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ASYMPTOTIC ANALYSIS OF A PARABOLIC SYSTEM ARISING IN COMBUSTION THEORY

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Abstract. We consider a one-dimensional parabolic system which describes the propagation of a flame with Lewis number close to one. We first prove estimates on the solutions that are independent of the activation energy. Next, we investigate the high activation energy asymptotics and derive the limiting free boundary problem. Our results rigorously justify the models currently used by physicists.

1. Introduction

The aim of this paper is to investigate the asymptotic behaviour as $\varepsilon \to 0$ of the following parabolic problem:

$$\frac{\partial \theta_\varepsilon}{\partial t} - \frac{\partial^2 \theta_\varepsilon}{\partial x^2} = \psi_\varepsilon f_\varepsilon(\theta_\varepsilon), \tag{1.1a}$$

$$\frac{\partial \psi_\varepsilon}{\partial t} - \Lambda \frac{\partial^2 \psi_\varepsilon}{\partial x^2} = -\psi_\varepsilon f_\varepsilon(\theta_\varepsilon). \tag{1.1b}$$

This system describes the propagation of a flame in a mixture at rest in the limit of small Mach numbers and for a one-step reaction $R \to P$. The unknowns are the renormalized temperature $\theta_\varepsilon \geqslant 0$ and the concentration of the reactant ψ_ε, $0 \leqslant \psi_\varepsilon \leqslant 1$.

The reaction rate $f_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ is given by the Arrhenius law and depends on some parameter $0 < \varepsilon \ll 1$ (the inverse of the reduced activation energy). It takes the form

$$f_\varepsilon(s) = \varepsilon^{-2} f(\varepsilon^{-1}(s-1)) \tag{1.2a}$$

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where the function $f\colon \mathbb{R} \to \mathbb{R}$ has an exponential growth:

$$\exists\, c_0 > 0 \text{ such that } 0 \leqslant f(\sigma) \leqslant c_0\, e^{\sigma},\ 0 \leqslant f'(\sigma) \leqslant c_0\, e^{\sigma} \quad \forall \sigma \in \mathbb{R}, \tag{1.2b}$$

and also satisfies:

$$f(\sigma) \equiv 0 \text{ for } \sigma \leqslant -1,\ f(\sigma) > 0 \text{ for } \sigma > -1, \tag{1.2c}$$

$$f \text{ is of class } C^{\infty} \text{ on } [-1, +\infty). \tag{1.2d}$$

Finally, the parameter $\Lambda > 0$ is the inverse of the Lewis number.

For $\Lambda = 1$, the system reduces to a scalar equation ($\psi_\varepsilon = 1 - \theta_\varepsilon$). Many papers have been concerned with this equation. Of particular interest is the singular limit $\varepsilon \to 0$ corresponding to the high activation energy asymptotics extensively used by the physicists. It was initially proposed by Zeldovich and Frank-Kamenetski [19] and was first rigorously studied in the case of one-dimensional travelling waves solutions (cf [4, 15]). Next, the singular limit in the elliptic case is addressed by Berestycki and his collaborators (see [3, 5] and the references therein). The parabolic equation (that is (1.1b) with $\theta_\varepsilon = 1 - \psi_\varepsilon$) is studied in Caffarelli-Vazquez [9]. Under appropriate assumptions on the initial data, the authors show that ψ_ε converges (through a subsequence) to a function u that satisfies the following free boundary problem (in a weak integral sense)

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u = 0 \text{ in } \{u > 0\}, \\ &u = 0, \quad \frac{\partial u}{\partial \nu} = -\sqrt{2M} \text{ on } \partial\{u > 0\}. \end{aligned}$$

Here ν is the outward unit spatial normal to the free boundary, and $M = \int_0^{+\infty} s f(-s)\, \mathrm{d}s$.

Also, the work of Bonder and Wolanski [6] deals with system (1.1) for $\Lambda = 1$ but with $\theta_\varepsilon + \psi_\varepsilon \neq 1$ due to the initial data. Assuming that $\varepsilon^{-1}(\theta_\varepsilon + \psi_\varepsilon - 1)$ converges to some given function, the authors are able to determine the limit free boundary problem.

Very few results are available in the general case $\Lambda \neq 1$. They only concern one-dimensional travelling waves (see [4, 16, 11]).

Note that, for the parabolic system (1.1), the existence of a solution is not guaranteed in general. The maximum principle does not apply to the temperature and it is known that the temperature on the flame front may exceed the one of the burnt gases (equal to one for our renormalized unknowns), see [7, 17, 10] and the computations in [2].

The system is considered by physicists in the so-called near equidiffusional limit, that is, under the condition (see [7, 10, 17])

$$|\Lambda_\varepsilon - 1| \leqslant \ell\,\varepsilon, \text{ with } \ell > 0 \text{ independent on } \varepsilon. \tag{1.3}$$

From a mathematical point of view, under this condition, it follows from Barabanova [1] that Problem (1.1) associated to homogeneous Neumann conditions in a bounded interval $I \subset \mathbb{R}$ has a solution at least when ε is small enough (see Section 2.1 below for more details). The aim of the paper is to investigate the asymptotic behaviour of the solutions as $\varepsilon \to 0$. The study requires careful estimates of the solutions, that are independent of ε. We addressed this question in an earlier work [14] but here we improve our previous results. For the asymptotic analysis, it is important to consider the enthalpy function $H_\varepsilon = \theta_\varepsilon + \psi_\varepsilon - 1$ and to show that

$$|H_\varepsilon| = O(\varepsilon). \tag{1.4}$$

Since the concentration ψ_ε varies between 0 and 1, (1.4) ensures that $\theta_\varepsilon(x,t) \leqslant 1 + c\,\varepsilon$. This precise bound for the temperature is the basic *assumption* in the formal asymptotic methods used by physicists. We derive it rigorously under condition (1.3) (while we make a stronger assumption in [14], namely $\Lambda - 1 = O(\varepsilon^r)$ for some $r > 1$). In [14], we obtained the estimates by considering the equations for ψ_ε and H_ε. A new tool here is to introduce the function $G_\varepsilon = \theta_\varepsilon + \Lambda_\varepsilon\psi_\varepsilon - 1$ and to deal with the equations for θ_ε and G_ε. Sections 2 and 3 are devoted to estimates in various spaces, and in particular in $L^\infty(\mathbb{R}^+; W^{1,\infty}(I))$.

Then, in Sections 4 and 5, we study the asymptotic limit $\varepsilon \to 0$ and the corresponding free boundary problem. For this purpose it is important to re-write the system in terms of θ_ε and the reduced enthalpy $h_\varepsilon = \varepsilon^{-1}H_\varepsilon$, *i.e.,* to consider the problem

$$\frac{\partial\theta_\varepsilon}{\partial t} - \frac{\partial^2\theta_\varepsilon}{\partial x^2} = (1 - \theta_\varepsilon + \varepsilon h_\varepsilon) f_\varepsilon(\theta_\varepsilon), \tag{1.5a}$$

$$\frac{\partial h_\varepsilon}{\partial t} - \Lambda_\varepsilon \frac{\partial^2 h_\varepsilon}{\partial x^2} = \varepsilon^{-1}(1 - \Lambda_\varepsilon)\frac{\partial^2\theta_\varepsilon}{\partial x^2}. \tag{1.5b}$$

Section 4 is devoted to the convergence of the functions θ_ε and h_ε and the resulting equations, whereas, in Section 5, we obtain the free boundary conditions.

Under appropriate assumptions, we prove that the sequences θ_ε and h_ε converge (through a subsequence) to some functions Θ and S which are solutions of the following problem:

$$\frac{\partial\Theta}{\partial t}-\frac{\partial^2\Theta}{\partial x^2}=0 \text{ in } \{\Theta<1\}, \quad \Theta\equiv 1 \text{ in } \{\Theta\geqslant 1\}, \tag{1.6a}$$

$$[\Theta]=0, \quad \left|\frac{\partial\Theta}{\partial x}\right|=\sqrt{2M} \text{ on } \partial\{\Theta<1\}, \tag{1.6b}$$

$$\frac{\partial S}{\partial t}-\frac{\partial^2 S}{\partial x^2}=-\lambda\frac{\partial^2\Theta}{\partial x^2} \text{ in } \{\Theta<1\} \text{ and } \{\Theta \overset{\circ}{\frown}{=} 1\}, \tag{1.6c}$$

$$[S]=0, \quad \left[\frac{\partial S}{\partial x}\right]=\lambda\left[\frac{\partial\Theta}{\partial x}\right] \text{ on } \partial\{\Theta<1\}, \tag{1.6d}$$

where the brackets denote the jump in the quantity from both sides of the free boundary, M is given by

$$M(x,t)=\int_{-\infty}^{S(x,t)}(S(x,t)-y)\,f(y)\,\mathrm{d}y, \tag{1.7}$$

and $\lambda\in[-\ell;\ell]$ is a cluster point of the sequence $\varepsilon^{-1}(\Lambda_\varepsilon-1)$.

This free boundary problem is the one predicted by physicists in the near equi-diffusional limit, that is, under assumption (1.3) (see [17, 7, 10]). Also our result generalizes the one of Bonder and Wolanski for $\Lambda_\varepsilon=1$. However, in contrast with [6], we do not make any assumption on the convergence of h_ε that turns out to be weaker than the one in [6]. Then, the techniques in [6] (see also [8, 9]) to derive the free boundary condition for θ_ε can be extended to our case (Section 5).

2. Uniform L^∞ estimates

2.1. **Preliminary results.** We investigate problem (1.1) in $I\times\mathbb{R}^+$, where I is a bounded open interval of $\mathbb{R}$. These equations are supplemented with homogeneous Neumann boundary conditions

$$\frac{\partial\theta_\varepsilon}{\partial x}=\frac{\partial\psi_\varepsilon}{\partial x}=0 \text{ on } \partial I\times\mathbb{R}^+, \tag{2.1}$$

and the initial conditions

$$\theta_\varepsilon(x,0)=\theta_0(x) \quad \psi_\varepsilon(x,0)=\psi_0(x) \quad \text{in } I. \tag{2.2}$$

We assume that

$$\theta_0,\ \psi_0\in L^\infty(I), \text{ with } 0\leqslant\theta_0(x) \text{ and } 0\leqslant\psi_0(x)\leqslant 1 \text{ for a.e. } x\in I. \tag{2.3}$$

The existence of a global solution of Problem (1.1), (2.1), (2.2) for ε sufficiently small follows from Barabanova [1]. Indeed an existence result is derived in [1] under the following restriction on ψ_0 :

$$\|\psi_0\|_{L^\infty(I)} < \frac{4\varepsilon\Lambda_\varepsilon}{|\Lambda_\varepsilon - 1|^2}. \tag{2.4}$$

In view of (1.3) and (2.3), this condition is necessarily satisfied if ε is sufficiently small, that is,

$$\varepsilon < \frac{4\Lambda_\varepsilon}{\ell^2}. \tag{2.5}$$

This condition is assumed hereafter. For convenience, we also assume that $\varepsilon < 1$.

Throughout the paper, we denote by c any constant that is independent of ε but may depend on the data (I, f, θ_0, ψ_0) and the parameter ℓ. Also we will use the following notation:

- D is the spatial derivative $D = \dfrac{\partial}{\partial x}$,
- χ_A is the indicator function of A,
- $C^{\alpha,\frac{\alpha}{2}}(I \times \mathbb{R}^+)$, with $\alpha \in (0,1]$, is the space of functions u that satisfy

  $\exists L > 0$ such that $|u(x,t) - u(y,s)| \leqslant L\big(|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}\big)$

  for all (x,t), $(y,s) \in I \times \mathbb{R}^+$,
- $C^{\alpha,\frac{\alpha}{2}}_{loc}(I \times \mathbb{R}^+)$ $(0 < \alpha \leqslant 1)$ is the space of functions that are $C^{\alpha,\frac{\alpha}{2}}(K)$ for all compact subsets K of $I \times \mathbb{R}^+$,
- $\|.\|_{p,I}$ denotes the norm in the space $L^p(I)$,
- $\|.\|_{2,I,\tau,T}$ denotes the norm in the space $L^2(I \times (\tau, T))$,
- for $(x_0, t_0) \in I \times \mathbb{R}^+$ and $\tau > 0$

$$Q_\tau(x_0,t_0) = (x_0 - \tau; x_0 + \tau) \times (t_0 - \tau^2; t_0 + \tau^2),$$
$$Q^-_\tau(x_0,t_0) = (x_0 - \tau; x_0 + \tau) \times (t_0 - \tau^2; t_0],$$

- and for any subset K of $I \times \mathbb{R}^+$

$$\mathcal{N}_\tau(K) = \bigcup_{(x_0,t_0)\in K} Q_\tau(x_0,t_0), \qquad \mathcal{N}^-_\tau(K) = \bigcup_{(x_0,t_0)\in K} Q^-_\tau(x_0,t_0).$$

As already mentioned, in a previous paper [14], we derived various estimates of the solution $(\theta_\varepsilon, \psi_\varepsilon)$ of (1.1), (2.1), (2.2) in terms of ε. Some of them will be useful here and we first recall them. The first one concerns the term $\exp\big(\frac{\theta_\varepsilon - 1}{\varepsilon}\big)$, closely related to $f_\varepsilon(\theta_\varepsilon)$. The first step consists in deriving some bounds in $L^\infty(\mathbb{R}^+; L^p(I))$ for $1 \leqslant p < +\infty$.

