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AN IDENTIFICATION PROBLEM WITH EVOLUTION ON THE BOUNDARY OF PARABOLIC TYPE

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Abstract. We consider an equation of the type $A(u + k * u) = f$, where A is a linear second-order elliptic operator, k is a scalar function depending on time only and $k * u$ denotes the standard time convolution of functions defined on $\mathbf{R}$ with their supports in $[0, T]$. The previous equation is endowed with dynamical boundary conditions.

Assuming that the kernel k is unknown and information is given, under suitable additional conditions k can be recovered and global existence, uniqueness and continuous dependence results can be shown.

0. Introduction

Recovering unknown functions in equations describing physical, chemical, geological phenomena has become a routine requirement in applied sciences, though such problems are, in general, ill posed from the mathematical point of view. Due to this intrinsic difficulty, up to now less interest has been devoted to biological problems, since their formulation is generally more complex and less usual to mathematicians.

The present work is inspired by the paper [3] which is concerned with an equation arising in the study of the electrical conduction in biological tissues. After a homogenization procedure, the authors in [3] deduce the following equation:

$$-\mathrm{div}\left[(A_0 + \sigma_0 I)\nabla_x u_0(t,x) + \int_0^t A_1(t-s)\nabla_x u_0(s,x)\,ds\right] = \widetilde{f}(t,x), \quad (0.1)$$

where σ_0 is a positive constant and $A_0 + \sigma_0 I$ and A_1 are symmetric matrices, the first being constant and positive definite and the latter depending on

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time, only. Equation (0.1) turns to be of elliptic type depending on time, meant as a parameter, and is subject to a "boundary dynamic condition;" i.e., the evolution with respect to time occurs only in the boundary condition.

The identification problem we deal with in the present paper is meant as a first step to solve the inverse problem consisting in recovering the matrix A_1 in equation (0.1). For the sake of simplicity we assume that $A_0(t) = \widetilde{A} - \sigma_0 I$, $\widetilde{A}$ being a positive definite constant matrix, and $A_1(t) = k(t)\widetilde{A}$, so that we are concerned with identifying the unknown (time) convolution *scalar* kernel k appearing in a linear second–order elliptic equation of this type

$$A(u + k * u) = f \qquad \text{in} \quad (0,T) \times \Omega, \tag{0.2}$$

where $k * u(t,x) = \int_0^t k(t-s)u(s,x)\,ds$ and Ω is a bounded (smooth) domain in $\mathbf{R}^n$.

Equation (0.2) is endowed with a dynamical boundary condition of the flux form, as in [3],

$$D_t u + bD_{\nu_A}[u + k * u] = g, \qquad \text{on} \quad (0,T) \times \partial\Omega, \tag{0.3}$$

or of the following form

$$D_t u + bD_{\nu_A}[u] = g, \qquad \text{on} \quad (0,T) \times \partial\Omega. \tag{0.4}$$

To our knowledge the problem of determining the memory kernel k is new under dynamical boundary conditions.

On the contrary, direct problems with $k = 0$ and dynamical boundary conditions are well studied, from the mathematical point of view (cf. e.g. the papers [1]-[9], [11]-[16], [21]-[26] and the references therein).

Various applications of direct problems with dynamical boundary conditions can be found in different applied sciences (cf. the introductions in [3], [4], [5] and [13]). More exactly, the first two papers deal with transmission problems for biological tissues, where one of the phases is the extracellular space, the latter being the intracellular space, while the interface is the cell membrane and the unknown u is the electrical potential. Moreover, in [5] the authors consider the quasi-static problem of a viscoelastic body occupying a bounded region Ω (Ω is an open connected subset of $\mathbf{R}^3$) and subject to an elastic force on the boundary $\partial\Omega$ (which is supposed to be sufficiently regular).

We conclude this introduction by observing that our treatment to solve our identification problem leads to the (equivalent) identification problem (1.9)–(1.13) (cf. Section 1) related to the boundary $\partial\Omega$ of Ω, where the unknown kernel h convolves not only with the space operator $D_{\nu_A}v$ (D_{ν_A} standing for the conormal derivative), but also with the time derivative $D_t v$.

For such a kind of identification problem, to the authors' knowledge, the global in time existence of the solution (v, b) is not a standard result.

In this paper this difficulty is overcome by using a suitable estimate strategy and a fixed-point approach strictly related to the problem itself - we could say "suggested" by the problem itself (cf. Sections 4 and 5).

Finally, we observe that a simple treatment of the elliptic problem (0.2) (endowed with auxiliary nonhomogeneous Dirichlet conditions) forces us to choose for u - and consequently for $v = u_{|\partial\Omega}$ - an $L^p(\Omega)$-framework with $p \in (1, +\infty)$.

In contrast to this, we have chosen for the time dependence of v and h the Hölder spaces C^α, $\alpha \in (0,1)$, to ensure a good regularity for the unknown kernel h - and consequently for k related to h by the linear convolution equation (1.8).

The plan of the paper is the following. Section 1 will be devoted to the (formal) derivation of several problems. After introducing the suitable functional framework in Section 2, we will state our global existence, uniqueness, and continuous dependence results - Theorems 2.1 and 2.2 - and will prove also four equivalence theorems related to the auxiliary problems in Section 1 - Theorems 2.3–2.6. Section 3 will be devoted to proving several basic lemmata. Finally, in Section 4 we will give the proofs of Theorems 2.1 and 2.2, assuming the existence of the solution to an auxiliary nonlinear convolution fixed-point system. Such a result will be proved in Section 5.

1. The identification problem and the auxiliary ones

Let Ω be an open bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega$ of class C^2. Let $A : W^{2,p}(\Omega) \to L^p(\Omega)$, $p \in (1, +\infty)$, be a linear (uniformly) elliptic operator of the following form:

$$A = \sum_{i,j=1}^{n} D_{x_i}[a_{i,j}(x) D_{x_j}] + \sum_{j=1}^{n} a_j(x) D_{x_j} + a_0(x), \tag{1.1}$$

where $a_{i,j}, a_j \in C^2(\overline{\Omega})$, $i, j = 1, \dots, n$, and

$$a_0(x) - \frac{1}{2} \sum_{j=1}^{n} a_j(x) \leq 0, \quad x \in \overline{\Omega}, \tag{1.2}$$

$$\sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \geq \rho |\xi|^2, \quad (x, \xi) \in \overline{\Omega} \times \mathbf{R}^n,$$

for some positive constant ρ.

Note that we will be dealing with a generalization of the problem discussed in the Introduction, since the coefficients of the operator A *may* depend on space.

Consider the problem consisting of determining $u : [0,T] \times \Omega \to \mathbf{R}$ and $k : [0,T] \to \mathbf{R}$ such that

$$A(u + k * u) = f, \quad \text{in } (0,T) \times \Omega, \tag{1.3}$$

$$D_t u + b(x) D_{\nu_A}[u + ik * u] = g, \quad \text{on } (0,T) \times \partial\Omega, \tag{1.4}$$

$$u(0) = u_0, \quad \text{on } \partial\Omega, \tag{1.5}$$

$$\Phi[u(t)] = l(t), \quad t \in [0,T], \tag{1.6}$$

where $i \in \{0,1\}$ (cf. the two dynamic condition (0.3) and (0.4) in Section 0),

$$b \in C^2(\partial\Omega), \quad b(x) > 0, \quad \forall x \in \partial\Omega, \tag{1.7}$$

$$\Phi[z] = \int_\Omega \big[\varphi_0(x) z(x) + \varphi(x) \cdot \nabla z(x)\big]\, dx,$$

φ_0 and φ being two given functions in $L^{p'}(\Omega)$ and $L^{p'}(\Omega)^n$, $1/p + 1/p' = 1$.

Agreement. *From now on in order to simplify our notation we will write $F(t)$ instead of $F(t,\cdot)$, the latter representing the function $x \to F(t,x)$.*

The first step for determining the unknown pair (u,k) consists of changing the unknown u to $v := u + k * u$. For this purpose we recall a well-known result (cf. e.g. [18]): with any $k \in L^p((0,T))$, $p \in [1,+\infty]$, we can associate a unique solution $h \in L^p((0,T))$ solving the convolution equation

$$h + k + h * k = 0 \quad \text{in } (0,T). \tag{1.8}$$

Concerning equation (1.8) the following classic regularity result holds:

Lemma 1.1. *Let $h \in C^\alpha([0,T])$, $\alpha \in (0,1)$. Then equation* (1.8) *admits a unique solution $h \in C^\alpha([0,T])$, $\alpha \in (0,1)$, continuously depending on k with respect to the norm of $C^\alpha([0,T])$.*

Consequently, we get

$$v = u + k * u \quad \iff \quad u = v + h * v. \tag{1.9}$$

It is easy to check that the pair (v,k) solves the following problem:

$$Av = f \quad \text{in } (0,T) \times \Omega, \tag{1.10}$$

$$D_t v + h * D_t v + b D_{\nu_A} v + b_i h * D_{\nu_A} v + h u_0 = g \quad \text{on } (0,T) \times \partial\Omega, \tag{1.11}$$

$$v(0) = u_0 \quad \text{on } (0,T) \times \partial\Omega, \tag{1.12}$$

$$\Phi[v(t)] + h * \Phi[v(t)] = l(t), \quad t \in [0,T], \tag{1.13}$$

where

$$b_i(x) := (1-i)b(x), \quad x \in \partial\Omega, \quad i = 0, 1.$$

According to Theorem 9.11 in [10], from our assumptions on the coefficients $a_{i,j}$ and a_0 we can conclude that, for any pair $(f, w) \in L^p(\Omega) \times W^{2-1/p,p}(\partial\Omega)$, $p \in (1, +\infty)$, the Dirichlet problem

$$Av = f, \quad \text{in } \Omega, \quad v = w \quad \text{on } \partial\Omega, \tag{1.14}$$

admits a unique solution $v \in W^{2,p}(\Omega)$ satisfying the estimate

$$\|v\|_{W^{2,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|w\|_{W^{2-1/p,p}(\partial\Omega)}).$$

Moreover, according to the fundamental result in [17, Theorem 4.2], the solution to the elliptic boundary-value problem (1.14) admits the representation

$$v = L_0 w + L_1 f, \tag{1.15}$$

where

$$L_0 \in \mathcal{L}(W^{s-1/p,p}(\partial\Omega); W^{s,p}(\Omega)), \quad 1/p < s \leq 2, \ s \neq 1 + 1/p, \tag{1.16}$$

$$L_1 \in \mathcal{L}(L^p(\Omega); W^{2,p}(\Omega)). \tag{1.17}$$

We emphasize that $L_0 w$ stands for the solution to problem (1.14) with $f = 0$, while $L_1 f$ stands for the solution to problem (1.14) with $w = 0$; i.e.,

$$\begin{cases} AL_0 w = 0 & \text{in } \Omega, \\ L_0 w = w & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} AL_1 f = f & \text{in } \Omega, \\ L_1 f = 0 & \text{on } \partial\Omega. \end{cases}$$

Then problem (1.10)-(1.13) is equivalent to the following:

$$\begin{aligned} D_t w + bD_{\nu_A} L_0 w = g - h * D_t w - bD_{\nu_A} L_1 f - b_i h * D_{\nu_A} L_0 w \\ - b_i h * D_{\nu_A} L_1 f - h u_0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.18}$$

$$w(0) = u_0, \quad \text{on } \partial\Omega, \tag{1.19}$$

$$\Phi[L_0 w(t)] + \Phi[L_1 f(t)] + h * \Phi[L_0 w(t)] + h * \Phi[L_1 f(t)] = l(t), \quad t \in [0, T], \tag{1.20}$$

where

$$w := v|_{\partial\Omega}. \tag{1.21}$$

Set $B := -bD_{\nu_A} L_0$; according to (1.7) and the proof of Proposition 1.6 in [11], we deduce the generation estimate

$$\|w\|_{W^{1-1/p,p}(\partial\Omega)} \leq C|\lambda|^{-1} \|\lambda w - Bw\|_{W^{1-1/p,p}(\partial\Omega)},$$

$$\forall w \in W^{1-1/p,p}(\partial\Omega), \quad \forall \lambda \in \{\mu \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\lambda| > r\}.$$

Then, from Propositions 2.1.1 and 2.1.11 in [20], the linear operator B generates an analytic semigroup $\{S(t)\}_{t\geq 0}$ of linear bounded operators in $W^{1-1/p,p}(\partial\Omega)$, $p \in (1, +\infty)$.

From now on we will proceed formally, postponing the rigorous statements of the functional framework and the equivalences of the following auxiliary identification problems to Section 2, and their proofs to Section 4.

