Advances in Differential Equations Volume 14, Numbers 9-10 (2009), 985–1018

EXISTENCE OF SOLUTIONS FOR A NONLINEAR BOUSSINESQ-STEFAN SYSTEM

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Abstract. In this paper, we consider a class of nonlinear Boussinesq-Stefan type systems: a Navier-Stokes equation for the velocity u and the pressure p with second member $F(\theta)$ where θ is the temperature field, the incompressibility condition and a scalar equation for θ having a convection term and a nonlinear diffusion operator, in which the right-hand side $\mu(\theta)|Du|^2$ is the dissipation energy. The function $F(\theta)$ is the buoyancy force which satisfies a growth assumption in dimension 2 and is bounded in dimension 3. We present some existence results through a fixed-point argument. We use the traditional results of Navier-Stokes equations and those of renormalized solutions. One of the difficulties is the coupling between the two equations for u and θ through the dissipation energy $\mu(\theta)|Du|^2$. This prevents us from showing compactness, at least if we use the classical results of renormalized solutions for a Stefan problem with L^1 data.

1. Introduction

In this paper, we deal with existence of a weak-renormalized solution for a class of nonlinear Boussinesq-Stefan systems of the type

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - 2 \text{ div } (\mu(\theta)Du) + \nabla p = F(\theta) \text{ in } Q, \tag{1.1}$$

$$\frac{\partial b(\theta)}{\partial t} - \text{ div } (a(x, \theta, \nabla\theta)) + \text{ div } (\Phi(\theta)) = 2\mu(\theta)|Du|^2 \text{ in } Q, \tag{1.2}$$

$$\text{div } u = 0 \text{ in } Q, \tag{1.3}$$

$$u = 0 \text{ and } \theta = 0 \text{ on } \Sigma_T, \tag{1.4}$$

$$u(t = 0) = u_0 \text{ and } b(\theta)(t = 0) = b_0 \text{ in } \Omega, \tag{1.5}$$

Accepted for publication: May 2009.
AMS Subject Classifications: 35K55, 35D05, 35B30, 76R05, 76R50.

where Ω is an open, Lipschitz and bounded subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$), with boundary $\partial\Omega$, $T > 0$, $Q = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. The unknowns are the displacement field $u : \Omega \times (0,T) \longrightarrow \mathbb{R}^N$ and the temperature field $\theta : \Omega \times (0,T) \longrightarrow \mathbb{R}$. The field $Du = \frac{1}{2}(\nabla u + (\nabla u)^t)$ is the so-called rate-deformation tensor field. Equation (1) is the conservation equation of momentum.

In this equation, the quantities μ and p respectively denote the kinematic viscosity and the pressure of the fluid. In the right-hand side of equation (1), the function $F(\theta)$, where F is continuous from $\mathbb{R}$ into $\mathbb{R}^N$, represents gravity's force which is proportional to the variations of density through the temperature. The function μ is assumed to be continuous and bounded on $\mathbb{R}$. The initial data u_0 belongs to $(L^2(\Omega))^N$, with a null divergence and $u_0.n = 0$ on $\partial\Omega$, where we denote by n the unit outward normal to $\partial\Omega$. Equation (2) is the temperature equation, in which the right-hand side $\mu(\theta)|Du|^2$ is the dissipation energy. In (2), b is a monotone function defined from $\mathbb{R}$ into $\mathcal{P}(\mathbb{R})$ such that $b^{-1} \in C^0(\mathbb{R})$. Moreover, the function $a(x,s,\xi)$ is monotone and coercive with respect to ξ. The function Φ is locally Lipschitz continuous from $\mathbb{R}$ into $\mathbb{R}^N$. The initial data b_0 belongs to $L^1(\Omega)$.

The Boussinesq-Stefan system $(1.1) - (1.5)$ of hydrodynamics equations arises from the coupling between a Navier-Stokes equation for the velocity u and the pressure p and an additional scalar equation for the temperature having a convection term and a nonlinear diffusion operator. In the case where $b \in C^1$ and under additional assumptions on b, there are solutions of systems similar to $(1),(5)$ but within a particular framework: F does not depend on θ, $b(\theta) = \theta$, $\Phi \equiv 0$, $a(x,s,\xi) = \xi$, $\mu(\theta)|Du|^2$ is replaced by a function given in L^1 and the convection term is of type $u \cdot \nabla\theta$ (see, e.g.[2], [11], [14], [19]). In the same framework but with a nonlinear internal energy b and a null dissipation energy, an existence and uniqueness result of a weak solution has been established in [14].

Under the assumptions adopted on b in our paper (possible discontinuities of b), Equation (2) completed with standard boundary and initial conditions is a Stefan-problem with L^1 data (see [6] and the references given there). When System $(1),(5)$ is investigated, the analysis developed in [6] cannot be used, since the dissipation energy is not stable in $L^1(Q)$ with respect to approximations. Loosely speaking, we cannot deduce directly from [6] the pointwise convergence of θ with respect to approximations. We will proceed in another way. We define a new smooth function f (at least twice differentiable) such that $f(x) = x^3$ on (a,b) $(a < 0 < b)$ and $b - a$ is small enough. The function f is linear out of (a,b). We set $\gamma_1 = f^+$ and

$\gamma_2 = -f^-$. The functions $\gamma_1(b(r) - \varepsilon - 1)$ and $\gamma_2(b(r) + \varepsilon)$ are C^1-functions. By an Aubin's type argument, we show that $\gamma_1(b(\theta) - \varepsilon - 1)$ and $\gamma_2(b(\theta) + \varepsilon)$ converge almost everywhere with respect to approximations and we deduce the pointwise convergence of θ (with respect to approximations). The model studied in this paper is more general than those which are described in the above references:

The viscosity coefficient and the external force are temperature-dependent (with nonlinear dependence).

