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CONCENTRATION OF SOLUTIONS OF A SEMILINEAR PDE WITH SLOW SPATIAL DEPENDENCE

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Abstract. The problem $-\epsilon^2\nabla\cdot(P(x)\nabla u)+F(V(x),u)=0$ is studied in the whole of $\mathbb{R}^n$, where $V(x)$ is a multidimensional potential and $P(x)$ a matrix function. Under general conditions solutions are constructed for small positive ϵ having simple concentration properties. The asymptotic form of the solution is studied as $\epsilon \to 0$ as well as its positivity.

1. Introduction

The purpose of this paper is to investigate the existence of solutions for small positive ϵ of the semilinear equation

$$-\epsilon^2\nabla\cdot(P(x)\nabla u)+F(V(x),u)=0, \quad x\in\mathbb{R}^n \tag{1.1}$$

in the space $H^2(\mathbb{R}^n)$ that concentrate at a single point $b\in\mathbb{R}^n$ as $\epsilon\to 0$. This may be viewed as a generalization of the problem of stationary semiclassical states of the non-linear Schrödinger equation, this being the problem of solutions for small ϵ to

$$-\epsilon^2\Delta u+V(x)u-u^p=0, \quad x\in\mathbb{R}^n \tag{1.2}$$

about which there is an extensive literature, not only on solutions that concentrate at a single point but also on those that concentrate at multiple points. The novel features of the treatment to be presented here are twofold:

1) *The function $V(x)$ is vector-valued with values in $\mathbb{R}^r$.*

2) *The positive-definite matrix function $P(x)$ is introduced.*

The solutions of (1.1) that concentrate for small ϵ are related to the ground state solution $\phi_{c,a}$ (the non-trivial solution in $H^1(\mathbb{R}^n)$ of lowest energy) of

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the equation

$$-\nabla \cdot (c\nabla u) + F(a,u) = 0, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where a is an r-vector and c a positive definite matrix. The connection between concentration of solutions and the ground state was studied for the problem (1.2) in an influential early paper [2] with methods related to those of the present paper.

With this generalization it becomes possible in principle to study problems of the form

$$-\epsilon^2 \nabla \cdot (P(x)\nabla u) + \sum_{k=1}^{r} V_k(x) u^k = 0, \quad x \in \mathbb{R}^n$$

provided we have sufficient knowledge of the ground state solution of

$$-\nabla \cdot (c\nabla u) + \sum_{k=1}^{r} a_k u^k = 0, \quad x \in \mathbb{R}^n$$

for a range of values of $a = (a_1, \ldots, a_r)$ and the positive-definite matrix c. In our study of (1.1) the existence of the ground state for a range of values of a and c, with the non-degeneracy properties set out in Section 2, will simply be assumed. Ground states are known to exist for a wide range of problems, but non-degeneracy of the sort needed here is so far only known for some special classes of equations.

There are two principal cases where the ground state solution is known to satisfy all the conditions needed for applying the results of this paper. The first is the one-dimensional problem

$$-c\,u'' + F(a,u) = 0$$

with vector parameter $a \in \mathbb{R}^r$ and scalar $c > 0$, which can be studied by phase-plane arguments. If we assume that $F(a,0) = 0$, $\frac{\partial F}{\partial u}(a,0) > 0$, the function $G(a,u) = \int_0^u F(a,s)\,ds$ is positive on an interval $]0, \lambda(a)[$ and has a simple zero at $u = \lambda(a)$, all for a range of values of a in some open set $A \subset \mathbb{R}^r$, then the solution satisfying $u(0) = \lambda(a)$, $u'(0) = 0$ is an appropriate ground state.

The second case is the problem

$$-\nabla \cdot (c\nabla u) + a_1 u - a_2 u^p = 0, \quad x \in \mathbb{R}^n$$

for $2 \le p < (n+2)/(n-2)$, for which the existence and non-degeneracy of the ground state was established in a series of papers, see for example [8],

[4], [11]. The ground state is given by

$$\phi_{c,a}(x) = \left(\frac{a_1}{a_2}\right)^{\frac{1}{p-1}} \psi(a_1^{1/2} c^{-1/2} x),$$

where ψ is the ground state of

$$-\Delta u + u - u^p = 0.$$

Applying results of the present paper we obtain conclusions about the concentration of solutions of

$$-\epsilon^2 \nabla \cdot (P(x)\nabla u) + V_1(x)u - V_2(x)u^p = 0$$

(see Section 5).

In order to summarize our conclusions and compare them with previous results we subject (1.1) to a preliminary transformation $x = \epsilon y$ to obtain the problem *with slow spatial dependence.* From now on the problem will only be considered in this form. Therefore we replace "y" by "x" and write it as

$$-\nabla \cdot (P(\epsilon x)\nabla u) + F(V(\epsilon x), u) = 0, \quad x \in \mathbb{R}^n. \tag{1.4}$$

The case of scalar V and $P = I$ (the unit matrix) was the subject of a previous paper [6]. The main conclusion in that case was that under some further technical assumptions (1.4) had a solution u_ϵ for all sufficiently small $\epsilon > 0$ of the form

$$u_\epsilon(x) = \phi_a\Big(x - \frac{b}{\epsilon} + s_\epsilon\Big) + \epsilon^2 w_\epsilon\Big(x - \frac{b}{\epsilon} + s_\epsilon\Big)$$

provided b was a non-degenerate critical point of V. Here, $a = V(b)$, the vector s_ϵ has the limit 0, ϕ_a is the ground state (that is, $\phi_{I,a}$ as introduced above), and the function w_ϵ is orthogonal to the partial derivatives of ϕ_a. Moreover, w_ϵ has a computable limit in H^2 which is not in general the zero-function.

It will be seen that merely allowing V to be vector-valued while still keeping $P = I$ already requires a treatment substantially different from that of [6], and the conclusions, set out in Theorem 7 of the present paper, are different in two significant ways.

1. *The critical points of V are no longer relevant.* This might be predicted by viewing (1.4) as a small perturbation of (1.3). A perturbation scheme should pick out a discrete set of points $b \in \mathbb{R}^n$ such that a solution exists looking like a translate of $\phi_{V(b)}$. But if V has values in $\mathbb{R}^r$ its critical points (if by this we mean points where $DV(b)$ has rank less than r) are not generically

isolated except when $r = 1$ or $r = n - 1$ so they are not what we seek. It turns out that we must seek critical points of the composition $M \circ V$ where

$$M(a) = \int |\nabla \phi_a(x)|^2 \, dx.$$

A further modification is then required to handle the case when $P \neq I$. We let

$$M(c, a) = \int \nabla \phi_{c,a}(x) \cdot c \nabla \phi_{c,a}(x) \, dx$$

and seek critical points of $b \mapsto M(P(b), V(b))$.

2. *The asymptotic form is of first order in* ϵ. This is already apparent for the vector V while still keeping $P = I$. As will be shown in Theorem 7, if b is a non-degenerate critical point of $M \circ V$ the correct asymptotic form for the solution is

$$u_\epsilon(x) = \phi_a\Big(x - \frac{b}{\epsilon} + s_\epsilon\Big) + \epsilon w_\epsilon\Big(x - \frac{b}{\epsilon} + s_\epsilon\Big),$$

the second term being now of first order in ϵ. It will be clear that the function w_ϵ appearing here need not have the limit zero. This difference between vector-valued V and scalar-valued V is rather unexpected. In the case $P \neq I$ b is a non-degenerate critical point of $M(P(b), V(b))$ and $\phi_{c,a}$ replaces ϕ_a.

2. Principal assumptions and preliminary lemmas

Assumptions on F. We assume the existence of the following derivatives with the stated growth conditions:

$$|F(a,u)|, \ |D_a F(a,u)|, \ \big|D_a^2 F(a,u)\big| \leq C(|u| + |u|^{\alpha_1}),$$

$$\Big|\frac{\partial F}{\partial u}(a,u)\Big|, \ \Big|D_a \frac{\partial F}{\partial u}(a,u)\Big| \leq C(1 + |u|^{\alpha_2}),$$

$$\Big|\frac{\partial^2 F}{\partial u^2}(a,u)\Big| \leq C(1 + |u|^{\alpha_3}),$$

where the constant C can be chosen uniformly for a in each bounded set and the exponents are non-negative and satisfy $\alpha_k \geq 2 - k$.

In addition we assume $1 \leq n \leq 7$, with no upper limit placed on $\alpha_1, \alpha_2, \alpha_3$ if $n \leq 4$, whereas for $n = 5, 6, 7$ we assume

$$\alpha_1 < \frac{n}{n-4}, \quad \alpha_2 < \frac{4}{n-4}, \quad \alpha_3 < \frac{8-n}{n-4}.$$

Note that it is obviously enough to assume the bound for α_3; the others then follow.

Assumptions on V. We assume that V and DV are bounded, while D^2V is polynomially bounded.

Assumptions on P. We assume that $P(x)$ is a real, symmetric and positive-definite matrix for each x, such that the functions P, DP, D^2P are bounded, and D^3P is polynomially bounded.

We will denote by $M_s^{n\times n}$ the manifold of real, symmetric and positive definite $n \times n$-matrices.

Positivity assumptions on P and V. We assume there exists $h > 0$ such that

$$\frac{\partial F}{\partial u}(a, 0) > h$$

for a in the range of V and

$$P(b) > hI$$

for all $b \in \mathbb{R}^n$.

Ground state assumptions. We assume the existence of a ground state solution $u = \phi_a(x)$ of

$$-\Delta u + F(a, u) = 0, \quad x \in \mathbb{R}^n \tag{2.1}$$

for a range A of values of a in $\mathbb{R}^r$. A ground state is techniquely a non-trivial solution of lowest energy (this refers to the variational formulation of the problem; use of the term ground state is though usually reserved for when it is unique up to translation). For our purposes it has the following properties:

(1) $\phi_a(x)$ is a strictly positive, radial element of H^2.

(2) $\phi_a(x)$ is a *quasi-non-degenerate solution.* By this we mean that the operator

$$-\triangle + \frac{\partial F}{\partial u}(a, \phi_a(x)) : H^2 \to L^2$$

has as kernel the space spanned by the n partial derivatives $D_j\phi_a(x)$, which are assumed to be independent, and its range is the space orthogonal in L^2 to its kernel.

(3) $\phi_a(x)$ and its first-order derivatives decay exponentially as $|x| \to \infty$.

(4) $\phi_a(x)$ is a continuously differentiable function of a.

These assumptions are by no means independent; for example exponential decay is more or less a consequence of the positivity assumption, but we take it as an assumption to avoid overburdening the exposition. That these assumptions hold for the problem (1.2) for $2 \leq p < (n+2)/(n-2)$ was established in a series of papers; see [8, 11, 4].

Now we can construct the ground state solution of

$$-\nabla\cdot c\nabla u + F(a,u) = 0, \quad x\in\mathbb{R}^n, \tag{2.2}$$

where $c\in M_s^{n\times n}$. It is given by

$$u = \phi_{c,a}(x) := \phi_a(c^{-\frac{1}{2}}x). \tag{2.3}$$

Preliminary lemmas. We shall rely heavily on the material of [6, Section 3], particularly Lemma 3.3, Lemma 3.4, Lemma 3.5 and Lemma 3.6, all of which hold for F once we assume the above growth conditions.

We need an addendum to [6, Lemma 3] to handle the matrix function P.

