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A BILLIARD-BASED GAME INTERPRETATION OF THE NEUMANN PROBLEM FOR THE CURVE SHORTENING EQUATION

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Abstract. This paper constructs a family of discrete games, whose value functions converge to the unique viscosity solution of the Neumann boundary problem of the curve shortening flow equation. To derive the boundary condition, a billiard semiflow is introduced and its basic properties near the boundary are studied for convex and more general domains. It turns out that Neumann boundary problems of mean curvature flow equations can be intimately connected with purely deterministic game theory.

1. Introduction

In this paper, we study a deterministic game interpretation for the Neumann boundary problem of the two-dimensional curvature flow equation backward in time:

$$\begin{cases} \partial_t u + |\nabla u| \mathrm{div}(\frac{\nabla u}{|\nabla u|}) = 0 & \text{in } \Omega \times (0,T), \\ \langle \nabla u(x,t), \nu(x) \rangle = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,T) = u_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a C^2 boundary $\partial\Omega$, $\nu(x)$ is the unit outward normal to $\partial\Omega$ at x and $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^2$.

The deterministic game-theoretic approach to second-order elliptic and parabolic equations with geometric nonlinearity is first proposed in [17], where a family of discrete-time games are constructed so that their value functions u^ε converge to the unique viscosity solution of the level-set mean curvature flow equation. The level-set formulation for geometric evolution with an application of viscosity solutions is established by [3] and [7] (see also a detailed book [9]). The results in [17] are unexpected at a first glance,

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since deterministic differential games are known to be connected with first-order Hamilton-Jacobi-Isaacs equations (see, e.g., [1]). The point however lies in the singularity caused by the limit. This idea is close to [2, 23] on the morphological operators in image processing and is developed for more general anisotropic geometric flow equations [10] and even a much larger class including the heat equation [18]; see [15] for further generalization.

As for the Neumann boundary, its relaxation in the viscosity sense is first proposed by Lions [19]. Applications to deterministic optimal control and differential game theory in [19] rely much on a reflecting process, the solution of the deterministic Skorokhod problem. The Skorokhod problem, especially in its stochastic version, is of great importance in stochastic system theory. Its properties in different circumstances are clarified in a great deal of literature such as [26, 20, 6].

Our present work serves a two-fold purpose: (a) We show that the game-theoretic approach can be extended to the Neumann boundary problem of mean curvature flow, whose well-posedness is studied in [24] for a convex domain and in [11] for a nonconvex one. See also [14, 25] for more general boundary problems. However, it is worth mentioning that in our interpretation the game values u^ε may not be continuous in the space variable while those in [17] are. The discontinuity essentially arises in our reconciliation between the interior and boundary motion. (b) What is more important is that we give a discrete way of realizing Neumann boundary conditions. Instead of discretizing the Skorokhod problem, we use the billiard dynamics, which is a classical but quite different type of reflection on the boundary $\partial\Omega$. It turns out that billiard motion, usually studied in dynamical systems, can be adapted to our game interpretation with values approximating the solution of (1.1). Moreover, in two dimensions the billiard law seems more explicit than the Skorokhod reflection.

The adoption of the billiard dynamics is not straightforward. It actually requires a little modification so that we can handle its boundary behavior. We follow early study due to Halpern [12], Katok and Strelcyn [16], and a recent book by Chernov and Markarian [4] to define a billiard semiflow on the whole closure of the domain and investigate some of its properties. A more general billiard semiflow for oblique boundary problems is treated in [21]. See also a forthcoming work [22].

To explain heuristically in more detail how to characterize the solution of (1.1) by a family of games, we first recall the fundamental planar billiard law: the angle of incidence equals the angle of reflection, based on which we denote by $S^t(x, v)$ the position of unit-speed billiard motion at time t for a starting point $x \in \overline{\Omega}$ and a direction v. If we assume

(D1) Ω is a bounded and convex domain in $\mathbb{R}^2$ with C^2 boundary,

it then can be shown that

$$S^t(x,v) = x + tv - \alpha^t(x,v) \text{ with } |\alpha^t(x,v)| \le 2t, \tag{1.2}$$

where $\alpha^t(x,v)$ is an adjusting vector, identified as a (possibly infinite) sum of outward normals to the boundary near x. Such a representation actually demonstrates that billiards are similar in form to the Skorokhod problem, though it is not clear in what sense the solution of the Skorokhod problem can be understood as a direct limit of billiards without putting them in our game or optimal control setting as follows.

Let us play a discrete two-person game from an initial position $x \in \overline{\Omega}$ and the time $t > 0$. The maturity time is denoted by T and $T \ge t$. For any $\varepsilon > 0$, each step brings about a movement of length $\sqrt{2}\varepsilon$ but costs time ε^2. Then the total number of game steps N^ε equals $[\frac{T-t}{\varepsilon^2}]$, where $[a]$ stands for the largest integer less than or equal to each real number a. We suppress the superscript ε of N^ε for simplicity and denote the k-step state by $y(k) \in \overline{\Omega}$, which is controlled by both players for $k = 0, 1, 2, \ldots, N$. Suppose one game player, Paul, wants to minimize the value $u_0(y(N))$ while the other, Carol, is to maximize it. In the k-th round:

(1) Paul chooses a direction v_k, i.e., $|v_k| = 1$;
(2) Carol has the right to reverse Paul's choice, which determines $b_k = \pm 1$;
(3) The game state is moved from $y(k-1)$ to $y(k) = S^{\sqrt{2}\varepsilon}(y(k-1), b_k v_k)$.

We implement a mirror-like reflection on the boundary so as to keep the game proceeding. The state equations are:

$$y(k) = S^{\sqrt{2}\varepsilon}(y(k-1), b_k v_k), \quad \text{for all } k = 1, 2, \ldots, N;$$
$$y(0) = x \in \overline{\Omega}.$$

For every starting position $x \in \overline{\Omega}$ and time $t \in [0, T]$, we define in accordance to [8] the value function $u^\varepsilon(x,t)$ as

$$u^\varepsilon(x,t) = \inf_{|v_1|=1} \sup_{b_1=\pm 1} \ldots \inf_{|v_N|=1} \sup_{b_N=\pm 1} u_0(y(N)), \tag{1.3}$$

which, in particular, implies $u^\varepsilon(x,t) = u_0(x)$ when $t \in (T - \varepsilon^2, T]$. By definition, u^ε satisfies the dynamic programming principle

$$u^\varepsilon(x,t) = \inf_{|v|=1} \sup_{b=\pm 1} u^\varepsilon\left(S^{\sqrt{2}\varepsilon}(x, bv), t + \varepsilon^2\right) \text{ for all } t \in (0, T - \varepsilon^2]. \tag{1.4}$$

It follows formally by Taylor's expansion that at (x,t)

$$0 \approx \inf_{|v|=1} \sup_{b=\pm 1} \left\{ \varepsilon^2 u_t^\varepsilon + \left\langle \nabla u^\varepsilon, S^{\sqrt{2}\varepsilon}(x,t) - x \right\rangle \right. \\ \left. + \frac{1}{2} \left\langle \nabla^2 u^\varepsilon \left(S^{\sqrt{2}\varepsilon}(x,t) - x \right), S^{\sqrt{2}\varepsilon}(x,t) - x \right\rangle \right\}. \tag{1.5}$$

Noticing that u_t^ε is independent of b and v, we apply our billiard expression (1.2) to obtain from above that

$$0 \approx \varepsilon^2 u_t^\varepsilon + \inf_{|v|=1} \sup_{b=\pm 1} \left\{ \left\langle \nabla u^\varepsilon, \sqrt{2}\varepsilon b v - \alpha^{\sqrt{2}\varepsilon} \right\rangle \right. \\ \left. + \frac{1}{2} \left\langle \nabla^2 u^\varepsilon \left(\sqrt{2}\varepsilon b v - \alpha^{\sqrt{2}\varepsilon} \right), \sqrt{2}\varepsilon b v - \alpha^{\sqrt{2}\varepsilon} \right\rangle \right\} \quad \text{at } (x,t). \tag{1.6}$$

To emphasize our particular interest in the boundary condition, we assume $x \in \partial\Omega$. Viewing for the moment that $u^\varepsilon(x,t)$ has bounded derivatives and converges in some sense to a function $u(x,t)$ and

$$\lim_{\varepsilon \to 0} \frac{\alpha^{\sqrt{2}\varepsilon}}{|\alpha^{\sqrt{2}\varepsilon}|} = \nu(x) \quad \text{uniformly in } b \text{ and } v, \tag{1.7}$$

we discuss two cases for every subsequence, still indexed by ε:

1. Boundary condition dominant case: There exists $C > 0$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\alpha^{\sqrt{2}\varepsilon}| = C.$$

We then divide both sides of (1.6) by ε, pass to the limit $\varepsilon \to 0$ and get via (1.7) that

$$0 = \sqrt{2} \inf_{|v|=1} \sup_{b=\pm 1} |\langle \nabla u(x,t), bv \rangle| - C \langle \nabla u(x,t), \nu(x) \rangle.$$

Since the first term on the right-hand side is always zero, the classical Neumann boundary condition remains.

2. Mixed type case: Assume contrary to the former case

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\alpha^{\sqrt{2}\varepsilon}| = 0.$$

Then the same first-order operation as above yields that the "inf sup" is attained at $v = \frac{\nabla^\perp u}{|\nabla u|}$, where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. We here assume that $\nabla u(x,t) \neq 0$ for otherwise we realize the Neumann boundary condition again. Despite uncertainty about the order of $|\alpha^{\sqrt{2}\varepsilon}|$, we can always get the boundary condition desired. Indeed, if $\frac{1}{\varepsilon^2}|\alpha^{\sqrt{2}\varepsilon}| \to \infty$ as $\varepsilon \to 0$, we divide both sides of (1.6) by $|\alpha^{\sqrt{2}\varepsilon}|$ and send $\varepsilon \to 0$ to get

$$\langle \nabla u(x,t), \nu(x) \rangle = 0.$$

If $|\alpha^{\sqrt{2}\varepsilon}|$ is of order $o(\varepsilon^2)$, we in turn use ε^2 as the divisor and obtain the limit equation

$$u_t + \left\langle \nabla^2 u \frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|} \right\rangle = 0 \quad \text{at } (x,t),$$

or in other words,

$$u_t + |\nabla u| \mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) = 0 \quad \text{at } (x,t).$$

The most sophisticated case is ascribed to the occasion when $|\alpha^{\sqrt{2}\varepsilon}|$ exactly has the order ε^2. The limit of the divided equation then is

$$-u_t(x,t) - \left\langle \nabla^2 u(x,t) \frac{\nabla^\perp u(x,t)}{|\nabla u(x,t)|}, \frac{\nabla^\perp u(x,t)}{|\nabla u(x,t)|} \right\rangle + M \langle \nabla u(x,t), \nu(x) \rangle = 0,$$

where M is a positive constant. It follows that either

$$-u_t - \left\langle \nabla^2 u \frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|} \right\rangle \geq 0 \quad \text{and} \quad \langle \nabla u, \nu(x) \rangle \leq 0 \quad \text{at } (x,t)$$

or

$$-u_t - \left\langle \nabla^2 u \frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|} \right\rangle \leq 0 \quad \text{and} \quad \langle \nabla u, \nu(x) \rangle \geq 0 \quad \text{at } (x,t),$$

which still shows a good chance that u fulfills the Neumann boundary condition in the viscosity sense.

The preceding mechanism gives rise to the pivotal result of this paper.

Theorem 1.1. *Assume that* Ω *satisfies* (D1). *Assume that* u_0 *is a continuous function in* $\overline{\Omega}$. *Let* u^ε *be the value function of the game defined by* (1.3). *Then* u^ε *converges, as* $\varepsilon \to 0$, *to the unique viscosity solution of* (1.1) *uniformly on compact subsets of* $\overline{\Omega} \times (0,T]$.

If one keeps above formal argument, translation into viscosity language is similar to [17] for $\Omega = \mathbb{R}^2$, although we need several properties of a billiard semiflow. We shall prove that the upper relaxed limit $\overline{u} = \limsup{}^* u^\varepsilon(x,t)$ is a subsolution while the lower relaxed limit $\underline{u} = \liminf{}_* u^\varepsilon(x,t)$ is a supersolution (see Section 3 for the precise definitions of $\overline{u}$, $\underline{u}$ and viscosity solutions). If there is no terminal layer so that $\overline{u} = \underline{u}$ at $t = T$, the comparison principle yields $\overline{u} = \underline{u}$, which implies local uniform convergence.

To show the nonexistence of terminal layers, the value functions u^ε are proved in [17] to be equicontinuous for a C^2 terminal datum u_0 when $\Omega = \mathbb{R}^2$. However, this method is not directly applicable for our general Ω because of an extra difficulty about the possible discontinuity of billiard motion. We instead use the barrier argument to prove the convergence without assuming that u_0 is C^2. (Our method provides a strong result even when $\Omega = \mathbb{R}^2$.)

With several technical modifications, the convergence result in Theorem 1.1 is still valid even when Ω is not convex under some additional assumptions on the boundary $\partial\Omega$. Our approach works at least for nonconvex domains which have finite bumps. In this paper we do not intend to give any generalization of equations, dimensions, or games, though it is possible, as suggested in [17], so as to clarify the essential part of the problem. Moreover, applications of these billiard dynamics to the Neumann or general oblique problem of first-order Hamilton-Jacobi equations are also possible and even simpler; see [22].

