Advances in Differential Equations Volume 13, Numbers 11-12 (2008), 1109–1129

MORSE THEORY AND A SCALAR FIELD EQUATION ON COMPACT SURFACES

Andrea Malchiodi
SISSA, via Beirut 2-4, 34014 Trieste, Italy

(Submitted by: Jean-Michel Coron)

Abstract. The aim of this paper is to study a nonlinear scalar field equation on a surface Σ via a Morse-theoretical approach, based on some of the methods in [25]. Employing these ingredients, we derive an alternative and direct proof (plus a clear interpretation) of a degree formula obtained in [18], which used refined blow-up estimates from [34] and [17]. Related results are derived for the prescribed Q-curvature equation on four manifolds.

1. Introduction

Given a real parameter ρ, we consider the equation

$$-\Delta_g u + \rho = \rho \frac{h(x)e^u}{\int_\Sigma h(x)e^u dV_g} \qquad \text{on } \Sigma, \tag{1.1}$$

where Σ is a two-dimensional (orientable, closed) compact surface with metric g and $h : \Sigma \to \mathbb{R}$ a smooth positive function. The above equation arises in mathematical physics as a mean field equation of Euler flows or for the description of self-dual condensates of some Chern-Simons-Higgs model, see for example [9, 20, 31, 47] and the book [54]. (1.1) also appears in conformal geometry, since it is the PDE version of the *Kazdan-Warner problem* (known as the *Nirenberg problem* when Σ is the standard sphere), consisting in finding a conformal metric for which the Gauss curvature is a prescribed function on Σ, see the (far from complete) list of references [3, 12, 14, 19, 33, 50]. The literature on (1.1) is broad, and we will be able to review only some of the existing results.

Accepted for publication: July 2008.
AMS Subject Classifications: 35B33, 53A30, 53C21, 58E05.

Problem (1.1) has a variational structure and solutions can be found as critical points of the functional

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + \rho \int_\Sigma u\, dV_g - \rho \log \int_\Sigma h(x) e^u dV_g, \quad u \in H^1(\Sigma). \tag{1.2}$$

Since both (1.1) and I_ρ stay invariant if we add a constant to u, we can also restrict ourselves to the subspace $\overline{H}^1(\Sigma) \subseteq H^1(\Sigma)$ of the functions with zero average, so we will sometimes neglect the second term in I_ρ. By the Moser-Trudinger inequality, see (2.1), I_ρ is coercive if $\rho < 8\pi$, and one can find solutions of (1.1) by minimization. If $\rho = 8\pi$ the situation is more delicate, since I_ρ still has a lower bound but it is not coercive anymore, see [10, 22, 46] for these and related questions.

If $\rho > 8\pi$ the functional I_ρ is unbounded from below, and hence solutions to (1.1) have to be found by other methods. In [23, 53] the authors tackled the problem using min-max arguments, and obtained existence results for $\rho \in (8\pi, 16\pi)$ (or some subset of this interval), see also [41]. A general feature of equation (1.1) is a compactness property when ρ is not an integr multiple of 8π: this is given by the following theorem, proven in [35], which sharpens some quantization results in [8].

Theorem 1.1. ([35]) *Suppose $h : \Sigma \to \mathbb{R}$ is smooth positive, and that ρ belongs to a compact subset of $\mathbb{R} \setminus \cup_{k \in \mathbb{N}} 8\pi k$. Then the solutions of* (1.1) (*up to adding constants*) *are bounded in $C^{2,\alpha}(\Sigma)$ for any $\alpha \in (0,1)$.*

If $\rho < 8\pi$, or if $\rho \in (8k\pi, 8(k+1)\pi)$ for some $k \in \mathbb{N}$, by elliptic regularity this theorem allows us to define the global Leray-Schauder degree of equation (1.1), fixing a large ball in $\overline{H}^1(\Sigma)$ (or in $C^{2,\alpha}(\Sigma)$) which includes all the solutions. In [34] it was shown that the degree is 1 when $\rho < 8\pi$ (consistently with the coercivity condition), and noticed that for $\rho \in (8k\pi, 8(k+1)\pi)$ it depends only on k and the topology of Σ. It was also suggested to compute it in general by studying the loss of compactness of (1.1) when ρ approaches (positive) multiples of 8π and analyzing the jump values of the degree after crossing the critical thresholds: then some blow-up estimates in this spirit were derived, sharpening the result in [35]. This approach was indeed used for studying the prescribed scalar curvature problem on spheres, see [33], Part II.

Pushing this analysis further in [17], this program was completed in [18] using a finite-dimensional reduction to compute the jump values. The authors obtained the following general result, extending the degree-counting formula in [36], where the case $\Sigma = S^2$ and $k = 1$ was studied.

Theorem 1.2. ([18]) *Let $8k\pi < \rho < 8(k+1)\pi$, $k \in \mathbb{N}$, and let $d(\rho)$ be the Leray-Schauder degree of* (1.1). *Then, letting $\chi(\Sigma)$ denote the Euler characteristic of Σ, one has*

$$d(\rho) = (-\chi(\Sigma)+1)\cdots(-\chi(\Sigma)+k)/k!. \tag{1.3}$$

When the genus of Σ is positive, one finds the existence of solutions to (1.1) for all k: using blow-up analysis, it was also possible to deduce it for some integr values of $\rho/8\pi$.

In four-dimensional geometry there is a higher-order analogue of (1.1), involving the *Paneitz operator* (introduced in [48]) and the *Q-curvature*, see [4, 5, 6]. Letting P_g and Q_g denote those corresponding to a given manifold (M,g), whose expressions are

$$P_g(\varphi) = \Delta_g^2\varphi + div_g\left(\tfrac{2}{3}R_g g - 2Ric_g\right)d\varphi;$$
$$Q_g = -\tfrac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|Ric_g|^2\right),$$

and considering the conformal change of metric $\tilde{g} = e^{2w}g$, $Q_{\tilde{g}}$ is given by

$$P_g w + 2Q_g = 2Q_{\tilde{g}}e^{4w}. \tag{1.4}$$

Apart from the analogy with the prescribed Gauss curvature equation, there is an extension of the Gauss-Bonnet formula involving the Weyl tensor and the integral q_P of Q_g, which is a conformal invariant. We refer to the survey [16] or to [13], [29] for details and applications.

As for the uniformization theorem, one can ask whether every four-manifold (M,g) carries a conformal metric $\tilde{g}$ for which the corresponding Q-curvature $Q_{\tilde{g}}$ is a constant. Writing $\tilde{g} = e^{2w}g$, the question amounts to solving (1.4) in w with $Q_{\tilde{g}}$ constant. A positive answer was given in [7, 15] under the assumptions $P_g \geq 0$ and $q_P < 8\pi^2$, which are naively the counterpart of $\rho < 8\pi$ for (1.1) (also in this case there is a variant of the Moser-Trudinger inequality which makes the problem coercive).

In [25] an extension of this result was obtained for a large class of manifolds, assuming $q_P \neq 8k\pi^2$, $k \in \mathbb{N}$, and that P_g has no kernel. The proof relies on a direct min-max method (extending the one in [23] and based on some improvements of the Moser-Trudinger inequality) and on some compactness results in [26, 42], which are the counterpart of Theorem 1.1. See also [32] for the case $q_P = 8\pi^2$.

The latter min-max method was shown to be useful (see [24] and [43]) in deriving existence results for (1.1) as well (and actually for some higher-dimensional analogues of the Q-curvature equation, see [28] and [45]). The aim of this paper is to show its connection with Theorem 1.2 by giving a simple and direct proof of it, based only on results which can be found in Section 3 of [25] and Subsection 4.1 of [43].

