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INSTANTANEOUS SUPPORT SHRINKING PHENOMENON IN THE CASE OF FAST DIFFUSION FOR A DOUBLY NONLINEAR PARABOLIC EQUATION WITH ABSORPTION

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Abstract. We study the instantaneous support shrinking phenomenon for a doubly nonlinear parabolic equation in the fast diffusion case. The initial data of the Cauchy problem are locally finite Radon measures. We obtain for nonnegative solutions necessary and sufficient condition for instantaneous support shrinking phenomenon in terms of local behavior of the initial data. In the same terms we express sharp with respect to rate bilateral estimates for the size of the support.

1. Statement of the problem and the main result

In the domain $R^N \times [0,T]$, where N is the dimension of the space R^N, $T>0$, we consider the following Cauchy problem:

$$\frac{\partial}{\partial t}(|u|^{\beta-1}u(x,t)) - \nabla(|\nabla u|^{p-2}\nabla u) + |u|^{r-1}u(x,t) = 0, \quad x \in R^N, t>0, \tag{1.1}$$

$$|u|^{\beta-1}u(x,0) = |u_0|^{\beta-1}u_0(x), \quad x \in R^N, \tag{1.2}$$

where $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_N})$, $|u_0|^{\beta-1}u_0(x)$ is a given initial datum which is a locally finite Radon measure. In this paper we deal with the case of fast diffusion and strong absorption which means the following restriction on the parameters of the problem

$$\beta > 0, \qquad 0 < r < p-1 < \beta. \tag{1.3}$$

In addition, we restrict our parameters in the following way:

$$k \equiv N(p-1-\beta) + \beta p = Nd + \beta p > 0, \ d \equiv p-1-\beta < 0, \tag{1.4}$$

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and everywhere in this paper we use letters k and d only in the sense of this definition.

It is important to remark, that we are naturally forced to the last restriction, because in the opposite case, i.e., $k < 0$, the problem (1.1), (1.2) with locally finite Radon measure as initial datum has no solutions in general, as was shown in [11].

By a *weak solution* of the problem (1.1), (1.2) on the time interval $[0, T]$ we mean a measurable function $u(x, t)$ with the following properties:

1) For any function $\zeta \in C_0^\infty(R^N)$ the mapping

$$t \in [0, T] \to \int_{R^N} |u|^{\beta-1} u(x, t)\zeta(x)dx$$

is continuous.

2) For any finite with respect to x and sufficiently regular function $\eta(x, t)$ the following integral identity is satisfied:

$$\int_{R^N} |u|^{\beta-1} u(x, t)\eta dx + \sum_{i=1}^{N} \int_0^t \int_{R^N} |\nabla u|^{p-2} u_{x_i} \eta_{x_i} dx d\tau + \int_0^t \int_{R^N} |u|^{r-1} u\eta dx d\tau$$

$$= \int_{R^N} |u_0|^{\beta-1} u_0(x)\eta(x, 0)dx + \int_0^t \int_{R^N} |u|^{\beta-1} u(x, \tau)\eta_\tau(x, \tau)dx d\tau. \quad (1.5)$$

The existence of a weak solution for the problem (1.1), (1.2) under our assumptions about the data follows from [10], [11] (and also from [20] in the case $\beta = 1$). Moreover, it follows from the papers [10], [11], that a weak solution of the problem (1.1), (1.2) is locally bounded for $t > 0$ and, besides, $u_{x_i} \in L_{p,loc}(R^N \times (0, T))$. This in particular means that the integral identity (1.5) is still valid for finite with respect to x testing functions $\eta(x, t) \in L_{p,loc}((0, T), W^1_{p,loc}(R^N))$. Let us remark also, that the author is not aware of results about the uniqueness of weak solutions of the problem (1.1), (1.2) in the case when simultaneously $\beta \neq 1$, $p \neq 2$, and the initial data do not belong to $L_1(R^N)$ (as in our case when the initial data are locally finite Radon measures). At the same time the uniqueness of strong solutions of the problem follows from [21], where by *strong solution* we mean a weak solution $u(x, t)$ for which all terms of equation (1.1) belong to $L_{2,loc}(R^N \times [0, T])$ and the equation is satisfied almost everywhere. Taking into account this circumstance, below in the proof of the estimate (1.18) we consider our weak solution as a limit of strong solutions with smooth and finite initial data (according to the method by which a weak solution was obtained in the papers [10], [11]).

It is known from the papers [1]-[11], [20] that, if the initial function $|u_0|^{\beta-1}u_0(x)$ is sufficiently regular and condition (1.4) is satisfied, then the problem (1.1), (1.2) is solvable in the weak sense and possesses the instantaneous support shrinking property. This property means that the support of the solution is compact at any arbitrary small moment $t > 0$ regardless of the support of the initial function (the initial support may coincide with all of R^N). We study this phenomenon for the problem (1.1), (1.2) with a Radon measure as the initial datum and establish sharp with respect to rate bilateral estimates of the support of a weak solution.

In the present time there are two types of results concerning instantaneous support shrinking phenomenon. Those of the first type, which are mainly based on barrier technique, consider locally bounded initial functions in (1.2) which are monotonic decreasing at infinity (or have a monotonic majorizing function) - see papers [5], [7], [9]. Let us formulate briefly the result of these papers as applied to our statement of the problem. Let here and below

$$D(t) = \inf_{\rho>0} \{\rho : u(x,t) \equiv 0, |x| > \rho\} \tag{1.6}$$

be the upper bound of the support of the solution $u(x,t)$ for the problem (1.1), (1.2) at a time $t > 0$. Let the initial function $u_0(x)$ be continuous and decrease monotonically at infinity and let

$$f_\infty(\rho) \equiv \max_{|y|=\rho} |u_0(y)| \to 0, \rho \to \infty. \tag{1.7}$$

Then, as it follows, for example, from the papers [5], [7], [9],

$$D(t) \sim C f_\infty^{-1}(\gamma t^{\frac{\beta}{\beta-r}}). \tag{1.8}$$

The second type of results concern the initial functions which are locally integrable with some exponent $q > \beta$ - see the papers [1]-[4]. In particular, let

$$f_q(\rho) \equiv \sup_{|y|=\rho} \int_{|x-y|<1} |u_0(x)|^q \, dx$$

and let $f_q(\rho) \to 0$ as $\rho \to \infty$. Then the problem (1.1), (1.2) possesses the instantaneous support shrinking property and

$$D(t) \le f_q^{-1}\Big(\gamma t^{\frac{qp+N(p-1-r)}{p(\beta-r)}}\Big) \tag{1.9}$$

with sufficiently small $\gamma > 0$. Here, in the case of a nonstrictly monotonic function $f_q(\rho)$,

$$f_q^{-1}(s) \equiv \inf \{\rho : f_q(\rho) < s\}.$$

As can be easily seen, the last estimate does not reduce to estimate (1.8) when improving the regularity properties of the initial function from local integrability towards continuity and monotonicity. However, as we will demonstrate, estimate (1.9) is sharp "for the functional class as a whole" and is reached for initial functions, which with their properties approach locally finite measures like δ - functions. In particular, let $\overline{n} = (n_1, n_2, ..., n_N)$ be a point with integer coordinates, $\mu_{\overline{n}}\delta(x - \overline{n})$ be the Dirac delta function concentrated at the point $\overline{n}$ and with a mass $\mu_{\overline{n}}$, $\mu_{\overline{n}} \to 0, |\overline{n}| \to \infty$. Then we will show that, for the initial datum

$$|u_0|^{\beta-1} u_0(x) = \sum_{\overline{n} \in R^N} \mu_{\overline{n}}\delta(x - \overline{n}), \tag{1.10}$$

the following asymptotic takes place:

$$D(t) \sim C f_0^{-1}(\gamma t^{\frac{N(p-1-r)+\beta p}{p(\beta-r)}}), \tag{1.11}$$

where

$$f_0^{-1}(\gamma t^{\frac{N(p-1-r)+\beta p}{p(\beta-r)}}) = \inf\left\{\rho : |\mu_{\overline{n}}| < \gamma t^{\frac{N(p-1-r)+\beta p}{p(\beta-r)}}, |\overline{n}| > \rho\right\}. \tag{1.12}$$

