with initial data $u_0^+(r)$ and $u_0^-(r)$, respectively. By applying the above argument for radially symmetric solutions and the comparison principle, we have

$$(t+1)^{\frac{l}{2}}u(x,t) \le (t+1)^{\frac{l}{2}}u^+(|x|,t) \le kw_l\big((t+1)^{-\frac{1}{2}}|x|\big) + \varepsilon$$

and

$$(t+1)^{\frac{l}{2}}u(x,t) \ge (t+1)^{\frac{l}{2}}u^-(|x|,t) \ge kw_l\big((t+1)^{-\frac{1}{2}}|x|\big) - \varepsilon$$

for all $x \in \mathbb{R}^N$ and $t \ge e^{s_\varepsilon} - 1$. Thus we obtain

$$(t+1)^{\frac{l}{2}}\|u(\cdot,t) - kW_l(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \varepsilon, \qquad t \ge e^{s_\varepsilon} - 1.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. $\square$

4. Proof of Theorem 1.3

The next lemmas will be used in the proof of Theorem 1.3.

Lemma 4.1. *Let $p > (N+2)/N$, and let u be a radially symmetric global solution of* (2.1). *If*

$$u_0(r) \ge kw_l(r) + \kappa(r+1)^{-\lambda}, \qquad r \ge 0, \tag{4.1}$$

for some $l \in (m,N)$, $\lambda \in (l,N)$ and positive constants k and κ, then

$$u(r,t) \ge (t+1)^{-\frac{l}{2}}kw_l\big((t+1)^{-\frac{1}{2}}r\big) + c(t+1)^{-\frac{\lambda}{2}}, \quad r \ge 0,\ t \ge 0, \tag{4.2}$$

holds for some positive constant c.

Proof. Define

$$\underline{v}(\rho,s) := kw_l(\rho) + Be^{-\beta s}w_\lambda(\rho), \qquad \rho \ge 0,\ s \ge 0,$$

with $\beta = \frac{\lambda - l}{2}$. Since $\lambda > l$, it follows from (4.1) and (1.10) that

$$v(\rho,0) = u_0(\rho) \ge kw_l(\rho) + Bw_\lambda(\rho), \qquad \rho \ge 0,$$

if $B > 0$ is sufficiently small. Moreover, by using (1.9), we compute

$$\begin{aligned}
&\underline{v}_s - \underline{v}_{\rho\rho} - \tfrac{N-1}{\rho}\underline{v}_\rho - \tfrac{\rho}{2}\underline{v}_\rho - \tfrac{l}{2}\underline{v} - e^{-\frac{l-m}{m}s}\underline{v}^p \\
&\quad < -\beta Be^{-\beta s}w_\lambda - k\Big\{(w_l)_{\rho\rho} + \tfrac{N-1}{\rho}(w_l)_\rho + \tfrac{\rho}{2}(w_l)_\rho + \tfrac{l}{2}w_l\Big\} \\
&\qquad - Be^{-\beta s}\Big\{(w_\lambda)_{\rho\rho} + \tfrac{N-1}{\rho}(w_\lambda)_\rho + \tfrac{\rho}{2}(w_\lambda)_\rho + \tfrac{l}{2}w_\lambda\Big\} \\
&\quad = \big(-\beta + \tfrac{\lambda-l}{2}\big)\,Be^{-\beta s}w_\lambda = 0.
\end{aligned}$$

Hence $\underline{v}$ is a subsolution of (2.4). Then by the comparison principle, we have

$$v(\rho,s) \quad \ge \quad kw_l(\rho) + Be^{-\beta s}w_\lambda(\rho)$$

$$\geq \quad kw_l(\rho) + c_\lambda^- B e^{-\beta s}, \qquad \rho \geq 0,\ s \geq 0.$$

By (2.3), this is equivalent to (4.2) with $c = c_\lambda^- B$. □

Lemma 4.2. *Let $p > (N+2)/N$, and let u be a radially symmetric global solution of* (2.1). *If*

$$u_0(r) > kw_l(r), \qquad r \geq 0,$$

for some $l \in (m, N)$ and $k > 0$, then there exists $c > 0$ such that

$$u(0,t) \geq (t+1)^{-\frac{l}{2}} k + c(t+1)^{-\frac{pl-2}{2}}, \qquad t \geq 0. \tag{4.3}$$

Proof. Fix $\lambda > N$ arbitrarily and define

$$\tilde{w}_\lambda(\rho) = \begin{cases} w_\lambda & \text{if } 0 \leq \rho < z, \\ 0 & \text{if } \rho \geq z, \end{cases} \tag{4.4}$$

where $z \in (0, \infty)$ is the first zero of w_λ. Set

$$\underline{v}(\rho, s) := kw_l(\rho) + ce^{-\beta s}\tilde{w}_\lambda(\rho), \qquad \rho \geq 0,\ s \geq 0.$$