A crucial observation, as noticed in [19], is that the constant in the Moser-Trudinger inequality (2.1) can be roughly divided by the number of regions where e^u is supported (for a precise statement see Lemma 2.1). As a consequence, if $\rho \in (8k\pi, 8(k+1)\pi)$ and if $I_\rho(u)$ attains large negative values, e^u has to *concentrate* near at most k points of Σ, in the sense specified by Lemma 2.2. From these considerations one is led naturally to associate to e^u a probability measure $\sum_{i=1}^k t_i \delta_{x_i}$ with $(x_i)_i \subseteq \Sigma$, and $\sum_{i=1}^k t_i = 1$. The set of such objects, denoted here by Σ_k, is known in literature as the *formal set of barycenters of* Σ *of order* k, and it has been used in [1, 2] to study the loss of compactness of the Yamabe equation. It is indeed possible to prove, see Proposition 2.3, that $\{I_\rho \leq -L\}$, has the same homology as Σ_k for L large (positive); here Σ_k is naturally endowed with the weak topology of distributions.

At this point, choosing L possibly larger so that all the solutions of (1.1) lie in $\{I_\rho > -L\}$ (this can be done by Theorem 1.1) it is tempting to use Morse theory and compute the degree $d(\rho)$ as

$$\begin{aligned} d(\rho) &= \chi\left(\overline{H}^1(\Sigma), \{I_\rho \leq -L\}\right) = \chi\left(\overline{H}^1(\Sigma)\right) - \chi(\{I_\rho \leq -L\}) \\ &= 1 - \chi(\Sigma_k). \end{aligned} \tag{1.5}$$

The first equality is suggested by the Poincaré-Hopf theorem, see Proposition 2.6, while the second follows from the exactness of the homology sequence of a pair. This argument is clearly purely intuitive and heuristic but actually it can be made rigorous, giving an interpretation of (1.3). A precise version of the latter formula, see Proposition 2.6, requires usually that the whole Hilbert space be replaced by a bounded sublevel of the functional (here we have none) and also the Palais-Smale condition to hold, which is not known for I_ρ. Anyhow, following the classical arguments of Morse theory, we can tackle these problems using a generalized notion of degree and a deformation lemma from [39].

Another application of the above method regards the Q-curvature equation (1.4). As already mentioned, when $q_P \neq 8k\pi^2$ and $k \in \mathbb{N}$ (and if P_g has no nontrivial kernel), from a theorem in [42] the equation is compact (see also [26]), so it still makes sense to define the Leray-Schauder degree. We have the following theorem.

Theorem 1.3. *Let* (M, g) *be a compact four-manifold such that the Paneitz operator* P_g *has* $\overline{k}$ *negative eigenvalues and only trivial kernel* (*the constant* *functions), and such that*

$$q_P := \int_M Q_g dV_g \neq 8k\pi^2 \quad \text{for } k \in \mathbb{N}.$$

Suppose $\overline{Q}$ *is a constant (or a smooth function) with the same sign as* q_P*. Then the degree of* (1.4) *(with* $\overline{Q}$ *replacing* $Q_{\tilde{g}}$*) is given by*

$$\begin{cases} (-1)^{\overline{k}}, & \text{if } q_P < 8\pi^2; \\ (-1)^{\overline{k}}(-\chi(\Sigma)+1)\cdots(-\chi(\Sigma)+k)/k!, & \text{if } q_P \in (8k\pi^2, 8(k+1)\pi^2), k \in \mathbb{N}. \end{cases} \tag{1.6}$$

A related result can be found in [37, 38] (see also [49]): here the authors use blow-up analysis techniques, as in [17, 18], to study the biharmonic equation in domains of $\mathbb{R}^4$.

Remark 1.4. Morse theory gives in general more information about the structure of critical points compared to degree theory, because one includes the other as a particular case. For example, in [24] some existence results are obtained when Σ has the topology of S^2 and $k > 1$, using the fact that Σ_k is not contractible. Notice that in this case $d(\rho)$ vanishes by (1.3). The same considerations hold for (1.4), and further applications are under current investigation. However, the above purely topological arguments cannot deal with the cases $\rho = 8k\pi$, which also need refined analytic estimates as in [17], [18] and [34].

In Section 2, we recall some useful facts concerning the improved Moser-Trudinger inequality, some tools in nonlinear functional analysis and a deformation lemma for I_ρ. In Section 3, we give our proof of Theorem 1.2 and that of Theorem 1.3: a key step is the computation of the Euler characteristic of Σ_k, which is done using Morse theory in the opposite spirit as before, to match the formulas (1.3) and (1.5).

2. Preliminaries

In this section we collect some useful facts. First of all we consider the Moser-Trudinger inequality and discuss some of its applications to the study of the topological structure of the sublevels of I_ρ. Next, we recall some abstract tools in nonlinear functional analysis concerning the Leray-Schauder degree, the Poincaré-Hopf theorem and some of their extensions. Finally we state a deformation lemma from [39], describing the main ingredients of the proof and some consequences.

2.1. **The Moser-Trudinger inequality and some applications.** It is well known, see for example Theorem 1.7 in [27], that the integral

$$\int_\Sigma e^{\frac{4\pi(u-\overline{u})^2}{\int_\Sigma |\nabla u|^2 dV_g}} dV_g$$

is bounded, uniformly for $u \in H^1(\Sigma)$, by a constant C_Σ depending only on (Σ, g), where $\overline{u}$ stands for the average of u on Σ. By elementary inequalities then one finds

$$\log \int_\Sigma e^{(u-\overline{u})} dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla_g u|^2 dV_g + C'_\Sigma, \quad \text{for every } u \in H^1(\Sigma) \qquad (2.1)$$

for another fixed constant C'_Σ depending only on Σ and g. Following an argument in [19], one can prove that when the function e^u is *spread* into different regions inequality (2.1) improves.

Lemma 2.1. *Let δ_0, γ_0 be positive real numbers, and for a fixed integer ℓ, let $\Omega_1, \dots, \Omega_{\ell+1}$ be subsets of Σ satisfying $dist(\Omega_i, \Omega_j) \geq \delta_0$ for $i \neq j$. Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\ell, \tilde{\varepsilon}, \delta_0, \gamma_0)$ such that*

$$\log \int_\Sigma e^{(u-\overline{u})} dV_g \leq C + \frac{1}{16(\ell+1)\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla_g u|^2 dV_g$$

for all the functions $u \in H^1(\Sigma)$ satisfying

$$\int_{\Omega_i} e^u dV_g \geq \gamma_0 \int_\Sigma e^u dV_g$$

for every $i \in \{1, \dots, \ell+1\}$.

For a proof of the lemma we refer the reader to Section 2 in [25] (see also Section 3 in [43]). Assume next that $\rho \in (8\pi k, 8\pi(k+1))$ for some $k \in \mathbb{N}$; as a consequence of Lemma 2.1 one has that if $\ell \geq k$, then the functional I_ρ stays uniformly bounded from below. Loosely speaking, if $I_\rho(u)$ attains large negative values, e^u has to concentrate near at most k points of Σ. Indeed, using Lemma 2.1 and a covering argument, one obtains the following result, whose proof can also be found in the aforementioned sections.

Lemma 2.2. *Assuming $\rho \in (8k\pi, 8(k+1)\pi)$ with $k \in \mathbb{N}$, the following property holds. For any $\varepsilon > 0$ and any $r > 0$ there exists a large positive $L = L(\varepsilon, r)$ such that for every $u \in H^1(\Sigma)$ with*

$$I_\rho(u) \leq -L \quad \textit{and} \quad \int_\Sigma e^u dV_g = 1,$$

there exist k points $p_{1,u}, \dots, p_{k,u} \in \Sigma$ such that

$$\int_{\Sigma \setminus \cup_{i=1}^k B_r(p_{i,u})} e^u dV_g < \varepsilon.$$

Lemma 2.2 means qualitatively that the probability measure $e^u dV_g$ is *concentrated* near at most k points of Σ, and hence

$$e^u \simeq \sum_{i=1}^k t_i \delta_{x_i} = \sigma,$$

where $t_i \geq 0$, $x_i \in \Sigma$ for every $i \in \{1, \dots, k\}$, $\sum_{i=1}^k t_i = 1$ and where δ_{x_i} stands for the Dirac mass at x_i. Therefore, using the terminology of the introduction, e^u is *close* to some formal barycenter $\sigma \in \Sigma_k$. It was indeed shown in [25] (see also Section 4 in [43] for the specific case of I_ρ) that there exists a continuous and nontrivial map Ψ from low sublevels of the Euler functional into Σ_k. The fact that Ψ is nontrivial follows from the noncontractibility of Σ_k (see Lemma 3.7 in [25] and also Lemma 4.1 in [43]) and the fact that there exists also a family of maps Φ_λ from Σ_k into $H^1(\Sigma)$

$$\Phi_\lambda = \{\varphi_{\lambda,\sigma}(\cdot)\}_{\sigma \in \Sigma_k}; \qquad \varphi_{\lambda,\sigma}(y) = \log \sum_{i=1}^k t_i \Big(\frac{\lambda}{1 + \lambda^2 d_i^2(y)} \Big)^2 - \log \pi, \quad (2.2)$$

for which $I_\rho(\Phi_\lambda(\sigma)) \to -\infty$ and $\Psi(\Phi_\lambda(\sigma)) \to \sigma$ uniformly for $\sigma \in \Sigma_k$, see Proposition 4.4 in [43].

