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THE EVOLUTION OF DEAD CORES IN STRONGLY DEGENERATE DIFFUSION PROBLEMS

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Abstract. This paper studies the large time behavior of nonnegative solutions of

$$u_t = u^p \Delta u - u^{-q} \chi_{\{u>0\}} \qquad \text{in } \Omega \times (0,\infty), \qquad p \ge 1, q > -1,$$

with prescribed boundary values $u|_{\partial\Omega} = 1$ in smoothly bounded domains $\Omega \subset \mathbb{R}^n$. The particular interest here is in the question of how the interplay of strongly degenerate diffusion and strong absorption influences the evolution of the dead core set $\{u(t) = 0\}$ (which is enforced to be nontrivial for all time by the assumption that the initial data u_0 vanish in some ball). The main results state that there exist four parameter regimes in which, widely independent of u_0, all solutions undergo

(i) a complete extinction at time $t = 0$ ($q > p - 1$)
(ii) a total extinction in finite but positive time ($0 < q \le p - 1$),
(iii) an extinction in infinite time ($1 - p < q < 0$), or
(iv) no extinction ($q < 1 - p$),

respectively. Specifically, the first two types of behavior are in sharp contrast to the non-degenerate and weakly degenerate cases $p = 0$ and $p \in (0,1)$, respectively, where only (iv) occurs.

Introduction

This paper deals with nonnegative solutions of the problem

$$\begin{aligned} u_t &= u^p \Delta u - u^{-q} \chi_{\{u>0\}} \qquad \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} &= 1, \\ u|_{t=0} &= u_0, \end{aligned} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with boundary of class C^3,

$$p \ge 1, \qquad q > -1,$$

and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points where u is positive.

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For general $p \geq 0$, there are several methods to prove sufficient criteria for the finite-time onset of a *dead core* of initially positive solutions: In particular, if u_0 is small enough, for instance, then $\inf_{x\in\Omega} u(x,t) \to 0$ as $t \nearrow T$ occurs for some finite T – see [16], [17], [1] or [24], for instance. One field of interest is then to study the behavior of such "quenching solutions" near their (more precisely: shortly *before* their) quenching time T; various results of this type, e.g. on quenching rates and asymptotic profiles, can be found in [8], [6], [24], [13], [14] and [15]. As a classical survey of earlier results on the interplay between diffusion and strong absorption we recommend [19].

If one wishes to analyze the behavior of solutions *after* such a quenching, one is first led to the question whether a reasonable extension beyond the onset of the singularity can be found at all. Situations where this is possible are usually referred to as *incomplete extinction*, whereas on the contrary *complete extinction* is said to occur if only the trivial extension $u \equiv 0$ is adequate beyond the quenching time. Satisfactory results concerning complete versus incomplete extinction and life beyond quenching have up to now, as it seems, only been found for the non-degenerate case $p = 0$ and in the weakly degenerate case $p \in (0,1)$. Observe that in the latter, via the substitution $v = u^{1-p}$, the PDE in (0.1) actually transforms into the porous medium equation with strong absorption,

$$v_t = \nabla \cdot \left(v^{\frac{p}{1-p}} \nabla v\right) - (1-p)v^{-\frac{p+q}{1-p}}. \tag{0.2}$$

Restated in terms of our parameters p and q, it was shown in [3] and later on in [2] that for such p and all $q < -p$, quenching solutions with positive initial data can be extended beyond their quenching time in a natural way so as to exist as global weak solutions. That extinction is incomplete for such p actually throughout the whole range $q < 1-p$ is indicated by the results in [9], which assert this in the spatially one-dimensional setting for bell-shaped initial data; within the same framework, extinction was shown in [9] to be complete if $q \geq 1-p$.

Concerning qualitative properties beyond quenching, it was proved in [3] and [2] that if u is decreasing with time then the *dead core set*

$$\{u(t) = 0\} \equiv \{x \in \Omega : u(x,t) = 0\}$$

of the natural weak extension satisfies

$$\{u(t) = 0\} \nearrow D_\infty \subset\subset \Omega$$

as $t \to \infty$. More precisely, a re-inspection and straightforward extension of the arguments in [22] shows that for all $p \in [0,1)$ and $q \in (-1, 1-p)$ the set $\{u(t)=0\}$, no matter whether it converges or not as $t\to\infty$, remains uniformly bounded away from $\partial\Omega$ for all times.

It is the purpose of the present paper to demonstrate that this situation becomes completely different in the *strongly degenerate* case $p\geq 1$. Then, namely, we encounter a much larger variety of possible asymptotic behaviors of the dead core set. More specifically, supposing henceforth for convenience that $u_0\in W^{1,\infty}(\Omega)$ satisfies $u_0|_{\partial\Omega}=1$ and the technical assumption

$$u_0\equiv 0 \quad \text{in some ball contained in } \Omega, \tag{0.3}$$

we shall see that

- if $p\in[1,2)$ and $q<1-p$ then $\{u(t)=0\}$ remains bounded away from $\partial\Omega$ uniformly with respect to t (Theorem 4.2);
- if $p>1$ and $q\in(1-p,0)$ then $\{u(t)=0\}$ approaches all of Ω, but only in infinite time (Theorem 3.8);
- when $p>1$ and $q\in(0,p-1]$ then the set $\{u(t)=0\}$ reaches the whole of Ω in some finite positive time (Theorem 2.3);
- in the case $p\geq 1$ and $q>p-1$, we have $u\equiv 0$ in $\Omega\times(0,\infty)$ (Lemma 1.2).

We thus have 'persistence' as before in the case of mildly strong absorption and not too large degeneracies; but we also find total extinction both in finite and in infinite time, and even complete extinction at $t=0$. These phenomena, a little surprising in view of the nonzero Dirichlet data, clearly reflect the decreasing influence of diffusion in the presence of strong degeneracies and sufficiently strong absorption.

In particular we note that the qualitative behavior of the dead core set does not, except through (0.3), depend on u_0. This may give rise to the conjecture that our results continue to hold for all quenching solutions (not necessarily fulfilling (0.3)), but we do not pursue this here.

Let us mention that also *without* the absorption term in (0.1), the exponent $p=1$ appears to be critical in respect of positivity and other properties such as regularity and uniqueness (cf. [5], [4], [26]). This is already indicated by the dramatic changes that the ingredients of (0.2) undergo when p crosses the value $p=1$: Namely, for $p>1$ the function $v=u^{1-p}$ actually becomes an *inverse* power of u and satisfies the *very fast diffusion equation* with superlinear *source*,

$$v_t=\nabla\cdot\left(v^{m-1}\nabla v\right)+(p-1)v^r \tag{0.4}$$

with $m=-\frac{1}{p-1}<0$ and $r=\frac{p+q}{p-1}>1$. Clearly, incomplete and complete extinction of u translate to the (similarly defined) phenomena of incomplete and complete blow-up for v. In light of this transformation, one might draw a parallel between one of our results and those in [9]: It was shown there that blow-up for one-dimensional solutions of (0.4) emanating from bell-shaped data is incomplete if and only if $r+m \leq 2$, provided that $m > 0$ and $r > 1$. If one formally extends this statement to arbitrary $m \in \mathbb{R}$ and chooses m and r as specified above so as to fit (0.1), one exactly rediscovers our condition $q \leq p-1$ necessary and sufficient for u to be nontrivial.

The paper is organized as follows: In Section 1, we obtain a function u as the limit of a decreasing family of solutions u_ε to suitably regularized versions of (0.1), and show that u solves (0.1) in some (very weak) sense. Moreover, we find that the exponent $q = p-1$ is critical in the sense that u is nontrivial for all initial data (satisfying (0.3)) if and only if $q \leq p-1$. After that, in Sections 2–4 we establish the mentioned (non-)extinction results for the cases $q < 1-p$, $q \in (0, 1-p)$ and $q > 0$, respectively. The proofs are mainly based on maximum principle arguments applied to u_ε and explicit sub- and supersolutions; therefore our method may in part be compared to travelling wave techniques that have been fruitfully applied to establish (in-)complete blow-up or extinction results ([9]), but also to illuminate the evolution of positivity sets in several contexts (see [11], [12], for instance).

1. Preliminaries

1.1. **Approximating a very weak solution.** When searching for an appropriate generalized solution concept for (0.1), one faces several difficulties arising from the degeneracy in the diffusion part and the singularity in the reaction term; various results on the loss of regularity near points where u is small can be found for equations like (0.1) in [25] and – even without any absorption – in [4]. But apart from these, a new problem results from the fact that u is supposed to satisfy the *nonhomogeneous* boundary condition in (0.1): In view of the results announced in the introduction, there is no hope for continuity or Sobolev regularity up to the boundary – not even for smooth initial data – because for $q > 0$ we expect our 'solutions' to vanish identically in Ω beyond some finite time.

Let us therefore start our "existence theory" by following a natural way of seeking solutions to problems of type (0.1), which consists of regularizing the problem suitably (cf. [25] for more details): For $\varepsilon \in (0,1)$, we let $u_{0\varepsilon} := u_0 + \varepsilon$ and $g_\varepsilon(s) := \chi(\frac{s}{\varepsilon}) s^{-q}$ for $s > 0$, where $\chi : [0,\infty) \to \mathbb{R}$ is smooth and nondecreasing with $\chi \equiv 0$ on $(0,1)$ and $\chi \equiv 1$ in $(2,\infty)$. Then the boundary value problems

$$u_{\varepsilon t} = u_\varepsilon^p \Delta u_\varepsilon - g_\varepsilon(u_\varepsilon) \qquad \text{in } \Omega \times (0,\infty),$$

$$u_\varepsilon|_{\partial\Omega} = 1+\varepsilon, \qquad (1.1)$$
$$u|_{t=0} = u_{0\varepsilon},$$

have unique classical solutions u_ε which in view of parabolic comparison satisfy

$$\varepsilon \le u_\varepsilon \le \|u_0\|_{L^\infty(\Omega)} + \varepsilon \qquad \text{in } \Omega \times (0,\infty). \qquad (1.2)$$

Again by comparison, the fact that $g_\varepsilon(s) \nearrow s^{-q}\chi_{\{s>0\}}$ as $\varepsilon \searrow 0$ implies that the u_ε are ordered and hence

$$u_\varepsilon \searrow u \qquad \text{as } \varepsilon \to 0 \qquad (1.3)$$

is valid for some nonnegative, bounded and upper semicontinuous function u. In order to find out in which way u might "solve" (0.1), we let $\varphi \in C^2(\bar\Omega)$ be nonnegative such that $\varphi|_{\partial\Omega} = 0$ and multiply (1.1) by $\chi_\delta(u_\varepsilon)u_\varepsilon^{-p}\varphi$ for arbitrary but fixed $\delta > 0$. Writing

$$H_\delta(s) := \int_1^s \chi_\delta(\sigma)\sigma^{-p}d\sigma \quad \text{and} \quad \rho_\delta(s) := \int_0^s \chi_\delta(\sigma)d\sigma$$

for $s \ge 0$, we obtain after some integration by parts over $\Omega\times(s,t)$, $0 \le s < t$, the identity

$$\begin{aligned}\int_\Omega H_\delta(u_\varepsilon(t))\varphi &= \int_\Omega H_\delta(u_\varepsilon(s))\varphi + \int_s^t\int_\Omega \chi_\delta(u_\varepsilon)\Delta u_\varepsilon\varphi - \int_s^t\int_\Omega u_\varepsilon^{-p}g_\varepsilon(u_\varepsilon)\varphi \\ &= \int_\Omega H_\delta(u_\varepsilon(s))\varphi - \int_s^t\int_\Omega \chi'_\delta(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi + \int_s^t\int_\Omega(\rho_\delta(u_\varepsilon) - \rho_\delta(1+\varepsilon))\Delta\varphi \\ &\quad - \int_s^t\int_\Omega \chi_\delta(u_\varepsilon)u_\varepsilon^{-p}g_\varepsilon(u_\varepsilon)\varphi. \end{aligned} \qquad (1.4)$$