Lemma 1. *Let $P_\nu(x)$ be a sequence of matrix functions such that $P_\nu(x)\to P(x)$, $DP_\nu(x)\to DP(x)$ pointwise, P_ν and DP_ν being uniformly bounded in L^∞. Let $u_\nu\in H^2$ converge in norm to $u\in H^2$. Then*

$$\nabla\cdot(P_\nu\nabla u_\nu)\to\nabla\cdot(P\nabla u)$$

in L^2.

Proof. It suffices to show that

$$D_i(c_\nu D_j u_\nu)\to D_i(cD_j u)$$

assuming that c_ν is a sequence of scalar functions such that $c_\nu(x)\to c(x)$, $Dc_\nu(x)\to Dc(x)$ pointwise, with c_ν, Dc_ν bounded uniformly in L^∞. But it obviously follows from these assumptions that

$$(D_i c_\nu)(D_j u_\nu)\to(D_i c)(D_j u), \qquad c_\nu(D_i D_j u_\nu)\to c(D_i D_j u)$$

in L^2. □

It is useful to state here a convergence criterion that we shall often have need of. It is an obvious consequence of Lebesgue's dominated convergence theorem.

Lemma 2. *Let $f_\nu(t,x)$ be a sequence, or generalized sequence, of functions of the variables $t\in T$ and $x\in\mathbb{R}^n$, where T is a finite measure space, typically $[0,1]$ or $[0,1]\times[0,1]$. Suppose that $\lim f_\nu = f$ pointwise. Suppose that there exists a function g such that*

(a) *$|f_\nu(t,x)|\le g(x)$ for each ν, t and x;*

(b) *$\int |g(x)|^2\,dx<\infty$.*

Then

$$\lim\int_T f_\nu(t,\cdot)\,dt = \int_T f(t,\cdot)\,dt$$

in $L^2(\mathbb{R}^n)$.

The first use of Lemma 2 is to prove an addendum to [6, Lemma 3.6].

Lemma 3. *The map $s \mapsto \nabla \cdot (P(x+s)\nabla u)$ from $\mathbb{R}^n$ to L^2 is differentiable for each $u \in H^2$. Its derivative is the linear map $\sigma \mapsto \nabla \cdot (DP(x+s)\sigma\nabla u)$. The derivative depends continuously on (s,u).*

Proof. Let e_i be a basis vector in $\mathbb{R}^n$. Then

$$h^{-1}\Big(\nabla \cdot (P(x+s+he_i)\nabla u) - \nabla \cdot (P(x+s)\nabla u)\Big) - \nabla \cdot \Big(\frac{\partial P}{\partial b_i}(x+s)\sigma\nabla u\Big)$$
$$= \int_0^1 \Big[\nabla \cdot \Big(\frac{\partial P}{\partial b_i}(x+s+the_i)\nabla u\Big) - \nabla \cdot \Big(\frac{\partial P}{\partial b_i}(x+s)\nabla u\Big)\Big]\,dt.$$

The integrand tends pointwise to 0 as $h \to 0$, and is bounded, uniformly for $0 < h < 1$ and $0 < t < 1$, by a function in L^2, thanks to the boundedness of P, DP and D^2P. By Lemma 2 this establishes the formula for the derivative. That the derivative is continuous follows from Lemma 1. □

The device, used in the above proof, of writing an increment of a function as an integral over the interval $[0,1]$ in anticipation of applying Lemma 2, that is, of writing

$$f(v) - f(v_0) = \int_0^1 Df(v_0 + t(v - v_0))(v - v_0)\,dt,$$

will be used often and without further explanation.

We will need a version of Wang's lemma (see [10] for the original version and its context).

Lemma 4. *Let $f_\nu(x)$ be a family of bounded measurable scalar functions of $x \in \mathbb{R}^n$ (indexed by ν in some index set) and let $P_\nu(x)$ be a uniformly bounded sequence of functions with values in $M_s^{n\times n}$ such that $DP_\nu(x)$ is uniformly bounded. Assume that there exists $\delta > 0$ such that*

$$f_\nu(x) > \delta, \quad P_\nu(x) > \delta I$$

for all ν and all $x \in \mathbb{R}^n$. Suppose that v_ν is a sequence in H^2 such that

$$-\nabla \cdot (P_\nu(x)\nabla v_\nu) + f_\nu(x)v_\nu \to 0$$

in L^2. Then $v_\nu \to 0$ in H^2.

The proof of Lemma 4 depends on the case $p = 2$ of the following global *a priori* bound.

Lemma 5. *Let L be the elliptic operator over all $\mathbb{R}^n$ given by*

$$Lu = \nabla \cdot (P(x)\nabla u),$$

where $P(x)$ is $M_s^{n\times n}$-valued and satisfies

$$hI < P(x) < KI, \quad |DP(x)| < K, \quad x \in \mathbb{R}^n$$

for constants $K > h > 0$. Then there exists a constant C depending only on h, K, n and $p > 1$ such that

$$||u||_{W^{2,p}} \le C(||Lu||_{L^p} + ||u||_{L^p})$$

for all $u \in W^{2,p}$.

This is more or less standard though hard to find in the literature stated for the whole of $\mathbb{R}^n$. A proof for bounded domains can be found in [3, Theorem 9.13]. The bound can be extended to $\mathbb{R}^n$ under the assumptions we have given by using a partition of unity, necessarily infinite, but consisting of translates of a fixed function by the points of a lattice, much as in a similar proof of Gårding's inequality for an unbounded domain that can be found in [1].

To prove Lemma 4 let A_ν be the operator

$$A_\nu u = -\nabla \cdot (P_\nu(x)\nabla u) + f_\nu(x)u.$$

It follows from Lemma 5 and standard theory (see for example [7, Theorem VIII.15]) that A_ν is self-adjoint with domain H^2 for each ν and is moreover the self-adjoint operator induced by the quadratic form

$$Q_\nu(u) = \int (\nabla u \cdot P_\nu(x)\nabla u + f_\nu(x)u^2)\, dx, \quad u \in H^1.$$

Moreover, from the inequality

$$Q_\nu(u) \ge \delta \int |\nabla u|^2\, dx + \delta \int |u|^2\, dx > \delta \|u\|_{L^2}, \tag{2.4}$$

which holds for all $u \in H^1$ and all ν, we see that the spectrum of A_ν lies in the interval $[\delta, \infty[$ independently of ν, so that A_ν^{-1}, regarded as a map from L^2 to itself, has norm uniformly bounded by δ^{-1}. The sequence v_ν of the lemma, which satisfies $A_\nu v_\nu \to 0$ in L^2, therefore converges to 0 in L^2. Finally we apply Lemma 5 to show that v_ν converges to 0 in H^2, completing the proof of Lemma 4.

The solutions of problem (1.4) will be obtained by an abstract device combining elements of the implicit function theorem and Newton's method. We reproduce the following from [5].

Lemma 6. *Let E and F be real Banach spaces, and let $f : \mathbb{R}_+ \times E \to F$. Assume that*

(1) *$f(\epsilon, \cdot)$ is C^1 for each $\epsilon \geq 0$;*

(2) *there exists $x_0 \in E$ such that $f(0, x_0) = 0$;*

(3) *$D_x f(0, x_0)$ is invertible;*

(4) *$\lim_{\epsilon \to 0^+} f(\epsilon, x_0) = 0$;*

(5) *for all sufficiently small $\epsilon > 0$ the operator $D_x f(\epsilon, x_0)$ is invertible and $||D_x f(\epsilon, x_0)^{-1}||$ is uniformly bounded as $\epsilon \to 0^+$;*

(6) *$\lim_{\epsilon \to 0^+, x \to x_0} ||D_x f(\epsilon, x) - D_x f(\epsilon, x_0)|| = 0$.*

Then there exist $\epsilon_0 > 0$ and a neighbourhood U of x_0 in E such that for each ϵ in the range $0 < \epsilon < \epsilon_0$ there exists a unique solution $x = x_\epsilon$ of $f(\epsilon, x) = 0$ in U. Moreover, $\lim_{\epsilon \to 0} x_\epsilon = x_0$.

If, furthermore, $f(\epsilon, x)$ is a continuous function of ϵ for $\epsilon > 0$ and for all x in a neighbourhood of x_0, and the map $\epsilon \mapsto D_x f(\epsilon, x_0)$ is continuous in the strong operator topology, then the solution x_ϵ is a continuous function of ϵ.

We refer to the problem $f(0, x) = 0$ as the *limit problem*. As pointed out in [5] Lemma 6 remains true without any assumptions on the limit problem, or even without assuming its existence. It will however be seen in the ensuing application that the limit problem plays a strong role, firstly in providing the limit solution x_0 and secondly in verifying the difficult condition (5).

3. Main results

In this section we state the two results the proofs of which will occupy most of this paper. We adopt the assumptions on F, V, P and $\phi_{c,a}$ of the previous section.

Introduce the function

$$M(c, a) = \int \nabla\phi_{c,a}(x) \cdot c\nabla\phi_{c,a} \, dx, \quad c \in M_s^{n\times n}, \ a \in \mathbb{R}^r.$$

Let $\tilde{V} : \mathbb{R}^n \to M_s^{n\times n} \times \mathbb{R}^r$ be the function

$$\tilde{V}(b) = (P(b), V(b)).$$

Theorem 7. *Let $b \in \mathbb{R}^n$ be a non-degenerate critical point of the function $M \circ \tilde{V}$ and let $(c, a) = \tilde{V}(b)$. Let*

$$W = \Big\{ w \in H^2 : \int w D_j \phi_{c,a} = 0, \ j = 1, \dots, n = \Big\}.$$

Then:

(i) *For all sufficiently small* $\epsilon > 0$ *the equation*

$$-\nabla \cdot (P(\epsilon x)\nabla u) + F(V(\epsilon x), u) = 0 \tag{3.1}$$

has a solution $u_\epsilon(x)$ *with asymptotic form*

$$u_\epsilon(x) = \phi_{c,a} = \Big(x - \frac{b}{\epsilon} + s_\epsilon = \Big) + \epsilon w_\epsilon = \Big(x - \frac{b}{\epsilon} + s_\epsilon = \Big)$$

such that $s_\epsilon \in \mathbb{R}^n$, $w_\epsilon \in W$ *and* s_ϵ *and* w_ϵ *depend continuously on* ϵ. *Moreover,* $s_\epsilon \to 0$ *and* w_ϵ *converges to a function* η *in the* H^2*-norm as* $\epsilon \to 0+$.

(ii) *The function* η *is the unique solution in* W *to the equation*

$$\begin{aligned} -\nabla \cdot (c\nabla w) + \frac{\partial F}{\partial u}\Big(a, \phi_{c,a}(x)\Big)w - \nabla \cdot ((DP(b)x)\nabla \phi_{c,a}(x)) \\ + (D_a F)(a, \phi_{c,a}(x))DV(b)x = 0. \end{aligned} \tag{3.2}$$

Theorem 8. *Let* b *be a non-degenerate critical point of* $M \circ \tilde{V}$ *and let* u_ϵ *be the solution given by Theorem* 7. *Then if* $\epsilon > 0$ *is sufficiently small* u_ϵ *is strictly positive.*

4. Proof of Theorem 7

The proof of Theorem 7 will occupy the whole of this section and will require a series of lemmas. Headed subsections are intended for ease of reading.

