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INITIAL-BOUNDARY-VALUE PROBLEMS FOR THE BONA-SMITH FAMILY OF BOUSSINESQ SYSTEMS

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Abstract. In this paper we consider the one-parameter family of Bona-Smith systems, which belongs to the class of Boussinesq systems modelling two-way propagation of long waves of small amplitude on the surface of water in a channel. We study three initial-boundary-value problems for these systems, corresponding, respectively, to nonhomogeneous Dirichlet, reflection, and periodic boundary conditions posed at the endpoints of a finite spatial interval, and establish existence and uniqueness of their solutions. We prove that the initial-boundary-value problem with Dirichlet boundary conditions is well posed in appropriate spaces locally in time, while the analogous problems with reflection and periodic boundary conditions are globally well posed under mild restrictions on the initial data.

1. Introduction

We consider the following family of Boussinesq type systems of water wave theory, introduced in [8],

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0,\\ &u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0, \end{aligned} \tag{1.1}$$

where $\eta = \eta(x,t)$, $u = u(x,t)$ are real-valued functions of $x \in \mathbb{R}$ and $t \geq 0$. The *dispersion coefficients* a, b, c, d are given by

$$\begin{aligned} a &= \tfrac{1}{2}\big(\theta^2 - \tfrac{1}{3}\big)\nu, \qquad & b &= \tfrac{1}{2}\big(\theta^2 - \tfrac{1}{3}\big)\big(1-\nu\big),\\ c &= \tfrac{1}{2}\big(1-\theta^2\big)\mu, \qquad & d &= \tfrac{1}{2}\big(1-\theta^2\big)\big(1-\mu\big), \end{aligned} \tag{1.2}$$

where $0 \leq \theta \leq 1$ and ν, μ are real constants.

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The systems (1.1) are approximations of the two-dimensional Euler equations and model the irrotational, free surface flow of an incompressible, inviscid fluid in a uniform horizontal channel in the absence of cross-channel disturbances. As opposed to one-way models, like the KdV or the BBM equations, the systems (1.1) describe *two-way propagation*. The variables in (1.1) are dimensionless and unscaled; x and t are proportional to position along the channel and time, respectively, while $\eta(x,t)$ and $u(x,t)$ are proportional to the excursion of the free surface at (x,t) above an undisturbed level, and to the horizontal velocity of the fluid at a height $y=-1+\theta(1+\eta(x,t))$, respectively. (In terms of these variables the bottom of the channel is at $y=-1$, while the free surface corresponds to $\theta=1$.)

The Boussinesq systems (1.1) are long-wave, small-amplitude approximations of the Euler equations. Specifically, they are valid when $\varepsilon := A/h \ll 1$, $\lambda/h \gg 1$ and the Stokes number $A\lambda^2/h^3$ is of order 1, [8]. Here A measures the maximum surface elevation above an undisturbed level of fluid of depth h, and λ is a typical wavelength. Appropriate expansions in powers of ε and substitution into the Euler equations leads to a system of the form (1.1) with dimensionless but scaled variables, wherein the nonlinear and dispersive terms (the third-order derivatives) are balanced, being both multiplied by ε, while the right-hand side is of order ε^2. When the right-hand side is replaced by zero, it is expected the solutions of the resulting system with suitable initial data will approximate for $t>0$ appropriate smooth solutions of the Euler equations in the same scaling with an error of $O(\varepsilon^2 t)$. In [7] and [5] it was proved that the error of this approximation is indeed of order $O(\varepsilon^2 t)$, uniformly for $t\in[0,T_\varepsilon]$, where $T_\varepsilon = O(1/\varepsilon)$, provided the Cauchy problem for the Boussinesq system under consideration is locally well posed. (In the present paper we will for the most part consider the systems in their dimensionless, unscaled form (1.1).)

In particular, we will focus on the *Bona-Smith family of systems*, [11], which are of the form (1.1) when the parameters in (1.2) satisfy $\nu=0$ and $b=d$. The latter condition implies that $\mu=(4-6\theta^2)/3(1-\theta^2)$ for $\theta\neq 1$. Hence, the constants of the Bona-Smith systems are given by the formulas

$$a=0,\ b=d=\frac{3\theta^2-1}{6},\ c=\frac{2-3\theta^2}{3}. \tag{1.3}$$

We will also consider the limiting case obtained by setting $\theta=1$ in (1.3), i.e., the system with $a=0$, $b=d=1/3$, $c=-1/3$. The value $\theta^2=2/3$ in (1.3) yields the *BBM-BBM system* $a=c=0$, $b=d=1/6$, [6].

From the analysis of [8] one may infer that the Bona-Smith systems are linearly ill posed if $\theta^2 < 2/3$. Hence, in the sequel, we will only consider the systems with $\theta^2 \in [2/3, 1]$. It also follows from [8] that the Bona-Smith systems are linearly well posed for (η, u) in $H^{s+1} \times H^s$ for $s \geq 0$ if $\theta^2 > 2/3$, and in $H^s \times H^s$ for $s \geq 0$ if $\theta^2 = 2/3$. (By $H^s = H^s(\mathbb{R})$ for real $s \geq 0$ we denote the usual, L^2-based Sobolev classes on $\mathbb{R}$, and put $H^0 = L^2$.) It also follows from [8] that the linearized Bona-Smith systems are well posed in the L^p-based Sobolev space pairs $W_p^{s+1} \times W_p^s$ for $s \geq 0$ and $1 < p < \infty$, and ill posed for $p = 1$ or ∞. (The BBM-BBM system, i.e. the case where $\theta^2 = 2/3$, is well posed in L^p for $1 \leq p \leq \infty$, as explained in [8].)

We turn now to the Cauchy problem for the nonlinear Bona-Smith systems, that we rewrite again for ease in referencing as

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x - b\eta_{xxt} = 0, \\ &u_t + \eta_x + uu_x + c\eta_{xxx} - bu_{xxt} = 0, \\ &b = (3\theta^2 - 1)/6, \;\; c = (2 - 3\theta^2)/3, \;\; 2/3 \leq \theta^2 \leq 1, \end{aligned} \tag{1.4}$$

for $x \in \mathbb{R}$, $t > 0$. The system is to be solved under the initial conditions

$$\eta(x, 0) = \eta_0(x), \;\; u(x, 0) = u_0(x), \tag{1.5}$$

where η_0, u_0 are given functions on $\mathbb{R}$. This problem has been studied by Bona and Smith, [11], in the case $\theta^2 = 1$, but it is easy to extend their theory (cf. also [9]) to the general case $\theta^2 \in (2/3, 1]$ and obtain the following result: If $\eta_0 \in H^1 \cap C_b^3$, $u_0 \in L^2 \cap C_b^2$ (where $C_b^k = C_b^k(\mathbb{R})$ is the space of bounded, continuous functions on $\mathbb{R}$ whose first k derivatives are also continuous and bounded), and if $\inf_{x \in \mathbb{R}} \eta_0(x) > -1$ and

$$E(0) := \int_{-\infty}^{\infty} \Big(\eta_0^2 + |c|(\eta_0')^2 + (1 + \eta_0)u_0^2\Big) dx < 2|c|^{1/2}, \tag{1.6}$$

then, there is a unique, global classical solution (η, u) of (1.4)–(1.5), which, for each $T > 0$, is a continuous map from $[0, T]$ into $(H^1 \cap C_b^3) \times (L^2 \cap C_b^2)$. If further regularity is assumed for the initial data, then (1.4–(1.5) is well posed in $H^{s+1} \times H^s$ for $s \geq 0$ or in $(H^1 \cap C_b^{s+1}) \times (L^2 \cap C_b^s)$ for an integer $s \geq 0$.

The key step in the proof of this result in [11] is establishing an *a priori* $H^1 \times L^2$ estimate for the solution (η, u). This follows from the fact that (1.4) is a Hamiltonian system, whose Hamiltonian or "energy" functional

$$E(t) := \int_{-\infty}^{\infty} \Big(\eta^2 + |c|\eta_x^2 + (1 + \eta)u^2\Big) dx, \tag{1.7}$$

is invariant for $t \geq 0$. The restrictions $\theta^2 > 2/3$ and $\eta_0 > -1$ ensure that $1 + \eta(x, t)$ and, consequently, $E(t)$, remain positive for $x \in \mathbb{R}$, $t > 0$. In the case of the BBM-BBM system, where $c = 0$, the global theory breaks down and the Cauchy problem is well posed in $H^{s+1} \times H^s$ for $s \geq 0$ only locally in time, [6], [9].

Our aim in the paper at hand is to study the well-posedness of three types of *initial-boundary-value problems* (*ibvp's*) for systems of Bona-Smith type on bounded spatial intervals $[-L, L]$, $L > 0$, in which the system and the initial conditions $\eta(x, 0) = \eta_0(x)$, $u(x, 0) = u_0(x)$, $x \in [-L, L]$, are supplemented by three types of boundary conditions posed at $x = \pm L$ for $t \geq 0$. In our study we follow the general scheme of proof of [11] adapting it to the case of the boundary-value-problems at hand. First, in Paragraph 2.1, we analyze the ibvp with *nonhomogeneous Dirichlet* boundary conditions, wherein η and u are prescribed as given functions of t at both endpoints. We prove that this ibvp is well posed, for example in pairs of appropriate spaces of smooth functions, locally in time. (An analogous result for the BBM-BBM system was proved in [6].) Having in hand such a well-posedness result enables one to consider these types of boundary conditions, realized, for example, from temporal records of experimental measurements of η and u at two stations along a channel where a two-way surface wave flow has been established, to solve the resulting ibvp by an accurate and stable numerical scheme, and compare the numerical solution with experimental values at points $x \in (-L, L)$ in order to assess the effectiveness of the particular Boussinesq system to model the flow. An energy-type proof yields that, in general, the temporal interval of existence of solutions of this ibvp, written in dimensionless but scaled variables, wherein the nonlinear and dispersive terms in (1.4) are multiplied by the small parameter ε, is of the form $[0, T_\varepsilon]$, where T_ε may be taken independent of ε. (As was previously mentioned, for the Cauchy problem $T_\varepsilon = O(\frac{1}{\varepsilon})$.) In Paragraph 2.2 we study the ibvp with *reflection boundary conditions*, i.e., with $\eta_x = 0$, $u = 0$ prescribed at $x = \pm L$ for all $t \geq 0$. These are useful for studying the (ideal) reflection of a wave impinging on a rigid wall, vertical to the direction of the motion. With these boundary conditions the ibvp for the Bona-Smith systems for $\theta^2 \in (2/3, 1]$ is *globally* well posed, under mild restrictions on the initial data, analogous to those required for the proof of the Cauchy problem previously outlined. Global well posedness follows from the fact that the analog on $[-L, L]$ of the Hamiltonian (1.7) is conserved under the reflection boundary conditions. Finally, in Section 2.3 we prove that the *periodic initial-value-problem* for the Bona-Smith systems with $\theta^2 \in (2/3, 1]$ is globally well posed under similar restrictions on the (periodic) initial data. For the BBM-BBM system ($\theta^2 = 2/3$) we can only prove local well-posedness for the reflection and the periodic ibvp.