Dropping the nonpositive term containing χ' and applying the dominated convergence theorem in letting $\varepsilon \to 0$, we see that

$$\int_\Omega H_\delta(u(t))\varphi \le \int_\Omega H(u(s))\varphi + \int_s^t\int_\Omega(\rho_\delta(u)-\rho_\delta(1))\Delta\varphi - \int_s^t\int_\Omega \chi_\delta(u)u^{-q-p}\varphi. \qquad (1.5)$$

Using the monotone convergence theorem, we may take $\delta \searrow 0$ to end up with

$$\int_\Omega H(u(t))\varphi \le \int_\Omega H(u(s))\varphi + \int_s^t\int_\Omega (u-1)\Delta\varphi - \int_s^t\int_\Omega \chi_{\{u>0\}}u^{-q-p}\varphi, \qquad (1.6)$$

where

$$H(s) := \int_1^s \frac{ds}{s^p} = \begin{cases} \ln s & \text{if } p = 1, \\ -\frac{1}{p-1}\left(s^{1-p} - 1\right) & \text{if } p > 1. \end{cases}$$

Observe that since $H(0) := \lim_{s\searrow 0} H(s) = -\infty$ due to the fact that $p \geq 1$, both sides in (1.6) may attain the value $-\infty$, so that (1.6) is to be understood as an inequality in $[-\infty, \infty)$.

It is not very satisfactory to have an *in*equality as the only feature that relates u to (0.1); but equality in (1.6) only seems to be achievable if one can control the second term on the right-hand side of (1.4) when letting ε and δ tend to zero. This at least amounts to deriving spatial $W^{1,2}(\Omega)$ estimates on u_ε which, as mentioned above, are definitely not available in general. However, if we restrict φ so as to satisfy additionally $\operatorname{supp}\varphi \subset \bigcap_{\sigma\in[s,t]}\{u(t) > \eta\}$ for some $\eta > 0$, then the second term on the right of (1.4) vanishes whenever δ is small enough, so that we finally obtain (1.6) with equality.

Lemma 1.1. *The function* $u := \lim_{\varepsilon\to 0} u_\varepsilon$ *is a bounded and upper semicontinuous* very weak solution *of* (0.1) *in the sense that for any nonnegative* $\varphi \in C^2(\bar{\Omega})$ *satisfying* $\varphi|_{\partial\Omega} = 0$*, the inequality* (1.6) *holds in* $[-\infty, \infty)$*, and equality in* (1.6) *holds whenever* φ *additionally satisfies*

$$\operatorname{supp}\varphi \subset \bigcap_{\sigma\in[s,t]} \{u(t) > \eta\} \qquad \textit{for some } \eta > 0. \tag{1.7}$$

Note that, unfortunately, the function $u \equiv 0$ *always* is a very weak solution in the above sense, because in this case the left-hand side in (1.6) equals $-\infty$ for all $t > 0$ and the set of test functions satisfying (1.7) is empty. It would be interesting to study further properties of $u = \lim u_\varepsilon$ and thereby possibly obtain a more selective characterization of how it solves (0.1), possibly being *maximal* among all "solutions" (which is true in many less degenerate cases, cf. [21]). In one space dimension, it can be shown that u is continuous in $\Omega \times [0, \infty)$ and that u^p in fact belongs to $L^\infty((0,\infty); BV(\bar{\Omega}))$, provided that $q < p - 1$. As we are primarily interested in the asymptotic properties of u, however, we confine ourselves in this paper to the result that "$u = \lim_{\varepsilon\to 0}$ solves (0.1) in *some* very weak sense."

Thus, omitting further discussions on (non-)uniqueness (which appears to be quite a delicate topic, cf. [27] or [20]), in the sequel by u we shall exclusively mean the pointwise limit $u = \lim_{\varepsilon\to 0} u_\varepsilon$ just constructed.

1.2. **On the criticality of the exponent $q = p-1$.** Let us start our study of u by stating some elementary positivity properties of this function. The first lemma indicates that in general u is not very meaningful for $q > p - 1$.

Lemma 1.2. *If $q > p-1$, $\Omega = B_R$ is a ball and u_0 is radially symmetric and nondecreasing satisfying $u_0 \equiv 0$ in B_ρ for some $\rho \in (0,R)$ then*

$$u \equiv 0 \qquad a.e.\ in\ \Omega \times (0,\infty).$$

Proof. Multiplying (1.1) by $u_\varepsilon^{-p}\varphi(x)$ where $0 \le \varphi \in C_0^\infty(B_\rho \setminus \bar{B}_{\frac{\rho}{2}})$ is arbitrary but nontrivial, we find that

$$\int_\Omega Hu_\varepsilon(t)\cdot\varphi - \int_\Omega H(\varepsilon)\cdot\varphi = \int_0^t\int_\Omega u_\varepsilon\Delta\varphi - \int_0^t\int_\Omega g_\varepsilon(u_\varepsilon)\cdot\varphi \le c(\varphi)$$

for all ε and hence, since $H(0) = -\infty$, $\int_\Omega Hu(t)\cdot\varphi = -\infty$ for all $t > 0$. In particular, for all $t > 0$ there exists $r_1(t) \in (\frac{\rho}{2},\rho)$ such that $u(x,t) = 0$ holds for all $|x| \le r_1(t)$, so that $r_\star(t) := \sup\{r \in (0,R) : u(t) \equiv 0 \text{ in } B_r\}$ is well defined and positive for all t.

Next, multiplying (1.1) by $u_\varepsilon^{\gamma-1}\cdot\psi^2(x)$ by $\gamma := \frac{q-p-1}{2} > 0$ and $\psi \in C_0^\infty(\Omega)$ we obtain, applying Young's inequality and Fatou's lemma in taking $\varepsilon \to 0$, that

$$\frac{1}{\gamma}\int_\Omega u^\gamma(t)\cdot\psi^2 + \frac{p+\gamma-1}{2}\int_0^t\int_\Omega u^{p+\gamma-2}|\nabla u|^2\psi^2 + \int_0^t\int_\Omega \chi_{\{u>0\}}u^{-q+\gamma-1}\psi^2$$
$$\le \frac{1}{\gamma}\int_\Omega u_0^\gamma\cdot\psi^2 + \frac{2}{p+\gamma-1}\int_0^t\int_\Omega u^{p+\gamma}|\nabla\psi|^2$$

for all $t > 0$. In particular, this yields

$$\int_{B_{R_1}} |\nabla u^{\frac{p+\gamma}{2}}|^2(t) \le c_1(R_1,t) \quad\text{and}\quad \int_{B_{R_1}} \chi_{\{u>0\}}u^{-q+\gamma-1}(t) < \infty \tag{1.8}$$

for almost every $t > 0$ and any $R_1 < R$. The first of these estimates together with the Sobolev inequality gives in the case of $R_\star(t) < R_1$

$$u(x,t) \le c\cdot(c_1(R_1,t))^{\frac{1}{p+\gamma}}\cdot\Big(|x| - r_\star(t)\Big)^{\frac{1}{p+\gamma}}, \qquad r_\star(t) < |x| < R_1,$$

which means that

$$u^{-q+\gamma-1}(t) \ge \frac{c(R_1,t)}{|x| - r_\star(t)}$$

for such t, contradicting the second inequality in (1.8). This shows that $r_\star(t) = R$ for almost every $t > 0$ and thus the lemma has been proved. □

Remark. The assertion of the lemma is actually valid for arbitrary u_0 satisfying (0.3) in any domain Ω. This fact can be proved by connecting any $x_0 \in \Omega$ to the ball in which u_0 vanishes by a finite series of overlapping balls, and iterating Lemma 1.2 using the comparison principle. Since a precise documentation of such a procedure is quite involved (cf. Section 2 for a similar technique), we do not carry this out here.

As some sort of complement of Lemma 1.2, let us make sure that u in fact is nontrivial whenever $q \le p-1$.

Lemma 1.3. *If $q \le p-1$, then $u \not\equiv 0$ in $\Omega \times (0,\infty)$.*

Proof. Since Ω is bounded, we may assume after a translation of coordinates that the hyperplane $\{x_n = 0\}$ is tangent to Ω. By continuity of u_0, there exists a nonnegative $v_0 \in C_0^\infty(\mathbb{R})$ such that $v_0(0) > 0$, $v_0 \le 1$ on $\mathbb{R}$ and $u_0(x_1,...,x_n) \ge v_0(x_n)$ holds for all $x = (x_1,...,x_n) \in \Omega$. By Theorem 1.4 in [25], the solutions of the one-dimensional Cauchy problem

$$\begin{aligned} v_{\varepsilon t} &= v_\varepsilon^p v_{\varepsilon\xi\xi} - g_\varepsilon(v_\varepsilon) \qquad \text{in } \mathbb{R}\times(0,\infty), \\ v_\varepsilon|_{t=0} &= v_0 + \varepsilon, \end{aligned}$$

decrease to a *continuous* limit v in $\mathbb{R}\times[0,\infty)$ as $\varepsilon \to 0$. (This is where the assumption $q \le p-1$ is needed.) As the comparison principle implies that $v_\varepsilon \le 1+\varepsilon$ in $\mathbb{R}\times(0,\infty)$ and thus $u_\varepsilon(x,t) \ge v_\varepsilon(x_n,t)$ in $\Omega\times(0,\infty)$, this shows that u cannot coincide almost everywhere with the trivial function. □

2. Total extinction in finite time for $q > 0$

In this section we consider the case where absorption is very strong in the sense that $q > 0$. We start with the observation that, roughly speaking, once the solution $u = \lim u_\varepsilon$ vanishes identically in some ball B, it will eventually become zero in any other ball B' with the same center, provided that $\bar{B}'$ is still contained in Ω.

Note carefully that the assumptions in the lemma generalize the situation at $t = 0$, where u_ε tends to zero uniformly in a ball.

Lemma 2.1. *Let $p > 1$ and $q \in (0, p-1]$. Suppose that there are $t_0 \ge 0$, $x_0 \in \Omega$ and $r_0 > 0$ such that $B_{r_0}(x_0) \subset \Omega$ and*

$$u_\varepsilon(t_0) \to 0 \qquad \text{uniformly in } B_{r_0}(x_0) \text{ as } \varepsilon \to 0. \tag{2.1}$$

Then for all $R > r_0$ with $B_R(x_0) \subset \Omega$ there is $T_{ext} > 0$ (depending on p, q, n, r_0, R and $\sup_{\varepsilon\in(0,1)} \|u_\varepsilon\|_{L^\infty(\Omega\times(t_0,\infty))}$ only) such that

$$u_\varepsilon \to 0 \qquad \text{uniformly in } B_{\kappa R}(x_0)\times(t_0+T_{ext},\infty) \text{ as } \varepsilon \to 0 \tag{2.2}$$

is satisfied for any $\kappa \in (0,1)$.