Then, differentiating (1.20), we deduce that the pair (w, h) solves the following problem:

$$\begin{aligned} D_t w + bD_{\nu_A} L_0 w &= g - h * D_t w - bD_{\nu_A} L_1 f - b_i h * D_{\nu_A} L_0 w \\ &\qquad - b_i h * D_{\nu_A} L_1 f - h u_0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.22}$$

$$w(0) = u_0, \quad \text{on } \partial\Omega, \tag{1.23}$$

$$\begin{aligned} &\Phi[L_0 D_t w(t)] + \Phi[L_1 D_t f(t)] + h * \Phi[L_0 D_t w(t)] + h * \Phi[L_1 D_t f(t)], \\ &+ h(t)\{\Phi[L_0 u_0] + \Phi[L_1 f(0)]\} = l'(t), \quad t \in [0, T]. \end{aligned} \tag{1.24}$$

Then, according to formula (1.23) and well-known results in semigroup theory, problem (1.22) and (1.23) is equivalent to the following fixed-point equation:

$$\begin{aligned} w(t) = &\Big\{ S(t) u_0 + \int_0^t S(t-s)\big[g(s) - bD_{\nu_A} L_1 f(s)\big]\, ds \\ &\quad - h(0) \int_0^t S(t-s) b_i (1 * D_{\nu_A} L_1 f(s))\, ds \Big\} \\ &- h(0) \int_0^t S(t-s) u_0\, ds - \int_0^t q(s) S(t-s) u_0\, ds \\ &- \Big\{ \int_0^t S(t-s) \big\{ (h(0)+q) * D_t w + (1-i)(h(0)+q) * Bw \big\}\, ds \Big\} \\ &- \int_0^t S(t-s) b_i q * D_{\nu_A} L_1 f(s)\, ds \\ =: &\big[\widetilde{\zeta}(t) - h(0) 1 * S(\cdot) u_0\big] - q * S(\cdot) u_0 - N_1(w, h(0)+q)(t) - N_2(q, \mathbf{d})(t) \\ =: &\,\zeta(t) - q * S(\cdot) u_0 - N_1(w, h(0)+q)(t) - N_2(q, \mathbf{d})(t), \end{aligned} \tag{1.25}$$

where $\mathbf{d} = (f, u_0, l, g)$ and

$$q(t) = h(t) - h(0). \tag{1.26}$$

Differentiating the first and the last sides in (1.25), we get

$$\begin{aligned} D_t w(t) = D_t \widetilde{\zeta}(t) &- [h(0) + q(t)] u_0 - [h(0) + q] * S(\cdot) B u_0 \\ &- D_t N_1(w, h(0)+q)(t) - D_t N_2(q, \mathbf{d})(t), \end{aligned} \tag{1.27}$$

since, according to our assumption on u_0 (cf. Section 2), we have

$$D_t[h * S(\cdot)u_0](t) = \int_0^t h(s)S'(t-s)u_0\,ds + h(t)u_0$$
$$= \int_0^t h(s)S(t-s)Bu_0\,ds + h(t)u_0.$$

After placing the expression for $D_t w$ from (1.27) into the first term in (1.24), we obtain the following equation, for $t \in [0,T]$:

$$\begin{aligned}&\Phi[L_0 D_t\widetilde{\zeta}(t)] - \Phi[(h(0)+q) * L_0 S(\cdot)Bu_0] - \Phi[L_0 D_t N_1(w, h(0)+q)(t)]\\ &- \Phi[L_0 D_t N_2(q,\mathbf{d})(t)] + \Phi[L_1 D_t f(t)]\\ &+ (h(0)+q) * \{\Phi[L_0 D_t w(t)] + \Phi[L_1 D_t f(t)]\}\\ &+ (h(0)+q(t))\Phi[L_1 f(0)] = l'(t).\end{aligned} \tag{1.28}$$

Remark 1.1. We observe that if we had additional information of the form

$$\Phi[z] = \int_{\partial\Omega} \varphi(x)z(x)\,d\sigma(x),$$

where σ stands for Lebesgue measure on $\partial\Omega$, a single differentiation in time would not allow us to derive a fixed-point equation for q. In that case q would appear only as a convolved term. A possible, but successful, different choice involving surface integrals is

$$\Phi[z] = \alpha \int_\Omega \big[\varphi_1(x)z(x) + \varphi_2(x)\cdot\nabla z(x)\big]\,dx + \beta\int_{\partial\Omega}\varphi_3(x)z(x)\,d\sigma(x),$$

where α, β are given constants with $\alpha \neq 0$.

Assume now that

$$\chi^{-1} := \Phi[L_1 f(0)] \neq 0. \tag{1.29}$$

Consequently, we deduce the following (equivalent) fixed-point equation for q :

$$\begin{aligned}q(t) = &\Big\{\chi\big[l'(t) - \Phi[L_0 D_t\widetilde{\zeta}(t)] + h(0)\Phi[1 * L_0 S(\cdot)Bu_0] - \Phi[L_1 D_t f(t)]\big] - h(0)\Big\}\\ &+ \chi\Phi[L_0 D_t N_1(w, h(0)+q)(t)] + \chi\Phi[L_0 D_t N_2(q,\mathbf{d})(t)]\\ &- \chi(h(0)+q) * \Phi[L_0 D_t w(t)] - \chi(h(0)+q) * \Phi[L_1 D_t f(t)]\\ &+ \chi\Phi[q * L_0 S(\cdot)Bu_0] =: \kappa(t) - N_3(w,q,\mathbf{d})(t), \quad t\in[0,T].\end{aligned} \tag{1.30}$$

To solve our identification problem in a simpler way we need to perform the following additional change of unknowns:

$$z(t) = w(t) - u_0 - t[Bu_0 + g_0(0)],$$

where

$$g_0(s) = g(s) - bD_{\nu_A}L_1f(s) - h(0)b_i(1 * D_{\nu_A}L_1f(s)) - h(0)u_0. \tag{1.31}$$

We stress that

$$z(0) = 0 \quad \text{and} \quad D_tz(0) = 0. \tag{1.32}$$

According to (1.25) and (1.30) it is easy to check that the pair (z, q) solves, for all $t \in [0, T]$, the fixed-point system

$$\begin{aligned}
z(t) &= -u_0 - t[Bu_0 + g_0(0)] + \zeta(t) - q * S(\cdot)u_0 \\
&\qquad - N_1(z + u_0 + t[Bu_0 + g_0(0)], h(0) + q)(t) - N_2(q, \mathbf{d})(t) \\
&= \Big\{ - u_0 - t[Bu_0 + g_0(0)] + \zeta(t) - N_1(u_0 + t[Bu_0 + g_0(0)], h(0))(t)\Big\} \\
&\quad - q * S(\cdot)u_0 - \Big\{N_1(z, h(0))(t) + N_1(u_0 + t[Bu_0 + g_0(0)], q)(t) \\
&\quad + N_1(z, q)(t) + N_2(q, \mathbf{d})(t)\Big\} =: \zeta_0(t) - q * S(\cdot)u_0 - N_4(z, q, \mathbf{d})(t),
\end{aligned} \tag{1.33}$$

$$\begin{aligned}
q(t) &= \Big\{\chi\Big[l'(t) - \Phi[L_0D_t\widetilde{\zeta}(t)] + h(0)\Phi[1 * L_0S(\cdot)Bu_0] - \Phi[L_1D_tf(t)]\Big] \\
&\qquad - h(0) - \chi h(0) * \Phi[L_0(Bu_0 + g_0(0))] - \chi h(0) * \Phi[L_1D_tf(t)] \\
&\quad + \chi\Phi[L_0D_tN_1(u_0 + t[Bu_0 + g_0(0)], h(0))(t)]\Big\} + \chi\Phi[L_0D_tN_1(z, h(0))(t)] \\
&\quad + \chi\Phi[L_0D_tN_1(u_0 + t[Bu_0 + g_0(0)], q)(t)] + \chi\Phi[L_0D_tN_1(z, q)(t)] \\
&\quad + \chi\Phi[L_0D_tN_2(q, \mathbf{d})(t)] - \chi q * \Phi[L_0(Bu_0 + g_0(0))] \\
&\quad - \chi(h(0) + q) * \Phi[L_0D_tz(t)] - \chi q * \Phi[L_1D_tf(t)] + \chi\Phi[q * L_0S(\cdot)Bu_0] \\
&\qquad =: \kappa_0(t) + N_5(z, h(0), q, \mathbf{d})(t),
\end{aligned} \tag{1.34}$$

where

$$\begin{aligned}
&N_5(z, h(0), q, \mathbf{d})(t) \\
&= \chi\Big\{\Phi\Big[L_0D_t\Big[S * \Big((h(0) + q) * D_tz + (1 - i)(h(0) + q) * Bz \\
&\quad + q * [b_iD_{\nu_A}L_1f + Bu_0 + g_0(0)](s) + (1 - i)(t * q)(s)B(Bu_0 + g_0(0))\Big)\Big]\Big] \\
&\quad - (h(0) + q) * \Phi[L_0D_tz(t)] - q * \Phi[L_1D_tf(t) + L_0(Bu_0 + g_0(0))] \\
&\quad + \Phi[q * L_0S(\cdot)Bu_0]\Big\} := \chi\widetilde{K}(z, h(0), q, \mathbf{d})(t), \quad t \in [0, T].
\end{aligned} \tag{1.35}$$

2. CONSISTENCY CONDITIONS AND STATEMENTS OF THE MAIN RESULTS

Our consistency conditions are a consequence of

$$\Phi[v(0)] = l(0), \qquad \Phi[D_t v(0)] + h(0)\Phi[v(0)] = l'(0). \tag{2.1}$$

Consequently, in order to make explicit conditions (2.1), we need to compute $v(0)$ and $D_t v(0)$ on Ω.

First we observe that from equations (1.10) and (1.15) we get

$$v = L_1 f + L_0(v|_{\partial\Omega}), \quad \text{in } [0,T]\times\Omega, \tag{2.2}$$

$$D_{\nu_A} v = D_{\nu_A} L_1 f + D_{\nu_A} L_0(v|_{\partial\Omega}), \quad \text{in } [0,T]\times\Omega. \tag{2.3}$$

Setting $t=0$ in (2.3), we get

$$D^n_{\nu_A} v(0) = D^n_{\nu_A} L_1 f(0) + D^n_{\nu_A} L_0 u_0 := D^n_{\nu_A}\widetilde{u}_0, \quad \text{in } \Omega, \quad n = 0,1.$$

Consequently, the first consistency condition is given by

$$l(0) = \Phi[L_1 f(0) + L_0 u_0]. \tag{2.4}$$

To derive the second consistency condition from the latter equality in (2.1) we observe that, from equations (1.10) and (1.11), we deduce

$$\begin{aligned} &A D_t v(0) = D_t f(0) \quad \text{in } \Omega, \\ &D_t v(0) = -b D_{\nu_A} v(0) - h(0)u_0 + g(0), \quad \text{on } \partial\Omega. \end{aligned}$$

Since

$$D_t v(0) = L_0(D_t v|_{\partial\Omega}(0)) + L_1 D_t f(0),$$

from the latter equality in (2.1) we obtain

$$-\Phi[L_0 b D_{\nu_A} v(0)] + \Phi[L_0 g(0)] + \Phi[L_1 D_t f(0)] + h(0)\Phi[L_1 f(0)] = l'(0).$$

Under the assumption (1.29) the previous equation allows us to determine the initial value of h:

$$\begin{aligned} h(0) = \chi\Big\{ &l'(0) + \Phi[L_0 b D_{\nu_A} L_1 f(0) + L_0 b D_{\nu_A} L_0 u_0] - \Phi[L_0 g(0)] \\ &- \Phi[L_1 D_t f(0)]\Big\} =: h_0. \end{aligned} \tag{2.5}$$

Remark 2.1. According to (2.5) it is easy to check that equations (1.20) and (1.24) are equivalent.

Remark 2.2. Since $h(0) =: h_0$ depends on the data (cf. (2.5)), from now on, we will drop the dependence on $h(0)$ in the operators N_5 and $\widetilde{K}$.

To state our main results we introduce the following spaces depending on the parameters $s_1, s_2 \in \mathbf{R}$ and $p \in (1, +\infty)$:

$$\mathcal{U}_T^{s_1,s_2,p}(\Omega) := C^{s_1}([0,T]; W^{s_2,p}(\Omega)), \tag{2.6}$$

$$\mathcal{V}_T^{s_1,s_2,p}(\partial\Omega) := C^{s_1}([0,T]; W^{s_2-1/p,p}(\partial\Omega)). \tag{2.7}$$

We list now our assumptions on the data (f, u_0, l, g) where $\alpha \in (0,1)$, $\varepsilon \in (0, \min\{\alpha, 1-\alpha\})$:

H1 $u_0 \in W^{2-1/p,p}(\partial\Omega)$;
H2 $f \in \mathcal{U}_T^{1+\alpha+\varepsilon,0,p}(\Omega)$, $g \in \mathcal{V}_T^{1+\alpha+\varepsilon,1,p}(\partial\Omega)$;
H3 $l \in C^{1+\alpha+\varepsilon}([0,T])$;
H4 $Bu_0 + g(0) - bD_{\nu_A}L_1 f(0) - h_0 u_0 \in W^{2-1/p,p}(\partial\Omega)$,
H5 $\chi^{-1} := \Phi[L_1 f(0)] \neq 0$,
H6 $l(0) = \Phi[L_1 f(0) + L_0 u_0]$,
H7 $-\Phi[L_0 b D_{\nu_A} v(0)] + \Phi[L_0 g(0)] + \Phi[L_1 D_t f(0)] + h_0 \Phi[L_1 f(0)] = l'(0)$.

The *admissible space* for our data will consist of vector functions $\mathbf{d} = (f, u_0, l, g)$ possessing properties H1-H5. Such a space will be denoted by $\mathbf{D}_T^{\alpha,\varepsilon}$.

Theorem 2.1. *Let* $\mathbf{d} = (f, u_0, l, g) \in \mathbf{D}_T^{\alpha,\varepsilon}$ *and satisfy conditions H1-H5. Then there exists a unique pair* $(u,k) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$ *solving problem* (1.3)–(1.6) *and satisfying* (2.5).

We now state a continuous dependence result for (u,k) in terms of the data (f, u_0, l, g). For this purpose we denote by H_j1-H_j5 assumptions H1-H5, where $(f, u_0, l, g, \chi, h_0)$ is replaced by $(f_j, u_{0,j}, l_j, g_j, \chi_j, h_{0,j})$, $(\chi_j, h_{0,j})$ being defined as (χ, h_0) is.

Theorem 2.2. *Let* $\mathbf{d}_j := (f_j, u_{0,j}, l_j, g_j)$ *satisfy assumptions* H_j1–H_j7, $j = 1,2$*, as well as the following additional assumption:*

H_j8 $|\chi_j| \le \mu$, $j = 1,2$, *for some given* $\mu > 0$.

Let R_0 *be a positive constant such that* $R_0 \ge \max_{j=1,2}\{\|\mathbf{d}_j\|\}$*, where*

$$\|\mathbf{d}\| = \|(f, u_0, l, g)\| = \|f\|_{\mathcal{U}_T^{1+\alpha,0,p}(\Omega)} + \|u_0\|_{W^{2-1/p,p}(\partial\Omega)} + \|l\|_{C^{1+\alpha+\varepsilon}([0,T])}$$

$$+\|g\|_{\mathcal{V}_T^{1+\alpha+\varepsilon,1,p}(\partial\Omega)} + \|Bu_0 + g(0) - bD_{\nu_A}L_1 f(0) - h_0 u_0\|_{W^{2-1/p,p}(\partial\Omega)}. \tag{2.8}$$

Let $(u_j, k_j) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \times \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$ *be solutions to problems* (1.3)–(1.6) *with* (f, u_0, l, g) *being replaced with* $(f_j, u_{0,j}, l_j, g_j)$, $j = 1,2$. *Then*

$$\|u_2 - u_1\|_{\mathcal{U}_T^{1+\alpha,1,p}(\Omega)\cap\mathcal{U}_T^{\alpha,2,p}(\Omega)} + \|k_2 - k_1\|_{C^{1+\alpha}([0,T])} \le C(R_0, T)\|\mathbf{d}_2 - \mathbf{d}_1\|. \tag{2.9}$$

We will conclude this section by proving the equivalence of problems (1.3)–(1.6) with each of the auxiliary problems in Section 1 by the following four theorems.