The internal energy is also assumed to be nonlinear with respect to the temperature and this affects the time derivative term in the temperature equation.

There is a right-hand side in the temperature equation which is quadratic in the spatial gradient of the velocity field.

The diffusion term in the temperature equation is of the Leray-Lions kind.

Existence of solutions of (1) $-$ (5) is based on the stability of equations (1) and (2) when approximation arguments are used, or on the uniqueness of solutions of these equations when we use fixed-point arguments. We are thus constrained to distinguish the case $N = 2$ and $N = 3$.

Case $N = 2$**.** It is known that if $F(\theta) \in L^2(0, T; H^{-1}(\Omega))$, then the Navier-Stokes equation (1) has a unique weak solution for $u_0 \in (L^2(\Omega))^2$ and the dissipation energy $\mu(\theta)|Du|^2$ is stable in $L^1(Q)$ with respect to approximations. Equation (2) is thus considered naturally within the L^1 framework. There are many works on parabolic equations with L^1 data (see, e.g. [4], [5], [6], [12], [22]). To guarantee the uniqueness and the stability of the solution of (2), we use the framework of renormalized solutions which have these properties contrary to the weak solutions. This notion has been introduced by R.-J. DiPerna and P.-L. Lions in [15] and [16] for the study of Boltzmann equations (see also P-L. Lions [19] for applications to fluid mechanics models).

It was then adapted by Boccardo et al [9] and Murat [20], [21] to nonlinear elliptic problems. At the same time, the equivalent notion of entropy solutions have been developed independently by Bénilan et al [8] for the study of nonlinear elliptic problems. The concept of renormalized solutions was also adapted to the parabolic version for equations of type (2) with L^1 data (see, e.g [1], [6], [17]). The type of solutions which we obtain depends on the behavior of the function F. If, for example, F is bounded, we obtain solutions for all given initial data $u_0 \in (L^2(\Omega))^2$ and $b_0 \in L^1(\Omega)$. To study the case of more general functions F, it is necessary to investigate the regularity of the solutions of (2). Under the assumptions adopted on b, the renormalized solutions of equation (2) satisfy the following regularities:

$$\theta \in L^{\infty}(0,T,L^1(\Omega)), \quad \forall\, k>0, \quad \int_0^T\int_\Omega |DT_k(\theta)|^2\,dx\,dt \le C\,k,$$

with $T_k(r)$ =min(k,max($r,-k$)) for all $r \in \mathbb{R}$. We show in a first step that $\theta \in L^r(0,T;L^q(\Omega))$ with $1<q<\infty$ and $1 \le r < \frac{q}{q-1}$ (a similar result is shown in [23] for $N>2$ but it cannot be used as such for $N=2$). In order to have $F(\theta) \in L^2(0,T;H^{-1}(\Omega))$, we are constrained to state the following growth assumption on F:

$$\forall r \in \mathbb{R}, \qquad |F(r)| \le a + M|r|^{\alpha},$$

with $a \ge 0$, $M \ge 0$ and $2\alpha \in [0,3[$. We show in a second step that $F(\theta)$ is identified with an element of $L^2(0,T;H^{-1}(\Omega))$ with

$$\|F(\theta)\|_{L^2(0,T;H^{-1}(\Omega))} \le C(a + \|\theta\|^{\alpha}_{L^r(0,T;L^q(\Omega))}).$$

These arguments allow us, thanks to fixed-point methods, to show that $(1)-(5)$ has solutions for small initial data.

Case $N=3$**.** The uniqueness of solution of the Navier-Stokes equation (1) and the stability of the dissipation energy are open problems if u_0 belongs only to $(L^2(\Omega))^3$. If, for example, $u_0 \in (H_0^1(\Omega))^3$, and F is bounded such that $\|u_0\|_{H_0^1(\Omega)} + \|F\|_{L^\infty(\mathbb{R})} \le \eta$, with η a small enough constant, we obtain the existence of a solution of (1)-(5) by the same techniques as in the case $N=2$.

The paper is organized as follows. Section 2 introduce the usual Navier-Stokes functional setting (according to the variational formulation introduced by Leray [18] within the framework of free divergence functional spaces), presents the assumptions needed in the present study and gives the definition of a weak-renormalized solution of $(1)-(5)$. In Section 3, we describe the method used to prove existence of a solution through a fixed-point argument with respect to the unknown θ. In Section 4, firstly, we recall the existence, uniqueness and stability results of the solution of parabolic problem $(3.5)-(3.7)$ resulting from $(1)-(5)$ in Section 3. We assume in this section that u is given in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with div $u=0$ in Q. Secondly, we show a few regularity results of any renormalized solution of $(3.5)-(3.7)$. In Section 5, we deal with existence of a solution of $(1)-(5)$ for $N=2$. We distinguish four cases according to the values of α. For $\alpha=0$ (F is bounded), we show that the studied system admits a weak-renormalized solution for all initial data by using Schauder's fixed-point theorem in $L^1(Q)$. For $0<2\alpha\le 1$, we introduce an approximate system of $(1)-(5)$ by regularizing the function F by F^ε (F^ε being continuous and bounded). Then, we can use the result of the first case ($\alpha=0$) to deduce that there exists a weak-renormalized solution of this approximate system for all initial data and we pass to the limit in this system to obtain the existence of a solution of $(1)-(5)$. For the last cases where $1<2\alpha<2$ and $2\le 2\alpha<3$, Schauder's fixed-point theorem ensures the existence of a weak-renormalized solution of the coupled system for small initial data. In Section 6, we deal with dimension $N=3$. In the particular case, where F is bounded in $L^\infty(\mathbb{R})$ and $u_0\in(H_0^1(\Omega))^3$, we prove the existence of a weak-renormalized solution of the coupled system for small data F and u_0.