Proof. The proof proceeds in two steps.
Step 1. We first claim that there is $\nu > 0$ such that (2.2) is true under the additional assumption

$$r_0 \ge (1-\nu)R. \tag{2.3}$$

In view of (2.1), we may restrict ourselves to the case $(x_0, t_0) = (0, 0)$ and hence suppose that $u_\varepsilon(\cdot, 0) \to 0$ uniformly in B_{r_0} as $\varepsilon \to 0$. Since $q > 0$, we can fix some $\alpha \in (\frac{1}{q+1}, 1)$ and then some large $k \in \mathbb{N}$ such that $k > \max\{(p+q+1)\alpha - 2, 2\}$. We set

$$\nu := \begin{cases} \frac{1}{2} & \text{if } n = 1, \\ \frac{1-\alpha}{n-1} & \text{if } n \ge 2, \end{cases}$$

$$M := \frac{R - r_1}{r_0 - r_1} \cdot \sup_{\varepsilon \in (0,1)} \|u_\varepsilon\|^{\frac{1}{\alpha}}_{L^\infty(\Omega \times (0,\infty))}, \quad \text{and} \quad \varphi_0 := \frac{M}{R - r_1},$$

where $r_1 := (1 - 2\nu)R$. Furthermore, let $\psi \in C_0^\infty([0, \infty))$ be nonincreasing such that $\psi \equiv 1$ in $[0, \frac{1}{2}]$ and $\psi \equiv 0$ in $[1, \infty)$. We claim that if we let $C_0 = C_0(p, q, \alpha, M)$ be sufficiently large and, for $\eta \in (0, 1)$, φ_η denote the solution of

$$\begin{aligned} \varphi_\eta'(t) &= C_0 \cdot \psi\Big(\sqrt{2\alpha(k-\alpha)}\,\eta^{\frac{(p+q+1)\alpha-2}{2k}}\varphi_\eta\Big) \cdot \varphi_\eta^{\,1+2\frac{(q+1)\alpha-1}{(p+q+1)\alpha-2}}, \qquad t > 0, \\ \varphi_\eta(0) &= \varphi_0, \end{aligned} \tag{2.4}$$

then for all η there is $\varepsilon(\eta) > 0$ such that

$$v_\eta(x,t) := \Big[\Big(M - \varphi_\eta(t) \cdot (R - |x|)\Big)_+^k + \eta\Big]^{\frac{\alpha}{k}}, \qquad (x,t) \in B_R \times (0, \infty),$$

satisfies

$$v_\eta \ge u_{\varepsilon(\eta)} \qquad \text{in } B_R \times (0, \infty). \tag{2.5}$$

To see what will result from this, note that in the limit $\eta \searrow 0$, we have $\varphi_\eta \nearrow \varphi$, where φ is the solution of

$$\begin{aligned} \varphi'(t) &= C_0 \varphi^{1+2\frac{(q+1)\alpha-1}{(p+q+1)\alpha-2}}, \qquad t > 0, \\ \varphi(0) &= \varphi_0, \end{aligned} \tag{2.6}$$

which blows up at

$$T_{ext} := \Big(2\frac{(q+1)\alpha - 1}{(p+q+1)\alpha - 2} \cdot C_0\Big)^{-1} \cdot \Big(\frac{M}{R - r_1}\Big)^{-2\frac{(q+1)\alpha-1}{(p+q+1)\alpha-2}}$$

where, evidently, T_{ext} depends on p, q, n, r_0, R and a uniform bound for u_ε only. For $\kappa \in (0,1)$, let T_κ be defined by

$$\varphi(T_\kappa) = \frac{M}{(1-\kappa)R}.$$

It is easy to see that $T_\kappa \nearrow T_{ext}$ as $\kappa \nearrow 1$ and, apart from that, if η is sufficiently small such that $\sqrt{2\alpha(k-\alpha)}\,\eta^{\frac{(p+q+1)\alpha-2}{2k}} \cdot \frac{M}{(1-\kappa)R} \le \frac{1}{2}$ then $\varphi_\eta \equiv \varphi$ on $(0, T_\kappa)$. Thus, once we have proved (2.5), we infer from this that for any $t \ge T_{ext}$

$$u_\varepsilon(t) \le u_{\varepsilon(\eta)}(t) \le v_\eta(t) \le v_\eta(T_\kappa) \equiv \eta^{\frac{\alpha}{\kappa}} \qquad \text{in } B_{\kappa R} \qquad \forall\, \varepsilon \le \varepsilon(\eta)$$

holds for all small η and hence $u_\varepsilon \to 0$ uniformly in $B_{\kappa R} \times (T_{ext}, \infty)$, as asserted.

To check (2.5), we first note that for $|x| = R$

$$v_\eta(x,t) \ge M^\alpha \ge \sup_{\varepsilon\in(0,1)} \|u_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))} \ge u_\varepsilon(x,t) \quad \forall\, t>0,\ \forall\, \varepsilon \in (0,1).$$

At $t = 0$, for $|x| \ge r_0$ we have

$$\begin{aligned} v_\eta(x,0) &\ge \Big(M - \varphi_0 \cdot (R-|x|)\Big)_+^\alpha \ge \Big(M - \varphi_0 \cdot (R - r_0)\Big)^\alpha \\ &= \Big(M - \frac{M}{R-r_1}(R-r_0)\Big)^\alpha = \Big(\frac{r_0 - r_1}{R - r_1} M\Big)^\alpha \\ &\ge \sup_{\varepsilon\in(0,1)} \|u_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))} \ge u_\varepsilon(x,0) \qquad \forall\, \varepsilon \in (0,1), \end{aligned}$$

while if $|x| < r_0$ then, since $u_\varepsilon(\cdot,0) \to 0$ uniformly in B_{r_0},

$$v_\eta(x,0) \ge \eta^{\frac{\alpha}{k}} \ge u_\varepsilon(x,0) \quad \forall\, \varepsilon \in (0, \varepsilon_1(\eta)),$$

provided that $\varepsilon_1(\eta)$ is small enough. We thereby have proved that

$$v_\eta \ge u_\varepsilon \qquad \forall\, \varepsilon \in (0, \varepsilon_1(\eta))$$

holds on the parabolic boundary of $B_R \times (0,\infty)$. We will next show that if $\varepsilon \le \varepsilon(\eta) := \min\{\varepsilon_1(\eta), \frac{1}{2}\eta^{\frac{\alpha}{k}}\}$, then

$$v_{\eta t} - v_\eta^p \Delta v_\eta + g_\varepsilon(v_\eta) \ge 0 \qquad \text{in } B_R \times (0,\infty), \tag{2.7}$$

from which (2.5) results by means of the comparison principle.

To see (2.7), we use the fact that $v_\eta \ge \eta^{\frac{\alpha}{k}} \ge 2\varepsilon$ for $\varepsilon \le \varepsilon(\eta)$ in calculating

$$v_{\eta t} - v_\eta^p \Delta v_\eta + g_\varepsilon(v_\eta) = v_{\eta t} - v_\eta^p \Delta v_\eta + v_\eta^{-q} \tag{2.8}$$
$$= -\alpha\Big[\Big(M - \varphi_\eta(R-|x|)\Big)_+^k + \eta\Big]^{\frac{\alpha-k}{k}} \Big(M - \varphi_\eta(R-|x|\Big)_+^{k-1}(R-|x|)\varphi_\eta'$$

$$
\begin{aligned}
&+\alpha(1-\alpha)\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{\frac{p\alpha+\alpha-k}{k}}\Big(M-\varphi_\eta(R-|x|\Big)_+^{k-2}\varphi_\eta^2\\
&-\alpha(k-\alpha)\eta\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{\frac{p\alpha+\alpha-2k}{k}}\Big(M-\varphi_\eta(R-|x|\Big)_+^{k-2}\varphi_\eta^2\\
&-\frac{n-1}{|x|}\alpha\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{\frac{p\alpha+\alpha-k}{k}}\Big(M-\varphi_\eta(R-|x|\Big)_+^{k-1}\varphi_\eta\\
&+\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{-\frac{q\alpha}{k}}\\
&=:-I_1+I_2-I_3-I_4+I_5 \qquad \text{in } B_R\times(0,\infty).
\end{aligned}
$$

First, since $R-|x|\le\frac{M}{\varphi_\eta}$ in the set where $\big(M-\varphi_\eta\cdot(R-|x|)\big)_+$ is positive, we find

$$
\begin{aligned}
I_1&\le\alpha\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{\frac{\alpha-k}{k}}\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{\frac{k-1}{k}}\frac{M}{\varphi_\eta}\varphi_\eta'\\
&\le\frac12 I_5,
\end{aligned}
$$

provided that

$$
\alpha M\cdot\Big[\Big(M-\varphi_\eta\cdot(R-|x|)\Big)_+^k+\eta\Big]^{\frac{(q+1)\alpha-1}{k}}\cdot\frac{\varphi_\eta'}{\varphi_\eta}\le\frac12,
$$

that is, if

$$
\Big(M-\varphi_\eta\cdot(R-|x|)\Big)_+^k+\eta\ge(2\alpha M)^{-\frac{k}{(q+1)\alpha-1}}\cdot\Big(\frac{\varphi_\eta'}{\varphi_\eta}\Big)^{-\frac{k}{(q+1)\alpha-1}}. \tag{2.9}
$$

On the other hand,

$$
\begin{aligned}
\frac{I_1}{I_2}&=\frac{1}{1-\alpha}\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{-\frac{p\alpha}{k}}\Big(M-\varphi_\eta(R-|x|)\Big)_+(R-|x|)\frac{\varphi_\eta'}{\varphi_\eta^2}\\
&\le\frac{M}{1-\alpha}\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{-\frac{p\alpha-1}{k}}\frac{\varphi_\eta'}{\varphi_\eta^3}.
\end{aligned}
$$

Since $\alpha>\frac{1}{q+1}\ge\frac1p$, we have in the complement of the set where (2.9) is valid

$$
\begin{aligned}
\frac{I_1}{I_2}&\le\frac{M}{1-\alpha}\cdot(2\alpha M)^{\frac{p\alpha-1}{(q+1)\alpha-1}}\cdot\Big(\frac{\varphi_\eta'}{\varphi_\eta}\Big)^{\frac{p\alpha-1}{(q+1)\alpha-1}}\cdot\frac{\varphi_\eta'}{\varphi_\eta^3}\\
&=\frac{M}{1-\alpha}\cdot(2\alpha M)^{\frac{p\alpha-1}{(q+1)\alpha-1}}\cdot\varphi_\eta^{-3-\frac{p\alpha-1}{(q+1)\alpha-1}}\cdot(\varphi_\eta')^{\frac{(p+q+1)\alpha-2}{(q+1)\alpha-1}}\le\frac12,
\end{aligned}
$$

provided that

$$\varphi_\eta' \le \Big(\frac{1-\alpha}{2M}\Big)^{\frac{(q+1)\alpha-1}{(p+q+1)\alpha-2}} \cdot (2\alpha M)^{-\frac{p\alpha-1}{(p+q+1)\alpha-2}} \cdot \varphi_\eta^{1+\frac{(q+1)\alpha-1}{(p+q+1)\alpha-2}},$$

which is fulfilled upon an obvious choice of $C_0(p,q,\alpha,M)$ in (2.4). Consequently,

$$I_1 \le \frac{1}{2}I_2 + \frac{1}{2}I_5 \qquad \text{in } B_R\times(0,\infty). \tag{2.10}$$

Next, since $(p+q+1)\alpha-k-2<0$ according to the selection of k,

$$\begin{aligned}
\frac{I_3}{I_5} &\le \alpha(k-\alpha)\eta\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{\frac{(p+q+1)\alpha-2k}{k}} \\
&\qquad \times\Big(M-\varphi_\eta(R-|x|)\Big)_+^{k-2}\varphi_\eta^2 \\
&\le \alpha(k-\alpha)\eta\Big[\Big(M-\varphi_\eta(R-|x|)\Big)_+^k+\eta\Big]^{\frac{(p+q+1)\alpha-k-2}{k}}\varphi_\eta^2 \\
&\le \alpha(k-\alpha)\eta^{\frac{(p+q+1)\alpha-2}{k}}\varphi_\eta^2 \le \frac{1}{2} \qquad \text{in } B_R\times(0,\infty),
\end{aligned} \tag{2.11}$$

because $\varphi_\eta(t)\le \frac{1}{\sqrt{2\alpha(k-\alpha)}}\eta^{-\frac{(p+q+1)\alpha-2}{2k}}$ on $(0,\infty)$ by (2.4).