Proposition 2.1. *Let* $(\theta_\varepsilon, \psi_\varepsilon)$ *be solutions of* (1.1), (2.1), (2.2). *Suppose that* θ_0 *and* ψ_0 *satisfy* (2.3) *and moreover*

$$\|\theta_0 + \psi_0 - 1\|_{\infty,I} \leqslant c\,\varepsilon. \tag{2.6}$$

Let $p \in [1, +\infty)$. *Then the function* $\exp\left(\frac{\theta_\varepsilon - 1}{\varepsilon}\right)$ *is bounded independently of* $\varepsilon < \varepsilon_0$ *in* $L^\infty(\mathbb{R}^+; L^p(I))$ *where* $\varepsilon_0 = \varepsilon_0(p)$ *is given by*

$$\varepsilon_0(p) = \frac{2\Lambda_\varepsilon}{p\ell^2}.$$

For a proof of this result, we refer the reader to Theorem 3.1 in [14].

The following estimates concern ψ_ε and the function $H_\varepsilon = \theta_\varepsilon + \psi_\varepsilon - 1$.

Proposition 2.2. *Suppose that* (1.2), (1.3) *and* (2.3) *hold; then the following estimates are satisfied for all* $\varepsilon > 0$, $t \geqslant 0$ *and* $\eta \geqslant 0$*:*

$$\int_t^{t+\eta} \int_I |D\psi_\varepsilon|^2 (x,s)\, dx\, ds \leqslant c, \tag{2.7}$$

$$\int_t^{t+\eta} \int_I |DH_\varepsilon|^2 (x,s)\, dx\, ds \leqslant c\,\varepsilon^2, \tag{2.8}$$

$$\int_I H_\varepsilon^2(x,t)\, dx \leqslant c\,\varepsilon^2. \tag{2.9}$$

Moreover, if θ_0 *and* ψ_0 *are bounded independently of* ε *in* $H^1(I)$, *with*

$$\|D(\theta_0 + \psi_0 - 1)\|_{2,I} \leqslant c\,\varepsilon^{\frac{1}{2}}, \tag{2.10}$$

then the following estimate holds for all $0 < \varepsilon < \hat{\varepsilon}$, $t \geqslant 0$ *and* $\eta > 0$:

$$\int_t^{t+\eta} \int_I \Big(\frac{\partial H_\varepsilon}{\partial t}\Big)^2 (x,s)\, dx\, ds \leqslant c\,\varepsilon^{\frac{1}{2}} + c\eta + c\eta^{\frac{1}{2}}. \tag{2.11}$$

These estimates are derived by considering the equations for ψ_ε and H_ε and using energy methods. The reader is referred to [14] for the details of the proof.

In the sequel it will be important to introduce the function $G_\varepsilon = \theta_\varepsilon + \Lambda_\varepsilon \psi_\varepsilon - 1$. Note that it satisfies $G_\varepsilon = H_\varepsilon + (\Lambda_\varepsilon - 1)\psi_\varepsilon$ while $\theta_\varepsilon = H_\varepsilon + 1 - \psi_\varepsilon$. Therefore the above estimates on H_ε and ψ_ε yield similar bounds for the functions G_ε and θ_ε, namely the following.

Corollary 2.3. *Under the assumptions of Proposition* 2.2, *the following estimates hold for all* $\varepsilon > 0$, $t \geqslant 0$ *and* $\eta \geqslant 0$:

$$\int_t^{t+\eta} \int_I |D\theta_\varepsilon|^2 (x,s)\, dx\, ds \leqslant c, \tag{2.12}$$

$$\int_t^{t+\eta}\int_I |DG_\varepsilon|^2(x,s)\,dx\,ds \leqslant c\,\varepsilon^2, \tag{2.13}$$

$$\int_I G_\varepsilon^2(x,t)\,dx \leqslant c\,\varepsilon^2. \tag{2.14}$$

If moreover (2.10) *is satisfied, then we have for all* $0<\varepsilon<\hat{\varepsilon}$, $t \geqslant 0$ *and* $\eta > 0$

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,s)\,dx\,ds \leqslant c\varepsilon^{\frac{1}{2}} + c\eta + c\eta^{\frac{1}{2}}. \tag{2.15}$$

From now on, we derive new estimates on θ_ε and G_ε. Hereafter, we make the following assumption on the initial data:

$$\theta_0,\ \psi_0 \in H^2(I) \text{ with} \tag{2.16a}$$

$$D\psi_0 = 0, \qquad D\theta_0 = 0 \text{ on } \partial I, \tag{2.16b}$$

$$\theta_0(x) \geqslant 0, \qquad 0 \leqslant \psi_0(x) \leqslant 1, \tag{2.16c}$$

$$\begin{aligned}&\|D\theta_0\|_{\infty,I} \leqslant c, \quad \|\theta_0\|_{H^1(I)} \leqslant c, \quad \|\theta_0\|_{H^2(I)} \leqslant c\,\varepsilon^{-1},\\ &\|H_0\|_{\infty,I} \leqslant c\varepsilon \quad \|H_0\|_{H^1(I)} \leqslant c\varepsilon, \quad \|H_0\|_{H^2(I)} \leqslant c.\end{aligned} \tag{2.16d}$$

Due to the relations between the functions H_ε and G_ε, these assumptions on θ_0 and H_0 yield similar bounds on ψ_0 and G_0.

Also, the results below hold for ε sufficiently small (like Proposition 2.1 or (2.15)) but we will not specify it.

2.2. **Estimates in** $L^\infty(\mathbb{R}^+;H^1(I))$**.** Here, we derive some bounds for G_ε and θ_ε in the norm of $L^\infty(\mathbb{R}^+;H^1(I))$.

Proposition 2.4. *Suppose that* (1.2), (1.3) *and* (2.16) *hold. Then the following estimates hold for all* $t \geqslant 0$:

$$\int_I |D\theta_\varepsilon(x,t)|^2\,dx \leqslant c, \tag{2.17}$$

$$\int_I |DG_\varepsilon(x,t)|^2\,dx \leqslant c\,\varepsilon^2. \tag{2.18}$$

Moreover, we have for all $\eta > 0$

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2(x,s)\,dx\,ds \leqslant c(1+\eta), \tag{2.19}$$

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,s)\,dx\,ds \leqslant c\,\varepsilon^2(1+\eta). \tag{2.20}$$

We first state some lemmas that will be used throughout the proof of Proposition 2.4.

Lemma 2.5. *Define*

$$\mathcal{F}_\varepsilon(u,x,t) = \int_0^u \Lambda_\varepsilon^{-1}\big(G_\varepsilon(x,t)+1-s\big) f_\varepsilon(s)\, ds \tag{2.21}$$

and

$$F_\varepsilon(u) = \int_0^u f_\varepsilon(s)\, ds. \tag{2.22}$$

Then under hypotheses (1.2), (1.3) *and* (2.16)*, we have for all* $t \geqslant 0$

$$0 \leqslant \int_I \mathcal{F}_\varepsilon(\theta_\varepsilon,x,t)\, dx \leqslant c, \tag{2.23}$$

and for all $\eta > 0$

$$\left| \int_t^{t+\eta} \int_I F_\varepsilon(\theta_\varepsilon) \frac{\partial G_\varepsilon}{\partial t}\, dx\, ds \right| \leqslant c_0 \varepsilon^{-1} \eta^{\frac{1}{2}} \left\| \frac{\partial G_\varepsilon}{\partial t} \right\|_{2,I,t,t+\eta}. \tag{2.24}$$

Proof. Clearly, since $G_\varepsilon + 1 - \theta_\varepsilon = \Lambda_\varepsilon \psi_\varepsilon \geqslant 0$, we have $\mathcal{F}_\varepsilon(\theta_\varepsilon,x,t) \geqslant 0$. Next, thanks to (1.2), we see that

$$\mathcal{F}_\varepsilon(\theta_\varepsilon,x,t) \leqslant c_0 \Lambda_\varepsilon^{-1} \int_{-1}^{\varepsilon^{-1}(\theta_\varepsilon - 1)} \big(\varepsilon^{-1} G_\varepsilon(x,t) - y\big) e^y\, \mathrm{d}y$$

and therefore

$$\begin{aligned} 0 &\leqslant \int_I \mathcal{F}_\varepsilon(\theta_\varepsilon,x,t)\, \mathrm{d}x \leqslant c \int_I \varepsilon^{-1} \left|G_\varepsilon(x,t)\right|\, \mathrm{d}x + c\int_I \varepsilon^{-1} G_\varepsilon(x,t) \exp\frac{\theta_\varepsilon - 1}{\varepsilon}\, \mathrm{d}x \\ &\quad + c\int_I \exp\frac{\theta_\varepsilon - 1}{\varepsilon}\, \mathrm{d}x + c\int_I \varepsilon^{-1}(1-\theta_\varepsilon)\exp\frac{\theta_\varepsilon - 1}{\varepsilon}\, \mathrm{d}x \\ &\leqslant c\,\varepsilon^{-1} \|G_\varepsilon\|_{2,I} + c\,\varepsilon^{-1}\|G_\varepsilon\|_{2,I} \left\| \exp\frac{\theta_\varepsilon - 1}{\varepsilon} \right\|_{2,I} + c \left\| \exp\frac{\theta_\varepsilon - 1}{\varepsilon} \right\|_{2,I} + c. \end{aligned}$$

According to Proposition 2.1, $\exp\big(\frac{\theta_\varepsilon - 1}{\varepsilon}\big)$ is uniformly bounded in $L^2(I)$ for $t \geqslant 0$ and $\varepsilon < \varepsilon_0(2)$, while (2.14) provides an estimate of $\|G_\varepsilon\|_{2,I}$. These bounds combined with the above inequality yield (2.23).

Next, due to (1.2), we see that

$$0 \leqslant F_\varepsilon(\theta_\varepsilon) = \int_0^{\theta_\varepsilon(x,t)} f_\varepsilon(s)\, \mathrm{d}s = \varepsilon^{-1} \int_{-1}^{\varepsilon^{-1}(\theta_\varepsilon - 1)} f(y)\, \mathrm{d}y \leqslant c_0\, \varepsilon^{-1} \exp\frac{\theta_\varepsilon - 1}{\varepsilon}.$$

Hence, thanks to Proposition 2.1 and Hölder estimates,

$$\begin{aligned}\left|\int_t^{t+\eta}\int_I F_\varepsilon(\theta_\varepsilon)\frac{\partial G_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}s\right| &\leqslant c_0\,\varepsilon^{-1}\int_t^{t+\eta}\int_I \exp\Big(\frac{\theta_\varepsilon-1}{\varepsilon}\Big)\left|\frac{\partial G_\varepsilon}{\partial t}\right|\mathrm{d}x\,\mathrm{d}s\\ &\leqslant c\,\varepsilon^{-1}\eta^{\frac12}\Big(\int_t^{t+\eta}\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2\mathrm{d}x\,\mathrm{d}s\Big)^{\frac12},\end{aligned}$$

which is (2.24). □

Lemma 2.6. *Assume that* (1.2), (1.3) *and* (2.16) *hold. Suppose moreover that the function G_ε satisfies*

$$\int_t^{t+\eta}\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,s)\,dx\,ds \leqslant a^2, \tag{2.25}$$

for all $t\geqslant 0$ and $\eta>0$ with $a=a(\eta)$ independent on t.