2. Assumptions and definition of a weak-renormalized solution

Throughout the paper, we assume that the following assumptions hold true: Ω is an open, Lipschitz and bounded subset of $\mathbb{R}^N$ ($N=2$ or $N=3$) with boundary $\partial\Omega$, $T>0$ is given and we set $Q=\Omega\times(0,T)$ and $\Sigma_T=\partial\Omega\times(0,T)$. We introduce the usual Navier-Stokes functional setting

$$C_\sigma^\infty(\Omega)=\{u\in C_0^\infty(\Omega;\mathbb{R}^N):\operatorname{div}u=0\},$$
$$L_\sigma^q(\Omega)=\text{closure of } C_\sigma^\infty(\Omega) \text{ in } L^q(\Omega;\mathbb{R}^N),$$
$$H_\sigma^1(\Omega)=\text{closure of } C_\sigma^\infty(\Omega) \text{ in } H_0^1(\Omega;\mathbb{R}^N),$$
$$L_\sigma^q(Q)=L^q(0,T;L_\sigma^q(\Omega)),$$

when $q\ge 1$. We will need the following assumptions:

$$b:\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R}) \text{ is a monotone function such that } 0\in b(0);\ \text{i.e., } b^{-1}(0)=0, \tag{2.1}$$

where b^{-1} denotes the inverse function of b,

$$|b(r)|\ge C|r|\,\forall r\in\mathbb{R}, \tag{2.2}$$

where $C>0$ is a constant,

$$b^{-1} \text{ is continuous on } \mathbb{R}, \tag{2.3}$$

$$\begin{aligned}&\text{for any } n\in\mathbb{N}, \text{the function } b \text{ has a}\\&\text{finite number of discontinuity points in } [-n,n],\end{aligned} \tag{2.4}$$

and b is a C^1-function out of these points,

$$\mu \text{ is continuous on } \mathbb{R},\ \text{such that } m_0\le\mu(s)\le m_1,\forall s\in\mathbb{R}, \tag{2.5}$$

with $0<m_0\le m_1$,

$$F \text{ is continuous from } \mathbb{R} \text{ into } \mathbb{R}^N \text{ and satisfies the growth assumption} \tag{2.6}$$

$$\forall r \in \mathbb{R} \qquad |F(r)| \le a + M|r|^{\alpha} \text{ with } a \ge 0, M \ge 0 \text{ and } 0 \le 2\alpha < 3,$$

$a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,s,\xi)\xi \ge \alpha'|\xi|^2, \tag{2.7}$$

$$|a(x,s,\xi) - a(x,r,\xi)| \le L(s,r)(K(x) + |\xi|)|s-r|, \tag{2.8}$$

$$|a(x,s,\xi)| \le M(|s|)(K(x) + |\xi|), \tag{2.9}$$

$$(a(x,s,\xi) - a(x,s,\eta))(\xi - \eta) \ge 0, \tag{2.10}$$

for every ξ, η in $\mathbb{R}^N$, every s, r in $\mathbb{R}$ and almost every x in Ω, where $\alpha' > 0$, $K(x) \in L^2(\Omega)$ and L: $\mathbb{R}^2 \longrightarrow \mathbb{R}, M : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions,

$$\text{the function } \Phi : \mathbb{R} \longrightarrow \mathbb{R}^N \text{ is locally Lipschitz continuous,} \tag{2.11}$$

$$u_0 \in (L^2(\Omega))^N, \text{ div } u_0 = 0 \text{ in } Q, \text{ and } u_0 \cdot n = 0 \text{ on } \partial\Omega, \tag{2.12}$$

$$b_0 \in L^1(\Omega). \tag{2.13}$$

In the case $N = 3$ (Section 6), we adopt stronger assumptions than (2.6) and (2.12); i.e., F is bounded in $L^\infty(\mathbb{R})$ ($\alpha = 0$) and $u_0 \in (H_0^1(\Omega))^3$.

As usual, the pressure p is eliminated in the system (1) – (5). Rham's lemma [13] allows us to recover this unknown. In the sequel we study the following system:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - 2 \text{ div } (\mu(\theta)Du) = F(\theta) \text{ in } (H_\sigma^1)'(\Omega), \tag{2.14}$$

for almost every t $\in (0,T)$,

$$\frac{\partial b(\theta)}{\partial t} - \text{ div } (a(x,\theta,\nabla\theta)) + \text{ div } (\Phi(\theta)) = 2\mu(\theta)|Du|^2 \text{ in } Q, \tag{2.15}$$

$$\text{div } u = 0 \text{ in } Q, \tag{2.16}$$

$$u = 0 \text{ and } \theta = 0 \text{ on } \Sigma_T, \tag{2.17}$$

$$u(t=0) = u_0 \text{ and } b(\theta)(t=0) = b_0 \text{ in } \Omega. \tag{2.18}$$

To introduce the weak-renormalized solution of problem (2.14) – (2.18), we consider the usual bilinear and trilinear forms defined by

$$A_\theta(u,v) = \frac{1}{2}\sum_{i,j=1}^{N} \int_\Omega \mu(\theta)(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\frac{\partial v_j}{\partial x_i}\, dx,$$

$$d(u,v,w) = \sum_{i,j=1}^{N} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_j \, dx = \int_\Omega (u \cdot \nabla)v \cdot w \, dx,$$

for all $u, v \in H_\sigma^1(\Omega), w \in H_\sigma^1(\Omega)$, where θ is a measurable function.