Finally, since $\varphi_\eta(t)\ge\varphi_0\ge\frac{M}{R}$, the expression $\frac{(M-\varphi_\eta\cdot(R-|x|))_+}{|x|}$ is nondecreasing with respect to $|x|$ and thus

$$\begin{aligned}
\frac{I_4}{I_2} &= \frac{1}{1-\alpha}\cdot\frac{n-1}{|x|}\cdot\Big(M-\varphi_\eta\cdot(R-|x|)\Big)_+\cdot\frac{1}{\varphi_\eta} \\
&\le \frac{n-1}{1-\alpha}\cdot\frac{\Big(M-\varphi_\eta\cdot(R-|x|)\Big)_+}{R}\cdot\frac{1}{\varphi_\eta} = \frac{n-1}{1-\alpha}\cdot\frac{M}{R}\cdot\frac{1}{\varphi_\eta} \\
&\le \frac{n-1}{1-\alpha}\cdot\frac{M}{R}\cdot\frac{1}{\varphi_0} = \frac{n-1}{1-\alpha}\cdot\frac{R-r_1}{R} = \frac{n-1}{1-\alpha}\cdot 2\nu\le\frac{1}{2} \quad \text{in } B_R\times(0,\infty).
\end{aligned} \tag{2.12}$$

Now (2.8), (2.10), (2.11) and (2.12) prove (2.7).

Step 2. We next assert that (2.2) is valid without any restriction on r_1 and R. Indeed, choose any integer $N\ge\frac{R}{r_0}(\ln\frac{1}{1-\nu})^{-1}$ and let $\tilde r_0 := (1-\nu)^N R$. Then $\tilde r_0\le r_0$ (whence (2.1) is true with r_0 replaced by $\tilde r_0$) and $R_k := (1-\nu)^{-k}\tilde r_0, k=1,2,...,$ defines an increasing sequence of radii with $R_N=R$. Now a repeated application of Step 1 enables us to find $T_1,...,T_N$

such that for all $\kappa \in (0,1)$ and all $k = 1, ..., N$ we have

$$u_\varepsilon \to 0 \qquad \text{uniformly in } B_{\kappa R_k} \times \Big(t_0 + \sum_{j=1}^{k} T_j\,,\, \infty\Big).$$

Employing this result for $k = N$, we complete the proof with the choice $T_{ext} := \sum_{j=1}^{N} T_j$. □

Without further comment we state the following.

Corollary 2.2. *Under the assumptions and with the notation of Lemma 2.1, we have $u \equiv 0$ in $B_R(x_0) \times (t_0 + T_{ext}, \infty)$.*

Suppose u_0 vanishes in some ball $B^I \subset \Omega$. Loosely speaking, we now connect any ball B^{II} with B^I by a chain of finitely many balls $B^1, ..., B^N$ in such a way that each center of B^j is contained in B^{j-1}. Iterating then Lemma 2.1 along $B^1, ..., B^N$ and finally using Corollary 2.2, we obtain the fact that u will vanish in all of B^{II} beyond some time T. Since $\partial\Omega$ is smooth, the corresponding radii and therefore also T can be controlled uniformly with respect to the choice of B^{II}. This will be the main idea in the proof of the following.

Theorem 2.3. *Suppose $p > 1$, $q \in (0, p-1]$ and u_0 vanishes in the ball $B_{r_0}(x_0) \subset \Omega$. Then there is $T_{ext} > 0$ such that $u(t) \not\equiv 0$ for all $t < T_{ext}$ but*

$$u \equiv 0 \qquad \text{in } \Omega \times (T_{ext}, \infty). \tag{2.13}$$

Proof. For $R > 0$, let $\Omega_R := \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > R\}$. Since $\partial\Omega \in C^3$, there is some small $R \le 16 r_0$ such that $\Omega_{\frac{R}{2}}$ is connected with $x_0 \in \Omega_{\frac{R}{2}}$ and

$$\Omega \setminus \Omega_R \subset \bigcup_{x\in\Omega,\ \text{dist}\,(x,\partial\Omega)=R} B_R(x). \tag{2.14}$$

(In fact, we can choose R small such that for all $y \in \partial\Omega$ and for all $r < R$ there exists precisely one $x(r,y) \in \Omega$ with $\bar{B}_r(x(r,y)) \cap \partial\Omega = \{y\}$ – namely, we have $x(r,y) = y - rN(y)$, where $N(y)$ denotes the outward unit normal vector of $\partial\Omega$ at y. Now for any $z \in \Omega \setminus \Omega_R$ there is $y \in \partial\Omega$ such that $|x - y| = \text{dist}\,(z, \partial\Omega) =: r$ and hence $z = y - rN(y) = x(r,y)$. As $|z - x(y)| = R - r < R$, z lies in $\bigcup_{y\in\partial\Omega} B_R(x(R,y))$ and thus also in the set on the right of (2.14).)

Next, the compactness of $\bar{\Omega}_{\frac{R}{2}}$ allows us to extract $x_1, ..., x_N \in \bar{\Omega}_{\frac{R}{2}}$ such that

$$\bar{\Omega}_{\frac{R}{2}} \subset \bigcup_{k=0}^{N} B_{\frac{R}{16}}(x_k). \tag{2.15}$$

Since $\Omega_{\frac{R}{2}}$ is connected, it is easily seen by induction that the $x_1, ..., x_N$ may be renumbered in such a way that

$$\forall\, k \in \{1, ..., N\}\ \exists\, j(k) \in \{0, ..., k-1\} \text{ such that } B_{\frac{R}{16}}(x_{j(k)}) \cap B_{\frac{R}{16}}(x_k) \neq \emptyset. \tag{2.16}$$

We claim that there exist $T_0, ..., T_N$ such that

$$u_\varepsilon \to 0 \qquad \text{uniformly in } B_{\frac{R}{4}}(x_k) \times [T_k, \infty), \tag{2.17}$$

which in view of (2.15) will particularly give

$$u \equiv 0 \qquad \text{in } \Omega_{\frac{R}{2}} \times (T, \infty) \tag{2.18}$$

with $T := \max_{k \in \{0,...,N\}} T_k$.

In fact, (2.17) for $k = 0$ follows from Lemma 2.1, because $u_{0\varepsilon} \to 0$ uniformly in $B_{r_0}(x_0) \supset B_{\frac{R}{16}}(x_0)$ as $\varepsilon \to 0$ and $B_{\frac{R}{2}}(x_0) \subset \Omega$. If $T_0, ..., T_{k-1}$ have already been found for some $k \leq N$, we use (2.16) to obtain $B_{\frac{R}{16}}(x_k) \subset B_{\frac{R}{4}}(x_{j(k)})$ and hence, by (2.17), $u_\varepsilon(T_{j(k)}) \to 0$ uniformly in $B_{\frac{R}{4}}(x_{j(k)})$ as $\varepsilon \to 0$. Now Lemma 2.1 says that there is $T_k > T_{j(k)}$ such that (2.17) is valid, for we have $B_{\frac{R}{2}}(x_k) \subset \Omega$.

Having proved (2.17), we proceed to show that there is $T' > 0$ such that for all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) = R$ we have

$$u \equiv 0 \quad \text{in } B_R(x) \times \Big(T + T'\,,\, \infty\Big). \tag{2.19}$$

To see this, we note that such an x belongs to $\Omega_{\frac{R}{2}}$ and therefore to $B_{\frac{R}{16}}(x_k)$ for some $k \in \{0, ..., N\}$. As $u_\varepsilon(T_k) \to 0$ uniformly in $B_{\frac{R}{4}}(x_k) \supset B_{\frac{R}{16}}(x)$ as $\varepsilon \to 0$ by (2.17), Corollary 2.2 provides some $T' > 0$ independent of x such that (2.19) holds.

Finally, since

$$\Omega = \Omega_{\frac{R}{2}} \bigcup \Big(\bigcup_{x \in \Omega,\ \operatorname{dist}(x,\partial\Omega) = R} B_R(x) \Big),$$

(2.13) is a consequence of (2.18) and (2.19). In view of Lemma 1.3, the proof is complete. □

3. Extinction in infinite time for $q \in (1-p, 0)$

In the regime $q \in (1-p, 0)$, we shall see that u still undergoes an asymptotic total extinction. However, contrary to the case $q > 0$ considered before, extinction (in convex domains) now *always* occurs in infinite time;

that is, $u(t) \not\equiv 0$ for all $t \in (0,\infty)$, but $\{u(t)=0\} \to \Omega$ as $t \to \infty$. In other words, we will show for the dead core set $\{u(t)=0\}$ that

$$\{u(t)=0\} \neq \Omega \quad \forall\, t \in (0,\infty), \qquad \text{but} \qquad \{u(t)=0\} \nearrow \Omega \text{ as } t \to \infty.$$

3.1. **Absence of extinction in finite time.** Let us first make sure that complete extinction in finite time cannot occur in convex domains.

Lemma 3.1. *Suppose Ω is convex, $p>1$, and that $q>-1$ is such that $q \in (1-p,0)$. Then for all $\nu>0$ there exists $c_\nu>0$ such that*

$$u(x,t)>0 \qquad \textit{whenever } t \geq 0 \textit{ and } \operatorname{dist}(x,\partial\Omega) < c_\nu e^{-(q+1+\nu)t}. \tag{3.1}$$

Proof. From the Lipschitz continuity of u_0 and the fact that $u_0|_{\partial\Omega}=1$ we gain a constant $c_0>0$ such that

$$u_0^{q+1}(x) \geq 1 - c_0 \operatorname{dist}(x,\partial\Omega) \qquad \forall\, x \in \Omega. \tag{3.2}$$

For fixed $\nu>0$, let

$$\varphi(t) := \varphi_0 e^{(q+1+\nu)t}, \qquad t>0,$$

with

$$\varphi_0 := \max\Big\{ c_0\,,\ \frac{q+1}{\sqrt{-q}} \cdot \Big(\frac{q+1+\nu}{\nu}\Big)^{\frac{p-q-1}{2(q+1)}} \Big\}. \tag{3.3}$$

Next, given $y \in \partial\Omega$, let $N(y)$ denote the outward unit normal vector of $\partial\Omega$ at y. As $\partial\Omega \in C^3$, there is $\delta>0$ such that the 'δ-neighborhood' $S_\delta \subset \Omega$ of $\partial\Omega$ is characterized, using this normal vector field, in the following sense: We have

$$S_\delta := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \delta\} = \{y - \kappa N(y) : y \in \partial\Omega, \kappa \in (0,\delta)\}.$$