This paper is organized in the following way. We introduce convex planar billiards including the definition and properties in Section 2. A comparison between billiard and Skorokhod types of reflection is discussed as well. Section 3 is the major part, devoted to the rigorous proof of Theorem 1.1. Section 4 is for an extension to nonconvex domains. We conclude this paper with an appendix consisting of a review of billiards' singularity and some investigation of its influence on the continuity of our value functions.

2. A Billiard Semiflow

This section provides a discrete system for the games to be introduced in Section 3 and Section 4. But it is also interesting to investigate billiards as an independent topic.

2.1. **General Planar Billiards.** We start with a few preliminaries of classical billiards. A domain $\Omega \subset \mathbb{R}^2$, piecewise C^1 but not necessarily convex, is said to be a *billiard table*. We hereafter use a standard notation $\mathbf{S}^1$ to denote the set of all unit vectors in $\mathbb{R}^2$, i.e., $\mathbf{S}^1 = \{v \in \mathbb{R}^2 : |v| = 1\}$. The billiard flow in Ω, denoted by $T^t : \overline{\Omega} \times \mathbf{S}^1 \to \overline{\Omega}$ ($t \in \mathbb{R}$), describes the billiard motion in the table. To be more precise, imagine there is a little ball (mass point) starting from a point $x \in \overline{\Omega}$ moving at unit speed along a straight line directed by $v \in \mathbf{S}^1$ inside Ω. When it hits the boundary, the tangential component of the original direction v keeps still but its normal component changes sign immediately. After that, the ball goes on its straight-line motion until a next collision takes place. For a fixed pair (x, v), $T^t(x, v)$ represents the ball's position at time t. The set $\{T^t(x,v) \in \overline{\Omega} : t \geq 0\}$ is called a *billiard trajectory* starting from (x, v) and is deemed to record the whole track of movement. We call the hitting points on the boundary *vertices* of the trajectory.

We remark that a billiard flow should be rigorously defined as a group of maps on the phase space $\Omega \times \mathbf{S}^1$ with a parameter $t \in \mathbb{R}$, however, since its second component is discontinuous in t and of little concern here, we will pay more attention to T^t, its natural projection onto the first component, concealing the second one but referring to it as P_2T^t whenever needed.

As a flow, T^t clearly satisfies the group property restricted in $\Omega \times \mathbf{S}^1$ with the identity T^0 and $T^{-t}(x,v) = T^t(x,-v)$ for any $x \in \Omega$ and $v \in \mathbf{S}^1$.

It is easy to understand such a trajectory of a billiard motion but there are three types of situations invalidating its reasonableness:

1. **Singularity:** The billiard ball hits a non-differentiable point of $\partial\Omega$.

2. **Entrance:** The billiard trajectory contains an interval in $\partial\Omega$ or in other words, the ball hits an inflection point in a direction right equal to the tangential to $\partial\Omega$.

3. **Termination:** The sequence of vertices $\{p_n\}_{n\geq 1}$ may converge to a point on $\partial\Omega$, called a *terminating point*. For a convex table, an equivalent definition is $\sum_{n\geq 1} |p_{n+1} - p_n| < \infty$, which reveals that the trajectory will break down by finite time. Halpern [12] provides an example. We review the results related to it in Appendix A.1.

From the point of view of dynamical systems, it is unnecessary to define the billiard flow for these three types, just because they rarely happen. It is known that the Lebesgue measure of their corresponding starting points and directions is zero, provided that Ω is of finite measure (Theorem A.4). Nevertheless, for billiards in $\overline{\Omega}$ and further applications to games, we will fill the gaps by appending proper definitions.

2.2. **A Modified Convex Billiard.** From now on, the ball motion will be turned into another style, which is no longer a billiard in the classical sense. Let us call it a *modified billiard* and use the notation $S^t(x, v)$ to express its position in $\overline{\Omega}$ at time $t \geq 0$ for the motion starting from $x \in \overline{\Omega}$ with the initial direction $v \in \mathbf{S}^1$. It will be seen that S^t is not a flow but a semiflow; i.e., it satisfies associativity only for time $t \geq 0$. We need a better table fulfilling (D1). On this occasion, "singularity," among those situations listed in the last subsection, is immediately ruled out. Moreover, due to the convexity, the situation of entrance hardly takes place unless the motion starts on the boundary.

We hereafter utilize the arc-length parametrization $\Gamma(\cdot) : \mathbb{R} \to \mathbb{R}^2$, a function of class C^2, to represent $\partial\Omega$. Its derivative with respect to s is denoted by Γ_s. We also use $\theta(s)$ to stand for the normal angle of $\Gamma(s)$. Its relationship with the normal is described by

$$\nu(\Gamma(s)) = \big(\cos(\theta(s)),\ \sin(\theta(s))\big). \tag{2.1}$$

We first prepare two lemmas. The first one is elementary, just like the mean value theorem.

Lemma 2.1. *Assume that Ω satisfies* (D1) *and is parameterized as $\Gamma(\cdot)$ by its arc length. Then for any $s_2 > s_1$, there exists $\xi \in [s_1, s_2]$ such that*

$$Y(s_1, s_2) = -|Y(s_1, s_2)|\nu(\Gamma(\xi)), \tag{2.2}$$

where $Y(s_1, s_2) = \Gamma(s_2) - \Gamma(s_1) - (s_2 - s_1)\Gamma_s(s_1) \in \mathbb{R}^2$; and

$$\langle \Gamma_s(s_1), \nu(\Gamma(\xi))\rangle \geq 0. \tag{2.3}$$

Proof. We first fix s_1 and consider the curve locally near $\Gamma(s_1)$ so that the graph of a convex function $f \in C^2(\mathbb{R})$ can be employed to represent it. There is no loss of generality in assuming $\Gamma(s_1) = 0$ and $\Gamma_s(s_1) = (1, 0)$, or, equivalently, $f(0) = 0$ and $f'(0) = 0$. We thus obtain

$$\Gamma(s_2) - \Gamma(s_1) - (s_2 - s_1)\Gamma_s(s_1) = \left(x - \int_0^x \sqrt{1 + f'(t)^2}\, dt, f(x)\right),$$

where x is the unique value satisfying

$$\int_0^x \sqrt{1 + f'(t)^2}\, dt = s_2 - s_1.$$

Setting the slope

$$k(s_1, s_2) := \frac{f(x)}{x - \int_0^x \sqrt{1 + f'(t)^2}\, dt} = \frac{f(x)}{-\int_0^x \frac{f'(t)^2}{1+\sqrt{1+f'(t)^2}}\, dt}$$

and noticing that $\frac{f'(t)}{1+\sqrt{1+f'(t)^2}}$ and $f'(t)$ are both nondecreasing because of the convexity of f, we are led to

$$k(s_1, s_2) \leq \frac{f(x)}{-\frac{f'(x)}{1+\sqrt{1+f'(x)^2}} \int_0^x f'(t)\, dt} \leq -\frac{1}{f'(x)}.$$

Since the normal angle $\theta(\cdot)$ is continuous and nondecreasing with $\theta(s_1) = -\frac{\pi}{2}$ and $\theta(s_2) = -\arctan \frac{1}{f'(x)}$, the mean value theorem yields the existence of $\xi \in [s_1, s_2]$ such that

$$\tan(\theta(\xi)) = k(s_1, s_2).$$

Consequently we get the equality (2.2) immediately by applying (2.1).

Now it remains to show (2.2) for a global situation. Since there is $\bar{x} > 0$ such that $f'(x) \to \infty$ as $x \to \bar{x}$, the graph representation is not allowed for $s_2 \geq \bar{s} := \int_0^{\bar{x}} \sqrt{1 + f'(t)^2}\, dt$. However, the statement we are concerned with still holds. Indeed, in terms of the convexity of the boundary, $\Gamma(s_2)$ must be bounded in $[0, \bar{x}] \times [0, +\infty)$ even if $s_2 \geq \bar{s}$. Therefore $k(s_1, s_2)$ can still be computed to get

$$k(s_1, s_2) \leq 0, \quad \text{if } s_2 \geq \bar{s}.$$

Noting that $\theta(s_1) = -\frac{\pi}{2}$ and $\theta(\bar{s}) = 0$, we again have some $\xi \in [s_1, \bar{s}] \subset [s_1, s_2]$ such that $\tan(\theta(\xi)) = k(s_1, s_2)$ and in turn obtain (2.2).

Observing from above that $\theta(\xi) \in [-\frac{\pi}{2}, 0]$, we easily get the inequality (2.3) since $\langle \Gamma_s(s_1), \nu(\Gamma(\xi)) \rangle = \cos(\theta(\xi))$. □

The second important lemma is for the case of termination. It is due to Halpern [12] (see Theorem A.1 in Appendix A.1).

Lemma 2.2. *Suppose that Ω satisfies* (D1)*. If a trajectory terminates at a point $\Gamma(s_\infty) \in \partial\Omega$, with a sequence of vertices $\{\Gamma(s_n)\}_{n\geq 1}$ arranged in order, then there exists $N > 0$ such that, for $n \geq N$, s_n monotonically converges to s_∞ and $(\Gamma(s_\infty) - \Gamma(s_n))/|s_\infty - s_n|$ converges to a unit tangent, denoted by v_∞, to the boundary at $\Gamma(s_\infty)$.*

We are now in a position to give a definition for our modified billiard as follows.

Definition 2.1. *Let Ω satisfy (D1).*

(i) *If $x \in \partial\Omega$, and v equals to the tangent of $\partial\Omega$, then*

$$S^t(x, v) := \Gamma(t), \quad \text{for any } t \geq 0,$$

where $\Gamma(\cdot)$ is the arc-length parametrization of $\partial\Omega$ such that $\Gamma(0) = x$ and $\Gamma_s(0) = v$.

(ii) *If $x \in \Omega$ and v is such that $T^t(x, v)$ terminates on $\partial\Omega$ at time t_0, then*

$$S^t(x, v) := \begin{cases} T^t(x, v) & \text{if } 0 \leq t < t_0, \\ S^{t-t_0}(T^{t_0}(x, v), v_\infty) & \text{if } t \geq t_0, \end{cases}$$

where v_∞ is obtained from Lemma 2.2.

(iii) *If $x \in \partial\Omega$ and v points inside Ω, then*

$$S^t(x, v) := \begin{cases} x & \text{if } t = 0, \\ S^{t-\varepsilon}(x + \varepsilon v, v) & \text{if } t > 0, \end{cases}$$

where $\varepsilon > 0$ *is such that* $x + \delta v \in \Omega$ *for all* $\delta \in (0, \varepsilon)$.

Remark 2.1. The definition of (i) says that when $x \in \partial\Omega$ and v lies on the tangential line of $\partial\Omega$, the ball will slide along the boundary curve. It corresponds to a special case in (ii) when the value of the terminating time t_0 equals zero.

In (ii), since $T^{t_0}(x, v) \in \partial\Omega$ and v_∞ is a tangent of $\partial\Omega$, we are able to put (i) to use. If the trajectory never terminates on the boundary, we just adhere to the convention $t_0 = \infty$.

Heuristically speaking, although the termination phenomenon stops the trajectory, a final direction remains. The billiard ball is set to inherit the direction and roll along the boundary. Ambiguity may be caused if we abuse the notation $P_2S^t(x, v)$ for $x \in \partial\Omega$. We however can overcome it by always taking the reflected-off direction for the points on $\partial\Omega$, for the natural reason that the billiard law will instantaneously switch any outward pointing unit vector into an inward one. S^t then can be easily verified to be a semiflow in this case.

For non-tangential motion starting on the boundary, it also suffices to give a definition for a reflected-off direction v, pointing inside the domain, as we have done in (iii). Moreover, S^t in (iii) is well defined, since it does not depend on the choice of ε, which is implied by the associativity of the semiflow defined in (ii).

For a convex domain, we have so far completed the definition of S^t. We next present a few basic properties, which will play an essential role in the game interpretation later on.

2.3. **Basic Properties.** We first see that billiard motion in the table can be identified as one conducted outside the table. For $t \geq 0$, $x \in \overline{\Omega}$ and $v \in \mathbf{S}^1$, we set

$$\alpha^t(x, v) = x + tv - S^t(x, v) \tag{2.4}$$

and call it the *boundary adjustor*. We next give a more specific representation of the boundary adjustor of our billiard semiflow. Let $B_r(x)$ denote the closed ball in $\mathbb{R}^2$ centered at x with radius r.

Lemma 2.3. *Assume that* Ω *satisfies* (D1). *For any fixed* $t \geq 0$, $x \in \overline{\Omega}$ *and* $v \in \mathbf{S}^1$, *let* $\alpha^t(x, v)$ *be the boundary adjustor of* $S^t(x, v)$. *Then there exist* $d_l \geq 0$ *and* $y_l \in \partial\Omega \cap B_t(x)$, $l = 1, 2, \ldots$ *such that*

$$\alpha^t(x, v) = \sum_{l=0}^{\infty} d_l \nu(y_l), \tag{2.5}$$

where the convergence on the right-hand side is in $\mathbb{R}^2$*. In addition, the following estimates hold:*

$$\left|\alpha^t(x,v)\right| \le 2t. \tag{2.6}$$

$$\Big|\sum_{l=k}^{\infty} d_l \nu(y_l)\Big| \le 4t,\ \textit{for all } k = 1, 2, \dots \tag{2.7}$$

$$\sum_{l=1}^{\infty} |y_{l+1} - y_l| \le 2t. \tag{2.8}$$

Lemma 2.3 is implied by Lemma 2.1 provided that the terminating time t_0 equals 0. In essence, for a mere boundary sliding case, a singleton $y_0 = \Gamma(\xi)$ given in Lemma 2.1 is sufficient for Lemma 2.3 with no need to find y_l for $l \ge 1$.