The well posedness of these ibvp's provides a firm theoretical foundation for deriving and rigorously analyzing numerical methods for the Bona-Smith systems. In a sequel to this paper, [4], we discretize the three ibvp's using the standard Galerkin-finite element method with piecewise polynomial functions in space and the classical fourth-order Runge-Kutta scheme in time, analyze the convergence of the resulting semidiscrete and fully discrete schemes, and use them in numerical experiments to study phenomena of interactions of *solitary-wave* solutions of these systems and their interactions with the boundary.

Many of the theoretical results of the paper at hand were first proved in [1]. Some were announced in preliminary form in [2] and [3].

2. Initial-boundary-value problems

Let $I = (-L, L)$ for some $L > 0$. In this section we study three initial-boundary-value problems (ibvp's) for some Boussinesq systems of the form (1.1). Specifically, we seek functions η and u defined for $x \in \bar{I}$ and $t \geq 0$ and satisfying for $x \in \bar{I}$, $t > 0$, the system

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0, \\ &u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0, \end{aligned} \tag{2.1}$$

with given initial conditions

$$\eta(x,0) = \eta_0(x), \qquad u(x,0) = u_0(x), \qquad x \in \bar{I}, \tag{2.2}$$

and several types of boundary conditions for u and η at $x = \pm L$, $t \geq 0$. Specific hypotheses about the coefficients a, b, c, d will be made in each case.

2.1. **Nonhomogeneous Dirichlet boundary conditions.**

We start by considering the subclass of systems of the form (2.1) with $b = d > 0$, $a = 0$, $c < 0$, i.e., the systems

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x - b\eta_{xxt} = 0, \\ &u_t + \eta_x + u u_x + c\eta_{xxx} - b u_{xxt} = 0, \end{aligned} \qquad x \in \bar{I}, \quad t \geq 0, \tag{2.3}$$

which include the Bona-Smith family (1.4). We supplement (2.3) by the initial conditions (2.2) and nonhomogeneous boundary conditions of Dirichlet type for η and u at both ends of the interval $\bar{I}$, given for $t \geq 0$ by

$$\eta(-L,t) = h_1(t),\ \eta(L,t) = h_2(t),\ u(-L,t) = v_1(t),\ u(L,t) = v_2(t), \tag{2.4}$$

where h_i, v_i, $i=1,2$, are given continuous functions.

To analyze the ibvp consisting of (2.3), (2.2), (2.4) we shall write it first in integral form, cf., e.g. [11], [10], [6]. With this aim in mind, given a suitable function f defined on $\bar{I}$ we consider the following two-point boundary-value problem

$$\begin{aligned} &w - bw'' = -f',\ x \in \bar{I},\\ &w(-L) = w(L) = 0. \end{aligned} \tag{2.5}$$

Let G be the Green's function for (2.5) defined for $x, \xi \in \bar{I}$ by

$$G(x,\xi) := -\frac{1}{bW}\begin{cases} w_1(\xi)w_2(x), & -L \leqslant \xi \leqslant x,\\ w_1(x)w_2(\xi), & x < \xi \leqslant L, \end{cases}$$

where $w_1(x) := \sinh\frac{L+x}{\sqrt{b}}$, $w_2(x) := \sinh\frac{L-x}{\sqrt{b}}$, and $W := w_1w_2' - w_1'w_2 = -\frac{1}{\sqrt{b}}\sinh\frac{2L}{\sqrt{b}}$. Then, the solution of (2.5) may be expressed in the form

$$w(x) = (A_Df)(x) := \int_{-L}^{L} G_\xi(x,\xi)f(\xi)d\xi. \tag{2.6}$$

In what follows, for a nonnegative integer k, we denote by $C^k = C^k(\bar{I})$ the Banach space of real-valued, k-times continuously differentiable functions defined on $\bar{I}$, equipped with the norm

$$\|v\|_{C^k} := \max_{0\leqslant j\leqslant k}\max_{x\in\bar{I}}\left|v^{(j)}(x)\right|.$$

We also let $H^k = H^k(I)$ denote the usual (Hilbert) Sobolev space of (classes of) real-valued functions on I having generalized derivatives of order up to k in $L^2 = L^2(I)$. We equip H^k with the norm

$$\|v\|_k := \Big(\sum_{j=0}^{k}\|v^{(j)}\|^2\Big)^{1/2},$$

where $\|\cdot\|$ denotes the norm on $L^2 = H^0$. (The inner product on L^2 will be denoted by $(\cdot,\cdot)$.) In addition, H_0^1 will denote the subspace of H^1 whose elements vanish at $x = \pm L$. We denote norms of the spaces $L^p = L^p(I)$, $1 \leqslant p \leqslant \infty$, by $\|\cdot\|_{L^p}$. On occasion we shall also use the Sobolev space $W_p^k = W_p^k(I)$ for $p=1$ and $p=\infty$, whose usual norm we will denote by $\|\cdot\|_{W_p^k}$.

The following lemma gives some properties of the linear operator A_D that will be useful in the sequel.

Lemma 2.1. *Let A_D be defined by* (2.6) *and M denote positive constants depending on b, not necessarily the same at any two places. Then the following hold:*

(i) *If $v \in L^1$, then $A_D v \in W_1^1$ and $\|A_D v\|_{L^\infty} \leqslant M\|v\|_{L^1}$, $\|A_D v\| \leqslant M\|v\|_{L^1}$.*
(ii) *If $v \in L^2$, then $A_D v \in H_0^1$ and $\|A_D v\|_1 \leqslant M\|v\|$.*
(iii) *If $v \in H^1$, then $A_D v \in H^2$ and $\|A_D v\|_2 \leqslant M\|v\|_1$.*
(iv) *If $v \in C^m$, $m \geq 0$, then $A_D v \in C^{m+1}$ and $\|A_D v\|_{C^{m+1}} \leqslant M\|v\|_{C^m}$, where $M = M(m, b)$.*

Proof. (i): Let $v \in L^2$. Then, if $I_1(x) := \int_{-L}^x w_1' v$ and $I_2 := \int_x^L w_2' v$, $x \in \bar{I}$, we have, by (2.6),

$$A_D v := -\frac{1}{bW}\left[w_2 I_1 + w_1 I_2\right]. \tag{2.7}$$

It is clear that I_i, and therefore $A_D v$, belong to the space W_1^1. Moreover, for $x \in \bar{I}$, using the definitions of the w_i, we have

$$\begin{aligned} |(A_D v)(x)| &\leqslant \frac{1}{b|W|}\Big[w_2(x)\int_{-L}^x w_1'|v| - w_1(x)\int_x^L w_2'|v|\Big] \\ &\leqslant \frac{1}{b|W|}(w_2 w_1' - w_1 w_2')\|v\|_{L^1} = \frac{1}{b}\|v\|_{L^1}, \end{aligned}$$

from which the two estimates in (i) follow. To prove (ii) note that it follows by the definition of I_i that, for $v \in L^2$, $A_D v \in H_0^1$. In addition, for $\phi \in H_0^1$, integration by parts gives

$$\begin{aligned} &(A_D v, \phi) + b((A_D v)', \phi') \\ &= (v, \phi') - \frac{1}{bW}(w_2 I_1 + w_1 I_2, \phi) - \frac{1}{W}(w_2' I_1 + w_1' I_2, \phi') = (v, \phi'). \end{aligned}$$

Putting $\phi = A_D v$ we obtain the estimate in (ii). To prove (iii) note that for $v \in H^1$ we have, in L^2, that

$$A_D v - b(A_D v)'' = -v',$$

from which (iii) follows in view of (ii). Finally, note that if $v \in C^0$, then $A_D v \in C^1$ by (2.7). Moreover, for $x \in \bar{I}$, by the properties of the w_i

$$|(A_D v)(x)| \leqslant -\frac{1}{bW}\Big(w_2(x)\int_{-L}^x w_1' - w_1(x)\int_x^L w_2'\Big)\|v\|_{C^0} \leqslant \frac{1}{\sqrt{b}}\|v\|_{C^0},$$

and

$$|(A_D v)'(x)| \leqslant \tfrac{1}{b}\|v\|_{C^0} - \frac{1}{bW}\Big(-w_2'(x)\int_{-L}^{x} w_1' - w_1'(x)\int_{x}^{L} w_2'\Big)\|v\|_{C^0} = \tfrac{2}{b}\|v\|_{C^0},$$

and the required estimate follows for $m = 0$. For $m \geq 1$ use the relation

$$(A_D v)^{(m+1)} - b(A_D v)^{(m+1)} = -v^{(m+1)}, \tag{2.8}$$

and obtain the result by induction. □

Consider now the ibvp (2.3), (2.2), (2.4). Inverting the operator $1 - b\partial_x^2$ under the boundary conditions in (2.4) we obtain from the first p.d.e. of (2.3), after one integration by parts in space, the following differential-integral equation valid for $x \in \bar{I}$, $t \geq 0$:

$$\eta_t(x,t) = \frac{w_2(x)}{\mu}h_1'(t) + \frac{w_1(x)}{\mu}h_2'(t) + A_D(u + \eta u)(x,t), \tag{2.9}$$

where $\mu := \sinh(2L/\sqrt{b})$. If we perform analogous operations on the second p.d.e. of (2.3) we encounter the term $A_D\eta_{xx}$. Now, using the definition of A_D and integration by parts, it is not hard to see, for a twice differentiable function v on $\bar{I}$, that

$$A_D v'' = \frac{1}{b}(v' + Bv' + A_D v), \tag{2.10}$$

where, for $v \in C^0$, the operator B is defined as

$$(Bv)(x) = \frac{1}{\sqrt{b}W}(v(-L)w_2(x) + v(L)w_1(x)), \quad x \in \bar{I}. \tag{2.11}$$