Theorem 2.3. *Let* $\mathbf{d} = (f, u_0, l, g) \in \mathbf{D}_T^{\alpha,\varepsilon}$*. Then problems* (1.3)–(1.6) *and* (1.10)–(1.13) *are equivalent and their solutions* $(u,k) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$ *and* $(v,h) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$ *are related by formulae* (1.8) *and* (1.9)*. Moreover, the map* $(v,h) \to (u,k)$*, defined by* (1.8), (1.9)*, is continuous from* $[\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$ *into itself.*

Theorem 2.4. *Let* $\mathbf{d} = (f, u_0, l, g) \in \mathbf{D}_T^{\alpha,\varepsilon}$*. Then problems* (1.10)–(1.13) *and* (1.18)–(1.20) *are equivalent and their solutions* $(v,h) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$ *and* $(w,h) \in [\mathcal{V}_T^{1+\alpha,1,p}(\partial\Omega) \cap \mathcal{V}_T^{\alpha,2,p}(\partial\Omega)] \times C^\alpha([0,T])$ *are related by formulae* (1.15) *and* (1.21)*. Moreover, the map* $v \to w$*, defined by* (1.21)*, is continuous from* $[\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)]$ *into* $[\mathcal{V}_T^{1+\alpha,1,p}(\partial\Omega) \cap \mathcal{V}_T^{\alpha,2,p}(\partial\Omega)]$*.*

Theorem 2.5. *Let* $\mathbf{d} = (f, u_0, l, g) \in \mathbf{D}_T^{\alpha,\varepsilon}$*. Then problems* (1.18)–(1.20) *and* (1.22)–(1.24) *are equivalent.*

Theorem 2.6. *Let* $\mathbf{d} = (f, u_0, l, g) \in \mathbf{D}_T^{\alpha,\varepsilon}$*. Then problems* (1.22)–(1.24) *and* (1.25), (1.30) *are equivalent and their solutions* $(w,h) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$ *and* $(w,q) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$ *are related by formula* (1.26)*.*

Proof of Theorem 2.3. Let $(u,k) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$. Then equation (1.8) for h yields - as is well known - a unique solution $h \in C^\alpha([0,T])$ (cf. Lemma 1.1). Moreover, v defined by (1.9) belongs to $\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)$, according to well-known properties concerning convolutions.

Furthermore, as we have shown in Section 1, if (u,k) solves (1.3)–(1.6), then (v,h) solves (1.10)–(1.13). The converse relation depends on the formula for the time derivative of a convolution.

The continuous dependence result, concerning the map defined by (1.8) and (1.9), easily follows from its definition. □

Proof of Theorem 2.4. Let $(v,h) \in [\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)] \times C^\alpha([0,T])$. Then $w = v|_{\partial\Omega}$ belongs to $\mathcal{V}_T^{1+\alpha,1,p}(\partial\Omega) \cap \mathcal{V}_T^{\alpha,2,p}(\partial\Omega)$.

Conversely, if $(w,h) \in [\mathcal{V}_T^{1+\alpha,1,p}(\partial\Omega) \cap \mathcal{V}_T^{\alpha,2,p}(\partial\Omega)] \times C^\alpha([0,T])$, then v defined by (1.15), according to the properties of the operators L_0 and L_1 (cf. (1.16), (1.17)), belongs to $\mathcal{U}_T^{1+\alpha,1,p}(\Omega) \cap \mathcal{U}_T^{\alpha,2,p}(\Omega)$.

Furthermore, as we have shown in Section 1, if (v,h) solves (1.10)–(1.13), then (w,h) solves (1.18)–(1.20). Conversely, if (w,h) solves (1.18)–(1.20) and v is defined by (1.15), then, taking advantage of well-known properties of the Dirichlet problem for elliptic operators, we deduce that (v,h) solves (1.10)–(1.13). Indeed v solves the elliptic problem (1.14).

The continuous dependence result, concerning the map defined by (1.21), easily follows from its definition. □

Proof of Theorem 2.5. It suffices to observe that, by integration, from (1.24) we deduce equation (1.20), where $l(t)$ is replaced by $l(t)+c$, c being a constant. Finally, from the consistency condition (2.4) we conclude that $c = 0$. □

Proof of Theorem 2.6. It suffices to show that problem (1.25), (1.30) implies (1.22)–(1.24). First we observe that equation (1.25) is equivalent to (1.22). Then, according to condition (1.29), we immediately deduce that equation (1.30) is equivalent to (1.28). Finally, using formula (1.27), from (1.28) we easily deduce equation (1.24). □

3. Basic Lemmas

In this section we will use the shorthand $X := W^{1-1/p,p}(\partial\Omega)$, while $\mathcal{U}_T^{s_1,s_2,p}(\Omega)$ and $\mathcal{V}_T^{s_1,s_2,p}(\partial\Omega)$ are defined by (2.6) and (2.7).

Lemma 3.1. *Let $h \in \mathcal{V}_T^{\alpha,1,p}(\partial\Omega)$ and $l \in L^{1/(1-\alpha)}(0,T)$, $\alpha \in (0,1)$. Then $l * h \in \mathcal{V}_T^{\alpha,1,p}(\partial\Omega)$ and satisfies, for all $\tau \in (0,T]$, the estimate*

$$\|l * h\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \le (1+\tau^\alpha)\Big\{2\int_0^\tau |l(s)|[h]_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)}\,ds + \|h(0)\|_X\|l\|_{L^{1/(1-\alpha)}(0,\tau)}\Big\}, \tag{3.1}$$

where

$$[h]_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} := \sup_{0\le t_1<t_2\le\tau}(t_2-t_1)^{-\alpha}\|h(t_2)-h(t_1)\|_X.$$

Moreover, let $h \in L^{1/(1-\alpha)}((0,T);X)$ and $l \in C^\alpha([0,T])$, $\alpha \in (0,1)$. Then $l * h \in \mathcal{V}_T^{\alpha,1,p}(\partial\Omega)$ and satisfies, for $\tau \in (0,T]$, the estimate

$$\|l * h\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \le (1+\tau^\alpha)\Big\{2\int_0^\tau \|h(s)\|_X[l]_{C^\alpha([0,\tau-s])}\,ds + |l(0)|\|h\|_{L^{1/(1-\alpha)}((0,\tau);X)}\Big\}, \tag{3.2}$$

where

$$[l]_{C^\alpha([0,\tau])} := \sup_{0\le t_1<t_2\le\tau} (t_2-t_1)^{-\alpha}|l(t_2)-l(t_1)|.$$

Proof. Let $t_1, t_2 \in [0,T]$, $t_2 \ge t_1$. Then

$$\begin{aligned}
&\|l*h(t_2)-l*h(t_1)\|_X = \Big\| \int_0^{t_2} l(s)h(t_2-s)ds - \int_0^{t_1} l(s)h(t_1-s)ds \Big\|_X \\
&\le \Big\| \int_{t_1}^{t_2} l(s)h(t_2-s)ds \Big\|_X + \Big\| \int_0^{t_1} l(s)[h(t_2-s)-h(t_1-s)]ds \Big\|_X \\
&=: I_1(t_1,t_2) + I_2(t_1,t_2).
\end{aligned}$$

Now we estimate $I_1(t_1,t_2)$ and $I_2(t_1,t_2)$:

$$\begin{aligned}
I_1(t_1,t_2) &\le \int_{t_1}^{t_2} \|l(s)h(t_2-s)\|_X ds \\
&\le \int_{t_1}^{t_2} |l(s)|\|h(t_2-s)-h(0)\|_X\, ds + \|h(0)\|_X \int_{t_1}^{t_2} |l(s)|\, ds \\
&\le \int_{t_1}^{t_2} |l(s)|[h]_{\mathcal{V}^{\alpha,1,p}_{t_2-s}(\partial\Omega)}(t_2-s)^\alpha\, ds + \|h(0)\|_X \int_{t_1}^{t_2} |l(s)|\, ds \\
&\le (t_2-t_1)^\alpha \Big(\int_{t_1}^{t_2} |l(s)|[h]_{\mathcal{V}^{\alpha,1,p}_{t_2-s}(\partial\Omega)}\, ds + \|h(0)\|_X \|l\|_{L^{1/(1-\alpha)}(0,T)} \Big) \\
&\le (t_2-t_1)^\alpha \Big(\int_0^\tau |l(s)|[h]_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\, ds + \|h(0)\|_X \|l\|_{L^{1/(1-\alpha)}(0,T)} \Big), \\
I_2(t_1,t_2) &\le \int_0^{t_1} \|l(s)[h(t_2-s)-h(t_1-s)]\|_X ds \\
&\le (t_2-t_1)^\alpha \int_0^{t_1} |l(s)|[h]_{\mathcal{V}^{\alpha,1,p}_{t_2-s}(\partial\Omega)}\, ds \\
&\le (t_2-t_1)^\alpha \int_0^{t_2} |l(s)|[h]_{\mathcal{V}^{\alpha,1,p}_{t_2-s}(\partial\Omega)}\, ds \\
&\le (t_2-t_1)^\alpha \int_0^{\tau} |l(s)|[h]_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\, ds.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\|l*h(t_2)-l*h(t_1)\|_X \\
&\le (t_2-t_1)^\alpha \Big(2\int_0^\tau |l(s)|[h]_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\, ds + \|h(0)\|_X \|l\|_{L^{1/(1-\alpha)}(0,T)} \Big).
\end{aligned} \tag{3.3}$$

Setting $t_2 = t$ and $t_1 = 0$ in (3.3), for all $t \in [0, \tau]$ we get

$$\|l * h(t)\|_X \leq \tau^\alpha \Big(2 \int_0^\tau |l(s)| [h]_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\, ds + \|h(0)\|_X \|l\|_{L^{1/(1-\alpha)}(0,T)}\Big).$$

Consequently we obtain (3.1). The latter assertion of the lemma and estimate (3.2) can be proved similarly. □

From now on let $\{S(t)\}_{t\geq 0}$ be the semigroup of linear bounded operators in $\mathcal{L}(X)$ generated by $B = -b(x) D_{\nu_A} L_0$ with $D(B) = W^{2-1/p,p}(\partial\Omega)$. Moreover, we will use the following well-known estimates (cf., e.g., Proposition 2.1.1 in [20]), holding for positive and continuous functions M_j, $j \in \mathbf{N}$, defined on $[0, +\infty)$:

$$\|S^{(j)}(t)\|_{\mathcal{L}(X)} \leq M_j(T) t^{-j}, \qquad \forall t \in (0, T]. \tag{3.4}$$

Lemma 3.2. *Let $h \in \mathcal{V}^{0,1,p}_T(\partial\Omega)$ with $\alpha \in (0,1)$. Then, for all $\tau \in [0, T]$,*

$$\begin{aligned} &\|S * h\|_{\mathcal{V}^{\alpha,1,p}_\tau(\partial\Omega)} \\ &\leq \Big[M_0(T)(1 + \tau^\alpha) + M_1(T)\alpha^{-1}\Big] \int_0^\tau (\tau - s)^{-\alpha} \|h\|_{\mathcal{V}^{0,1,p}_s(\partial\Omega)}\, ds \\ &\leq C_1(\tau) \tau^{1-\alpha} \|h\|_{\mathcal{V}^{0,1,p}_\tau(\partial\Omega)}, \end{aligned} \tag{3.5}$$

C_1 being a positive, continuous and nondecreasing function in $[0, +\infty)$.

Proof. Let $0 \leq t_1 < t_2 \leq \tau \leq T$. Then we have

$$\begin{aligned} &\|S * h(t_2) - S * h(t_1)\|_X \leq \int_{t_1}^{t_2} \|S(t_2 - s)\|_{\mathcal{L}(X)} \|h(s)\|_X\, ds \\ &+ \int_{t_1}^{t_2} \|S(t_2 - s) - S(t_1 - s)\|_{\mathcal{L}(X)} \|h(s)\|_X\, ds =: \sum_{j=1}^{2} I_j(t_1, t_2). \end{aligned}$$

Observe now that

$$\begin{aligned} I_1(t_1, t_2) &\leq M_0(T) \int_{t_1}^{t_2} (t_2 - s)^\alpha (t_2 - s)^{-\alpha} \|h\|_{\mathcal{V}^{0,1,p}_s(\partial\Omega)}\, ds \\ &\leq M_0(T)(t_2 - t_1)^\alpha \int_0^{t_2} (t_2 - s)^{-\alpha} \|h\|_{\mathcal{V}^{0,1,p}_s(\partial\Omega)}\, ds \\ &\leq M_0(T)(t_2 - t_1)^\alpha \int_0^\tau (\tau - s)^{-\alpha} \|h\|_{\mathcal{V}^{0,1,p}_s(\partial\Omega)}\, ds \\ I_2(t_1, t_2) &\leq M_1(T) \int_0^{t_1} \|h\|_{\mathcal{V}^{0,1,p}_s(\partial\Omega)} ds \int_{t_1 - s}^{t_2 - s} t^{-1}\, dt \end{aligned}$$

$$\leq M_1(T)\int_0^{t_1}(t_1-s)^{-\alpha}\|h\|_{\mathcal{V}_s^{0,1,p}(\partial\Omega)}\,ds\int_{t_1-s}^{t_2-s}t^{-1+\alpha}\,dt$$
$$\leq M_1(T)\alpha^{-1}(t_2-t_1)^{\alpha}\int_0^{\tau}(\tau-s)^{-\alpha}\|h\|_{\mathcal{V}_s^{0,1,p}(\partial\Omega)}\,ds.$$

Hence the seminorm of $S*h$ in $\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)$ satisfies

$$[S*h]_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}\leq\Big[M_0(T)+M_1(T)\alpha^{-1}\Big]\int_0^{\tau}(\tau-s)^{-\alpha}\|h\|_{\mathcal{V}_s^{0,1,p}(\partial\Omega)}\,ds. \quad (3.6)$$

Then, from (3.6) and

$$\begin{aligned}\|S*h\|_{\mathcal{V}_\tau^{0,1,p}(\partial\Omega)} &\leq M_0(T)\int_0^{t}(t-s)^{\alpha}(t-s)^{-\alpha}\|h\|_{\mathcal{V}_s^{0,1,p}(\partial\Omega)}\,ds\\ &\leq M_0(T)\tau^{\alpha}\int_0^{\tau}(\tau-s)^{-\alpha}\|h\|_{\mathcal{V}_s^{0,1,p}(\partial\Omega)}\,ds,\end{aligned}$$

we derive (3.5). □

Lemma 3.3. *Let $h\in\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)$ with $\alpha\in(0,1)$ and $h(0)=0$. Then, for all $\tau\in[0,T]$,*

$$\|D_t(S*h)\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}+\|B(S*h)\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}\leq C_2(\tau)\|h\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)},$$

C_2 being a positive, continuous and nondecreasing function in $[0,+\infty)$.