If alternatively the assumption $t_0 > 0$ is made, we will see that, for every $l \ge 1$, the boundary point y_l above is actually the vertex $\Gamma(s_l)$ for the billiard trajectory while y_0 is taken as a "mean value point" on the sliding piece over the time t_0. Indeed, we define all the vertices in order as $y_1, y_2, \dots$ Suppose $y_1 \ne x$ and $y_l \ne y_{l+1}$ for $l \ge 1$. Correspondingly, we refer to the terminating point, if it exists, as y_∞ as well as $\Gamma(s_\infty)$ in Lemma 2.2. The point y_0 could again be determined through Lemma 2.1. It is then sufficient to show these y_l satisfy (2.5)-(2.8).

It is quite transparent that the broken line connecting them in succession forms the first portion of the trajectory before time t_0. Moreover, to express the length and direction of each segment of the broken line, we set

$$a_0 := |y_1 - x|, \quad v_0 := v = \frac{y_1 - x}{a_0},$$

and for any $k \ge 1$

$$a_k := |y_{k+1} - y_k|, \quad v_k := \frac{y_{k+1} - y_k}{a_k}.$$

The billiard law immediately yields

$$v_{k+1} - v_k = -c_k \nu(y_{k+1}),\ \text{for all } k \ge 0, \tag{2.9}$$

where $c_k = 2\langle v_k, \nu(y_{k+1})\rangle \ge 0$. Thanks to Lemma 2.2 about the monotonicity of s_k for large k, it is true that

$$v_k = \frac{y_{k+1} - y_k}{|y_{k+1} - y_k|} = \frac{y_{k+1} - y_k}{s_{k+1} - s_k} \cdot \frac{|s_{k+1} - s_k|}{|y_{k+1} - y_k|} \to v_\infty,\ \text{as } k \to \infty,$$

which through (2.9) amounts to saying

$$v_\infty - v = \sum_{k=1}^{\infty}(v_k - v_{k-1}) = -\sum_{k=1}^{\infty} c_{k-1}\nu(y_k). \tag{2.10}$$

Proof of Lemma 2.3. We construct a path via y_l only for the case $x \in \Omega$ and thus $t_0 > 0$. The case $x \in \partial\Omega$ can be handled similarly.

If $t < t_0$, then $S^t(x,v) = T^t(x,v)$ and there exists $n \geq 0$ such that either
(a) $t < a_0$ or
(b) $\sum_{k=0}^{n} a_k \leq t < \sum_{k=0}^{n+1} a_k$.

In case of (a), it is trivial. Both sides of (2.5) are equal to 0. If instead (b) holds, we have

$$S^t(x,v) = x + \sum_{k=0}^{n} a_k v_k + \Big(t - \sum_{k=0}^{n} a_k\Big) v_{n+1}. \tag{2.11}$$

Observe also that, for each $j = 1, 2, \ldots, n$,

$$\sum_{k=0}^{n} a_k v_k = t v_0 + \sum_{k=j}^{n} a_k v_k - \Big(t - \sum_{k=0}^{j-1} a_k\Big) v_j + \sum_{l=1}^{j} \Big(t - \sum_{k=0}^{l-1} a_k\Big)(v_l - v_{l-1}).$$

Plugging (2.9) and the equality above with $j = n$ into (2.11), we get

$$S^t(x,v) = x + t v_0 - \sum_{l=1}^{n+1} c_{l-1}\Big(t - \sum_{k=0}^{l-1} a_k\Big)\nu(y_l), \tag{2.12}$$

which together with the definition of α^t in (2.4) implies (2.5) with $d_l = c_{l-1}(t - \sum_{k=0}^{l-1} a_k)$ for $l = 1, 2, \ldots, n+1$ and $d_l = 0$ for the other l.

Now if $t \geq t_0$, then $\lim_{k\to\infty} s_k = s_\infty$, $\sum_{k=0}^{\infty} a_k = t_0$ and $\sum_{k=0}^{\infty} a_k v_k = y_\infty$. By the argument above, simply taking the limit $n \to \infty$ in (2.12), we are led to

$$\alpha^{t_0}(x,v) = \sum_{l=1}^{\infty} d_l' \nu(y_l), \tag{2.13}$$

where

$$d_l' = c_{l-1}(t - \sum_{k=0}^{l-1} a_k) \quad (l \geq 1).$$

Noticing further that (2.4) gives

$$\alpha^t(x,v) - \alpha^{t_0}(x,v) = (t - t_0)v + S^{t_0}(x,v) - S^t(x,v),$$

we only need to show its right-hand side is a sum of normals. Recall the arc-length parametrization $\Gamma(s_l)$ for y_l and assume without loss of generality that $\{s_l\}_{l\geq 1}$ is increasing when l is sufficiently large. We therefore obtain

$$S^t(x,v) - S^{t_0}(x,v) = \Gamma(s_\infty + t - t_0) - \Gamma(s_\infty).$$

In terms of Lemma 2.1, this yields the existence of some constant $C \geq 0$ and a point $\Gamma(\xi)$ on the arc between $S^t(x,v)$ and $S^{t_0}(x,v)$ such that

$$S^{t_0}(x,v) - S^t(x,v) = (t_0 - t)v_\infty + C\nu(\Gamma(\xi)). \tag{2.14}$$

Noting (2.10) and combining the equations (2.13) and (2.14), we finally deduce (2.5) provided that we take $y_0 = \Gamma(\xi)$, $d_0 = C$ and $d_l = d_l' + (t - t_0)c_{l-1}$ for every $l \geq 1$.

The estimate (2.6) is implied directly by an observation that

$$\left|S^t(x,v) - x\right| \leq t.$$

Moreover, if we think of the billiard motion starting from y_k for any $k \geq 1$, we obtain

$$\left|S^{t_k}(y_k, v_k) - y_k\right| \leq 2t_k \leq 2t,$$

where t_k denotes the remaining length the billiard ball needs to cover from y_k to $S^t(x,v)$. It follows that

$$\left|d_0\nu(y_0) + \sum_{l=k}^{\infty} d_l\nu(y_l)\right| \leq 2t,$$

and thus (2.7) is verified since $|d_0|$ is bounded by $2t$.

The estimate (2.8) is obvious due to our choice of y_l. □

We next pause to compare our modified billiard with the deterministic Skorokhod problem, which is known to play a central role in the Neumann boundary problems for continuous optimal control and differential games. Let us review the definition of a solution of the two-dimensional Skorokhod problem for a normal reflection. For any $T > 0$, let $|\psi|(t)$ denote the total variation over the interval $[0,t]$ $(t \leq T)$ whenever ψ is of bounded variation in $[0,T]$; i.e, $\psi \in BV(0,T)$.

Definition 2.2 ([6]). *For any fixed $T \geq 0$, if for each $w(t) \in C([0,T];\mathbb{R}^2)$ with $w(0) \in \overline{\Omega}$, there exists a pair of functions $(x(t), \eta(t))$ fulfilling*

(1) $x(t) = w(t) - \eta(t)$ *for* $t \in [0,T]$ *with* $x(0) = w(0)$,
(2) $x(t) \in \overline{\Omega}$ *for* $t \in [0,T]$,
(3) $|\eta|(T) < \infty$,
(4) $|\eta|(t) = \int_{(0,t]} 1_{\{x(s)\in\partial\Omega\}}\, d|\eta|(s)$,

(5) $\eta(t) = \int_{(0,t]} \nu(x(s))\, d|\eta|(s)$,

then $(x(t, \eta(t)))$ *is said to be the solution of the Skorokhod problem in* $[0, T]$ *for* w *with respect to the domain* Ω *and direction* ν*.*

The dynamics $x(t)$ is determined one part after another by the rule that it is always abiding by $w(t)$ while it lies in Ω but is pushed back in $\overline{\Omega}$ along the inward normal whenever it is about to cross the boundary. In this sense, we convince ourselves that the Skorokhod problem and billiards are very similar. Indeed, our explicit representation in (2.4) and (2.5) is an analogue of the condition (1) in Definition 2.2. Therefore, it is no wonder that differential games based on billiard dynamics can also be linked with Neumann boundary problems. However, they are not exactly the same:

(1) Definition 2.2 is valid for every continuous function w while billiards are defined only for straight-line movement.
(2) An advantage of billiards rests on their explicit form and natural uniqueness. In contrast, the solution of the Skorokhod problem in two dimensions usually has no explicit representation and showing its existence is often troublesome, if not difficult, though the uniqueness might be easier sometimes (see [26, 20]).
(3) The solution of the Skorokhod problem enjoys a high level of stability. Indeed, the map $w(\cdot) \mapsto x(\cdot)$ is 1/2-Hölder continuous on compact subsets of $C([0, T]; \mathbb{R}^2)$, as is clarified in [20, Theorem 2.2]. By comparison, the lack of stability could be a drawback of our billiard semiflow. We shall give an example in Appendix A, indicating that the billiard trajectories can vary drastically even if the initial position and direction are slightly perturbed. Since our example is made from the terminating phenomenon, it is a question whether the billiard motion is still unstable in a more regular domain with no terminating trajectory in it. Consult Appendix A for such restrictions on the domain, under which the billiard semiflow is promoted to a flow.
(4) The definition of billiards requires C^1 smoothness of the boundary even if the domain is convex (we here assume $\partial\Omega$ is of class C^2). However, the Skorokhod problem is solvable under slightly weaker assumptions for a convex domain [26]. Nonconvex cases are complicated for both. As far as our knowledge goes, the Skorokhod problem needs smoothness of $\partial\Omega$ better than the C^1 class and close to the "positive reach" condition. Our billiard dynamics works for a general domain too but with a stronger assumption on its number of bumps, which will be explained in detail in Section 4.

We continue to study the properties of our billiards. The next one holds exclusively for convex billiards.

Lemma 2.4. *Assume that* Ω *satisfies* (D1)*. Then*

$$|x_0 - S^t(x,v)| \le |x_0 - (x+tv)| \text{ for all } x, x_0 \in \overline{\Omega},\ v \in \mathbf{S}^1 \text{ and } t \ge 0. \tag{2.15}$$

Remark 2.2. This lemma can be considered as a generalization of the separation or support theorem of convex sets in $\mathbb{R}^2$. One can easily find (2.15) is also sufficient for the convexity assumption.

To prove Lemma 2.4, we again divide the argument into two parts, one for the boundary slide and the other for the mirror-like reflection.

Lemma 2.5. *Let* Ω *be a domain satisfying* (D1)*. Assume its boundary is parametrized by arc length as* $\Gamma(\cdot)$ *with* $\Gamma(0) = x \in \partial\Omega$*. Then, for every* $x_0 \in \overline{\Omega}$*,*

$$|x_0 - \Gamma(s)| \le |x_0 - (x + t\Gamma_s(0))|\,. \tag{2.16}$$

Proof. In virtue of Lemma 2.1, we can find $\xi \in [0,t]$ such that

$$\begin{aligned} x + t\Gamma_s(s) - \Gamma(t) = C\nu(\Gamma(\xi)) \ \text{ and} \\ \langle \Gamma_s(0), \nu(\Gamma(\xi))\rangle \ge 0. \end{aligned} \tag{2.17}$$

According to an elementary separation theorem for convex sets, the tangential line to $\partial\Omega$ at $\Gamma(\xi)$ breaks the plane into halves, one of which has no intersection with $\overline{\Omega}$. We establish orthogonal coordinates and assume, without loss of generality,

$$\Gamma(\xi) = (0,0) \in \mathbb{R}^2 \text{ and } \overline{\Omega} \subset \{(x,y) \in \mathbb{R}^2 : y \ge 0\}.$$

Let us use $\vec{i}$ and $\vec{j}$ to denote respectively the unit vectors for the x and y positive axes; i.e., $\vec{i} = x/|x|$ and $\vec{j} = y/|y|$, and we consequently have $\vec{j} = -\nu(\Gamma(\xi))$. We are led from the convexity of Ω to

$$\left\langle \Gamma(\xi) - \Gamma(0), \vec{j} \right\rangle \ge \xi \left\langle \Gamma_s(0), \vec{j} \right\rangle;$$

that is, $\langle x + \xi\Gamma_s(0), \vec{j}\rangle \le 0$. We then get from (2.17) that $\langle x + t\Gamma(0), \vec{j}\rangle \le 0$, since $\xi \le t$. Hence, it becomes quite evident that

$$\left\langle x_0 - \Gamma(t), \vec{i} \right\rangle = \left\langle x_0 - (x + t\Gamma_s(0)), \vec{i} \right\rangle;$$

$$\left|\left\langle x_0 - \Gamma(t), \vec{j} \right\rangle\right| \le \left|\left\langle x_0 - (x + t\Gamma_s(0)), \vec{j} \right\rangle\right|.$$

In consequence, (2.16) holds. ∎

Lemma 2.6. *Let* Ω *satisfy* (D1). *For any* $x_0, x \in \overline{\Omega}$, *consider the billiard* $S^t(x,v)$ *with positive terminating time and* $t > a_0$, *and set*

$$z := x + tv - 2(t - a_0)\,\langle v, \nu(\Gamma(s_1))\rangle\,\nu(\Gamma(s_1)).$$

Then

$$|x_0 - z| \le |x_0 - (x + tv)|. \tag{2.18}$$

Proof. As in the proof of Lemma 2.5, we apply the separation theorem, using the tangential at $\Gamma(s_1)$ to support $\overline{\Omega}$. More precisely, we set $\Gamma(s_1) = (0,0)$ and $\overline{\Omega} \subset \{(x,y) \in \mathbb{R}^2 : y \ge 0\}$, and then by taking the unit vectors $\vec{i}$ and $\vec{j}$, we have $\langle x + tv, \vec{j}\rangle < 0$, for otherwise no collision takes place, which contradicts the assumption $t > a_0$. Then we easily obtain

$$\left|\left\langle x_0 - (x+tv), \vec{i}\right\rangle\right| = \left|\left\langle x_0 - z, \vec{i}\right\rangle\right|$$

and

$$\left|\left\langle x_0 - (x+tv), \vec{j}\right\rangle\right| \ge \left|\left\langle x_0 - z, \vec{j}\right\rangle\right|,$$

which imply the inequality (2.18). □

We prove Lemma 2.4 below.