Hence, $A_D\eta_{xx} = \frac{1}{b}(\eta_x + A_D\eta + B\eta_x)$, and it follows from the second p.d.e. of (2.3) that for $x \in \bar{I}$, $t \geq 0$,

$$u_t(x,t) = \frac{w_2(x)}{\mu}v_1'(t) + \frac{w_1(x)}{\mu}v_2'(t) + \frac{c}{b}(\widetilde{A}_D\eta + B\eta_x)(x,t) + A_D(\eta + \tfrac{1}{2}u^2)(x,t), \tag{2.12}$$

where $\widetilde{A}_D = A_D + \partial_x$. Integrating now (2.9) and (2.12) with respect to t yields the system

$$\eta(x,t) = \eta_0(x) + \frac{w_2(x)}{\mu}(h_1(t) - h_1(0)) + \frac{w_1(x)}{\mu}(h_2(t) - h_2(0)) + \int_0^t A_D(u + \eta u)d\tau, \tag{2.13}$$

$$u(x,t) = u_0(x) + \frac{w_2(x)}{\mu}(v_1(t) - v_1(0)) + \frac{w_1(x)}{\mu}(v_2(t) - v_2(0)) \qquad (2.14)$$
$$+ \int_0^t \Big[\frac{c}{b}(\widetilde{A}_D\eta + B\eta_x) + A_D(\eta + \tfrac{1}{2}u^2)\Big]\, d\tau.$$

It is clear that any classical solution (η, u) of the initial boundary value problem (2.3), (2.2), (2.4) satisfies the system of integral equations (2.13)-(2.14) in its temporal interval of existence.

In the sequel, given a Banach space X, and some positive number T we shall denote by $C(0,T;X)$ the Banach space of continuous maps $v : [0,T] \to X$ with norm $\|v\|_{C(0,T;X)} = sup_{0\leqslant t\leqslant T}\|v(t)\|_X$. In particular, when $X = C^m$ we shall frequently denote $C^m_T = C(0,T;C^m)$ and the corresponding norm by $\|v\|_{C^m_T}$.

We proceed now to establish uniqueness and local existence of solutions of the system of integral equations (2.13)-(2.14).

Proposition 2.1. *Let $0 < T < \infty$, $\eta_0 \in C^1$, $u_0 \in C^0$, $h_i, v_i \in C([0,T])$, $i = 1, 2$. Then, the system* (2.13)-(2.14) *has at most one solution $(\eta, u) \in C^1_T \times C^0_T$.*

Proof. Let (η_i, u_i), $i = 1, 2$, be two solutions of (2.13)-(2.14) in $C^1_T \times C^0_T$. Then, with $\eta := \eta_1 - \eta_2$, $u := u_1 - u_2$, we have for $0 \leqslant t \leqslant T$

$$\eta(t) = \int_0^t A_D(u + \eta_1 u + u_2\eta)d\tau, \qquad (2.15)$$

$$u(t) = \int_0^t \Big[\frac{c}{b}(\widetilde{A}_D\eta + B\eta_x) + A_D\eta + \frac{1}{2}A_D((u_1 + u_2)u)\Big]d\tau. \qquad (2.16)$$

Using Lemma 2.1(iv) we conclude from (2.15) that for some positive constant $M_1 = M_1(b)$ there holds for $t \in [0,T]$

$$\|\eta(t)\|_{C^1} \leqslant M_1\Big[(1 + \|\eta_1\|_{C^0_T})\int_0^t \|u(\tau)\|_{C^0}d\tau + \|u_2\|_{C^0_T}\int_0^t \|\eta(\tau)\|_{C^0}d\tau\Big].$$

Since from (2.11) for any integer $m \geq 0$ there exists a constant $c_1 = c_1(m,b)$ such that for $v \in C^0$

$$\|Bv\|_{C^m} \leqslant c_1 \max(|v(L)|, |v(-L)|) \leqslant c_1\|v\|_{C^0},$$

it follows from (2.16) and Lemma 2.1(iv) that for $t \in [0,T]$

$$\|u(t)\|_{C^0} \leqslant M_2\Big[\|u_1 + u_2\|_{C^0_T}\int_0^t \|u(\tau)\|_{C^0}d\tau + \int_0^t \|\eta(\tau)\|_{C^1}d\tau\Big],$$

for some positive constant $M_2 = M_2(b,c)$. We conclude therefore that for $t \in [0,T]$ and some constant c_2 we have

$$\|\eta(t)\|_{C^1} + \|u(t)\|_{C^0} \leqslant c_2 \int_0^t (\|\eta(\tau)\|_{C^1} + \|u(\tau)\|_{C^0}) d\tau,$$

from which, by Gronwall's lemma, we infer that $\eta = u = 0$. □

Proposition 2.2. *Let* $0 < T < \infty$, $\eta_0 \in C^1$, $u_0 \in C^0$, $h_i, v_i \in C([0,T])$, $i = 1, 2$, *and*

$$\beta_0 := \|\eta_0\|_{C^1} + \|u_0\|_{C^0} + \max_{t\in[0,T]} \sum_{i=1}^{2} (|h_i(t)| + |v_i(t)|).$$

Then, there exists a $T_0 = T_0(T, \beta_0) \in (0,T]$ *such that the system* (2.13)-(2.14) *has a unique solution* $(\eta, u) \in C^1_{T_0} \times C^0_{T_0}$.

Proof. With T_0 to be suitably chosen, let E be the Banach space $C^1_{T_0} \times C^0_{T_0}$ with norm $\|(v,w)\|_E := \|v\|_{C^1_{T_0}} + \|w\|_{C^0_{T_0}}$. Consider the mapping $\Gamma : E \to E$ defined by

$$\Gamma(v,w) = \Big(H + \int_0^t A_D(w + vw) d\tau, U + \int_0^t \Big[\frac{c}{b}(\widetilde{A}_D v + B v_x) + A_D(v + \frac{1}{2} w^2) \Big] d\tau \Big),$$

where, cf. (2.13)-(2.14),

$$H(x,t) := \eta_0(x) + \frac{w_2(x)}{\mu}(h_1(t) - h_1(0)) + \frac{w_1(x)}{\mu}(h_2(t) - h_2(0)),$$

$$U(x,t) := u_0(x) + \frac{w_2(x)}{\mu}(v_1(t) - v_1(0)) + \frac{w_1(x)}{\mu}(v_2(t) - v_2(0)).$$

Let B_R be the closed ball in E with center 0 and radius $R > 0$. Let $(\eta_i, u_i) \in B_R$, $i = 1, 2$. As in the proof of Proposition 2.1 and with notation introduced therein, we have

$$\begin{aligned} &\|\Gamma(\eta_1,u_1) - \Gamma(\eta_2,u_2)\|_E \\ &\leqslant M_1 T_0 [(1 + \|\eta_1\|^0_{C_{T_0}})\|u_1 - u_2\|_{C^0_{T_0}} + \|u_2\|_{C^0_{T_0}} \|\eta_1 - \eta_2\|_{C^0_{T_0}}] \\ &\qquad + M_2 T_0 [\|u_1 + u_2\|_{C^0_{T_0}} \|u_1 - u_2\|_{C^0_{T_0}} + \|\eta_1 - \eta_2\|_{C^1_{T_0}}] \\ &\leqslant \Theta \|(\eta_1,u_1) - (\eta_2,u_2)\|_E, \end{aligned} \tag{2.17}$$

where $\Theta := T_0[M_1 + M_2 + R(M_1 + 2M_2)]$. Moreover, if $(\eta, u) \in B_R$ it holds that

$$\|\Gamma(\eta,u)\|_E \quad \leqslant \quad \|\Gamma(\eta,u) - \Gamma(0,0)\|_E + \|\Gamma(0,0)\|_E$$

$$\leqslant \quad \Theta R + \|(H,U)\|_E \leqslant \Theta R + c_1\beta_0, \tag{2.18}$$

where c_1 is a constant depending on β. If we choose now $R = 2c_1\beta$ and $T_0 \leqslant \frac{1}{2[M_1+M_2+R(M_1+2M_2)]} \in (0,T]$, we see that $\Theta \leqslant \frac{1}{2}$ and $\Theta R + c_1\beta_0 \leqslant R$. Hence, in view of (2.17) and (2.18), the contraction mapping theorem applies to Γ considered as a mapping of B_R into itself. Consequently, Γ has a unique fixed point $(\eta, u) \in B_R$, which is the required solution of (2.13)-(2.14). □

We are now ready to establish a local existence-uniqueness result for the classical solutions of the ibvp (2.3)-(2.2)-(2.4).