We recall that a_θ is continuous and coercive in $H^1_\sigma(\Omega)\times H^1_\sigma(\Omega)$ for almost every $t\in[0,T]$ and that d is anti-symmetric and continuous in $H^1_\sigma(\Omega)\times H^1_\sigma(\Omega)\times H^1_\sigma(\Omega)$.

Definition 2.1 (Renormalized solution)**.** *Let G be a function in $L^1(Q)$ and assume b_0 belongs to $L^1(\Omega)$. A measurable function θ defined on Q is a renormalized solution of the problem*

$$P_1(G,\theta_0)\begin{cases} \dfrac{\partial b(\theta)}{\partial t} - \operatorname{div}(a(x,\theta,\nabla\theta)) + \operatorname{div}(\Phi(\theta)) = G & \text{in } Q,\\ \theta = 0 & \text{on } \Sigma_T,\\ b(\theta)(t=0) = b_0 & \text{in } \Omega,\end{cases}$$

if

$$\theta\in L^\infty(0,T;L^1(\Omega)); \tag{2.19}$$

$$T_K(\theta)\in L^2(0,T;H^1_0(\Omega)) \text{ for any } K\geq 0; \tag{2.20}$$

$$\frac{1}{n}\int_{\{(x,t)\in Q: n\leq|\theta(x,t)|\leq 2n\}} a(x,\theta,\nabla\theta)\nabla\theta\,dx\,dt \longrightarrow 0 \text{ as } n\to+\infty; \tag{2.21}$$

there exists $\beta_\theta\in L^1(Q)$ such that $\beta_\theta\in b(\theta)$ almost everywhere in Q, and the following equation holds:

$$\begin{aligned} &-\int_0^T\int_\Omega \varphi_t\int_0^{\beta_\theta} S'(b^{-1}(r))dr\,dx\,dt\\ &\quad-\int_\Omega\varphi(0)\Big(\int_0^{b_0} S'(b^{-1}(r))dr\Big)dx + \int_0^T\int_\Omega a(x,\theta,\nabla\theta)\nabla\varphi S'(\theta)dx\,dt\\ &+\int_0^T\int_\Omega a(x,\theta,\nabla\theta)\nabla\theta S''(\theta)\,\varphi dx\,dt - \int_0^T\int_\Omega\nabla\varphi\Big(\int_0^\theta \Phi'(\xi)S'(\xi)d\xi\Big)dx\,dt\\ &= \int_0^T\int_\Omega GS'(\theta)\varphi dx\,dt, \end{aligned} \tag{2.22}$$

for any $S\in W^{2,\infty}(\mathbb{R})$ with S' having a compact support, and any $\varphi\in C_0^\infty([0,T)\times\bar\Omega)$ such that $S'(\theta)\varphi\in L^2(0,T;H^1_0(\Omega))$.

Remark 2.1. Since we supposed that b^{-1} is continuous on $\mathbb{R}$, we have $\theta = b^{-1}(\beta_\theta)$.

We point out the interesting results shown in [6].

Theorem 2.2. *Under the above assumptions on the data G and b_0, the problem $P_1(G,\theta_0)$ admits a unique renormalized solution.*

If θ^ε and θ^η are two renormalized solutions relating to the data $(G^\varepsilon, b_0^\varepsilon)$ and (G^η, b_0^η), then

$$\|\beta_{\theta^\varepsilon} - \beta_{\theta^\eta}(t)\|_{L^1(\Omega))} \leq \|G^\varepsilon - G^\eta\|_{L^1(Q)} + \|b_0^\varepsilon - b_0^\eta\|_{L^1(\Omega)}, \ \ \forall t \in (0,T).$$

The following definition specifies the concept of a weak-renormalized solution of the system (2.14) – (2.18).

Definition 2.2. *A couple of functions (θ, u) defined on $\Omega \times (0,T)$ is called a weak-renormalized solution of* (2.14) – (2.18) *if u and θ satisfy*

$$u \in L^2(0,T; H^1_\sigma(\Omega)) \cap L^\infty(0,T; L^2_\sigma(\Omega)), \tag{2.23}$$

$$T_K(\theta) \in L^2(0,T; H^1_0(\Omega)) \text{ for any } K \geq 0 \text{ and } \theta \in L^\infty(0,T; L^1(\Omega)), \tag{2.24}$$

$$\frac{1}{n}\int_{\{(x,t)\in Q: n \leq |\theta(x,t)| \leq 2n\}} a(x,\theta,\nabla\theta)\nabla\theta \, dx\, dt \longrightarrow 0 \text{ as } n \to +\infty, \tag{2.25}$$

$$u \text{ is a weak solution of the Navier-Stokes equation } (2.14), \tag{2.26}$$

$$u(t=0) = u_0 \text{ a.e. in } \Omega, \tag{2.27}$$

there exists $\beta_\theta \in L^1(Q)$ such that $\beta_\theta \in b(\theta)$ almost everywhere in Q, and the following equation holds:

$$\begin{aligned}
&-\int_0^T\int_\Omega \varphi_t \int_0^{\beta_\theta} S'(b^{-1}(r))dr\, dx\, dt \\
&\quad - \int_\Omega \varphi(0)\Big(\int_0^{b_0} S'(b^{-1}(r))dr\Big)dx + \int_0^T\int_\Omega a(x,\theta,\nabla\theta)\nabla\varphi S'(\theta)dx\, dt \\
&+ \int_0^T\int_\Omega a(x,\theta,\nabla\theta)\nabla\theta S''(\theta)\,\varphi dx\, dt - \int_0^T\int_\Omega \nabla\varphi\Big(\int_0^\theta \Phi'(\xi)S'(\xi)d\xi\Big)dx\, dt \\
&\qquad = \int_0^T\int_\Omega 2\mu(\theta)|Du|^2\, S'(\theta)\varphi dx\, dt,
\end{aligned} \tag{2.28}$$

for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support, and any $\varphi \in C_0^\infty([0,T) \times \bar\Omega)$ such that $S'(\theta)\varphi \in L^2(0,T; H^1_0(\Omega))$.

3. The fixed-point argument

In this section, we describe the (standard) method used to prove the existence of a solution through a fixed-point argument with respect to the unknown θ.

Let L be a Lebesgue's space of the type $L = L^r(0,T,L^q(\Omega))$ $(r, q \geq 1)$. For a fixed $\theta \in L$, let us consider the Navier-Stokes equations

$$u_t + (u\cdot\nabla)u - 2 \text{ div } (\mu(\theta)Du) = F(\theta) \text{ in } (H^1_\sigma)'(\Omega), \tag{3.1}$$

for almost every t $\in (0,T)$,

$$\text{div } u = 0 \text{ in } Q, \tag{3.2}$$

$$u = 0 \text{ on } \Sigma_T, \tag{3.3}$$

$$u(t=0) = u_0 \text{ in } \Omega. \tag{3.4}$$

Suppose that (3.1) – (3.4) admit a unique solution $u \in L^2(0,T;H^1_\sigma(\Omega))$ so that $\mu(\theta)|Du|^2 \in L^1(Q)$. Indeed, this is the case if $F(\theta) \in L^2(0,T;H^{-1}(\Omega))$.

Then, we consider the parabolic problem

$$\frac{\partial b(\hat{\theta})}{\partial t} - \text{ div } (a(x,\hat{\theta},\nabla\hat{\theta})) + \text{ div } (\Phi(\hat{\theta})) = 2\mu(\theta)|Du|^2 \text{ in } Q, \tag{3.5}$$

$$\hat{\theta} = 0 \text{ on } \Sigma_T, \tag{3.6}$$

$$b(\hat{\theta})(t=0) = b_0 \text{ in } \Omega. \tag{3.7}$$

Assume that the assumptions on the data insure that (3.5) – (3.7) admit a unique renormalized solution $\hat{\theta}$. In order to apply a fixed-point argument, it is first necessary to have $\hat{\theta} \in L$ so that we can consider the mapping $\psi : \theta \longrightarrow \hat{\theta}$ from L into L.

As a consequence, the value of α must be such that the regularity of the renormalized solution of (3.5) – (3.7) implies $F(\theta) \in L^2(0,T;H^{-1}(\Omega))$. This leads to different choices of L depending on the range of α. Secondly, we use the stability of the renormalized solution with respect to the data (see, e.g. [6]) and the stability of the quantity $\mu(\theta)|Du|^2$ (with respect to approximations) to show that ψ is continuous and compact. At last, in order to show that there exists a ball B of L such that $\psi(B) \subset B$, we distinguish two cases: if $0 \leq 2\alpha \leq 1$, this is proved for any data satisfying (2.12)–(2.13), while if $1 < 2\alpha < 3$, we are led to assume that a, $\|b_0\|_{L^1(\Omega)}$ and $\|u_0\|_{L^2(\Omega)}$ are small enough.

4. The parabolic problem

In this section, firstly, we recall the existence and uniqueness results of the solution of (3.5) – (3.7). There are now a large number of papers on the properties of renormalized (or entropic) solutions for this type of problem (see, e.g. [4], [5], [6], [7], [12], [19], [22], [23]). We assume in this section that u is given in $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with div $u = 0$ in Q. Secondly, we show a few regularity results of any renormalized solution of (3.5) – (3.7). Note that since L is a Lebesgue's space, assumption (2.6) implies that $\mu(\theta)|Du|^2 \in L^1(Q)$. We find a detailed proof of the following lemma in [6].

Lemma 4.1. *Under the assumptions* (2.1)-(2.5), (2.7)-(2.11) *and* (2.13), *there exists a unique renormalized solution* $\hat{\theta}$ *of* (3.5)−(3.7) *such that* $T_K(\hat{\theta}) \in L^2(0,T;H_0^1(\Omega))$, *for all* $K \geq 0$ *and* $\beta_{\hat{\theta}} \in L^\infty(0,T;L^1(\Omega))$.

By (2.2), we also have $\hat{\theta} \in L^\infty(0,T;L^1(\Omega))$. We can now prove the regularity results of the renormalized solution of (3.5) − (3.7) for $N \geq 1$.