Proof of Lemma 2.4. We only need to prove this for the cases $t_0 > 0$ and $t > a_0$, since the others are either trivial or given already in Lemma 2.5.

If $t < t_0$, then there exists $n \ge 0$ such that

$$\sum_{k=0}^{n} a_k \le t < \sum_{k=0}^{n+1} a_k.$$

We apply Lemma 2.6 inductively for every k with a change $x \curvearrowright y_k$, $v \curvearrowright v_k$, $t \curvearrowright t - \sum_{j=0}^{k-1} a_j$ and $a_0 \curvearrowright a_k$, and as a result have

$$\begin{aligned} |x_0 - S^t(x,v)| &= \left|x_0 - y_{n+1} + \Big(t - \sum_{k=0}^{n} a_k\Big) v_{n+1}\right| \\ &\le \left|x_0 - y_n + \Big(t - \sum_{k=0}^{n-1} a_k\Big) v_n\right| \le \ldots \le |x_0 - x + tv|. \end{aligned}$$

For the case $t \ge t_0$, our discussion in Lemma 2.3 asserts

$$\begin{aligned} &S^t(x,v) = \Gamma(s_\infty + t - t_0), \\ &x + tv - \sum_{l=1}^{\infty} d_l \nu(y_l) = \Gamma(s_\infty) + (t - t_0) v_\infty. \end{aligned}$$

Then, by Lemma 2.5, we get

$$\left|x_0 - S^t(x,v)\right| \leq \left|x_0 - \Big(x + tv - \sum_{l=1}^{\infty} d_l \nu(y_l)\Big)\right|. \tag{2.19}$$

On the other hand, we argue in the same way as in the case $t < t_0$ and have, for every $n \in \mathbb{N}$,

$$\left|x_0 - \Big(x + tv - \sum_{l=1}^{n} d_l \nu(y_l)\Big)\right| \leq |x_0 - (x + tv)|.$$

Sending $n \to \infty$, we complete our proof by taking (2.19) into account. □

3. Neumann Boundary of Motion by Curvature

We now apply our billiard semiflow S^t to games. The billiard involved games for the Neumann boundary problem have been introduced in Section 1. This section is devoted to the proof of our main result Theorem 1.1.

Let us rewrite the equations as

$$\partial_t u + \Delta u - \left\langle \nabla^2 u \tfrac{\nabla u}{|\nabla u|}, \tfrac{\nabla u}{|\nabla u|} \right\rangle = 0 \quad \text{in } \Omega \times (0,T), \tag{3.1a}$$

$$\langle \nabla u(x,t), \nu(x) \rangle = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3.1b}$$

$$u(x,T) = u_0(x) \quad \text{in } \overline{\Omega}. \tag{3.1c}$$

We give a definition of a solution of this Neumann boundary problem, handling the boundary condition in the viscosity sense. The reader is referred to [5] and [9] for more details about the viscosity solution theory.

Definition 3.1. *An upper semicontinuous* (*respectively, lower semicontinuous*) *function* u *on* $\overline{\Omega} \times (0,T)$ *is a viscosity subsolution* (*respectively, viscosity supersolution*) *of* (3.1*a*)-(3.1*b*) *if, whenever there are* $(\hat{x}, \hat{t}) \in \overline{\Omega} \times (0,T)$, *a neighborhood* $\mathcal{O}$ *relative to* $\overline{\Omega} \times (0,T)$ *of* $(\hat{x}, \hat{t})$ *and a function* $\varphi \in C^2(\mathcal{O})$ *such that*

$$\max_{\mathcal{O}} (u - \varphi) = (u - \varphi)(\hat{x}, \hat{t})$$
$$\Big(\textit{respectively, } \min_{\mathcal{O}} (u - \varphi) = (u - \varphi)(\hat{x}, \hat{t})\Big),$$

the following holds:

(i) $\hat{x} \in \Omega$ *and* $\nabla\varphi(\hat{x},\hat{t}) \neq 0$ *imply that*

$$\partial_t\varphi + \Delta\varphi - \left\langle \nabla^2\varphi \frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \geq 0 \quad \text{at } (\hat{x},\hat{t})$$

$$\left(\text{respectively, } \partial_t\varphi + \Delta\varphi - \left\langle \nabla^2\varphi \frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \leq 0 \quad \text{at } (\hat{x},\hat{t})\right).$$

(ii) $\hat{x} \in \Omega$, $\nabla\varphi(\hat{x},\hat{t}) = 0$ *and* $\nabla^2\varphi(\hat{x},\hat{t}) = 0$ *imply that*

$$\partial_t\varphi(\hat{x},\hat{t}) \geq 0 \quad \left(\text{respectively, } \partial_t\varphi(\hat{x},\hat{t}) \leq 0\right).$$

(iii) $\hat{x} \in \partial\Omega$ *and* $\nabla\varphi(\hat{x},\hat{t}) \neq 0$ *imply either*

$$\partial_t\varphi + \Delta\varphi - \left\langle \nabla^2\varphi \frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \geq 0 \quad \text{at } (\hat{x},\hat{t})$$

$$\left(\text{respectively, } \partial_t\varphi + \Delta\varphi - \left\langle \nabla^2\varphi \frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \leq 0 \quad \text{at } (\hat{x},\hat{t})\right)$$

or

$$\left\langle \nabla\varphi(\hat{x},\hat{t}), \nu(\hat{x}) \right\rangle \leq 0 \quad \left(\text{respectively, } \left\langle \nabla\varphi(\hat{x},\hat{t}), \nu(\hat{x}) \right\rangle \geq 0\right).$$

Definition 3.2. *A function u on $\overline{\Omega} \times (0,T)$ is called a viscosity solution of* (3.1*a*)-(3.1*b*) *if it is both a viscosity subsolution and a viscosity supersolution.*

Let us recall the *upper* and *lower relaxed limits* of $\{u^\varepsilon\}_{\epsilon>0}$ as

$$\begin{aligned}\overline{u}(x,t) :&= \limsup_{\varepsilon\to 0}{}^* u^\varepsilon(x,t) \\ &= \lim_{\delta\to 0} \sup\{u^\varepsilon(y,s) : 0 < \varepsilon < \delta,\ |x-y| + |t-s| < \delta\}\end{aligned}$$

and

$$\begin{aligned}\underline{u}(x,t) :&= \liminf_{\varepsilon\to 0}{}_* u^\varepsilon(x,t) \\ &= \lim_{\delta\to 0} \inf\{u^\varepsilon(y,s) : 0 < \varepsilon < \delta,\ |x-y| + |t-s| < \delta\}.\end{aligned}$$

The strategy for the proof of Theorem 1.1 is as follows: We first show the upper and lower relaxed limits coincide at the terminal time. We then present two propositions to indicate that they are respectively a subsolution and a supersolution of (3.1a)-(3.1b) and thereby give the locally uniform convergence through the comparison principle established by Sato [24].

Proposition 3.1. *Under all the assumptions of Theorem* 1.1*,*

$$\overline{u}(x,T) = \underline{u}(x,T) = u_0(x) \quad \text{for all } x \in \overline{\Omega}. \tag{3.2}$$

Results like Proposition 3.1 are often shown via the construction of "barrier solutions." We here follow this idea by playing "barrier games." One may think that the proof could be easier if u^ε were "continuous in discrete time," but we cannot count on that since the continuity of our billiard semiflow $S^t(x, v)$ in x is not known.

Lemma 3.2. *Assume that $x_0, x \in \overline{\Omega}$. Let the game start from x. Then, for any $k \in \mathbb{N}$, there exists a strategy for the minimizing player Paul such that the resulting k-step state y_k satisfies, indifferent to the decisions of his opponent Carol,*

$$|x_0 - y_k|^2 \le |x_0 - x|^2 + 2k\varepsilon^2. \tag{3.3}$$

Proof. The strategy is one of feedback. It only requires that the proceeding direction be perpendicular to the line connecting x_0 and the position at that time. More precisely, we take $w_1 \in \mathbf{S}^1$ such that $w_1 \perp (x - x_0)$ and inductively, for j-step state $y_j \in \overline{\Omega}$, $w_{j+1} \in \mathbf{S}^1$ $(j \ge 1)$ satisfying $w_{j+1} \perp (y_j - x_0)$. It is quite easy to see that this strategy fulfills (3.3). Indeed, let b_j denote the j-step response of the minimizing player for each $j \ge 1$ and then inequality (2.15) in Lemma 2.4 yields

$$\begin{aligned} |x_0 - y_{j+1}|^2 &= \left| x_0 - S^{\sqrt{2}\varepsilon}(y_j, b_{j+1} w_{j+1}) \right|^2 \\ &\le \left| x_0 - (y_j + \sqrt{2}\varepsilon b_{j+1} w_{j+1}) \right|^2 = |x_0 - y_j|^2 + 2\varepsilon^2. \end{aligned}$$ □

Lemma 3.3. *Let $x_0, x \in \overline{\Omega}$ and the game start at x. Then, for any $k \in \mathbb{N}$, the maximizing player Carol can adopt a strategy which leads to the k-step state satisfying*

$$|x_0 - y_k|^2 \le |x_0 - x|^2 + 2k\varepsilon^2, \tag{3.4}$$

no matter how the minimizing player Paul chooses the directions.

Proof. We give the strategy by taking $b_j = \pm 1$ such that

$$b_1 \langle v_1, x - x_0 \rangle \le 0,$$

$$b_j \langle v_j, y_{j-1} - x_0 \rangle \le 0, \text{ for all } j \ge 2,$$

where v_j is Paul's choice at the j-th round. Again, in view of Lemma 2.4, we have

$$\begin{aligned} |x_0 - y_{j+1}|^2 &= \left| x_0 - S^{\sqrt{2}\varepsilon}(y_j, b_{j+1} v_{j+1}) \right|^2 \le \left| x_0 - (y_j + \sqrt{2}\varepsilon b_{j+1} v_{j+1}) \right|^2 \\ &= |x_0 - y_j|^2 + 2\varepsilon^2 + 2\sqrt{2}\varepsilon b_{j+1} \langle v_{j+1}, y_j - x_0 \rangle \le |x_0 - y_j|^2 + 2\varepsilon^2, \end{aligned}$$

and thus complete the proof. □

Proof of Proposition 3.1. We fix an arbitrary point $x_0 \in \overline{\Omega}$ and prove in the first place $\overline{u}(x_0, T) \leq u_0(x_0)$.

Since u_0 is continuous in $\overline{\Omega}$, we may set a function for every $\lambda > 0$:

$$\overline{V}_\lambda(x) = \lambda + u_0(x_0) + C|x - x_0|^2$$

with a sufficiently large constant C depending on λ such that

$$\overline{V}_\lambda(x) \geq u_0(x), \text{ for all } x \in \overline{\Omega}. \tag{3.5}$$

We next play an upper barrier game, starting at $(x, t) \in \overline{\Omega} \times (0, T]$ and conforming to the same rules but aiming at a new objective $\overline{V}_\lambda$. It is clear that though the exact optimal strategy for Paul is unknown, we can guarantee him a value not more than

$$\lambda + u_0(x_0) + C\big(|x - x_0|^2 + 2(T - t)\big)$$

if he takes a strategy introduced in Lemma 3.2. To see this, we recall that the total number of steps in the game is $\left[\frac{T-t}{\varepsilon^2}\right]$. Then, following that strategy, Paul gets to a final position, say y, decided also by Carol's response, such that

$$|x_0 - y|^2 \leq |x_0 - x|^2 + 2\varepsilon^2 \left[\frac{T-t}{\varepsilon^2}\right] \leq |x_0 - x|^2 + 2(T - t),$$

which implies

$$\overline{V}_\lambda(y) \leq \lambda + u_0(x_0) + C\big(|x - x_0|^2 + 2(T - t)\big). \tag{3.6}$$

We also learn from (3.5) that, for any such y,

$$u^\varepsilon(x, t) \leq u_0(y) \leq \overline{V}_\lambda(y). \tag{3.7}$$

We plug (3.6) into (3.7) and then have, in terms of the definition of upper relaxed limits,

$$\overline{u}(x_0, T) \leq \lim_{\substack{|x - x_0| \to 0 \\ T - t \to 0}} \Big(\lambda + u_0(x_0) + C\Big(|x - x_0|^2 + 2(T - t)\Big)\Big) = \lambda + u_0(x_0).$$

Letting $\lambda \downarrow 0$, we get $\overline{u}(x_0, T) \leq u_0(x_0)$.