Theorem 2.1. *Let* $0 < T < \infty$, $\eta_0 \in C^3$, $u_0 \in C^2$, $h_i, v_i \in C^1([0,T])$, $i = 1,2$, *and suppose that the compatibility conditions*

$$\eta_0(-L) = h_1(0), \eta_0(L) = h_2(0), u_0(-L) = v_1(0), u_0(L) = v_2(0), \tag{2.19}$$

are satisfied. Then, the solution $(\eta, u) \in C^1_{T_0} \times C^0_{T_0}$ *of* (2.13)-(2.14), *where existence and uniqueness were established in Proposition* 2.2, *lies in* $C^3_{T_0} \times C^2_{T_0}$, *and is a classical solution of the ibvp* (2.3), (2.2), (2.4) *in the temporal interval* $[0, T_0]$.

Proof. Consider the system (2.13)-(2.14) of integral equations. It is straightforward, by repeating the contraction mapping argument of Proposition 2.2, using our hypothesis on the regularity of the data, and Lemma 2.1(iv) to establish the existence and uniqueness of a solution (η, u) in the Banach space $C^3_{T_2} \times C^2_{T_2}$ for a suitably small $T_2 \in (0,T]$, depending on

$$\beta_2 := \|\eta_0\|_{C^3} + \|u_0\|_{C^2} + \max_{t\in[0,T]} \sum_{i=1}^{2} |h_i(t)| + |v_i(t)|.$$

In addition, it is not hard to see that η and u, given by (2.13), (2.14), respectively, are differentiable with respect to t and that their derivatives η_t, u_t are by the formulas (2.9) and (2.12), respectively. Since we assumed that $h_i, v_i \in C^1([0,T])$, $i = 1,2$, use of Lemma 2.1(iv) gives now that $(\eta_t, u_t) \in C^3_{T_2} \times C^2_{T_2}$.

Suppose that $T_2 < T_0$. Then, by Proposition 2.1, the solution pair $(\eta, u) \in C^3_{T_2} \times C^2_{T_2}$ coincides, on $[0, T_2]$, with that in $C^1_{T_0} \times C^0_{T_0}$ guaranteed by Proposition 2.2. By a standard argument, cf. [11] Section 5, its existence interval may be extended to $[0, T_0]$. For this purpose, it is necessary to establish an *a priori* estimate of $\|\eta\|_{C^3_{T_0}} + \|u\|_{C^2_{T_0}}$ independent of T_2. Since $(\eta, u) \in C^2_{T_0} \times C^0_{T_0}$, from (2.13) and using Lemma 2.1(iv) we have for

$t \in [0, T_0]$

$$\|\eta\|_{C^2} \leqslant \|\eta_0\|_{C^2} + C \max_{0\leqslant t\leqslant T} \sum_{i=1}^{2} |h_i(t)| + C \int_0^t (1 + \|\eta\|_{C^1})\|u\|_{C^1} d\tau,$$

where, here and in what follows, C will denote various constants independent of t and T_2 not necessarily the same in any two places. Since $\eta \in C^2_{T_0}$, we conclude that for $t \in [0, T_0]$

$$\|\eta\|_{C^2} \leqslant C(\beta_1 + \int_0^t \|u\|_{C^1} d\tau), \tag{2.20}$$

where

$$\beta_1 := \|\eta_0\|_{C^2} + \|u_0\|_{C^1} + \max_{t\in[0,T]} \sum_{i=1}^{2} |h_i(t)| + |v_i(t)|.$$

Similarly, from (2.14) we obtain, for $t \in [0, T_0]$,

$$\begin{aligned} \|u\|_{C^1} &\leqslant \|u_0\|_{C^1} + C \max_{t\in[0,T]} \sum_{i=1}^{2} |v_i(t)| + C \int_0^t (\|\eta\|_{C^2} + \|u\|^2_{C^0}) d\tau \\ &\leqslant C\Big[\beta_1 + \int_0^t (\|\eta\|_{C^2} + \|u\|_{C^0}) d\tau\Big]. \end{aligned} \tag{2.21}$$

From (2.20), (2.21) and Gronwall's lemma we infer that

$$\|\eta\|_{C^2_{T_0}} + \|u\|_{C^1_{T_0}} \leqslant C\beta_1 \mathrm{e}^{CT_0},$$

which is the required *a priori* estimate.

At this point we have established the existence of a unique solution (η, u) of (2.13)-(2.14) with the properties that $(\eta, u) \in C^3_{T_0} \times C^2_{T_0}$ and $(\eta_t, u_t) \in C^3_{T_0} \times C^2_{T_0}$. This solution is a classical solution of the ibvp (2.3), (2.2), (2.4) in the temporal interval $[0, T_0]$; that it satisfies the initial conditions (2.2) is obvious from (2.13)-(2.14). Using the compatibility conditions (2.19), the definitions of the operators $\widetilde{A}_D$, B and the functions w_1, w_2 , and the fact that $G_\xi(\pm L, \xi) = 0$, we may also check that the boundary conditions (2.4) are satisfied for $t \in [0, T_0]$. In addition, differentiating (2.9) twice with respect to x and using the definition of w_1, w_2 and (2.8) for $m = 1$, we see that for $x \in \bar{I}$, $t \in [0, T_0]$ we have $\eta_t - b\eta_{xxt} = -\partial_x(u + \eta u)$, which is the first p.d.e. in (2.2). Finally, from (2.12), differentiating twice with respect to x we have, similarly, that for $x \in \bar{I}$, $t \in [0, T_0]$

$$u_t - bu_{xxt} = \frac{c}{b}\left[A_D\eta + \eta_x - b(A_D\eta)_{xx}\right] - c\eta_{xxx} + (I - b\partial_x^2)A_D(\eta + \tfrac{1}{2}u^2)$$

$$= -\partial_x(\eta + \tfrac{1}{2}u^2) - c\eta_{xxx},$$

where, in the last equality, we used twice (2.8) for $m = 1$. Hence, (η, u) is also a classical solution of the second p.d.e. in (2.2). $\square$

Remark 2.1. One may readily establish higher regularity of the solution provided the data are more regular. For example, by a straightforward extension of the proof of Theorem 2.1 one may show that if $\eta_0 \in C^{m+1}$, $u_0 \in C^m$ and $h_i, v_i \in C^l([0,T])$, $i = 1, 2$, for integers $m \geq 2$, $\ell \geq 1$, and some $0 < T < \infty$, and if the compatibility conditions (2.19) hold, then the classical solution (η, u) of the ibvp (2.3), (2.2), (2.4) has the properties that $(\eta, u) \in C^{m+1}_{T_0} \times C^m_{T_0}$ and $(\partial_t^k \eta, \partial_t^k u) \in C^{m+1}_{T_0} \times C^m_{T_0}$, for $1 \leqslant k \leqslant \ell$.

Remark 2.2. Local in time well-posedness of the ibvp at hand may also be established in appropriate pairs of Sobolev classes. For example, it is straightforward to prove, by a modification of Proposition 2.1, that for any $T > 0$ the integral equations (2.13)-(2.14) have at most one solution $(\eta, u) \in C(0, T; H^2) \times C(0, T; H^1)$ provided $\eta_0 \in H^2$, $u_0 \in H^1$ and, e.g., $h_i, v_i \in C([0, T])$. (For this purpose use the properties (ii) and (iii) of A_D, cf. Lemma 2.1.) In addition, the proof of Proposition 2.2 may be adapted to yield local existence of a solution $(\eta, u) \in C(0, T'; H^2) \times C(0, T'; H^1)$ of (2.13)-(2.14) for some $T' \leqslant T$, depending on

$$\beta' := \|\eta_0\|_2 + \|u_0\|_1 + \max_{t\in[0,T]} \sum_{i=1}^{2} |h_i(t)| + |v_i(t)|.$$

This solution coincides with the classical solution of the ibvp established in Theorem 2.1 if the data satisfy the additional regularity and compatibility conditions in the statement of that theorem.

Remark 2.3. Consider, for simplicity, the case of homogeneous boundary conditions and write the system (1.4) in dimensionless but *scaled* variables to obtain the ibvp

$$\begin{aligned} &\eta_t + u_x + \varepsilon(\eta u)_x - \varepsilon b \eta_{xxt} = 0, \\ &u_t + \eta_x + \varepsilon u u_x + \varepsilon c \eta_{xxx} - \varepsilon b u_{xxt} = 0, \end{aligned} \quad x \in \bar{I},\ t \geq 0, \tag{2.22}$$

$$\eta(x, 0) = \eta_0(x),\ u(x, 0) = u_0(x),\ x \in \bar{I}, \tag{2.23}$$

$$\eta(\pm L, t) = u(\pm L, t) = 0,\ t \geq 0, \tag{2.24}$$

where $\varepsilon = A/h \ll 1$, cf. the remarks in the Introduction. If $b > 0$ and $c < 0$ we may derive local *a priori* estimates in $H^2 \cap H^1_0 \times H^1_0$ for the solution (η, u) of (2.22)–(2.24) by the energy method. These estimates are valid on a temporal interval $[0, T_\varepsilon]$, where T_ε is independent on ε. To see this, multiply the first p.d.e. in (2.22) by η and η_{xx} and the second by u, and use integration by parts and the boundary conditions (2.24) to obtain, for $t > 0$,

$$\frac{1}{2}\frac{d}{dt}\int_{-L}^{L}(\eta^2+\varepsilon b\eta_x^2)dx+\int_{-L}^{L}(u_x\eta+\varepsilon(\eta u)_x\eta)dx=0,$$
$$\frac{1}{2}\frac{d}{dt}\int_{-L}^{L}(\eta_x^2+\varepsilon b\eta_{xx}^2)dx-\int_{-L}^{L}(u_x\eta_{xx}+\varepsilon(\eta u)_x\eta_{xx})dx=0,$$
$$\frac{1}{2}\frac{d}{dt}\int_{-L}^{L}(u^2+\varepsilon bu_x^2)dx+\int_{-L}^{L}(\eta_x u-\varepsilon c\eta_{xx}u_x)dx=0.$$