Conversely, given $L > 0$ large, one can construct a homotopy between the identity on $\{I_\rho \leq -L\}$ and $\Phi_\lambda \circ \Psi$ for some λ sufficiently large; the proof, which is rather technical in nature, is postponed to an appendix. The latter facts and the invariance of homology groups under homotopy equivalences imply the following result.

Proposition 2.3. *If $k \in \mathbb{N}$ and $\rho \in (8k\pi, 8(k+1)\pi)$, there exists $L > 0$ such that $\{I_\rho \leq -L\}$ has the same homology as Σ_k.*

2.2. **Some tools in nonlinear functional analysis and Morse theory.** We recall first (a particular case of) the Sard-Smale theorem, see for example [21], page 91, for the statement and the references for the proof.

Theorem 2.4. (Sard-Smale). *Let Ω be an open subset of a Hilbert space X. Suppose $\mathcal{G} \in C^1(\Omega; X)$ is proper when restricted to any bounded subset of Ω. Suppose $\nabla \mathcal{G}(x)$ is of the form Identity – compact for every $x \in \Omega$. Then the set of regular values of $\mathcal{G}$ is dense in X.*

We will apply this result to $X = \overline{H}^1(\Sigma)$ and $\mathcal{G} = \nabla I_\rho$; since both ∇I_ρ and its Frechet derivative (identified with an X-valued map via the Riesz theorem) are of the form $Identity - compact$, the assumptions of Theorem 2.4 hold true in this case.

We recall next an extension of the classical Leray-Schauder degree: given a bounded subset B of a Hilbert space X, the *Kuratowski measure of non-compactness of* B, $\alpha(B)$, is defined as the infimum of the positive d such that B admits a finite cover by subsets of X of diameter less than or equal to d. For example, $\overline{B}$ is compact if and only if $\alpha(B) = 0$.

Let Ω be an open subset of X and $\mathcal{F} : \Omega \to X$ a continuous map; then $\mathcal{F}$ is said to be a strict α-contraction if $\alpha(\mathcal{F}(B)) < k\,\alpha(B)$ for some fixed $k < 1$. Examples of such maps are k-Lipschitz maps, compact maps (for which $k = 0$) and sums of a compact and a k-Lipschitz map.

Suppose $\mathcal{F} : \Omega \to X$ is a strict α-contraction, that $y \in X \setminus (Id - \mathcal{F})(\partial\Omega)$ and that $(Id - \mathcal{F})^{-1}(\{y\})$ is compact. Then, see [21], Sections 9 and 10.1, we can define a *generalized degree*

$$(Id - \mathcal{F}, \Omega, y) \quad \longmapsto \quad D(Id - \mathcal{F}, \Omega, y),$$

which coincides with the classical Leray-Schauder degree $D_{LS}(Id - \mathcal{F}, \Omega, y)$ if $\mathcal{F}$ is compact and if Ω is bounded, and enjoys all its classical properties.

Suppose now that N is a compact n-dimensional manifold with smooth boundary, and that $f : \overline{N} \to \mathbb{R}$ is a function satisfying the *general boundary conditions*, namely $f|_N$ and $f|_{\partial N}$ are Morse functions and f has no critical points on ∂N. Then, denoting by $\tilde{C}_i$ the number of critical points of f in N with index i plus the number of critical points $f|_{\partial N}$ with index i where ∇f points inward, one has the Morse-Van Schaak relation

$$\sum_{i=0,\dots,n} (-1)^i \tilde{C}_i = \chi(N).$$

By the density of nondegenerate functions in $C^2(\overline{N})$, see [30], page 147, one finds the following result.

Proposition 2.5. *Suppose N is a compact n-dimensional manifold N with smooth boundary $\partial N \neq \emptyset$, and that $f : \overline{N} \to \mathbb{R}$ satisfies the general boundary conditions. Assume that, in each component of ∂N, ∇f points either inward or outward. Then, if C_i is the number of critical points of f in N with index i and if N_- is the subset of ∂N where ∇f points inward, we have*

$$\sum_{i=0,\dots,n} (-1)^i C_i = \chi(N, N_-). \tag{2.3}$$

We recall next a variant of the classical Poincaré-Hopf index theorem, deeply related to the above proposition, which can be found in [11], pages 99-104 (here we adapt the statement to our purposes).

Proposition 2.6. *Let X be a Hilbert space X, and let $f : X \to \mathbb{R}$ be of class C^2. Suppose that $\nabla f(x)$ is of the form $Identity - compact$ for every $x \in H$, and that f satisfies the Palais-Smale condition. Assume also that, for some $a, b \in \mathbb{R}$, $a < b$, $\Omega := \{f \leq b\} \setminus \{f < a\}$ is bounded, and that f has no critical points at the levels a, b. Then, letting $\Omega_- = \{f = a\}$, one has*

$$D_{LS}(\nabla f, \Omega, 0) = \chi(\Omega, \Omega_-). \tag{2.4}$$

We will actually need a slightly more general version of this result, since we have to deal with unbounded sets and we do not know whether the Palais-Smale condition holds true. However, our argument will still follow the same (classical) proof in [11], see Section 3 below.

2.3. **A deformation lemma.** Here we recall a result in [39], where a vector field which deforms suitable sublevels of the functional I_ρ is constructed, by-passing the Palais-Smale condition (which is known to hold only for bounded sequences). In [52] a related argument was previously used, which exploited a monotonicity property with respect to a parameter in the problem. Below, we set

$$X = \overline{H}^1(\Sigma), \qquad J(u) = \log \int_\Sigma h(x) e^u dV_g, \quad u \in X, \tag{2.5}$$

so we have $I_\rho(u) = \frac{1}{2}\|u\|^2 - \rho J(u)$. The result in [39] we need is the following.

Lemma 2.7. *Given $a, b \in \mathbb{R}$, $a < b$, the following alternative holds: either there exists $(\rho_l, u_l) \subseteq \mathbb{R} \times X$ satisfying*

$$I'_{\rho_l}(u_l) = 0 \text{ for every } l; \qquad a \leq I_\rho(u_l) \leq b; \qquad \rho_l \to \rho,$$

or the set $\{I_\rho \leq a\}$ is a deformation retract of $\{I_\rho \leq b\}$.

By *deformation retract onto $A \subseteq X$* we mean a continuous map $\eta : [0,1] \times X \to X$ such that $\eta(t, u_0) = u_0$ for every $(t, u_0) \in [0,1] \times A$ and such that $\eta(1, \cdot)$ is contained in A.