Lemma 4.2. *Any renormalized solution* $\hat{\theta}$ *of* (3.5) − (3.7) *satisfies the following estimates:*

For $N \geq 1$ *and all* $p \in [1, \frac{N+2}{N})$, *there exists a constant* c *(depending only on* p, N, Ω, *and* T*) such that*

$$\|\hat{\theta}\|_{L^p(Q)} \leq c\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b_0\|_{L^1(\Omega)}).$$

For $N = 2$, *for all* q, r *such that* $1 < q < \infty$, *and* $1 \leq r < \frac{q}{q-1}$, *we have* $\hat{\theta} \in L^r(0,T;L^q(\Omega))$, *and there exists a constant* C *(depending on* T, r, q, Ω*) such that*

$$\|\hat{\theta}\|_{L^r(0,T;L^q(\Omega))} \leq C\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b_0\|_{L^1(\Omega)}).$$

Proof of Lemma 4.2. We start by showing the first part of Lemma 4.2. We mainly use the standard method of [6]. Indeed, by Theorem 2.2, we have

$$\|\beta_{\hat{\theta}}\|_{L^\infty(0,T;L^1(\Omega))} \leq c\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b_0\|_{L^1(\Omega)}),$$

with c a generic constant depending on p, N, Ω and T. Assumption (2.2) gives thus the following estimate:

$$\|\hat{\theta}\|_{L^\infty(0,T;L^1(\Omega))} \leq c\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b_0\|_{L^1(\Omega)}), \tag{4.1}$$

and by using lemma 3.5 of [6], we obtain

$$\int_Q |\nabla T_K(\hat{\theta})|^2\,dx\,dt \leq K\,c\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b_0\|_{L^1(\Omega)}). \tag{4.2}$$

Estimate (4.2) allows us to conclude from lemma 1 of [3] that, for all $p \in [1, \frac{N+2}{N})$, there exists a constant C (depending only on p, N, Ω, and T) such that

$$\|\hat{\theta}\|_{L^p(Q)} \leq C\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b_0\|_{L^1(\Omega)})^{\frac{N}{N+2}} \cdot \|\hat{\theta}\|^{\frac{2}{N+2}}_{L^\infty(0,T;L^1(\Omega))}. \tag{4.3}$$

Estimates (4.1) and (4.3) allow us to achieve the proof of the first part of Lemma 4.2.

Now, we turn to the proof of the second part of Lemma 4.2. A similar result was shown in [23] in the case where $N > 2$ but it cannot be used as such for $N = 2$.

Since $\hat{\theta}$ is a renormalized solution of (3.5) – (3.7) and in view of (4.1) and (4.2), we have

$$\hat{\theta} \in L^\infty(0,T;L^1(\Omega)), \|\hat{\theta}\|_{L^\infty(0,T;L^1(\Omega))} \le M, \tag{4.4}$$

$$\forall K>0, T_K(\hat{\theta}) \in L^2(0,T;H_0^1(\Omega)), \text{with} \int_Q |DT_K(\hat{\theta})|^2\,dx\,dt \le K\,M. \tag{4.5}$$

Our goal is to show that $\|\hat{\theta}\|_{L^r(0,T;L^q(\Omega))} \le C\,M$, for all q,r such that $1 < q < \infty$, and $1 \le r < \frac{q}{q-1}$, where C is a generic constant. By Gagliardo-Nirenberg's inequality, we have

$$\int_\Omega |T_K(\hat{\theta})|^3\,dx \le C\int_\Omega |T_K(\hat{\theta})|\,dx \int_\Omega |DT_K(\hat{\theta})|^2\,dx \le C\,M\int_\Omega |DT_K(\hat{\theta})|^2\,dx,$$

for almost any t in $(0,T)$. This yields

$$\text{meas}\{(x,t) : |\hat{\theta}| > K\} \le C\,\frac{M}{K^3}\int_\Omega |DT_K(\hat{\theta})|^2\,dx. \tag{4.6}$$

We write

$$\int_0^T \Big(\int_\Omega |\hat{\theta}|^q\,dx\Big)^{\frac{r}{q}}dt = \int_0^T \Big(\int_0^\infty \text{meas}\{x\in\Omega : |\hat{\theta}|^q > s\}\,ds\Big)^{\frac{r}{q}}dt,$$

and by (4.4), we have

$$\text{meas}\{x\in\Omega : |\hat{\theta}| > K\} \le \frac{M}{K}, \tag{4.7}$$

for almost any t in $(0,T)$.

For almost any t in $(0,T)$, we have for any $0 \le \sigma \le 1$

$$\text{meas}\{x\in\Omega : |\hat{\theta}(x,t)|^q > s\} = \text{meas}\{x\in\Omega : |\hat{\theta}(x,t)| > s^{\frac{1}{q}}\}$$
$$= \Big(\text{meas}\{x\in\Omega : |\hat{\theta}(x,t)| > s^{\frac{1}{q}}\}\Big)^\sigma \Big(\text{meas}\{x\in\Omega : |\hat{\theta}(x,t)| > s^{\frac{1}{q}}\}\Big)^{1-\sigma}.$$

In view of (4.6) and (4.7), we obtain

$$\text{meas}\{x\in\Omega : |\hat{\theta}(x,t)|^q > s\} \le CM^\sigma\Big(\frac{\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat{\theta})|^2\,dx}{s^{\frac{3}{q}}}\Big)^\sigma \frac{M^{1-\sigma}}{s^{\frac{1-\sigma}{q}}},$$

from which we deduce that

$$\text{meas}\{x\in\Omega : |\hat{\theta}(x,t)|^q > s\} \le CM\Big(\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat{\theta})|^2\,dx\Big)^\sigma \frac{1}{s^{\frac{1+2\sigma}{q}}}, \tag{4.8}$$

for almost any t in $(0,T)$.

In the sequel, the proof will be divided into three steps.