Multiplying the second identity by $-\varepsilon c$ and adding all three we finally obtain the energy identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{-L}^{L}(\eta^2+\varepsilon(b-c)\eta_x^2-\varepsilon^2 bc\eta_{xx}^2+u^2+\varepsilon bu_x^2)dx\\&=-\frac{\varepsilon}{2}\int_{-L}^{L}\eta^2u_xdx-c\varepsilon^2\int_{-L}^{L}(\eta u)_x\eta_{xx}dx.\end{aligned}\tag{2.25}$$

By the Cauchy-Schwarz and the Sobolev-Poincaré inequalities we have, for some generic constant C independent of ε,

$$\Big|\int_{-L}^{L}\eta^2u_xdx\Big|\le\|\eta\|_{L^\infty}\|\eta\|\|u_x\|\le C\|\eta\|\|\eta_x\|\|u_x\|,$$
$$\begin{aligned}\Big|\int_{-L}^{L}(\eta u)_x\eta_{xx}dx\Big|&=\Big|\int_{-L}^{L}(\eta_x u\eta_{xx}+\eta u_x\eta_{xx})dx\Big|\\&\le\|u\|_{L^\infty}\|\eta_x\|\|\eta_{xx}\|+\|\eta\|_{L^\infty}\|u_x\|\|\eta_{xx}\|\le C\|\eta_x\|\|\eta_{xx}\|\|u_x\|.\end{aligned}$$

Using now Hölder's inequality we get

$$\begin{aligned}\varepsilon\|\eta\|\|\eta_x\|\|u_x\|&=\|\eta\|(\varepsilon^{1/2}\|\eta_x\|)(\varepsilon^{1/2}\|u_x\|)\\&\le C(\|\eta\|^3+\varepsilon^{3/2}\|\eta_x\|^3+\varepsilon^{3/2}\|u_x\|^3)\\&\le C(\|\eta\|^2+\varepsilon\|\eta_x\|^2+\varepsilon\|u_x\|^2)^{3/2},\end{aligned}$$

and

$$\begin{aligned}\varepsilon^2\|\eta_x\|\|\eta_{xx}\|\|u_x\|&=(\varepsilon^{1/2}\|\eta_x\|)(\varepsilon\|\eta_{xx}\|)(\varepsilon^{1/2}\|u_x\|)\\&\le C(\varepsilon\|\eta_x\|^2+\varepsilon^2\|\eta_{xx}\|^2+\varepsilon\|u_x\|^2)^{3/2}.\end{aligned}$$

Therefore, if

$$I_\varepsilon=I_\varepsilon(t):=\int_{-L}^{L}(\eta^2+\varepsilon(b-c)\eta_x^2-\varepsilon^2bc\eta_{xx}^2+u^2+\varepsilon bu_x^2)dx,$$

(2.25) implies that

$$\frac{dI_\varepsilon}{dt} \le C I_\varepsilon^{3/2},$$

from which

$$I_\varepsilon(t) \le \frac{I_\varepsilon(0)}{(1 - Ct\sqrt{I_\varepsilon(0)})^2},$$

for t sufficiently small. Therefore we have local in time *a priori* $H^2 \cap H_0^1 \times H_0^1$ estimates for the solution of (2.22)–(2.24) on a temporal interval $[0, T_\varepsilon]$, where $T_\varepsilon = O\big(\frac{1}{I_\varepsilon(0)^{1/2}}\big)$. Since

$$I_\varepsilon(0) = \int_{-L}^{L} [\eta_0^2 + \varepsilon(b-c)(\eta_0')^2 - \varepsilon^2 bc(\eta_0'')^2 + u_0^2 + \varepsilon b(u_0')^2]dx,$$

and $0 < \varepsilon \ll 1$, it is clear that T_ε may be taken to be independent of ε. If $c = 0$ (the BBM-BBM case) a similar proof yields *a priori* $H_0^1 \times H_0^1$ estimates on $[0, T_\varepsilon]$ where T_ε is independent of ε.

Finally, let us note that the generalization of the analysis of this section to systems of the form (2.1) with $a = 0$, $c < 0$ and $b > 0$, $d > 0$ with $b \ne d$ is immediate.

In addition, a similar analysis may be applied to the analogous ibvp for Boussinesq systems of "reverse" Bona-Smith type, i.e., systems with $a < 0$, $c = 0$, $b > 0$, $d > 0$, with Dirichlet boundary conditions of the type (2.4), to yield e.g., well-posedness with $(\eta, u) \in C(0, T_0; C^2) \times C(0, T_0; C^3)$ for small enough T_0.

In the case of systems of BBM-BBM type ($a = c = 0$, $b > 0$, $d > 0$) we have local well-posedness in the balanced spaces $C(0, T_0; C^2) \times C(0, T_0; C^2)$ as Bona and Chen have shown in [6]. In this case, the proof of the analog of Theorem 2.1 is simpler.

2.2. **Reflection boundary conditions.** In this paragraph we shall study the well-posedness of ibvp's for some systems of the form (2.1) in the case of *reflection* boundary conditions at the endpoints of $\bar{I}$, that is, in the case of homogeneous boundary conditions of Neumann type for η and of Dirichlet type for u.

First we consider ibvp's of this kind for systems of Bona-Smith type, rewriting them here for the convenience of the reader:

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x - b\eta_{xxt} = 0, \\ &u_t + \eta_x + uu_x + c\eta_{xxx} - bu_{xxt} = 0, \end{aligned} \qquad x \in \bar{I}, \quad t \ge 0, \tag{2.26}$$

where $b > 0$, $c < 0$. The systems are supplemented by initial conditions

$$\eta(x,0) = \eta_0(x), \qquad u(x,0) = u_0(x), \qquad x \in \bar{I}, \tag{2.27}$$

and the boundary conditions

$$\eta_x(-L,t) = \eta_x(L,t) = 0, \qquad u(-L,t) = u(L,t) = 0, \qquad t \geq 0. \tag{2.28}$$

As was done in the previous section, we shall convert first this ibvp into a system of integral equations. For this purpose, consider the two-point boundary-value problem with Neumann boundary conditions

$$\begin{aligned} &w - bw'' = -f', \ x \in \bar{I}, \\ &w'(-L) = w'(L) = 0. \end{aligned} \tag{2.29}$$

Let G_1 be the Green's function for (2.29) defined for $x, \xi \in \bar{I}$ by

$$G_1(x,\xi) := -\frac{1}{bW} \begin{cases} \omega_1(\xi)\omega_2(x), & -L \leqslant \xi \leqslant x, \\ \omega_1(x)\omega_2(\xi), & x < \xi \leqslant L, \end{cases}$$

where $\omega_1(x) := \cosh\left(\frac{L+x}{\sqrt{b}}\right)$, $\omega_2(x) := \cosh\left(\frac{L-x}{\sqrt{b}}\right)$, $W = \omega_1\omega_2' - \omega_1'\omega_2 = -\frac{1}{\sqrt{b}} \sinh\frac{2L}{\sqrt{b}}$. Define the linear operator A_N as

$$(A_N f)(x) := \int_{-L}^{L} G_{1,\xi} f(\xi) d\xi. \tag{2.30}$$

Note that if, e.g., $f \in C^1$ with $f(-L) = f(L) = 0$, then $w = A_N f$ is a classical solution of the boundary-value problem (2.29). In the following lemma we prove two properties of A_N that will be needed in the sequel.

Lemma 2.2.

(i) *If $v \in L^2$, then $A_N v \in H^1$ and $\|A_N v\|_1 \leqslant \max\left(1, \frac{1}{b}\right)\|v\|$.*

(ii) *If $v \in C^m$ for $m \geq 0$, then $A_N v \in C^{m+1}$ and $\|A_N v\|_{C^{m+1}} \leqslant M\|v\|_{C^m}$, for some constant M depending on m and b.*

Proof. Let $v \in L^2$. Then, if $I_1(x) := \int_{-L}^{x} \omega_1' v$ and $I_2(x) := \int_{x}^{L} \omega_2' v$ we have, by the definition of A_N,

$$A_N v = -\frac{1}{bW}(\omega_2 I_1 + \omega_1 I_2), \tag{2.31}$$

and

$$(A_N v)' = \frac{1}{b}\Big[v - \frac{1}{W}(\omega_2' I_1 + \omega_1' I_2)\Big]. \tag{2.32}$$

Hence, using the definitions of ω_i, we see that $A_N v \in H^1$ and $v - b(A_N v)' \in H_0^1$. It follows that for any $\phi \in H^1$

$$(v - b(A_N v)', \phi') = \frac{1}{W}(\omega_2' I_1 + \omega_1' I_2, \phi') = (A_N v, \phi),$$

which yields the required estimate in (i) if we put $\phi = A_N v$. To prove (ii), note that, for $v \in C^0$, it follows from (2.31) that $A_N v \in C^1$. In addition, $\|A_N v\|_{C^0} \leqslant M_1(b)\|v\|_{C^0}$ and $\|(A_N v)'\|_{C^0} \leqslant M_2(b)\|v\|_{C^0}$, from which (ii) follows for $m = 0$.

To prove the result for general m, note that (2.31) and (2.32) imply, if $v \in C^1$, that $b(A_N v)'' = v' + (A_N v)$. By induction, if $v \in C^m$, $m \geq 1$, we obtain that

$$(A_N v)^{(m+1)} = \frac{1}{b}\left[(A_N v)^{(m-1)} + v^{(m)}\right], \tag{2.33}$$

from which the required estimate in (ii) follows. □

Consider now the ibvp (2.26)–(2.28). Inverting the operator $1 - b\partial_x^2$ under the boundary conditions in (2.29) and taking into account the fact that u satisfies homogeneous Dirichlet boundary conditions at $x = \pm L$, we obtain from the first p.d.e. in (2.26), for $x \in \bar{I}$ and $t \geq 0$, that

$$\eta_t = A_N(u + \eta u). \tag{2.34}$$

Inverting now the operator $1 - b\partial_x^2$ under the boundary conditions in (2.5) we obtain from the second p.d.e. in (2.26) that $u_t = A_D(c\eta_{xx} + \eta + \frac{1}{2}u^2)$. Taking into account (2.10) and the fact that (2.28) and (2.11) imply that $B\eta_x = 0$, we see that

$$u_t = \frac{c}{b}\tilde{A}_D\eta + A_D(\eta + \tfrac{1}{2}u^2), \tag{2.35}$$

where we put again $\tilde{A}_D = A_D + \partial_x$. Integrating (2.34) and (2.35) with respect to t yields the system of integral equations

$$\eta(x,t) = \eta_0(x) + \int_0^t A_N(u + \eta u)d\tau, \tag{2.36}$$

$$u(x,t) = u_0(x) + \int_0^t \left[\frac{c}{b}\tilde{A}_D\eta + A_D(\eta + \tfrac{1}{2}u^2)\right] d\tau, \tag{2.37}$$

where $x \in \bar{I}$, $t \geq 0$. It is clear that any classical solution of the ibvp (2.26)–(2.28) satisfies in its temporal interval of existence the system of integral equations (2.36)–(2.37).