Since $\overline{u}(x_0, T) \geq \underline{u}(x_0, T)$, it remains to prove $\underline{u}(x_0, T) \geq u_0(x_0)$. We choose for arbitrary $\lambda > 0$ a function

$$\underline{V}_\lambda(x) = -\lambda + u_0(x_0) - C(|x - x_0|^2)$$

with a constant C satisfying $\underline{V}_\lambda(x) \leq u_0(x)$ for $x \in \overline{\Omega}$. We now play a lower barrier game whose terminal data are given by $\underline{V}_\lambda$. For the same reason stated above, Carol can use the strategy in Lemma 3.3 this time to ensure herself a value

$$-\lambda + u_0(x_0) - C\big(|x - x_0|^2 + 2(T - t)\big).$$

Hence, noting that u^ε is always larger than the value of this lower barrier game, we obtain

$$u^\varepsilon(x,t) \geq -\lambda + u_0(x_0) - C\big(|x-x_0|^2 + 2(T-t)\big)$$

and therefore $\underline{u}(x_0,T) \geq u_0(x_0)$ by sending λ to 0. □

Proposition 3.4. *Assume that Ω satisfies* (D1). *Let $\{u^\varepsilon\}$ be uniformly bounded in $\overline{\Omega} \times (0,T)$ for $0<\varepsilon<1$. Assume that u^ε satisfies the dynamic programming principle* (1.4). *Then $\overline{u}$ is a viscosity subsolution of* (3.1a)-(3.1b).

Proof. We argue by contradiction only for $x_0 \in \partial\Omega$. For an interior point x_0, a contradiction will be reached by the same argument in [17]. Suppose to the contrary that there exist a smooth function ϕ and a δ-neighborhood of (x_0,t_0) such that $\overline{u}-\phi$ attains a unique maximum at (x_0,t_0) and both relations below hold:

$$\partial_t\phi(x,t) + \Delta\phi(x,t) - \left\langle \nabla^2\phi(x,t)\frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|}, \frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|}\right\rangle \leq -\eta_0, \quad \text{for all } (x,t)\in B_\delta((x_0,t_0)) \text{ and } x\in\overline{\Omega}, \tag{3.8}$$

$$\langle \nabla\phi(x,t), \nu(y)\rangle \geq \eta_0, \quad \text{for all } (x,t)\in B_\delta((x_0,t_0)),\ x\in\overline{\Omega} \text{ and } y \in B_\delta(x_0)\cap\partial\Omega, \tag{3.9}$$

where η_0 is a certain positive constant.

We are allowed to take a sequence $(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}) \to (x_0,t_0)$ such that

$$u^{\varepsilon_n}(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}) \to \overline{u}(x_0,t_0) \quad \text{and } \varepsilon_n \to 0 \quad \text{as } n\to\infty.$$

Next let us construct a specific path of the game, as,

$$X_1 = (x_1,t_1) = (x^0_{\varepsilon_n}, t^0_{\varepsilon_n}), \text{ and}$$

$$X_{k+1} = (x_{k+1},t_{k+1}) = \left(S^{\sqrt{2}\varepsilon_n}(x_k, b_k v_k), t_k + \varepsilon_n^2\right)$$

with $v_k = \frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}$ for $k\geq 1$. By (1.4), there is $b_k = \pm 1$ such that

$$u^{\varepsilon_n}(X_k) \leq u^{\varepsilon_n}(X_{k+1}),$$

and by induction,

$$u^{\varepsilon_n}(X_1) \leq u^{\varepsilon_n}(X_k). \tag{3.10}$$

We fix such b_k.

Now we estimate $\phi(X_{k+1}) - \phi(X_k)$ by applying Taylor's formula. Since $x_k \in \overline{\Omega}$, Lemma 2.3 states that there exist $d_l^{(k)} \geq 0$ and $y_l^{(k)} \in \partial\Omega \cap B_{\sqrt{2}\varepsilon_n}(x_k)$ such that

$$\alpha_n^k = \alpha^{\sqrt{2}\varepsilon_n}(x_k, b_k v_k) = \sum_{l=0}^{\infty} d_l^{(k)} \nu(y_l^{(k)}) \tag{3.11}$$

and $|\alpha_n^k| \leq 2\sqrt{2}\varepsilon_n$. Since $\langle v_k, \nabla\phi(X_k)\rangle = 0$, we observe that

$$\begin{aligned} \phi(X_{k+1}) - \phi(X_k) &= \left\langle -\nabla\phi(X_k), \alpha_n^k \right\rangle + \varepsilon_n^2 \partial_t \phi(X_k) \\ &+ \frac{1}{2}\left\langle \nabla^2\phi(X_k)\left(\sqrt{2}\varepsilon_n b_k v_k - \alpha_n^k\right), \sqrt{2}\varepsilon_n b_k v_k - \alpha_n^k \right\rangle + o(\varepsilon_n^2). \end{aligned} \tag{3.12}$$

Assuming for the time being that $X_k \in B_\delta((x_0, t_0))$ and $B_{\sqrt{2}\varepsilon_n}(x_k) \subset B_\delta(x_0)$, which makes sense at least for sufficiently small k, we handle two cases and utilize (3.8) or (3.9) respectively for each k.

Case A. (Boundary condition dominant case) Suppose that, for a given k,

$$\liminf_{\varepsilon_n \to 0} \frac{|\alpha_n^k|}{\varepsilon_n} \geq C$$

with some constant C which may depend on k. Then (3.11) gives

$$\left|\left\langle \nabla\phi(X_k), \alpha_n^k \right\rangle\right| \geq \sum_{l=0}^{\infty} d_l^{(k)} \eta_0 \geq \eta_0 |\alpha_n^k| \geq C\eta_0\varepsilon_n. \tag{3.13}$$

We therefore deduce from (3.12)

$$\phi(X_{k+1}) - \phi(X_k) \leq -C\eta_0\varepsilon_n \text{ for sufficiently large } n. \tag{3.14}$$

Case B. (Mixed type case) Suppose

$$\liminf_{\varepsilon_n \to 0} \frac{|\alpha_n^k|}{\varepsilon_n} = 0;$$

i.e., we can take a subsequence, still denoted by ε_n, such that

$$|\alpha_n^k| = o(\varepsilon_n).$$

In this case, we have, from (3.9) and (3.11),

$$\left\langle \nabla\phi(X_k), \alpha_n^k \right\rangle \geq 0, \tag{3.15}$$

which implies through (3.12) that

$$\begin{aligned}&\phi(X_{k+1}) - \phi(X_k)\\ &\leq \varepsilon_n^2 \partial_t \phi(X_k) + \varepsilon_n^2 \left\langle \nabla^2 \phi(X_k) \frac{\nabla^\perp \phi(X_k)}{|\nabla \phi(X_k)|}, \frac{\nabla^\perp \phi(X_k)}{|\nabla \phi(X_k)|} \right\rangle + o(\varepsilon_n^2) \\ &\leq -\frac{1}{2}\eta_0 \varepsilon_n^2 \text{ for sufficiently large } n,\end{aligned} \tag{3.16}$$

where the last inequality is obtained by adopting (3.8). (In fact, this inequality can be deduced by the same argument for the case of $x_0 \in \Omega$.)

Therefore, it can be concluded that

$$\phi(X_k) \leq \phi(X_1) - \frac{1}{4} k \eta_0 \varepsilon_n^2 \tag{$*$}$$

for $n > n_0$ with some n_0 independent of k as long as

$$X_j \in B_\delta((x_0, t_0)) \text{ and } B_{\sqrt{2}\varepsilon_n}(x_j) \subset B_\delta(x_0) \text{ for any } j = 1, 2, \ldots, k. \tag{3.17}$$

Combining ($*$) with (3.10), we obtain

$$u^{\varepsilon_n}(X_1) - \phi(X_1) \leq u^{\varepsilon_n}(X_k) - \phi(X_k). \tag{3.18}$$

Since ϕ is smooth, there exists $\sigma > 0$ satisfying

$$|\phi(x, t) - \phi(x_0, t_0)| \leq \sigma \text{ for all } (x, t) \in B_\delta((x_0, t_0)).$$

We may also assume that $n > n_0$ is sufficiently large such that

$$|\phi(X_1) - \phi(x_0, t_0)| \leq \sigma \text{ and} \tag{3.19}$$

$$\varepsilon_n \leq \min\{\delta/4, 1\}. \tag{3.20}$$

Now let k_n be the maximal k satisfying (3.17). (If (3.17) holds for all natural number k, we set $k_n = \infty$.) We claim that $\{k_n\}$ is a nontrivial divergent sequence, or more precisely,

$$\frac{\delta}{4\varepsilon_n} \leq k_n \leq \frac{8\sigma}{\eta_0 \varepsilon_n^2}. \tag{3.21}$$

Indeed, by definition, $X_{k_n+1} \notin B_\delta((x_0, t_0))$ or $B_{\sqrt{2}\varepsilon_n}(x_{k_n+1}) \backslash B_\delta(x_0) \neq \emptyset$. A direct computation with an application of (3.20) consequently yields

$$|X_{k_n} - (x_0, t_0)| \geq \frac{\delta}{2} \text{ or } |x_{k_n} - x_0| \geq \frac{\delta}{2}. \tag{3.22}$$

By (3.20) our construction of X_k thus implies the lower bound in (3.21). To show k_n's bound from above, we suppose to the contrary that $k_n > \frac{8\sigma}{\eta_0 \varepsilon_n^2}$. Then, by ($*$) and (3.19), we are led to

$$\phi(X_{k_n}) < \phi(x_0, t_0) - \sigma,$$

which contradicts the fact that $k = k_n$ satisfies (3.17).

We conclude from the definition of k_n and (3.22) that $\{X_k\}$ with $k = k_n$ admits a convergent subsequence in $B_\delta((x_0, t_0))$ such that its limit (x', t') as $n \to \infty$ does not equal (x_0, t_0). It follows easily that

$$(\overline{u} - \phi)(x_0, t_0) \leq \limsup_{n\to\infty}(u^{\varepsilon_n} - \phi)(X_{k_n}) \leq (\overline{u} - \phi)(x', t'),$$

which is a contradiction to the uniqueness of maximizers of $\overline{u} - \phi$ in the δ-neighborhood. □

The distinct two cases above, remarked also in Section 1, have a geometric meaning. $\left|\alpha_n^k\right|$ essentially quantifies the influence of boundary reflections on the game path commencing from the point in question. The effect indicated by case A is stronger than that in case B. In this sense, case A is supposed to be consistent with the boundary condition while case B may instead give rise to curvature motion in the interior, as is shown in our proof.

We deal with the supersolution part in the same way but the next preliminary fact is needed.

Lemma 3.5. *Let ξ be a unit vector in $\mathbb{R}^2$ and X be a real symmetric 2×2 matrix. Then there exists a constant $M > 0$, that depends only on the norm of X, such that for any unit vector $v \in \mathbb{R}^2$,*

$$|\langle X\xi^\perp, \xi^\perp\rangle - \langle Xv, v\rangle| \leq M|\langle \xi, v\rangle|, \tag{3.23}$$

where $\xi^\perp$ denotes a unit orthonormal vector of ξ.

Proof. It is clear that any v can be composed as

$$v = \langle v, \xi\rangle\xi + \langle v, \xi^\perp\rangle\xi^\perp,$$

which yields by computation that

$$\begin{aligned}
&|\langle X\xi^\perp, \xi^\perp\rangle - \langle Xv, v\rangle| \\
&= |\langle v, \xi\rangle^2\langle X\xi^\perp, \xi^\perp\rangle - \langle v, \xi\rangle^2\langle X\xi, \xi\rangle - 2\langle v, \xi\rangle\langle v, \xi^\perp\rangle\langle X\xi, \xi^\perp\rangle| \\
&\leq 4\|X\||\langle v, \xi\rangle|,
\end{aligned}$$

as desired. □

Proposition 3.6. *Assume that Ω satisfies* (D1). *Let $\{u^\varepsilon\}$ be uniformly bounded in $\overline{\Omega} \times (0, T)$ for $0 < \varepsilon < 1$. Assume that u^ε satisfies the dynamic programming principle* (1.4). *Then $\underline{u}$ is a viscosity supersolution of* (3.1a)-(3.1b).

Proof. We still assume by contradiction that there exist a constant $\eta_0 > 0$, a smooth function ϕ and a δ-neighborhood of $(x_0, t_0) \in \partial\Omega \times (0, T)$ in which (x_0, t_0) is the unique minimizer of $\overline{u} - \phi$ with

$$\begin{aligned}\partial_t \phi(x,t) + \Delta\phi(x,t) - \left\langle \nabla^2\phi(x,t)\frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|}, \frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|}\right\rangle \geq \eta_0 > 0,\\ \text{for all } (x,t) \in B_\delta((x_0,t_0)) \text{ and } x \in \Omega,\end{aligned} \tag{3.24}$$

$$\begin{aligned}&\langle \nabla\phi(x,t), \nu(y)\rangle \leq -\eta_0 < 0,\\ &\quad \text{for all } (x,t) \in B_\delta((x_0,t_0)),\ x \in \overline{\Omega} \text{ and } y \in B_\delta(x_0) \cap \partial\Omega.\end{aligned} \tag{3.25}$$

It is possible to find a sequence $\left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right) \to (x_0, t_0)$ such that

$$u^{\varepsilon_n}\left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right) \to \underline{u}(x_0,t_0) \text{ and } \varepsilon_n \to 0, \text{ as } n \to \infty.$$

We define a sequence of points in $\overline{\Omega} \times (0, T)$ in the following way:

$$X_1 = (x_1, t_1) = \left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right),$$

$$X_{k+1} = (x_{k+1}, t_{k+1}) = X_k + \left(S^{\sqrt{2}\varepsilon_n}(x_k, b_k v_k), \varepsilon_n^2\right) \text{ for } k \geq 1, \tag{3.26}$$

where v_k can be immediately determined through dynamic programming (1.4) so as to satisfy

$$u^{\varepsilon_n}(X_k) \geq \sup_{b_k=\pm 1} u^{\varepsilon_n}(X_{k+1}) - \varepsilon_n^3.$$

Note that $S^t(x, v)$ may not be continuous in x and v. There might be no minimizer v_k achieving the infimum in dynamic programming principle. We thus allow an error ε_n^3 which actually causes no problem in our further calculation. We fix v_k in such a way that

$$u^{\varepsilon_n}(X_k) \geq u^{\varepsilon_n}(X_{k+1}) - \varepsilon_n^3, \text{ for both } b_k = \pm 1, \tag{3.27}$$

leaving b_k to be selected through the estimate of $\phi(X_{k+1}) - \phi(X_k)$ below.