To prove the lemma, one argues as follows: assuming the first alternative false, let $\tau > 0$ be such that $I_{\tilde\rho}$ has no critical point $\overline{u}$ for $\tilde\rho \in (\rho - \tau, \rho)$ with $I_{\tilde\rho}(\overline{u}) \in [a, b]$. Then set

$$Z_\rho(u) = -\left[|\nabla J(u)| \nabla I_\rho(u) + |\nabla I_\rho(u)| \, \nabla J(u) \right],$$

and for a smooth nondecreasing cutoff function $\omega_\tau : \mathbb{R} \to [0,1]$ satisfying $\omega_\tau(t) = 0$ for $t \leq \tau$ and $\omega_\tau(t) = 1$ for $t \geq 2\tau$, define the vector field[1]

$$W(u) = -\omega_\tau\Big(\frac{|\nabla I_\rho(u)|}{|\nabla J(u)|}\Big)\nabla I_\rho(u) + Z_\rho(u). \tag{2.6}$$

The strategy of the proof consists in showing that, as long as I_ρ stays between a and b, every integral curve of W remains bounded (with bounds depending on the initial datum, a, b and τ) and that along the flow the derivative of I_ρ is bounded above by a negative constant. A key point is to notice that $\langle Z_\rho(u), \nabla I_\rho(u)\rangle \leq 0$, and if $\langle Z_\rho(u_l), \nabla I_\rho(u_l)\rangle$ tends to zero along some sequence $(u_l)_l$, then

$$\lim_{l\to+\infty} \frac{Z_\rho(u_l)}{1 + |\nabla J(u_l)|} = 0.$$

As a consequence of Lemma 2.7 we get a useful corollary.

Corollary 2.8. *If $\rho \in (8k\pi, 8(k+1)\pi)$ for some $k \in \mathbb{N}$ and if b is sufficiently large and positive, the sublevel $\{I_\rho \leq b\}$ is a deformation retract of X, and hence it has the homology of a point.*

Proof. By Theorem 1.1 and by the local boundedness of J in X (which follows from (2.1)), choosing $\tilde{\rho}$ sufficiently close to ρ we see that the first alternative in Lemma 2.7 cannot hold if $a \geq n_0$ and $b = a+1$ for a sufficiently large integer n_0. Therefore, if $n \geq n_0$ the sublevel $\{I_\rho \leq n\}$ is a deformation retract of $\{I_\rho \leq n+1\}$. As we remarked, $\langle \nabla I_\rho(u), W(u)\rangle$ is negative and stays away from zero in $\{n \leq I_\rho \leq n+1\}$, so the vector field

$$\hat{W}(u) := -\frac{W(u)}{\langle \nabla I_\rho(u), W(u)\rangle}$$

is well defined and smooth in this set, and we have, clearly,

$$\langle \nabla I_\rho(u), \hat{W}(u)\rangle = -1.$$

We can choose the retraction η_n from $\{I_\rho \leq n+1\}$ onto $\{I_\rho \leq n\}$ to be simply the flow generated by $\hat{W}$, and arrested as soon as $I_\rho = n$.

Given any $x \in X$ such that $I_\rho(x) > n_0$, let n be the unique integer satisfying $n \leq I_\rho(x) < n+1$. Then, for $s \in [0, n-n_0]$, we consider the following map:

$$\hat{\eta}(s,x) = \eta_n(s,x), \quad s \in [0,1]; \qquad \hat{\eta}(s,x) = \eta_{n-1}(s, \eta_n(1,x)), \quad s \in [1,2];$$

$$\hat{\eta}(s,x) = \eta_{n-i}(s, \eta_{n-i-1}(1, \eta_{n-1-2}(1, \cdots))), \; s \in [i, i+1]; \; \; i = 0, 1, \dots, n-n_0.$$

[1] There are some typos in the definition of Z and W in [39], fixed in [40]: we report here the correct formulas

Letting s_x be the first s for which $I_\rho(\hat\eta(s,x)) = n_0$, for $t \in [0,1]$ we define $\eta(t,x) = \hat\eta(s_x t, x)$. This map is clearly continuous in both t and x, and realizes the required deformation retract. □

3. Proof of Theorems 1.2 and 1.3

First of all we compute the Euler characteristic of Σ_k; our proof is related to that of Lemma 5.2 in [18].

Proposition 3.1. *For any natural number k we have*

$$\chi(\Sigma_k) = 1 - (1-\chi(\Sigma)) \cdots (k - \chi(\Sigma))/k!. \tag{3.1}$$

Proof. We proceed by induction in k. Formula (3.1) is clearly true for $k=1$. If $k>1$ we consider the pair (Σ_k, Σ_{k-1}) and notice that

$$\chi(\Sigma_k) = \chi(\Sigma_k, \Sigma_{k-1}) + \chi(\Sigma_{k-1}).$$

We claim that the following formula holds for any natural number k :

$$\chi(\Sigma_k, \Sigma_{k-1}) = (-1)^{k-1} \frac{\chi(\Sigma)(\chi(\Sigma)-1)(\chi(\Sigma)-2)\cdots(\chi(\Sigma)-(k-1))}{k!}. \tag{3.2}$$

Once we have this formula, using the relation

$$\sum_{l=m}^{n} \binom{l}{m} = \binom{n+1}{m+1}$$

(which can be derived from Pascal's rule for binomial coefficients), the proposition follows immediately.

We compute $\chi(\Sigma_k, \Sigma_{k-1})$ using Morse theory: the set $\Sigma_k \setminus \Sigma_{k-1}$ is an open manifold (with boundary Σ_{k-1}) of dimension $3k-1$ consisting of the $2k$-tuples $(t_i, x_i)_{i=1,\dots,k}$ satisfying the constraints $t_i \in (0,1)$ for all i, $\sum_{i=1}^k t_i = 1$ and $x_i \neq x_j$ for $i \neq j$. A point in $\Sigma_k \setminus \Sigma_{k-1}$ approaches the boundary of this manifold if either $\min_{i\neq j} dist(x_i, x_j)$ tends to zero, or if $(t_1, \dots, t_k)$ approaches the boundary of the $(k-1)$-simplex $\sigma_{k-1} := \{\sum_{i=1}^k t_i = 1\}$.

Given a positive δ smaller than the injectivity radius of Σ, we consider a nonincreasing function $G : (0,+\infty) \to (0,+\infty)$ satisfying

$$G(t) = \frac{1}{t}, \text{ for } t \in (0,\delta]; \qquad G(t) = \frac{1}{2\delta} \text{ for } t > 2\delta. \tag{3.3}$$

For $\varepsilon > 0$ small (to be fixed later) we define $F : \Sigma_k \setminus \Sigma_{k-1} \to \mathbb{R}$ as

$$F\Big(\sum_{i=1}^k t_i \delta_{x_i}\Big) = \underbrace{-\sum_{i\neq j} G(d(x_i,x_j))}_{\mathfrak{F}((x_i)_i)} - \underbrace{\sum_{i=1}^k \frac{1}{t_i(1-t_i)}}_{\mathfrak{G}((t_i)_i)}, \tag{3.4}$$

$$d(x_i, x_j) = dist(x_i, x_j).$$

Clearly, F is well defined on $\Sigma_k \setminus \Sigma_{k-1}$ (is invariant under permutation of the couples $(t_i, x_i)_{i=1,\dots,k}$) and tends to $-\infty$ as the argument approaches Σ_{k-1}. Also, the gradient of this function with respect to the metric of $\Sigma^k \times \sigma_{k-1}$ (where the permutation group is acting) tends to $+\infty$ in norm as $\sum_{i=1}^k t_i \delta_{x_i}$ tends to Σ_{k-1}. By this reason F satisfies the Palais-Smale condition on $\Sigma_k \setminus \Sigma_{k-1}$, and Σ_{k-1} is a deformation retract of $F_{-L} := \{F \leq -L\} \cup \Sigma_{k-1}$ for L sufficiently large and positive (the retract can be obtained simply letting $L \to +\infty$). By Proposition 2.5 (and by excision of $\{F < -L\}$), if $\hat{F} : \{F \geq -L\} \to \mathbb{R}$ is a nondegenerate function sufficiently close to F in C^2 norm, then we have

$$\chi(\Sigma_k, \Sigma_{k-1}) = \sum_{i=0,\dots,3k-1} (-1)^i \hat{C}_i, \tag{3.5}$$

where $\hat{C}_i$ is the number of critical points of $\hat{F}$ with index i. To compute the latter sum we define the set

$$\Lambda_k = \left\{ (x_1, \dots, x_k) \in \Sigma^k \ : \ x_i \neq x_j \text{ for } i \neq j \right\}. \tag{3.6}$$

We notice that, reasoning as for F, both $\mathfrak{F} : \Lambda_k \to \mathbb{R}$ and $\mathfrak{G} : \sigma_{k-1} \to \mathbb{R}$ satisfy the Palais-Smale condition and tend to $-\infty$ at the boundaries of their domains. Hence, since $\mathfrak{G}$ has a single critical point (which is a nondegenerate maximum, of index $k-1$), if $\hat{\mathfrak{F}}$ is a nondegenerate function close in the C^2 norm to $\mathfrak{F}$ (and invariant under permutation), (3.5) implies

$$\chi(\Sigma_k, \Sigma_{k-1}) = \frac{(-1)^{k-1}}{k!} \sum_{i=0,\dots,2k} (-1)^i \hat{\mathfrak{C}}_i, \tag{3.7}$$

where $\hat{\mathfrak{C}}_i$ is the number of critical points of $\hat{\mathfrak{F}}$ with index i. The factorial of k in the denominator appears since we have invariance under the group of permutations.