As in the previous section, we study first the uniqueness and existence of solutions of (2.36)-(2.37). For this purpose we shall denote $H^1_T = C(0,T;H^1)$, $L^2_T = C(0,T;L^2)$.

Proposition 2.3. *Let* $0 < T < \infty$, $\eta_0 \in H^1$, $u_0 \in L^2$. *Then the system* (2.36)–(2.37) *has at most one solution* $(\eta, u) \in H^1_T \times L^2_T$.

Proof. Let (η_i, u_i), $i = 1, 2$, be two solutions of (2.36)–(2.37) in $H^1_T \times L^2_T$. Then, with $\eta := \eta_1 - \eta_2$, $u := u_1 - u_2$, we have for $0 \leqslant t \leqslant T$

$$\eta(t) = \int_0^t A_N(u + \eta_1 u + u_2 \eta) d\tau, \tag{2.38}$$

$$u(t) = \int_0^t \Big[\frac{c}{b}\tilde{A}_D \eta + A_D(\eta + \frac{1}{2}(u_1 + u_2)u)\Big] d\tau. \tag{2.39}$$

Using Lemma 2.2(i) and Sobolev's theorem we have from (2.38) for some positive constant $M_1 = M_1(b)$ that, for $t \in [0, T]$,

$$\|\eta(t)\|_1 \leqslant M_1 \Big[(1 + \|\eta_1\|_{H^1_T}) \int_0^t \|u(\tau)\| d\tau + \|u_2\|_{L^2_T} \int_0^t \|\eta(\tau)\|_1 d\tau\Big].$$

In addition, using (2.39) and Lemma 2.1(i) we have, for some constant $M_2(b)$ and for $t \in [0, T]$,

$$\|u(t)\| \leqslant M_2 \Big[\int_0^t \|\eta\|_1 d\tau + \|u_1 + u_2\|_{L^2_T} \int_0^t \|u(\tau)\| d\tau\Big].$$

By our hypotheses on η_i, u_i it follows for $t \in [0, T]$ and for some constant c_1 that

$$\|\eta(t)\|_1 + \|u(t)\| \leqslant c_1 \int_0^t (\|\eta(\tau)\|_1 + \|u(\tau)\|) d\tau,$$

which, by Gronwall's lemma, leads to $\eta = u = 0$. □

Remark 2.4. Suppose that $(\eta_0, u_0) \in C^{m+1} \times C^m$, for some integer $m \geq 0$. Then, using (2.38) and (2.39) and the properties of A_D, A_N given by Lemma 2.1(iv) and Lemma 2.2(ii), we may prove, in a manner similar to the proof of the previous proposition, that given $0 < T < \infty$, the system (2.36)–(2.37) has at most one solution $(\eta, u) \in C^{m+1}_T \times C^m_T$.

Proposition 2.4. *Suppose that* $(\eta_0, u_0) \in H^1 \times L^2$ *and* $\beta := \|\eta_0\|_1 + \|u_0\|$. *Then, there exists a positive number* $T = T(\beta)$ *such that the system* (2.36)–(2.37) *has a unique solution* $(\eta, u) \in H^1_T \times L^2_T$.

Proof. With T to be suitably chosen, let E be the Banach space $H^1_T \times L^2_T$ with norm $\|(v,w)\|_E := \|v\|_{H^1_T} + \|w\|_{L^2_T}$. The mapping $\Gamma : E \to E$ given for $(v,w) \in E$ by

$$\Gamma(v,w) = \Big(\eta_0 + \int_0^t A_N(w+vw)d\tau, u_0 + \int_0^t \Big[\frac{c}{b}\tilde{A}_D v + A_D(v + \frac{1}{2}w^2)\Big]d\tau\Big)$$

is well defined, as may be seen by the properties of A_D and A_N in Lemma 2.1(i) and Lemma 2.2(i), respectively. Let B_R be the closed ball in E with center 0 and radius $R > 0$ and let $(\eta_i, u_i) \in B_R$, $i = 1, 2$. As in the proof of Proposition 2.3, we have, using the properties of A_D and A_N and Sobolev's theorem, that

$$\begin{aligned}
&\|\Gamma(\eta_1,u_1) - \Gamma(\eta_2,u_2)\|_E \\
&\leqslant M_1 T[(1 + \|\eta_1\|_{H^1_T})\|u_1 - u_2\|_{L^2_T} + \|u_2\|_{L^2_T}\|\eta_1 - \eta_2\|_{H^1_T}] \\
&\quad + M_2 T[\|\eta_1 - \eta_2\|_{H^1_T} + \|u_1 + u_2\|_{L^2_T}\|u_1 - u_2\|_{L^2_T}] \\
&\leqslant \Theta\|(\eta_1,u_1) - (\eta_2,u_2)\|_E, \qquad (2.40)
\end{aligned}$$

where $\Theta := T[M_1 + M_2 + R(M_1 + 2M_2)]$. In addition, if $(\eta, u) \in B_R$ it follows that

$$\begin{aligned}
\|\Gamma(\eta,u)\|_E &\leqslant \|\Gamma(\eta,u) - \Gamma(0,0)\|_E + \|\Gamma(0,0)\|_E \\
&\leqslant \Theta R + \|(\eta_0,u_0)\|_E \leqslant \Theta R + \beta. \qquad (2.41)
\end{aligned}$$

Choosing $R = 2\beta$, $T = \frac{1}{2[M_1+M_2+R(M_1+2M_2)]}$, we see that $\Theta = 1/2$, and $\Theta R + \beta = R$. Consequently, (2.40) and (2.41) imply that the contraction mapping theorem applies to the mapping $\Gamma : B_R \to B_R$. Therefore, Γ has a unique fixed point $(\eta, u) \in B_R$, which is the required solution of (2.36)–(2.37). □

It will be found useful in the sequel to have local in time existence results for solutions of (2.36)–(2.37) in spaces of smooth functions as well. With this aim in mind, we define for integer $m \geq 0$

$$C_0^m := \{v \in C^m : v(-L) = v(L) = 0\}$$

and

$$\widetilde{C}_0^{m+1} := \{v \in C^{m+1} : v'(-L) = v'(L) = 0\}.$$

The spaces C_0^m, $\widetilde{C}_0^{m+1}$ are obviously closed subspaces of the Banach spaces C^m and C^{m+1}, respectively. With this notation in mind, we have the following.

Proposition 2.5. *Given a nonnegative integer m, and $(\eta_0, u_0) \in \widetilde{C}_0^{m+1} \times C_0^m$, let $\gamma_m := \|\eta_0\|_{C^{m+1}} + \|u_0\|_{C^m}$. Then, there exists $T_m = T_m(\gamma_m) > 0$, such that the system* (2.36)–(2.37) *has a unique solution*

$$(\eta, u) \in C(0, T_m; \widetilde{C}_0^{m+1}) \times C(0, T_m; C_0^m).$$

Proof. With T_m to be suitably chosen, let E_m be the Banach space $C(0, T_m; \widetilde{C}_0^{m+1}) \times C(0, T_m; C_0^m)$ with norm $\|(v, w)\|_{E_m} := \|v\|_{C_{T_m}^{m+1}} + \|w\|_{C_{T_m}^m}$. Consider the mapping Γ_m given for $(v, w) \in E_m$ by

$$\Gamma_m(v, w) = (\eta_0 + \int_0^t A_N(w + vw)d\tau, u_0 + \int_0^t [\frac{c}{b}\tilde{A}_D v + A_D(v + \tfrac{1}{2}w^2)]d\tau.$$

That Γ_m is well defined on E_m, and has values in $C_{T_m}^{m+1} \times C_{T_m}^m$, follows from our hypotheses on the initial data and by Lemma 2.1(iv) and Lemma 2.2(ii). Let

$$\phi(x, t) := \eta_0(x) + \int_0^t A_N(w + vw)(x, t)d\tau.$$

Differentiating ϕ with respect to x, using (2.32) and the facts that $w \in C(0, T_m; C_0^m)$, $\eta_0 \in \widetilde{C}_0^{m+1}$, yields that $\phi_x(\pm L, t) = 0$, $0 \leqslant t \leqslant T_m$. In addition, it follows from the hypotheses $u_0 \in C_0^m$, $v \in C(0, T_m; \widetilde{C}_0^{m+1})$, and the definition of A_D that

$$\psi := u_0 + \int_0^t [\frac{c}{b}\widetilde{A}_D v + A_D(v + \tfrac{1}{2}w^2)]d\tau$$

vanishes at $x = \pm L$. Therefore, $\Gamma_m : E_m \to E_m$. The rest of the proof follows, *mutatis mutandis*, that of Proposition 2.4, when use is made of the estimates in Lemma 2.1(iv) and Lemma 2.2(ii) to establish that Γ_m is a contraction map from B to B, where B is a closed ball of center zero and suitable radius in E_m, provided T_m is taken sufficiently small. □

We are now in position to prove a local well-posedness result for classical solutions of the ibvp (2.26)–(2.28).