Then, for all $t\geqslant 0$ and $\eta>0$, we have

$$\int_I |D\theta_\varepsilon|^2(x,t)\,dx \leqslant c + c\,\eta^{-1} + c\,\varepsilon^{-1}\eta^{\frac12}a, \tag{2.26}$$

$$\int_t^{t+\eta}\int_I\Big(\frac{\partial\theta}{\partial t}\Big)^2(x,s)\,dx\,ds \leqslant \frac12\int_I|D\theta_\varepsilon|^2(x,t)\,dx + c + c\,\varepsilon^{-1}\eta^{\frac12}a. \tag{2.27}$$

Proof. By multiplying the equation (1.1a) for θ_ε by $\dfrac{\partial\theta_\varepsilon}{\partial t}$ and integrating over I, we obtain that

$$\int_I\Big(\frac{\partial\theta_\varepsilon}{\partial t}\Big)^2\mathrm{d}x + \frac12\frac{d}{dt}\int_I|D\theta_\varepsilon|^2\,\mathrm{d}x = \int_I\psi_\varepsilon f_\varepsilon(\theta_\varepsilon)\frac{\partial\theta_\varepsilon}{\partial t}\,\mathrm{d}x. \tag{2.28}$$

Next, we have

$$\psi_\varepsilon f_\varepsilon(\theta_\varepsilon)\frac{\partial\theta_\varepsilon}{\partial t} = \frac{\partial}{\partial t}\mathcal{F}_\varepsilon(\theta_\varepsilon,x,t) - \Lambda_\varepsilon^{-1}F_\varepsilon(\theta_\varepsilon)\frac{\partial G_\varepsilon}{\partial t}. \tag{2.29}$$

Setting

$$y(t) = \int_I\Big(\frac12|D\theta_\varepsilon|^2(x,t) - \mathcal{F}_\varepsilon(\theta_\varepsilon,x,t)\Big)\,\mathrm{d}x, \tag{2.30}$$

equation (2.28) may be rewritten

$$y'(t) \leqslant -\Lambda_\varepsilon^{-1}\int_I F_\varepsilon(\theta_\varepsilon)\frac{\partial G_\varepsilon}{\partial t}\,\mathrm{d}x. \tag{2.31}$$

Observe that for $t\geqslant 0$ and $\eta>0$ we have

$$y(t+\eta) \leqslant \frac1\eta\int_t^{t+\eta}y(s)\,\mathrm{d}s + c\,\varepsilon^{-1}\eta^{\frac12}a. \tag{2.32}$$

Indeed, for $t \geqslant 0$ and $\eta > 0$, there exists $\tau \in (t, t+\eta)$ such that

$$y(\tau) = \frac{1}{\eta} \int_t^{t+\eta} y(s)\,\mathrm{d}s.$$

Then, by integrating (2.31) between τ and $t+\eta$, we see that

$$y(t+\eta) \leqslant y(\tau) - \Lambda_\varepsilon^{-1} \int_\tau^{t+\eta} \int_I F_\varepsilon(\theta) \frac{\partial G_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}s;$$

that yields (2.32) thanks to (2.24) and (2.25).

Next, by the definition of y, we have

$$\int_t^{t+\eta} y(s)\,\mathrm{d}s \leqslant \frac{1}{2} \int_t^{t+\eta} \int_I |D\theta_\varepsilon|^2 (x,s)\,\mathrm{d}x\,\mathrm{d}s$$

and thanks to (2.12)

$$\int_t^{t+\eta} y(s)\,\mathrm{d}s \leqslant c.$$

Therefore, we infer from (2.32) that

$$y(t+\eta) \leqslant \frac{c}{\eta} + c\,\varepsilon^{-1} \eta^{\frac{1}{2}} a.$$

Recalling the definition (2.30) of y and the estimate (2.23), we obtain finally that, for all $t \geqslant \eta$,

$$\int_I |D\theta_\varepsilon|^2 (x,t)\,\mathrm{d}x \leqslant c + \frac{c}{\eta} + c\,\varepsilon^{-1} \eta^{\frac{1}{2}} a.$$

This bound also holds for $0 \leqslant t \leqslant \eta$. This is easily shown by integrating (2.31) between 0 and t and using (2.23)-(2.25). Estimate (2.26) is derived.

Finally, the estimate (2.27) follows from the integration of equation (2.28) between t and $t+\eta$, that gives

$$\int_t^{t+\eta}\!\!\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 (x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{1}{2} \int_I |D\theta_\varepsilon|^2 (x,t)\,\mathrm{d}x + \int_t^{t+\eta}\!\!\int_I \psi_\varepsilon f_\varepsilon(\theta_\varepsilon) \frac{\partial \theta_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}s.$$

The relation (2.29) and the estimates (2.23) and (2.24) provide (2.27). □

We aim now to prove Proposition 2.4.

Proof of Proposition 2.4. The main step consists in proving the estimate (2.18) and (2.20) thanks to some iterative argument. The function G_ε satisfies the equation

$$\frac{\partial G_\varepsilon}{\partial t} - \Lambda_\varepsilon D^2 G_\varepsilon = (1-\Lambda_\varepsilon) \frac{\partial \theta_\varepsilon}{\partial t}. \tag{2.33}$$

By multiplying by $\frac{\partial G_\varepsilon}{t}$ and integrating on I, we easily obtain the differential inequality

$$\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x + \Lambda_\varepsilon \frac{d}{dt}\int_I |DG_\varepsilon|^2\,(x,t)\,\mathrm{d}x \leqslant (\Lambda_\varepsilon - 1)^2 \int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x. \tag{2.34}$$

We will make repeated use of the uniform Gronwall lemma (see for instance Temam [18]). Recall that the estimate (2.13) holds. Also assuming (2.25), we have the estimates (2.26)-(2.27) and the uniform Gronwall lemma applied to (2.34) gives that for $t \geqslant \eta > 0$ (here we use (1.3) which implies that $(\Lambda_\varepsilon - 1)^2 \leqslant c\,\varepsilon^2$)

$$\int_I |DG_\varepsilon|^2\,(x,t)\,\mathrm{d}x \leqslant c\,\varepsilon^2\eta^{-1} + c\,\varepsilon^2 + c\,\varepsilon\eta^{\frac{1}{2}}a. \tag{2.35}$$

It is easy to check that (2.35) also holds for $0 \leqslant t \leqslant \eta$ by integrating (2.34) and using the assumption on the initial data (see (2.16)).

Next, the integration of (2.34) between t and $t+\eta$ provides the following new estimate of $\frac{\partial G_\varepsilon}{\partial t}$ in $L^2(I\times(t,t+\eta))$:

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant \hat{a}^2 \tag{2.36}$$

with

$$\hat{a}^2 = c\,\varepsilon^2 + c\,\varepsilon^2\eta^{-1} + \frac{c}{2}\varepsilon^2\eta + \frac{1}{2}a^2.$$

To start the iterative argument, we take $a = a_0$ given by (2.15). Then (2.36) provides $a_1 = \hat{a}_0$. We recursively define $a_{n+1} = \hat{a}_n$ and conclude that

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial G}{\partial t}\Big)^2(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant a_n^2 \text{ for all } n \geqslant 0,$$

where $a_{n+1}^2 = c\,\varepsilon^2 + c\,\varepsilon^2\eta^{-1} + \frac{c}{2}\varepsilon^2\eta + \frac{1}{2}a_n^2$.

Now the sequence $(a_n)_{n\geqslant 0}$ converges to a_∞ as $n \to +\infty$ and

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant a_\infty^2 = 2c\,\varepsilon^2 + 2c\,\varepsilon^2\eta^{-1} + c\,\varepsilon^2\eta.$$

To conclude, we again use Lemma 2.6 with $a = a_\infty$; thanks to (2.26), we obtain that, for all $t \geqslant 0$, $\eta > 0$,

$$\int_I |D\theta_\varepsilon|^2\,(x,t)\,\mathrm{d}x \leqslant c + c\,\eta^{-1} + c\,\eta^{\frac{1}{2}} + \varepsilon^2\eta.$$

The particular choice $\eta = 1$ provides the estimate (2.17).

Next, the estimate (2.35) with $a = a_\infty$ reads

$$\int_I |DG|^2 (x,t)\,\mathrm{d}x \leqslant c\,\varepsilon^2 + c\,\varepsilon + c\,\varepsilon^2\eta^{-1} + c\,\varepsilon\eta^{\frac{1}{2}} + c\,\varepsilon^2\eta$$

which gives (2.18) for $\eta = 1$.

Finally, by integrating (2.34) between t and $t+\eta$, we see that

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant c\,\varepsilon^2 + \varepsilon^2\int_t^{t+\eta}\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x\,\mathrm{d}s.$$

Besides, estimate (2.27) for $a = a_\infty$ gives that

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant c + c\,\eta.$$

Combining these inequalities, we obtain that

$$\int_t^{t+\eta}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant c\,\varepsilon^2 + c\,\varepsilon^2\eta,$$

which is (2.20). The proof of Proposition 2.4 is now complete. □

Remark 2.7. The estimates of this section hold in any space dimension, that is, for problem (1.1) in $\Omega \times \mathbb{R}^+$ associated to homogeneous Neumann boundary conditions, where Ω is an open bounded regular set of $\mathbb{R}^N$.

2.3. **Estimates in** $L^\infty(I \times \mathbb{R}^+)$**.** We can now derive some L^∞ bounds.

Proposition 2.8. *Under the assumptions of Proposition* 2.4, *the following estimates hold for all* $t \geqslant 0$ *and* $x \in I$:

$$|G_\varepsilon(x,t)| \leqslant c\,\varepsilon, \tag{2.37}$$

$$0 \leqslant \theta_\varepsilon(x,t) \leqslant 1 + c\,\varepsilon. \tag{2.38}$$

In particular, the function $\varepsilon^{-1}G_\varepsilon$ *is bounded independently of* ε *in* $L^\infty(I \times \mathbb{R}^+)$.

Proof. The bound (2.37) is a straightforward consequence of (2.18) due to the Sobolev injection $H^1(I) \subset L^\infty(I)$. Next, estimate (2.38) follows easily since $\theta_\varepsilon = 1 + G_\varepsilon - \Lambda_\varepsilon\psi_\varepsilon \leqslant 1 + G_\varepsilon$. □

The above L^∞ bounds yield some simple estimates on the nonlinear terms.

Corollary 2.9. *Under the assumptions of Proposition* 2.4, *we have that*

$$\exp\Big(\frac{\theta_\varepsilon - 1}{\varepsilon}\Big) \leqslant c, \qquad 0 \leqslant f_\varepsilon(\theta_\varepsilon) \leqslant c\,\varepsilon^{-2} \tag{2.39}$$

$$0 \leqslant \psi_\varepsilon f_\varepsilon'(\theta_\varepsilon) \leqslant c\,\varepsilon^{-2}, \tag{2.40}$$

$$\psi_\varepsilon^2 f_\varepsilon(\theta_\varepsilon)^2 \leqslant c\,\varepsilon^{-2}. \tag{2.41}$$

Proof. The bounds (2.39) follow immediately from (2.38). Next, since $\psi_\varepsilon = \Lambda_\varepsilon^{-1}(1-\theta_\varepsilon+G)$, we have that

$$\psi_\varepsilon \leqslant \begin{cases} \Lambda_\varepsilon^{-1}G_\varepsilon & \text{if } \theta_\varepsilon \geqslant 1, \\ \Lambda_\varepsilon^{-1}(1-\theta_\varepsilon)+\Lambda_\varepsilon^{-1}G_\varepsilon & \text{if } 0 \leqslant \theta_\varepsilon \leqslant 1. \end{cases}$$

Therefore, due to (1.2),

$$\psi_\varepsilon f_\varepsilon'(\theta_\varepsilon) \leqslant c_0\Lambda_\varepsilon^{-1}\varepsilon^{-3}G_\varepsilon \exp\Big(\frac{\theta_\varepsilon-1}{\varepsilon}\Big) + c_0\Lambda_\varepsilon^{-1}\varepsilon^{-3}(1-\theta_\varepsilon)\chi_{\{0\leqslant\theta_\varepsilon\leqslant 1\}} \exp\Big(\frac{\theta_\varepsilon-1}{\varepsilon}\Big). \tag{2.42}$$

The first term in the right-hand side of (2.42) is estimated thanks to (2.39) and (2.37) while the function $\varepsilon^{-1}(1-s)\exp(\frac{s-1}{\varepsilon})$ is bounded on $[0,1]$. This gives (2.40). The proof of (2.41) is similar. □

Remark 2.10. An estimate like (2.37) can be derived in higher space dimension but under a stronger condition on Λ_ε. This condition depends on the space dimension. For example, if $\Omega \subset \mathbb{R}^2$, it reads $\Lambda_\varepsilon - 1 = O(\varepsilon/\ln(\varepsilon^{-1}))$; in that case, (2.37) is obtained thanks to estimates in $L^\infty(\mathbb{R}^+;H^2(\Omega))$ as in Section 3 below.