Now notice that $\hat{\mathfrak{F}}$ tends to $-\infty$ at the boundary of Λ_k (since $\mathfrak{F}$ does), so reversing its sign we obtain a nondegenerate function with the same total index, and $-\nabla \hat{\mathfrak{F}}$ points outward $\{-\hat{\mathfrak{F}} < L\}$ for L large and positive. Since the latter set is a deformation retract of Λ_k, which can be realized by letting $L \to +\infty$, still by Proposition 2.5 we find that (see (3.6))

$$\sum_{i=0,\dots,2k} (-1)^i \hat{\mathfrak{C}}_i = \chi(\Lambda_k) = \frac{\chi(\Sigma)(\chi(\Sigma)-1)(\chi(\Sigma)-2)\cdots(\chi(\Sigma)-(k-1))}{k!}.$$

The latter Euler characteristic has been computed in [18], page 1705, and is deduced rather easily from the Hopf theorem for fibrations, see [51], page 481. The conclusion follows from (3.7) and the above formula. □

We are now in position to prove the two theorems.

Proof of Theorem 1.2. First of all we reduce ourselves to the nondegenerate case, following [11], page 103. We choose τ so small that $[\rho-\tau,\rho+\tau] \subseteq (8k\pi, 8(k+1)\pi)$, and $a<b$ so that all the critical points $\overline{u}$ of $I_{\tilde{\rho}}$ for $\tilde{\rho} \in [\rho-\tau,\rho+\tau]$ satisfy $I_\rho(\overline{u}) \in (a,b)$. This is possible by Theorem 1.1 and by the local boundedness of J, see (2.1), (2.5). Choosing also the number a sufficiently large and negative and b sufficiently large and positive, we can assume the conclusions of Proposition 2.3 (with $a=-L$) and Corollary 2.8 to be true.

We let K denote the set of critical points of I_ρ, which by Theorem 1.1 is compact. We also let $(I_\rho)_a^b := \{a < I_\rho < b\}$, fix a small $\delta>0$ so that $dist(K, X \setminus (I_\rho)_a^b) > 4\delta$, and define the set $K_\delta = \{x \in X \ : \ dist(x,K) < \delta\}$. We next choose a smooth cutoff function $p(x)$ such that

$$p(x) = 1 \text{ for every } x \in K_\delta; \qquad\qquad p(x) = 0 \text{ for every } x \in X \setminus K_{2\delta}.$$

We can also choose p so that its derivative is uniformly bounded in $K_{2\delta}$. Considering the restriction of ∇I_ρ to K_δ, we see that it is of the form $Id-T$ with T compact, and therefore by Theorem 2.4 there exists an arbitrarily small x_0 such that $\nabla^2 I_\rho$ is nondegenerate on $\{\nabla I_\rho = x_0\} \cap K_\delta$. We also notice that $\|\nabla I_\rho\| \geq \gamma_\delta > 0$ on $\overline{K_{2\delta}} \setminus K_\delta$ for some fixed constant γ_δ; as a consequence we have that for $\|x_0\|$ sufficiently small the function

$$\tilde{I}(x) = I_\rho(x) + p(x)(x, x_0)$$

is nondegenerate in $(I_\rho)_a^b$, coincides with I_ρ in $X \setminus K_{2\delta}$, and has the same critical points as I_ρ.

For $\|x_0\|$ small, $\nabla\tilde{I} - Id$ is a strict α-contraction, see Subsection 2.2, and since $(\nabla\tilde{I})^{-1}(\{0\}) \subseteq K_\delta$, $D(\nabla\tilde{I}, (I_\rho)_a^b, 0)$ is well defined. Suppose now that R is so large that all the critical points of $I_{\tilde{\rho}}$ for $\tilde{\rho} \in [\rho-\tau,\rho+\tau]$ are in $B_{\frac{R}{2}}(0) \subseteq X$. By the excision property and the homotopy invariance of D we have

$$D_{LS}(\nabla I_\rho, B_R, 0) = D(\nabla\tilde{I}, (I_\rho)_a^b, 0). \tag{3.8}$$

We notice that bounded Palais-Smale sequences of $\tilde{I}$ are precompact (since those of I_ρ are, see the comments at the beginning of Subsection 2.3), and in particular $\tilde{I}$ has only finitely many critical levels, each of which consists only of finitely many critical points. Choosing R possibly larger we can also assume that $K_{2\delta} \subseteq B_{\frac{R}{2}}$. Then we define a smooth radial cutoff function $\theta : X \to [0,1]$ satisfying

$$\theta(u) = 1 \text{ for } u \in B_R; \qquad \theta(u) = 0 \text{ for } u \in X \setminus B_{2R}.$$

Recalling the definition of W in (2.6), we then consider the following vector field:

$$\tilde{W}(u) = -\theta(u)\nabla\tilde{I}(u) + (1-\theta(u))W(u).$$

Since $\tilde{I}$ coincides with I_ρ outside B_R, $\tilde{W}$ decreases $\tilde{I}$ in the complement of K. Moreover, noticing that $I_{\tilde{\rho}}$, $\tilde{\rho} \in [\rho-\tau, \rho+\tau]$, has no critical points outside $B_R(0)$ and using the same argument as in [39], one can show that every integral curve of $\tilde{W}$ inside $(I_\rho)_a^b$ stays uniformly bounded (depending only on the initial datum, a, b and τ). By the compactness of bounded Palais-Smale sequences of $\tilde{I}$ and by the properties of the vector field Z_ρ (see the comments after (2.6)) one finds that if $\tilde{I}$ has no critical levels inside some interval $[\tilde{a}, \tilde{b}]$, then $\{\tilde{I} \leq \tilde{b}\}$ can be deformed into $\{\tilde{I} \leq \tilde{a}\}$.

Finally, let $\tilde{c}_i$ be the number of critical points of $\tilde{I}$ with index i. By classical Morse-theoretical arguments in the spirit of Proposition 2.6, see [11], page 103, excising $\{I_\rho < a\}$ one sees that

$$D(\nabla\tilde{I}, (I_\rho)_a^b, 0) = \sum_i (-1)^i \tilde{c}_i = \chi(\{I_\rho \leq b\}, \{I_\rho \leq a\}).$$

The conclusion follows from formula (3.8), Proposition 2.3, Corollary 2.8 and Proposition 3.1. □

Proof of Theorem 1.3. We can reason as for the previous case: the main difference is that the presence of negative eigenvalues for P_g affects the topology of the sublevels of the Euler functional. In [25], Sections 3 and 4, it was shown that the counterpart of Proposition 2.3 holds true replacing Σ_k with

$$S^{\overline{k}-1} \text{ if } q_P < 8\pi^2; \qquad A_{k,\overline{k}} = \widetilde{M_k \times B_1^{\overline{k}}} \text{ if } q_P \in (8k\pi^2, 8(k+1)\pi^2).$$

Here M_k is the set of k-barycenters of M, $B_1^{\overline{k}}$ the closed unit ball in $\mathbb{R}^{\overline{k}}$, $S^{\overline{k}-1}$ the unit sphere in $\mathbb{R}^{\overline{k}}$ while the equivalence relation $\sim$ means that $M_k \times \partial B_1^{\overline{k}}$ is identified with $\partial B_1^{\overline{k}}$, namely $M_k \times \{y\}$, for every fixed $y \in \partial B_1^{\overline{k}}$, is collapsed to a single point.