Proposition 2.6. *Suppose that $(\eta_0, u_0) \in \widetilde{C}_0^3 \times C_0^2$. Let $(\eta, u) \in H_T^1 \times L_T^2$ be the solution of the system* (2.36)–(2.37), *whose existence and uniqueness was established in Proposition* 2.4. *Then, $\eta, \eta_t \in C(0, T; \widetilde{C}_0^3)$, $u, u_t \in C(0, T; C_0^2)$ and (η, u) is a classical solution of the ibvp* (2.26)–(2.28) *in* $[0, T]$.

Proof. Consider the system of integral equations (2.36)–(2.37). By Proposition 2.5, there exists a positive $T_2 = T_2(\gamma_2)$ where $\gamma_2 = \|\eta_0\|_{C^3} + \|u_0\|_{C^2}$, such that a unique solution (η, u) of this system exists in $C(0, T_2; \widetilde{C}_0^3) \times$

$C(0,T_2;C_0^2)$. (It is not hard to see from (2.36) and (2.37) that η and u are differentiable with respect to t and that η_t and u_t are given by (2.34) and (2.35), respectively. Use of Lemma 2.1(iv) and Lemma 2.2(ii) gives that $(\eta_t,u_t)\in C(0,T_2;\widetilde{C}_0^3)\times C(0,T_2;C_0^2)$.)

Suppose that $T_2<T$. By Proposition 2.3 the solution pair (η,u) coincides on $[0,T_2]$ with that in $H_T^1\times L_T^2$ guaranteed by Proposition 2.4. As was done, e.g., in Theorem 2.1, if an *a priori* estimate of $\|\eta\|_{C_T^3}+\|u\|_{C_T^2}$ independent of T_2 is established, then the argument of Proposition 2.5 may be repeated to extend the solution from T_2 to $T_2^{(1)}>T_2$ and then for $T_2^{(1)}$ to $T_2^{(2)}>T_2^{(1)}$ and so on, reaching T in a finite number of steps.

To establish this *a priori* estimate, note that (2.36), Lemma 2.2 and Sobolev's theorem yield for $t\in[0,T]$

$$\|\eta\|_{C^1}\leqslant\|\eta_0\|_{C^1}+C(1+\|\eta\|_{H_T^1})\int_0^t\|u\|_{C^0}d\tau, \tag{2.42}$$

where C in the sequel will denote generically various constants independent of t and T_2. In addition, from (2.37), Lemma 2.1(iv) and (i), we have for $t\in[0,T]$

$$\|u\|_{C^0}\leqslant\|u_0\|_{C^0}+C(T\|u\|_{L_T^2}^2+\int_0^t\|\eta\|_{C^1}d\tau). \tag{2.43}$$

From (2.42) and (2.43) and Gronwall's lemma we conclude that, for some constant C depending on T, $\|\eta\|_{H_T^1}$, $\|u\|_{L_T^2}$ and $\|\eta_0\|_{C^1}+\|u_0\|_{C^0}$, we have

$$\|\eta\|_{C_T^1}+\|u\|_{C_T^0}\leqslant C.$$

Using this estimate and similar arguments we may easily obtain the required *a priori* estimate of $\|\eta\|_{C_T^2}+\|u\|_{C_T^1}$ and finally the needed estimate of $\|\eta\|_{C_T^3}+\|u\|_{C_T^2}$.

At this point we have established the existence of a unique solution (η,u) of (2.36)–(2.37) with the property that $(\eta,u)\in C(0,T,\widetilde{C}_0^3)\times C(0,T;C_0^2)$ (and such that $(\eta_t,u_t)\in C(0,T,\widetilde{C}_0^3)\times C(0,T;C_0^2))$. This solution is a classical solution of the ibvp (2.26)–(2.28). Indeed, differentiating twice (2.34) with respect to x we have, by (2.33), for $x\in\bar{I}$, $t\in[0,T]$, that

$$\eta_t+b\eta_{xxt}=A_N(u+\eta u)-b[A_N(u+\eta u)]_{xx}=(u+\eta u)_x.$$

In addition, differentiating twice in (2.35) with respect to x and using the last argument of the proof of Theorem 2.1, we conclude that (η,u) also satisfies the second p.d.e. of (2.26) for $x\in\bar{I}$, $t\in[0,T]$. We conclude that (η,u) is a classical solution of the ibvp (2.26)–(2.28) for $t\in[0,T]$. □

Consider a classical solution (η, u) of the system (2.26). If we write the latter in the form

$$\eta_t + P_x = 0,$$
$$u_t + Q_x = 0,$$

with $P := u + \eta u - b\eta_{xt}$, $Q := \eta + \frac{1}{2}u^2 + c\eta_{xx} - bu_{xt}$, it follows that

$$\eta_t Q + u_t P + (PQ)_x = 0. \tag{2.44}$$

The boundary conditions (2.28) imply that $P(\pm L, t) = 0$. In addition, we may easily check that

$$\eta_t Q + u_t P = \frac{1}{2}\partial_t(\eta^2 + u^2 + \eta u^2 - c\eta_x^2).$$

Using these observations and integrating (2.44) with respect to x on I we conclude that classical solutions of (2.26)–(2.28) conserve the "energy" functional

$$E(t) := \int_{-L}^{L} (\eta^2 + |c|\eta_x^2 + (1+\eta)u^2)(x,t)dx; \tag{2.45}$$

i.e., they satisfy $E(t) = E(0)$ in the temporal interval of their existence. The functional E is analogous to the Hamiltonian of the Cauchy problem for the same system, [11], [9].

The conservation of E leads to a global existence result for classical solutions of the ibvp (2.26)–(2.28) under mild restrictions on the initial data:

Theorem 2.2. *Suppose that* $(\eta_0, u_0) \in \widetilde{C}_0^3 \times C_0^2$ *and that* $\eta_0(x) > -1$, $x \in \bar{I}$. *Suppose also that*

$$E(0) = \int_{-L}^{L} (\eta_0^2 + |c|(\eta_0')^2 + (1+\eta_0)u_0^2)dx < \frac{L|c|^{1/2}}{L + |c|^{1/2}}. \tag{2.46}$$

Then, given any $T^* \in (0, \infty)$ *there exists a unique classical solution of the ibvp* (2.26)–(2.28) *in* $[0, T^*]$.

Proof. From our hypotheses and Propositions 2.4 and 2.6 it follows that there exists a $T = T(\beta) > 0$, where $\beta = \|\eta_0\|_1 + \|u_0\|$, such that the ibvp (2.26)–(2.28) has a classical solution in $[0, T]$. It also follows that there is a $t_0 > 0$ such that $1 + \eta > 0$ in $[-L, L] \times [0, t_0]$. Consider now the Sobolev type inequality

$$\|v\|_{C^0}^2 \leqslant \frac{\gamma + L}{\gamma L}(\|v\|^2 + \gamma^2\|v'\|^2), \tag{2.47}$$

which is valid for any $\gamma > 0$ and $v \in H^1$. To prove (2.47) consider the set $\{\phi_n\}_{n\geq 0}$ of functions on $\bar{I}$ given by $\phi_0(x) = (2L)^{-1/2}$,

$$\phi_n(x) = \Big(\frac{4L}{4L^2 + n^2\pi^2}\Big)^{1/2} \cos\frac{n\pi(x+L)}{2L},$$

$n = 1, 2, \ldots$. It is straightforward to check that $\{\phi_n\}_{n\geq 0}$ is an orthonormal basis of H^1 and that the ϕ_n are also orthogonal in L^2. It follows that for any $v \in H^1$ we have $v = \sum_{n\geq 0} a_n\phi_n$ with $a_n = (v, \phi_n)_1$, where $(\cdot,\cdot)_1$ denotes the usual inner product in H^1. Hence, for any $\gamma > 0$ we have

$$\|v\|^2 + \gamma^2\|v'\|^2 = \sum_{n\geq 0} \gamma_n |a_n|^2, \tag{2.48}$$

where $\gamma_n := \frac{4L^2+\gamma^2 n^2\pi^2}{4L^2+n^2\pi^2}$, $n \geq 0$. Moreover, for $x \in \bar{I}$

$$|v(x)| \leqslant \frac{1}{\sqrt{2L}}|a_0| + \sum_{n\geq 1}\Big(\frac{4L}{4L^2+n^2\pi^2}\Big)^{1/2}|a_n| \leqslant \sum_{n\geq 0}\Big(\frac{4L}{4L^2+n^2\pi^2}\Big)^{1/2}|a_n|.$$

Therefore,

$$\|v\|_{C^0}^2 \leqslant \sum_{n\geq 0}\frac{4L}{(4L^2+n^2\pi^2)\gamma_n^2}\cdot\sum_{n\geq 0}\gamma_n^2|a_n|^2,$$

and (2.47) follows from (2.48) and the estimate

$$\begin{aligned}\sum_{n\geq 0}\frac{4L}{(4L^2+n^2\pi^2)\gamma_n^2} &= \frac{1}{L} + \frac{4L}{\gamma^2\pi^2}\sum_{n\geq 1}\frac{1}{n^2 + \left(\frac{2L}{\gamma\pi}\right)^2}\\ &\leqslant \frac{1}{L} + \frac{4L}{\gamma^2\pi^2}\cdot\frac{\gamma\pi}{2L}\cdot\frac{\pi}{2} = \frac{1}{L} + \frac{1}{\gamma}.\end{aligned}$$

From (2.47) with $\gamma = |c|^{1/2}$ we infer for $t \in [0, t_0]$ that

$$\|\eta(t)\|_{C^0}^2 \leqslant \frac{L+|c|^{1/2}}{L|c|^{1/2}}E(t) = \frac{L+|c|^{1/2}}{L|c|^{1/2}}E(0) =: \lambda^2 < 1,$$

using the invariance of E and the hypothesis (2.46). Therefore, $\|\eta(t)\|_{C^0} \leqslant \lambda < 1$ in $[0, t_0]$, implying that $\min_{x\in\bar{I}}\eta(x, t_0) \geq -\lambda > -1$. So this argument may continue up to $t = T$ yielding

$$1 + \eta(x,t) \geq 1 - \lambda > 0,$$

for $x \in \bar{I}$, $t \in [0, T]$. Hence,

$$E(0) = E(t) \geq \int_{-L}^{L}(\eta^2 + |c|\eta_x^2 + (1-\lambda)u^2)dx \geq M(\|\eta(t)\|_1^2 + \|u(t)\|^2),$$

where $M = \min(1, |c|, 1-\lambda) > 0$, implying that the quantity $\|\eta(t)\|_1 + \|u(t)\|$ is bounded by a constant independent of t. We conclude that the contraction mapping argument of Proposition 2.3 may be repeated a finite number of times to reach any $T^* < \infty$, thus proving the theorem. □

Remark 2.5. The condition $1+\eta_0 > 0$ in $\bar{I}$ is a natural one and means that initially there is water in the channel at all points $x \in \bar{I}$, since the bottom in this choice of variables is at depth -1. As we saw in the proof of the above theorem, this condition and the assumption (2.46) imply that the channel never runs dry. As a consequence, $E(t)$ remains positive for all t and global existence of classical solutions follows. The specific form of the constants in the Sobolev inequality (2.47) allows us to recover the bound $|c|^{1/2}$ (cf. [9]) as $L \to \infty$ in the right-hand side of (2.46).