3. UNIFORM GRADIENT ESTIMATES

In order to investigate the asymptotic behaviour of the system (1.1) as $\varepsilon \to 0$, we need some further estimates on the gradient of G_ε and θ_ε in L^∞ norm. This is the aim of the present section. We start by deriving some bounds in $L^\infty(\mathbb{R}^+;H^2(I))$.

3.1. **Preliminary estimates.** We first derive some estimates for θ_ε.

Proposition 3.1. *Assume that* (1.2), (1.3), (2.16) *hold. Then, the following estimate is satisfied for all* $t \geqslant 0$:

$$\int_I \Big(\frac{\partial\theta_\varepsilon}{\partial t}\Big)^2(x,t)\,dx \leqslant c\,\varepsilon^{-2}. \tag{3.1}$$

Furthermore for all $t \geqslant 0$ *and* $\eta > 0$, *we have*

$$\int_t^{t+\eta}\int_I \Big|D\frac{\partial\theta_\varepsilon}{\partial t}\Big|^2(x,s)\,dx\,ds \leqslant c\,\varepsilon^{-2}(1+\eta). \tag{3.2}$$

Proof. By differentiating the equation for θ_ε, since $\psi_\varepsilon = \Lambda_\varepsilon^{-1}(G_\varepsilon - \theta_\varepsilon + 1)$, we have that

$$\frac{\partial}{\partial t}\frac{\partial \theta_\varepsilon}{\partial t} - D^2\frac{\partial \theta_\varepsilon}{\partial t} = \Lambda_\varepsilon^{-1} f_\varepsilon(\theta_\varepsilon)\frac{\partial G_\varepsilon}{\partial t} + \Big(\psi_\varepsilon f_\varepsilon'(\theta_\varepsilon) - \Lambda_\varepsilon^{-1} f_\varepsilon(\theta_\varepsilon)\Big)\frac{\partial \theta_\varepsilon}{\partial t}. \tag{3.3}$$

Hence, by multiplying this equation by $\dfrac{\partial \theta_\varepsilon}{\partial t}$ and integrating over I,

$$\frac{1}{2}\frac{d}{dt}\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x + \int_I \Big|D\frac{\partial \theta_\varepsilon}{\partial t}\Big|^2 \mathrm{d}x \leqslant \Lambda_\varepsilon^{-1}\int_I f_\varepsilon(\theta_\varepsilon)\frac{\partial G_\varepsilon}{\partial t}\frac{\partial \theta_\varepsilon}{\partial t}\,\mathrm{d}x + \int_I \psi_\varepsilon f_\varepsilon'(\theta_\varepsilon)\Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x, \tag{3.4}$$

and thanks to (2.39), (2.40),

$$\frac{d}{dt}\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x + \int_I \Big|D\frac{\partial \theta_\varepsilon}{\partial t}\Big|^2 \mathrm{d}x \leqslant c\,\varepsilon^{-2}\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x + c\,\varepsilon^{-2}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x. \tag{3.5}$$

In view of (2.19) and (2.20), we can apply the uniform Gronwall lemma to (3.5) and obtain for $t \geqslant \eta$, $\eta > 0$ that

$$\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 (x,t)\,\mathrm{d}x \leqslant c(\eta^{-1} + 1 + \eta)\exp(\eta\varepsilon^{-2}).$$

For $0 \leqslant t \leqslant \eta$, the usual Gronwall lemma provides

$$\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 (x,t)\,\mathrm{d}x \leqslant \exp(\eta\varepsilon^{-2})\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 (x,0)\,\mathrm{d}x + c\,\varepsilon^{-2}\exp(\varepsilon^{-2}\eta)\int_0^\eta\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2 (x,s)\,\mathrm{d}x\,\mathrm{d}s. \tag{3.6}$$

Next the equation for θ_ε at $t = 0$ yields that

$$\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 (x,0)\,\mathrm{d}x \leqslant 2\int_I \Big(D^2\theta_0\Big)^2 (x)\,\mathrm{d}x + 2\int_I \psi_0^2 f_\varepsilon(\theta_0)^2\,\mathrm{d}x$$

and thanks to (2.16) and (2.41),

$$\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 (x,0)\,\mathrm{d}x \leqslant c\,\varepsilon^{-2}.$$

Consequently, we infer from (3.6) that for $0 \leqslant t \leqslant \eta$

$$\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 (x,t)\,\mathrm{d}x \leqslant c(\varepsilon^{-2} + \eta)\exp(\eta\varepsilon^{-2}).$$

Combining the two bounds, we conclude that for all $t \geqslant 0$ and $\eta > 0$

$$\int_I \Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2 (x,t)\,\mathrm{d}x \leqslant c\big(\varepsilon^{-2} + \eta^{-1} + \eta\big)\exp(\eta\varepsilon^{-2}).$$

In particular, choosing $\eta = \varepsilon^2$ in this estimate provides the bound (3.1).

Next, returning to (3.5), we integrate between t and $t+\eta$, $\eta > 0$, and using (2.19) and (2.20), we see that

$$\int_t^{t+\eta}\int_I \left| D\frac{\partial \theta_\varepsilon}{\partial t}\right|^2 (x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant c\,\varepsilon^{-2}(1+\eta),$$

which provides (3.2). □

We can now state some bounds on G_ε.

Proposition 3.2. *Under the assumptions of Proposition* 3.1, *the following estimate holds for all $t \geqslant 0$:*

$$\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2 (x,t)\,dx \leqslant c. \tag{3.7}$$

Proof. Let us multiply the equation for $\frac{\partial G_\varepsilon}{\partial t}$ by $\frac{\partial G_\varepsilon}{\partial t}$ and integrate over I. This gives

$$\frac{1}{2}\frac{d}{dt}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x + \Lambda_\varepsilon \int_I \left| D\frac{\partial G_\varepsilon}{\partial t}\right|^2 \mathrm{d}x = (1-\Lambda_\varepsilon)\int_I \frac{\partial^2 \theta_\varepsilon}{\partial t^2}\frac{\partial G_\varepsilon}{\partial t}\,\mathrm{d}x. \tag{3.8}$$

On the other hand, by using (3.3), we have that

$$\int_I \frac{\partial^2 \theta_\varepsilon}{\partial t^2}\frac{\partial G_\varepsilon}{\partial t}\,\mathrm{d}x = -\int_I D\frac{\partial G_\varepsilon}{\partial t}\cdot D\frac{\partial \theta_\varepsilon}{\partial t}\,\mathrm{d}x + \Lambda_\varepsilon^{-1}\int_I f_\varepsilon(\theta_\varepsilon)\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x$$
$$+ \int_I (\psi_\varepsilon f_\varepsilon'(\theta_\varepsilon) - \Lambda_\varepsilon^{-1} f_\varepsilon(\theta_\varepsilon))\frac{\partial \theta_\varepsilon}{\partial t}\frac{\partial G_\varepsilon}{\partial t}\,\mathrm{d}x. \tag{3.9}$$

Combining these identities and recalling the bounds (2.39), (2.40), we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x + \Lambda_\varepsilon \int_I \left| D\frac{\partial G_\varepsilon}{\partial t}\right|^2 \mathrm{d}x \leqslant (\Lambda_\varepsilon - 1)\int_I D\frac{\partial \theta_\varepsilon}{\partial t}\cdot D\frac{\partial G_\varepsilon}{\partial t}\,\mathrm{d}x$$
$$+ c\,\varepsilon^{-1}\int_I \Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2 \mathrm{d}x + c\,\varepsilon^{-1}\int_I \left|\frac{\partial \theta_\varepsilon}{\partial t}\right|\left|\frac{\partial G_\varepsilon}{\partial t}\right|$$

and thanks to (1.3) and the Cauchy-Schwarz inequality

$$\frac{d}{dt}\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2\,\mathrm{d}x+\Lambda_\varepsilon\int_I\Big|D\frac{\partial G_\varepsilon}{\partial t}\Big|^2\,\mathrm{d}x\leqslant c\,\varepsilon^2\int_I\Big|D\frac{\partial \theta_\varepsilon}{\partial t}\Big|^2\,\mathrm{d}x + c\int_I\Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2\,\mathrm{d}x+c\,\varepsilon^{-2}\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2\,\mathrm{d}x. \quad (3.10)$$

Recall that estimates (3.2), (2.19) and (2.20) hold. Hence we can apply the uniform Gronwall lemma to (3.10), and obtain for all $t\geqslant\eta$, $\eta>0$

$$\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x\leqslant c(\varepsilon^2\eta^{-1}+\varepsilon^2+1+\eta)\exp(\eta\varepsilon^{-2}).$$

For $0\leqslant t\leqslant\eta$, the usual Gronwall lemma provides that

$$\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x\leqslant c(1+\eta)\exp(\eta\varepsilon^{-2}).$$

Combining the two bounds, we get finally for all $t\geqslant 0$, $\eta>0$,

$$\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x\leqslant c(\varepsilon^2\eta^{-1}+1+\varepsilon^2\eta)\exp(\eta\varepsilon^{-2}).$$

Now, we choose $\eta=\varepsilon^2$. Using (1.3), we obtain for all $t\geqslant 0$

$$\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x\leqslant c,$$

which is (3.7). □

As a consequence of Proposition 3.1 and 3.2, we have the following result.

Corollary 3.3. *Under the assumptions of Proposition* 3.1, *we have, for all* $t\geqslant 0$,

$$\int_I (D^2G_\varepsilon)^2(x,t)\,dx\leqslant c. \quad (3.11)$$

Proof. The equation for G_ε may be rewritten

$$\Lambda_\varepsilon D^2G_\varepsilon=\frac{\partial G_\varepsilon}{\partial t}+(1-\Lambda_\varepsilon)\frac{\partial\theta_\varepsilon}{\partial t}$$

so that

$$\int_I (D^2G_\varepsilon)^2(x,t)\,\mathrm{d}x\leqslant 2\Lambda_\varepsilon^{-2}\int_I\Big(\frac{\partial G_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x+2\Lambda_\varepsilon^{-2}\varepsilon^2\int_I\Big(\frac{\partial \theta_\varepsilon}{\partial t}\Big)^2(x,t)\,\mathrm{d}x$$

that gives (3.11) in view of the estimates (3.1) and (3.7). □

3.2. **Some L^∞ estimates for the gradients.** We first state an estimate for DG_ε.

Proposition 3.4. *Under the assumptions of Proposition 3.1, the following estimate holds:*

$$\|DG_\varepsilon\|_{L^\infty(\mathbb{R}^+\times I)} \leqslant c\,\varepsilon^{\frac{1}{2}}. \tag{3.12}$$

In particular, DG_ε is bounded independently of ε in $L^\infty(\mathbb{R}^+ \times I)$.

Proof. The bound (3.12) follows readily from (2.18) and (3.11) thanks to the Gagliardo-Nirenberg inequality

$$\|DG_\varepsilon\|_{L^\infty(I)} \leqslant c\,\|G_\varepsilon\|^{\frac{1}{2}}_{H^1(I)}\,\|G_\varepsilon\|^{\frac{1}{2}}_{H^2(I)}\,. \qquad \square$$

We next turn to estimates for $D\theta_\varepsilon$.