A mixture of notations will be used for derivatives as was already used in Section 1. This is intended to aid comprehension. Thus D_1, D_2 ... denote partial differentiation with respect to coordinates of $\mathbb{R}^n$. On the other hand D_a denotes differentiation with respect to the r-vector a. An expression such as $D_aF(a,u)$ is usually followed by an r-vector, which is usually of the form $DV(b)x$. We use $\frac{\partial}{\partial a_k}$ to denote partial differentiation with respect to the kth coordinate of a. The expression $D\nabla\phi_a$ denotes the derivative of the vector-field $\nabla\phi_a$ and is usually followed by an n-vector. As usual $\nabla\cdot$ followed by a vector denotes divergence. An expression $DP(b)$ is followed by an n-vector and, for example, $DP(b)x$ is an n by n symmetric matrix (not necessarily positive) and is usually itself followed by an n-vector. The derivative of $F(a,u)$ with respect to u is always denoted by $\frac{\partial F}{\partial u}$ with a corresponding notation for the higher derivatives.

Over and over again we use the equation satisfied by $\phi_{c,a}$, that is, equation (1.3), and the fact that the partial derivatives $D_m\phi_{c,a}$ lie in the kernel of the differential operator $-\nabla\cdot c\nabla + \frac{\partial F}{\partial u}(a, \phi_{c,a})$. Heavy use is made of the radiality of ϕ_a. This implies, for example, that $D_m\phi_a$ is an odd function, invariant under permutations of the coordinates that leave x_m fixed. In particular $D_i\phi_a$ and $D_j\phi_a$ are orthogonal if $i \neq j$.

Although the proof is lengthy, in outline it is simple enough. We first develop a formula for the gradient of $M \circ \tilde{V}$. Then we convert our problem into an equivalent one, called the rescaled problem and denoted by $\Gamma_\epsilon(s,w) = 0$, with new unknowns $s \in \mathbb{R}^n$ and w lying in a certain subspace of H^2. Finally we solve $\Gamma_\epsilon(s,w) = 0$ for small ϵ using Lemma 6.

Study of the gradient of $M \circ \tilde{V}$.

Lemma 9.

$$-\frac{1}{n}\nabla(M \circ \tilde{V})(b) = \int \Big[-\nabla \cdot ((DP(b)x)\nabla\phi_{c,a}(x)) + (D_aF)(a, \phi_{c,a}(x))DV(b)x\Big]\nabla\phi_{c,a}(x)\,dx,$$

where $a = V(b)$ and $c = P(b)$.

Proof. We have

$$\phi_{c,a}(x) = \phi_a(c^{-\frac{1}{2}}x)$$

and so

$$\nabla\phi_{c,a}(x) = c^{-\frac{1}{2}}(\nabla\phi_a)(c^{-\frac{1}{2}}x)$$

and

$$M(c,a) = \int c^{-\frac{1}{2}}(\nabla\phi_a)(c^{-\frac{1}{2}}x) \cdot c^{\frac{1}{2}}(\nabla\phi_a)(c^{-\frac{1}{2}}x)\,dx = (\det c)^{\frac{1}{2}}M_I(a), \quad (4.1)$$

where $M_I(a) = M(I,a)$.

Letting $c = P(b)$ and $a = V(b)$ and differentiating we have

$$\nabla(M \circ \tilde{V})(b) \cdot s = \frac{1}{2}(\det c)^{\frac{1}{2}}\operatorname{tr}(c^{-1}DP(b)s)M_I(a) + (\det c)^{\frac{1}{2}}\nabla M_I(a) \cdot DV(b)s.$$

We shall prove the formulas

$$\frac{1}{2n}(\det c)^{\frac{1}{2}}\operatorname{tr}(c^{-1}DP(b)s)M_I(a) = \int \nabla \cdot \Big((DP(b)x)\nabla\phi_{c,a}(x)\Big)\Big(\nabla\phi_{c,a}(x) \cdot s\Big)\,dx, \quad (4.2)$$

$$-\frac{1}{n}(\det c)^{\frac{1}{2}}\nabla M_I(a) \cdot DV(b)s = \int \Big((D_aF)(a, \phi_{c,a}(x))(DV(b)x)\Big)\Big(\nabla\phi_{c,a}(x) \cdot s\Big)\,dx. \quad (4.3)$$

Reduction of (4.2) *to the case* $c = I$.

$$\int \nabla \cdot ((DP(b)x)\nabla\phi_{c,a}(x))\nabla\phi_{c,a}(x)\cdot s\,dx$$
$$= -(\det c)^{\frac{1}{2}}\int (Q_c(y)(\nabla\phi_a)(y))\cdot((D\nabla\phi_a)(y)c^{-\frac{1}{2}}s)\,dy,$$

where $Q_c \in L(\mathbb{R}^n, M_s^{n\times n})$ is the linear matrix function

$$Q_c(y) = c^{-\frac{1}{2}}DP(b)(c^{\frac{1}{2}}y)c^{-\frac{1}{2}}.$$

Hence (4.2) is established if we show that

$$\frac{1}{2n}\mathrm{tr}\,(Q(s))M_I(a) = -\int (Q(y)(\nabla\phi_a)(y))\cdot((D\nabla\phi_a)(y)s)\,dy$$

for all s and all $Q \in L(\mathbb{R}^n, M_s^{n\times n})$. The proof depends heavily on the radiality of ϕ_a. Let $Q(y)$ be the matrix $\sum_{k=1}^n q_{ij,k}y_k$, where $y = (y_1,\dots,y_n)$; note that $q_{ij,k} = q_{ji,k}$.

We first prove the formula

$$\int y_k(D_j\phi_a(y))(D_iD_l\phi_a(y))\,dy = \frac{1}{2n}(\delta_{kj}\delta_{il} - \delta_{ki}\delta_{jl} - \delta_{kl}\delta_{ij})M_I(a). \quad (4.4)$$

Repeated integration by parts gives

$$\int y_k(D_j\phi_a(y))(D_iD_l\phi_a(y))\,dy = -\int D_i(y_kD_j\phi_a(y))D_l\phi_a(y)\,dy$$
$$= \frac{1}{n}(-\delta_{ik}\delta_{jl} - \delta_{kl}\delta_{ij} + \delta_{kj}\delta_{li})M_I(a) - \int y_k(D_j\phi_a(y))(D_iD_l\phi_a(y))\,dy$$

and this implies (4.4).

Now we have

$$\int (Q(y)\nabla\phi_a(y))\cdot(D\nabla\phi_a(y)s)\,dy$$
$$= \sum_{1\le i,j,k,l\le n}\int q_{ij,k}y_k(D_j\phi_a(y))(D_iD_l\phi_a(y))s_l\,dy$$
$$= \frac{1}{2n}\sum_{1\le i,j,k,l\le n} q_{ij,k}s_l(-\delta_{ik}\delta_{jl} - \delta_{kl}\delta_{ij} + \delta_{kj}\delta_{li})$$
$$= \frac{1}{2n}M_I(a)(-\sum_{i,j} q_{ij,i}s_j - \sum_{i,k} q_{ii,k}s_k + \sum_{i,j} q_{ij,j}s_i)$$

$$= -\frac{1}{2n} M_I(a) \sum_{i,k} q_{ii,k} s_k \quad \text{(by symmetry)}$$
$$= -\frac{1}{2n} M_I(a) \operatorname{tr}(Q(s)).$$

Reduction of (4.3) *to the case* $c = I$.

$$\int \Big((D_aF)(a, \phi_{c,a}(x))(DV(b)x)\Big)\Big(\nabla\phi_{c,a}(x)\cdot s\Big)\,dx$$
$$= \int \Big((D_aF)(a, \phi_a(c^{-\frac{1}{2}}x))(DV(b)x)\Big)\Big((\nabla\phi_a)(c^{-\frac{1}{2}}x)\cdot c^{-\frac{1}{2}}s\Big)\,dx$$
$$= (\det c)^{\frac{1}{2}} \int \Big((D_aF)(a, \phi_a(y))DV(b)(c^{\frac{1}{2}}y)\Big)\Big(\nabla\phi_a(y)\cdot c^{-\frac{1}{2}}s\Big)\,dy$$

and so (4.3) is equivalent to

$$\Big(-\frac{1}{n}\Big)\nabla M_I(a)\cdot DV(b)s$$
$$= \int \Big((D_aF)(a, \phi_a(y))DV(b)(c^{\frac{1}{2}}y)\Big)\Big(\nabla\phi_a(y)\cdot c^{-\frac{1}{2}}s\Big)\,dy$$

for all s, or equivalently

$$\Big(-\frac{1}{n}\Big)\nabla M_I(a)\cdot Ts = \int (D_aF)(a, \phi_a(y))(Ty)\nabla\phi_a(y)\cdot s\,dy$$

for all s, where $T = DV(b)c^{\frac{1}{2}}$. We prove this holds for an arbitrary $(r \times n)$-matrix $T = (T_{ij})$.

We first note that since

$$-\Delta\phi_a(y) + F(a, \phi_a(y)) = 0$$

we must have

$$-\Delta D_a\phi_a(y) + \frac{\partial F}{\partial u}(a, \phi_a(y))D_a\phi_a(y) + (D_aF)(a, \phi_a(y)) = 0$$

and so (by Leibniz's rule)

$$(D_aF)(a, \phi_a(y))Ty = \Big(\Delta - \frac{\partial F}{\partial u}(a, \phi_a(y))\Big)(D_a\phi_a(y)Ty)$$
$$- 2\sum_{j,k} T_{kj}\Big(D_j\frac{\partial}{\partial a_k}\Big)\phi_a(y).$$

By radiality and integration by parts we now find

$$\int \Big((D_aF)(a,\phi_a(y))(Ty)\Big)\Big(\nabla\phi_a(y)\cdot s\Big)\,dy$$
$$= -2\sum_{j,k} T_{kj}\int \Big(D_j\frac{\partial}{\partial a_k}\Big)\phi_a(y)\Big(\nabla\phi_a(y)\cdot s\Big)\,dy = -\frac{1}{n}\nabla M_I(a)\cdot Ts.$$

This concludes the proof of Lemma 9. □

The rescaled problem. Let

$$u(x) = \phi_{c,a}(x+\xi) + \epsilon w(x+\xi),$$

where $w \in W$ and $\xi = -\frac{b}{\epsilon} + s$, $s \in \mathbb{R}^n$.

Substitute into (3.1), replace x by $x-\xi$, use the equation satisfied by $\phi_{a,c}$ and divide by ϵ. This leads to the problem

$$-\nabla\cdot(c\nabla w) + \epsilon^{-1}\Big(F(V(\epsilon(x-\xi)),\phi_{c,a}(x)+\epsilon w) - F(a,\phi_{c,a}(x))\Big) \tag{4.5}$$
$$-\epsilon^{-1}\nabla\cdot((P(\epsilon(x-\xi))-c)\nabla\phi_{c,a}(x)) - \nabla\cdot((P(\epsilon(x-\xi))-c)\nabla w) = 0.$$

This is a problem for the unknown $(s,w) \in \mathbb{R}^n \times W$. The left-hand side maps this space into L^2. Let $\mathbb{P}$ be the orthogonal projection of L^2 onto the subspace W_0 orthogonal to the space spanned by the partial derivatives $D_m\phi_{c,a}$. We transform (4.5) into an equivalent problem in the following way. Firstly for $m = 1,\dots,n$ we multiply by $D_m\phi_{c,a}$, integrate and divide by ϵ. This gives the n equations

$$\epsilon^{-2}\int\Big[-\nabla\cdot((P(\epsilon(x-\xi))-c)\nabla\phi_{c,a}(x)) - \epsilon\nabla\cdot((P(\epsilon(x-\xi))-c)\nabla w)$$
$$+F(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w) - F(a,\phi_{c,a}) - \epsilon\frac{\partial F}{\partial u}(a,\phi_{c,a})w\Big]\,D_m\phi_{c,a}(x)\,dx = 0$$
$$m = 1,\dots,n. \tag{4.6}$$

In addition to these n equations we have the result of projecting (4.5) to W_0

$$\mathbb{P}\Big[-\nabla\cdot(c\nabla w) + \epsilon^{-1}\Big(F(V(\epsilon(x-\xi)),\phi_{c,a}(x)+\epsilon w) - F(a,\phi_{c,a}(x))\Big)$$
$$-\epsilon^{-1}\nabla\cdot((P(\epsilon(x-\xi))-c)\nabla\phi_{c,a}(x)) - \nabla\cdot((P(\epsilon(x-\xi))-c)\nabla w)\Big] = 0. \tag{4.7}$$

It is clear that (4.5) is equivalent to the system (4.6) and (4.7).