Proof. We recall now that the following formulae hold true (cf. [24]):

$$D_t\int_0^t S(t-s)h(s)\,ds=\int_0^t S'(t-s)[h(s)-h(t)]\,ds+S(t)h(t),$$
$$B\int_0^t S(t-s)h(s)\,ds=\int_0^t S'(t-s)[h(s)-h(t)]\,ds+S(t)h(t)-h(t).$$

Consequently,

$$D_t(S*h)(t)=\int_0^t S'(t-s)[h(s)-h(t)]\,ds+S(t)h(t)=:\sum_{j=1}^{2}H_jh(t).$$

From well-known results in [24], taking the assumption $h(0)=0$ into account, we easily deduce that $L_1h\in\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)$ and satisfies the estimate

$$\|H_1h\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}\leq C_3(\tau)\widetilde{M}(T)\|h\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)},$$

where C_3 and $\widetilde{M}$ denote two positive, continuous and nondecreasing functions in $[0,+\infty)$.

Then, to estimate H_2h in $\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)$ we need the identity

$$\begin{aligned} S(t_2)h(t_2) - S(t_1)h(t_1) &= S(t_2)[h(t_2) - h(t_1)] + [S(t_2) - S(t_1)]h(t_1) \\ &=: \sum_{j=1}^{2} S_j(t_1, t_2). \end{aligned}$$

From estimate (3.4) with $j = 0$, we get

$$\|S_1(t_1, t_2)\|_X \le M_0(T)\|h\|_{C^\alpha([0,\tau];X)}(t_2 - t_1)^\alpha, \quad 0 \le t_1 \le t_2 \le \tau.$$

From the assumption $h(0) = 0$ and estimate (3.4) with $j = 1$, we obtain

$$\begin{aligned} \|S_2(t_1, t_2)\|_X &\le \|S(t_2) - S(t_1)\|_{\mathcal{L}(X)}|h(t_1) - h(0)| \\ &\le \int_{t_1}^{t_2} \|S'(s)\|_{\mathcal{L}(X)}|h(t_1) - h(0)|ds \\ &\le M_1(T)\|h\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \int_{t_1}^{t_2} s^{-1}t_1^\alpha ds \\ &\le M_1(T)\|h\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \int_{t_1}^{t_2} s^{\alpha-1} ds. \\ &\le \frac{M_1(T)}{\alpha}\|h\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}(t_2 - t_1)^\alpha, \quad 0 \le t_1 \le t_2 \le \tau. \end{aligned}$$

Likewise we can deduce the estimate for $\|B(S * h)\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}$. □

Lemma 3.4. *Let $\zeta_0 \in D(B) = W^{2-1/p,p}(\partial\Omega)$. Then, for all $\alpha \in (0,1)$, and $\tau \in (0, T]$,*

$$\|[S(\cdot) - I]\zeta_0\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \le M_0(T)\tau^{1-\alpha}(1 + \tau^\alpha)\|B\zeta_0\|_X. \tag{3.7}$$

Proof. To prove (3.7), we need the following identities:

$$\begin{aligned} \sigma(t) &:= [S(t) - I]\zeta_0 = [S(t) - S(0)]\zeta_0, \quad t \in [0, \tau], \\ \sigma(t_2) - \sigma(t_1) &= [S(t_2) - S(t_1)]\zeta_0, \quad 0 \le t_1 \le t_2 \le \tau. \end{aligned}$$

We observe that (3.7) immediately follows from the chain of estimates

$$\begin{aligned} &\|\sigma(t_2) - \sigma(t_1)\|_X \le \Big\| \int_{t_1}^{t_2} S'(s)\zeta_0 \, ds\Big\|_X = \Big\| \int_{t_1}^{t_2} S(s)B\zeta_0 \, ds\Big\|_X \\ &\le M_0(T)(t_2 - t_1)\|B\zeta_0\|_X \le M_0(T)(t_2 - t_1)^\alpha\tau^{1-\alpha}\|B\zeta_0\|_X, \\ &\qquad\qquad\qquad 0 \le t_1 \le t_2 \le \tau. \quad \square \end{aligned}$$

We now introduce the following space for our vector unknown (z,q) in problem (1.33), (1.34) (cf. the proof of Proposition 1.6 in [11]):

$$Z_T^\alpha \times Q_T^\alpha := [C^{1+\alpha}([0,T];W^{1-1/p,p}(\partial\Omega)) \cap C^\alpha([0,T];W^{2-1/p,p}(\partial\Omega))] \times C^\alpha([0,T]), \tag{3.8}$$

endowed with the natural norm:

$$\|z\|_{Z_T^\alpha} = \|z\|_{\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)} + \|D_t z\|_{\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)} + \|Bz\|_{\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)},$$
$$\|q\|_{Q_T^\alpha} = \|q\|_{C^\alpha([0,T])}.$$

Remark 3.1. According to the definitions of $N_1(w,h)$, $h = h_0 + q$, and $N_2(q,\mathbf{d})$ in (1.25) and of $N_5(z,q,\mathbf{d})$ in (1.35) and well-known results in [24], we easily deduce, for $(w,h,\mathbf{d},a) \in Z_T^\alpha \times Q_T^\alpha \times \mathbf{D}_T^{\alpha,\varepsilon} \times \mathbf{R}$:

$$N_1(w,h)(0) = D_t N_1(w,h)(0) = N_2(q,\mathbf{d})(0) = D_t N_2(q,\mathbf{d})(0) = 0,$$
$$N_5(z,q,\mathbf{d})(0) = 0.$$

Remark 3.2. Note that $N_1(w,h)$, $h = h_0 + q$, and $N_2(q,\mathbf{d})$ satisfy, respectively, the following Cauchy problems, $j = 1,2$:

$$\begin{cases} u'(t) = Bu(t) + f_j(t) & \text{in } \Omega, \\ u(0) = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$f_1 = h * D_t w + b_i h * D_{\nu_A} L_0 w, \quad f_2 = b_i q * D_{\nu_A} L_1 f.$$

Lemma 3.5. *Let the data $\mathbf{d}_j = (f_j, u_{0,j}, l_j, g_j)$ satisfy the assumptions H_j1–H_j8, $j = 1,2$. Let $(z_j,q_j) \in Z_T^\alpha \times Q_T^\alpha$, $j = 1,2$, $\alpha \in (0,1)$. Then*

$$\begin{aligned} &\|N_1(z_2,q_2) - N_1(z_1,q_1)\|_{Z_\tau^\alpha} \\ &\le C_5(\tau) \int_0^\tau \Big\{ \|q_2 - q_1\|_{C^0([0,s])} \Big[\|D_t z_1\|_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)} + \|Bz_1\|_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)} \Big] \\ &\quad + \|q_2\|_{C^0([0,s])} \Big[\|D_t(z_2 - z_1)\|_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)} + \|B(z_2 - z_1)\|_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)} \Big] \Big\}\, ds, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\|N_2(q_2,\mathbf{d}_2) - N_2(q_1,\mathbf{d}_1)\|_{Z_\tau^\alpha} \le C_6(\tau) \|\mathbf{d}_2\| \int_0^\tau \|(q_2 - q_1)\|_{C^0([0,s])}\, ds \\ &\quad + C_6(T) \|\mathbf{d}_2 - \mathbf{d}_1\| \int_0^\tau \|q_1\|_{C^0([0,s])}\, ds, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\|N_5(z_2,q_2,\mathbf{d}_2) - N_5(z_1,q_1,\mathbf{d}_1)\|_{C^\alpha([0,\tau])} \le C_7(\tau)\mu \\ &\times \int_0^\tau \Big\{ \big[\|\mathbf{d}_2 - \mathbf{d}_1\| + \|q_2 - q_1\|_{C^0([0,s])} \big] \big[\|D_t z_2\|_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)} + \|Bz_2\|_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)} \big] \end{aligned}$$

$$+ \left[\|\mathbf{d}_1\| + \|q_1\|_{C^0([0,s])} \right] \left[\|D_t(z_2 - z_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)} + \|B(z_2 - z_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)} \right]$$

$$+ \|q_2 - q_1\|_{C^0([0,s])} \|\mathbf{d}_2\| + \|q_1\|_{C^0([0,s])} \|\mathbf{d}_2 - \mathbf{d}_1\| \Big\} \, ds$$

$$+ \mu^2 \|\mathbf{d}_2 - \mathbf{d}_1\| C_8(\tau) \int_0^\tau \Big\{ \left[\|\mathbf{d}_2\| + \|q_2\|_{C^0([0,s])} \right]$$

$$\times \left[\|D_t z_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)} + \|B z_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)} \right] + \|q_2\|_{C^0([0,s])} \|\mathbf{d}_2\| \Big\} \, ds, \qquad (3.11)$$

C_j, $j = 5, ..., 8$, *being positive, continuous and nondecreasing functions in* $[0, +\infty)$.

Proof. From the identities

$$\begin{aligned}
& N_1(z_2, q_2)(t) - N_1(z_1, q_1)(t) \\
= \; & \int_0^t S(t-s) \big\{ (q_2 - q_1) * D_t z_1 - (1-i)(q_2 - q_1) * B z_1 \big\} \, ds \\
& + \int_0^t S(t-s) \big\{ q_2 * D_t(z_2 - z_1) - (1-i) q_2 * B(z_1 - z_2) \big\} \, ds, \\
& D_t N_1(z_2, q_2)(t) - D_t N_1(z_1, q_1)(t) \\
= \; & D_t \int_0^t S(t-s) \big\{ (q_2 - q_1) * D_t z_1 - (1-i)(q_2 - q_1) * B z_1 \big\} \, ds \\
& + D_t \int_0^t S(t-s) \big\{ q_2 * D_t(z_2 - z_1) - (1-i) q_2 * B(z_1 - z_2) \big\} \, ds, \\
& B N_1(z_2, q_2)(t) - B N_1(z_1, q_1)(t) \\
= \; & B \int_0^t S(t-s) \big\{ (q_2 - q_1) * D_t z_1 - (1-i)(q_2 - q_1) * B z_1 \big\} \, ds \\
& + B \int_0^t S(t-s) \big\{ q_2 * D_t(z_2 - z_1) - (1-i) q_2 * B(z_1 - z_2) \big\} \, ds,
\end{aligned}$$

and from Lemmata 3.1, 3.2 and 3.3 we get

$$\begin{aligned}
& \|N_1(z_2, q_2) - N_1(z_1, q_1)\|_{Z^\alpha_\tau} \\
\le \; & C_1(\tau) \tau^{1-\alpha} \Big[\|(q_2 - q_1) * D_t z_1\|_{C^0([0,\tau],X)} + \|(q_2 - q_1) * B z_1\|_{C^0([0,\tau],X)} \\
& + \|q_2 * D_t(z_2 - z_1)\|_{C^0([0,\tau],X)} + \|q_2 * B(z_1 - z_2)\|_{C^0([0,\tau],X)} \Big] \\
& + C_2(\tau) \Big[\|(q_2 - q_1) * D_t z_1\|_{\mathcal{V}^{\alpha,1,p}_\tau(\partial\Omega)} + \|(q_2 - q_1) * B z_1\|_{\mathcal{V}^{\alpha,1,p}_\tau(\partial\Omega)} \\
& + \|q_2 * D_t(z_2 - z_1)\|_{\mathcal{V}^{\alpha,1,p}_\tau(\partial\Omega)} + \|q_2 * B(z_1 - z_2)\|_{\mathcal{V}^{\alpha,1,p}_\tau(\partial\Omega)} \Big]
\end{aligned}$$

$$\leq C_5(\tau)\int_0^\tau \Big\{\|q_2-q_1\|_{C^0([0,s])}\Big[\|D_t z_1\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}+\|Bz_1\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\Big]$$
$$+\|q_2\|_{C^0([0,s])}\Big[\|D_t(z_2-z_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}+\|B(z_2-z_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\Big]\Big\}\,ds.$$

Likewise, from the identities

$$N_2(q_2,\mathbf{d}_2)(t)-N_2(q_1,\mathbf{d}_1)(t)=\int_0^t S(t-s)b_i(q_2-q_1)*D_{\nu_A}L_1f_2(s)\,ds$$
$$+\int_0^t S(t-s)b_iq_1*D_{\nu_A}L_1(f_2-f_1)(s)\,ds,$$
$$D_tN_2(q_2,\mathbf{d}_2)(t)-D_tN_2(q_1,\mathbf{d}_1)(t)$$
$$=D_t\int_0^t S(t-s)b_i(q_2-q_1)*D_{\nu_A}L_1f_2(s)\,ds$$
$$+D_t\int_0^t S(t-s)b_iq_1*D_{\nu_A}L_1(f_2-f_1)(s)\,ds$$
$$BN_2(q_2,\mathbf{d}_2)(t)-BN_2(q_1,\mathbf{d}_1)(t)$$
$$=B\int_0^t S(t-s)b_i(q_2-q_1)*D_{\nu_A}L_1f_2(s)\,ds$$
$$+B\int_0^t S(t-s)b_iq_1*D_{\nu_A}L_1(f_2-f_1)(s)\,ds,$$

we obtain

$$\|N_2(q_2,\mathbf{d}_2)(t)-N_2(q_1,\mathbf{d}_1)\|_{Z_T^\alpha}$$
$$\leq C_6(\tau)\int_0^\tau\Big[\|(q_2-q_1)\|_{C^0([0,s],X)}\|D_{\nu_A}L_1f_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}$$
$$+\|q_1\|_{C^0([0,s])}\|D_{\nu_A}L_1(f_2-f_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\Big]\,ds$$
$$\leq C_6(\tau)|\!|\!|\mathbf{d}_2|\!|\!|\int_0^\tau\|(q_2-q_1)\|_{C^0([0,s])}\,ds+C_6(\tau)|\!|\!|\mathbf{d}_2-\mathbf{d}_1|\!|\!|\int_0^\tau\|q_1\|_{C^0([0,s])}\,ds.$$