The "limiting" accuracy of estimate (1.9) for $q = \beta$ follows from (1.11) when the initial datum behaves as in (1.10). Let us mention also that, as it turns out, the relation (1.11) takes place for initial data of the form (1.10) only: for initial data, which are uniformly locally integrable in R^N, the rate of the size of the support is always less than that in (1.11).

Consequently, the goals of this paper are, on the one hand, to include in the class of possible initial data locally bounded Radon measures, and, on the other hand, to obtain sharp with respect to rate estimates of the support of the solution such that the estimates (1.8) and (1.9) follow from them as particular cases.

We use the methods of local integral estimates, which were developed in the papers [12]-[14] by D. Andreucci and A.F. Tedeev. Moreover, with the idea itself of applying these methods to our problem the author was advised by one of the authors [12]-[14] A.F. Tedeev.

Let us introduce the following notation. Let $B_\rho(x_0)$ denote an open ball of radius ρ centered at a point x_0. When the initial function is a Radon measure the integral over $B_\rho(x_0)$ of the absolute value of the initial function means total variation of this measure over the ball $B_\rho(x_0)$. Denote also

$$d_r \equiv p - 1 - r > 0. \tag{1.13}$$

In addition, in what follows we will denote by C, b, γ different absolute constants or constants which depend only on given parameters N, p, r, β.

To formulate the main result we need to introduce the following important exponent:

$$\varkappa \equiv \frac{p-1-r}{p(\beta-r)} > 0. \tag{1.14}$$

It is worth noting, that $\varkappa$ is analogous to the exponent, which was obtained in [15] in the estimation of the support in the setting which is opposite to our case in some sense, i.e., $r > p > 1+\beta$ when initially compact support starts to expand itself.

In addition, for a fixed $L > 0$ we define the function

$$\begin{aligned} \varphi_t(x_0) &= \frac{1}{\omega_N L^N t^{N\varkappa}} \int_{|x-x_0|<Lt^{\varkappa}} |u_0(x)|^{\beta}\, dx \\ &\equiv \fint_{B_{Lt^{\varkappa}}(x_0)} |u_0(x)|^{\beta}\, dx, \end{aligned} \tag{1.15}$$

where ω_N is the volume of the unit ball of R^N, and also we define the function

$$\varphi_t(\rho) \equiv \sup_{|x_0|=\rho} \varphi_t(x_0). \tag{1.16}$$

Theorem 1. *Let restrictions* (1.3), (1.4) *be satisfied. If the initial function in* (1.2) *is nonnegative* (*nonpositive*), *then the solution of the problem* (1.1), (1.2) *possesses the instantaneous support shrinking property if and only if for the initial function* $|u_0|^{\beta-1}u_0(x)$ (*which may be a Radon measure*) *the condition*

$$\varphi_t(\rho) \to 0, \qquad \rho \to \infty,$$

is satisfied for some fixed $t > 0$ (*as it is easy to see, in this case the above condition is satisfied for any fixed* $t > 0$). *Moreover, there are constants* t_0, L, γ_0, γ_1 M_1, *such that on the interval* $[0, t_0]$ *the following estimates are valid:*

$$D(t) \le \varphi_t^{-1}(\gamma_0 t^{\frac{\beta}{\beta-r}}), \tag{1.17}$$

$$D(t) \ge \varphi_{M_1 t}^{-1}(\gamma_1 t^{\frac{\beta}{\beta-r}}), \tag{1.18}$$

where, for a nonstrictly monotone function $\varphi_t(\rho)$,

$$\varphi_t^{-1}(s) \equiv \inf_{\rho} \left\{\rho : \varphi_t(k) < s, k > \rho\right\}. \tag{1.19}$$

The estimate (1.17) *from above is still valid for an initial function which changes its sign.*

Remark 1. From estimate (1.17) in view of the definition of the function φ_t it is easy to infer estimates (1.8) and (1.9) by applying the Hölder inequality to obtain (1.9) and by applying the mean value theorem to obtain (1.8). Moreover, relation (1.11) also easily follows from estimates (1.17) and (1.18).

Remark 2. Though the present paper treats the case of fast diffusion, when $d = p - 1 - \beta < 0$, all calculations and estimates below are valid, in fact, in the slow diffusion case as well, that is when $d = p - 1 - \beta > 0$ (it is important that $k > 0$ in (1.4)). The main difference between these two cases is the estimate of local mass of the solution in terms of local mass of the initial datum (see Section 4 and Remark 4).

Remark 3. Passing to the proof of Theorem 1, we note that according to the formulation of the theorem, we will assume that the initial datum, and hence the solution itself, are nonnegative in Sections 2-4 of the paper and we will not mention this each time separately (though all reasoning and proofs of Sections 2 and 3 are still valid for initial data and solutions of arbitrary sign).

Note also, that to obtain some integral relations we need we will multiply equation (1.1) by different testing functions with subsequent integration. Such operations are based, in fact, on the choice in the integral identity (1.5) of some cutoffs of the Steklov mollifiers of solutions as testing functions. Then a suitable limiting procedure is needed. This process is quite standard and it is described, for example, in the classical monograph [16], so we skip the technical details.

2. A condition on local energy for the solution to become zero locally

We will prove the following lemma in this section.

Lemma 2.1. *Let restrictions* (1.3), (1.4) *be satisfied. Let further* $0 < R_1 < R_2,\ R_2 = Rt^{\varkappa}, R_1 = (1-\sigma)R_2,\ B_{R_i} = B_{R_i}(x_0) = \{x : |x - x_0| < R_i\}$, $x_0 \in R^N$, $i = 1, 2$. *Then there is a constant* $\gamma_2 = \gamma_2(R, \sigma)$ *such that, if*

$$Y(t/2, R_2) \equiv \sup_{t/2<\tau<t} \int_{B_{R_2}} u^{1+\beta}(x,\tau)dx + \int_{t/2}^{t}\int_{B_{R_2}} |\nabla u|^p \, dxd\tau$$

$$+ \int_{t/2}^{t}\int_{B_{R_2}} u^{1+r} dxd\tau \le \gamma_2 t^{\frac{Nd_r + p(1+\beta)}{p(\beta - r)}},$$

then $u(x,t) \equiv 0$ *on the set* $B_{R_1}(x_0) \times [3t/4, t]$.