Let us denote the left-hand side of (4.6) by $\Gamma_\epsilon^{(1)}(s,w)$ and the left-hand side of (4.7) by $\Gamma_\epsilon^{(2)}(s,w)$. Define

$$\Gamma_\epsilon(s,w) = (\Gamma_\epsilon^{(1)}(s,w), \Gamma_\epsilon^{(2)}(s,w)), \quad s \in \mathbb{R}^n,\ w \in W,\ \epsilon > 0.$$

The *rescaled problem* has the concise form

$$\Gamma_\epsilon(s,w) = 0, \quad \epsilon > 0. \tag{4.8}$$

Lemma 10. *The map $\Gamma_\epsilon : \mathbb{R}^n \times W \to \mathbb{R}^n \times W_0$ is C^1 for $\epsilon > 0$.*

Proof. This follows from [6, Lemma 3.6], and Lemma 3. □

The limit problem. The basis of the existence proof is an examination of the limit $\epsilon \to 0$ of the rescaled problem (4.8). This requires the introduction of second derivatives as bilinear maps that act on a pair of vectors, such as $DV(b)(s_1, s_2)$. We then use the abbreviated notation $DV(b)s^{\otimes 2}$ for $DV(b)(s,s)$ in order to improve the readability of lengthy equations.

Lemma 11. *The limit*

$$\lim_{\epsilon \to 0+} \Gamma_\epsilon(s,w) = \Gamma_0(s,w) = (\Gamma_0^{(1)}(s,w), \Gamma_0^{(2)}(s,w))$$

exists, being attained in the L^2-topology for the second coordinate $\Gamma_\epsilon^{(2)}$. The limit is given by the following formulas:

$$\begin{aligned}\Big(\Gamma_0^{(1)}(s,w)\Big)_m = \int \Big[&-\frac{1}{2}\nabla \cdot \Big(D^2P(b)(x-s)^{\otimes 2}\nabla\phi_{c,a}(x)\Big) \\ &- \nabla \cdot \Big(DP(b)(x-s)\nabla w\Big) + \Big(D_a\frac{\partial F}{\partial u}\Big)(a,\phi_{c,a})DV(b)(x-s)\, w \\ &+ \frac{1}{2}\frac{\partial^2 F}{\partial u^2}(a,\phi_{c,a})w^2 + \frac{1}{2}(D_a^2F)(a,\phi_{c,a})\Big(DV(b)(x-s)\Big)^{\otimes 2} \\ &+ \frac{1}{2}(D_aF)(a,\phi_{c,a})\Big(D^2V(b)(x-s)^{\otimes 2}\Big)\Big] D_m\phi_{c,a}(x)\, dx \\ &\qquad m = 1,\ldots,n\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\Gamma_0^{(2)}(s,w) &= -\nabla \cdot (c\nabla w) + \frac{\partial F}{\partial u}\Big(a, \phi_{c,a}(x)\Big)w \\ &+ \mathbb{P}\Big[-\nabla \cdot (DP(b)(x-s)\nabla\phi_{c,a}(x)) + (D_aF)(a,\phi_{c,a}(x))DV(b)(x-s)\Big].\end{aligned} \tag{4.10}$$

Proof. Integration by parts indicates that

$$\int \nabla\cdot(DP(b)s\nabla\phi_{c,a}(x))D_m\phi_{c,a}(x)\,dx = 0 \tag{4.11}$$

even without any radial assumption, and

$$\int (D_aF)(a,\phi_{c,a}(x))(DV(b)s)D_m\phi_{c,a}(x)\,dx = 0 \tag{4.12}$$

as the integrand is an odd function. Using these equations, the assumption $\nabla(M\circ\tilde{V})(b) = 0$ and Lemma 9 we can write $\Gamma_\epsilon^{(1)}(s,w)$ (see (4.6)) as the sum of multiple integrals in order to eliminate negative powers of ϵ. The following expression for $\Gamma_\epsilon^{(1)}(s,w)$ is obtained:

$$\begin{aligned}
&-\int\int_{0<t_1,t_2<1}\nabla\cdot(D^2P(t_2t_1\epsilon(x-s)+b)(x-s)^{\otimes 2}\nabla\phi_{c,a})D_m\phi_{c,a}\,t_1\,dt_1\,dt_2\,dx\\
&-\int\int_0^1\nabla\cdot(DP(t\epsilon(x-s)+b)(x-s)\nabla w)D_m\phi_{c,a}\,dt\,dx\\
&+\int\int_{0<t_1,t_2<1}(D_a^2F)(V(t_2t_1\epsilon(x-s)+b),\phi_{c,a})\big(DV(t_2t_1\epsilon(x-s)+b)(x-s)\big)^{\otimes 2}\\
&\qquad\qquad D_m\phi_{c,a}\,t_1\,dt_1\,dt_2\,dx\\
&+\int\int_{0<t_1,t_2<1}(D_aF)(V(t_2t_1\epsilon(x-s)+b),\phi_{c,a})\big(D^2V(t_2t_1\epsilon(x-s)+b)(x-s)^{\otimes 2}\big)\\
&\qquad\qquad D_m\phi_{c,a}\,t_1\,dt_1\,dt_2\,dx\\
&+\int\int_{0<t_1,t_2<1}\frac{\partial^2F}{\partial u^2}(V(\epsilon(x-s)+b),\phi_{c,a}+t_2t_1\epsilon w)w^2D_m\phi_{c,a}\,t_1\,dt_1\,dt_2\,dx\\
&+\int\int_0^1\left(D_a\frac{\partial F}{\partial u}\right)(V(t\epsilon(x-s)+b),\phi_{c,a})DV(t\epsilon(x-s)+b)(x-s)w\\
&\qquad\qquad D_m\phi_{c,a}\,dt\,dx.
\end{aligned}\tag{4.13}$$

In a similar way we can obtain an expression for $\Gamma_\epsilon^{(2)}(s,w)$ without negative powers of ϵ:

$$\begin{aligned}
\mathbb{P}\Big[&-\nabla\cdot(P(\epsilon(x-s)+b)\nabla w)+\int_0^1\frac{\partial F}{\partial u}(V(\epsilon(x-s)+b),\phi_{c,a}+t\epsilon w)w\,dt\\
&+\int_0^1 D_aF(V(t\epsilon(x-s)+b),\phi_{c,a})(DV(t\epsilon(x-s)+b)(x-s))\,dt\\
&+\int_0^1\nabla\cdot(DP(t\epsilon(x-s)+b)(x-s)\nabla\phi_{c,a}\,dt\Big].
\end{aligned}\tag{4.14}$$

The expressions (4.13) and (4.14) make Lemma 11 (for the most part) a fairly straightforward consequence of Lebesgue's dominated convergence theorem. We break the limit into pieces beginning with fragments of (4.14). As it suffices to prove the required limits before projecting we drop the projection $\mathbb{P}$ in the following arguments.

Firstly

$$\int_0^1 D_a F(V(t\epsilon(x-s)+b), \phi_{c,a})(DV(t\epsilon(x-s)+b)(x-s))\,dt \xrightarrow{L^2} (D_aF)(a,\phi_{c,a})DV(b)(x-s).$$

This is a consequence of Lemma 2. To bound the integrand by an L^2-function of x we must use the exponential decay of $\phi_{c,a}$. Next

$$\int_0^1 \nabla\cdot(DP(t\epsilon(x-s)+b)(x-s)\nabla\phi_{c,a})\,dt \xrightarrow{L^2} \nabla\cdot\big(DP(b)(x-s)\nabla\phi_{c,a}\big).$$

Again this is by Lemma 2 using the exponential decay of $\phi_{c,a}$ and its derivatives. This completes the proof of the limit of (4.7).

We turn to the limit of (4.6) which we consider in the form (4.13). It is clear that the integrands converge pointwise (that is, for each x, t_1, t_2) to the appropriate quantities, so that we obtain (4.9) if we can justify the convergence of the integrals as $\epsilon \to 0$. This is a question of bounding each integrand by an integrable function of x, t_1, t_2, subject to $0 < t_1, t_2 < 1$, and independent of ϵ, subject to $0 < \epsilon < 1$. This follows from the exponential decay of $D_m\phi_{c,a}$ together with the boundedness or polynomial growth of V, P and their derivatives. □

Solution of the limit equation. The analysis of the limit equations (4.10) and (4.9) depends strongly on the radiality of ϕ_a. This implies, in particular, that $\phi_{c,a}$ is an even function and $D_m\phi_{c,a}$ an odd function.

By (4.11) and (4.12) the function

$$-\nabla\cdot(DP(b)s\nabla\phi_{c,a}(x)) + (D_aF)(a,\phi_{c,a}(x))DV(b)s$$

is in the range of the operator $-\nabla\cdot c\nabla + \frac{\partial F}{\partial u}(a,\phi_{c,a})$ for every s. Let $w = \chi(x)$ be the unique solution in W to

$$-\nabla\cdot(c\nabla w) + \frac{\partial F}{\partial u}(a,\phi_{c,a})w - \nabla\cdot(DP(b)s\nabla\phi_{c,a}(x)) + (D_aF)(a,\phi_{c,a}(x))DV(b)s = 0.$$

Although we do not explicitly indicate it in our notation χ depends linearly on the vector $s \in \mathbb{R}^n$. Now from the equation

$$-\nabla \cdot (c\nabla\phi_{c,a}) + F(a, \phi_{c,a}) = 0,$$

writing $c = P(b)$, $a = V(b)$ and differentiating with respect to b in the direction of the vector s we obtain

$$\begin{aligned} &-\nabla \cdot (c\nabla D_b\phi_{P(b),V(b)}s) + \frac{\partial F}{\partial u}(V(b), \phi_{P(b),V(b)})(D_b\phi_{P(b),V(b)}s) \\ &-\nabla \cdot (DP(b)s\nabla\phi_{P(b),V(b)}(x)) + (D_aF)(V(b), \phi_{P(b),V(b)}(x))DV(b)s = 0. \end{aligned} \tag{4.15}$$

Since $D_b\phi_{P(b),V(b)}(x)s$ is an even function, and hence in W, we deduce that

$$\chi = D_b\phi_{P(b),V(b)}s. \tag{4.16}$$

By Lemma 9 and the assumption that b is a critical point of $M \circ \tilde{V}$ we see that there exists a unique solution $w = \eta(x)$ in W to

$$\begin{aligned} -\nabla \cdot (c\nabla w) + \frac{\partial F}{\partial u}\Big(a, \phi_{c,a}(x)\Big)w - \nabla \cdot ((DP(b)x)\nabla\phi_{c,a}(x)) \\ + (D_aF)(a, \phi_a(x))DV(b)x = 0. \end{aligned} \tag{4.17}$$

Now we can solve the first limit equation (4.10) for each s by

$$w = \eta - \chi = \eta - D_b\phi_{P(b),V(b)}s.$$

We note that η is an odd function while $D_b\phi_{P(b),V(b)}s$ is an even function. Next we substitute this formula for w into the second limit equation (4.9) taking into account symmetries of the integrands which force some integrals to be zero. The result is

$$\begin{aligned} &\int \Big[\nabla \cdot (D^2P(b)(x, s)\nabla\phi_{c,a}(x)) + \nabla \cdot (DP(b)s\nabla\eta) \\ &+\nabla \cdot (DP(b)x\nabla\chi) - \Big(D_a\frac{\partial F}{\partial u}\Big)(a, \phi_{c,a})DV(b)s\,\eta - \Big(D_a\frac{\partial F}{\partial u}\Big)(a, \phi_{c,a})DV(b)x\,\chi \\ &- \frac{\partial^2 F}{\partial u^2}(a, \phi_{c,a})\eta\chi - (D_a^2F)(a, \phi_{c,a})(DV(b)x, DV(b)s) \\ &- (D_aF)(a, \phi_{c,a})(D^2V(b)(x, s))\Big] D_m\phi_{c,a}(x)\, dx = 0 \\ &m = 1, \ldots, n. \end{aligned} \tag{4.18}$$

The equation set (4.18) is linear and homogeneous in s (since χ is a linear function of s); that is, it has the form $Ts = 0$ where T is an $n \times n$-matrix.