Then, from (1.35), we get

$$\begin{aligned}&N_5(z_2,q_2,\mathbf{d}_2)(t)-N_5(z_1,q_1,\mathbf{d}_1)(t)\\ =\;&(\chi_2-\chi_1)\widetilde{K}(z_2,q_2,\mathbf{d}_2)(t)+\chi_1\Big[\widetilde{K}(z_2,q_2,\mathbf{d}_2)(t)-\widetilde{K}(z_1,q_1,\mathbf{d}_1)(t)\Big].\end{aligned}$$

From the definition of operator $\widetilde{K}$ (cf. (1.35) and Lemmata 3.1 and 3.3) we easily obtain the estimates

$$\|\widetilde{K}(z_2,q_2,\mathbf{d}_2)-\widetilde{K}(z_1,q_1,\mathbf{d}_1)\|_{C^\alpha([0,\tau])}$$

$$
\begin{aligned}
&\le C_9(\tau)\int_0^\tau \Big\{\big[|h_{0,2}-h_{0,1}|+\|q_2-q_1\|_{C^0([0,s])}\big]\|D_t z_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\\
&\quad+\big[|h_{0,2}-h_{0,1}|+\|q_2-q_1\|_{C^0([0,s])}\big]\|Bz_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\\
&\quad+\big[|h_{0,1}|+\|q_1\|_{C^0([0,s])}\big]\Big[\|D_t(z_2-z_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}+\|B(z_2-z_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\Big]\\
&\quad+\|q_2-q_1\|_{C^0([0,s])}\big[\|f_2\|_{\mathcal{U}^{1+\alpha,0,p}_{\tau-s}(\Omega)}+\|Bu_{0,2}+g_{0,2}(0)\|_X\big]\\
&\quad+\|q_2-q_1\|_{C^\alpha([0,s])}\|B[Bu_{0,2}+g_{0,2}(0)]\|_X\\
&\quad+\|q_1\|_{C^0([0,s])}\big[\|f_2-f_1\|_{\mathcal{U}^{1+\alpha,0,p}_{\tau-s}(\Omega)}+\|B(u_{0,2}-u_{0,1})+(g_{0,2}-g_{0,1})(0)\|_X\big]\\
&\quad+\|q_1\|_{C^\alpha([0,s])}\|B[(u_{0,2}-u_{0,1})+(g_{0,2}-g_{0,1})(0)]\|_X\\
&\quad+\|q_2-q_1\|_{C^0([0,s])}\|SBu_{0,2}\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\\
&\quad+\|q_1\|_{C^0([0,s])}\|SB(u_{0,2}-u_{0,1})\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\Big\}\,ds\\
&\le C_{10}(\tau)\int_0^\tau \Big\{\big[\|\mathbf{d}_2-\mathbf{d}_1\|+\|q_2-q_1\|_{C^0([0,s])}\big]\|D_t z_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\\
&\quad+\big[\|\mathbf{d}_2-\mathbf{d}_1\|+\|q_2-q_1\|_{C^0([0,s])}\big]\|Bz_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\\
&\quad+\big[\|\mathbf{d}_1\|+\|q_1\|_{C^0([0,s])}\big]\Big[\|D_t(z_2-z_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}+\|B(z_2-z_1)\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\Big]\\
&\quad+\|q_2-q_1\|_{C^0([0,s])}\|\mathbf{d}_2\|+\|q_1\|_{C^0([0,s])}\|\mathbf{d}_2-\mathbf{d}_1\|\Big\}\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
\|\widetilde{K}(z_2,q_2,\mathbf{d}_2)\|_{C^\alpha([0,\tau])}\le C_{11}(\tau)\int_0^\tau\Big\{&\Big[\|\mathbf{d}_2\|+\|q_2\|_{C^0([0,s])}\Big]\\
&\times\Big[\|D_t z_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}+\|Bz_2\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\Big]+\|q_2\|_{C^0([0,s])}\|\mathbf{d}_2\|\Big\}\,ds.
\end{aligned}
$$

Consequently we deduce estimate (3.11). □

The following Lemma 3.6 estimates the function ζ_0 defined by (1.33).

Lemma 3.6. *Let* $\mathbf{d}=(f,u_0,l,g)\in\mathbf{D}^{\alpha,\varepsilon}_T$ *and let* $\alpha\in(0,1)$ *and* $\varepsilon\in(0,\min\{\alpha,1-\alpha\})$. *Then the function* ζ_0, *defined in (1.33), belongs to* $Z^\alpha_T(\partial\Omega)$ *and satisfies the estimate*

$$
\begin{aligned}
&\|\zeta_0\|_{Z^\alpha_\tau}:=\|\zeta_0\|_{\mathcal{V}^{\alpha,1,p}_T(\partial\Omega)}+\|D_t\zeta_0\|_{\mathcal{V}^{\alpha,1,p}_T(\partial\Omega)}+\|B\zeta_0\|_{\mathcal{V}^{\alpha,1,p}_T(\partial\Omega)}\\
\le\ &C_{12}(\tau)\tau^\varepsilon[\|B(Bu_0+g_0(0))\|_X+[g_0]_{\mathcal{V}^{\alpha+\varepsilon,1,p}_T(\partial\Omega)}\\
&+\|Bu_0+g_0(0)\|_X+\|Bu_0\|_X], \qquad (3.12)
\end{aligned}
$$

where (cf. (1.31) and (2.5))

$$g_0(s) = g(s) - bD_{\nu_A}L_1 f(s) - h_0 b_i(1 * D_{\nu_A}L_1 f(s)) - h_0 u_0,$$

and C_{12} is a positive, continuous and nondecreasing function on $[0,+\infty)$.

Proof. From the formula

$$\begin{aligned}\zeta_0(t) &= -t[Bu_0 + g_0(0)] + [S(t) - I]u_0 + \int_0^t S(t-s)[g_0(s) - g_0(0)]\,ds \\ &\quad + \int_0^t S(t-s)g_0(0)\,ds - N_1(u_0 + t[Bu_0 + g_0(0)], h_0)(t) \\ &= -t[Bu_0 + g_0(0)] + [S(t) - I]u_0 + \int_0^t S(t-s)g_0(s)\,ds \\ &\quad - \int_0^t sS(t-s)\big\{h_0[Bu_0 + g_0(0)] \\ &\qquad - (1-i)h_0 B[u_0 + s/2(Bu_0 + g_0(0))]\big\}\,ds, \end{aligned} \tag{3.13}$$

we get

$$\begin{aligned} D_t\zeta_0(t) &= [S(t) - I][Bu_0 + g_0(0)] + D_t\int_0^t S(t-s)[g_0(s) - g_0(0)]\,ds \\ &\quad - D_t N_1(u_0 + t[Bu_0 + g_0(0)], h_0)(t), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} &D_t N_1(u_0 + t[Bu_0 + g_0(0)], h_0)(t) \\ = &\int_0^t S(s)\big\{h_0[Bu_0 + g_0(0)] - (1-i)h_0 B[u_0 + s/2(Bu_0 + g_0(0))]\big\}\,ds. \end{aligned}$$

We have used here, for all $j \in \mathbf{N}$, the identities

$$D_t\int_0^t s^j S(t-s)\,ds = D_t\int_0^t (t-s)S(s)\,ds = j\int_0^t s^{j-1}S(t-s)\,ds.$$

From (3.13) we easily deduce the following identity:

$$\begin{aligned} B\zeta_0(t) &= -tB[Bu_0 + g_0(0)] + [S(t) - I]Bu_0 \\ &\quad + B\int_0^t S(t-s)[g_0(s) - g_0(0)]\,ds + B\int_0^t S(s)g_0(0)\,ds \\ -B&\int_0^t sS(t-s)\big\{h_0[Bu_0 + g_0(0)] - (1-i)h_0 B[u_0 + s/2(Bu_0 + g_0(0))]\big\}\,ds \end{aligned}$$

$$
\begin{aligned}
&= -tB[Bu_0 + g_0(0)] + [S(t) - I]Bu_0 + D_t \int_0^t S(t-s)[g_0(s) - g_0(0)]\, ds \\
&\quad - [g_0(t) - g_0(0)] + [S(t) - I]g_0(0) \\
&\quad - \int_0^t S(s)\big\{h_0[Bu_0 + g_0(0)] - (1-i)h_0B[u_0 + s/2(Bu_0 + g_0(0))]\big\}\, ds \\
&\quad - t\big\{h_0[Bu_0 + g_0(0)] - (1-i)h_0B[u_0 + t/2(Bu_0 + g_0(0))]\big\} \\
&= -tB[Bu_0 + g_0(0)] - t\big\{h_0[Bu_0 + g_0(0)] \\
&\quad - (1-i)h_0B[u_0 + t/2(Bu_0 + g_0(0))]\big\} + D_t\zeta_0(t) := \zeta_1(t) + D_t\zeta_0(t).
\end{aligned}
$$

Recalling that $\zeta_0(\tau) = \int_0^\tau D_t\zeta_0(s)\, ds$, we easily get the estimate

$$
\|\zeta_0\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \le \tau^{1-\alpha}(1+\tau^\alpha)\|D_t\zeta_0\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}. \tag{3.15}
$$

Consequently, from formula (3.15) we deduce

$$
\|\zeta_0\|_{Z_\tau^\alpha} \le [2 + \tau^{1-\alpha}(1+\tau^\alpha)]\|D_t\zeta_0\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} + \|\zeta_1\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}. \tag{3.16}
$$

Then from Lemma 3.5 we obtain

$$
\begin{aligned}
&\|D_tN_1(u_0 + t[Bu_0 + g_0(0)], h_0)\|_{\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)} \\
\le\ & C_5(\tau)\int_0^\tau |h_0|[\|Bu_0 + g_0(0)\|_X + \|Bu_0\|_X \\
&\qquad + (\tau - s)^{1-\alpha}\|B(Bu_0 + g_0(0))\|_X]\, ds \\
\le\ & C_{13}(\tau)\tau[\|Bu_0 + g_0(0)\|_X + \|Bu_0\|_X + T^{1-\alpha}\|B(Bu_0 + g_0(0))\|_X].
\end{aligned} \tag{3.17}
$$

As a consequence, from (3.14), applying Lemmata 3.3 and 3.4 and observing that

$$
\|g_0 - g_0(0)\|_{\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)} \le \tau^\varepsilon(1+T^\alpha)[g_0]_{\mathcal{V}_T^{\alpha+\varepsilon,1,p}(\partial\Omega)},
$$

we get

$$
\begin{aligned}
\|D_t\zeta_0\|_{\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)} \le\ & M_0(T)\tau^{1-\alpha}(1+\tau^\alpha)\|B(Bu_0 + g_0(0))\|_X + C_2(\tau)\tau^\varepsilon \\
&\times (1+T^\alpha)[g_0]_{\mathcal{V}_T^{\alpha+\varepsilon,1,p}(\partial\Omega)} + C_{13}(\tau)\tau[\|Bu_0 + g_0(0)\|_X \\
&+ \|Bu_0\|_X + T^{1-\alpha}\|B(Bu_0 + g_0(0))\|_X].
\end{aligned} \tag{3.18}
$$

Indeed,

$$
\begin{aligned}
&\|\lambda - \lambda(0)\|_{C^\alpha([0,\tau])} = \|\lambda - \lambda(0)\|_{C^0([0,\tau])} + [\lambda - \lambda(0)]_{C^\alpha([0,\tau])} \\
&\le \tau^{\alpha+\varepsilon}[\lambda]_{C^{\alpha+\varepsilon}([0,\tau])} + \tau^\varepsilon[\lambda]_{C^{\alpha+\varepsilon}([0,\tau])} = \tau^\varepsilon(1+\tau^\alpha)[\lambda]_{C^{\alpha+\varepsilon}([0,\tau])}.
\end{aligned} \tag{3.19}
$$

Moreover,

$$\|\zeta_1\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \le \tau^{1-\alpha}(1+|h_0|)[\|B(Bu_0+g_0(0))\|_X + \|Bu_0+g_0(0)\|_X + \|Bu_0\|_X + T\|B(Bu_0+g_0(0))\|_X]. \tag{3.20}$$

From estimates (3.16), (3.18) and (3.20) we easily deduce (3.12). □

The following Lemma 3.7 estimates function κ_0 defined in (1.34).

Lemma 3.7. *Let* $\mathbf{d} = (f, u_0, l, g) \in \mathbf{D}_T^{\alpha,\varepsilon}$ *and let* $\alpha \in (0,1)$ *and* $\varepsilon \in (0, \min\{\alpha, 1-\alpha\})$. *Then the function* κ_0 *belongs to* Q_T^α *and satisfies the estimate*

$$\|\kappa_0\|_{C^\alpha([0,\tau])} \le C_{14}(\tau)\tau^\varepsilon\Big[\ [l]_{C^{1+\varepsilon+\alpha}([0,T])} + \|B(Bu_0+g_0(0))\|_X + [g_0]_{\mathcal{V}_T^{\alpha+\varepsilon,1,p}(\partial\Omega)} + \|Bu_0+g_0(0)\|_X + \|Bu_0\|_X + [f]_{\mathcal{U}_\tau^{1+\varepsilon+\alpha,0,p}(\Omega)}\Big], \tag{3.21}$$

where C_{14} *is a positive, continuous and nondecreasing function in* $[0,+\infty)$.