Just as we did in Proposition 3.4, it is desired that there exist $b_k = \pm 1$ leading to

$$\phi(X_{k+1}) - \phi(X_k) \geq w(\varepsilon_n),$$

for a modulus $w(\cdot)$. Again we use Lemma 2.3 and develop $\phi(X_{k+1}) - \phi(X_k)$ as follows:

$$\begin{aligned}\phi(X_{k+1}) - \phi(X_k) &= \left\langle \nabla\phi(X_k), \sqrt{2}\varepsilon_n b_k v_k - \alpha_n^k\right\rangle + \varepsilon_n^2 \partial_t\phi(X_k)\\ &+ \frac{1}{2}\left\langle \nabla^2\phi(X_k)\left(\sqrt{2}\varepsilon_n b_k v_k - \alpha_n^k\right), \sqrt{2}\varepsilon_n b_k v_k - \alpha_n^k\right\rangle + o(\varepsilon_n^2),\end{aligned} \tag{3.28}$$

where α_n^k is defined as in (3.11). We bear in mind that till now X_k, $d_l^{(k)}$ and $y_l^{(k)}$ are still depending on the choice of b_k, rather than express them too carefully by making our notation heavier.

Noticing (2.6), we again meet two cases for each k.

Case A. (Boundary condition dominant case) Assume there exists $b_k = 1$ or -1 such that

$$\liminf_{\varepsilon_n \to 0} \frac{|\alpha_n^k|}{\varepsilon_n} \geq C,$$

where $C > 0$ is a constant depending on k. For the same reason as in (3.13) and (3.15), we are led from (3.25) and (3.28) to

$$\sum_{b_k = \pm 1} (\phi(X_{k+1}) - \phi(X_k)) \geq C\eta_0 \varepsilon_n + o(\varepsilon_n), \tag{3.29}$$

which enables us to pick $b_k = 1$ or -1 in (3.26) so that

$$\phi(X_{k+1}) - \phi(X_k) \geq \frac{C}{2}\eta_0 \varepsilon_n + o(\varepsilon_n).$$

Case B. (Mixed type case) Assume now that

$$\liminf_{\varepsilon_n \to 0} \frac{|\alpha_n^k|}{\varepsilon_n} \to 0, \text{ for both values } b_k = \pm 1.$$

By taking a subsequence, we may replace "lim inf" by "lim" in the above equation. Then by virtue of (3.25) there exists $b_k = \pm 1$ such that

$$\begin{aligned} \phi(X_{k+1}) - \phi(X_k) \geq & \sqrt{2}\varepsilon_n \left|\langle \nabla \phi(X_k), v_k \rangle\right| + \varepsilon_n^2 \partial_t \phi(X_k) \\ & + \varepsilon_n^2 \langle \nabla^2 \phi(X_k) v_k, v_k \rangle + o(\varepsilon_n^2). \end{aligned} \tag{3.30}$$

Observing that $\nabla^2 \phi(X_k)$ is locally bounded, we employ Lemma 3.5 to learn that the right-hand side of the inequality above is greater than

$$\frac{1}{2}\Big(\varepsilon_n^2 \partial_t \phi(X_k) + \varepsilon_n^2 \left\langle \nabla^2 \phi(X_k) \frac{\nabla^\perp \phi(X_k)}{|\nabla \phi(X_k)|}, \frac{\nabla^\perp \phi(X_k)}{|\nabla \phi(X_k)|} \right\rangle \Big)$$

when ε_n is sufficiently small. Hence, by (3.24), we actually get

$$\phi(X_{k+1}) - \phi(X_k) \geq \frac{\eta_0}{2}\varepsilon_n^2 \text{ for } n \text{ sufficiently large.} \tag{3.31}$$

The rest of the proof is almost the same as that of Proposition 3.4. We take a subsequence of ε_n and its corresponding k satisfying the condition (3.17) and having X_k converge to a point $(x', t') \neq (x_0, t_0)$. After summing up both

(3.27) and (3.31) for k and sending ε_n to zero, we conclude the proof with a contradiction of

$$\underline{u}(x_0, t_0) - \phi(x_0, t_0) \geq \underline{u}(x', t') - \phi(x', t'). \qquad \square$$

We eventually complete our proof of Theorem 1.1 by gathering Propositions 3.1, 3.4 and 3.6 and employing a comparison principle presented in [24, Theorem 2.1]. We adapt that theorem to our backward mean curvature flow case.

Theorem 3.7. *Suppose* Ω *satisfies* (D1)*. Let* u *and* v *be, respectively, sub- and supersolutions of* (3.1a)-(3.1b)*. If* $u^*(x,T) \leq v_*(x,T)$*, then there is a modulus* m *such that*

$$u^*(x,t) - v_*(x,t) \leq m(|x-y|), \text{ for all } (x,y,t) \in \overline{\Omega} \times \overline{\Omega} \times (0,T).$$

In particular, $u^* \leq v_*$ *on* $\overline{\Omega} \times (0,T)$*.*

4. Relaxation to Nonconvex Domains

In this section, we intend to extend the convex billiard semiflow defined in Section 2.2 to a nonconvex one so that the deterministic game setting in Section 2.3 can be relaxed to connect the Neumann boundary problem (1.1) for more general domains. In what follows, we shall not repeat the whole process but merely note the difference.

We content ourselves with tackling nonconvex domains of special shapes. We write the curvature of $\partial\Omega$ at any point $z \in \partial\Omega$ as $\kappa(z)$. A point $z \in \partial\Omega$ is called an *exit* if any small neighborhood of z contains two points $z_1, z_2 \in \partial\Omega$ such that $\kappa(z_1) < 0$ and $\kappa(z_2) \geq 0$. Let I denote the set of all the exits. Such a notion is close to that of inflection points, but we reestablish it here for a particular purpose to be explained later. Now we assume

$$\text{(D2)} \quad \begin{cases} \Omega \text{ is a bounded domain in } \mathbb{R}^2\text{, and its boundary is of class } C^2 \\ \text{and has at most finitely many exits.} \end{cases}$$

Implying that the boundary has finite curve bumps, (D2) grants great convenience to classifying those zero-curvature points. It prevents the boundary from oscillating too much, so the curvature of $\partial\Omega$ around any $z \in I$, treated as a function, should be negative on one side of z and nonnegative on the other side.

We perceive about billiards that although the convexity of Ω is dropped, terminating takes place still on convex pieces of $\partial\Omega$; this is more rigorously stated as the lemma below.

Lemma 4.1. *Under the assumption* (D2)*, if a trajectory terminates at a point* $\Gamma(s_\infty) \in \Omega$ *with collision points, in order,* $\Gamma(s_1)$, $\Gamma(s_2)$, *...*, *then there is* $N > 0$ *such that, for all* $n \geq N$,

$$\kappa(\Gamma(s_n)) \geq 0.$$

Proof. In accordance with the definition of termination, we only pay attention to a neighborhood of the limit point, where $\partial\Omega$ can be represented by the graph of a function. More explicitly, we assume without loss of generality that there are $\delta > 0$, a C^2 function $f(\cdot) : [-\delta, \delta] \to \mathbb{R}$ and a sequence $\{x_j\} \subset \mathbb{R}$ such that

$$\begin{aligned}
&f(0) = 0,\ f'(0) = 0;\\
&\Gamma(s_j) = (x_j, f(x_j)) \text{ and } |x_j| \leq \delta,\ j = 1, 2, \dots;\\
&\Gamma(s_\infty) = (0, f(0)) = (0, 0).
\end{aligned}$$

In addition, it is quite clear that our assumption (D2) admits only three possible cases:

(1) $f''(x) \geq 0$ for all $x \in [-\delta, 0) \cup (0, \delta]$;
(2) $f''(x) < 0$ for all $x \in [-\delta, 0) \cup (0, \delta]$;
(3) $z \in I$; or expressed with no loss of generality as, $f''(x) < 0$ if $x \in (0, \delta]$ and $f''(x) \geq 0$ if $x \in [-\delta, 0)$.

(1) directly gives us what we need, so it suffices to look into the occasions (2) and (3).

We claim that (2) is impossible. If this is not the case, there is sufficiently large n such that

$$f''(x_n) < 0, \quad f''(x_{n+1}) < 0,$$

and thus

$$f''(x) \leq 0, \text{ for all } x \in [x_n, x_{n+1}].$$

It follows that $\Gamma(s_{n-1})$ and $\Gamma(s_n)$ are connected by a straight line segment outside $\overline{\Omega}$, contradicting the definition of a billiard.

At last, in the case of (3), we assume by contradiction that for any large N there is $n > N$ such that $f''(x_n) < 0$. For the same reason explained for the case (2), both $f''(x_{n-1})$ and $f''(x_{n+1})$ are nonnegative, which forces $x_{n-1}, x_{n+1} \in [-\delta, 0)$ and $x_n \in (0, \delta]$. In consequence, after the trajectory hits the boundary at $(x_n, f(x_n))$, its direction v satisfies

$$v \cdot (1, 0) \geq 0.$$

This convinces us that the ball will never hit the arc between $(-\delta, f(-\delta))$ and (0, 0) before it collides on the other part of the boundary, which clearly contradicts the property $x_{n+1} \in [-\delta, 0)$. $\square$

We have actually shown the following.

Lemma 4.2. *The statement of Lemma* 2.2 *holds under the assumption of* (D2) *in place of* (D1)*.*

All the above amounts to saying that relaxing a convex domain into such a nonconvex one does not largely affect the billiard motion in the interior; we just need to supplement our definition of S^t for the boundary sliding case. In view of (D2), the tangent to the boundary at each $z \in I$ induces a secant line segment of the domain, whose length will be uniformly bounded by some constant C. Take $L := \frac{C}{2}$. We just let the sliding ball leave the boundary along the secant and restart a straight line motion with a moving distance no more than L in the interior. This is the exact reason why elements of I are named exits. The following definition of the nonconvex billiard semiflow $\tilde{S}^t$ is an adaptation of Definition 2.1 based on Lemma 4.2.

Definition 4.1. *Assume that Ω satisfies (D2).*

(i) *If $x \in \partial\Omega$, and v equals the tangent of $\partial\Omega$, then*

$$\tilde{S}^t(x,v) := \begin{cases} S^t(x,v) = \Gamma(t) & \text{if } \ 0 \le t \le \tilde{t}, \\ \Gamma(\tilde{t}) + (t-\tilde{t})\Gamma_s(\tilde{t}) & \text{if } \ \tilde{t} \le t \le \tilde{t} + L, \end{cases}$$

where $\tilde{t} := \inf\{s \ge 0 : \Gamma(s) \in I\}$ and $\Gamma(\cdot)$ is the arc-length parametrization of $\partial\Omega$ such that $\Gamma(0) = x$ and $\Gamma_s(0) = v$.

(ii) *If $x \in \Omega$ and v is such that $T^t(x,v)$ terminates on $\partial\Omega$ at time t_0, then*

$$\tilde{S}^t(x,v) = \begin{cases} T^t(x,v) & \text{if } \ 0 \le t < t_0, \\ \tilde{S}^{t-t_0}(T^{t_0}(x,v), v_\infty) & \text{if } \ t \ge t_0, \end{cases}$$

where v_∞ is obtained from Lemma 4.2.

(iii) *If $x \in \partial\Omega$ and v points inside Ω, then*

$$\tilde{S}^t(x,v) := \begin{cases} x & \text{if } \ t = 0, \\ \tilde{S}^{t-\varepsilon}(x+\varepsilon v, v) & \text{if } \ t, \end{cases}$$

where $\varepsilon > 0$ is such that $x + \delta v \in \Omega$ for all $\delta \in (0,\varepsilon)$.

Through the foregoing definitions, the orbit of a billiard ball in the domain of (D2) can be clearly understood for all $t \in [0,L]$. This is actually adequate to sustain our application to games. The reason why we impose the assumption (D2) is that we would like to avoid the situation that termination, entrance and exit circulate infinitely many times in a finite time.

The new billiard trajectories keep convex even though the domain is not convex. Since $\tilde{S}^t$ ends up with a straight line segment after an exit, we still have the same results in Lemma 2.1 and hence the key Lemma 2.3 on the boundary adjustor; the only job here is to restrict t in $[0, L]$. We hereafter still use $S^t(x, v)$ to denote our general billiard $\tilde{S}^t(x, v)$ for simplicity in notation.

Lemma 4.3. *The statement of Lemma* 2.3 *holds under the assumption* (D2) *in place of* (D1) *but for* $0 \leq t \leq L$.