It therefore has the solution $s = 0$ from which we obtain a solution of the limit equations, $(s, w) = (0, \eta)$.

Lemma 12. *The solution $s = 0$ of (4.18) is non-degenerate. In other words the matrix T is non-singular.*

From this follows:

Lemma 13. *The solution $(s, w) = (0, \eta)$ of (4.10), (4.9) is nondegenerate.*

Lemma 12 is a consequence of the following.

Lemma 14. *The matrix T is, up to a constant multiplier, the Hessian matrix of $M \circ \tilde{V}$.*

Proof. We differentiate the right-hand side of the formula in Lemma 9 (more precisely its mth coordinate) with respect to b in the direction of the vector s. Throughout we must understand $a = V(b)$, $c = P(b)$. It will be seen that we obtain the negative of the expression in (4.18).

We can already account for three terms in (4.18). The result of differentiating $DP(b)$ in the first term of Lemma 9 gives the first term of (4.18) (with change of sign; we won't mention this point from now on). The result of differentiating $DV(b)$ in the second term of Lemma 9 gives the last term of (4.18). The result of differentiating $D_aF(a, \phi_{c,a})$ in the second term of Lemma 9 with respect to b through its first argument a (remembering that $a = V(b)$ so we must use the chain rule) gives the penultimate term of (4.18).

The remaining terms obtained by differentiating the formula of Lemma 9 can all be expressed through the derivative of $\phi_{c,a}$ with respect to b. They consist of

$$\int \Big[-\nabla\cdot((DP(b)x)\nabla D_b\phi_{c,a}s) + \left(D_a\frac{\partial F}{\partial u}\right)(a, \phi_{c,a}(x))(DV(b)x)(D_b\phi_{c,a}s)\Big] D_m\phi_{c,a}(x)\,dx$$
$$+ \int \Big[-\nabla\cdot((DP(b)x)\nabla\phi_{c,a}(x)) + (D_aF)(a, \phi_{c,a}(x))DV(b)x\Big] D_m(D_b\phi_{c,a}s)\,dx$$

and using (4.16) this becomes

$$\int \Big[-\nabla\cdot((DP(b)x)\nabla\chi) + \left(D_a\frac{\partial F}{\partial u}\right)(a, \phi_{c,a}(x))(DV(b)x)\chi\Big] D_m\phi_{c,a}(x)\,dx$$
$$+ \int \Big[-\nabla\cdot((DP(b)x)\nabla\phi_{c,a}(x)) + (D_aF)(a, \phi_{c,a}(x))DV(b)x\Big] D_m\chi\,dx. \tag{4.19}$$

The first two terms here give the third and the fifth terms of (4.18) respectively. We still have to account for the three terms of (4.18) that contain the function η using the second integral of (4.19). Using (4.17) and (4.16), and integrating by parts several times, we can write this second integral as

$$\begin{aligned}
&\int \Big(\nabla\cdot(c\nabla\eta) - \frac{\partial F}{\partial u}(a,\phi_{c,a}(x))\eta\Big) D_m\chi\,dx \\
&= \int \Big[\Big(-\nabla\cdot(c\nabla\chi) + \frac{\partial F}{\partial u}(a,\phi_{c,a}(x))\chi\Big) D_m\eta + \frac{\partial^2 F}{\partial u^2}(a,\phi_{c,a}(x))\eta\chi D_m\phi_{c,a}\Big]\,dx \\
&= \int \Big[\Big(\nabla\cdot(DP(b)s\nabla\phi_{c,a}) - (D_aF)(a,\phi_{c,a})DV(b)s\Big) D_m\eta \\
&\qquad\qquad + \frac{\partial^2 F}{\partial u^2}(a,\phi_{c,a}(x))\eta\chi D_m\phi_{c,a}\Big]\,dx \\
&= \int \Big[-\nabla\cdot(DP(b)s\nabla\eta) + \Big(D_a\frac{\partial F}{\partial u}\Big)(a,\phi_{c,a})(DV(b)s)\eta + \frac{\partial^2 F}{\partial u^2}(a,\phi_{c,a}(x))\eta\chi\Big] \\
&\qquad\qquad D_m\phi_{c,a}\,dx
\end{aligned}$$

yielding the remaining three terms as required. □

The derivative of the rescaled equations. We study the derivative of the rescaled equation evaluated at the limit solution $s = 0$, $w = \eta$. The main objective (reached in the next subsection) is to show that the derivative has an inverse which remains bounded in the operator norm as $\epsilon \to 0$. In this section we study convergence of the derivative as $\epsilon \to 0$.

Lemma 15. *For each $s, \sigma \in \mathbb{R}^n$, $w, v \in W$ the limit*

$$\lim_{\epsilon\to 0} D\Gamma_\epsilon(s,w)(\sigma,v) = D\Gamma_0(s,w)(\sigma,v)$$

holds, uniformly with respect to (s,w) in a bounded set.

The lemma says that $D\Gamma_\epsilon(s,w)$ tends to $D\Gamma_0(s,w)$ in the strong operator topology; that the limit does not hold in the operator-norm is a difficulty that has to be overcome by use of Lemma 6 instead of the usual implicit function theorem at a later stage of the proof.

The first objective is to prove Lemma 15 and an accompanying lemma on weak convergence, so let us set out the derivative

$$D\Gamma_\epsilon(s,w) = (D\Gamma_\epsilon^{(1)}(s,w), D\Gamma_\epsilon^{(2)}(s,w))$$

as a linear map from $\mathbb{R}^n \times W$ to $\mathbb{R}^n \times W_0$ acting on (σ, v)

$$\begin{aligned}
&\Big(D\Gamma_\epsilon^{(1)}(s,w)(\sigma,v)\Big)_m \\
&\quad = \epsilon^{-1}\int\Big[\nabla\cdot((DP(\epsilon(x-\xi))\sigma)\nabla\phi_{c,a}) + \epsilon\nabla\cdot((DP(\epsilon(x-\xi))\sigma)\nabla w) \\
&- \nabla\cdot(P(\epsilon(x-\xi))-c)\nabla v) - D_aF(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)DV(\epsilon(x-\xi))\sigma \\
&\quad + \frac{\partial F}{\partial u}(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)v - \frac{\partial F}{\partial u}(a,\phi_{c,a})v\Big]D_m\phi_{c,a}(x)\,dx \qquad (4.20)
\end{aligned}$$

$$\begin{aligned}
&D\Gamma_\epsilon^{(2)}(s,w)(\sigma,v) \\
&\quad = \mathbb{P}\Big[-\nabla\cdot(c\nabla v) - D_aF(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)DV(\epsilon(x-\xi))\sigma \\
&\qquad + \frac{\partial F}{\partial u}(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)v + \nabla\cdot(DP(\epsilon(x-\xi))\sigma\nabla\phi_{c,a}) \\
&\qquad + \epsilon\nabla\cdot(DP(\epsilon(x-\xi))\sigma\nabla w) - \nabla\cdot((P(\epsilon(x-\xi))-c)\nabla v)\Big]. \qquad (4.21)
\end{aligned}$$

Lemma 15 asserts that this has the strong limit, uniformly for (s,w) in a bounded set,

$$\begin{aligned}
&\Big(D\Gamma_0^{(1)}(s,w)(\sigma,v)\Big)_m = \int\Big[\nabla\cdot(D^2P(b)(x-s,\sigma)\nabla\phi_{c,a}) + \nabla\cdot(DP(b)\sigma\nabla w) \\
&- \nabla\cdot(DP(b)(x-s)\nabla v) - D_a^2F(a,\phi_{c,a})(DV(b)(x-s),DV(b)\sigma) \\
&- D_a\frac{\partial F}{\partial u}(a,\phi_{c,a})DV(b)\sigma w - D_aF(a,\phi_{c,a})D^2V(b)(x-s,\sigma) \qquad (4.22) \\
&+ D_a\frac{\partial F}{\partial u}(a,\phi_{c,a})DV(b)(x-s)v + \frac{\partial^2F}{\partial u^2}(a,\phi_{c,a})wv\Big]D_m\phi_{c,a}(x)\,dx,
\end{aligned}$$

$$\begin{aligned}
D\Gamma_0^{(2)}(s,w)(\sigma,v) &= -\nabla\cdot(c\nabla v) + \frac{\partial F}{\partial u}(a,\phi_{c,a})v \\
&\quad + \mathbb{P}\Big[-D_aF(a,\phi_{c,a})DV(b)\sigma + \nabla\cdot(DP(b)\sigma\nabla\phi_{c,a})\Big]. \qquad (4.23)
\end{aligned}$$

Now we prove Lemma 15 by a close examination of the individual terms. We begin with (4.21), breaking it into pieces and dropping the projection. Firstly

$$\frac{\partial F}{\partial u}(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)v \xrightarrow{L^2} \frac{\partial F}{\partial u}(a,\phi_{c,a})v.$$

We break this into two limits

$$\frac{\partial F}{\partial u}(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)v-\frac{\partial F}{\partial u}(V(\epsilon(x-\xi)),\phi_{c,a})v\xrightarrow{L^2}0$$

$$\frac{\partial F}{\partial u}(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)v-\frac{\partial F}{\partial u}(a,\phi_{c,a})v\xrightarrow{L^2}0.$$

These are dealt with by writing each left-hand side as an integral over $0<t<1$ and applying Lemma 2, using the growth conditions on F to bound the integrands. Next

$$D_aF(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)DV(\epsilon(x-\xi))\sigma\xrightarrow{L^2}D_aF(a,\phi_{c,a})DV(b)\sigma.$$