Then we have for $\rho \neq z$ that

$$\begin{aligned}
&\underline{v}_s - \underline{v}_{\rho\rho} - \tfrac{N-1}{\rho}\underline{v}_\rho - \tfrac{\rho}{2}\underline{v}_\rho - \tfrac{l}{2}\underline{v} - e^{-\frac{l-m}{m}s}\underline{v}^p \\
&\quad = -\beta c e^{-\beta s}\tilde{w}_\lambda - k\Big\{(w_l)_{\rho\rho} + \tfrac{N-1}{\rho}(w_l)_\rho + \tfrac{\rho}{2}(w_l)_\rho + \tfrac{l}{2}w_l\Big\} \\
&\qquad\quad -ce^{-\beta s}\Big\{(\tilde{w}_\lambda)_{\rho\rho} + \tfrac{N-1}{\rho}(\tilde{w}_\lambda)_\rho + \tfrac{\rho}{2}(\tilde{w}_\lambda)_\rho + \tfrac{l}{2}\tilde{w}_\lambda\Big\} - e^{-\frac{l-m}{m}s}\underline{v}^p \\
&\quad = (-\beta + \tfrac{\lambda - l}{2})ce^{-\beta s}\tilde{w}_\lambda - e^{-\frac{l-m}{m}s}\underline{v}^p.
\end{aligned}$$

Since $\underline{v} \geq kw_l > 0$ for all $\rho > z$, if we take

$$\beta = \frac{l-m}{m} = \frac{pl-2-l}{2},$$

and $c > 0$ sufficiently small, then $\underline{v}$ is a subsolution of (2.4). Then, by the comparison principle, we obtain

$$v(0,s) \geq \underline{v}(0,s) = kw_l(0) + ce^{-\beta s}$$

for all $t \geq 0$. By (2.3) and $w_l(0) = 1$, this implies (4.3). □

Lemma 4.3. *Let $p > (N+2)/N$, and let u be a radially symmetric global solution of* (2.1). *If*

$$u_0(r) > kw_l(r), \qquad r \geq 0,$$

for some $l \in (m, N)$ and $k > 0$, then, for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$u(0,t) \geq (t+1)^{-\frac{l}{2}} k + c_\varepsilon(t+1)^{-\frac{N}{2}-\varepsilon}, \qquad t \geq 0. \tag{4.5}$$

Proof. For any $\varepsilon > 0$, we take $\lambda = N + 2\varepsilon$ and define

$$\underline{v}(\rho, s) := kw_l(\rho) + c_\varepsilon e^{-\beta s}\tilde{w}_\lambda(\rho), \qquad \rho \geq 0,\ s \geq 0,$$

where $\tilde{w}_\lambda$ is defined by (4.4), and c_ε is a sufficiently small number such that

$$u_0(r) > \underline{v}(r, 0) = kw_l(r) + c_\varepsilon \tilde{w}_\lambda(r), \qquad r \geq 0.$$

By direct substitution, we have for $\rho \neq z$ that

$$\begin{aligned}
&\underline{v}_s - \underline{v}_{\rho\rho} - \tfrac{N-1}{\rho}\underline{v}_\rho - \tfrac{\rho}{2}\underline{v}_\rho - \tfrac{l}{2}\underline{v} - e^{-\frac{l-m}{m}s}\underline{v}^p \\
&\quad < -\beta c_\varepsilon e^{-\beta s}\tilde{w}_\lambda - k\Big\{(w_l)_{\rho\rho} + \tfrac{N-1}{\rho}(w_l)_\rho + \tfrac{\rho}{2}(w_l)_\rho + \tfrac{l}{2}w_l\Big\} \\
&\qquad - c_\varepsilon e^{-\beta s}\Big\{(\tilde{w}_\lambda)_{\rho\rho} + \tfrac{N-1}{\rho}(\tilde{w}_\lambda)_\rho + \tfrac{\rho}{2}(\tilde{w}_\lambda)_\rho + \tfrac{l}{2}\tilde{w}_\lambda\Big\} \\
&\quad = (-\beta + \tfrac{\lambda - l}{2})c_\varepsilon e^{-\beta s}\tilde{w}_\lambda.
\end{aligned}$$

Hence, if we take

$$\beta = \frac{\lambda - l}{2} = \frac{N - l}{2} + \varepsilon,$$

then $\underline{v}$ is a subsolution. Then, by the comparison principle, we obtain

$$v(0, s) \geq \underline{v}(0, s) = kw_l(0) + c_\varepsilon e^{-(\frac{N-l}{2} + \varepsilon)s}$$

for all $t \geq 0$. By (2.3) and the fact that $w_l(0) = 1$, this implies (4.5). □

Now, let us complete the proof of Theorem 1.3.

Proof of Theorem 1.3. First we assume that u_0 satisfies

$$u_0(x) \geq kw_l(|x|) + \kappa(|x| + 1)^{-\lambda}, \qquad x \in \mathbb{R}^N,$$

for some $\lambda \in (l, N)$ and $k, \kappa > 0$. We take a radially symmetric function $u_0^-(r)$ satisfying (4.1) and

$$u_0^-(|x|) \leq u_0(x), \qquad x \in \mathbb{R}^N.$$

Then the assertion (i) immediately follows from Lemma 4.1 and the comparison principle.

The proofs of (ii) and (iii) are obtained in the same manner by using Lemmas 4.2 and 4.3, respectively. □

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