Proof. First we recall that (cf. (1.25), (1.25))

$$D_t\widetilde{\zeta}(t) = D_t\zeta(t) + h_0 S(t)u_0. \tag{3.22}$$

Consequently

$$\begin{aligned}\kappa_0(t) \;=\; &\chi\Big[l'(t) - \Phi[L_0 D_t\zeta(t) + h_0 L_0 S(t)u_0] \\ &+ h_0\Phi[1 * L_0 S(\cdot)Bu_0] - \Phi[L_1 D_t f(t)]\Big] \\ &- h_0 - \chi h_0 * \Phi[L_0(Bu_0+g_0(0))] - \chi h_0 * \Phi[L_1 D_t f(t)] \\ &+ \chi\Phi[L_0 D_t N_1(u_0 + t[Bu_0+g_0(0)], h_0)(t)].\end{aligned}$$

From (2.5) and (3.22), we immediately obtain the formulae

$$D_t\widetilde{\zeta}(0) = D_t\zeta(0) + h_0u_0 = -bD_{\nu_A}u_0 + g(0) - bD_{\nu_A}L_1 f(0), \tag{3.23}$$

$$\begin{aligned}\kappa_0(0) &= \chi\Big\{l'(0) - \Phi[L_0(D_t\zeta(0)+h_0u_0)] - \Phi[L_1 D_t f(0)]\Big\} - h_0 \\ &= \chi\Big\{l'(0) + \Phi[L_0 bD_{\nu_A}u_0 - L_0 g(0) + L_0 bD_{\nu_A}L_1 f(0)] - \Phi[L_1 D_t f(0)]\Big\} \\ &\quad - h_0 = 0.\end{aligned} \tag{3.24}$$

Then, from formulae (1.34) and (3.24) we easily deduce the following representation for κ_0:

$$\begin{aligned}\kappa_0(t) &= \kappa_0(t) - \kappa_0(0) \\ &= \chi\Big\{l'(t) - l'(0) - \Phi[L_0(D_t\zeta(t) - D_t\zeta(0))] - h_0\Phi[L_0(S(t)-I)u_0]\end{aligned}$$

$$+h_0\Phi[1 * L_0 S(\cdot)Bu_0] - \Phi[L_1(D_t f(t) - D_t f(0))]\Big\} - \chi t h_0 \Phi[L_0(Bu_0 + g_0(0))]$$
$$-\chi h_0 * \Phi[L_1 D_t f(t)] + \chi\Phi[L_0 D_t N_1(u_0 + t[Bu_0 + g_0(0)], h_0)(t)]. \quad (3.25)$$

Moreover, we recall (cf. (1.33)) that

$$\zeta(t) = \zeta_0(t) + u_0 + t[Bu_0 + g_0(0)] + N_1(u_0 + t[Bu_0 + g_0(0)], h_0)(t). \quad (3.26)$$

Since $\zeta_0(0) = D_t N_1(u_0 + t[Bu_0 + g_0(0)]$ and $h_0)(0) = 0$, (3.26) implies

$$D_t\zeta(t) - D_t\zeta(0) = D_t\zeta_0(t) + D_t N_1(u_0 + t[Bu_0 + g_0(0)], h_0)(t).$$

Whence, taking advantage of formula (3.25), Lemmata 3.2, 3.4 and 3.5 and estimate (3.17), we obtain the following estimate, from which (3.21) easily follows:

$$\begin{aligned}
\|\kappa_0\|_{C^\alpha([0,\tau])} &\le C_{15}(\tau)\mu\Big[\|l' - l'(0)\|_{C^\alpha([0,\tau])} + \|D_t\zeta - D_t\zeta(0)]\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \\
&\quad + \|D_t f - D_t f(0)]\|_{\mathcal{U}_\tau^{\alpha,0,p}(\Omega)} + \tau^{1-\alpha}(1+\tau^\alpha)\|Bu_0\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \\
&\quad + \tau^{2-\alpha}\|Bu_0\|_X + \tau^{1-\alpha}(1+\tau^\alpha)\|Bu_0 + g_0(0))\|_X \\
&\quad + \tau^{1-\alpha}(1+\tau^\alpha)\|D_t f\|_{\mathcal{U}_T^{\alpha,0,p}(\Omega)} + \|D_t N_1(u_0 + t[Bu_0 + g_0(0)], h_0)\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)}\Big] \\
&\le C_{16}(\tau)\mu + \Big[\tau^\varepsilon[l]_{C^{1+\varepsilon+\alpha}([0,T])} + \tau^\varepsilon\|B(Bu_0 + g_0(0))\|_X + \tau^\varepsilon[g_0]_{\mathcal{V}_T^{\alpha+\varepsilon,1,p}(\partial\Omega)} \\
&\quad + \tau^\varepsilon\|Bu_0 + g_0(0)\|_X + \tau^\varepsilon\|Bu_0\|_X + \tau^\varepsilon\|Bu_0 + g_0(0)\|_X + \tau^\varepsilon\|Bu_0\|_X \\
&\quad + \tau^\varepsilon[f]_{\mathcal{U}_\tau^{1+\varepsilon+\alpha,0,p}(\Omega)} + \tau^{1-\alpha}\|Bu_0\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \\
&\quad + \tau^{2-\alpha}\|Bu_0\|_X + \tau^{1-\alpha}\|Bu_0 + g_0(0))\|_X + \tau^{1-\alpha}\|f\|_{\mathcal{U}_T^{1+\alpha,0,p}(\Omega)} \\
&\quad + \tau\|Bu_0 + g_0(0)\|_X + \tau\|Bu_0\|_X + \tau T^{1-\alpha}\|B(Bu_0 + g_0(0))\|_X\Big]. \quad \square
\end{aligned}$$

Lemma 3.8. *Let the data* $\mathbf{d}_j = (f_j, u_{0,j}, l_j, g_j)$ *satisfy assumptions* H_j1–H_j8, $j = 1, 2$, *and set*

$$\begin{aligned}
\zeta_{0,j}(t) &:= -t[Bu_{0,j} + g_{0,j}(0)] + [S(t) - I]u_{0,j} + \int_0^t S(t-s)g_{0,j}(s)\,ds \\
&\qquad - N_1(u_{0,j} + t[Bu_{0,j} + g_{0,j}(0)], h_{0,j})(t), \\
\kappa_{0,j}(t) &:= \chi_j K_j(t),
\end{aligned}$$

where

$$\chi_j^{-1} := \Phi[L_1 f_j(0)],$$
$$g_{0,j}(s) = g_j(s) - bD_{\nu_A}L_1 f_j(s) - h_j(0)b_i(1 * D_{\nu_A}L_1 f_j(s)) - h_j(0)u_{0,j},$$
$$h_{0,j} = \chi_j\Big\{l_j'(0) + \Phi[L_0 bD_{\nu_A}L_1 f_j(0) + L_0 bD_{\nu_A}L_0 u_{0,j}]$$

$$- \Phi[L_0 g_j(0)] - \Phi[L_1 D_t f_j(0)]\Big\},$$

$$\begin{aligned} K_j(t) = {} & l_j'(t) - l_j'(0) - \Phi[L_0(D_t\zeta_j(t) - D_t\zeta_j(0))] - h_{0,j}\Phi[L_0(S(t) - I)u_{0,j}] \\ & + h_{0,j}\Phi[1 * L_0 S(\cdot)Bu_{0,j}] - \Phi[L_1(D_t f_j(t) - D_t f_j(0))] \\ & - h_{0,j} * \Phi[L_0(Bu_{0,j} + g_{0,j}(0))] - \chi_j h_{0,j} * \Phi[L_1 D_t f_j(t)] \\ & + \Phi[L_0 D_t N_1(u_{0,j} + t[Bu_{0,j} + g_{0,j}(0)], h_{0,j})(t)]. \end{aligned}$$

Let $\alpha \in (0,1)$ and $\varepsilon \in (0, \min\{\alpha, 1-\alpha\})$. Then the functions $\zeta_{0,2} - \zeta_{0,1}$ and $\kappa_{0,2} - \kappa_{0,1}$ satisfy the estimates

$$\|\zeta_{0,2} - \zeta_{0,1}\|_{Z_\tau^\alpha} \le \tau^\varepsilon \|\mathbf{d}_2 - \mathbf{d}_1\| C_{17}(\tau)[R_0(R_0\mu^2 + \mu + 1) + 1], \tag{3.27}$$

$$\|\kappa_{0,2} - \kappa_{0,1})\|_{C^\alpha([0,T])} \le \tau^\varepsilon \|\mathbf{d}_2 - \mathbf{d}_1\| C_{18}(\tau)[R_0(R_0\mu^2 + \mu + 1) + 1], \tag{3.28}$$

where $\|\mathbf{d}\|$ is defined in (2.8), $R_0 \ge \max_{j=1,2}\{\|\mathbf{d}_j\|\}$ and C_{17}, C_{18} are positive, continuous and nondecreasing functions in $[0, +\infty)$.

Proof. First we note that

$$\begin{aligned} & \zeta_{0,2}(t) - \zeta_{0,1}(t) \\ & = -t[B(u_{0,2} - u_{0,1}) + (g_{0,2} - g_{0,1})(0)] + [S(t) - I](u_{0,2} - u_{0,1}) \\ & \quad + \int_0^t S(t-s)(g_{0,2} - g_{0,1})(s)\,ds - N_1(u_{0,2} + t[Bu_{0,2} + g_{0,2}(0)], h_{0,2})(t) \\ & \quad + N_1(u_{0,1} + t[Bu_{0,1} + g_{0,1}(0)], h_{0,1})(t), \end{aligned} \tag{3.29}$$

$$\begin{aligned} & D_t\zeta_{0,2}(t) - D_t\zeta_{0,1}(t) = [S(t) - I][B(u_{0,2} - u_{0,1}) + g_{0,2}(0) - g_{0,1}(0)] \\ & \quad + D_t \int_0^t S(t-s)[g_{0,2}(s) - g_{0,2}(0) - g_{0,1}(s) + g_{0,1}(0)]\,ds \\ & \quad - D_t N_1(u_{0,2} + t[Bu_{0,2} + g_{0,2}(0)], h_{0,2})(t) \\ & \quad + D_t N_1(u_{0,1} + t[Bu_{0,1} + g_{0,1}(0)], h_{0,1})(t), \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} h_{0,2} - h_{0,1} = {} & (\chi_2 - \chi_1)\Big\{l_2'(0) + \Phi[L_0 b D_{\nu_A} L_1 f_2(0) + L_0 b D_{\nu_A} L_0 u_{0,2}] \\ & - \Phi[L_0 g_2(0)] - \Phi[L_1 D_t f_2(0)]\Big\} + \chi_1\Big\{(l_2' - l_1')(0) \\ & + \Phi[L_0 b D_{\nu_A}(L_1 f_2(0) - L_1 f_1(0)) + L_0 b D_{\nu_A}(L_0 u_{0,2} - L_0 u_{0,1})] \\ & - \Phi[L_0(g_2(0) - g_1(0))] - \Phi[L_1 D_t(f_2(0) - f_1(0))]\Big\}, \end{aligned}$$

and

$$\chi_2 - \chi_1 := -\chi_2\chi_1\Phi[L_1(f_2(0) - f_1(0))].$$

Consequently,

$$|\chi_2 - \chi_1| \le C\mu^2 \|\mathbf{d}_2 - \mathbf{d}_1\|, \qquad |h_{0,2} - h_{0,1}| \le C(R_0\mu^2 + \mu)\|\mathbf{d}_2 - \mathbf{d}_1\|.$$

With considerations similar to the ones used in Lemma 3.6, we get

$$\begin{aligned}\|\zeta_{0,2} - \zeta_{0,1}\|_{Z_\tau^\alpha} \le [2 + \tau^{1-\alpha}(1+\tau^\alpha)]\|D_t\zeta_{0,2} - D_t\zeta_{0,1}\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \\ + \|\zeta_{1,2} - \zeta_{1,1}\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)},\end{aligned}$$

where

$$\begin{aligned}\zeta_{1,j} \quad := \quad & -tB[Bu_{0,j} + g_{0,j}(0)] - t\big\{h_{0,j}[Bu_{0,j} + g_{0,j}(0)] \\ & +(1-i)h_{0,j}B[u_{0,j} + t/2(Bu_{0,j} + g_{0,j}(0))]\big\}, j = 1,2.\end{aligned}$$

Then from Lemma 3.5 we obtain

$$\begin{aligned}&\|D_tN_1(u_{0,2} + t[Bu_{0,2} + g_{0,2}(0)], h_{0,2}) \\ &\qquad - D_tN_1(u_{0,1} + t[Bu_{0,1} + g_{0,1}(0)], h_{0,1})\|_{\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)} \\ &\le C_5(\tau)\int_0^\tau |h_{0,2} - h_{0,1}|[\|Bu_{0,1} + g_{0,1}(0)\|_X + \|Bu_{0,1}\|_X \\ &\qquad + (\tau - s)^{1-\alpha}\|B(Bu_{0,1} + g_{0,1}(0))\|_X]\,ds \\ &\quad + C_5(\tau)\int_0^\tau |h_{0,2}|[\|B(u_{0,2} - u_{0,1}) + g_{0,2}(0) - g_{0,1}(0)\|_X + \|B(u_{0,2} - u_{0,1})\|_X \\ &\quad +(\tau - s)^{1-\alpha}\|B[B(u_{0,2} - u_{0,1}) + g_{0,2}(0) - g_{0,1}(0)]\|_X]\,ds \\ &\le C_{19}(\tau)\tau\|\mathbf{d}_2 - \mathbf{d}_1\|R_0(R_0\mu^2 + \mu + 1). \qquad (3.31)\end{aligned}$$

As a consequence, from (3.30), (3.31) and (3.19) we get

$$\|D_t\zeta_{0,2} - D_t\zeta_{0,1}\|_{\mathcal{V}_T^{\alpha,1,p}(\partial\Omega)} \le \tau^\varepsilon\|\mathbf{d}_2 - \mathbf{d}_1\|C_{20}(\tau)[R_0(R_0\mu^2 + \mu + 1) + 1].$$

Moreover, we have

$$\|\zeta_{1,2} - \zeta_{1,1}\|_{\mathcal{V}_\tau^{\alpha,1,p}(\partial\Omega)} \le \tau^{1-\alpha}CR_0(R_0\mu^2 + \mu + 1)\|\mathbf{d}_2 - \mathbf{d}_1\|.$$

By means of computations similar to the ones in the proof of Lemma 3.7 and by virtue of estimates (3.27), (3.31) we deduce (3.28). □

4. PROOFS OF THEOREMS 2.1 AND 2.2.

We will solve the fixed-point system (1.33), (1.34) for (z, q) in a *suitable closed subset* of a ball with center at $(z, q) = (0, 0)$ in the product space $Z_T^\alpha \times Q_T^\alpha$ (cf. (3.8)).