Therefore, following exactly the previous proof, one finds that the global degree of the equation is given by the numbers

$$1 - \chi(S^{\overline{k}-1}) \text{ if } q_P < 8\pi^2; \qquad 1 - \chi(A_{k,\overline{k}}) \text{ if } q_P \in (8k\pi^2, 8(k+1)\pi^2).$$

In the first alternative we are done. To compute the latter Euler characteristic one can use the Mayer-Vietoris theorem, see for example [44], page 207. We can cover $A_{k,\overline{k}}$ with the two sets

$$\mathcal{A} = M_k \times B^{\overline{k}}_{\frac{3}{4}}; \qquad \mathcal{B} = M_k \times \left(B^{\overline{k}}_1 \setminus B^{\overline{k}}_{\frac{1}{4}} \right),$$

where $B^{\overline{k}}_\mu$ stands for the closed ball of radius μ in $\mathbb{R}^{\overline{k}}$. Clearly $\mathcal{A}$ has the homotopy type of M_k, $\mathcal{B}$ that of $S^{\overline{k}-1}$ and $\mathcal{A}\cap\mathcal{B}$ that of $M_k \times S^{\overline{k}-1}$. Therefore, by the exactness of the Mayer-Vietoris sequence and the Künneth theorem we find the relation

$$\chi(A_{k,\overline{k}}) = \chi(M_k) + \chi(S^{\overline{k}-1}) - \chi(M_k)\chi(S^{\overline{k}-1}),$$

which implies

$$1 - \chi(A_{k,\overline{k}}) = (1 - \chi(M_k))(1 - \chi(S^{\overline{k}-1})).$$

By the same arguments as in the proof of Proposition 3.1 one finds that

$$1 - \chi(M_k) = (-\chi(M) + 1) \cdots (-\chi(M) + k)/k!,$$

so we have the conclusion from the last two formulas. □

Remark 3.2. Reasoning as for Theorem B.2 in [33] Part I, one can show that the Leray-Schauder degree in the Sobolev space $\overline{H}^1(\Sigma)$ for (1.1) coincides with the degree in every Hölder space $C^{2,\alpha}(\Sigma)$, $\alpha \in (0,1)$. Similar considerations hold for equation (1.4).

4. Appendix

This appendix is devoted to the proof of the last statement before Proposition 2.3. For $t \in [0,1]$, $L > 0$ so large that I_ρ has no critical points in $\{I_\rho \leq -L\}$ (see Theorem 1.1), $s \in [0,1]$, and $\tilde{\tau} > 0$ small (so small that $\frac{\rho}{1+s\tilde{\tau}} \neq 8k\pi$ for $s \in [0,1]$), we consider the families of sets $(\mathcal{U}_s)_s, (\mathcal{S}_s)_s \subseteq X$

$$\mathcal{U}_s = \left\{ u \in X : I_\rho(u) \leq -L - \frac{1}{2} s\tilde{\tau}\|u\|^2 \right\}; \tag{4.1}$$
$$\mathcal{S}_s = \left\{ u \in X : I_\rho(u) = -L - \frac{1}{2} s\tilde{\tau}\|u\|^2 \right\},$$

see (2.5). It is clear that $\mathcal{U}_s \subseteq \mathcal{U}_{s'}$ for $s \geq s'$ and that, by a simple manipulation, $\mathcal{S}_s = \{I_{\frac{\rho}{1+s\tilde{\tau}}} = -\frac{L}{1+s\tilde{\tau}}\}$. If $\tilde{\tau}$ is small and L is large, by Theorem 1.1 and the implicit function theorem $\mathcal{S}_s$ is a smooth manifold in X of codimension 1, so the $\mathcal{S}_s$'s foliate the closure of $\mathcal{U}_0 \setminus \mathcal{U}_1$ when s varies in $[0,1]$.

Lemma 4.1. *There exists a deformation retract* $H(\cdot, t)$, $t \in [0,1]$, *of* $\mathcal{U}_0$ *onto* $\mathcal{U}_1$.

Proof. For $u \in \overline{\mathcal{U}_0 \setminus \mathcal{U}_1}$, let $s(u)$ be the unique $s \in [0,1]$ such that $u \in \mathcal{S}_{s(u)}$; by the above comments $u \mapsto s(u)$ is continuous. Set also $\rho(u) = \frac{\rho}{1+s\tilde{\tau}}$, and recalling (2.6) define the vector field

$$\check{W}(u) = -\omega_\tau\Big(\frac{|\nabla I_{\rho(u)}|}{|\nabla J(u)|}\Big)\nabla I_{\rho(u)} + Z_{\rho(u)}(u).$$

An argument similar to [39] shows that every trajectory of $\check{W}$ stays bounded as long as $s(u) \in [0,1]$: to see this consider the Cauchy problem $\frac{d}{dt}u(t) = \check{W}(u(t))$, with $u(0) = u_0$, $s(u_0) \in [0,1]$. Then as in the formula between (11) and (12) of [39] one has that

$$\frac{d}{dt}|_{t=0}\left[J(u(t))\right] \leq -\frac{1}{\tau}\frac{d}{dt}|_{t=0}\left[I_{\rho(u_0)}(u(t))\right]. \tag{4.2}$$

Notice that in the right-hand side we are keeping the value of ρ fixed and equal to $\rho(u_0)$. On the other hand, recalling the definition of $\rho(u)$ we have that

$$\frac{d}{dt}|_{t=0}\left[I_{\rho(u(t))}(u(t))\right] = \frac{d}{dt}|_{t=0}\left[I_{\rho(u_0)}(u(t))\right] + \frac{\rho\tilde{\tau}\frac{d}{dt}|_{t=0}s(u(t))}{(1+s(u_0)\tilde{\tau})^2}J(u_0). \tag{4.3}$$

Since $\mathcal{S}_s = \{I_{\frac{\rho}{1+s\tilde{\tau}}} = -\frac{L}{1+s\tilde{\tau}}\}$ and since $\check{W}$ decreases $I_{\rho(u_0)}$, $\check{W}$ is transversal to $\mathcal{S}_s$ and hence $s(u(t))$ is nondecreasing in t. Hence by our choice of $s(u)$ we have that

$$\begin{aligned}\frac{d}{dt}|_{t=0}\left[I_{\rho(u(t))}(u(t))\right] &= -\frac{d}{dt}|_{t=0}\frac{L}{1+s(u(t))\tilde{\tau}} \\ &= \frac{L}{(1+s(u_0)\tilde{\tau})^2}\frac{d}{dt}|_{t=0}s(u(t)) \geq 0.\end{aligned}$$

From (4.2) and the last two formulas (which also hold for any choice of initial time) we get

$$\frac{\frac{d}{dt}J(u(t))}{J(u(t))} \leq \frac{\rho\tilde{\tau}}{\tau}\frac{d}{dt}s(u(t)) \quad \text{for any } t,$$

which implies

$$J(u(t)) \leq J(u(0))e^{\frac{\rho\tilde{\tau}}{\tau}(s(u(t))-s(u(0)))};$$

since we have $I_\rho(u(t)) \leq -L$, we get uniform bounds on each trajectory, depending on the initial value, as long as $s(u(t)) \in [0,1]$.

We claim next that, along each trajectory $u(t)$, $\frac{d}{dt}s(u(t))$ is bounded below by a positive constant. Assuming the contrary, since $\check{W}$ decreases $I_{\rho(u_0)}$, since $J(u(t))$ stays bounded, and since

$$\frac{d}{dt}\left[I_{\rho(u(t))}(u(t))\right] \geq 0,$$

from (4.3) we obtain a bounded subsequence u_n for which $\nabla I_{\rho(u_n)}[\check{W}(u_n)] \to 0$. The final arguments in [39] (after formula (14)) apply to this case as well, yielding a contradiction to Theorem 1.1.