We now design a game in the nonconvex domain, following the same rules in Section 3 and only adding a restraint $0 \leq \sqrt{2}\varepsilon \leq L$. Since the step size ε will eventually be sent to zero, there is no loss in spite of our restraint. We acquire in the new domain game values approximating the viscosity solution of the Neumann boundary problem of motion by curvature.

Theorem 4.4. *Assume that* Ω *satisfies* (D2). *Assume that* u_0 *is a continuous function on* $\overline{\Omega}$. *Let* u^ε *be the associated value function of the game defined by* (1.3) *with* $0 < \varepsilon < L/\sqrt{2}$ *and* u_0 *be a continuous function in* $\overline{\Omega}$. *Then* u^ε *converges, as* $\varepsilon \to 0$, *to the unique viscosity solution of* (1.1) *uniformly on compacta of* $\overline{\Omega} \times (0, T]$.

The proof of Theorem 4.4 is almost the same as that of Theorem 1.1. The next two propositions are variants of Propositions 3.4 and 3.6.

Proposition 4.5. *Assume that* Ω *satisfies* (D2). *Let* $\{u^\varepsilon\}$ *be uniformly bounded in* $\overline{\Omega} \times (0, T)$ *for* $0 < \varepsilon < L/\sqrt{2}$. *Assume that* u^ε *satisfies the dynamic programming principle* (1.4). *Then* $\overline{u}$ *is a viscosity subsolution of* (3.1a)-(3.1b).

Proposition 4.6. *Assume that* Ω *satisfies* (D2). *Let* $\{u^\varepsilon\}$ *be uniformly bounded in* $\overline{\Omega} \times (0, T)$ *for* $0 < \varepsilon < L/\sqrt{2}$. *Assume that* u^ε *satisfies the dynamic programming principle* (1.4). *Then* $\underline{u}$ *is a viscosity supersolution of* (3.1a)-(3.1b).

However, the results in Lemma 2.4 and Proposition 3.1 are no longer true because Ω here is not convex. We have to find another way to construct barrier games. A related problem was solved by Ishii and Sato [14] about the construction of sub- and supersolutions when they applied Perron's method to show the existence of solutions for nonlinear oblique boundary problems. It turns out that their results can meet our needs.

Theorem 4.7 ([14, Theorem 4.4])**.** *Assume that* Ω *is a bounded domain in* $\mathbb{R}^n$ *with* C^1 *boundary. Assume the boundary condition* B *satisfies:*

(B1) *$B \in C(\mathbb{R}^n \times \mathbb{R}^n) \cap C^{1,1}(\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}))$;*
(B2) *For each $x \in \mathbb{R}^n$ the function $p \mapsto B(x,p)$ is positively homogeneous of degree one in p; i.e.,*

$$B(x, \alpha p) = \alpha B(x,p), \text{ for all } \alpha \geq 0 \text{ and } p \in \mathbb{R}^n \backslash \{0\};$$

(B3) *There exists a positive constant θ such that*

$$\langle \nu(z), \nabla_p B(z,p) \rangle \geq \theta,$$

for all $z \in \partial\Omega$ and $p \in \mathbb{R}^n \backslash \{0\}$.

Then there are a function $w \in C^{1,1}(\overline{\Omega} \times \overline{\Omega})$ and positive constants C and δ such that, for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

(i) $|x-y|^4 \leq w(x,y) \leq C|x-y|^4$,
$|\nabla_x w(x,y)| \vee |\nabla_y (x,y)| \leq C|x-y|^3$,
(ii) $B(x, \nabla_x w(x,y)) \geq 0$ *if* $y \in \partial\Omega$,
$B(y, -\nabla_y w(x,y)) \leq 0$ *if* $y \in \partial\Omega$,
(iii) $|\nabla_x w(x,y) + \nabla_y w(x,y)| \leq C|x-y|^4$,
$\rho(\nabla_x w(x,y), -\nabla_y w(x,y)) \leq C|x-y|$ *if* $0 < |x-y| \leq \delta$, *and for a.e.* $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,
(iv) $\nabla^2 w(x,y) \leq C \left\{ |x-y|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + |x-y|^4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right\}$.

Here $\vee$ and $\wedge$ stand respectively for the usual maximizing and minimizing operations, and

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|z_1| \wedge |z_2|}.$$

Let w be such a function and then for any fixed $x_0 \in \overline{\Omega}$, we see that $W(x) := w(x_0, x)$ belongs to $C^{1,1}(\overline{\Omega})$ and satisfies

$$|x - x_0|^4 \leq W(x) \leq C|x - x_0|^4, \text{ for all } x \in \overline{\Omega}, \tag{4.1}$$

and

$$\langle \nabla W(y), \nu(y) \rangle \geq 0, \text{ for all } y \in \partial\Omega. \tag{4.2}$$

In order to estimate $\langle \nabla W(x), S^{\sqrt{2}\varepsilon}(x, bv) - x \rangle$ for any $x \in \overline{\Omega}$, $b = \pm 1$ and $v \in \mathbf{S}^1$, we calculate $\left\langle \nabla W(x), \alpha^t \right\rangle$, where

$$\alpha^t = \alpha^t(x, bv) = \sum_{l=0}^{\infty} d_l^b \nu(y_l^b)$$

with t sufficiently small. We here suppress the superscript b for convenience. Substituting y_l into y in (4.2), we are led to

$$\begin{aligned}&\left\langle\nabla W(x), \alpha^t\right\rangle\\ &\geq \left\langle\nabla W(x)-\nabla W(y_0), \alpha^t\right\rangle+\left\langle\nabla W(y_0)-\nabla W(y_1), \sum_{l=1}^{\infty} d_l \nu(y_l)\right\rangle\\ &\quad+\sum_{k=2}^{\infty}\left\langle\nabla W(y_{k-1})-\nabla W(y_k), \sum_{l=k}^{\infty} d_l \nu(y_l)\right\rangle.\end{aligned} \tag{4.3}$$

Since W is a $C^{1,1}$ function and Ω is assumed to be bounded, there is a constant M such that

$$|\nabla W(x)-\nabla W(y)| \leq M|x-y|, \text{ for all } x, y \in \overline{\Omega}.$$

By Lemma 4.3, we may still use (2.6) and (2.7) to get

$$\begin{cases}|\nabla W(x)-\nabla W(y_0)| \leq \sqrt{2}\varepsilon M, \qquad \left|\alpha^t\right| \leq 2\sqrt{2}\varepsilon,\\ |\nabla W(y_0)-\nabla W(y_1)| \leq \sqrt{2}\varepsilon M, \qquad \left|\sum_{l=1}^{\infty} d_l \nu(y_l)\right| \leq 4\sqrt{2}\varepsilon,\\ \sum_{k=2}^{\infty}|\nabla W(y_{k-1})-\nabla W(y_k)| \leq 2\sqrt{2}\varepsilon M,\\ \left|\sum_{l=k}^{\infty} d_l \nu(y_l)\right| \leq 4\sqrt{2}\varepsilon \text{ (for all } k \geq 2).\end{cases}$$

Anyway, we can get from above a constant $C_1>0$ to proceed with the calculation in (4.3),

$$\left\langle\nabla W(x), \alpha^t\right\rangle \geq -C_1 \varepsilon^2.$$

Hence, we conclude

$$\begin{aligned}&\left\langle\nabla W(x), S^{\sqrt{2}\varepsilon}(x, bv)-x\right\rangle \leq\left\langle\nabla W(x), \sqrt{2}\varepsilon bv\right\rangle+C_1 \varepsilon^2,\\ &\qquad\qquad \text{for all } x \in \overline{\Omega},\ b=\pm 1 \text{ and } v \in \mathbf{S}^1.\end{aligned} \tag{4.4}$$

Lemma 4.8. *Let u^ε be the associated game value and $(x,t) \in \overline{\Omega} \times [\varepsilon^2, T]$. Then the following relations hold.*

(i) $u^\varepsilon(x, t-\varepsilon^2)-u^\varepsilon(x, t) \leq \sup_{y \in \Omega}\{u^\varepsilon(y, T^\varepsilon-\varepsilon^2)-u_0(y)\}$;

(ii) $u^\varepsilon(x, t-\varepsilon^2)-u^\varepsilon(x, t) \geq \inf_{y \in \Omega}\{u^\varepsilon(y, T^\varepsilon-\varepsilon^2)-u_0(y)\}$.

The lemma above says that comparison between two game values could be postponed whenever they share the same starting point. Its proof, based on the dynamic programming equation, is given in [10].

Lemma 4.9. *Let* Ω *satisfy* (D2). *Then there exists a constant* $K > 0$ *such that, for all* $x \in \overline{\Omega}$, *the following two inequalities hold.*

(i) $\inf_{|v|=1} \sup_{b=\pm 1} W\left(S^{\sqrt{2}\varepsilon}(x, bv)\right) - W(x) \leq K\varepsilon^2$;

(ii) $\sup_{|v|=1} \inf_{b=\pm 1} W\left(S^{\sqrt{2}\varepsilon}(x, bv)\right) - W(x) \leq K\varepsilon^2$.

Proof. If $\nabla W(x) = 0$, then, by Taylor's formula, both inequalities in question are trivial, since Ω is bounded and W is a $C^{1,1}$ function in it.

If otherwise $\nabla W(x) \neq 0$, then, for either b, we can take $\hat{v} = \frac{\nabla^{\perp} W(x)}{|\nabla W(x)|}$ so that Taylor's formula, together with the preceding estimate (4.4), yields

$$\begin{aligned} W\left(S^{\sqrt{2}\varepsilon}(x, b\hat{v})\right) - W(x) &\leq \left\langle \nabla W(x), \left(S^{\sqrt{2}\varepsilon}(x, b\hat{v}) - x\right)\right\rangle + M\varepsilon^2 \\ &\leq \langle \nabla W(x), \sqrt{2}\varepsilon b|v|\rangle + C_1\varepsilon^2 + M\varepsilon^2 \leq (C_1 + M)\varepsilon^2. \end{aligned}$$

Setting $K = C_1 + M$, we conclude (i).

For the same reason, choose $\hat{b}$ for any $v \in \mathbf{S}^1$ such that $\hat{b}\langle \nabla W(x), v\rangle \leq 0$ and thus we have

$$W\left(S^{\sqrt{2}\varepsilon}(x, \hat{b}v)\right) - W(x) \leq K\varepsilon^2,$$

which implies (ii). □

We now use this W to construct barriers and prove rigorously the consistency of $\overline{u}$ and $\underline{u}$ at the terminal time.

Proposition 4.10. *Under the same assumptions in Theorem* 4.4, *there holds* $\overline{u}(x, T) = \underline{u}(x, T)$.

Proof. Fix $x_0 \in \overline{\Omega}$. For any $\lambda > 0$, there is a large constant $\mu > 0$ such that $\overline{V}_\lambda(x) = \lambda + u_0(x_0) + \mu W(x)$ fulfills

$$\overline{V}_\lambda(x) \geq u_0(x), \text{ for all } x \in \overline{\Omega}. \tag{4.5}$$

Let $\overline{U}^{\varepsilon}_{\lambda}$ denote the value of the upper barrier game with an objective function $\overline{V}_\lambda$. It is implied by (4.5) that

$$u^{\varepsilon} \leq \overline{U}^{\varepsilon}_{\lambda}. \tag{4.6}$$

(We just let Paul follow his optimal decisions in the upper barrier game.) By virtue of Lemma 4.8(i) and the dynamic programming equation, we have

$$\overline{U}_\lambda^\varepsilon(x, t-\varepsilon^2) - \overline{U}_\lambda^\varepsilon(x,t) \leq \sup_{y\in\overline{\Omega}} \left\{ \inf_{|v|=1} \sup_{b=\pm 1} \overline{V}_\lambda\left(S^{\sqrt{2}\varepsilon}(y, bv)\right) - \overline{V}_\lambda(y) \right\},$$

which by Lemma 4.9 gives

$$\overline{U}_\lambda^\varepsilon(x, t-\varepsilon^2) - \overline{U}_\lambda^\varepsilon(x,t) \leq K\mu\varepsilon^2.$$

We therefore obtain from (4.6)

$$\begin{aligned} u^\varepsilon(x,t) - u_0(x_0) &\leq \overline{U}_\lambda^\varepsilon(x,t) - \overline{U}_\lambda^\varepsilon(x, T^\varepsilon) + \lambda + u_0(x_0) + \mu W(x) - u_0(x_0) \\ &\leq K\mu N\varepsilon^2 + \lambda + \mu W(x) \leq \lambda + K\mu(T-t) + C\mu|x-x_0|^4. \end{aligned}$$

So we are led to $\overline{u}(x_0, T) \leq \lambda + u_0(x_0)$ for every λ and thus $\overline{u}(x_0, T) \leq u_0(x_0)$. On the other hand, for every λ, if we play a lower barrier game with the terminal cost

$$\underline{V}_\lambda(x) = -\lambda + u_0(x_0) - \mu W(x),$$

where μ is such that $\underline{V}_\lambda \leq u_0$, then it follows by the same argument but an application of Lemma 4.8(ii) that

$$u^\varepsilon(x,t) \geq -\lambda + u_0(x_0) - K\mu(T-t) - C\mu|x-x_0|^4.$$

Hence, we finally obtain $\underline{u}(x_0, T) \geq u_0(x_0)$. □

Our proof of Proposition 4.10 is in essence the same as that of Proposition 3.1. In the case of a convex domain, our auxiliary function $W(x)$ reduces to $|x-x_0|^2$, which in turn implies Lemma 2.4 by the argument in Lemma 4.9.