Proof. Let (for $n = 0, 1, ...$) $R_n = R_1 + (R_2 - R_1)2^{-n}$, $\overline{R}_n = (R_n + R_{n+1})/2$, $t_n = \frac{3t}{4} - \frac{t}{4}2^{-n}$, $\bar{t}_n = (t_n + t_{n+1})/2$, $B_n = B_{R_n}$ be shrinking concentric balls with center at x_0, $\widetilde{B_n} = B_{\overline{R}_n}$, $Q_n = B_n \times [t_n, t]$, $\widetilde{Q_n} = \widetilde{B_n} \times [\bar{t}_n, t]$. Let, further, $\zeta_n \in C^\infty(R^N \times [0, T])$ be a cutoff function for the cylinder Q_n such that $\zeta_n \equiv 1$ on Q_{n+1}, $\zeta_n \geq 1/2$ on $\widetilde{Q_n}$, $\zeta_n \equiv 0$ outside Q_n, $|\nabla \zeta| \leq C2^n(R_2 - R_1)^{-1}$, $|\zeta_t| \leq C2^n t^{-1}$. Let also ξ_n be smooth cutoff functions for the cylinders $\widetilde{Q_n}$ such that $\xi_n \equiv 1$ on Q_{n+1}, $\xi_n \equiv 0$ outside $\widetilde{Q_n}$, $|\nabla \xi| \leq C2^n(R_2 - R_1)^{-1}$, $|\xi_t| \leq C2^n t^{-1}$.

Multiply both sides of equation (1.1) by $u(x,\tau)\xi_n^s(x,\tau)$, $s > p$, and integrate over $\widetilde{Q_n}$. We obtain after integration by parts:

$$\frac{1}{1+\beta}\int_{\widetilde{B_n}} u^{1+\beta}(x,t)\xi_n^s dx + \int_{\bar{t}_n}^t \int_{\widetilde{B_n}} |\nabla u|^p \xi_n^s dx d\tau + \int_{\bar{t}_n}^t \int_{\widetilde{B_n}} u^{1+r}\xi_n^s dx d\tau$$
$$= \frac{s}{1+\beta}\int_{\bar{t}_n}^t \int_{\widetilde{B_n}} u^{1+\beta}\xi_n^{s-1}\xi_{n\tau} dx d\tau - s\sum_{i=1}^N \int_{\bar{t}_n}^t \int_{\widetilde{B_n}} |\nabla u|^{p-2} u_{x_i} u \xi_{n x_i}\xi_n^{s-1} dx d\tau$$
$$\equiv I_1 + I_2.$$

Let us estimate the sum I_2 by the Young inequality with $\varepsilon = 1/2$ as follows:

$$|I_2| \leq C \iint_{\widetilde{Q_n}} |\nabla u|^{p-1} \xi_n^s u |\nabla \xi_n| \xi_n^{-1} dx d\tau \tag{2.1}$$
$$\leq \frac{1}{2}\iint_{\widetilde{Q_n}} |\nabla u|^p \xi_n^s dx d\tau + C\iint_{\widetilde{Q_n}} u^p \xi_n^{s-p} |\nabla \xi_n|^p dx d\tau.$$

Substituting this estimate in the previous inequality and keeping in mind properties of the function $\xi_n(x,\tau)$, as $t > 0$ is arbitrary, we obtain the relation

$$\sup_{t_{n+1}<\tau<t} \int_{B_{n+1}} u^{1+\beta}(x,\tau)dx + \iint_{Q_{n+1}} |\nabla u|^p dx d\tau + \iint_{Q_{n+1}} u^{1+r} dx d\tau \tag{2.2}$$
$$\leq C2^n\Big(t^{-1}\iint_{\widetilde{Q_n}} u^{1+\beta} dx d\tau + (R_2 - R_1)^{-p}\iint_{\widetilde{Q_n}} u^p dx d\tau\Big).$$

Define the functions $v_n(x,\tau) = \zeta_n(x,\tau)u(x,\tau)$. Note that, in view of the properties of the function $\zeta_n(x,\tau)$,

$$\iint_{Q_{n+1}} |\nabla v_{n+1}|^p dx d\tau \leq C\iint_{Q_{n+1}} |\nabla u|^p dx d\tau + C2^{np}(R_2 - R_1)^{-p}\iint_{\widetilde{Q_n}} u^p dx d\tau.$$

Thus, bearing in mind that $\zeta_n \geq 1/2$ on $\widetilde{Q_n}$, from the two last relations we obtain, introducing quantities Y_n:

$$\begin{aligned} Y_{n+1} \equiv \sup_{t_{n+1}<\tau<t} \int_{B_{n+1}} v_{n+1}^{1+\beta}(x,\tau)dx + \iint_{Q_{n+1}} |\nabla v_{n+1}|^p \, dxd\tau \\ + \iint_{Q_{n+1}} v_{n+1}^{1+r} dxd\tau \\ \leq C2^{np}\Big(t^{-1} \iint_{Q_n} v_n^{1+\beta} dxd\tau + (R_2 - R_1)^{-p} \iint_{Q_n} v_n^p dxd\tau\Big) \\ \equiv C2^{np}(I_1 + I_2). \end{aligned} \tag{2.3}$$

Consider first the quantity I_1 in the right-hand side of the last inequality. Let us estimate this quantity as follows:

$$I_1 \leq t^{-1}\Big(\sup_{t_n<\tau<t} \int_{B_n} v_n^{1+\beta}(x,\tau)dx\Big)^{1-\alpha} \int_{t_n}^{t} \Big(\int_{B_n} v_n^{1+\beta}(x,\tau)dx\Big)^{\alpha} d\tau, \tag{2.4}$$

where $\alpha \in (0,1)$ will be chosen later. Apply to the integral over B_n at the end of the last relation the Nirenberg - Gagliardo inequality:

$$\begin{aligned} \Big(\int_{B_n} v_n^{1+\beta}(x,\tau)dx\Big)^{\alpha} \\ \leq C\Big(\int_{B_n} |\nabla v_n|^p dx\Big)^{\alpha\omega_0 \frac{1+\beta}{p}} \Big(\int_{B_n} v_n^{1+r}(x,\tau)dx\Big)^{\alpha(1-\omega_0)\frac{1+\beta}{1+r}}, \end{aligned} \tag{2.5}$$

where ω_0 is determined from the equality

$$\frac{1}{1+\beta} = \omega_0(\frac{1}{p} - \frac{1}{N}) + (1-\omega_0)\frac{1}{1+r}.$$

Note that under the conditions $k > 0$ and $r < \beta$, we have $\omega_0 \in (0,1)$. Subject now α to the following condition: the sum of the exponents of the integrals in the right-hand side of (2.5) must be equal to 1; i.e.,

$$\alpha\omega_0 \frac{1+\beta}{p} + \alpha(1-\omega_0)\frac{1+\beta}{1+r} = 1.$$

Direct calculations show that

$$\alpha = \Big(1 + \omega_0 \frac{1+\beta}{N}\Big)^{-1} = \frac{Nd_r + p(1+r)}{Nd_r + p(1+\beta)}, \quad 1 - \alpha = \frac{p(\beta - r)}{Nd_r + p(1+\beta)}.$$