In conclusion, each point in $\mathcal{S}_0$ reaches $\mathcal{S}_1$ in finite time; conversely, following the inverse flow, each point in $\mathcal{S}_1$ reaches $\mathcal{S}_0$ in finite time. In this way, by a suitable reparametrization in t, one can find a deformation retract of $\mathcal{U}_0$ onto $\mathcal{U}_1$. □

Lemma 4.2. *For $L, \lambda > 0$ sufficiently large, there exists a homotopy $\tilde{H}(u,t) : \mathcal{U}_1 \times [0,1] \to \mathcal{U}_0$ such that $\tilde{H}(u,0) = u$ and $\tilde{H}(u,1) = \Phi_\lambda(\Psi(u))$ for every $u \in \mathcal{U}_1$.*

Proof. We discuss the case $k = 1$ only (see some comments below for $k > 1$); we will also avoid normalizing functions in order to have zero average; this is not relevant since the functionals I_ρ are independent of this normalization.

Given $\varepsilon, \overline{r} > 0$, if L is sufficiently large, by Lemma 2.2 and by the construction of Ψ in [25] we have that if $u \in \mathcal{U}_1$ and if $\Psi(u) = \delta_{p(u)}$, then

$$\int_{\Sigma \setminus B_{\overline{r}}(p_u)} e^u dV_g \leq \varepsilon \int_\Sigma e^u dV_g.$$

We next use polar (geodesic normal) coordinates r, θ near p_u, and consider the following homotopy ($t \in [0,1]$)

$$v_t(r,\theta) = \begin{cases} \frac{r-\overline{r}}{\overline{r}}\left(tu_r(r) + (1-t)u(r,\theta)\right) & \text{for } r \in [\overline{r}, 2\overline{r}]; \\ tu_r(r) + (1-t)u(r,\theta) & \text{for } r \in [2\overline{r}, 4\overline{r}]; \\ \frac{5\overline{r}-r}{\overline{r}}\left(tu_r(r) + (1-t)u(r,\theta)\right) & \text{for } r \in [4\overline{r}, 5\overline{r}], \end{cases}$$

where u_r stands for the radial (Euclidean) average of u on $\partial B_r(0)$. It is easy to see that for $\overline{r}$ sufficiently small one has that

$$\|\nabla v_t\|^2_{L^2(M)} \leq (1 + o_{\overline{r},\varepsilon}(1))\|\nabla u\|^2_{L^2(M)} + C_{\overline{r},\varepsilon},$$

that

$$\overline{v_t} \leq \overline{u} + o_{\overline{r},\varepsilon}(1)\|\nabla u\|^2_{L^2(M)} + C_{\overline{r},\varepsilon},$$

and that

$$\log \int_M e^{v_t} dV_g \geq \log \int_M e^u dV_g + o_{\overline{r},\varepsilon}(1)\|\nabla u\|^2_{L^2(M)} - C_{\overline{r},\varepsilon},$$

which imply

$$I_\rho(v_t) \leq I_\rho(u) + o_{\overline{r},\varepsilon}(1) \int_\Sigma |\nabla u|^2 dV_g + C_{\overline{r},\varepsilon}$$

for every $t \in [0,1]$. Notice that the function v_1 is radial in $B_{3\overline{r}}(p_u) \setminus B_{2\overline{r}}(p_u)$ and in the Sobolev class $H^1_{rad}([2\overline{r}, 4\overline{r}])$, which allows us to get control of $\|v_1 - \overline{u}\|_{L^\infty([2\overline{r},4\overline{r}])}$, depending on $\overline{r}$, in terms of $\|\nabla u\|_{L^2}$.

We next consider another homotopy: for $\lambda \geq 1$ we define $\tilde{v}_\lambda$ as

$$\tilde{v}_\lambda(r,\theta) = \begin{cases} v_1(\lambda r, \theta) + 4\log\lambda, & \text{for } r \leq \frac{3\overline{r}}{\lambda}; \\ v_1(2\overline{r}) - 4\log r + 4\log(3\overline{r}) & \text{for } r \in \left(\frac{3\overline{r}}{\lambda}, 3\overline{r}\right); \\ v_1(r,\theta) & \text{for } r \geq 3\overline{r}. \end{cases}$$

Still using normal coordinates, one can see that

$$\|\nabla \tilde{v}_\lambda\|^2_{L^2(M)} = (1 + o_{\overline{r},\varepsilon}(1)) \left(\|\nabla v_1\|^2_{L^2(M)} + 16\pi \log\lambda \right),$$

that

$$\overline{\tilde{v}_\lambda} \leq \overline{v_1} + o_{\overline{r},\varepsilon}(1) \|\nabla v_1\|^2_{L^2(M)} + C_{\overline{r},\varepsilon},$$

and

$$\log \int_M e^{\tilde{v}_\lambda} dV_g \geq \log \int_M e^{v_1} dV_g + 2\lambda + o_{\overline{r},\varepsilon}(1) \|\nabla u\|^2_{L^2(M)} - C_{\overline{r},\varepsilon},$$

so we obtain that

$$I_\rho(\tilde{v}_\lambda) \leq I_\rho(u) + o_{\overline{r},\varepsilon}(1) \int_\Sigma |\nabla u|^2 dV_g - (2\rho - 16\pi + o_{\overline{r},\varepsilon}(1)) \log\lambda + C_{\overline{r},\varepsilon}.$$

Since we are assuming $u \in \mathcal{U}_1$, choosing $2\tilde{\tau} > o_{\overline{r},\varepsilon}(1)$ we have that

$$I_\rho(u) + o_{\overline{r},\varepsilon}(1) \|\nabla u\|^2_{L^2(M)} \leq -L,$$

and choosing also L so large that $C_{\overline{r},\varepsilon} < \frac{L}{2}$ we obtain that $I_\rho(\tilde{v}_\lambda) \leq -\frac{L}{2}$. For $\lambda = \lambda(u)$ sufficiently large (depending on u), it is clear that $\tilde{v}_\lambda$ can be deformed into $\varphi_{\lambda(u),\delta_{p_u}}$ (see (2.2), in which we are assuming $k = 1$) within $\{I_\rho \leq -\frac{L}{2}\}$; we can now apply the flow along the vector field W, see (2.6), until we reach the sublevel $\{I_\rho \leq -L\}$. One can finally deform $\varphi_{\lambda(u),\delta_{p_u}}$ into $\varphi_{\lambda,\delta_{p_u}}$ (for a fixed λ) within $\{I_\rho \leq -L\}$ (recall that $I_\rho(\lambda,\sigma) \to -\infty$ as $\lambda \to +\infty$), so the conclusion holds.

The case $k > 1$ can be treated using similar arguments; we refer in particular to Subsection 4.1 in [43] for energy estimates on the test functions $\varphi_{\lambda,\sigma}$ (see (2.2)) for $k \neq 1$. □

To prove our claim, we have to extend the homotopy in Lemma 4.2 to the sublevel $\mathcal{U}_0$ of I_ρ; for doing this one can simply apply first the deformation retract H in Lemma 4.1. The composition of $\tilde{H}(\cdot,1)\circ H(\cdot,1)$ deforms u into $\Phi_\lambda(\Psi(H(u,1)))$ though; however, to reach the conclusion it is sufficient to consider $\Phi_\lambda(\Psi(H(u,t)))$, and let t run from 1 to 0.

Acknowledgements. The author is grateful to L. Di Vizio and A. Bahri for some helpful comments. He is supported by MURST within the PRIN 2006 *Variational methods and nonlinear differential equations*.

REFERENCES

[1] A. Bahri, "Critical Points at Infinity in some Variational Problems," Research Notes in Mathematics, 182, Longman-Pitman, London, 1989.

[2] A. Bahri and J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math., 41 (1988), 253–294.