Step 1: $q = r$. We take $\sigma = 1$. For any positive real number β, we have

$$\int_0^T\int_\Omega |\hat\theta|^q dx\,dt \le \int_0^T\int_0^\beta \text{meas}\{x\in\Omega : |\hat\theta|^q > s\}\,ds\,dt + \int_0^T\int_\beta^\infty \text{meas}\{x\in\Omega : |\hat\theta|^q > s\}\,ds\,dt.$$

Due to (4.8), we obtain

$$\int_0^T\int_\Omega |\hat\theta|^q dx\,dt \le \beta T|\Omega| + CM\int_0^T\int_\beta^\infty\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat\theta)|^2\,dx\frac{1}{s^{\frac{3}{q}}}\,ds\,dt.$$

Fubini's theorem allows us to have

$$\begin{aligned}\int_0^T\int_\Omega |\hat\theta|^q dx\,dt &\le \beta T|\Omega| + CM\int_\beta^\infty\int_0^T\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat\theta)|^2\,dx\frac{1}{s^{\frac{3}{q}}}\,dt\,ds,\\ &\le \beta T|\Omega| + CM\int_\beta^\infty \frac{M}{s^{\frac{2}{q}}}\,ds.\end{aligned}$$

We recall that $q<2$, thus

$$\int_0^T\int_\Omega |\hat\theta|^q dx\,dt \le \beta T|\Omega| + CM^2\frac{\beta^{1-\frac{2}{q}}}{-1+\frac{2}{q}}.$$

We take $\beta = M^q$ and deduce that

$$\int_0^T\int_\Omega |\hat\theta|^q dx\,dt \le M^q(T|\Omega| + \frac{Cq}{2-q}).$$

Step 2: $q<r$. For any positive real number β, we have

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le \int_0^T\Big(\int_0^\beta \text{meas}\{x\in\Omega : |\hat\theta|^q > s\}\,ds\Big)^{\frac{r}{q}}dt + \int_0^T\Big(\int_\beta^\infty \text{meas}\{x\in\Omega : |\hat\theta|^q > s\}\,ds\Big)^{\frac{r}{q}}dt.$$

We take $\sigma = \frac{q}{r}$ in (4.8) and obtain

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q dx\Big)^{\frac{r}{q}}dt \le \beta^{\frac{r}{q}}|\Omega|^{\frac{r}{q}}T + C\int_0^T\Big(\int_\beta^\infty \frac{M\Big(\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat\theta)|^2\,dx\Big)^{\frac{q}{r}}}{s^{\frac{1}{q}+\frac{2}{r}}}\,ds\Big)^{\frac{r}{q}}\,dt.$$

For a real number $\gamma>1$, we write

$$\Big(\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat\theta)|^2\,dx\Big)^{\frac{q}{r}} = \Big(\frac{\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat\theta)|^2\,dx}{s^{\frac{1}{q}+\gamma}}\Big)^{\frac{q}{r}} s^{\frac{1}{r}+\frac{\gamma q}{r}},$$

for almost any t in $(0,T)$.

Using Hölder's inequality, we have

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le \beta^{\frac{r}{q}}\,|\Omega|^{\frac{r}{q}}\,T$$

$$+CM^{\frac{r}{q}}\int_0^T\Big[\int_\beta^\infty \frac{\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat\theta)|^2\,dx}{s^{\frac{1}{q}+\gamma}}\Big]\,ds\cdot\Big[\int_\beta^\infty \frac{ds}{s^{[\frac{1}{q}+\frac{1}{r}(1-q\gamma)]\frac{r}{r-q}}}\Big]^{\frac{r-q}{q}}\,dt.$$

The term $\Big[\int_\beta^\infty \frac{ds}{s^{[\frac{1}{q}+\frac{1}{r}(1-q\gamma)]\frac{r}{r-q}}}\Big]^{\frac{r-q}{q}}$ is independent of t, and is finite if and only if we have $[\frac{1}{q}+\frac{1}{r}(1-q\gamma)]\frac{r}{r-q}>1$.

Fubini's theorem implies that

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q dx\Big)^{\frac{r}{q}}dt \le \beta^{\frac{r}{q}}|\Omega|^{\frac{r}{q}}T+CM^{1+\frac{r}{q}}\int_\beta^\infty \frac{1}{s^\gamma}ds\Big[\int_\beta^\infty \frac{ds}{s^{[\frac{1}{q}+\frac{1}{r}(1-q\gamma)]\frac{r}{r-q}}}\Big]^{\frac{r-q}{q}}.$$

We know that $r<\frac{q}{q-1}$ (by the hypothesis of Lemma 4.2), thus $1+\frac{1}{q}+\frac{r}{q^2}-\frac{r}{q}>1$ and we can choose a real number γ such that $1<\gamma<1+\frac{1}{q}+\frac{r}{q^2}-\frac{r}{q}$. It is easy to check that $[\frac{1}{q}+\frac{1}{r}(1-q\gamma)]\frac{r}{r-q}>1$.