Taking into account, that the sum of the exponents of the integrals in the right-hand side of (2.5) is equal to 1, integrating the inequality (2.5) with respect to time t, and applying first the Hölder inequality and then the Young inequality, we obtain

$$\int_{t_n}^{t}\Big(\int_{B_n} v_n^{1+\beta}(x,\tau)dx\Big)^{\alpha} d\tau$$
$$\leq C\Big(\int_{t_n}^{t}\int_{B_n} |\nabla v_n|^p\, dxd\tau + \int_{t_n}^{t}\int_{B_n} v_n^{1+r}(x,\tau)dxd\tau\Big).$$

Thus, it follows from the last inequality applied to estimate (2.4) that

$$I_1 \leq Ct^{-1}Y_n^{1+(1-\alpha)} = Ct^{-1}Y_n^{1+\frac{p(\beta-r)}{Nd_r+p(1+\beta)}}. \tag{2.6}$$

Consider now the quantity I_2 in (2.3). To estimate I_2 we apply to the integral with respect to dx over B_n the Nirenberg-Gagliardo inequality of the form (which is a simple consequence of the standard Nirenberg-Gagliardo inequality)

$$\int_{B_n} v_n^p dx \leq C\Big(\int_{B_n} |\nabla v_n|^p\, dx\Big)^{\omega_1}\Big(\int_{B_n} v_n^{1+\beta}(x,\tau)dx\Big)^{\omega_2\frac{p}{1+\beta}}$$
$$\times\Big(\int_{B_n} v_n^{1+r}(x,\tau)dx\Big)^{\omega_3\frac{p}{1+r}}, \tag{2.7}$$

where numbers $\omega_i \in (0,1)$, $i = 1,2,3$ are subject to the conditions

$$\omega_1+\omega_2+\omega_3 = 1, \quad \frac{1}{p} = \omega_1(\frac{1}{p}-\frac{1}{N}) + \omega_2\frac{1}{1+\beta} + \omega_3\frac{1}{1+r}. \tag{2.8}$$

Subject now the numbers ω_i to the following condition: the sum of the exponents of the first and the last integrals in (2.7) must be equal to 1,

$$\omega_1 + \omega_3\frac{p}{1+r} = 1. \tag{2.9}$$

From the system (2.8)-(2.9) the numbers ω_i are uniquely determined and satisfy the condition $\omega_i \in (0,1)$. By direct calculations we have

$$\omega_2\frac{p}{1+\beta} = \frac{p(p-1-r)}{N(p-1-r)+p(1+\beta)} = \frac{pd_r}{Nd_r+p(1+\beta)}.$$

Integrating (2.7) with respect to time, factoring out

$$\Big(\sup_{t_n<\tau<t}\int_{B_n} v_n^{1+\beta}(x,\tau)dx\Big)^{\omega_2\frac{p}{1+\beta}},$$

and applying the Hölder and the Young inequalities, taking into account (2.9), we obtain

$$\int_{t_n}^{t}\int_{B_n} v_n^p dxd\tau \le C\Big(\sup_{t_n<\tau<t}\int_{B_n} v_n^{1+\beta}(x,\tau)dx\Big)^{\omega_2\frac{p}{1+\beta}}$$
$$\times\Big(\int_{t_n}^{t}\int_{B_n}|\nabla v_n|^p\,dxd\tau+\int_{t_n}^{t}\int_{B_n} v_n^{1+r}dxd\tau\Big).$$

Hence, using the definitions of R_1 and R_2, for the quantity I_2 in the right-hand side of (2.3) we have the estimate

$$I_2 \le C(R\sigma)^{-p}t^{-d_r/(\beta-r)}Y_n^{1+\frac{pd_r}{Nd_r+p(1+\beta)}}. \tag{2.10}$$

It follows from (2.3), (2.6), and (2.10) that

$$Y_{n+1}\le C2^{np}\Big(t^{-1}Y_n^{1+\frac{p(\beta-r)}{Nd_r+p(1+\beta)}}+(R\sigma)^{-p}t^{-d_r/(\beta-r)}Y_n^{1+\frac{pd_r}{Nd_r+p(1+\beta)}}\Big).$$

On the basis of the iterative Lemma 5.6 in [16, Chapter II], from the last inequality we infer that $Y_n\to 0$ as $n\to\infty$ if the following quantity is sufficiently small:

$$C\Big(t^{-1}Y_0^{\frac{p(\beta-r)}{Nd_r+p(1+\beta)}}+(R\sigma)^{-p}t^{-d_r/(\beta-r)}Y_0^{\frac{pd_r}{Nd_r+p(1+\beta)}}\Big)$$
$$=C\Big\{t^{-1}Y_0^{\frac{p(\beta-r)}{Nd_r+p(1+\beta)}}+(R\sigma)^{-p}\Big[t^{-1}Y_0^{\frac{p(\beta-r)}{Nd_r+p(1+\beta)}}\Big]^{\frac{d_r}{\beta-r}}\Big\}.$$

It is clear that this quantity will be small if and only if the quantity in brackets $t^{-1}Y_0^{\frac{p(\beta-r)}{Nd_r+p(1+\beta)}}$ is small, that is if

$$Y_0\le\gamma_2 t^{\frac{Nd_r+p(1+\beta)}{p(\beta-r)}}, \tag{2.11}$$

where the number $\gamma_2=\gamma_2(R,\sigma)$ is sufficiently small. Thus, in view of the definition of the quantities Y_n, Lemma 2.1 is proved. □

3. An estimate of the local energy in terms of the local mass of the solution

In this section we will obtain some estimates of the local energy of the solution from Lemma 2.1 of the previous section in terms of the local mass of the solution. The following lemma is reasonably analogous to [12, Lemma 4.2], but we give its proof here for the sake of completeness.

Lemma 3.1. (compare with [12, Lemma 4.2].) *Let restrictions* (1.3), (1.4) *be satisfied. Let also* $0<r_1<r_2$, $r_1<C(r_2-r_1)$, $0<t_2<t_1<t$, *and let*

B_{r_i}, $i=1,2$, be balls with the centers at x_0 and with the radii r_i. Then for a solution $u(x,\tau)$ of the problem (1.1), (1.2) *the following estimate is valid:*

$$Y(t_1,r_1)\equiv \sup_{t_1<\tau<t}\int_{B_{r_1}} u^{1+\beta}(x,\tau)dx+\int_{t_1}^{t}\int_{B_{r_1}}|\nabla u|^p\,dxd\tau+\int_{t_1}^{t}\int_{B_{r_1}}u^{1+r}dxd\tau \tag{3.1}$$
$$\le C\Big[\frac{t-t_2}{(t-t_2)^{\frac{k+N}{k}}}\Big(\sup_{t_2<\tau<t}\int_{B_{r_2}}u^\beta dx\Big)^{\frac{k+p}{k}}+\frac{t-t_2}{(r_2-r_1)^{\frac{k+N}{\beta}}}\Big(\sup_{t_2<\tau<t}\int_{B_{r_2}}u^\beta dx\Big)^{\frac{p}{\beta}}\Big].$$

Proof. Define the quantities $t_n=t_2+(t_1-t_2)2^{-n}$, $\bar{t}_n=(t_n+t_{n+1})/2$, $r_n=r_2-(r_2-r_1)2^{-n}$, $\bar{r}_n=(r_n+r_{n+1})/2$ and the sequence of expanding (unlike Lemma 2.1) regions $B_n=B_{r_n}$, $\widetilde{B_n}=B_{\bar{r}_n}$, $Q_n=B_n\times[t_n,t]$, $\widetilde{Q_n}=\widetilde{B_n}\times[\bar{t}_n,t]$. Let further $\zeta_n(x,\tau)$ be smooth cutoff functions such that $\zeta_n\equiv 1$ on Q_n, $\zeta_n\ge 1/2$ on $\widetilde{Q_n}$, $\zeta_n\equiv 0$ outside Q_{n+1}, $|\zeta_{nt}|\le C2^n(t_1-t_2)^{-1}$, $|\nabla\zeta_n|\le C2^n(r_2-r_1)^{-1}$.