Proposition 3.5. *Assume that the hypotheses of Proposition 3.1 hold. Then $D\theta_\varepsilon$ is bounded independently of ε in $L^\infty(\mathbb{R}^+ \times I)$.*

Proof. Our arguments use some techniques of Caffarelli and Vazquez [9]. We define the rescaled functions

$$u_\varepsilon(x,t) = \varepsilon^{-1}(1-\theta_\varepsilon(\varepsilon x, \varepsilon^2 t)), \tag{3.13}$$

$$g_\varepsilon(x,t) = \varepsilon^{-1} G_\varepsilon(\varepsilon x, \varepsilon^2 t). \tag{3.14}$$

They satisfy the following equations:

$$\frac{\partial u_\varepsilon}{\partial t} - D^2 u_\varepsilon = -\Lambda_\varepsilon^{-1}(g_\varepsilon + u_\varepsilon) f(-u_\varepsilon) \text{ in } I_\varepsilon \times \mathbb{R}^+, \tag{3.15a}$$

$$\frac{\partial g_\varepsilon}{\partial t} - \Lambda_\varepsilon D^2 g_\varepsilon = \lambda_\varepsilon \frac{\partial u_\varepsilon}{\partial t} \text{ in } I_\varepsilon \times \mathbb{R}^+, \tag{3.15b}$$

where $\lambda_\varepsilon = \varepsilon^{-1}(\Lambda_\varepsilon - 1)$, and $I_\varepsilon = \{x \in \mathbb{R} : \varepsilon x \in I\}$, together with homogeneous Neumann boundary conditions and the initial conditions

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x) = \varepsilon^{-1}(1-\theta_0(\varepsilon x)) \text{ in } I_\varepsilon,$$
$$g_\varepsilon(x,0) = g_{0\varepsilon}(x) = \varepsilon^{-1} G_0(\varepsilon x) \text{ in } I_\varepsilon.$$

Now, introduce the change of variables $u_\varepsilon = \phi(v_\varepsilon)$, where $\phi\colon \mathbb{R}^+ \to \mathbb{R}$ is a convex, increasing and C^2 function which satisfies

$$\phi(0) \leqslant -\|g_\varepsilon\|_\infty, \quad \phi'(0) = 1, \quad \phi''(0) = 1,$$

$$\phi(v_0) = 1, \quad \phi' \equiv b \text{ for } v \geqslant v_0, \quad \Big(\frac{\phi''}{\phi'}\Big)' \leqslant -a \text{ for } v \leqslant v_0, \tag{3.16}$$

for some positive constants v_0, a and b (for example, ϕ can be chosen as a piecewise polynomial function).

Using the Bernstein technique, we consider the equation satisfied by $w_\varepsilon = (Dv_\varepsilon)^2$ that reads

$$\frac{\partial w_\varepsilon}{\partial t} - D^2 w_\varepsilon + 2(D^2 v_\varepsilon)^2 = -2\Big(\frac{\phi''}{\phi'}\Big)' w_\varepsilon^2 - 2\frac{\phi''}{\phi'} Dv_\varepsilon Dw_\varepsilon$$
$$+ \Lambda_\varepsilon\Big((g_\varepsilon + \phi(v_\varepsilon))f'(-\phi(v_\varepsilon)) - f(-\phi(v_\varepsilon)) + (g_\varepsilon + \phi(v_\varepsilon))\frac{\phi''}{\phi'^2} f(-\phi(v_\varepsilon))\Big) w_\varepsilon$$
$$- \Lambda_\varepsilon \frac{Dg_\varepsilon}{\phi'} f(-\phi(v_\varepsilon)) Dv_\varepsilon. \quad (3.17)$$

Then, it suffices to show that there exists a constant $K > 0$ independent of ε such that

$$w(x,t) \leqslant K^2 \text{ for all } (x,t) \in \bar{I}_\varepsilon \times \mathbb{R}^+. \quad (3.18)$$

Indeed, assuming (3.18) is proved, we have

$$|Dv_\varepsilon(x,t)| \leqslant K \text{ for all } (x,t) \in I_\varepsilon \times \mathbb{R}^+.$$

Next, since $D\theta_\varepsilon(x,t) = -\phi'(v) Dv_\varepsilon(\varepsilon^{-1}x, \varepsilon^{-2}t)$ with $1 \leqslant \phi'(v) \leqslant b$, we infer that

$$|D\theta_\varepsilon(x,t)| \leqslant bK \text{ for all } (x,t) \in I \times \mathbb{R}^+ \text{ and } \varepsilon > 0$$

which concludes the proof of Proposition 3.5.

Let us now derive (3.18). Let $T > 0$ and $(x_0, t_0) \in \bar{I}_\varepsilon \times [0,T]$ be a point where w is maximized. Observe that $x_0 \in I_\varepsilon$ (unless $w \equiv 0$) because $Du_\varepsilon = 0$ on ∂I_ε. We aim to majorize $w(x_0, t_0)$ by some constant independent of ε.

First, assume that $t_0 = 0$. We have

$$w(x_0, 0) = (Dv_\varepsilon(x,0))^2 \leqslant \frac{1}{\phi'(v)^2} \|Du_{0\varepsilon}\|_\infty$$

and this bound is independent of ε since $\phi'(v) \geqslant 1$ and $\|Du_{0\varepsilon}\|_\infty = \|D\theta_0\|_\infty$.

Assume now that $t_0 > 0$. We first note that we can assume that (x_0, t_0) lies in the region $B = \{(x,t) \in I_\varepsilon \times \mathbb{R}^+ : v(x,t) \leqslant v_0\}$, where v_0 is given by (3.16). Indeed, assume $(x_0, t_0) \in A = \{(x,t) \in I_\varepsilon \times [0, T : v(x,t) > v_0\}$. Then, in this region, we have $\phi(v) \geqslant \phi(v_0) = 1$ and thus $f(-\phi(v)) = 0$. Therefore w satisfies

$$\frac{\partial w}{\partial t} - D^2 w \leqslant 0 \text{ in } A.$$

The maximum principle implies then that $w(x,t) = w(x_0, t_0)$ for all $(x,t) \in \overline{A}$ which can be connected with (x_0, t_0) by a path in A consisting only of horizontal segments and "upward" pointing vertical segments,so that the maximum is also reached in B.

Next, at the point (x_0, t_0), we have

$$\frac{\partial w}{\partial t} \geqslant 0, \quad Dw = 0, \quad D^2 w \leqslant 0.$$

Hence, the right-hand side E of (3.17) is nonnegative at (x_0, t_0).

Now, due to the hypotheses on the function ϕ, we see that

$$E \leqslant -2aw^2 + 2\Lambda_\varepsilon(|g_\varepsilon|+1)(f'(-\phi(v)) + f(-\phi(v)))w + \Lambda_\varepsilon\,|Dg_\varepsilon|\, f(-\phi(v))w^{\frac{1}{2}}.$$

Next, note that in view of (2.37)

$$-\phi(v) = -u_\varepsilon = \varepsilon^{-1}(\theta_\varepsilon - 1) = \varepsilon^{-1}(G_\varepsilon - \Lambda_\varepsilon\psi_\varepsilon) \leqslant \varepsilon^{-1}G_\varepsilon$$

is bounded independently of ε. Also, (3.12) provides a bound independent of ε of $Dg_\varepsilon(x,t) = DG_\varepsilon(\varepsilon x, \varepsilon^2 t)$. Consequently, there exists a constant c independent of ε such that

$$E \leqslant -2aw^2 + cw + cw^{\frac{1}{2}}.$$

Since $E \geqslant 0$ at (x_0, t_0), we conclude easily that $w(x_0, t_0) \leqslant \max(1, \frac{c}{a})$ which gives (3.18) and completes the proof of Proposition 3.5. □

3.3. **Regularity in time.** As usual for parabolic equations, Lipschitz regularity in space yields Hölder $\frac{1}{2}$ regularity in time. In the spirit of [6], we only state a local result.

Proposition 3.6. *Under the assumptions of Proposition* 3.5, *let K be a compact subset of $I \times \mathbb{R}^+$ and $\tau > 0$ such that $\mathcal{N}_\tau(K) \subset I \times \mathbb{R}^+$. Then there exists $C = C(\tau)$ such that*

$$|\theta_\varepsilon(x, t_2) - \theta_\varepsilon(x, t_1)| \leqslant C\,|t_2 - t_1|^{\frac{1}{2}},$$

for all (x, t_1), $(x, t_2) \in K$. Consequently, $\theta_\varepsilon \in C^{1,\frac{1}{2}}_{loc}(I \times \mathbb{R}^+)$.

Proof. The proof follows the argument in [6] and is omitted. □

Remark 3.7. The following global result can also be derived:

$$|\theta_\varepsilon(x, t_2) - \theta_\varepsilon(x, t_1)| \leqslant C\,|t_2 - t_1|^{\frac{1}{2}} + C\,\varepsilon$$

for all $x \in \bar{I}$, $t_2 \geqslant t_1 \geqslant 0$ (see [13, 9]).

4. Passage to the limit $\varepsilon \to 0$

For the asymptotic analysis, it is convenient to rewrite Problem (1.1) in terms of the unknowns θ_ε and $h_\varepsilon = \varepsilon^{-1} H_\varepsilon = \varepsilon^{-1}(\theta_\varepsilon + \psi_\varepsilon - 1)$. The new system reads

$$\frac{\partial \theta_\varepsilon}{\partial t} - D^2\theta_\varepsilon = (1 - \theta_\varepsilon + \varepsilon h_\varepsilon) f_\varepsilon(\theta_\varepsilon), \tag{4.1a}$$

$$\frac{\partial h_\varepsilon}{\partial t} - \Lambda_\varepsilon D^2 h_\varepsilon = \varepsilon^{-1}(1 - \Lambda_\varepsilon) D^2\theta_\varepsilon \tag{4.1b}$$

with the boundary conditions

$$D\theta_\varepsilon = Dh_\varepsilon = 0 \text{ on } \partial I \times \mathbb{R}^+, \tag{4.2}$$

and the initial conditions

$$\theta_\varepsilon(x,0) = \theta_0(x), \qquad h_\varepsilon(x,0) = \varepsilon^{-1}(\theta_0 + \psi_0 - 1). \tag{4.3}$$

We start by investigating the behaviour of θ_ε as $\varepsilon \to 0$. We first derive a convergence result (through a subsequence).

Proposition 4.1. *Assume that the hypotheses of Proposition* 3.5 *hold. Then there exists a sequence* $\varepsilon_n \to 0$ *and* $\Theta\colon I \times \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\Theta \in C^{1,\frac{1}{2}}_{loc}(I \times \mathbb{R}^+)$ *such that*

$$\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \Theta \text{ uniformly on compact subsets of } I \times \mathbb{R}^+.$$

Moreover, Θ *is a solution of the heat equation in* $I \times \mathbb{R}^+ \cap \{\Theta < 1\}$.

Proof. We consider a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets in $I \times \mathbb{R}^+$ such that

$$I \times \mathbb{R}^+ = \bigcup_{n\in\mathbb{N}} K_n,$$

for all compact $K \subset I \times \mathbb{R}^+$,

$K \cap K_n = \emptyset$ for all but a finite number of values of n.

By applying Arzela-Ascoli's theorem in each K_n and using a diagonal process of Cantor type, we obtain a subsequence still denoted by ε_n and a function Θ such that $\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \Theta$ uniformly on K_l for all $l \in \mathbb{N}$. Clearly the choice of the compact sequence K_l ensures that $\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \Theta$ uniformly on compact subsets of $I \times \mathbb{R}^+$.

We now prove that Θ solves the heat equation in $I \times \mathbb{R}^+ \cap \{\Theta < 1\}$. Let $(x_0, t_0) \in \{\Theta < 1\}$. Since Θ is continuous, there exists a neighborhood V

of (x_0, t_0) such that $\Theta(x,t) \leqslant 1 - 2\delta$ for all $(x,t) \in V$. Then the uniform convergence of θ_{ε_n} implies $\theta_{\varepsilon_n}(x,t) \leqslant 1 - \delta$ for all $(x,t) \in V$ and $n \in \mathbb{N}$. Therefore, due to the assumption (1.2), we have $f_{\varepsilon_n}(\theta_{\varepsilon_n}) = 0$ in V for $n \geqslant n_0$. Hence the function θ_{ε_n} satisfies the heat equation in V for $n \geqslant n_0$ and the same holds for Θ. □

Next we give more precise information on the convergence of θ_{ε_n}.