This we break into three limits

$$\Big(D_aF(V(\epsilon(x-\xi)),\phi_{c,a}+\epsilon w)-D_aF(V(\epsilon(x-\xi)),\phi_{c,a})\Big)DV(\epsilon(x-\xi))\sigma \xrightarrow{L^2}0$$

$$(D_aF(V(\epsilon(x-\xi)),\phi_{c,a})-D_aF(a,\phi_{c,a}))DV(\epsilon(x-\xi))\sigma\xrightarrow{L^2}0$$

$$(D_aF(V(a,\phi_{c,a})(DV(\epsilon(x-\xi))-DV(b))\sigma\xrightarrow{L^2}0.$$

These are handled in the same way, using Lemma 2. For the third limit we must use exponential decay of $\phi_{c,a}$ to quell factors with polynomial growth. Next

$$\epsilon\nabla\cdot(DP(\epsilon(x-\xi))\sigma\nabla w)\xrightarrow{L^2}0.$$

This is obvious bearing in mind the boundedness of DP and D^2P. Next

$$\nabla\cdot(P(\epsilon(x-\xi))-c)\nabla v)\xrightarrow{L^2}0.$$

This follows from Lemma 1, and similarly for

$$\nabla\cdot(DP(\epsilon(x-\xi))\nabla\sigma\phi_{c,a})\xrightarrow{L^2}\nabla\cdot(DP(b)\nabla\sigma\phi_{c,a}).$$

It remains to prove that (4.20) converges to (4.22). Using equations (4.11) and (4.12) we can write (4.20) as a sum of eight integrals which we rewrite as multiple integrals with the purpose of eliminating negative powers of ϵ:

$$\int\int_0^1\nabla\cdot(D^2P(t\epsilon(x-s)+b)(x-s,\sigma)\nabla\phi_{c,a})D_m\phi_{c,a}\,dt\,dx$$

$$+\int\nabla\cdot(DP(\epsilon(x-s)+b)\sigma\nabla w)D_m\phi_{c,a}\,dx$$

$$-\int\int_0^1\nabla\cdot(DP(t\epsilon(x-s)+b)(x-s)\nabla v)D_m\phi_{c,a}\,dt\,dx$$

$$
\begin{aligned}
&-\int\int_0^1 \Big(D_a\frac{\partial F}{\partial u}\Big)(V(\epsilon(x-s)+b),\phi_{c,a}+t\epsilon w)DV(\epsilon(x-s)+b)\sigma w D_m\phi_{c,a}\,dt\,dx\\
&-\int\int_0^1 D_a^2F(V(t\epsilon(x-s)+b),\phi_{c,a})(DV(t\epsilon(x-s)+b)(x-s),DV(\epsilon(x-s)+b)\sigma)\\
&\qquad\qquad D_m\phi_{c,a}\,dt\,dx\\
&-\int\int_0^1 D_aF(a,\phi_{c,a})D^2V(t\epsilon(x-s)+b)(x-s,\sigma)D_m\phi_{c,a}\,dt\,dx\\
&+\int\int_0^1 \frac{\partial^2 F}{\partial u^2}(V(\epsilon(x-s)+b),\phi_{c,a}+t\epsilon w)wvD_m\phi_{c,a}\,dt\,dx\\
&+\int\int_0^1 \Big(D_a\frac{\partial F}{\partial u}\Big)(V(t\epsilon(x-s)+b),\phi_{c,a})DV(t\epsilon(x-s)+b)(x-s)v\\
&\qquad\qquad D_m\phi_{c,a}\,dt\,dx. \qquad (4.24)
\end{aligned}
$$

This expression converges, integral by integral, to the following eight-integral expression constituting (4.22):

$$
\begin{aligned}
&\int \nabla\cdot(D^2P(b)(x-s,\sigma)\nabla\phi_{c,a}D_m\phi_{c,a}(x)\,dx\\
&+\int \nabla\cdot(DP(b)\sigma\nabla w)D_m\phi_{c,a}(x)\,dx\\
&-\int \nabla\cdot(DP(b)(x-s)\nabla v)D_m\phi_{c,a}(x)\,dx\\
&-\int \Big(D_a\frac{\partial F}{\partial u}\Big)(a,\phi_{c,a})DV(b)\sigma wD_m\phi_{c,a}(x)\,dx\\
&-\int D_a^2F(a,\phi_{c,a})(DV(b)(x-s),DV(b)\sigma)D_m\phi_{c,a}(x)\,dx\\
&-\int D_aF(a,\phi_{c,a})D^2V(b)(x-s,\sigma)D_m\phi_{c,a}(x)\,dx\\
&+\int \frac{\partial^2F}{\partial u^2}(a,\phi_{c,a})wvD_m\phi_{c,a}(x)\,dx\\
&+\int \Big(D_a\frac{\partial F}{\partial u}\Big)(a,\phi_{c,a})DV(b)(x-s)vD_m\phi_{c,a}(x)\,dx. \qquad (4.25)
\end{aligned}
$$

Convergence of the eight integrals as $\epsilon \to 0+$ is guaranteed by Lebesgue's dominated convergence theorem, using the exponential decay of $\phi_{c,a}$ and its derivatives, the boundedness of DV and the polynomial boundedness of D^2V. These limits are also consequences of the next lemma.

Lemma 16. *Let (σ_ν, v_ν) be a bounded sequence in $\mathbb{R}^n \times W$ and let $\epsilon_\nu \to 0$. Then*

$$D\Gamma^{(1)}_{\epsilon_\nu}(s,w)(\sigma_\nu, v_\nu) - D\Gamma^{(1)}_0(s,w)(\sigma_\nu, v_\nu) \to 0 \tag{4.26}$$

in the usual topology of $\mathbb{R}^n$, and

$$D\Gamma^{(2)}_{\epsilon_\nu}(s,w)(\sigma_\nu, v_\nu) - D\Gamma^{(2)}_0(s,w)(\sigma_\nu, v_\nu) \to 0 \tag{4.27}$$

in the weak-H^2 topology on W_0.

Note that in the weak-H^2 topology convergence of a sequence w_ν to w means that $\int w_\nu \chi \to \int w\chi$ for all $\chi \in H^2$.

Obviously (4.26) implies that (4.20) converges to (4.22). To obtain (4.26) we must study the result of subtracting (4.25) from (4.24), integral by integral, with ϵ_ν instead ϵ, σ_ν instead of σ and v_ν instead of v. Those integrals containing σ can be handled by the dominated convergence theorem (since σ is a finite-dimensional vector we could replace it by a basis vector e_i). There remain the third, seventh and eighth integrals.

Consider first the difference for the third integrals

$$\begin{aligned}
&\epsilon_\nu^{-1}\int \nabla\cdot(P(\epsilon_\nu(x-s)+b)-c)\nabla v_\nu) D_m\phi_{c,a}\,dx \\
&\qquad\qquad - \int \Big(\nabla\cdot(DP(b)(x-s)\nabla v_\nu\Big) D_m\phi_{c,a}(x)\,dx \\
&= \int\int_0^1 \Big(\nabla\cdot\big(DP(t\epsilon_\nu(x-s)+b)-DP(b)\big)(x-s)\nabla v_\nu\Big) D_m\phi_{c,a}\,dt\,dx \\
&= \sum_{i,j,k}\Bigg[\int\int_0^1 \Big(\frac{\partial P_{ij}}{\partial b_k}(t\epsilon_\nu(x-s)+b) - \frac{\partial P_{ij}}{\partial b_k}(b)\Big)(x_k-s_k)D_iD_jv_\nu D_m\phi_{c,a}\,dt\,dx \\
&\qquad + \int\int_0^1 \Big(\frac{\partial P_{ij}}{\partial b_i}(t\epsilon_\nu(x-s)+b) - \frac{\partial P_{ij}}{\partial b_i}(b)\Big)D_jv_\nu D_m\phi_{c,a}\,dt\,dx \\
&\quad + \epsilon_\nu\int\int_0^1 \frac{\partial^2 P_{ij}}{\partial b_i\partial b_k}\Big(t\epsilon_\nu(x-s)+b\Big)(x_k-s_k)D_jv_\nu D_m\phi_{c,a}\,t\,dt\,dx\Bigg].
\end{aligned}\tag{4.28}$$

The absolute value of the (i,j,k)-term in this sum is bounded by

$$||v_\nu||_{H^2}(||f_{1,\nu}||_{L^2} + ||f_{2,\nu}||_{L^2}),$$

where

$$f_{1,\nu}(x) = \int_0^1 \Big(\frac{\partial P_{ij}}{\partial b_k}(t\epsilon_\nu(x-s)+b) - \frac{\partial P_{ij}}{\partial b_k}(b)\Big)(x_k-s_k)D_m\phi_{c,a}\,dt$$

and

$$f_{2,\nu}(x) = \int_0^1 \Big(\frac{\partial P_{ij}}{\partial b_i}\big(t\epsilon_\nu(x-s)+b\big) - \frac{\partial P_{ij}}{\partial b_i}(b) + \epsilon_\nu t \frac{\partial^2 P_{ij}}{\partial b_i \partial b_k}\big(t\epsilon_\nu(x-s)+b\big)(x_k - s_k)\Big) D_m\phi_{c,a}\, dt.$$

Both integrals tend to 0 in L^2 by Lemma 2, in virtue of the boundedness and continuity of DP and D^2P, and the exponential decay of $D_m\phi_{c,a}$. Since v_ν is bounded in H^2 we conclude that (4.28) tends to 0.

Consider next the difference for the seventh integrals:

$$\int\int_0^1 \Big[\frac{\partial^2 F}{\partial u^2}(V(\epsilon_\nu(x-s)+b), \phi_{c,a}+t\epsilon_\nu w) - \frac{\partial^2 F}{\partial u^2}(a,\phi_{c,a})\Big] w v_\nu D_m\phi_{c,a}\, dt\, dx.$$

The absolute value of this is bounded by $\|v_\nu\|_{L^2}\|g_\nu\|_{L^2}$, where

$$g_\nu(x) = \int_0^1 \Big[\frac{\partial^2 F}{\partial u^2}(V(\epsilon_\nu(x-s)+b), \phi_{c,a}+t\epsilon_\nu w) - \frac{\partial^2 F}{\partial u^2}(a,\phi_{c,a})\Big] w D_m\phi_{c,a}\, dt.$$

This tends to 0 in L^2 by Lemma 2.

Finally, the difference for the eighth integrals:

$$\int\int_0^1 \Big[\Big(D_a\frac{\partial F}{\partial u}\Big)(V(t\epsilon_\nu(x-s)+b), \phi_{c,a}+t\epsilon_\nu w) DV(t\epsilon_\nu(x-s)+b) - \Big(D_a\frac{\partial F}{\partial u}\Big)(a,\phi_{c,a})DV(b)\Big](x-s)v_\nu D_m\phi_{c,a}\, dt\, dx.$$

The absolute value of this is bounded by $\|v_\nu\|_{L^2}\|h_\nu\|_{L^2}$, where

$$h_\nu(x) = \int_0^1 \Big[\Big(D_a\frac{\partial F}{\partial u}\Big)(V(t\epsilon_\nu(x-s)+b), \phi_{c,a}+t\epsilon_\nu w) DV(t\epsilon_\nu(x-s)+b) - \Big(D_a\frac{\partial F}{\partial u}\Big)(a,\phi_{c,a})DV(b)\Big](x-s) D_m\phi_{c,a}\, dt.$$

Once again we invoke Lemma 2 using exponential decay to suppress polynomial terms.

Next we consider (4.27). From (4.21) and (4.23) we first consider the terms linear in σ. Here we may replace σ by a basis vector e_i; then the said terms from (4.21) converge to the corresponding ones of (4.23) in L^2. Finally, for the terms linear in v, consider first

$$\mathbb{P}\Big[\frac{\partial F}{\partial u}\Big(V(\epsilon_\nu(x-s)+b), \phi_{c,a}+\epsilon_\nu w\Big)v_\nu - \frac{\partial F}{\partial u}(a,\phi_{c,a})v_\nu\Big].$$

To show that this converges to 0 in the weak-H^2 topology on L^2 it suffices to do the same without the projection $\mathbb{P}$. But this was covered in [6, Lemma 3.4].