Completely analogously to the proof of Lemma 2.1 let us define auxiliary functions $v_n(x,\tau)=\zeta_n(x,\tau)u(x,\tau)$ and take into account that

$$|\nabla v_n|^p\le C\left(|\nabla u|^p+2^{np}u^p\right). \tag{3.2}$$

Then in the same way as inequality (2.3) was obtained, we get for these functions $v_n(x,\tau)$ the estimate

$$Y_n\equiv \sup_{t_{n+1}<\tau<t}\int_{B_{n+1}}v_n^{1+\beta}(x,\tau)dx+\iint_{Q_{n+1}}|\nabla v_n|^p\,dxd\tau+\iint_{Q_{n+1}}v_n^{1+r}dxd\tau \tag{3.3}$$
$$\le Cb^n\Big((t_1-t_2)^{-1}\iint_{Q_{n+2}}v_{n+1}^{1+\beta}dxd\tau+(r_2-r_1)^{-p}\iint_{Q_{n+2}}v_{n+1}^{p}dxd\tau\Big)\equiv I_1+I_2,$$

where b is some positive constant.

Let us estimate expressions I_1 and I_2 in the right-hand side of (3.3) applying to the integrals with respect to dx over B_{n+2} the Nirenberg-Gagliardo inequality. We have for I_1:

$$\int_{B_{n+2}}v_{n+1}^{1+\beta}(x,\tau)dx$$
$$\le C\Big(\int_{B_{n+2}}|\nabla v_{n+1}|^p\,dx\Big)^{\omega_1\frac{1+\beta}{p}}\Big(\int_{B_{n+2}}v_{n+1}^{\beta}(x,\tau)dx\Big)^{(1-\omega_1)\frac{1+\beta}{\beta}}$$

where ω_1 is determined from the relation

$$\frac{1}{1+\beta}=\omega_1\Big(\frac{1}{p}-\frac{1}{N}\Big)+(1-\omega_1)\frac{1}{\beta}.$$

Integrating the last inequality with respect to time, factoring out

$$\sup_{t_{n+2}<\tau<t}\int_{B_{n+2}} v_{n+1}^{\beta}(x,\tau)dx,$$

and applying the Hölder inequality, we obtain

$$I_1 \leq C\Big(\iint_{Q_{n+2}} |\nabla v_{n+1}|^p\,dx\Big)^{\omega_1\frac{1+\beta}{p}} b^n \frac{(t-t_{n+2})^{1-\omega_1\frac{1+\beta}{p}}}{(t_1-t_2)}$$
$$\times\Big(\sup_{t_{n+2}<\tau<t}\int_{B_{n+2}} v_{n+1}^{\beta}dx\Big)^{(1-\omega_1)\frac{1+\beta}{\beta}}.$$

Applying now to the right-hand side of the last relation the Young inequality with $\varepsilon=\delta/2$ (where δ is sufficiently small and will be chosen below), we obtain

$$I_1 \leq \frac{\delta}{2}\iint_{Q_{n+2}} |\nabla v_{n+1}|^p\,dx$$
$$+C_\delta\Big(b^{\frac{1}{1-\omega_1\frac{1+\beta}{p}}}\Big)^n \frac{(t-t_{n+2})}{(t_1-t_2)^{\frac{1}{1-\omega_1\frac{1+\beta}{p}}}}\Big(\sup_{t_{n+2}<\tau<t}\int_{B_{n+2}} v_{n+1}^{\beta}dx\Big)^{\frac{(1-\omega_1)\frac{1+\beta}{\beta}}{1-\omega_1\frac{1+\beta}{p}}}$$
$$\leq \frac{\delta}{2}\iint_{Q_{n+2}} |\nabla v_{n+1}|^p\,dx + C_\delta b^n\frac{(t-t_2)}{(t_1-t_2)^{\frac{1}{1-\omega_1\frac{1+\beta}{p}}}}E^{M_1}, \tag{3.4}$$

where we denote

$$E=\sup_{t_2<\tau<t}\int_{B_{r_2}} u^{\beta}(x,\tau)dx, \qquad M_1=\frac{(1-\omega_1)\frac{1+\beta}{\beta}}{1-\omega_1\frac{1+\beta}{p}}.$$

Let us now turn to the integral I_2 in the right-hand side of (3.3). If $p>\beta$, then, performing analogous estimates for the expression I_2 in the right-hand side of (3.3), we have step by step:

$$\int_{B_{n+2}} v_{n+1}^{p}dx \leq C\Big(\int_{B_{n+2}} |\nabla v_{n+1}|^p\,dx\Big)^{\omega_2}\Big(\int_{B_{n+2}} v_{n+1}^{\beta}dx\Big)^{(1-\omega_2)\frac{p}{\beta}},$$

where ω_2 is determined from the relation

$$\frac{1}{p}=\omega_2\Big(\frac{1}{p}-\frac{1}{N}\Big)+(1-\omega_2)\frac{1}{\beta}.$$

Further we integrate with respect to time:

$$I_2=b^n(r_2-r_1)^{-p}\iint_{Q_{n+2}} v_{n+1}^{p}dxd\tau$$

$$\leq \Big(\iint_{Q_{n+2}} |\nabla v_{n+1}|^p dx d\tau \Big)^{\omega_2} b^n \frac{(t-t_{n+1})^{1-\omega_2}}{(r_2-r_1)^p} \Big(\sup_{t_{n+2}<\tau<t} \int_{B_{n+2}} v_{n+1}^{\beta} dx \Big)^{(1-\omega_2)\frac{p}{\beta}}.$$

Applying, at last, the Young inequality with $\delta/2$, we obtain, as before,

$$I_2 \leq \frac{\delta}{2} \iint_{Q_{n+2}} |\nabla v_{n+1}|^p \, dx d\tau + C_\delta b^n \frac{(t-t_{n+1})}{(r_2-r_1)^{\frac{p}{1-\omega_2}}} E^{M_2}, \tag{3.5}$$

where $M_2 = \frac{p}{\beta}$.

If now $p \leq \beta$, a simple application of the Hölder inequality (taking into account the condition $r_1 < C(r_2 - r_1)$ and the value of the expression $p/(1-\omega_2)$ in (3.5)) gives the estimate (3.5) without the first term in the right-hand side and without b^n.