We here remark that the assumption (D2) is a little bit restrictive. A billiard semiflow is likely to be assigned to more general domains and so are our purely deterministic games. It is then important to understand more complicated billiards with the occurrences of termination, entrance and exit circulating.

Appendix A. Singular Phenomena of Billiards

A.1. **Termination.** We study in this appendix more details about the particular termination phenomenon and see the sufficient conditions under which billiards can be well defined up to time infinity. Most of the content here is taken from [12]. We give it here for completeness and readers' convenience.

Let us start with an example to show the phenomenon really exists if no specific assumption is imposed on the table. In this example, it can be shown that the domain is strictly convex but its third derivative is unbounded. We shall take a converging point sequence on a unit circle, and then find a C^2

curve through all those points to make the convergent points vertices of some billiard trajectory. To be more precise, we denote by S the unit circle centered at the origin. Set up polar coordinates (r, θ) and pick a sequence of points $p_n = (r_n, \theta_n) \in S$, $n = 1, 2, \ldots$ with $\theta_n = n^{-\frac{1}{2}}$. It is obvious that p_n converges to $(1, 0)$. We get a broken line by connecting them up and then choose pieces γ_n of different unit circles through p_n such that the broken line satisfies the billiard law at each p_n with respect to γ_n, which are written via polar equations as $r = r_n(\theta)$. We connect these pieces appropriately to obtain a smooth curve γ, whose polar equation indeed fulfills

$$r(\theta) = r_n(\theta) + \alpha_n(\theta)(r_{n+1}(\theta) - r_n(\theta)). \tag{A.1}$$

Here α_n are such that

$$\alpha_n(\theta) = \alpha\Big(\frac{\theta - \theta_n}{\theta_n - \theta_{n+1}}\Big),$$

where $\alpha : [0, 1] \to [0, 1]$ is an infinitely differentiable function such that $\alpha(t) = 1$ for $t \leq \frac{1}{3}$ and $\alpha(t) = 0$ for $t \geq \frac{2}{3}$. By constructing γ, we have given an example of the trajectory terminating in a C^2 domain.

From now on, we further focus on the regularity of γ, especially the piece near the terminating point $(1, 0)$. We calculate through Taylor's expansion and get estimates, when n is sufficiently large, say larger than some n_0,

$$\delta_n := \theta_n - \theta_{n-1} = a_n n^{-\frac{3}{2}}, \quad \omega_n := \delta_{n+1} - \delta_n = b_n n^{-\frac{5}{2}}, \tag{A.2}$$

where $a_n \to -\frac{1}{2}$ and $b_n \to -\frac{3}{4}$. Then, for any $\theta \in [\theta_{n+1}, \theta_n]$,

$$\big|\alpha_n'(\theta)\big| \leq 3cn^{\frac{3}{2}}, \quad \big|\alpha_n''(\theta)\big| \leq 5cn^3, \tag{A.3}$$

where c is a bound for both $|\alpha'|$ and $|\alpha''|$. We use $r = g(\theta, \omega)$ to denote the unit circle passing through $(1, 0)$ and making an angle of $\frac{\omega}{4}$ with S at $(1, 0)$. It is well defined and infinitely differentiable on $D = \{(\theta, \omega) : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4}\}$. We can thus write

$$r_n(\theta) = g(\theta - \theta_n, \omega_n). \tag{A.4}$$

Let us assume there is a bound C for the partial derivatives of g. Then, noticing $g(0, \omega) = g(\theta, 0) = 1$, we have

$$\begin{aligned} &|g_1(\theta, \omega)| = C|\omega|, \quad |g_{11}(\theta, \omega)| \leq C|\omega|, \\ &|g(\theta, \omega) - 1| \leq C|\omega||\theta|, \end{aligned} \tag{A.5}$$

where g_1 and g_{11} respectively denote the first- and second-order partial derivatives of g with respect to its first variable. Consequently, from (A.2), (A.4) and (A.5), it is true that, for every $\theta \in [\theta_{n+1}, \theta_n]$ and n sufficiently large,

$$\begin{aligned} &|r_n'(\theta)| \leq Cn^{-\frac{5}{2}}, \quad |r_n''(\theta)| \leq Cn^{-\frac{5}{2}}, \\ &|r_n(\theta) - 1| \leq C|\omega_n||\delta_n| \leq C|a_n b_n n^{-4}| \leq Cn^{-4}. \end{aligned} \tag{A.6}$$

Now we take a large n_0 and redefine γ_{n_0} to be given by $r_{n_0}(\theta) = 1$ and only consider the piece of γ corresponding to $\theta \in [-\pi, \pi)\backslash(0, \theta_{n_0})$. Then we are led from (A.1), (A.3) and (A.6) to

$$\begin{aligned} &|r(\theta) - 1| \leq 3Cn^{-4}, \quad |r'(\theta)| \leq (3C + 6cC)n^{-\frac{5}{2}}, \\ &|r''(\theta)| \leq 6cCn^{-1}, \quad \left|\frac{r''(\theta)}{\theta}\right| \leq \frac{6cCn^{-1}}{(n+1)^{-\frac{1}{2}}}, \end{aligned}$$

for $n \geq n_0$ and $\theta \in [\theta_{n+1}, \theta_n]$. It follows that $\lim_{\theta \to 0} r(\theta) = 1$, $\lim_{\theta \to 0} r'(\theta) = 0$, $\lim_{\theta \to 0} r''(\theta) = 0$. Therefore $r'(0)$ and $r''(0)$ exist and r, r' and r'' are continuous. Moreover, as $\theta \to 0$,

$$\frac{r''(\theta) - r''(0)}{\theta} = \frac{r''(\theta)}{\theta} \to 0.$$

Hence $r'''(0)$ exists. We also see that if n_0 is picked sufficiently large then the curvature of γ can be made arbitrarily close to the curvature of S and hence never vanishes. By Theorem A.3 below, the third derivative of $\partial\Omega$ must be unbounded.

There are also several sufficient conditions for nonoccurrence of termination given below for convex billiards. We assume the billiard table Ω is bounded and convex, and consider the Poincaré map of a billiard flow T^t. Given a point $p \in \partial\Omega$ and an angle θ, $0 < \theta < \pi$, set $G(p, \theta) = (p', \theta')$ where p' is the other intersection of the directed line L which passes through p making an angle θ with the tangent to $\partial\Omega$ at p ($\partial\Omega$ is oriented counterclockwise) and θ' is the positive angle between L and the tangent to $\partial\Omega$ at p'. That is, if the billiard ball leaves from p making an angle θ with $\partial\Omega$ then p' is the next point of contact with the boundary $\partial\Omega$ and θ' is the next angle of reflection. Since Ω is convex, $G : \partial\Omega \times (0, \pi) \to \partial\Omega \times (0, \pi)$ is well defined. Set $G^n = G \circ G \ldots \circ G$ n times. Let $P_1 : \partial\Omega \times (0, \pi) \to \partial\Omega$ and $P_2 : \partial\Omega \times (0, \pi) \to (0, \pi)$ be the natural projections.

Theorem A.1 ([12, Theorem 1]). *If Ω is any billiard table and $\lim_{n\to\infty} P_1 \circ G^n(p, \theta) = q$ then*

(a) $\lim_{n\to\infty} P_2 G^n(p, \theta) = 0$ *or* π.
(b) *Either* $\sum_{n=1}^{\infty} P_2 G^n(p, \theta) < \infty$ *or* $\sum_{n=1}^{\infty} \left(\pi - P_2 G^n(p, \theta)\right) < \infty$.
(c) $\sum_{n=0}^{\infty} \left|P_1 G^{n+1}(p, \theta) - P_1 G^n(p, \theta)\right| < \infty$.

Proof. Set up a Cartesian coordinate system with the origin at q, the positive x-axis in the direction of the tangent to γ at q and the positive y-axis in the direction of the normal of γ at q. We may express the curve γ locally about q as $y = g(x)$ for $|x| < \varepsilon$ where $\varepsilon > 0$ and g is continuously differentiable and satisfies $g(0) = 0$, $g'(0) = 0$. We denote this portion of γ by $\overline{\gamma}$. By picking ε smaller if necessary we may further require that $|g'(x)| \le 10^{-6}$ for $|s| < \varepsilon$. Since $\lim_{n\to\infty} P_1G^n(p,\theta) = q$, there is an n_0 such that $q_n = P_1G^n(p,\theta) \in \overline{\gamma}$ for $n \ge n_0$. By replacing (p,θ) with $G^{n_0}(p,\theta)$ we may assume $n_0 = 0$. Set $\theta_n = P_2G^n(p,\theta)$ and let x_n and y_n be the x and y coordinates of q_n. It follows from the mean value theorem that the slope s of the line $\overline{q_nq_{n+1}}$ must satisfy $|s| \le 10^{-6}$.

We may now show that the sequence x_n is monotone, for if the x_n's should change direction $x_n < x_{n+1} > x_{n+2}$ or $x_n > x_{n+1} < x_{n+2}$ for some n, then clearly the tangent to γ at q_{n+1} must be nearly vertical. But this is ruled out because $|g'(x)| \le 10^{-6}$ for each x, $|x| < \varepsilon$. Hence the sequence x_n is monotone. By reversing the direction of the x-axis if necessary we may assume that x_n is monotone increasing. Since $\lim_{n\to\infty} x_n = 0$, we must have $x_n \le 0$. Since the direction of motion monotonically changes by $2\theta_i$ at each point of contact, $\sum_{i=1}^{\infty} 2\theta_i$ is equal to the angle between the initial direction of motion and the tangent at q. Conclusions (a), (b), the fact that

$$\sum_{n=1}^{\infty} |q_{n+1} - q_n| \le \text{arc length of } \gamma \text{ from } q_1 \text{ to } q,$$

and (c) follows easily. □

The next corollary is a direct consequence of the above theorem.

Corollary A.2 ([12, Corollary 2]). *Given $(p,\theta) \in \partial\Omega \times (0,\pi)$ and p_0 strictly between p and $q = P_1G(p,\theta)$ on the straight line segment from p to q, set $v = \frac{q-p}{|q-p|}$; then the following are equivalent:*

(a) *$T^t(p_0, v)$ is well defined for all $t \ge 0$.*
(b) *$\sum_{n=0}^{\infty} \left|P_1G^{n+1}(p,\theta) - P_1G^n(p,\theta)\right| = \infty$.*
(c) *$P_1G^n(p,\theta)$ diverges.*

An important sufficient condition of non-terminating is the following.

Theorem A.3 ([12, Theorem 3]). *If $\partial\Omega$ has a bounded third derivative and nowhere vanishing curvature, then $T^t(x,v)$ is well defined for all $(x,v) \in \Omega \times \mathbf{S}^1$.*

It is well known that G preserves the measure μ on $\partial\Omega \times (0,\pi)$:

$$d\mu = \sin\theta \, dx d\theta.$$

We then get to understand that termination rarely happens in measure.

Theorem A.4 ([12, Theorem 4]). *On any billiard table* Ω*, for almost all initial conditions* $(x, v) \in \Omega \times \mathbf{S}^1$*,* $T^t(x, v)$ *is well defined for all* $t \geq 0$*.*

The proofs of Theorem A.3 and Theorem A.4 are omitted here.

A.2. **Discontinuity of the Billiard Semiflow.** We now come to the last problem about the continuity of $S^t(x, v)$ with respect to its variables t, x and v. It is quite clear that $S^t(x, v)$ is continuous in time t but neither the continuity in x nor in v alone is known. But if we ask whether $(x, v) \mapsto S^t(x, v)$ is continuous, the answer will be "No" in general, as mentioned in Section 4. A good counterexample could be made from the termination phenomenon discussed above.

Suppose there is a terminating billiard trajectory in $\overline{\Omega}$ with a sequence of vertices $\{y_n\}_{n=1}^{\infty}$ and a terminating point y_∞ on $\partial\Omega$. From the results we have gotten before, we know that, as $k \to \infty$, $a_k = |y_{k+1} - y_k| \to 0$ and $v_k = (y_{k+1} - y_k)/a_k \to v_\infty$, where v_∞ is a unit tangent of $\partial\Omega$ at y_∞. Take a point x on the straight line segment $\overline{y_1 y_2}$ such that the distance $d(x, \partial\Omega) > 0$. Moreover, it is possible to pick an open interval (x_1, x_2) on the segment $\overline{y_1 y_2}$, satisfying $x \in (x_1, x_2)$ and

$$\inf_{z \in (x_1, x_2)} d(z, \partial\Omega) > 0. \tag{A.7}$$

We assume the total length of the billiard trajectory from x to y_∞ is τ_0. Then it is not hard to see that $S^{\tau_0}(\cdot, \cdot)$ is not continuous at $(y_\infty, -v_\infty)$. Indeed, since $y_\infty \in \partial\Omega$ and $-v_\infty$ is a tangent, we have by our definition $S^{\tau_0}(y_\infty, -v_\infty) \in \partial\Omega$. On the other hand, $S^{\tau_0}(y_k, -v_k) \in (x_1, x_2)$ when k is large. So discontinuity is obtained immediately due to (A.7).

The example may deprive each value function $u^\varepsilon(x, v)$ in Section 3 and Section 4 of its continuity in x. It is interesting to know whether our modified billiards are continuous under stronger assumptions for nonoccurrence of terminating, for instance, a sufficient condition given in Theorem A.3: $\partial\Omega$ has a bounded third derivative and nowhere vanishing curvature. A positive answer might enable us to simplify the proof of game characterization. This problem to some extent determines how wide the application of billiards can be for Neumann type boundary.

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