Moreover, since Ω is convex, each hyperplane spanned by the tangent vectors of $\partial\Omega$ at some $y \in \partial\Omega$ does not intersect Ω. Hence, for each $x^0 \in S_\delta$ (which we keep fixed from now on), there exists $y^0 \in \partial\Omega$ such that, after rotating and translating the coordinate axes, we may assume $y^0=0$, $\Omega \subset \{z=(z_1,...,z_n) \in \mathbb{R}^n : z_n>0\}$ and $x^0=(0,...,0,x_n^0)$, where $x_n^0 = \operatorname{dist}(x^0,\partial\Omega)$. In these translated coordinates, we define a comparison function v by

$$v(x,t) := (1-\varphi(t)x_n)^{\frac{1}{q+1}}$$

in

$$Q := \Big\{(x,t)=(x_1,...,x_n,t) \in \Omega \times (0,\infty) : x_n < \frac{1}{\varphi(t)}\Big\}.$$

Then, at $t=0$, for all $x\in\Omega$ with $x_n<\frac{1}{\varphi_0}$ we have $\operatorname{dist}(x,\partial\Omega)\le x_n$ and thus

$$v(x,0)=\Big(1-\varphi_0x_n\Big)^{\frac{1}{q+1}}\le\Big(1-c_0x_n\Big)^{\frac{1}{q+1}}$$
$$\le\Big(1-c_0\operatorname{dist}(x,\partial\Omega)\Big)^{\frac{1}{q+1}}\le u_0(x)<u_\varepsilon(x,0).$$

Moreover, if $(x,t)\in\partial Q$ is such that $x\in\partial\Omega$, then $v(x,t)\le 1<1+\varepsilon=u_\varepsilon(x,t)$, while for $(x,t)\in\partial Q$ with $x_n=\frac{1}{\varphi(t)}$ we trivially have $v(x,t)=0<u_\varepsilon(x,t)$. Consequently, $v<u_\varepsilon$ on the parabolic boundary of Q. In addition,

$$v_t-v^p\Delta v+g_\varepsilon(v)\le v_t-v^p\Delta v+v^{-q}$$
$$=-\frac{1}{q+1}\Big(1-\varphi(t)x_n\Big)^{\frac{-q}{q+1}}\cdot\varphi'(t)x_n-\frac{-q}{(q+1)^2}\cdot\Big(1-\varphi(t)x_n\Big)^{\frac{p-2q-1}{q+1}}\cdot\varphi^2(t)$$
$$+\Big(1-\varphi(t)x_n\Big)^{\frac{-q}{q+1}}=:-I_1-I_2+I_3.$$

Here, $I_3\le I_1$ whenever $(x,t)\in Q$ is such that $x_n\ge\frac{q+1}{\varphi'(t)}$, whereas if $(x,t)\in Q$ satisfies $x_n<\frac{q+1}{\varphi'(t)}$ then, since $q<p-1$,

$$\frac{I_3}{I_2}=\frac{(q+1)^2}{-q}\cdot\Big(1-\varphi(t)x_n\Big)^{-\frac{p-q-1}{q+1}}\cdot\varphi^{-2}(t)$$
$$<\frac{(q+1)^2}{-q}\cdot\Big(1-(q+1)\frac{\varphi(t)}{\varphi'(t)}\Big)^{-\frac{p-q-1}{q+1}}\cdot\varphi^{-2}(t)$$
$$=\frac{(q+1)^2}{-q}\cdot\Big(\frac{\nu}{q+1+\nu}\Big)^{-\frac{p-q-1}{q+1}}\cdot\varphi^{-2}(t)$$
$$\le\frac{(q+1)^2}{-q}\cdot\Big(\frac{\nu}{q+1+\nu}\Big)^{-\frac{p-q-1}{q+1}}\cdot\varphi_0^{-2}\le 1$$

in virtue of (3.3). Hence, v is a subsolution of (1.1) in Q and thus, by comparison, lies below u_ε in all of Q. As a result, as long as the inequality $x_n^0=\operatorname{dist}(x^0,\partial\Omega)<\frac{1}{\varphi(t)}=\frac{1}{\varphi_0}e^{-(q+1+\nu)t}$ holds, we have $(x^0,t)\in Q$ and therefore

$$u_\varepsilon(x^0,t)\ge v(x^0,t)>0\qquad\forall\,\varepsilon>0,$$

which implies $u(x^0,t)>0$ and thereby proves the lemma. $\square$

Remark. In general (non-convex) domains the following weakened version of Lemma 3.1 is proved quite similarly: Let $q>-1$ be such that $q\in(1-p,0)$, and suppose $y\in\partial\Omega$ is an extremal point of Ω; that is, y belongs to the boundary of the convex hull of Ω. Then for all $\nu > 0$ there exists $c_\nu > 0$ with the property that

$$u(x,t) > 0 \quad \text{whenever } t \geq 0 \text{ and } x \in \Omega \text{ is such that } |x-y| < c_\nu e^{-(q+1+\nu)t}.$$

3.2. **Infinite time extinction in the radial case.** In order to prove that actually u vanishes everywhere at least asymptotically, we first deal with an auxiliary problem with more general boundary values in a ball, the solutions of which will serve as comparison functions in Lemma 3.7. Consider the problem

$$\begin{aligned} v_{\varepsilon t} &= v_\varepsilon^p \Delta v_\varepsilon - g_\varepsilon(v_\varepsilon) \qquad \text{in } B_R \times (0,\infty), \\ v_\varepsilon|_{\partial B_R} &= M + \varepsilon, \\ v_\varepsilon|_{t=0} &= v_0 + \varepsilon, \end{aligned} \tag{3.4}$$

where $R > 0$, $M \geq 1$ and $\varepsilon \in (0,1)$. The initial data v_0 are assumed to be in $W^{1,\infty}(B_R)$ with $v_0|_{\partial B_R} = M$, and we suppose that v_0 is radially symmetric and nondecreasing with respect to $|x| \in (0,R)$. We also assume that

$$v_0 \equiv 0 \qquad \text{in } B_\rho$$

for some $\rho \in (0,R)$. Clearly, under these assumptions v_ε will also be radially symmetric and nondecreasing with respect to $|x| \in (0,R)$.

We shall need two auxiliary lemmata. The first one is a variant of (part of) Lemma 2.3 in [25].

Lemma 3.2. *Under the assumptions $p \geq 1$ and $q \leq 0$, the following statements hold.*

i) We have

$$v_\varepsilon \searrow v \qquad \text{in } B_R \times (0,\infty) \qquad \text{as } \varepsilon \searrow 0. \tag{3.5}$$

ii) For all $\beta \in (0,1)$ there exists $C_0 > 0$ such that

$$\left|\nabla v^{\frac{1}{\beta}}(x,t)\right| \leq \frac{C_0}{R-|x|} \tag{3.6}$$

holds for $0 < |x| < R$ and $t \geq 0$.

iii) The free boundary function

$$\zeta(t) := \sup\{r \in (0,R) : v(x,t) = 0 \text{ for } 0 < |x| < r\}$$

is positive and nondecreasing for $t \in [0,\infty)$.

Proof. i) This is an immediate consequence of the comparison principle.

ii) Let us first show that for any $\beta \in (0\frac{1}{p}, 1)$ we have

$$\left|\nabla v_\varepsilon^{\frac{1}{\beta}}(x,t)\right| \le \frac{C_1}{|x|(R-|x|)}, \qquad 0<|x|<R,\ t>0, \tag{3.7}$$

with C_1 independent of ε. The proof of this is a straightforward application of a Bernstein-type technique (as applied in [25] to a similar problem), and so we may restrict ourselves to an outline of the main steps:

Introducing $w \equiv w_\varepsilon := v_\varepsilon^{\frac{1}{\beta}}$, we see that $w = w(r,t)$ satisfies

$$\begin{aligned} w_{rt} &= w^{p\beta}\Big(w_{rrr} + \frac{n-1}{r}w_{rr} - \frac{n-1}{r^2}w_r\Big) \\ &\quad + w^{p\beta-1}w_r \cdot \Big((p\beta+2\beta-2)w_{rr} + p\beta\cdot\frac{n-1}{r}w_r\Big) \\ &\quad -(1-\beta)(p\beta-1)w^{p\beta-2}w_r^3 - G_\varepsilon'(w)w_r, \end{aligned} \tag{3.8}$$

where $G_\varepsilon(s) := \frac{1}{\beta}s^{1-\beta}g_\varepsilon(s^\beta)$ is nondecreasing due to $\beta < 1 \le \frac{1}{q+1}$. Given $r_0 \in (0,R)$, let $\zeta \in C_0^\infty((\frac{r_0}{2}, R-\frac{r_0}{2}))$ be a cut-off function such that $0 \le \zeta \le 1$ and $\zeta \equiv 1$ in $(r_0, R-r_0)$ and such that $r_0|\zeta_r| + r_0^2|\zeta_{rr}| \le 4$, and consider $z(r,t) := \zeta^2(r)w_r^2(r,t)$. At any point where z takes its positive maximum M over $\bar{B}_R \times [0,T]$, where $T>0$ is fixed, we either have $t=0$ or $z_r = 0$ and $z_t - w^{p\beta}z_{rr} \ge 0$. Calculating the corresponding derivatives of z and using (3.8), we obtain in the latter case

$$\begin{aligned} 0 &\le \frac{1}{2}(z_t - w^{p\beta}z_{rr}) = -(1-\beta)(p\beta-1)\zeta^2 w^{p\beta-2}w_r^4 \\ &- \Big((p\beta+2\beta-2)\zeta\zeta_r - p\beta\zeta^2\cdot\frac{n-1}{r}\Big)w^{p\beta-1}w_r^3 \\ &+ \Big((2\zeta_r^2 - \zeta\zeta_{rr})w^{p\beta} - \zeta^2 G_\varepsilon'(w) - \zeta\zeta_r\cdot\frac{n-1}{r}w^{p\beta} - \zeta^2\cdot\frac{n-1}{r^2}w^{p\beta}\Big)w_r^2, \end{aligned}$$

from which we conclude that

$$\begin{aligned} &(1-\beta)(p\beta-1)\zeta^2 w_r^2 + \Big((p\beta+2\beta-2)\zeta_r - p\beta\zeta\cdot\frac{n-1}{r}\Big)w\cdot(\zeta w_r) \\ &\le \Big((2\zeta_r^2 - \zeta\zeta_{rr}) - \zeta\zeta_r\cdot\frac{n-1}{r}\Big)w^2 \end{aligned}$$

holds at such maximum points. This implies $z \le \frac{c}{r_0^2}$ and thereby (3.7) follows.

(By a standard argument basically due to Gilding ([10]), this implies that the convergence $v_\varepsilon \to v$ actually is locally uniform in $(B_R \setminus \{0\}) \times [0,\infty)$, so that v is continuous for $0 < |x| < R$ *and* $t \geq 0$; but we do not need this here.)