Thus, applying estimates (3.4) and (3.5) to inequality (3.3), we obtain

$$Y_n \equiv \sup_{t_{n+1}<\tau<t} \int_{B_{n+1}} v_n^{1+\beta}(x,\tau) dx + \iint_{Q_{n+1}} |\nabla v_n|^p \, dx d\tau + \iint_{Q_{n+1}} v_n^{1+r} dx d\tau$$
$$\leq \delta \iint_{Q_{n+2}} |\nabla v_{n+1}|^p \, dx d\tau + C_\delta b^n A, \tag{3.6}$$

where

$$A \equiv \frac{(t-t_2)}{(t_1-t_2)^{\frac{1}{1-\omega_1 \frac{1+\beta}{p}}}} E^{M_1} + \frac{(t-t_{n+1})}{(r_2-r_1)^{\frac{p}{1-\omega_2}}} E^{M_2}.$$

Applying further inequality (3.6) in succession on n starting from Y_0, we obtain that

$$Y_0 \leq \delta^n \iint_{Q_{n+1}} |\nabla v_{n+1}|^p \, dx d\tau + \Big(\sum_{k=0}^{n} (b\delta)^k \Big) C_\delta A. \tag{3.7}$$

Note now, that

$$\iint_{Q_{n+1}} |\nabla v_{n+1}|^p \, dx d\tau \leq C b^n \Big(\int_{t_2}^{t} \int_{B_{r_2}} (|\nabla u|^p + u^p) \, dx d\tau \Big) \leq C(u) b^n.$$

Choosing, at last, δ from the condition $\delta b = 1/2$ and passing to the limit in (3.7), we obtain, that

$$Y_0 \leq CA.$$

Calculating now explicitly numbers ω_1 and ω_2 from the corresponding relations and calculating exponents $M_1 = \frac{k+p}{k}$, $M_2 = \frac{p}{\beta}$, $1/(1-\omega_1 \frac{1+\beta}{p}) = \frac{k+N}{k}$, $p/(1-\omega_2) = \frac{k+N}{\beta}$, we arrive at the assertion of the lemma. □

Lemma 3.2. *Let restrictions* (1.3), (1.4) *be satisfied. Let further* R_1, R_2 *and* $Y(t/2, R_2)$ *be the same as in Lemma* 2.1, $R_3 = R_2(1+\sigma)$. *Then there* *exists a constant* $\gamma_3 > 0$, *such that the condition of Lemma* 2.1 *is satisfied; that is,*

$$Y(t/2, R_2) \le \gamma_2 t^{\frac{Nd_r + p(1+\beta)}{p(\beta - r)}} \equiv \gamma_2 t^{\nu}, \quad \nu \equiv \frac{Nd_r + p(1+\beta)}{p(\beta - r)},$$

if

$$E \equiv E(t, R, \sigma) \equiv \sup_{t/4<\tau<t} \int_{B_{R(1+\sigma)t^{\varkappa}}(x_0)} u^{\beta} dx \le \gamma_3 t^{\frac{\beta}{\beta - r} + N\varkappa}.$$

Proof. Let us use Lemma 3.1 for the proof of this lemma. Let us put in estimate (3.1) $r_1 = R_2 = Rt^{\varkappa}$, $r_2 = R_3 = R(1+\sigma)t^{\varkappa}$, $t_1 = t/2$, $t_2 = t/4$. Then the estimate (3.1) becomes

$$Y(t/2, r_1) \le C\left(t^{-\frac{N}{k}} E^{\frac{k+p}{k}} + t^{1-\varkappa\frac{k+N}{\beta}} E^{\frac{p}{\beta}}\right) \equiv I_1 + I_2.$$

We wish to find conditions on E, for which the following is satisfied:

$$I_1 \le \frac{\gamma_2}{2} t^{\nu}, \quad I_2 \le \frac{\gamma_2}{2} t^{\nu}. \tag{3.8}$$

The first condition in (3.8) is satisfied if for sufficiently small $\overline{\gamma}_3$

$$E^{\frac{k+p}{k}} \le \overline{\gamma}_3 t^{\frac{N}{k} + \nu};$$

that is, for some γ_3,

$$E \le \gamma_3 t^{\left(\frac{N}{k} + \nu\right)\frac{k}{k+p}} = \gamma_3 t^{\frac{\beta}{\beta - r} + N\varkappa}, \tag{3.9}$$

as will follow from elementary calculations with the use of the definitions of k, ν, $\varkappa$.

Let us turn now to the second condition in (3.8). It looks like

$$E^{\frac{p}{\beta}} \le \widetilde{\gamma}_3 t^{(-1+\varkappa\frac{k+N}{\beta})} t^{\nu},$$

with some sufficiently small $\widetilde{\gamma}_3$. Hence, the second condition in (3.8) is satisfied if for some sufficiently small γ_3

$$E \le \gamma_3 t^{\frac{\beta}{p}[(-1+\varkappa\frac{k+N}{\beta})+\nu]} = \gamma_3 t^{\frac{\beta}{\beta - r} + N\varkappa},$$

as will follow from an elementary calculation of the exponent. Decreasing, if we need, the constant γ_3 in (3.9), we see, that under this condition both of the estimates in (3.8) are valid. So Lemma 3.2 is proved. □

4. An estimate of the local mass of the solution in terms of the local mass of the initial datum

In this section we will obtain an estimate of the local mass of the solution $E(0,R)$ in terms of the local mass of the initial datum.

Lemma 4.1. *Let restrictions* (1.3), (1.4) *be satisfied. Let also* $t > 0$, γ_3 *and* $E(t,R,\sigma)$ *be quantities from Lemma* 3.2. *There exist sufficiently large number* $L > 0$, *such that*

$$E(t,R,\sigma) \le C \int_{B_{Lt^{\varkappa}}(x_0)} u_0^{\beta}(x)dx + \frac{\gamma_3}{2} t^{\frac{\beta}{\beta-r}+N\varkappa}. \tag{4.1}$$

Proof. Let $\rho > 0$, $B_\rho = B_\rho(x_0)$. For the solution $u(x,t)$ of the problem (1.1), (1.2), under the accepted assumptions (fast diffusion), in the papers [10, Theorem 1.4] and [20, Lemma III.3.1] the following estimate was proved:

$$\sup_{0<\tau<t} \int_{B_\rho} u^{\beta}(x,\tau)dx \le C \int_{B_{2\rho}} u_0^{\beta}(x)dx + C(\frac{t}{\rho^k})^{\frac{1}{-d}}.$$

Choosing in this inequality $\rho = \frac{L}{2} t^{\varkappa}$ with sufficiently large $L = L(R,\sigma,\gamma_3) > 2R(1+\sigma)$ (to make the factor at t in the second term sufficiently small) and calculating the total exponent of t in the second term in the right-hand side of the last inequality we obtain the assertion of the lemma. □

Remark 4. It is possible to show that an estimate analogous to (4.1) is valid and in the case of slow diffusion and the second term in the right-hand side of this estimate being absent, see [22].