Proofs of Theorems 2.1 and 2.2. The assertions easily follow from Theorems 2.3, 2.4 and Theorem 4.1 stated below. □

Theorem 4.1. *Let* $\mathbf{d} = (f, u_0, l, g) \in \mathbf{D}_T^{\alpha,\varepsilon}$ *and* $\|\mathbf{d}\| \leq R_0$. *Then there exists a unique pair* $(z, q) \in Z_T^\alpha \times Q_T^\alpha$ *solving problem (1.33) and (1.34).*

Let $\mathbf{d}_j := (f_j, u_{0,j}, l_j, g_j)$ *satisfy assumptions* $H_j 1$–$H_j 6$, $j = 1, 2$, *and be such that* $\max_{j=1,2}\{\|\mathbf{d}_j\|\} \leq R_0$. *Denote by* $(z_j, q_j) \in Z_T^\alpha \times Q_T^\alpha$ *the solution to problem (1.33), (1.34) corresponding to the data* $(f_j, u_{0,j}, l_j, g_j)$, $j = 1, 2$. *Then*

$$\|z_2 - z_1\|_{Z_T^\alpha} + \|q_2 - q_1\|_{Q_T^\alpha} \leq C(R_0, T)\|\mathbf{d}_2 - \mathbf{d}_1\|.$$

Proof. We introduce the following complete metric space:

$$\mathcal{X}^\alpha := \{(z, q) \in Z_T^\alpha \times Q_T^\alpha : z(0) = 0,\ D_t z(0) = 0,\ q(0) = 0\}.$$

We recall (cf. (1.32)) that any solution (z, q) to system (1.33), (1.34) necessarily satisfies

$$z(0) = 0, \quad D_t z(0) = 0, \quad q(0) = 0,$$

and (cf. (1.34) and (3.24))

$$\kappa_0(0) = 0.$$

Let us now introduce the nonlinear operator

$$\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2) = (\zeta_0 - q * S(\cdot)u_0 - N_4, \kappa_0 + N_5).$$

We are going to prove that $\mathcal{N}$ maps $\mathcal{X}^\alpha$ into itself. For this purpose we need to estimate the operator N_4 (cf. (1.33)). We will use estimate (3.9), in Lemma 3.4, where (z_2, q_2) is any of the following three pairs: (z, h_0), $(u_0 + t[Bu_0 + g_0(0)], q)$, (z, q), while $(z_1, q_1) = (0, 0)$. Similarly we use estimate (3.10) with $(q_2, \mathbf{d}_2) = (q, \mathbf{d})$ and $(q_1, \mathbf{d}_1) = (0, \mathbf{0})$). Since $N_1(0, 0) = D_t N_1(0, 0) = 0$, $N_2(0, \mathbf{0}) = D_t N_2(0, \mathbf{0}) = 0$, we obtain

$$\begin{aligned}
&\|N_4(z, q, \mathbf{d})\|_{Z_\tau^\alpha} \leq \|N_1(z, h_0)\|_{Z^\alpha} + \|N_1(u_0 + t[Bu_0 + g_0(0)], q)\|_{Z_\tau^\alpha} \\
&\qquad\qquad + \|N_1(z, q)\|_{Z_\tau^\alpha} + \|N_2(q, \mathbf{d})\|_{Z_\tau^\alpha} \\
&= \|N_1(z, h_0) - N_1(0, 0)\|_{Z_\tau^\alpha} + \|N_1(u_0 + t[Bu_0 + g_0(0)], q) - N_1(0, 0)\|_{Z_\tau^\alpha} \\
&\quad + \|N_1(z, q) - N_1(0, 0)\|_{Z_\tau^\alpha} + \|N_2(q, \mathbf{d}) - N_2(0, \mathbf{0})\|_{Z_\tau^\alpha} \\
&\leq 2C_5(\tau) \int_0^\tau \Big\{ |h_0| \Big[\|D_t z\|_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)} + \|Bz\|_{\mathcal{V}_{\tau-s}^{\alpha,1,p}(\partial\Omega)} \Big] \Big\}\, ds \\
&\quad + 2C_5(\tau) \int_0^\tau \Big\{ \|q\|_{C^0([0,s])} \Big[\|Bu_0 + g_0(0)\|_X + \|Bu_0\|_X
\end{aligned}$$

$$+ \tau^{1-\alpha}\|B[Bu_0 + g_0(0)]\|_X\Big]\Big\}\, ds$$

$$+ 2C_5(\tau)\int_0^\tau \Big\{\|q\|_{C^0([0,s])}\Big[\|D_t z\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)} + \|Bz\|_{\mathcal{V}^{\alpha,1,p}_{\tau-s}(\partial\Omega)}\Big]\Big\}\, ds$$

$$+ 2C_6(\tau)\|\!|\mathbf{d}|\!\|\int_0^\tau \|q\|_{C^0([0,s])}\, ds$$

$$\le C_{19}(T)\int_0^\tau \big[\|q\|_{Q^\alpha_s}\|z\|_{Z^\alpha_{\tau-s}} + \|\!|\mathbf{d}|\!\|\big(\|z\|_{Z^\alpha_s} + \|q\|_{Q^\alpha_s}\big)\big]\, ds. \tag{4.1}$$

Likewise we obtain

$$\|N_5(z,q,\mathbf{d})\|_{C^\alpha([0,T])} = \|N_5(z,q,\mathbf{d}) - N_5(0,0,\mathbf{0})\|_{C^\alpha([0,T])}$$

$$\le \quad C_{20}(T)\int_0^\tau \big[\|q\|_{Q^\alpha_s}\|z\|_{Z^\alpha_{\tau-s}} + \|\!|\mathbf{d}|\!\|\big(\|z\|_{Z^\alpha_s} + \|q\|_{Q^\alpha_s}\big)\big]\, ds. \tag{4.2}$$

Moreover, from Lemma 3.2 we deduce the estimate

$$\|q * S(\cdot)u_0\|_{Z^\alpha_\tau} \le C_1(T)\|\!|\mathbf{d}|\!\|\int_0^\tau (\tau - s)^{-\alpha}\|q\|_{Q^\alpha_s}\, ds. \tag{4.3}$$

From (4.1), (4.3) and Lemma 3.6, we easily get

$$\|\mathcal{N}_1(z,q,\mathbf{d})\|_{Z^\alpha_\tau} \le C_{12}(T)\tau^\varepsilon\|\!|\mathbf{d}|\!\| + C_1(T)\|\!|\mathbf{d}|\!\|\int_0^\tau (\tau - s)^{-\alpha}\|q\|_{Q^\alpha_s}\, ds$$

$$+ C_{19}(T)\int_0^\tau \big[\|q\|_{Q^\alpha_s}\|z\|_{Z^\alpha_{\tau-s}} + \|\!|\mathbf{d}|\!\|\big(\|z\|_{Z^\alpha_s} + \|q\|_{Q^\alpha_s}\big)\, ds$$

$$\le C_{12}(T)\tau^\varepsilon\|\!|\mathbf{d}|\!\| + C_{21}(T)\|\!|\mathbf{d}|\!\|\int_0^\tau [1 + (\tau - s)^{-\alpha}]\big(\|z\|_{Z^\alpha_s} + \|q\|_{Q^\alpha_s}\big)\, ds$$

$$+ C_{22}(T)\int_0^\tau \|q\|_{Q^\alpha_s}\|z\|_{Z^\alpha_{\tau-s}}\, ds.$$

Moreover, from (4.2) and Lemma 3.7 we easily get

$$\|\mathcal{N}_2(z,q,\mathbf{d})\|_{Z^\alpha_\tau} \quad \le \quad C_{14}(T)\tau^\varepsilon\|\!|\mathbf{d}|\!\| + C_{20}(T)\|\!|\mathbf{d}|\!\|\int_0^\tau \big(\|z\|_{Z^\alpha_s} + \|q\|_{Q^\alpha_s}\big)\, ds$$

$$+ C_{20}(T)\int_0^\tau \|q\|_{Q^\alpha_s}\|z\|_{Z^\alpha_{\tau-s}}\, ds.$$

Now we introduce the following complete metric space:

$$\mathcal{Y}^\alpha := \{(z,q) \in Z^\alpha_T \times Q^\alpha_T : z(0) = 0,\ D_t z(0) = 0,\ q(0) = 0,$$
$$\|z\|_{Z^\alpha_\tau} \le \varphi_1(\tau), \quad \|q\|_{Q^\alpha_\tau} \le \varphi_2(\tau), \quad \tau \in (0,T]\},$$

where φ_i, $i = 1, 2$, will be determined by the requirement that $\mathcal{N}$ should map $\mathcal{Y}^\alpha$ into itself. Observe now that $\mathcal{N}$ satisfies in $\mathcal{Y}^\alpha$ the following estimates:

$$\begin{aligned}\|\mathcal{N}_1(z, q, \mathbf{d})\|_{Z_\tau^\alpha} \le{}& C_{12}(T)\tau^\varepsilon \|\mathbf{d}\| \\ &+ C_{21}(T)\|\mathbf{d}\| \int_0^\tau [1 + (\tau - s)^{-\alpha}]\big[\varphi_1(s) + \varphi_2(s)\big]\, ds \\ &+ C_{22}(T) \int_0^\tau \varphi_1(\tau - s)\varphi_2(s)\, ds, \qquad (4.4)\end{aligned}$$

$$\begin{aligned}\|\mathcal{N}_2(z, q, \mathbf{d})\|_{Z_\tau^\alpha} \le{}& C_{14}(T)\tau^\varepsilon \|\mathbf{d}\| + C_{20}(T)\|\mathbf{d}\| \int_0^\tau \big[\varphi_1(s) + \varphi_2(s)\big]\, ds \\ &+ C_{20}(T) \int_0^\tau \varphi_1(\tau - s)\varphi_2(s)\, ds. \qquad (4.5)\end{aligned}$$

Consequently, $\mathcal{N}$ maps $\mathcal{Y}^\alpha$ into itself if and only if the pair (φ_1, φ_2) satisfies the integral system of inequalities

$$\begin{aligned}\varphi_1(\tau) \le{}& C_{12}(T)\tau^\varepsilon \|\mathbf{d}\| + C_{21}(T)\|\mathbf{d}\| \int_0^\tau [1 + (\tau - s)^{-\alpha}]\big[\varphi_1(s) + \varphi_2(s)\big]\, ds \\ &+ C_{22}(T) \int_0^\tau \varphi_1(\tau - s)\varphi_2(s)\, ds \\ \varphi_2(\tau) \le{}& C_{14}(T)\tau^\varepsilon \|\mathbf{d}\| + C_{20}(T)\|\mathbf{d}\| \int_0^\tau \big[\varphi_1(s) + \varphi_2(s)\big]\, ds \\ &+ C_{20}(T) \int_0^\tau \varphi_1(\tau - s)\varphi_2(s)\, ds.\end{aligned}$$

The existence of such a pair (φ_1, φ_2) is ensured by Theorem 5.1 in Section 5.

To conclude our proof we need some additional estimates for the increments of $\mathcal{N}$. From the definition of N_4 (cf. (1.33)) and estimates (3.9) and (3.10) in Lemma 3.4 we easily get

$$\begin{aligned}&\|N_4(z_2, q_2, \mathbf{d}_2) - N_4(z_1, q_1, \mathbf{d}_1)\|_{Z_\tau^\alpha} \le \|N_1(z_2, h_{0,2}) - N_1(z_1, h_{0,1})\|_{Z_\tau^\alpha} \\ &+ \|N_1(u_{0,2} + t[Bu_{0,2} + g_{0,2}(0)], q_2) - N_1(u_{0,1} + t[Bu_{0,1} + g_{0,1}(0)], q_1)\|_{Z_\tau^\alpha} \\ &+ \|N_1(z_2, q_2) - N_1(z_1, q_1)\|_{Z_\tau^\alpha} + \|N_2(q_2, \mathbf{d}_2) - N_2(q_1, \mathbf{d}_1)\|_{Z_\tau^\alpha} \\ \le{}& C_{23}(T)\|\mathbf{d}_2 - \mathbf{d}_1\| \sum_{j=1}^2 \|\varphi_j\|_{L^\infty(0,T)} \\ &+ C_{24}(T)\Big(\sum_{j=1}^2 \|\varphi_j\|_{L^\infty(0,T)} + R_0\Big) \int_0^\tau \Big[\|z_2 - z_1\|_{Z_s^\alpha} + \|q_2 - q_1\|_{Q_s^\alpha}\Big]\, ds, \\ &\|N_5(z_2, q_2, \mathbf{d}_2) - N_5(z_1, q_1, \mathbf{d}_1)\|_{C^\alpha([0,\tau])}\end{aligned}$$

$$\leq C_{25}(T)|||\mathbf{d}_2 - \mathbf{d}_1|||\Big(\sum_{j=1}^{2} \|\varphi_j\|_{L^\infty(0,T)} + \|\varphi_1\|_{L^\infty(0,T)}\|\varphi_2\|_{L^\infty(0,T)} + R_0\Big)$$
$$+ C_{26}(T)\Big(\sum_{j=1}^{2} \|\varphi_j\|_{L^\infty(0,T)} + R_0\Big)\int_0^\tau \big[\|q_2 - q_1\|_{C^0([0,s])} + \|z_2 - z_1\|_{Z_s^\alpha}\big]\, ds.$$

Moreover, by virtue of Lemma 3.2 we have

$$\begin{aligned}
&\|q_2 * S(\cdot)u_{0,2} - q_1 * S(\cdot)u_{0,1}\|_{Z_\tau^\alpha} \\
&\leq C_1(T)|||\mathbf{d}_2||| \int_0^\tau (\tau - s)^{-\alpha}\|q_2 - q_1\|_{Q_s^\alpha}\, ds \\
&\quad + C_1(T)|||\mathbf{d}_2 - \mathbf{d}_1||| \int_0^\tau (\tau - s)^{-\alpha}\|q_1\|_{Q_s^\alpha}\, ds \\
&\leq C_1(T)R_0 \int_0^\tau (\tau - s)^{-\alpha}\|q_2 - q_1\|_{Q_s^\alpha}\, ds + C_{27}(T)|||\mathbf{d}_2 - \mathbf{d}_1|||\|\varphi_1\|_{L^\infty(0,T)}.
\end{aligned}$$

From the previous estimates we deduce

$$\|\mathcal{N}_1(z_2, q_2, \mathbf{d}_2) - \mathcal{N}_1(z_1, q_1, \mathbf{d}_1)\|_{Z_\tau^\alpha} \leq C_{28}(R_0, T)|||\mathbf{d}_2 - \mathbf{d}_1|||$$
$$+ C_{29}(R_0, T)\int_0^\tau [1 + (\tau - s)^{-\alpha}]\Big[\|z_2 - z_1\|_{Z_s^\alpha} + \|q_2 - q_1\|_{Q_s^\alpha}\Big]\, ds, \quad (4.6)$$
$$\|\mathcal{N}_2(z_2, q_2, \mathbf{d}_2) - \mathcal{N}_2(z_1, q_1, \mathbf{d}_1)\|_{Q_\tau^\alpha} \leq C_{30}(R_0, T)|||\mathbf{d}_2 - \mathbf{d}_1|||$$
$$+ C_{31}(R_0, T)\int_0^\tau \big[\|q_2 - q_1\|_{C^0([0,s])} + \|z_2 - z_1\|_{Z_s^\alpha}\big]\, ds. \quad (4.7)$$

According to well-known results, to prove the existence and uniqueness of the solution (z, q) in $\mathcal{Y}^\alpha$ it suffices to take estimates (4.6) and (4.7) into account with $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{d}$.