Proposition 4.2. *Under the assumptions of Proposition* 3.5, *we have*

$$\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \Theta \ \textit{in}\ L^2_{loc}(\mathbb{R}^+; H^1(I)), \tag{4.4}$$

$$\frac{\partial \theta_{\varepsilon_n}}{\partial t} \xrightarrow[n\to+\infty]{} \frac{\partial \Theta}{\partial t} \ \textit{weakly in}\ L^2_{loc}(\mathbb{R}^+; L^2(I)). \tag{4.5}$$

Proof. By Proposition 2.4, θ_ε is bounded independently of ε in $L^\infty(\mathbb{R}^+; H^1(I))$ and $\frac{\partial \theta_\varepsilon}{\partial t}$ is bounded independently of ε in $L^2_{loc}(\mathbb{R}^+; L^2(I))$. Together with the uniform convergence of θ_{ε_n}, this readily implies that

$$\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \Theta \text{ in } L^2_{loc}(\mathbb{R}^+; L^2(I)),$$
$$\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \Theta \text{ weakly in } H^1_{loc}(I \times \mathbb{R}^+).$$

It remains to prove the strong convergence in $L^2_{loc}(\mathbb{R}^+; H^1(I))$. Since $D\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} D\Theta$ weakly in $L^2(Q_T)$, where $Q_T = I \times (0,T)$, it suffices to prove that

$$\iint_{Q_T} |D\Theta|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant \limsup_{n\to+\infty} \iint_{Q_T} |D\theta_{\varepsilon_n}|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Since $\psi_\varepsilon \geqslant 0$, $f_\varepsilon \geqslant 0$ and $\theta_\varepsilon \leqslant 1 + c\varepsilon$, by multiplying equation (1.1a) by $\theta_\varepsilon - 1 - c\varepsilon$ and integrating on Q_T, we see that

$$\iint_{Q_T} |D\theta_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\int_I \Big[(\theta_\varepsilon - 1 - c\varepsilon)^2(x,0) - (\theta_\varepsilon - 1 - c\varepsilon)^2(x,T)\Big]\,\mathrm{d}x.$$

Next, as $\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \Theta$ in $C^0([0,T]; L^2(I))$, we have that

$$\limsup_{n\to+\infty} \iint_{Q_T} |D\theta_{\varepsilon_n}|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\int_I \Big[(\Theta-1)^2(x,0) - (\Theta-1)^2(x,T)\Big]\,\mathrm{d}x. \tag{4.6}$$

Now, we have $\frac{\partial \Theta}{\partial t} - D^2\Theta = 0$ in $I \times \mathbb{R}^+ \cap \{\Theta < 1\}$. By multiplying this equation by $(\Theta - 1 + \delta)^-$, where $\delta > 0$, and integrating on $Q_T \cap \{\Theta < 1\}$ we obtain

$$\iint_{\{\Theta<1-\delta\ ;\ t\leqslant T\}} |D\Theta|^2\, \mathrm{d}x\, \mathrm{d}t = \frac{1}{2}\int_{I\cap\{\Theta<1-\delta\}} \Big[(\Theta-1+\delta)^2(x,0) - (\Theta-1+\delta)^2(x,T)\Big]\, \mathrm{d}x$$

so that in the limit $\delta\to 0$, since $0\leqslant\Theta\leqslant 1$,

$$\iint_{Q_T} |D\Theta|^2\, \mathrm{d}x\, \mathrm{d}t = \frac{1}{2}\int_I \Big[(\Theta-1)^2(x,0) - (\Theta-1)^2(x,T)\Big]\, \mathrm{d}x.$$

Coming back to (4.6), this inequality implies that

$$\iint_{Q_T} |D\Theta|^2\, \mathrm{d}x\, \mathrm{d}t \geqslant \limsup_{n\to+\infty}\int_{Q_T} |D\theta_{\varepsilon_n}|^2\, \mathrm{d}x\, \mathrm{d}t. \qquad \square$$

We aim now to describe the global equation satisfied by the limit Θ.

Proposition 4.3. *Under the assumptions of Proposition* 3.5, *there exists a finite measure μ supported on the free boundary $\Gamma = I\times\mathbb{R}^+\cap\partial\{\Theta<1\}$ such that $\psi_{\varepsilon_n} f_{\varepsilon_n}(\theta_{\varepsilon_n}) \xrightarrow[n\to+\infty]{} \mu$ weakly in $I\times\mathbb{R}^+$. Furthermore, Θ satisfies, for every $\zeta\in C_c^\infty(I\times[0,+\infty))$,*

$$\iint_{I\times\mathbb{R}^+} \Big[D\Theta D\zeta - \Theta\frac{\partial\zeta}{\partial t}\Big]\, dx\, dt - \int_I \Theta_0(x)\zeta(x,0)\, dx = \int_\Gamma \zeta\, d\mu.$$

Proof. Integrating (1.1b) gives readily that

$$\iint_{I\times\mathbb{R}^+} \psi_\varepsilon f_\varepsilon(\theta_\varepsilon)\, \mathrm{d}x\, \mathrm{d}t \leqslant \int_I \psi_0(x)\, \mathrm{d}x$$

so that the family $\{\psi_\varepsilon f_\varepsilon(\theta_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^1(I\times\mathbb{R}^+)$. Consequently, there exists a finite positive measure μ and a subsequence, still denoted by ε_n, such that $\psi_{\varepsilon_n} f_{\varepsilon_n}(\theta_{\varepsilon_n}) \xrightarrow[n\to+\infty]{} \mu$ weakly in $I\times\mathbb{R}^+$.

Now, multiply equation (1.1a) by $\zeta\in C_c^\infty(I\times\mathbb{R}^+)$ and integrate by parts on $I\times\mathbb{R}^+$. This gives

$$\iint_{I\times\mathbb{R}^+} \Big[-\theta_\varepsilon\frac{\partial\zeta}{\partial t} + D\theta_\varepsilon D\zeta\Big]\, \mathrm{d}x\, \mathrm{d}t - \int_I \theta_{0\varepsilon}(x)\zeta(x,0)\, \mathrm{d}x = \iint_{I\times\mathbb{R}^+} \psi_\varepsilon f_\varepsilon(\theta_\varepsilon)\zeta\, \mathrm{d}x\, \mathrm{d}t. \quad (4.7)$$

Recall that $\theta_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \Theta$ in $L^2_{loc}(\mathbb{R}^+; L^2(I))$ and that $\psi_\varepsilon f_\varepsilon(\theta_\varepsilon)$ converges to μ through a subsequence. Hence, passing to the limit, we obtain

$$\iint_{I\times\mathbb{R}^+} \Big[-\Theta\frac{\partial\zeta}{\partial t} + D\Theta D\zeta\Big]\, \mathrm{d}x\, \mathrm{d}t - \int_I \Theta_0(x)\zeta(x,0)\, \mathrm{d}x = \int_{I\times\mathbb{R}^+} \zeta\, \mathrm{d}\mu.$$

This proves in particular that the whole sequence $\psi_{\varepsilon_n} f_{\varepsilon_n}(\theta_{\varepsilon_n})$ converges to μ weakly in $I \times \mathbb{R}^+$. Then, since Θ satisfies the heat equation in $I \times \mathbb{R}^+ \cap \{\Theta < 1\}$, we have that μ is supported on $I \times \mathbb{R}^+ \cap \{\Theta = 1\}$.

On the other hand, if ζ is a test function with support in the interior of the set $\{\Theta = 1\}$, we have by passing to the limit in (4.7)

$$\iint_{\{\Theta=1\}} -\frac{\partial \zeta}{\partial t}\,\mathrm{d}x\,\mathrm{d}t = \int_{\{\Theta=1\}} \zeta\,\mathrm{d}\mu$$

which yields

$$\int_{\{\Theta=1\}} \zeta\,\mathrm{d}\mu = 0$$

so that μ is supported on $I \times \mathbb{R}^+ \cap \partial\{\Theta < 1\}$. □

Now we study the limit of the reduced enthalpy h_ε. For that purpose we first need to investigate the convergence of $\varepsilon^{-1} G_\varepsilon$.

Proposition 4.4. *Assume that the hypotheses of Proposition* 2.4 *hold. Then, there exists a sequence* ε_n *and a function* $g^* \colon I \times \mathbb{R}^+ \to \mathbb{R}$ *with* $g^* \in C^{\frac{1}{2},\frac{1}{4}}_{loc}(I \times \mathbb{R}^+)$, $g^* \in L^\infty(\mathbb{R}^+; H^1(I)) \cap L^2_{loc}(\mathbb{R}^+; H^2(I)) \cap H^1_{loc}(\mathbb{R}^+; L^2(I))$ *such that*

$$\varepsilon_n^{-1} G_{\varepsilon_n} \xrightarrow[n\to+\infty]{} g^* \text{ uniformly on compact subset of } I \times \mathbb{R}^+,$$

$$\varepsilon_n^{-1} G_{\varepsilon_n} \xrightarrow[n\to+\infty]{} g^* \text{ in } L^2(0,T; H^1(I)), \text{ for all } T > 0,$$

$$\varepsilon_n^{-1} G_{\varepsilon_n} \xrightarrow[n\to+\infty]{} g^* \text{ weakly in } H^1(0,T; L^2(I)), \text{ for all } T > 0.$$

Proof. The function $g_\varepsilon = \varepsilon^{-1} G_\varepsilon$ satisfies the following equation (see (2.33)):

$$\frac{\partial g_\varepsilon}{\partial t} - \Lambda_\varepsilon D^2 g_\varepsilon = \varepsilon^{-1}(1 - \Lambda_\varepsilon)\frac{\partial \theta_\varepsilon}{\partial t} \tag{4.8}$$

together with homogeneous Neumann boundary conditions and the initial condition

$$g_\varepsilon(x,0) = \varepsilon^{-1}(\theta_0(x) + \Lambda_\varepsilon \psi_0(x) - 1) \text{ for } x \in I. \tag{4.9}$$

Now, (1.3) and estimate (2.19) yield that $\varepsilon^{-1}(1-\Lambda_\varepsilon)\frac{\partial \theta_\varepsilon}{\partial t}$ is bounded independently of ε in $L^2(Q_T)$, $T > 0$. On the other hand, $\varepsilon^{-1}(\theta_0 + \Lambda_\varepsilon \psi_0 - 1)$ is bounded independently of ε in $H^1(I)$ (see (2.16)). Therefore, the corollary of Theorem IV.9.1. in [12] applies and yields that g_ε is bounded independently of ε in $C^{\frac{1}{2},\frac{1}{4}}(Q_T)$. Also, we have that g_ε is bounded independently of ε in $L^2(0,T; H^2(I))$. Moreover, note that, by (2.20), $\frac{\partial g_\varepsilon}{\partial t}$ is bounded in $L^2(0,T; L^2(I))$ for $T > 0$. These estimates yield the existence of a sequence ε_n and a function g^* satisfying the conclusions of Proposition 4.4. □

Combining the above results, we can now state convergence properties for h_ε.

Corollary 4.5. *Assume that the hypotheses of Proposition* 4.4 *hold. Then, there exists a sequence $\varepsilon_n \to 0$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \ell$, such that $\varepsilon^{-1}(\Lambda_\varepsilon - 1)$ converges to λ, and $h_\varepsilon = \varepsilon^{-1}H_\varepsilon$ converges to S through some subsequence ε_n uniformly on compact subsets of $I \times \mathbb{R}^+$. Moreover, $\varepsilon^{-1}H_\varepsilon \xrightarrow[\varepsilon\to 0]{} S$ strongly in $L^2(0,T;H^1(I))$ and weakly in $H^1(0,T;L^2(I))$.*

We have $S = g^* + \lambda(\Theta - 1)$, where Θ is given by Proposition 4.1 *and g^* by Proposition* 4.4. *In particular, we have $S \in C^{\frac{1}{2},\frac{1}{4}}_{loc}(I\times\mathbb{R}^+) \cap L^2_{loc}(\mathbb{R}^+;H^1(I)) \cap H^1_{loc}(\mathbb{R}^+;L^2(I))$. Finally, S satisfies the equation*

$$\frac{\partial S}{\partial t} - D^2 S = -\lambda D^2\Theta \tag{4.10}$$

in $\mathcal{D}'(I \times \mathbb{R}^+)$.