Lastly we consider

$$\mathbb{P}\Big[\nabla\cdot(P(\epsilon_\nu(x-s)+b)-c)\nabla v_\nu\Big]$$

with a view to showing it has the limit 0 in the weak-H^2 topology. It suffices to drop the projection and show that

$$\int\big(\nabla\cdot(P(\epsilon_\nu(x-s)+b)-c)\nabla v_\nu\big)\chi\,dx\to 0$$

for any given $\chi\in H^2$. We have

$$\begin{aligned}\Big|\int\big(\nabla\cdot(P(\epsilon_\nu(x-s)+b)-c)\nabla v_\nu\big)\chi\,dx\Big|&\\ &=\Big|\int\big(\nabla\cdot(P(\epsilon_\nu(x-s)+b)-c)\nabla\chi\big)v_\nu\,dx\Big|\\ &\leq\|\nabla\cdot(P(\epsilon_\nu(x-s)+b)-c)\nabla\chi\|_{L^2}\,\|v_\nu\|_{L^2}\\ &\leq\text{const.}\,\|\nabla\cdot(P(\epsilon_\nu(x-s)+b)-c)\nabla\chi\|_{L^2}\end{aligned}$$

and this converges to 0 by Lemma 1.

Bounding the inverse of the derivative.

Lemma 17. *For all sufficiently small ϵ the inverse $D\Gamma_\epsilon(0,\eta)^{-1}$ exists and there exists a constant C such that $||D\Gamma_\epsilon(0,\eta)^{-1}||<C$.*

Proof. By [6, Lemma 3.5] and the addition and subtraction of finite-rank operators we find that the linear mapping $D\Gamma_\epsilon(s,w)$ is a compact perturbation of the operator

$$(\sigma,v)\mapsto\bigg(\Big(\int(Lv)\,D_m\phi_{c,a}\,dx\Big)_{m=1}^{n},\mathbb{P}(Lv)\bigg),\tag{4.29}$$

where

$$L:H^2\to L^2;\quad Lv=-\nabla\cdot\big(P(\epsilon(x-s)+b)\nabla v\big)+\frac{\partial F}{\partial u}\big(V(\epsilon(x-s)+b),0\big)v$$

and the latter being invertible by the positivity assumption we conclude that (4.29) is invertible and hence that $D\Gamma_\epsilon(s,w)$ is a Fredholm operator of index zero. It is therefore enough to show that a contradiction ensues from the assumption that there exist sequences $\epsilon_\nu\to 0$, $\sigma_\nu\in\mathbb{R}^n$, $v_\nu\in W$ such that $|\sigma_\nu|+||v_\nu||_{H^2}=1$ and $D\Gamma_{\epsilon_\nu}(0,\eta)(\sigma_\nu,v_\nu)\to 0$ in L^2. Assuming the existence of these sequences we may assume that $\sigma_\nu \to \sigma_0$ and $v_\nu \to v_0$ weakly in H^2. Then $v_0 \in W$. We then have the following pairs of limits, respectively by assumption, by weak continuity and by Lemma 16:

$$D\Gamma^{(1)}_{\epsilon_\nu}(0,\eta)(\sigma_\nu, v_\nu) \to 0,$$
$$D\Gamma^{(2)}_{\epsilon_\nu}(0,\eta)(\sigma_\nu, v_\nu) \xrightarrow{L^2} 0,$$

$$D\Gamma^{(1)}_{0}(0,\eta)(\sigma_\nu, v_\nu) - D\Gamma^{(1)}_{0}(0,\eta)(\sigma_0, v_0) \to 0,$$
$$D\Gamma^{(2)}_{0}(0,\eta)(\sigma_\nu, v_\nu) - D\Gamma^{(2)}_{0}(0,\eta)(\sigma_0, v_0) \xrightarrow{\text{w. in } L^2} 0,$$

$$(D\Gamma^{(1)}_{\epsilon_\nu}(0,\eta) - D\Gamma^{(1)}_{0}(0,\eta))(\sigma_\nu, v_\nu) \to 0,$$
$$(D\Gamma^{(2)}_{\epsilon_\nu}(0,\eta) - D\Gamma^{(2)}_{0}(0,\eta))(\sigma_\nu, v_\nu) \xrightarrow{\text{w.-}H^2 \text{ in } L^2} 0.$$

The three topologies on L^2 are comparable so we deduce

$$D\Gamma_0(0,\eta)(\sigma_0, v_0) = 0,$$

which implies $\sigma_0 = 0$ and $v_0 = 0$ by Lemma 13. Now from (4.21) and [6, Lemma 3.5(i)] we deduce

$$\mathbb{P}\Big[-\nabla\cdot(P(\epsilon_\nu x + b)\nabla v_\nu) + \frac{\partial F}{\partial u}(V(\epsilon_\nu x + b), \phi_{c,a})v_\nu\Big] \to 0 \tag{4.30}$$

in L^2. But we also have, for $m = 1, \ldots, n$, using an argument very similar to that used to treat the limit of (4.28),

$$\int \Big[-\nabla\cdot(P(\epsilon_\nu x + b)\nabla v_\nu) + \frac{\partial F}{\partial u}(V(\epsilon_\nu x + b), \phi_{c,a})v_\nu\Big] D_m\phi_{c,a}\, dx$$
$$- \int \Big[-\nabla\cdot(c\nabla v_\nu) + \frac{\partial F}{\partial u}(a, \phi_{c,a})v_\nu\Big] D_m\phi_{c,a}\, dx \to 0,$$

but since

$$\int \Big[-\nabla\cdot(c\nabla v_\nu) + \frac{\partial F}{\partial u}(a, \phi_{c,a})v_\nu\Big] D_m\phi_{c,a}\, dx$$
$$= \int \Big[-\nabla\cdot(c\nabla D_m\phi_{c,a}) + \frac{\partial F}{\partial u}(a, \phi_{c,a})D_m\phi_{c,a}\Big] v_\nu\, dx = 0$$

we see that

$$\int \Big[-\nabla \cdot (P(\epsilon_\nu x + b)\nabla v_\nu) + \frac{\partial F}{\partial u}(V(\epsilon_\nu x + b), \phi_{c,a})v_\nu \Big] D_m \phi_{c,a}\, dx \to 0. \quad (4.31)$$

Now from (4.30) and (4.31) there results

$$-\nabla \cdot (P(\epsilon_\nu x + b)\nabla v_\nu) + \frac{\partial F}{\partial u}(V(\epsilon_\nu x + b), \phi_{c,a})v_\nu \to 0$$

and another application of [6, Lemma 3.5(i)] gives

$$-\nabla \cdot (P(\epsilon_\nu x + b)\nabla v_\nu) + \frac{\partial F}{\partial u}(V(\epsilon_\nu x + b), 0)v_\nu \to 0.$$

Finally, we deduce $||v_\nu||_{H^2} \to 0$ by Lemma 4, and, together with $\sigma_\nu \to 0$, this contradicts the assumption $|\sigma_\nu| + \|v_\nu\|_{H^2} = 1$. □

Conclusion of the proof of Theorem 7. We apply Lemma 6 with $f(\epsilon, \cdot) = \Gamma_\epsilon$, $x_0 = (0, \eta)$. The requisite parts (bar one) are in place: namely Lemmas 10, 11, 12 and 17 imply the conditions 1–5 of Lemma 6.

It remains to verify condition 6. For this it is sufficient to show that the second derivative $D_x^2 f(\epsilon, x)$ is uniformly bounded in norm as $\epsilon \to 0+$ and for x in a bounded set. This can be read directly from the expressions (4.20) and (4.21) for $D\Gamma_\epsilon(s, w)$, noting that the second differentiation eliminates all negative powers of ϵ and using the growth conditions on the derivatives of F, P and V.

To obtain continuity of the solution branch u_ϵ with respect to ϵ we must first consider the continuity of $\Gamma_\epsilon(s, w)$ with respect to ϵ. This is obvious from the expressions (4.6) and (4.7) (more precisely it follows from Lebesgue's dominated convergence theorem). Last of all we consider the continuity of $D\Gamma_\epsilon$ with respect to ϵ in the strong operator topology. This follows by the same arguments from (4.20) and (4.21).

This concludes the proof of Theorem 7.

5. Examples

The principal example of a ground state having the properties we have set out is provided by the equation

$$-\nabla \cdot c\nabla u + a_1 u - a_2 u^p = 0, \quad a = (a_1, a_2) \in \mathbb{R}^2, \quad c \in M_s^{n\times n}, \qquad (5.1)$$

where $p < (n+2)/(n-2)$. It will be seen that explicit knowledge of $\phi_{c,a}$ is not needed to compute critical points of $M \circ \tilde{V}$.

The ground state solution of (5.1) is given by

$$\phi_{c,a}(x) = \left(\frac{a_1}{a_2}\right)^{\frac{1}{p-1}} \psi(a_1^{1/2} c^{-1/2} x),$$

where ψ is the ground state of

$$-\Delta u + u - u^p = 0. \tag{5.2}$$

If we restrict p to be an integer greater than 1 (to obtain a differentiable non-linearity) we have the following cases all of which satisfy the growth conditions of Section 2:

$n = 1, 2$	no restriction on p
$n = 3$	$p = 2, 3, 4$
$n = 4$	$p = 2$
$n = 5$	$p = 2$.

A brief calculation gives

$$M(c, a) = K(\det c)^{\frac{1}{2}} a_1^{d_1} a_2^{d_2},$$

where

$$K = \int |\nabla\psi(x)|^2\, dx, \quad d_1 = \frac{p+1}{p-1} - \frac{n}{2}, \quad d_2 = \frac{2}{1-p}.$$

The problem

$$-\nabla \cdot P(\epsilon x)\nabla u + V_1(\epsilon x)u - V_2(\epsilon x)u^p = 0 \tag{5.3}$$

for small ϵ therefore leads to the search for non-degenerate critical points of

$$K^{-1}M(P(b), V_1(b), V_2(b)) = (\det P(b))^{\frac{1}{2}} V_1(b)^{d_1} V_2(b)^{d_2}$$

or equivalently of the function

$$\log K^{-1}M(P(b), V_1(b), V_2(b)) = \frac{1}{2}\log\det P(b) + d_1 \log V_1(b) + d_2 \log V_2(b)$$

giving rise to the equations

$$\frac{1}{2}\mathrm{tr}\Big(P(b)^{-1}\frac{\partial P}{\partial b_k}\Big) + \frac{d_1}{V_1(b)}\frac{\partial V_1}{\partial b_k} + \frac{d_2}{V_2(b)}\frac{\partial V_2}{\partial b_k} = 0, \quad k = 1, \ldots, n.$$

The positivity assumptions amount to $P(b) > hI$, $V_1(b) > h$ where $h > 0$ but we also require $V_2(b) > 0$ to avoid fractional powers of negative numbers.

6. Positivity

The object of this section is to prove Theorem 8. Many of the convergence arguments required in the proof are similar to those deployed in Section 4. We will therefore not always give full details of them.