Therefore, (3.6) will result from (3.7) and the assumption $v_0 \equiv 0$ in B_ρ as soon as we have shown iii).

iii) We need to show that if $t_0 \geq 0$ and $r_0 > 0$ are such that

$$v(x, t_0) = 0 \qquad \text{for all } 0 < |x| < r_0,$$

then

$$v(x, t) = 0 \qquad \text{for all } t > t_0 \text{ and } 0 < |x| < r_0.$$

If this were false, by continuity of $v(\cdot, t)$ for $0 < |x| < R$ (which evidently is a consequence of (3.7)) we would have $v(x, t) \geq \delta$ for some $\delta > 0$ and all x satisfying $r_1 < |x| < r_2$ with certain $0 < r_1 < r_2 < r$. Multiplying (3.4) by $v_\varepsilon^{-p}(x, t)\varphi(x)$ with any nontrivial nonnegative $\varphi \in C_0^\infty(B_{r_2} \setminus \bar{B}_{r_1})$ gives

$$\int_\Omega H v_\varepsilon(t) \cdot \varphi - \int_\Omega H v_\varepsilon(t_0) \cdot \varphi = \int_{t_0}^t \int_\Omega v_\varepsilon \Delta\varphi - \int_{t_0}^t \int_\Omega v_\varepsilon^{-p} g_\varepsilon(v_\varepsilon) \cdot \varphi \leq c(\varphi).$$

But the left-hand side tends to $+\infty$ as $\varepsilon \to 0$ by the Beppo-Levi theorem, because $H(0) = -\infty$, a contradiction. □

We also need the following variant of Gronwall's lemma, an elementary proof of which can be found in [25] (see Lemma 3.2 therein).

Lemma 3.3. *Suppose* $t_0 \geq 0, T > 0$ *and that a continuous function* y *is positive on* $[t_0, t_0 + T]$ *and fulfils*

$$y(t) \leq y_0 + A \cdot (t - t_0) - B \int_{t_0}^t y^{-\lambda}(s) ds \qquad \forall\, t \in [t_0, t_0 + T]$$

with given positive numbers y_0, A, B *and* λ. *If*

$$y_0 < \left(\frac{B}{2A}\right)^{\frac{1}{\lambda}},$$

then

$$T \leq \frac{2y_0^{1+\lambda}}{(1+\lambda)B}.$$

The key to our main result for the radial case is the next lemma which provides a lower bound for the speed at which – at least locally in time – the dead core set $\{u(t) = 0\}$ propagates.

Lemma 3.4. *Given $p \ge 1$ and $q \in (1-p, 0]$, let $\beta \in (\frac{2}{p+q+1}, 1)$. Then there are $C_1 > 0$ and $C_2 > 0$ (depending on n, ρ, R, M, p, q and β only) such that for all $d > 0$ and any $b \le C_1 d^{\frac{(p+q+1)\beta}{(p+q+1)\beta-2}}$ we have the following implication:*

If there are $r_0 \in [\rho, R-d]$ and $t_0 \ge 0$ such that

$$v(t_0) \equiv 0 \qquad \text{in } B_{r_0}, \tag{3.9}$$

then

$$v(t_0+T) \equiv 0 \qquad \text{in } B_{r_0+b} \tag{3.10}$$

for any T satisfying

$$T \ge C_2 d^{-(q+1)\beta} b^{(q+1)\beta}. \tag{3.11}$$

Proof. Given

$$b < \min\Big\{1, \frac{R - r_0 - \frac{d}{2}}{2}\Big\}, \tag{3.12}$$

we fix $\psi \in C_0^\infty(B_{r_0+2b})$ such that $o \le \psi \le 1$, $\psi \equiv 1$ in $B_{r_0+\frac{3b}{2}}$ and $|\nabla\psi| \le \frac{4}{b}$. With arbitrary $\gamma \in (0, q+1)$, we multiply (3.4) by $v_\varepsilon^{\gamma-1}\psi^2$ and integrate over $B_R \times (t_0, t)$, $t > t_0$, to see that

$$\frac{1}{\gamma}\int_{B_R} \psi^2 v_\varepsilon^\gamma(t) + (p+\gamma-1)\int_{t_0}^t\int_{B_R} \psi^2 v_\varepsilon^{p+\gamma-2}|\nabla v_\varepsilon|^2 + \int_{t_0}^t\int_{B_R}\psi^2 g_\varepsilon(v_\varepsilon)v_\varepsilon^{\gamma-1}$$
$$= \frac{1}{\gamma}\int_{B_R}\psi^2 v_\varepsilon^\gamma(t_0) - 2\int_{t_0}^t\int_{B_R}\psi\nabla\psi\cdot v_\varepsilon^{p+\gamma-1}\nabla v_\varepsilon.$$

Estimating the second term on the right by means of Young's inequality and then taking $\varepsilon \to 0$ using Beppo-Levi's theorem and Fatou's lemma, we arrive at

$$\frac{1}{\gamma}\int_{B_R}\psi^2 v^\gamma(t) + \int_{t_0}^t\int_{B_R}\psi^2\chi_{\{v>0\}}v^{-q+\gamma-1} \le \frac{1}{\gamma}\int_{B_R} v^\gamma(t_0) + c\int_{t_0}^t\int_{B_R}|\nabla\psi|^2 v^{p+\gamma}. \tag{3.13}$$

Assume (3.10) was false for some $T > 0$. Then $v(t_0+T) > 0$ outside B_{r_0+b} and therefore $v(x,t) > 0$ for $|x| > r_0 + b$ and all $t \in (t_0, t_0+T)$. Hence, employing Hölder's inequality shows that

$$\int_{t_0}^t\int_{B_R}\psi^2\chi_{\{v>0\}}v^{-q+\gamma-1} \ge \int_{t_0}^t\Big(\int_{B_R}\psi^2\chi_{\{v(s)>0\}}\Big)^{\frac{q+1}{\gamma}}\cdot\Big(\int_{B_R}\psi^2 v^\gamma\Big)^{-\frac{q+1-\gamma}{\gamma}}$$
$$\ge \Big|B_{r_0+\frac{3b}{2}} \setminus B_{r_0+b}\Big|^{\frac{q+1}{\gamma}}\cdot\int_{t_0}^t\Big(\int_{B_R}\psi^2 v^\gamma\Big)^{-\frac{q+1-\gamma}{\gamma}}$$
$$= \Big(\frac{r_0^n\omega_n}{n}\cdot\Big[\Big(1+\frac{3b}{2r_0}\Big)^n - \Big(1+\frac{b}{r_0}\Big)^n\Big]\Big)^{\frac{q+1}{\gamma}}\cdot\int_{t_0}^t\Big(\int_{B_R}\psi^2 v^\gamma\Big)^{-\frac{q+1-\gamma}{\gamma}}$$

$$\geq \Big(\frac{r_0^n\omega_n}{n}\Big)^{\frac{q+1}{\gamma}} \cdot \Big(\frac{n}{2r_0}b\Big)^{\frac{q+1}{\gamma}} \cdot \int_{t_0}^t \Big(\int_{B_R} \psi^2 v^\gamma\Big)^{-\frac{q+1-\gamma}{\gamma}}$$

$$\geq K_1 \cdot b^{\frac{q+1}{\gamma}} \cdot \int_{t_0}^t \Big(\int_{B_R} \psi^2 v^\gamma\Big)^{-\frac{q+1-\gamma}{\gamma}}, \tag{3.14}$$

where, as usual, ω_n denotes the $n-1$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^n$. Here and in the sequel we denote by $K_1, K_2, ...$ constants which depend only on n, ρ, R, M, p, q and β.

On the other hand, Lemma 3.2 implies that for $|x| < r_0 + 2b \leq R - \frac{d}{2}$ and all $s \geq t_0$

$$|\nabla v^{\frac{1}{\beta}}| \leq \frac{C_0}{R-|x|} \leq \frac{2C_0}{d}$$

and hence (3.9) entails

$$v(x,s) \leq \Big(\frac{2C_0}{d}\Big)^\beta (|x|-r_0)^\beta \leq \Big(\frac{4C_0 b}{d}\Big)^\beta \quad \forall\, |x| < r_0 + 2b,\ \forall\, s \geq t_0. \tag{3.15}$$

Accordingly,

$$\begin{aligned}\int_{t_0}^t \int_{B_R} |\nabla\psi|^2 v^{p+\gamma} \quad &\leq \quad \frac{16}{b^2} \cdot \Big(\frac{4C_0 b}{d}\Big)^{(p+\gamma)\beta} \cdot \Big|B_{r_0+2b} \setminus B_{r_0+\frac{3b}{2}}\Big| \cdot (t-t_0) \\ &\leq \quad K_2 d^{-(p+\gamma)\beta} b^{(p+\gamma)\beta-1}(t-t_0).\end{aligned}$$

Together with (3.13) and (3.14), this shows that $y(t) := \int_{B_R} \psi^2 v^\gamma(t)$ satisfies

$$y(t) \leq y(t_0) + K_2 d^{-(p+\gamma)\beta} b^{(p+\gamma)\beta-1}(t-t_0) - K_1 b^{\frac{q+1}{\gamma}} \cdot \int_{t_0}^t y^{-\frac{q+1-\gamma}{\gamma}}(s)ds$$

for all $t \in (t_0, t_0+T)$ and therefore Lemma 3.3 yields

$$T \leq \frac{2\gamma}{(q+1)K_1} \cdot b^{-\frac{q+1}{\gamma}} \cdot y^{\frac{q+1}{\gamma}}(t_0),$$

provided that

$$y(t_0) \leq \Big(\frac{K_1 b^{\frac{q+1}{\gamma}}}{2K_2 d^{-(p+\gamma)\beta} b^{(p+\gamma)\beta-1}}\Big)^{\frac{\gamma}{q+1-\gamma}}.$$

By (3.15), we have

$$y(t_0) \leq \Big(\frac{4C_0 b}{d}\Big)^{\beta\gamma} \cdot \Big|B_{r_0+2b} \setminus B_{r_0}\Big| \leq K_3 d^{-\beta\gamma} b^{\beta\gamma+1},$$

and thus the latter statement means that

$$T \leq \frac{2\gamma K_3^{\frac{q+1}{\gamma}}}{(q+1)K_1} \cdot d^{-(q+1)\beta} b^{(q+1)\beta} \tag{3.16}$$

holds under the assumption

$$K_3 d^{-\beta\gamma} b^{\beta\gamma+1} \le \Big(\frac{K_1}{2K_2}\Big)^{\frac{\gamma}{q+1-\gamma}} \cdot d^{\frac{(p+\gamma)\beta\gamma}{q+1-\gamma}} b^{(\frac{q+1}{\gamma}-(p+\gamma)\beta+1)\cdot\frac{\gamma}{q+1-\gamma}},$$

which is equivalent to

$$b < K_3^{-\frac{q+1-\gamma}{(p+q+1)\beta\gamma-2\gamma}} \Big(\frac{K_1}{2K_2}\Big)^{\frac{1}{(p+q+1)\beta-2}} d^{\frac{(p+q+1)\beta}{(p+q+1)\beta-2}} =: K_4 d^{\frac{(p+q+1)\beta}{(p+q+1)\beta-2}}. \tag{3.17}$$

Accordingly, if we choose

$$C_2 := 2 \cdot \frac{2\gamma K_3^{\frac{q+1}{\gamma}}}{(q+1)K_1}$$

and

$$C_1 := \frac{1}{2}\cdot\min\Big\{K_4, R^{-\frac{(p+q+1)\beta}{(p+q+1)\beta-2}}, \frac{1}{4}R^{-\frac{2}{(p+q+1)\beta-2}}\Big\},$$

then each $b \le C_1 d^{\frac{(p+q+1)\beta}{(p+q+1)\beta-2}}$ satisfies (3.12) and (3.17), while (3.11) is incompatible with (3.16). This contradiction shows that (3.10) must hold for such C_1 and C_2. □

Iterating this lemma, we obtain a global-in-time estimate for the free boundary $\zeta(t)$. As a tool, we shall need the following elementary fact.