Proof. The results of convergence follow immediately from Propositions 4.4, 4.1 and 4.2 since we have

$$h_\varepsilon = \varepsilon^{-1}H_\varepsilon = g_\varepsilon + \varepsilon^{-1}(\Lambda_\varepsilon - 1)(\theta_\varepsilon - 1)$$

and by (1.3), $\varepsilon^{-1}(\Lambda_\varepsilon - 1)$ is bounded independently of ε.

Then, recall that h_ε satisfies the equation

$$\frac{\partial h_\varepsilon}{\partial t} - \Lambda_\varepsilon D^2 h_\varepsilon = \varepsilon^{-1}(1-\Lambda_\varepsilon)D^2\theta_\varepsilon.$$

In view of the previous convergence results, we can take the limit $\varepsilon \to 0$ in this linear equation and this gives (4.10) at the limit. □

5. Free boundary conditions

In this section, we derive the boundary conditions for the limit problem. As in Bonder and Wolanski [6], this is done under two assumptions. The first one is that the free boundary has an outward unit spatial normal ν in a parabolic measure theoretic sense (see Definition 5.1 below); the other one is a particular assumption which means approximately that the mixture temperature is reaching the flame temperature only if some combustion is taking place; it reads:

For all compact subset K of $\{\Theta \overset{\circ}{\frown}{=} 1\}$, there exists $\eta \in (0,1)$ and $\varepsilon_0 > 0$ such that

$$\varepsilon^{-1}(\theta_\varepsilon - 1) \leqslant \eta \text{ in } K \text{ for all } \varepsilon < \varepsilon_0. \tag{5.1}$$

We now give the definition of a regular point.

Definition 5.1. *We say that ν is the outward unit spatial normal to the free boundary $\partial\{\Theta < 1\}$ at a point (x_0, t_0) in the parabolic measure theoretic sense, if $\nu \in \{\pm 1\}$ and*

$$\lim_{\rho \to 0} \frac{1}{\rho^3} \int_{Q_\rho(x_0,t_0)} \left| \chi_{\{\Theta<1\}} - \chi_{\{(x,t)\,|\,<x-x_0,\nu>>0\}} \right| \, dx\, dt = 0.$$

A point (x_0, t_0) of the free boundary $\partial\{\Theta < 1\}$ is regular *if there exists an exterior unit spatial normal to $\partial\{\Theta < 1\}$ at (x_0, t_0) in the parabolic measure theoretic sense.*

This definition is an extension of the usual definition of the outward normal to a boundary.

The following result states that Θ has an asymptotic development at any regular point of the free boundary.

Theorem 5.2. *Assume that* (5.1) *holds. Let (x_0, t_0) be a regular point of the free boundary $\partial\{\Theta < 1\}$ and ν denote the outward spatial normal to the free boundary at (x_0, t_0) in the parabolic measure theoretic sense. Then, Θ has the asymptotic development*

$$\Theta(x,t) = 1 + \alpha[(x - x_0)\nu]^- + o\Big(|x - x_0| + |t - t_0|^{\frac{1}{2}} \Big), \tag{5.2}$$

where $\alpha = \sqrt{2M(x_0,t_0)}$, with

$$M(x,t) = \int_{-\infty}^{S(x,t)} (S(x,t) - y) f(y)\, dy. \tag{5.3}$$

Here S is the limit of the reduced enthalpy given by Corollary 4.5.

Corollary 5.3. *Under the assumptions of Theorem* 5.2, *Θ satisfies the following boundary condition at a regular point:*

$$\frac{\partial \Theta}{\partial \nu} = \sqrt{2M},$$

where M is given by (5.3).

Before giving the proof of Theorem 5.2, let us study the jump condition on the free boundary for S; this will conclude the study of the free boundary problem.

Theorem 5.4. *Assume that the hypotheses of Theorem* 5.2 *hold. Then, if* (x_0, t_0) *is a regular point of the free boundary* $\partial\{\Theta < 1\}$*, we have*

$$\Big[\frac{\partial S}{\partial x}\Big] = \lambda\Big[\frac{\partial \Theta}{\partial x}\Big], \tag{5.4}$$

where the brackets denote the jump in the quantity along the free boundary.

Proof of Theorem 5.4. First, it is clear that Θ is smooth in $\{\Theta < 1\}$. Now, we have that S is a solution to (4.10) in $\mathcal{D}'(I \times \mathbb{R}^+)$, and thus, S is a smooth solution to (4.10) in $\{\Theta < 1\}$ and in the interior of $\{\Theta \equiv 1\}$.

Let $\zeta \in \mathcal{D}(\omega)$ where $\omega \subset I \times \mathbb{R}^+$ with $\omega \cap \partial\{\Theta < 1\} \neq \emptyset$. We have thus

$$\int_\omega \zeta \frac{\partial S}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\omega D\zeta DS\,\mathrm{d}x\,\mathrm{d}t = \lambda \int_\omega D\zeta D\Theta\,\mathrm{d}x\,\mathrm{d}t.$$

Denote by $\omega^+ = \omega \cap \{\Theta < 1\}$ and $\omega^- = \omega \cap \{\Theta \overset{\circ}{\overset{\frown}{=}} 1\}$. We see that

$$\int_\omega \zeta \frac{\partial S}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\omega^+} D\zeta DS\,\mathrm{d}x\,\mathrm{d}t + \int_{\omega^-} D\zeta DS\,\mathrm{d}x\,\mathrm{d}t = \int_{\omega^+} D\zeta D\Theta\,\mathrm{d}x\,\mathrm{d}t.$$

Then, Green's formula yields

$$\begin{aligned}\int_\omega \zeta \frac{\partial S}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_{\omega^+} \zeta D^2 S\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\omega^+} \zeta \frac{\partial S}{\partial \nu}\,\mathrm{d}x\,\mathrm{d}t - \int_{\omega^-} \zeta D^2 S\,\mathrm{d}x\,\mathrm{d}t \\ - \int_{\partial\omega^-} \zeta \frac{\partial S}{\partial \nu}\,\mathrm{d}x\,\mathrm{d}t = -\lambda \int_{\omega^+} \zeta D^2\Theta\,\mathrm{d}x\,\mathrm{d}t + \lambda \int_{\partial\omega^+} \zeta \frac{\partial \Theta}{\partial \nu}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since S is a smooth solution to (4.10) in ω^+ and ω^-, we obtain finally

$$\int_{\omega \cap \partial\{\Theta < 1\}} \zeta \Big(\Big[\frac{\partial S}{\partial \nu}\Big] - \lambda \Big[\frac{\partial \Theta}{\partial \nu}\Big] \Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \forall\, \zeta \in \mathcal{D}(\omega)$$

and this concludes the proof. $\square$

Proof of Theorem 5.2. Since many arguments are close to the ones in [6] (see also [9, 8]) and the proof is lengthy, we only give the main steps.

Let P_0 be a regular point of the free boundary $\partial\{\Theta < 1\}$. After translation, we may assume that $P_0 = (0, 0)$ and, for example, $\nu = -1$.

Let us perform a parabolic scaling around P_0; *i.e.,* define

$$'\Theta_\lambda(x, t) = 1 + \frac{1}{\lambda}\Big(\Theta(\lambda x, \lambda^2 t) - 1\Big).$$

If we start with $\Theta(x, t)$ restricted to $Q_r(0, 0)$, Θ_λ is defined at least on $Q_{\frac{r}{\lambda}}(P_0)$. Now, since $\Theta \in C^{1,\frac{1}{2}}_{loc}(I \times \mathbb{R}^+)$ and $\Theta(0, 0) = 1$, there exists a sequence $\lambda_n \to 0$ and $\mathcal{T} \in C^{1,\frac{1}{2}}_{loc}(\mathbb{R}^2)$ such that $\Theta_{\lambda_n} \to \mathcal{T}$ as $n \to +\infty$ uniformly on compact subsets of $\mathbb{R}^2$.

Clearly, we have $\mathcal{T} \leqslant 1$. Also, since Θ_{λ_n} satisfies the homogeneous heat equation in $\{\Theta_{\lambda_n} < 1\}$, the same holds true for $\mathcal{T}$ in $\{\mathcal{T} < 1\}$. Finally, P_0 being a regular point, necessarily $\mathcal{T}(x,t) = 1$ if $x < 0$. In view of the above properties, Corollary A.1. of [8] applies and for $\bar{t} \in \mathbb{R}$ provides the existence of $\alpha(= \alpha(\bar{t}))$ such that

$$\mathcal{T}(x,t) = 1 - \alpha x + o(|x| + |t-\bar{t}|^{\frac{1}{2}}), \text{ for } x > x_0 \text{ and } t < \bar{t}. \tag{5.5}$$

We now aim to show that $\alpha = \sqrt{2M(P_0)}$. The proof relies on the following identity:

$$\iint_{\{x>0\}} \zeta \frac{\partial \mathcal{T}}{\partial t} D\mathcal{T} \,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{2} \iint_{\{x>0\}} |D\mathcal{T}|^2 \, D\zeta \,\mathrm{d}x\,\mathrm{d}t + M(P_0) \iint_{\{x>0\}} D\zeta \,\mathrm{d}x\,\mathrm{d}t \quad \forall \zeta \in \mathcal{D}(\mathbb{R}^2). \tag{5.6}$$

To prove (5.6), we construct a sequence $j_n \to \infty$ such that

$$\delta_n = \frac{\varepsilon_{j_n}}{\lambda_n} \to 0 \text{ and } \theta_{\delta_n} \to \mathcal{T} \text{ as } n \to +\infty \text{ in } C^0_{loc}(I \times \mathbb{R}^+), \tag{5.7}$$

where we set

$$\theta_{\delta_n}(x,t) = 1 + \frac{1}{\lambda_n}\Big(\theta_{\varepsilon_{j_n}}(\lambda_n x, \lambda_n^2 t) - 1\Big), \quad \psi_{\delta_n}(x,t) = \frac{1}{\lambda_n}\psi_{\varepsilon_{j_n}}(\lambda_n x, \lambda_n^2 t).$$

Indeed, writing

$$\theta_{\delta_n}(x,t) - \mathcal{T}(x,t) = \frac{1}{\lambda_n}\Big(\theta_{\varepsilon_{j_n}}(\lambda_n x, \lambda_n^2 t) - \Theta(\lambda_n x, \lambda_n^2 t)\Big) + \Theta_{\lambda_n}(x,t) - \mathcal{T}(x,t),$$

we know that $\Theta_{\lambda_n} - \mathcal{T} \xrightarrow[n\to+\infty]{} 0$ in $C^0_{loc}(\mathbb{R}^2)$. Next, as $\theta_{\varepsilon_j} \xrightarrow[j\to+\infty]{} \Theta$ in $C^0_{loc}(I \times \mathbb{R}^+)$, given $r > 0$ such that $Q_r(0,0) \subset I \times \mathbb{R}^+$, there exists $j(n)$ such that $\left|\theta_{\varepsilon_j} - \Theta\right| \leqslant \frac{\lambda_n}{n}$ in $Q_r(0,0)$ for $j \geqslant j(n)$. Furthermore, let $j_n \geqslant j(n)$ be such that $\varepsilon_{j_n} < \lambda_n$ and take n large enough so that $\lambda_n \leqslant \frac{r}{k}$. Then $\theta_{\delta_n} \xrightarrow[n\to+\infty]{} \mathcal{T}$ uniformly in $Q_k(0,0)$.