5. Proof of Theorem 1

5.1. **The estimate from above of the support.** Assume at first $u_0(x) \ge 0$ and $\varphi_t(\rho) \to 0$ as $\rho \to \infty$. Fix some number $\sigma \in (0,1)$ and take $R = 1$ in Lemmas 4.1, 3.2 and 2.1. Fix $t < t_0$ and let $x_0 \in R^N$ be such that $|x_0| \ge \varphi_t^{-1}(\gamma_0 t^{\frac{\beta}{\beta-r}})$, where γ_0 is sufficiently small and will be chosen later.

Then for this x_0, by virtue of Lemma 4.1, the following will be satisfied (with the notation of Lemma 3.2):

$$\begin{aligned} E(t,1,\sigma) &\le C \int_{|x-x_0|<Lt^{\varkappa}} u_0^{\beta}(x)dx + \frac{\gamma_3}{2} t^{\frac{\beta}{\beta-r}+N\varkappa} \\ &\le (C\omega_N L^N \gamma_0 + \frac{\gamma_3}{2}) t^{\frac{\beta}{\beta-r}+N\varkappa}. \end{aligned} \tag{5.1}$$

Decreasing, if we need, the number γ_0, we see, that for this x_0 the conditions of Lemma 3.2 are satisfied. Thus, by virtue of Lemma 3.2, the conditions of

Lemma 2.1 are satisfied with $R_1 = (1-\sigma)(1+\sigma)^{-2}t^{\varkappa}$, $R_2 = (1+\sigma)^{-2}t^{\varkappa}$ and therefore, $u \equiv 0$ on the set $[3t/4, t] \times B_{(1-\sigma)(1+\sigma)^{-2}t^{\varkappa}}(x_0)$. So, for the upper bound of the support of the solution we get the estimate $D(t) \leq \varphi_t^{-1}(\gamma_0 t^{\frac{\beta}{\beta-r}})$, that is, the estimate (1.17) if t is sufficiently small.

If now the initial datum $u_0(x)$ is of arbitrary sign, then

$$-|u_0(x)|^{\beta} \leq |u_0(x)|^{\beta-1}u_0(x) \leq |u_0(x)|^{\beta}$$

and the estimate from above of the support follows from what was proved by comparison. The estimate (1.17) is proved.

5.2. **The estimate from below of the size of the support.** Turning now to the proof of the estimate (1.18) we note that we will use the comparison principle for solutions of the Cauchy problem (1.1), (1.2). Note also that for $p = 2$ or $\beta = 1$ it is well known (see, for example, [17]-[19] for the case $p = 2$ and [20] for $\beta = 1$), and for the general case $p \neq 2$ and simultaneously $\beta \neq 1$ it follows from the results of the paper [21]. So, we use the fact that for two initial data $|u_0(x)|^{\beta-1}u_0(x)$ and $|v_0(x)|^{\beta-1}v_0(x)$ in (1.2) such that $|u_0(x)|^{\beta-1}u_0(x) \leq |v_0(x)|^{\beta-1}v_0(x)$, for the corresponding solutions of the Cauchy problem (1.1), (1.2) we have $u(x,t) \leq v(x,t)$.

Let $|u_0(x)|^{\beta-1}u_0(x)$ be an arbitrary nonnegative locally integrable function or locally finite Radon measure. Let $t > 0$ be sufficiently small and fixed and let $|x_0| \leq \varphi_t^{-1}(\gamma_0 t^{\frac{\beta}{\beta-r}})$; that is,

$$\int_{|x-x_0|<Lt^{\varkappa}} u_0^{\beta}(x)dx \geq \omega_N L^N \gamma_0 t^{\frac{\beta}{\beta-r}+N\varkappa}. \tag{5.2}$$

Denote

$$v_0^{\beta}(x) = v_{0,t}^{\beta}(x) = \begin{cases} u_0^{\beta}(x), & |x-x_0| \leq Lt^{\varkappa}, \\ 0, & |x-x_0| > Lt^{\varkappa}, \end{cases}$$

and denote by $v(x,\tau)$ the corresponding solution of the Cauchy problem (1.1), (1.2) with the initial datum $v_0^{\beta}(x)$. Then, according to the comparison principle, $u(x,\tau) \geq v(x,\tau)$.

Consider now the point x_0 as the origin of the space R^N. In view of the definition of the function $v_0(x)$, from estimate (1.17) of the support from above, which is proved already, we infer that, for $\tau \in [0,t]$ and for all points x such that $|x-x_0| > 2Lt^{\varkappa}$, we have $v(x,\tau) \equiv 0$. Thus the support of $v(x,\tau)$ for $\tau \in [0,t]$ is a subset of $B_t = \{x : |x-x_0| < 3Lt^{\varkappa}\}$.

Let us now integrate equation (1.1) over the ball B_t and take into account that the solution is equal to zero in a neighborhood of the boundary of this ball. Integration by parts in the diffusive term gives

$$\frac{d}{d\tau}\int_{B_t} v^\beta dx + \int_{B_t} v^r dx = 0, \quad \tau \in [0,t]. \tag{5.3}$$

An application of the Hölder inequality gives

$$\int_{B_t} v^r dx \le \Big(\int_{B_t} v^\beta dx\Big)^{\frac{r}{\beta}}\Big(\int_{B_t} dx\Big)^{1-\frac{r}{\beta}} = \Big(\int_{B_t} v^\beta dx\Big)^{\frac{r}{\beta}} M t^{N\varkappa\frac{\beta-r}{\beta}}, \tag{5.4}$$

where $M = M(L)$. Denoting now

$$E(\tau) = \int_{B_t} v^\beta(x,\tau)dx,$$

we obtain from (5.3) and (5.4) that

$$\frac{dE}{d\tau} \ge -Mt^{N\varkappa\frac{\beta-r}{\beta}} E^{\frac{r}{\beta}},$$

and we note that

$$E(0) = \int_{B_t} u_0^\beta(x)dx \ge \omega_N L^N \gamma_0 t^{\frac{\beta}{\beta-r}+N\varkappa} \equiv \gamma t^{\frac{\beta}{\beta-r}+N\varkappa}.$$

Integrating this differential inequality, we arrive at the estimate

$$\begin{aligned} E(\tau)^{\frac{\beta-r}{\beta}} &\ge E(0)^{\frac{\beta-r}{\beta}} - \Big(\frac{\beta-r}{\beta}\Big) M t^{N\varkappa\frac{\beta-r}{\beta}}\tau \\ &\ge \Big(\gamma t^{\frac{\beta}{\beta-r}+N\varkappa}\Big)^{\frac{\beta-r}{\beta}} - \Big(\frac{\beta-r}{\beta}\Big) M t^{N\varkappa\frac{\beta-r}{\beta}}\tau \\ &= t^{N\varkappa\frac{\beta-r}{\beta}}\Big[\gamma^{\frac{\beta-r}{\beta}} t - \Big(\frac{\beta-r}{\beta}\Big) M\tau\Big]. \end{aligned}$$

Define $\tau_0 = \frac{1}{2}\frac{\gamma^{\frac{\beta-r}{\beta}}}{M}\frac{\beta}{\beta-r}t \equiv m_0 t$. Then

$$E(m_0 t) > \Big(\gamma^{\frac{\beta-r}{\beta}}/2\Big) t^{N\varkappa\frac{\beta-r}{\beta}+1} > 0.$$

Thus, we obtain that, for some $m_0 = m_0(N,\sigma,\beta,r)$,

$$E(m_0 t) \equiv \int_{B_t(x_0)} v^\beta(x, m_0 t) > 0.$$

Hence, $u(x, m_0 t) \ge v(x, m_0 t) > 0$ in the neighborhood of the point x_0. From this two conclusions follow.