Lemma 3.5. *Let $(\xi_k)_{k\in\mathbb{N}} \subset (0,\infty)$ be a sequence such that*

$$\xi_{k+1} = \xi_k - A\xi_k^{1+\lambda} \qquad \forall\, k\in\mathbb{N} \tag{3.18}$$

holds with some $A>0$ and $\lambda>0$. Then there is $B>0$ such that

$$\xi_k \le Bk^{-\frac{1}{\lambda}} \qquad \forall\, k\in\mathbb{N}. \tag{3.19}$$

Proof. We let $\bar\xi := ((1+\lambda)A)^{-\frac{1}{\lambda}}, \bar B := (\frac{2}{\lambda A})^{\frac{1}{\lambda}}$ and fix $k_0\in\mathbb{N}$ such that $k_0 \ge (\frac{\bar B}{\bar\xi})^\lambda$, $(1-x)^{\frac{1}{\lambda}} \ge 1-\frac{2}{\lambda}x$ for all $x\in[0,\frac{1}{k_0+1}]$ and $\xi_k \le \bar\xi$ for all $k\ge k_0$, where the latter condition can be fulfilled since evidently $\xi_k\to 0$ as $k\to\infty$.

With these definitions, we have $\xi_{k_0} \le \bar\xi = \bar B\cdot k_0^{-\frac{1}{\lambda}}$ and if $\xi_k \le \bar B\cdot k^{-\frac{1}{\lambda}}$ for some $k\ge k_0$ then, as $\xi\mapsto \xi - A\xi^{1+\lambda}$ is increasing on $[0,\bar\xi]$ and $\bar B\cdot k^{-\frac{1}{\lambda}} \le \bar B\cdot k_0^{-\frac{1}{\lambda}} = \bar\xi$,

$$\begin{aligned}\xi_{k+1} &= \xi_k - A\xi_k^{1+\lambda} \le \bar B\cdot k^{-\frac{1}{\lambda}} - A\bar B^{1+\lambda}k^{-\frac{1}{\lambda}-1}\\ &= \bar B\cdot(k+1)^{-\frac{1}{\lambda}}\cdot\Big(1-\frac{A\bar B^\lambda}{k}\Big)\cdot\Big(1-\frac{1}{k+1}\Big)^{-\frac{1}{\lambda}}\end{aligned}$$

$$\leq \bar{B} \cdot (k+1)^{-\frac{1}{\lambda}} \cdot \Big(1 - \frac{A\bar{B}^\lambda}{k}\Big) \cdot \Big(1 - \frac{2}{\lambda(k+1)}\Big)^{-1} \leq \bar{B} \cdot (k+1)^{-\frac{1}{\lambda}},$$

for $\frac{2}{\lambda(k+1)} \leq \frac{A\bar{B}^\lambda}{k}$ thanks to the choice of $\bar{B}$. Consequently, $\xi_k \leq \bar{B} \cdot k^{-\frac{1}{\lambda}}$ for all $k \geq k_0$ and the claim follows with $B := \max\{1^{\frac{1}{\lambda}}\xi_1, 2^{\frac{1}{\lambda}}\xi_2, ..., (k_0 - 1)^{\frac{1}{\lambda}}\xi_{k_0-1}, \bar{B}\}$. □

Now we give our main result on the radial problem.

Theorem 3.6. *Suppose $p \geq 1$, $q \in (1-p, 0]$ and $v_0 \equiv 0$ in B_{r_0} for some $r_0 \in (0, R)$. Then for all $\nu > 0$ there exists $C_3 = C_3(\nu, n, r_0, R, M, p, q) > 0$ such that*

$$v(x,t) = 0 \qquad \textit{for all } x \in B_R \textit{ with } |x| \leq R - C_3 t^{-\frac{p+q-1}{-2q+\nu}}$$

holds for all $t > 0$. In particular, we have $\{v(t) = 0\} \nearrow B_R$ as $t \nearrow \infty$.

Proof. With $\zeta(t)$ as defined before, we have to show that

$$\zeta(t) \geq R - ct^{-\frac{p+q-1}{-2q+\nu}}$$

for some c and all t large enough.

To this end, we fix $\beta \in (\frac{2}{p+q+1}, 1)$ such that $\frac{(p+q+1)\beta-2}{2(1-(q+1)\beta)} \geq \frac{p+q-1}{-2q+\nu}$.

Starting with the numbers r_0 and t_0 as given and $d_0 := R - r_0$, we let

$$\begin{aligned} r_{k+1} &:= r_k + C_1 d_k^{\frac{(p+q+1)\beta}{(p+q+1)\beta-2}}, \\ t_{k+1} &:= t_k + c_1 d_k^{\frac{2(q+1)\beta}{(p+q+1)\beta-2}} \qquad \text{and} \\ d_{k+1} &:= R - r_{k+1} \end{aligned}$$

for $k = 0, 1, 2, ...$, where $c_1 := C_1^{(q+1)\beta} C_2$ and C_1 and C_2 are taken from Lemma 3.4. A repeated application of this lemma shows that

$$\zeta(t) \geq r_k \qquad \forall\, t \in [t_k, t_{k+1}). \tag{3.20}$$

In particular, we see that $r_{k+1} \leq \zeta(t_k) \leq R$, so that the strictly increasing sequence $(r_k)_{k\in\mathbb{N}}$ must converge. By the recursive definition of r_k, this means that the positive distances d_k tend to zero as $k \to \infty$. As the latter ones also satisfy

$$d_{k+1} = R - r_{k+1} = R - r_k - C_1 d_k^{\frac{(p+q+1)\beta}{(p+q+1)\beta-2}} = d_k - C_1 d_k^{1+\frac{2}{(p+q+1)\beta-2}}$$

for all $k \in \mathbb{N}$, Lemma 3.5 provides a positive constant c_2 such that

$$d_k \leq c_2 k^{-\frac{(p+q+1)\beta-2}{2}} \qquad \forall\, k \in \mathbb{N}. \tag{3.21}$$

From the definition of t_k we thus obtain

$$t_k - t_2 = c_1 \sum_{j=2}^{k-1} d_j^{\frac{2(q+1)\beta}{(p+q+1)\beta-2}} \le c \sum_{j=2}^{k-1} j^{-(q+1)\beta} \le c \int_1^{k-2} s^{-(q+1)\beta} ds \le c k^{1-(q+1)\beta} \tag{3.22}$$

and, therefore,

$$d_k \le c_2 k^{-\frac{(p+q+1)\beta-2}{2}} \le c(t_k - t_2)^{-\frac{(p+q+1)\beta-2}{2(1-(q+1)\beta)}}$$

for all $k > 2$, implying that

$$d_k \le c_3 t_k^{-\frac{p+q-1}{-2q+\nu}} \qquad \forall\, k \ge k_1 \tag{3.23}$$

with k_1 large enough. Since the difference $t_{k+1} - t_k$ converges to zero as $k \to \infty$ due to the fact that $d_k \to 0$ and since $t_k \to \infty$ as $k \to \infty$ – otherwise we would have $\zeta(t) = a$ for all $t > \lim_{k\to\infty} t_k$ and the proof would be complete – there is $t_\star > 0$ such that for any given $t > t_\star$ we can find $k \ge k_1$ such that $\frac{t}{2} \le t_k \le t$. Consequently, (3.20) and (3.23) give

$$\zeta(t) \ge \zeta(t_k) \ge r_k = R - d_k \ge R - c_3 t_k^{-\frac{p+q-1}{-2q+\nu}} \ge R - c_3 \cdot 2^{\frac{p+q-1}{-2q+\nu}} t^{-\frac{p+q-1}{-2q+\nu}}$$

for all $t > t_\star$, which was to be proved. □

3.3. **Infinite time extinction in the general case.** Now we can deal with the case of arbitrary data in general domains. The first lemma is very much in the same spirit as Lemma 2.1, reflecting of course the fact that now u has to remain positive near $\partial\Omega$ at finite times. Its proof, however, is somewhat different and it relies on Theorem 3.6.

Lemma 3.7. *Let $p \ge 1$ and $q \in (1-p, 0]$. Suppose that there are $t_0 \ge 0$, $x_0 \in \Omega$ and $r_0 \in (0, R)$ such that $B_R(x_0) \subset \Omega$ and*

$$u_\varepsilon(t_0) \to 0 \qquad \text{uniformly in } B_{r_0}(x_0) \quad \text{as } \varepsilon \to 0.$$

Then for any $\nu > 0$ there is $C_4 > 0$ (depending on $\nu, n, r_0, \Omega, \|u_0\|_{L^\infty(\Omega)}, p$ and q only) such that

$$u_\varepsilon \to 0 \quad \text{uniformly in compact subsets of} \quad \Big\{(x,t) \in \Omega \times [t_0, \infty) : |x - x_0| \le R - C_4 (t-t_0)^{-\frac{p+q-1}{-2q+\eta}}\Big\}. \tag{3.24}$$

Proof. We may assume $(x_0, t_0) = (0, 0)$. With $M := \|u_0\|_{L^\infty(\Omega)} + 1$, we define a radially nondecreasing function $v_0 \in W^{1,\infty}(B_R)$ by

$$v_0(x) := \begin{cases} 0, & |x| \le \frac{r_0}{2}, \\ \frac{2M}{r_0}\Big(|x| - \frac{r_0}{2}\Big), & \frac{r_0}{2} < |x| \le r_0, \\ M, & |x| > r_0, \end{cases}$$

and let v_ε denote the corresponding solution of (3.4) in B_R. Then, since $u_{\varepsilon'}(0) \to 0$ uniformly in B_{r_0} as $\varepsilon' \to 0$ and $u_{\varepsilon'} \le M$ for all $\varepsilon' \in (0, 1)$, we find that

$$u_{\varepsilon'} \le v_\varepsilon \qquad \text{holds on the parabolic boundary of } B_R \times (0, \infty)$$

for sufficiently small $\varepsilon' \le \varepsilon_0'(\varepsilon)$. Thus, by comparison,

$$u_{\varepsilon'} \le v_\varepsilon \qquad \text{holds in } B_R \times (0, \infty)$$

for such ε'. Now the claim results from Theorem 3.6, which in combination with (3.5) says that $v_\varepsilon \to 0$ uniformly in compact subsets of $\big\{(x,t) \in B_R \times [0,\infty) : |x| \le R - C_3 t^{-\frac{p+q-1}{-2q+\eta}}\big\}$ with some $C_3 > 0$ depending on the parameters listed above. □

The proof of the main theorem, as that of Theorem 2.3, again uses the smoothness of $\partial\Omega$.

Theorem 3.8. *Suppose $p \ge 1$, $q \in (1-p, 0]$ and u_0 vanishes in the ball $B_{r_0}(x_0) \subset \Omega$. Then for all $\nu > 0$ there is $C_5 > 0$, depending only on $\nu, n, r_0, \Omega, \|u_0\|_{L^\infty(\Omega)}, p$ and q, such that*

$$u(x,t) = 0 \ \ \textit{for all } (x,t) \in \Omega \times (0,\infty) \textit{ with } \operatorname{dist}(x, \partial\Omega) \ge C t^{-\frac{p+q-1}{-2q+\nu}}. \tag{3.25}$$

Remark. Let us emphasize that (3.25) is indeed to be understood in the pointwise sense; that is, $u = \lim u_\varepsilon$ satisfies (3.25) even without being manipulated within sets of measure zero.