Next, note that $(\theta_{\delta_n}, \psi_{\delta_n})$ are solutions of system (1.1) with ε replaced by δ_n and $\Lambda = \Lambda_{\varepsilon_{j_n}}$. Moreover, we have $\left|\Lambda_{\varepsilon_{j_n}} - 1\right| \leqslant \ell\varepsilon_{j_n} \leqslant \ell\delta_n$ (if n is large enough). Hence, some of the convergence results of the previous sections hold for $(\theta_{\delta_n}, \psi_{\delta_n})$, but in a local form. In particular, we have that

$$D\theta_{\delta_n} \xrightarrow[n\to+\infty]{} D\mathcal{T} \text{ in } L^2_{loc}(\mathbb{R}^2), \tag{5.8}$$

$$\frac{\partial \theta_{\delta_n}}{\partial t} \xrightarrow[n\to+\infty]{} \frac{\partial \mathcal{T}}{\partial t} \text{ weakly in } L^2_{loc}(\mathbb{R}^2). \tag{5.9}$$

Indeed, for the first one, note that we have

$$D\theta_{\delta_n}(x,t) = D\theta_{\varepsilon_{jn}}(\lambda_n x, \lambda_n^2 t).$$

Hence, due to Proposition 3.5, $D\theta_{\delta_n}$ is bounded independently of ε in $L^\infty_{loc}(\mathbb{R}^2)$. This implies the weak convergence (through a subsequence) of $D\theta_{\delta_n}$ towards $D\mathcal{T}$ in $L^2_{loc}(\mathbb{R}^2)$. The strong convergence in L^2_{loc} is derived as in the proof of Proposition 4.2.

For the convergence of the time derivative, as in the proof of Lemma 2.6, it can be shown that

$$\iint_{Q_k(0,0)} \zeta^2 \Big(\frac{\partial \theta_{\delta_n}}{\partial t}\Big)^2 \mathrm{d}x\,\mathrm{d}t \leqslant c + c\delta_n^{-2} \iint \Big(\frac{\partial G_{\delta_n}}{\partial t}\Big)^2 \mathrm{d}x\,\mathrm{d}t,$$

while (2.20) yields that

$$\begin{aligned}\int_t^{t+\eta}\int_{x_1}^{x_2} \Big(\frac{\partial G_{\delta_n}}{\partial t}(x,t)\Big)^2 \mathrm{d}x\,\mathrm{d}t &\leqslant \frac{1}{\lambda_n}\int_{\lambda_n^2 t}^{\lambda_n^2 t+\lambda_n^2\eta}\int_I \Big(\frac{\partial G_{\varepsilon_{jn}}}{\partial t}\Big)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant c\,\frac{\varepsilon_{jn}^2}{\lambda_n}(1+\lambda_n^2\eta) \leqslant c\,\delta_n^2(1+\eta),\end{aligned}$$

so that $\delta_n^{-1}\frac{\partial G_{\delta_n}}{\partial t}$ and $\frac{\partial \theta_{\delta_n}}{\partial t}$ are bounded in $L^2_{loc}(Q_k(0,0))$.

Finally, writing $\Theta_{\lambda_n} - \mathcal{T} = \Theta_{\lambda_n} - \theta_{\delta_n} + \theta_{\delta_n} - \mathcal{T}$, the above convergence results combined with Proposition 4.2 imply that

$$D\Theta_{\lambda_n} \xrightarrow[n\to+\infty]{} D\mathcal{T} \text{ in } L^2_{loc}(\mathbb{R}^2), \tag{5.10}$$

$$\frac{\partial \Theta_{\lambda_n}}{\partial t} \xrightarrow[n\to+\infty]{} \frac{\partial \mathcal{T}}{\partial t} \text{ weakly in } L^2_{loc}(\mathbb{R}^2). \tag{5.11}$$

We can now derive the identity (5.6). Let $\zeta \in C_c^\infty(I\times(0,+\infty))$. Multiplying equation (1.1a) by $\zeta D\theta_\varepsilon$ and integrating over $I\times\mathbb{R}^+$, we obtain

$$\begin{aligned}\iint \zeta D\theta_\varepsilon \frac{\partial \theta_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t &= -\frac{1}{2}\iint |D\theta_\varepsilon|^2 D\zeta\,\mathrm{d}x\,\mathrm{d}t + \iint \mathcal{F}_\varepsilon(\theta_\varepsilon,x,t)D\zeta\,\mathrm{d}x\,\mathrm{d}t\\ &+ \iint (h_\varepsilon - S(x,t))f(S)\zeta DS\,\mathrm{d}x\,\mathrm{d}t + \iint Dh_\varepsilon\Big(\int_{\varepsilon^{-1}(\theta_\varepsilon-1)}^{S(x,t)} f(y)\,\mathrm{d}y\Big)\zeta\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{5.12}$$

where

$$\mathcal{F}_\varepsilon(u,x,t)=\int_u^{1+\varepsilon S(x,t)}\big(H_\varepsilon(x,t)+1-s\big)f(s)\,\mathrm{d}s.$$

Note that assumptions (1.2) on the function f and assumption (5.1) guarantee that

$$\int_{\varepsilon^{-1}(\theta_\varepsilon-1)}^{S(x,t)} f(y)\,\mathrm{d}y \underset{\varepsilon\to 0}{\longrightarrow} \Big(\int_{-1}^{S(x,t)} f(y)\,\mathrm{d}y\Big)\chi_{\{\Theta<1\}} \text{ a.e. in } I\times\mathbb{R}^+.$$

On the other hand, arguing as in the proof of Lemma 3.1. of [6], it can be shown that

$$\mathcal{F}_\varepsilon(\theta_\varepsilon,x,t) \underset{\varepsilon\to 0}{\longrightarrow} M(x,t)\chi_{\{\Theta<1\}} \text{ weakly in } L^2_{loc}(I\times\mathbb{R}^+).$$

Also using Proposition 4.2 and Corollary 4.5, we can take the limit as $\varepsilon\to 0$ in (5.12) and obtain that

$$\iint_{\{\Theta<1\}}\zeta D\Theta\frac{\partial\Theta}{\partial t}\,\mathrm{d}x\,\mathrm{d}t=-\frac{1}{2}\iint_{\{\Theta<1\}}|D\Theta|^2\,\zeta\,\mathrm{d}x\,\mathrm{d}t$$
$$+\iint_{\{\Theta<1\}}M(x,t)D\zeta\,\mathrm{d}x\,\mathrm{d}t+\iint_{\{\Theta<1\}}\Big(\int_{-1}^{S(x,t)}f(y)\,\mathrm{d}y\Big)\zeta DS\,\mathrm{d}x\,\mathrm{d}t.$$

Next, we apply this last identity to ζ^λ defined by $\zeta^\lambda(x,t)=\lambda\zeta(\lambda^{-1}x,\lambda^{-2}t)$ with $\zeta\in C_c^\infty(\mathbb{R}^2)$. This gives after a change of variables that

$$\iint_{\{\Theta_\lambda<1\}}\zeta D\Theta_\lambda\frac{\partial\Theta_\lambda}{\partial t}\,\mathrm{d}x\,\mathrm{d}t=-\frac{1}{2}\iint_{\{\Theta_\lambda<1\}}|D\Theta_\lambda|^2\,\zeta\,\mathrm{d}x\,\mathrm{d}t$$
$$+\iint_{\{\Theta_\lambda<1\}}M(\lambda x,\lambda^2 t)D\zeta\,\mathrm{d}x\,\mathrm{d}t+\iint_{\{\Theta<1\}}\Big(\int_{-1}^{S(x,t)}f(y)\,\mathrm{d}y\Big)\zeta^\lambda DS\,\mathrm{d}x\,\mathrm{d}t.$$

In view of (5.10) and (5.11), we can take the limit as $\lambda\to 0$ in this equality and obtain (5.6) at the limit.

Now, let $\zeta\in C_c^\infty(\mathbb{R}\times(-\infty,0))$. By applying (5.6) with ζ replaced by $\zeta^\lambda(x,t)=\lambda\zeta(\lambda^{-1}x,\lambda^{-2}(t-\bar{t}))$ and using a change of variables, we see that

$$\iint_{\{x>0\}}\frac{\partial\mathcal{T}_\lambda}{\partial t}D\mathcal{T}_\lambda\zeta\,\mathrm{d}x\,\mathrm{d}t=-\frac{1}{2}\iint_{\{x>0\}}|D\mathcal{T}_\lambda|^2\,D\zeta\,\mathrm{d}x\,\mathrm{d}t$$
$$+M(P_0)\iint_{\{x>0\}}D\zeta\,\mathrm{d}x\,\mathrm{d}t,\quad (5.13)$$

where we set

$$\mathcal{T}_\lambda(x,t)=1+\frac{1}{\lambda}\Big(\mathcal{T}(\lambda x,\bar{t}+\lambda^2 t)-1\Big).$$

Now, since $\mathcal{T} \in C^{1,\frac{1}{2}}(\mathbb{R}^2)$, there exists a sequence $\lambda_n \to 0$ and $\mathcal{T}_0 \in C^{1,\frac{1}{2}}(\mathbb{R}^2)$ such that $\mathcal{T}_{\lambda_n} \xrightarrow[n\to+\infty]{} \mathcal{T}_0$ as $n \to +\infty$ uniformly on compact subsets of $\mathbb{R}^2$. Observe that, by (5.5), we have $\mathcal{T}_0 = 1 - \alpha x$ in $\{x > 0,\ t < 0\}$. Arguing as for the proof of (5.10), (5.11), we can extract a subsequence still denoted by $(\lambda_n)_n$ such that

$$D\mathcal{T}_{\lambda_n} \xrightarrow[n\to+\infty]{} D\mathcal{T}_0 \text{ in } L^2_{loc}(\mathbb{R}^2),$$
$$\frac{\partial \mathcal{T}_{\lambda_n}}{\partial t} \xrightarrow[n\to+\infty]{} \frac{\partial \mathcal{T}_0}{\partial t} \text{ weakly in } L^2_{loc}(\mathbb{R}^2).$$

Taking the limit as $\lambda_n \to 0$ in (5.13), we conclude that

$$0 = -\frac{\alpha^2}{2} \iint_{\{x>0\}} D\zeta \,\mathrm{d}x\,\mathrm{d}t + M(P_0) \iint_{\{x>0\}} D\zeta \,\mathrm{d}x\,\mathrm{d}t,$$

which gives that

$$\alpha = \sqrt{2M(P_0)}. \tag{5.14}$$

Next, we aim to show that

$$\mathcal{T} = 1 - \alpha x^+. \tag{5.15}$$

Note that, due to the definition of $\mathcal{T}$, this will give readily (5.2) and conclude the proof of Theorem 5.2.

The main step for proving (5.15) is to show that

$$|D\mathcal{T}| \leqslant \sqrt{2M(P_0)} = \alpha \text{ in } \mathbb{R}^2. \tag{5.16}$$

For that purpose, we again consider the sequence θ_{δ_n} constructed above (see (5.7)). Setting $H_{\delta_n} = \theta_{\delta_n} + \psi_{\delta_n} - 1$, we have

$$\delta_n^{-1} H_{\delta_n}(x,t) = \varepsilon_{j_n}^{-1} H_{\varepsilon_{j_n}}(\lambda_n x, \lambda_n^2 t).$$

But, in view of Corollary 4.5, $\varepsilon_{j_n}^{-1} H_{\varepsilon_{j_n}} \xrightarrow[n\to+\infty]{} S$ in $C^0_{loc}(I \times \mathbb{R}^+)$, so that $\delta_n^{-1} H_{\delta_n}$ converges to the constant $S_0 = S(P_0)$ uniformly on compact subsets of $\mathbb{R}^2$. This crucial remark allows us to use the technique in [6] (see Lemma 2.2, 2.3 and 2.4) and derive (5.16). The proofs that are lengthy and similar to the ones in [6] are omitted.

To conclude, we now check that $\mathcal{T}(x,t) = 1 - \alpha x^+$. Recall that $\mathcal{T} = 1$ if $x \leqslant 0$. This property together with (5.16) enables us to say that $\mathcal{T}(x,t) \geqslant 1 - \alpha x$ for $x > 0$. Now, in view of (5.5), $V = \{\mathcal{T} < 1\} \neq \emptyset$. On this set, $\mathcal{T} - 1 + \alpha x$ is a nonnegative solution of the heat equation. By the maximum principle, either $\mathcal{T} = 1 - \alpha x$ in $\overline{V}$ or $\mathcal{T} > 1 - \alpha x$ in V. In the last case, Hopf's principle implies that $D\mathcal{T} < -\alpha$ at some point of ∂V which contradicts (5.16). Therefore we conclude that $\mathcal{T}(x,t) = 1 - \alpha x$ as long as $\mathcal{T} < 1$ which yields (5.15). The proof of Theorem 5.2 is complete. □

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