Let us point out that it is quite easy to generalize the analysis of this section in order to obtain local existence of classical solutions (i.e. up to Proposition 2.6) of ibvp's with reflection boundary conditions for systems of the form (2.1) with $a = 0$, $c < 0$ and $b > 0$, $d > 0$ with $b \neq d$. However, such systems are not Hamiltonian and global existence does not follow from our arguments. In the case of BBM-BBM type ($a = c = 0$, $b > 0$, $d > 0$) systems, we have local well-posedness in $C(0,T;\widetilde{C}_0^2) \times C(0,T;C_0^2)$, but even in the case $b = d$ global existence of smooth solutions does not follow from our arguments since $c = 0$ in (2.45).

2.3. **Periodic boundary conditions.** Finally, in this paragraph we shall study the well-posedness of the initial-periodic-boundary-value problem (ipvp) for some systems of the form (2.1). As the general scheme of proof resembles that of the two previous subsections we shall just state the results omitting the proofs. As usual we first consider the Bona-Smith system. We seek functions η, u, $2L$-periodic in x for $t \geq 0$, satisfying the system

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x - b\eta_{xxt} = 0, \\ &u_t + \eta_x + uu_x + c\eta_{xxx} - bu_{xxt} = 0, \end{aligned} \qquad x \in \bar{I}, \qquad t \geq 0, \tag{2.49}$$

where $b > 0$, $c < 0$. The systems are supplemented by the initial conditions

$$\eta(x,0) = \eta_0(x), \quad u(x,0) = u_0(x), \quad x \in \bar{I}, \tag{2.50}$$

where η_0, u_0 are given $2L$-periodic functions. As usual, we convert first this ipvp into a system of integral equations. For this purpose, consider the two-point boundary-value problem with periodic boundary conditions

$$\begin{aligned} &w - bw'' = -f', \quad x \in \bar{I}, \\ &w(-L) = w(L), \\ &w'(-L) = w'(L), \end{aligned} \tag{2.51}$$

where we suppose, for example, that $f \in C^1$ is $2L$-periodic. (In this section we shall denote by C_p^k the subspace of C^k consisting of $2L$-periodic functions v, whose derivatives $v', \cdots, v^{(k)}$ are $2L$-periodic. We shall also denote by H_p^1 the space of $2L$-periodic functions in H^1.) Define the linear operator

$$(A_p f)(x) := \int_{-L}^{L} G_{p,\xi}(x,\xi) f(\xi) d\xi, \tag{2.52}$$

where G_p is the Green's function for (2.51) defined for $x, \xi \in \bar{I}$ by

$$G_p(x,\xi) := \frac{1}{2b\omega'(L)} \begin{cases} \omega(x-\xi-L), & -L \leqslant \xi \leqslant x, \\ \omega(x-\xi+L), & x < \xi \leqslant L, \end{cases}$$

where $\omega(x) := \cosh \frac{x}{\sqrt{b}}$. If $f \in C^1$ with $f(-L) = f(L)$, then $w = A_p f$ is a classical solution of (2.51). It is proved in [1] using the representation (2.52) and Fourier analysis that the following lemma holds:

Lemma 2.3. *Let A_p be defined by* (2.52) *and M denote generic constants depending on b. Then, the following hold:*

(i) *If $v \in L^1$, then $A_p v \in W_1^1$ and $\|A_p v\|_{L^\infty} \leqslant M\|v\|_{L^1}$, $\|A_p v\| \leqslant M\|v\|_{L^1}$.*
(ii) *If $v \in L^2$, then $A_p v \in H_p^1$ and $\|A_p v\|_1 \leqslant M\|v\|$.*
(iii) *If $v \in C_p^m$, $m \geq 0$, then $A_p v \in C_p^{m+1}$ and $\|A_p v\|_{C^{m+1}} \leqslant M\|v\|_{C^m}$, where $M = M(m,b)$.*

In analogy to what was done in the previous two subsections, we may derive from (2.49)–(2.50) the system of integral equations

$$\eta(x,t) = \eta_0(x) + \int_0^t A_p(u + \eta u) d\tau, \tag{2.53}$$

$$u(x,t) = u_0(x) + \int_0^t \Big[\frac{c}{b}\tilde{A}_p \eta + A_p(\eta + \frac{1}{2}u^2)\Big] d\tau, \tag{2.54}$$

where $x \in \bar{I}$, $t \geq 0$, and $\tilde{A}_p := A_p + \partial_x$. Using Lemma 2.3, we see that the following analogs of similar previous results are valid:

Proposition 2.7. *Let $0 < T < \infty$, $\eta_0 \in H_p^1$, $u_0 \in L^2$. Then, the system* (2.53)–(2.54) *has at most one solution $(\eta, u) \in C(0,T; H_p^1) \times L_T^2$.*

Proposition 2.8. *Suppose that* $(\eta_0, u_0) \in H_p^1 \times L^2$ *and* $\beta := \|\eta_0\|_1 + \|u_0\|$. *Then, there exists* $T = T(\beta) > 0$ *such that the system* (2.53)–(2.54) *has a unique solution* $(\eta, u) \in C(0, T; H_p^1) \times L_T^2$.

Proposition 2.9. *Suppose that* $(\eta_0, u_0) \in C_p^{m+1} \times C_p^m$ *for a nonnegative integer* m, *and let* $\gamma_m := \|\eta_0\|_{C^{m+1}} + \|u_0\|_{C^m}$. *Then, there exists* $T_m = T_m(\gamma_m) > 0$ *such that the system* (2.53)–(2.54) *has a unique solution* $(\eta, u) \in C(0, T; C_p^{m+1}) \times C(0, T; C_p^m)$.

Combining these results we may prove the following local well-posedness result for the ipvp (2.49)–(2.50).

Proposition 2.10. *Suppose that* $(\eta_0, u_0) \in C_p^3 \times C_p^2$. *Let* $(\eta, u) \in C(0, T; H_p^1) \times L_T^2$ *be the solution of the system* (2.53)–(2.54), *whose existence and uniqueness was established in Proposition* 2.8. *Then,* $\eta, \eta_t \in C(0, T; C_p^3)$, $u, u_t \in C(0, T; C_p^2)$ *and* (η, u) *is a classical solution of the ipvp* (2.49)–(2.50) *in the temporal interval* $[0, T]$.

It is easily seen, integrating (2.44) with respect to x in $(-L, L)$ and using the periodic boundary conditions on u and η, that classical solutions of the ipvp (2.49)–(2.50) conserve the energy functional $E(t)$ given by (2.45). Therefore, one may prove again a global existence result under mild restrictions on the initial data, analogous to Theorem 2.2. The Sobolev inequality (2.47) should be replaced now by the inequality

$$\|v\|_{C^0}^2 \leqslant \Big(\frac{\gamma + L}{2\gamma L}\Big)(\|v\|^2 + \gamma^2 \|v'\|^2),$$

valid for any $v \in H_p^1$ and $\gamma > 0$; this inequality may be established again by Fourier expansions, cf. [1].

Theorem 2.3. *Suppose* $(\eta_0, u_0) \in C_p^3 \times C_p^2$ *and that* $\eta_0(x) > -1$, $x \in \bar{I}$. *In addition, suppose that*

$$E(0) = \int_{-L}^{L} (\eta_0^2 + |c|(\eta_0')^2 + (1 + \eta_0)u_0^2)dx < \frac{2L|c|^{1/2}}{L + |c|^{1/2}}. \tag{2.55}$$

Then, given any $T \in (0, \infty)$, *there exists a unique classical solution of the ipvp* (2.49)–(2.50) *in* $[0, T]$.

In closing, let us point out that, taking $L \to \infty$ in (2.55), one may recover the constant $\frac{2}{\sqrt{3}}$ that appears as a bound of $E(0)$ in the analogous theorem proved in [11] for the Cauchy problem for the Bona-Smith system with $c = -1/3$.

The local existence results of this section (i.e., Propositions 2.7 to 2.10) are easily generalized for systems of the form (2.1) with $a = 0$, $c < 0$ and $b > 0$, $d > 0$ with $b \neq d$. Similar local existence and uniqueness results may be proved by the same techniques for systems of BBM-BBM type ($a = c = 0$, $b, d > 0$), establishing that $(\eta, u) \in C(0, T; C_p^2) \times C(0, T; C_p^2)$ for small enough positive T, and also for the "reverse" Bona-Smith systems ($a < 0$, $c = 0$, $b > 0$, $d > 0$) for which it may be shown that $(\eta, u) \in C(0, T; C_p^2) \times C(0, T; C_p^3)$ for small enough T.

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