Proof. The proof follows the same basic ideas as that of Theorem 2.3: Using Lemma 3.7 (rather than Lemma 2.1), we proceed as in the derivation of (2.18) to gain the existence of $T > 0$ such that

$$u_\varepsilon(T) \to 0 \quad \text{uniformly in } \Big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \frac{R}{2}\Big\} \quad \text{as } \varepsilon \to 0 \tag{3.26}$$

and

$$u \equiv 0 \qquad \text{in } \Big\{(x,t) \in \Omega \times (T, \infty) : \operatorname{dist}(x, \partial\Omega) \ge \frac{R}{2}\Big\}, \tag{3.27}$$

where $R > 0$ is a fixed small number such that

$$\forall\, x \in \Omega \text{ with } \operatorname{dist}(x, \partial\Omega) < R \text{ there is } x_0 \in \Omega \text{ with } \operatorname{dist}(x_0, \partial\Omega) = R \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega) = \operatorname{dist}(x, \partial B_R(x_0)). \tag{3.28}$$

Next, from (3.26) together with Lemma 3.7 we easily infer

$$\forall\, x_0 \in \Omega \text{ with } \operatorname{dist}(x_0, \partial\Omega) = R: \tag{3.29}$$
$$u \equiv 0 \text{ in } \Big\{(x,t) \in B_R(x_0) \times (T, \infty) : \operatorname{dist}(x, \partial B_R(x_0)) \geq C_4 (t-T)^{-\frac{p+q-1}{-2q+\nu}}\Big\},$$

where C_4 is as in (3.24) and hence independent of x_0.

We claim that (3.29) yields

$$u \equiv 0 \quad \text{in } \Big\{(x,t) \in \Omega \times (T, \infty) : \operatorname{dist}(x, \partial\Omega) \geq C_4 (t-T)^{-\frac{p+q-1}{-2q+\nu}}\Big\}. \tag{3.30}$$

In fact, if (x,t) lies in the set S on the right of (3.30) then either $\operatorname{dist}(x, \partial\Omega) \geq \frac{R}{2}$ (whence $u(x,t) = 0$ by (3.26)), or, by (3.28), there exists $x_0 \in \Omega$ with $\operatorname{dist}(x_0, \partial\Omega) = R$ and $\operatorname{dist}(x, \partial\Omega) = \operatorname{dist}(x, \partial B_R(x_0))$. In the latter case, however, we have $\operatorname{dist}(x, \partial B_R(x_0)) \geq C_4 (T-t)^{-\frac{p+q-1}{-2q+\nu}}$, because $(x,t) \in S$, so that (3.29) ensures $u(x,t) = 0$.

Finally, (3.30) evidently implies (3.25). □

4. Persistence for $q < 1 - p$

If $p \in (1,2)$ then in fact the interval $(1-p, 0)$ considered in the last section is intermediate: Then there are also q satisfying $q \in (-1, 1-p)$, and it turns out that in this case of very mild absorption we will not have total extinction at all, neither in finite nor in infinite time – at least not in convex domains.

We first consider a slightly generalized version of problem (0.1) in a one-dimensional half-space.

Lemma 4.1. *Let $p \geq 0$, $q < 1 - p$ and suppose that for some $\delta > 0$ and $m > 0$, $v_0 \in C^\infty([0, \delta])$ is nonnegative with $v_0(0) = m$ and $v_0(\delta) = 0$. Then for all $\varepsilon \in (0,1)$ the problem*

$$\begin{aligned} v_{\varepsilon t} &= v_\varepsilon^p v_{\varepsilon\xi\xi} - g_\varepsilon(v_\varepsilon) \qquad \text{for } (\xi, t) \in (0, \delta) \times (0, \infty), \\ v_\varepsilon|_{\xi=0} &= m + \varepsilon, \qquad v_\varepsilon|_{\xi=\delta} = \varepsilon, \\ v_\varepsilon|_{t=0} &= v_{0\varepsilon} := v_0 + \varepsilon, \end{aligned} \tag{4.1}$$

has a unique positive classical solution $v_\varepsilon = v_\varepsilon(\xi, t)$ which fulfills

$$\varepsilon \leq v_\varepsilon \leq \|v_0\|_{L^\infty((0,\delta))} + \varepsilon. \tag{4.2}$$

Furthermore, for any $\varepsilon \in (0,1)$ we have the uniform estimate

$$\int_0^\delta v_{\varepsilon\xi}^2(t) \le \int_0^\delta v_{0\xi}^2 + \frac{2}{1-p-q}\int_0^\delta v_{0\varepsilon}^{1-p-q} \qquad \forall\, t>0. \tag{4.3}$$

Proof. Since, due to the properties of g_ε, the constants ε and $\|v_0\|_{L^\infty((0,\delta))}+\varepsilon$ are a sub- and a supersolution of (4.1), respectively, existence and uniqueness of v_ε satisfying (4.2) follow via standard arguments. To see (4.3), we multiply the PDE in (4.1) by $\frac{v_{\varepsilon t}}{v_\varepsilon^p}$ (which is continuous on $[0,\delta]\times(0,\infty)$ and vanishes for $\xi=0$ and $\xi=\delta$) and integrate over $(0,\delta)\times(\tau,t)$ for $0<\tau<t$. This yields

$$\int_\tau^t\int_0^\delta \frac{v_{\varepsilon t}^2}{v_\varepsilon^p} + \frac{1}{2}\int_0^\delta v_{\varepsilon\xi}^2(t) + \int_0^\delta \Gamma_\varepsilon(v_\varepsilon(t)) = \frac{1}{2}\int_0^\delta v_{\varepsilon\xi}^2(\tau) + \int_0^\delta \Gamma_\varepsilon(v_\varepsilon(\tau)),$$

where we have set

$$\Gamma_\varepsilon(s) := \int_0^s g_\varepsilon(\sigma)\sigma^{-p}d\sigma.$$

As $0 \le \Gamma_\varepsilon(s) \le \frac{1-p-q}{s}^{1-p-q}$ and $v_{\varepsilon\xi}$ is continuous and bounded in $(0,\delta)\times[0,\infty)$ (cf. Theorem V.5.3 and Theorem V.6.2 in [18]), we infer from this, after letting $\tau\to 0$, that

$$\frac{1}{2}\int_0^\delta v_{\varepsilon\xi}^2(t) \le \frac{1}{2}\int_0^\delta v_{0\xi}^2 + \frac{1}{1-p-q}\int_0^\delta v_{0\varepsilon}^{1-p-q},$$

which is (4.3). □

The solutions of the above auxiliary problem will serve as subsolutions for the case of general data in convex domains in $\mathbb{R}^n$. As we shall see, the v_ε are uniformly bounded away from zero near $\xi=0$, so that the same will be true for u in a suitable neighborhood of the boundary.

Theorem 4.2. *Suppose $p\ge 1$, $q\in(-1,1-p)$ and that Ω is convex. Then there is $d>0$ such that*

$$u(x,t)>0 \qquad \forall\,(x,t)\in\Omega\times(0,\infty) \textit{ with } \operatorname{dist}(x,\partial\Omega)<d. \tag{4.4}$$

Proof. As in the proof of Lemma 3.1, we find $\delta>0$ such that

$$S_\delta := \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)<\delta\} \;=\; \{y-\kappa N(y) : y\in\partial\Omega, \kappa\in(0,\delta)\},$$

where due to the continuity of u_0 we may assume δ to be so small that

$$u_0 \ge m \qquad \text{in } S_\delta \tag{4.5}$$

holds for some $m\in(0,1]$. Let us now fix $d\in(0,\delta)$ small enough such that

$$\frac{m^2}{\delta} + \frac{2}{1-p-q}(m+1)^{1-p-q}\delta \le \frac{m^2}{4d}. \tag{4.6}$$

Given $x^0 \in S_d$ (where S_d is defined in an obvious way), we perform an affine coordinate transformation such that $\Omega \subset \{z = (z_1, ..., z_n) \in \mathbb{R}^n : z_n > 0\}$ and $x^0 = (0, ..., 0, x_n^0)$ with $x_n^0 = \operatorname{dist}(x^0, \partial\Omega)$. We set

$$v_0(\xi) := \frac{m}{\delta}(\delta - \xi), \qquad \xi \in (0, \delta),$$

and let v_ε denote the solution of (4.1) corresponding to v_0. Defining

$$w_\varepsilon(x,t) := v_\varepsilon(x_n, t)$$

for $(x,t) = (x_1, ..., x_n, t) \in \bar{Q}$, where

$$Q := \Big\{(x,t) = (x_1, ..., x_n, t) \in \Omega \times (0, \infty) : x_n < \delta\Big\},$$

we see that w_ε solves

$$w_{\varepsilon t} = w_\varepsilon^p \Delta w_\varepsilon - g_\varepsilon(w_\varepsilon) \qquad \text{in } Q.$$

At $t = 0$, we have

$$w_\varepsilon(x, 0) = v_0(x_n) + \varepsilon \le m + \varepsilon \le 1 + \varepsilon \le u_{0\varepsilon}(x),$$

while for $(x,t) \in Q$ with $x \in \partial\Omega$,

$$w_\varepsilon(x,t) \le \|v_\varepsilon\|_{L^\infty((0,\delta)\times(0,\infty))} = m + \varepsilon \le u_\varepsilon(x,t).$$

However, if $(x,t) \in \partial Q$ satisfies $x_n = \delta$ then

$$w_\varepsilon(x,t) = \varepsilon \le u_\varepsilon(x,t),$$

so that $w_\varepsilon \le u_\varepsilon$ holds on the parabolic boundary of Q. Consequently, $w_\varepsilon \le u_\varepsilon$ is true in all of Q by comparison and in particular

$$u_\varepsilon(x^0, t) \ge v_\varepsilon(x_n^0, t) \qquad \forall\, t > 0. \tag{4.7}$$

In virtue of our choice (4.6) of d, we infer from Lemma 4.1 that

$$\begin{aligned}
\int_0^\delta v_{\varepsilon\xi}^2(t) &\le \int_0^\delta v_{0\xi}^2 + \frac{2}{1-p-q}\int_0^\delta (v_0 + \varepsilon)^{1-p-q} \\
&\le \int_0^\delta \frac{m^2}{\delta^2} + \frac{2}{1-p-q}\int_0^\delta (m+1)^{1-p-q} \\
&= \frac{m^2}{\delta} + \frac{2}{1-p-q}(m+1)^{1-p-q}\delta \le \frac{m^2}{4d}
\end{aligned}$$

is valid for all $t > 0$. Hence, for all $t > 0$ and any $\xi \in (0, d)$ we have

$$v_\varepsilon(\xi, t) = v_\varepsilon(0, t) + \int_0^\xi v_{\varepsilon\xi}(\sigma, t)\, d\sigma$$

$$\geq v_\varepsilon(0,t) - \left(\int_0^\delta v_{\varepsilon\xi}^2(\sigma,t)d\sigma\right)^{\frac{1}{2}} \cdot \xi^{\frac{1}{2}} \geq m - \left(\frac{m^2}{4d}\right)^{\frac{1}{2}} \cdot d^{\frac{1}{2}} = \frac{m}{2}$$

and thus (4.7) yields

$$u_\varepsilon(x^0,t) \geq v_\varepsilon(x_n^0,t) \geq \frac{m}{2} \qquad \forall\, t > 0,$$

because $x^0 \in S_d$ implies $x_n^0 < d$. As x^0 was an arbitrary element of S_d, the proof is thereby complete. □

Remark. Again (cf. the remark following Lemma 3.1), in arbitrary non-convex domains we can slightly modify the above proof to obtain 'persistence' in the sense that to any extremal point $y \in \partial\Omega$ of Ω there exists $d > 0$ such that

$$u(x,t) > 0 \qquad \text{for all } (x,t) \in \Omega \times (0,\infty) \quad \text{satisfying } |x-y| < d.$$

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