Let

$$G(a,u)=\frac{F(a,u)}{u}.$$

Then $u=\phi_{c,a}(x)$ satisfies the linear equation

$$-\nabla\cdot c\nabla u+G(a,\phi_{c,a}(x))u=0.$$

Since $G(a,0)=\frac{\partial F}{\partial u}(a,0)$ the operator $-\nabla\cdot c\nabla+G(a,\phi_{c,a}(x))$ is a compact perturbation of the operator $-\nabla\cdot c\nabla+\frac{\partial F}{\partial u}(a,0)$. As c is positive definite and $\frac{\partial F}{\partial u}(a,0)>h>0$ the spectrum of the latter operator is in the interval $[\delta,\infty[$, for some $\delta>0$, so that any spectrum that $-\nabla\cdot c\nabla+G(a,\phi_{c,a}(x))$ possesses in $]-\infty,\delta[$, and that includes the known eigenvalue 0, can consist only of eigenvalues of finite multiplicity. As $\phi_{c,a}$ is positive we have the following conclusion by standard theory of the linear Schrödinger equation.

Lemma 18. *0 is a simple eigenvalue of $-\nabla\cdot c\nabla+G(a,\phi_{c,a}(x))$ and is moreover its lowest eigenvalue and the infimum of its spectrum.*

Just as $\phi_{c,a}$ is an eigenfunction of a linear Schrödinger-type operator, so also is u_ϵ, the solution produced by Theorem 7.

Lemma 19. *Let b be a non-degenerate critical point of $M\circ\tilde V$ and let u_ϵ be the solutions given by Theorem 7. Then if $\epsilon>0$ is sufficiently small 0 is a simple eigenvalue and the lowest eigenvalue of the operator*

$$T_\epsilon v=-\nabla\cdot(P(\epsilon x)\nabla v)+G(V(\epsilon x),u_\epsilon(x))v.$$

Proof. Let Z_ϵ be the subspace

$$Z_\epsilon=\left\{v\in H^2:\int vu_\epsilon=0\right\}.$$

Define $S_\epsilon:\mathbb{R}\times Z_\epsilon\to L^2$ by

$$S_\epsilon(\sigma,z)=\sigma u_\epsilon+T_\epsilon z.$$

Step 1 : Show that S_ϵ is invertible and $||S_\epsilon^{-1}||$ bounded as $\epsilon\to 0$. The operator T_ϵ is a compact perturbation of the operator

$$T_\epsilon^{(\infty)}v=-\nabla\cdot(P(\epsilon x)\nabla v)+\frac{\partial F}{\partial u}(V(\epsilon x),0)v.$$

The latter is positive and therefore invertible. Hence T_ϵ is a Fredholm operator of index 0. It is enough therefore to show that, if $\epsilon_\nu > 0$, $(\sigma_\nu, z_\nu) \in \mathbb{R} \times Z_{\epsilon_\nu}$ are sequences such that $\epsilon_\nu \to 0+$, $|\sigma_\nu| + ||z_\nu||_{H^2} \leq 1$ and $S_{\epsilon_\nu}(\sigma_\nu, z_\nu) \to 0$ in L^2, then a subsequence of (σ_ν, z_ν) converges to 0 in $\mathbb{R} \times H^2$. Let ϵ_ν, (σ_ν, z_ν) be such sequences. We may replace them by subsequences and therefore assume that $\sigma_\nu \to \sigma_0$ and $z_\nu(\cdot - \xi_{\epsilon_\nu}) \to z_0$ weakly in H^2. Here we have

$$u_\epsilon(x) = \phi_{c,a}(x + \xi_\epsilon) + \epsilon w_\epsilon(x + \xi_\epsilon),$$

where

$$\xi_\epsilon = -\frac{b}{\epsilon} + s_\epsilon.$$

As a consequence we have $\int z_0 \phi_{c,a} = 0$. Now from

$$-\nabla \cdot P(\epsilon_\nu x)\nabla z_\nu + G(V(\epsilon_\nu x), u_{\epsilon_\nu}) z_\nu + \sigma_\nu u_{\epsilon_\nu} \stackrel{L^2}{\to} 0$$

we deduce

$$\begin{aligned}-\nabla \cdot P(\epsilon_\nu(x - \xi_{\epsilon_\nu}))\nabla z_\nu(x - \xi_{\epsilon_\nu})) + G(V(\epsilon_\nu(x - \xi_{\epsilon_\nu}), u_{\epsilon_\nu}(x - \xi_{\epsilon_\nu})) z_\nu(x - \xi_{\epsilon_\nu}) \\ + \sigma_\nu u_{\epsilon_\nu}(x - \xi_{\epsilon_\nu}) \stackrel{L^2}{\to} 0\end{aligned}$$

and going to the limit gives

$$-\nabla \cdot c\nabla z_0 + G(a, \phi_{c,a}) z_0 + \sigma_0 \phi_{c,a} = 0.$$

Now from $\int z_0 \phi_{c,a} = 0$ and Lemma 18 we have $\sigma_0 = 0$ and $z_0 = 0$. We deduce by the compact perturbation argument that

$$-\nabla \cdot P(\epsilon_\nu x)\nabla z_\nu + G(V(\epsilon_\nu x), 0) z_\nu \stackrel{L^2}{\to} 0$$

which implies, by Lemma 4, $||z_\nu||_{H^2} \to 0$.

Step 2 : *Show that there exists* $\rho_0 > 0$*, such that, for all sufficiently small* $\epsilon > 0$*, the only eigenvalue of* T_ϵ *in the interval* $[-\rho_0, \rho_0]$ *is* 0*, and that this eigenvalue is simple.* Applying step 1 we can choose $\rho_0 > 0$ so that the operator

$$(\sigma, z) \mapsto S_\epsilon(\sigma, z) - \lambda z$$

is invertible from H^2 to L^2 for $|\lambda| < \rho_0$ and sufficiently small ϵ. Suppose that λ is an eigenvalue with $|\lambda| < \rho_0$ and let $v = \sigma u_\epsilon + z$ be an eigenfunction. Then $T_\epsilon z = \lambda(\sigma u_\epsilon + z)$ which implies $S_\epsilon(-\lambda\sigma, z) = \lambda z$ so that $\lambda\sigma = 0$ and $z = 0$. Hence $\lambda = 0$ and v is a multiple of u_ϵ.

Step 3 : *Completion of the proof of Lemma* 19. By step 2 if T_ϵ has a negative eigenvalue for small ϵ it must lie in the interval $(-\infty, -\rho_0]$. Suppose such eigenvalues exist for a sequence $\epsilon_\nu \to 0$, denote the eigenvalue by λ_ν and let $v_\nu = \sigma_\nu u_{\epsilon_\nu} + z_\nu$ be an eigenfunction, where $|\sigma_\nu| + ||z_\nu||_{H^2} = 1$ and $z_\nu \in Z_{\epsilon_\nu}$. We may assume that $\lambda_\nu \to \lambda_\infty$ (where possibly $\lambda_\infty = -\infty$), that $\sigma_\nu \to \sigma_\infty$ and that $z_\nu(\cdot - \xi_{\epsilon_\nu}) \to z_\infty$ weakly in H^2. Note that $\int z_\infty \phi_{c,a} = 0$. Then we have

$$-\nabla \cdot (P(\epsilon_\nu x)\nabla z_\nu) + G(V(\epsilon_\nu x), u_{\epsilon_\nu}(x))z_\nu - \lambda_\nu(\sigma_\nu u_{\epsilon_\nu} + z_\nu) = 0.$$

Replacing x by $x - \xi_{\epsilon_\nu}$ and going to the limit we find

$$-\nabla \cdot c\nabla z_\infty + G(a, \phi_{c,a})z_\infty - \lambda_\infty(\sigma_\infty \phi_{c,a} + z_\infty) = 0.$$

It follows that $\lambda_\infty \neq -\infty$ and

$$0 = \int (\lambda_\infty(\sigma_\infty \phi_{c,a} + z_\infty))\phi_{c,a}\, dx = \lambda_\infty \sigma_\infty \int \phi_{c,a}^2\, dx$$

so that $\lambda_\infty \sigma_\infty = 0$. But step 2 tells us that $\lambda_\infty < 0$ so that $\sigma_\infty = 0$. We therefore find

$$-\nabla \cdot c\nabla z_\infty + G(a, \phi_{c,a})z_\infty = \lambda_\infty z_\infty$$

and since $\lambda_\infty < 0$ Lemma 18 gives $z_\infty = 0$. The compact perturbation argument gives

$$-\nabla \cdot (P(\epsilon_\nu x)\nabla z_\nu) + G(V(\epsilon_\nu x), 0)z_\nu - \lambda_\nu z_\nu \xrightarrow{L^2} 0$$

so that Lemma 4 returns the contradiction $||z_\nu||_{H^2} \to 0$ (recall we already have $\sigma_\nu \to 0$). □

Completion of the proof of Theorem 8. We can conclude that $u_\epsilon > 0$ under conditions that guarantee that the lowest eigenvalue of T_ϵ has a strictly positive eigenfunction. A straightforward adaptation of the proof that the lowest eigenvalue of the linear Schrödinger operator has a strictly positive eigenfunction (for a very readable version see [9]) shows that two things need to be established about T_ϵ to obtain the desired conclusion:

1. 0 is below the essential spectrum of T_ϵ.
2. The function $G(V(\epsilon x), u_\epsilon(x))$ is bounded.

The first follows from the fact that T_ϵ is a compact perturbation of the positive operator

$$T_\epsilon^{(\infty)} v = -\nabla \cdot (P(\epsilon x)\nabla v) + \frac{\partial F}{\partial u}(V(\epsilon x), 0)v.$$

The second follows from the fact that u_ϵ is bounded. To obtain this modicum of regularity we proceed as follows.

Firstly in the case $n < 4$ we have $u_\epsilon \in H^2 \subset C_0$ (the space of continuous functions that vanish at infinity) by the Sobolev embedding.

Secondly in the case $n = 4$ we have $u_\epsilon \in H^2 \subset L^r$ for $2 \le r < \infty$. Let L_ϵ be the operator $L_\epsilon v = \nabla \cdot P(\epsilon x)\nabla v$. Then $L_\epsilon u_\epsilon \in L^r$ for $2 \le r < \infty$. By Lemma 5 we have $u_\epsilon \in W^{2,r}$ for $2 \le r < \infty$ so we conclude $u_\epsilon \in C_0$ by the Sobolev embedding.

Finally, for $n = 5, 6, 7$ we recall that $1 \le \alpha_1 < \frac{n}{n-4}$. Suppose that $u_\epsilon \in W^{2,r}$ for some r in the range $2 \le r < \frac{n}{2}$. Then the Sobolev embedding gives $u_\epsilon \in L^2 \cap L^{nr/(n-2r)}$. Moreover

$$|L_\epsilon u_\epsilon| \le C(|u_\epsilon| + |u_\epsilon|^{\alpha_1}) \in L^{\frac{nr}{(n-2r)\alpha_1}}.$$

Since also $u_\epsilon \in L^{nr/(n-2r)\alpha_1}$ we find by Lemma 5 that $u_\epsilon \in W^{2,nr/(n-2r)\alpha_1}$. But the iteration ("bootstrap")

$$r_{k+1} = \frac{nr_k}{(n-2r_k)\alpha_1}, \quad r_0 = 2$$

will lead, after a finite number of steps, to $r_k > n/2$. So $u_\epsilon \in W^{2,r}$ for some $r > n/2$ and this gives $u_\epsilon \in C_0$.

The fact that u_ϵ tends to 0 at infinity gives another proof that 0 is below the essential spectrum of T_ϵ as it implies

$$\liminf_{|x|\to\infty} G(V(\epsilon x), u_\epsilon(x)) = \liminf_{|x|\to\infty} \frac{\partial F}{\partial u}(V(\epsilon x), 0) > h > 0.$$

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