[3] A. Bahri and J.M. Coron, *The scalar curvature problem on the standard three dimensional sphere*, J. Funct. Anal., 95 (1991), 106–172.

[4] T.P. Branson, "The Functional Determinant," Global Analysis Research Center Lecture Note Series, Number 4, Seoul National University (1993).

[5] T.P. Branson, *Differential operators canonically associated to a conformal structure*, Math. Scand., 57 (1985), 293–345.

[6] T.P. Branson and B. Ørsted, *Explicit functional determinants in four dimensions*, Proc. Amer. Math. Soc., 113 (1991), 669–682.

[7] S. Brendle, *Global existence and convergence for a higher order flow in conformal geometry*, Ann. of Math., 158 (2003), 323–343.

[8] H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions,* Commun. Partial Differ. Equations 16-8/9 (1991), 1223–1253.

[9] E. Caglioti, P.L Lions, C. Marchioro, and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description*, Comm. Math. Phys. 143 (1992), no. 3, 501-525. *Part II*, Comm. Math. Phys., 174 (1995), 229–260.

[10] L. Carleson and S.Y.A. Chang, *On the existence of an extremal function for an inequality of Moser*, Bull. Sci. Math., 110 (1986), 113–127.

[11] K.C. Chang, "Infinite-Dimensional Morse Theory and Multiple Solution Problems," PNLDE 6, Birkhäuser Boston, Inc., Boston, MA, 1993.

[12] S.Y.A. Chang, M.J. Gursky, and P.C. Yang, *The scalar curvature equation on 2- and 3- spheres*, Calc. Var. and Partial Diff. Eq., 1 (1993), 205–229.

[13] S.Y.A. Chang, M.J. Gursky, and P.C. Yang, *A conformally invariant sphere theorem in four dimensions*, Publ. Math. Inst. Hautes Études Sci., 98 (2003), 105–143.

[14] S.Y.A. Chang and P.C. Yang, *Prescribing Gaussian curvature on* S^2, Acta Math., 159 (1987), 215–259.

[15] S.Y.A. Chang and P.C. Yang, *Extremal metrics of zeta functional determinants on 4-manifolds*, Ann. of Math., 142 (1995), 171–212.

[16] S.Y.A. Chang and P.C. Yang, *On a fourth order curvature invariant*, Comtemp. Math. 237, Spectral Problems in Geometry and Arithmetic, Ed. T. Branson, AMS, 1999, 9–28.

[17] C.C. Chen and C.S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math., 55 (2002), 728–771.

[18] C.C. Chen and C.S. Lin, *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math., 56 (2003), 1667–1727.

[19] W. Chen and C. Li, *Prescribing Gaussian curvatures on surfaces with conical singularities*, J. Geom. Anal., 1 (1991), 359–372.

[20] S. Chanillo and M. Kiessling, *Conformally invariant systems of nonlinear PDE of Liouville type*, Geom. Funct. Anal., 5 (1995), 924–947.

[21] K. Deimling, "Nonlinear Functional Analysis," Springer-Verlag, Berlin, 1985.

[22] W. Ding, J. Jost, J. Li, and G. Wang, *The differential equation* $\Delta u = 8\pi - 8\pi h e^u$ *on a compact Riemann surface*, Asian J. Math., 1 (1997), 230–248.

[23] W. Ding, J. Jost, J. Li, and G. Wang, *Existence results for mean field equations*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 16 (1999), 653–666.

[24] Z. Djadli, *Existence result for the mean field problem on Riemann surfaces of all genuses*, Comm. Contemp. Math., to appear.

[25] Z. Djadli Z and A. Malchiodi, *Existence of conformal metrics with constant Q-curvature*, Ann. of Math., to appear.

[26] O. Druet and F. Robert, *Bubbling phenomena for fourth-order four-dimensional pdes with exponential growth*, Proc. Amer. Math. Soc., 134 (2006), 897–908.

[27] L. Fontana, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helvetici, 68 (1993), 415–454.

[28] C.R. Graham, R. Jenne, L.J. Mason, and G. Sparling, *Conformally invariant powers of the Laplacian. I. Existence*, J. London Math. Soc., 46 (1992) 557–565.

[29] M. Gursky, *The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE*, Comm. Math. Phys., 207 (1999), 131–143.

[30] M. Hirsch, "Differential Topology," Graduate Texts in Math., 33, Springer-Verlag, New York, 1994.

[31] M. Kiessling, *Statistical mechanics of classical particles with logarithmic interactions*. Comm. Pure Appl. Math., 46 (1993), 27–56.

[32] J. Li, Y. Li, and P. Liu, *The Q-curvature on a 4-dimensional Riemannian manifold* (M, g) *with* $\int_M Q_g dV_g = 8\pi^2$, preprint.

[33] Y.Y. Li, *Prescribing scalar curvature on* S^n *and related topics, Part I*, Journal of Differential Equations, 120 (1995), 319-410, *Part II, Existence and compactness*, Comm. Pure Appl. Math., 49 (1996), 437–477.

[34] Y.Y. Li, *Harnack type inequality: The method of moving planes*, Comm. Math. Phys., 200 (1999), 421–444.

[35] Y.Y. Li and I. Shafrir, *Blow-up analysis for solutions of* $-\Delta u = V e^u$ *in dimension two*, Indiana Univ. Math. J., 43 (1994), 1255–1270.

[36] C.S. Lin, *Topological degree for mean field equations on* S^2, Duke Math. J., 104 (2000), 501–536.

[37] C.S. Lin and J. Wei, *Sharp estimates for bubbling solutions of a fourth order mean field equation*, Annali Scuola Norm. Sup. Pisa, to appear.

[38] C.S. Lin, L. Wang, and J. Wei, *Topological Degree For Solutions of A Fourth Order Mean Field Equation*, preprint.

[39] M. Lucia, *A mountain pass theorem without Palais-smale condition*, C. R. Acad. Sci. Paris, Ser. I, 341 (2005) 287–291.

[40] M. Lucia, *A deformation lemma with an application to a mean field equation*, Topol. Methods Nonlinear Anal., 30 (2007), 113–138.

[41] M. Lucia, *A blowing-up branch of solutions for a mean field equation*, Calc. Var., 26 (2006), 313–330.

[42] A. Malchiodi, *Compactness of solutions to some geometric fourth-order equations*, J. Reine Angew. Math., 594 (2006), 137–174.

[43] A. Malchiodi, *Topological methods an elliptic equations with exponential nonlinearities*, Discrete Contin. Dyn. Syst., 21 (2008), 277–294.

[44] W. Massey, "A Basic Course in Algebraic Topology," Graduate Texts in Mathematics, 127, Springer-Verlag, New York, 1991.

[45] C.B. Ndiaye, *Constant Q-curvature metrics in arbitrary dimensions*, J. Funct. Anal., 251 (2007), 1–58.

[46] M. Nolasco and G. Tarantello, *On a sharp Sobolev-type inequality on two-dimensional compact manifolds*, Arch. Ration. Mech. Anal., 145 (1998), no. 2, 161–195.

[47] M. Nolasco and G. Tarantello, *Double vortex condensates in the Chern-Simons-Higgs theory*, Calc. Var. Partial Differential Equations, 9 (1999), 31–94.

[48] S. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.

[49] F. Robert and J. Wei, *Asymptotic behavior of a fourth order mean field equation with Dirichlet boundary condition*, Indiana University Mathematics Journal, to appear.

[50] R. Schoen and D. Zhang, *Prescribed scalar curvature on the n-sphere*, Calc. Var., 4 (1996), 1–25.

[51] E.M. Spanier, "Algebraic Topology," McGraw-Hill, New York-Toronto-London, 1966.

[52] M. Struwe, *The existence of surfaces of constant mean curvature with free boundaries*, Acta Math., 160 (1988), 19–64.

[53] M. Struwe and G. Tarantello, *On multivortex solutions in Chern-Simons gauge theory*, Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat., 8 (1998), 109–121.

[54] G. Tarantello, "Self-Dual Gauge Field Vortices: An Analytical Approach," PNLDE 72, Birkhäuser Boston, Inc., Boston, MA, 2007.