On the one hand, in the case when $\varphi_t(x_0)$ does not tend to zero as $|x_0| \to \infty$ we have points x_0 with the assumed property (5.2) at any small time t arbitrarily far from the origin. This proves the absence of the instantaneous support compactification: for any small moment of time of the form m_0t there is the point $x_0 = x_0(t)$ arbitrarily far from the origin, such that $u(x, m_0t) > 0$ in a neighborhood of this point.

On the other hand, if $\varphi_t(x_0) \to 0$ on $|x_0| \to \infty$, let us define $x_0 = x_0(t)$, where $|x_0| = \varphi_t^{-1}(\gamma_0 t^{\frac{\beta}{\beta-r}})$. Then

$$\oint_{|x-x_0|<m_0^{-\varkappa}L(m_0t)^{\varkappa}} u_0^{\beta}(x)dx \geq \gamma_0 m_0^{-\frac{\beta}{\beta-r}}(m_0t)^{\frac{\beta}{\beta-r}} \equiv M_0(m_0t)^{\frac{\beta}{\beta-r}},$$

and at the same time for all $y \in R^N$, such that $|y| > |x_0|$, according to the definition of the function $\varphi_t^{-1}(s)$ the following is satisfied:

$$\oint_{|x-y|<m_0^{-\varkappa}L(m_0t)^{\varkappa}} u_0^{\beta}(x)dx < \gamma_0 m_0^{-\frac{\beta}{\beta-r}}(m_0t)^{\frac{\beta}{\beta-r}} = M_0(m_0t)^{\frac{\beta}{\beta-r}}.$$

That is, according to the definition,

$$|x_0| = \varphi^{-1}_{m_0^{-1}(m_0t)}(M_0(m_0t)^{\frac{\beta}{\beta-r}}),$$

and $u(x_0, m_0t) \neq 0$. As t is arbitrary, denoting m_0t again by t, we see, that for any small $t > 0$ there is the point $x_0 = x_0(t/m_0)$ such that

$$|x_0(t/m_0)| = \varphi^{-1}_{m_0^{-1}t}(M_0 t^{\frac{\beta}{\beta-r}}), \qquad u(x_0(t/m_0), t) \neq 0.$$

Hence, estimate (1.18) is proved with $M_1 = m_0^{-1}$ and $\gamma_1 = M_0$, and, consequently, Theorem 1 is proved. □

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References

[1] R. Kersner and A. Shishkov, *Instantaneous shrinking of the support of energy solutions,* J. Math. Anal. Appl., 198 (1996), 729–750.

[2] A.E. Shishkov, *Dead cores and instantaneous compactification of the supports of energy solutions of quasilinear parabolic equations of arbitrary order,* Sbornik, Mathematics, 190 (1999), 1843–1869.

[3] S.N. Antontsev, J.I. Diaz, and S.I. Shmarev, *The support shrinking properties for solutions of quasilinear parabolic equations with strong absorption terms,* Ann. Fac. Sci. Toulouse Math., 6 (1995), 5–30.

[4] S.N. Antontsev, J.I. Diaz, and S.I. Shmarev, "Energy methods for the free boundary problems. Applications to nonlinear PDEs and fluid mechanics," 2002, Birkhauser, 334P.

[5] M. Ughi, *Initial behavior of the free boundary for a porous media equation with strong absorption,* Advances in Math. Sciences and Applications, Gakkotosho, Tokyo, 11 (2001), 333–345.

[6] A.S. Kalashnikov, *On the dependence of properties of solutions of parabolic equations in unbounded domains on the behavior of the coefficients at infinity,* Math. USSR Sb., 53 (1986), 399–410.

[7] A.S. Kalashnikov, *On the behavior of solutions of the Cauchy problem for parabolic systems with nonlinear dissipation near the initial hyperplane,* Trudy Sem. Petrovsk., 16 (1992), 106–117.

[8] U.G. Abdullaev, *Instantaneous shrinking of the support of solutions to a nonlinear degenerate parabolic equation,* Math. Notes, 63 (1998), 285–292.

[9] U.G. Abdullaev, *Exact local estimates for the supports of solutions in problems for nonlinear parabolic equations,* Mat. Sb., 186 (1995), 3–24.

[10] Kazuhiro Ishige, *On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation,* SIAM J. Math. Anal., 27 (1996), 1235–1260.

[11] H.J. Fan, *Cauchy Problem of Some doubly degenerate parabolic equations with initial datum a measure,* Acta Mathematica Sinica, English Series, 20 (2004), 663–682.

[12] D. Andreucci and A.F. Tedeev, *Universal bounds at the blow-up time for nonlinear parabolic equations,* Advances in Differential Equations, 10 (2005), 89–120.

[13] D. Andreucci and A.F. Tedeev, *Finite speed of propagation for the thin film equation and other higher order parabolic equations with general nonlinearity,* Interfaces and free boundaries, 3 (2001), 233–264.

[14] D. Andreucci and A.F. Tedeev, *A Fujita type result for a degenerate Neumann problem in domains with non compact boundary,* J. Math. Anal. Appl., 231 (1999), 543–567.

[15] F. Bernis, *Finite speed of propagation and asymptotic rates for some nonlinear higher order parabolic equations with absorption,* Proceedings of the Royal Society of Edinburgh, 104A (1986), 1–19.

[16] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Uraltseva, "Linear and Quasilinear Equations of Parabolic Type," Amer. Math. Soc., Providence, Rhode Island, 1968.

[17] M. Bertsch, *A class of degenerate diffusion equations with a singular nonlinear term,* Nonlinear Analysis, Methods&Applications, 7 (1983), 117–127.

[18] M. Bertsch, R. Kersner, and L.A. Peletier, *Positivity versus localization in degenerate diffusion equations,* Nonlinear Analysis, Methods&Applications, 9 (1985), 987–1008.

[19] D. Aronson, M.G. Crandall, and L.A. Peletier, *Stabilization of solutions of a degenerate nonlinear diffusion problem,* Nonlinear Analysis, Methods&Applications, 6 (1982), 1001–1022.

[20] E. Di Benedetto and M.A. Herrero, *Non-negative Solutions of the Evolution p-Laplacian Equation. Initial Traces and Cauchy Problem when* $1 < p < 2$*,* Archive Ration. Mech. and Anal., 111 (1990), 225–290.

[21] M. Tsutsumi, *On solutions of some doubly nonlinear degenerate parabolic equations with absorption,* J. Math. Anal. Appl, 132 (1988), 187–212.

[22] S.P. Degtyarev, *Conditions for instantaneous support shrinking and sharp estimates for the support of the solution of the Cauchy problem for a doubly non-linear parabolic equation with absorption,* Sbornik, Mathematics, 199, (2008), 511–538.