Finally, the continuous dependence result easily follows - via Gronwall's inequality - from estimates (4.6), (4.7) and the inequality

$$\begin{aligned}
\|z_2 - z_1\|_{Z_\tau^\alpha} + \|q_2 - q_1\|_{Q_\tau^\alpha} \quad \leq \quad & \|\mathcal{N}_1(z_2, q_2, \mathbf{d}_2) - \mathcal{N}_1(z_1, q_1, \mathbf{d}_1)\|_{Z_\tau^\alpha} \\
& + \|\mathcal{N}_2(z_2, q_2, \mathbf{d}_2) - \mathcal{N}_2(z_1, q_1, \mathbf{d}_1)\|_{Q_\tau^\alpha}.
\end{aligned}$$

5. Solving an auxiliary nonlinear convolution fixed-point system

We prove here the following Theorem 5.1.

Theorem 5.1. *Let* $c_j \in \mathbf{R}_+$, $k_j \in L^1((0,T);[0,+\infty))$ *and* $h_j \in L^\infty((0,T);[0,+\infty))$, *with* $h_j(\tau) \le \kappa_j \tau^\varepsilon$, $j=1,2$, *for some* $\varepsilon \in (0,1)$. *Then the convolution system*

$$\varphi_1(\tau) = h_1(\tau) + k_1 * \big(\varphi_1 + \varphi_2\big)(\tau) + c_1\varphi_1 * \varphi_2(\tau), \quad \text{for a.e. } \tau \in (0,T), \tag{5.1}$$

$$\varphi_2(\tau) = h_2(\tau) + k_2 * \big(\varphi_1 + \varphi_2\big)(\tau) + c_2\varphi_1 * \varphi_2(\tau), \quad \text{for a.e. } \tau \in (0,T), \tag{5.2}$$

admits a unique solution $(\varphi_1,\varphi_2) \in L^\infty((0,T);[0,+\infty))^2$ *such that*

$$\max_{i=1,2} \|\varphi_i\|_{L^\infty((0,T))} \le e^{\lambda_0 T}\Big\{\kappa_i T^\varepsilon + 2T^{-1}\|k_i\|_{L^1(0,T)}\Big[\sum_{j=1}^2 c_j\Big]^{-1} + c_i T^{-2}\Big[\sum_{j=1}^2 c_j\Big]^{-2}\Big\}. \tag{5.3}$$

Proof. Introduce, for all $r \in \mathbf{R}_+$, the following complete metric subspace in $L^\infty((0,T);[0,+\infty))^2$:

$$\mathcal{E} = \{(\varphi_1,\varphi_2) \in L^\infty((0,T);[0,+\infty))^2 : \|\varphi_1\|_{\lambda,\infty} + \|\varphi_2\|_{\lambda,\infty} \le r\},$$

where

$$\|\varphi\|_{\lambda,p} = \|e^{-\lambda\tau}\varphi\|_{L^p(0,T)}, \qquad \lambda \in \mathbf{R}_+,\ p \in [1,+\infty].$$

Introduce now the nonlinear operator $M = (M_1, M_2)$ where

$$M_i(\varphi_1,\varphi_2) = h_i(\tau) + k_i * \big(\varphi_1 + \varphi_2\big) + c_i\varphi_1 * \varphi_2(\tau), \quad \text{for a.e. } \tau \in (0,T).$$

It is immediate to check that M maps $L^\infty((0,T);[0,+\infty))^2$ into itself.

Observe now that

$$\begin{aligned}\sum_{i=1}^2 \|M_i(\varphi_1,\varphi_2)\|_{\lambda,\infty} &\le \sum_{i=1}^2 \|h_i\|_{\lambda,\infty} \\ &+ \sum_{i=1}^2 \|k_i\|_{\lambda,1}\big(\|\varphi_1\|_{\lambda,\infty} + \|\varphi_2\|_{\lambda,\infty}\big) + \|\varphi_1\|_{\lambda,\infty}\|\varphi_2\|_{\lambda,\infty} T \sum_{i=1}^2 c_i \\ &\le \sum_{i=1}^2 \|h_i\|_{\lambda,\infty} + 2r\sum_{i=1}^2 \|k_i\|_{\lambda,1} + r^2 T \sum_{i=1}^2 c_i.\end{aligned}$$

Consequently, the operator M maps $\mathcal{E}$ into itself if and only if the pair (λ, r) satisfies the inequality

$$\sum_{i=1}^2 \|h_i\|_{\lambda,\infty} + 2r\sum_{i=1}^2 \|k_i\|_{\lambda,1} + r^2 T \sum_{i=1}^2 c_i \le r. \tag{5.4}$$

We now prove that the operator M is Lipschitz-continuous on $\mathcal{E}$. Indeed, for (φ_1, φ_2) and (ψ_1, ψ_2) we have

$$\begin{aligned}
&\sum_{i=1}^{2} \|M_i(\varphi_1, \varphi_2) - M_i(\psi_1, \psi_2)\|_{\lambda,\infty} \\
&\le \sum_{i=1}^{2} \|k_i\|_{\lambda,1}\Big(\|\varphi_1 - \psi_1\|_{\lambda,\infty} + \|\varphi_2 - \psi_2\|_{\lambda,\infty} \\
&\quad + T\|\varphi_1 - \psi_1\|_{\lambda,\infty}\|\varphi_2\|_{\lambda,1} + T\|\varphi_2 - \psi_2\|_{\lambda,\infty}\|\psi_1\|_{\lambda,1}\Big) \\
&\le \Big(\sum_{i=1}^{2} \|k_i\|_{\lambda,1} + rT\Big) \sum_{i=1}^{2} \|\varphi_i - \psi_i\|_{\lambda,\infty}.
\end{aligned}$$

In conclusion, M is a contraction mapping in $\mathcal{E}$ if, in addition to estimate (5.4), we have

$$\sum_{i=1}^{2} \|k_i\|_{\lambda,1} + rT \sum_{i=1}^{2} c_i < 1. \tag{5.5}$$

We now observe that system (5.4) and (5.5) is solvable for any fixed $r \in (0, T^{-1}\big(\sum_{i=1}^{2}\, , c_i\big)^{-1})$, provided we choose large enough λ's. Indeed, $\|k_i\|_{\lambda,1} \to 0$ as $\lambda \to 0+$, while

$$\|h_i\|_{\lambda,\infty} \le \kappa_i \sup_{\tau\in[0,T]} \tau^{\varepsilon} e^{-\lambda\tau} \le \widetilde{\kappa}_i \lambda^{-\varepsilon}, \qquad \lambda > 0.$$

We can now choose a large enough λ_0 such that

$$\sum_{i=1}^{2} \|h_i\|_{\lambda_0,\infty} + 2r \sum_{i=1}^{2} \|k_i\|_{\lambda_0,1} \le r\Big(1 - rT \sum_{i=1}^{2} c_i\Big), \tag{5.6}$$

$$\sum_{i=1}^{2} \|k_i\|_{\lambda_0,1} < 1 - rT \sum_{i=1}^{2} c_i. \tag{5.7}$$

This proves the existence and the uniqueness of the solutions (φ_1, φ_2) to the fixed-point system (5.1) and (5.2). Moreover, recalling that

$$r \in \Big(0, T^{-1}\Big(\sum_{i=1}^{2} c_i\Big)^{-1}\Big),$$

from system (5.1) and (5.2) and the inequalities

$$e^{-\lambda_0 T}\|\varphi_i\|_{0,\infty} \le \|\varphi_i\|_{\lambda_0,\infty} \le \|h_i\|_{\lambda_0,\infty} + 2r\|k_i\|_{\lambda_0,1} + c_i T r^2, \qquad i = 1, 2,$$

we easily deduce estimate (5.3). □

Remark 5.1. In the case of our problem we have (cf. (4.4) and (4.5))

$$h_1(\tau) = C_{12}(T)\|\mathbf{d}\|\tau^{\varepsilon}, \qquad h_2(\tau) = C_{14}(T)\|\mathbf{d}\|,$$
$$k_1(\tau) = C_{21}(T)\|\mathbf{d}\|(1+\tau^{-\alpha}), \qquad k_2(\tau) = C_{20}(T)\|\mathbf{d}\|.$$

Consequently, system (5.6) simplifies to a system of the form

$$C_{32}(T)\lambda^{-\varepsilon} + 2r\lambda^{-1+\alpha}\big[C_{33}(T) + C_{34}(T)\lambda^{-\alpha}\big] \le r\Big(1 - rT\sum_{i=1}^{2} c_i\Big),$$

$$\lambda^{-1+\alpha}\big[C_{33}(T) + C_{34}(T)\lambda^{-\alpha}\big] < 1 - rT\sum_{i=1}^{2} c_i,$$

where the constants C_{32}, C_{33}, C_{34} can be easily computed in terms of those appearing in system (5.6).

REFERENCES

[1] T. Aiki and N. Sato, *Large-time behavior of solutions to phase field equations with constraints under nonlinear dynamic boundary conditions,* Progress in partial differential equations, Vol. 2 (Pont-Mousson, 1997), 92–103, Pitman Res. Notes Math. Ser., 384, Longman, Harlow, 1998.

[2] T. Aiki and N. Sato, *Phase field equations with constraints under nonlinear dynamic boundary conditions,* Commun. Appl. Anal., 5 (2001), 215–234.

[3] M. Amar, D. Andreucci, R. Gianni, and P. Bisegna, *Evolution and memory effects in the homogenization limit for electrical conduction in biological tissues,* Math. Models Methods Appl. Sci., 14 (2004), 1261–1295.

[4] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni, *Existence and uniqueness for an elliptic problem with evolution arising in electrodynamics,* Nonlinear Anal. Real World Appl., 6 (2005), 367–380.

[5] C. A. Bosello and G. Gentili, *Some quasi-elastic problems with elastic and viscous boundary conditions in linear viscoelasticity,* Quart. Appl. Math., 4 (1996), 687–696.

[6] V. Casarino, K.J. Engel, R. Nagel, and G. Nickel, *A semigroup approach to boundary feedback systems,* Integral Equations Operator Theory, 47 (2003), 289–306.

[7] I. A. Chernov, *Classical solution of a boundary value problem with dynamic boundary conditions,* (Russian) Tr. Petrozavodsk. Gos. Univ. Ser. Mat., No. 7 (2000), 109–121.

[8] J. Escher, *Smooth solutions of nonlinear elliptic systems with dynamic boundary conditions,* Evolution equations, control theory, and biomathematics (Han sur Lesse, 1991), 173–183, Lecture Notes in Pure and Appl. Math., 155, Dekker, New York, 1994.

[9] J. Escher, *Nonlinear elliptic systems with dynamic boundary conditions,* Math. Z., 210 (1992), 413–439.

[10] D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Classics in Mathematics, Springer-Verlag, Berlin, 2001, xiv+517.

[11] D. Guidetti, *On elliptic boundary value problems with dynamic boundary conditions of parabolic type,* Diff. Int. Eqs., 7 (1994), 873–884.
[12] D. Guidetti, *Abstract convolution system on the line,* Diff. Int. Eqs., 5 (1997), 853–892.
[13] T. Hintermann, *Evolution equations with dynamic boundary conditions,* Proc. Roy. Soc. Edinburgh Sect., A 113 (1989), 43–60.
[14] M. Kirane, È. Nabana, and S. I. Pokhozhaev, *On the absence of solutions for elliptic systems with dynamic boundary conditions,* (Russian) Differ. Uravn., 38 (2002), 768–774, 862; Translation in Differ. Equ., 38 (2002), 808–815.
[15] M. O. Korpusov, A. G. Sveshnikov, *On the existence of a solution of the Laplace equation with a nonlinear dynamic boundary condition,* (Russian) Zh. Vychisl. Mat. Mat. Fiz., 43 (2003), 95–110; translation in Comput. Math. Math. Phys., 43 (2003), 92–107.
[16] J.L. Lions, "Équations différentielles opérationnelles et problémes aux limites," Die Grundlehren der mathematischen Wissenschaften, Bd. 111 Springer-Verlag, Berlin-Göttingen-Heidelberg 1961 ix+292 pp.
[17] J. L. Lions and E. Magenes, *Problemi ai limiti non omogenei (V)*, (in Italian) Ann. Sc. Norm. Sup. Pisa, 16 (1962), 1–44.
[18] A. Lorenzi and A. Prilepko, *Fredholm type results for integrodifferential identification parabolic problems,* Diff. Int. Equations, 6 (1993), 535–552.
[19] A. Lorenzi and A. I. Prilepko, *Global existence results for first–order integrodifferential identification results,* Rend. Sem. Mat. Padova, 95 (1996), 51–84.
[20] A. Lunardi, "Analytic Semigroups and Optimal Regularity in Parabolic Problems," Birkhäuser Basel, 1995.
[21] J. Petersson, *A note on quenching for parabolic equations with dynamic boundary conditions,* Nonlinear Anal., 58 (2004), no. 3-4, 417–423.
[22] R. Racke and S. Zheng, *The Cahn-Hilliard equation with dynamic boundary conditions,* Adv. Differential Equations, 8 (2003), no. 1, 83–110.
[23] N. Sauer, *Dynamical processes associated with dynamic boundary conditions for partial differential equations,* Differential equations and applications, Vol. I, II (Columbus, OH, 1988), 374–378, Ohio Univ. Press, Athens, OH, 1989.
[24] E. Sinetrari, *On the abtract Cauchy problem of parabolic type in spaces of continuous functions*, J. Math. Anal. Appl., 107 1985, 16–66.
[25] H. Wu and S. Zheng, *Asymptotic behavior of solution to the Cahn-Hilliard equation with dynamic boundary conditions,* Nonlinear partial differential equations and their applications, 382–390, GAKUTO Internat. Ser. Math. Sci. Appl., 20, Gakkōtosho, Tokyo, 2004.
[26] Z. Yin, *Global existence for elliptic equations with dynamic boundary conditions,* Arch. Math., (Basel) 81 (2003), 567–574.