We also have

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le \beta^{\frac{r}{q}}|\Omega|^{\frac{r}{q}}T+CM^{1+\frac{r}{q}}\,\frac{\beta^{1-\gamma}}{\gamma-1}\cdot\frac{\beta^{\big[1-[\frac{1}{q}+\frac{1}{r}(1-q\gamma)]\frac{r}{r-q}\big]\frac{r-q}{q}}}{[\frac{1}{q}+\frac{1}{r}(1-q\gamma)]\frac{r}{r-q}-1};$$

taking $\beta=M^q$, we conclude that

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le C(\gamma,r,q,N)M^r.$$

Step 3: $r<q$. If $q<2$, then $r<2$. By Hölder's inequality, we obtain

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le C(r,q,\Omega,T)\Big(\int_0^T\int_\Omega |\hat\theta|^q\,dx\,dt\Big)^{\frac{r}{q}}.$$

According to the first step, we deduce that

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le C(r,q,\Omega,T)M^r.$$

If $q\ge 2$, for $0<\sigma<1$, Hölder's inequality implies that

$$\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}} \le \Big(\int_\Omega |\hat\theta|\,dx\Big)^{\frac{\sigma r}{q}}\Big(\int_\Omega |\hat\theta|^{\frac{q-\sigma}{1-\sigma}}\,dx\Big)^{\frac{(1-\sigma)r}{q}},$$

for almost any t in $(0,T)$.

In view of (4.4), we obtain

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le M^{\frac{\sigma r}{q}}\int_0^T\Big(\int_\Omega |\hat\theta|^{\frac{q-\sigma}{1-\sigma}}\,dx\Big)^{\frac{1-\sigma}{q-\sigma}\,\frac{r(q-\sigma)}{q}}dt. \tag{4.9}$$

In the sequel, we show that for any $K>0$

$$\int_0^T\Big(\int_\Omega |T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}}\,dx\Big)^{\frac{1-\sigma}{q-\sigma}\,\frac{r(q-\sigma)}{q}}dt \le CM. \tag{4.10}$$

Recalling that $N=2$, by Sobolev's embedding theorem, we have

$$\int_\Omega |T_K(\hat\theta)|^p\,dx \le C(p,|\Omega|)\Big(\int_\Omega |DT_K(\hat\theta)|^2\,dx\Big)^{\frac{p}{2}}, \tag{4.11}$$

for any $p\ge 2$.

We take $p=\frac{2q}{(1-\sigma)r}$, and we write

$$|T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}} \le |T_K(\hat\theta)|^p\cdot K^{\frac{1}{1-\sigma}(q-\sigma-\frac{2q}{r})},$$

for almost any (x,t) in Q. Then

$$\Big(\int_\Omega |T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}}\,dx\Big)^{\frac{1-\sigma}{q-\sigma}\,\frac{r(q-\sigma)}{q}} \le K^{\frac{r}{q}(q-\sigma-\frac{2q}{r})}\Big(\int_\Omega |T_K(\hat\theta)|^p\,dx\Big)^{\frac{2}{p}},$$

for almost any t in $(0,T)$.

In view of (4.11), we deduce that

$$\Big(\int_\Omega |T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}}\,dx\Big)^{\frac{1-\sigma}{q-\sigma}\,\frac{r(q-\sigma)}{q}} \le c\,K^{\frac{r}{q}(q-\sigma-\frac{2q}{r})}\int_\Omega |DT_K(\hat\theta)|^2\,dx,$$

for almost any t in $(0,T)$. This implies that

$$\int_0^T\Big(\int_\Omega |T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}}\,dx\Big)^{\frac{(1-\sigma)r}{q}}dt \le CK^{\frac{r}{q}(q-\sigma-\frac{2q}{r})}K\,M.$$

Since $r<\frac{q}{q-1}$, we obtain $0<\frac{q}{r}(r-1)<1$ and we can take $\sigma=\frac{q}{r}(r-1)$. The proof of (4.10) is then complete.

Fatou's lemma and estimate (4.10) allow us to conclude that

$$\int_0^T\Big(\int_\Omega |\hat\theta|^{\frac{q-\sigma}{1-\sigma}}\,dx\Big)^{\frac{1-\sigma}{q-\sigma}\,\frac{r(q-\sigma)}{q}}dt \le CM. \tag{4.12}$$

The above inequality together with (4.9) make it possible to obtain

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le C(\gamma,r,q,N)M^r.$$

This achieves the proof of Lemma 4.2.

5. Existence of a solution for N=2

This section is devoted to establishing the following existence theorem.

Theorem 5.1. *Under the assumptions* (2.1) – (2.13) *on the data, assume that the continuous function* F *satisfies* $|F(r)| \leq a + M|r|^\alpha$ *for all* $r \in \mathbb{R}$, *with* $a \geq 0$, $M \geq 0$ *and* $2\alpha \in [0, 3[$. *Then we have the following.*

If $0 \leq 2\alpha \leq 1$, *there exists at least a weak-renormalized solution of the system* (2.14) – (2.18) *for* $N = 2$ (*in the sense of Definition* 2.2).

If $1 < 2\alpha < 3$, *there exists a real positive number* η, *such that if* $a + \|u_0\|_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)} \leq \eta$, *then there exists at least a weak-renormalized solution of the system* (2.14) – (2.18) *for* $N = 2$.

Proof of Theorem 5.1. We use the fixed-point argument described in Section 3 and we distinguish four cases according to the values of α.

Case 1: $\alpha = 0$. For a fixed $\theta \in L^1(Q)$, since F is bounded ($\alpha = 0$), we can denote by u the unique weak solution of (3.1) – (3.4) in $L^2(0, T; H^1_\sigma(\Omega)) \cap L^\infty(0, T; L^2_\sigma(\Omega))$ (see, e.g., [19], [26]). We denote by $\hat{\theta}$ (see Lemma 4.1) the unique renormalized solution of (3.5) – (3.7). The regularity of $\hat{\theta}$ (see Lemma 4.2) indeed implies that $\hat{\theta} \in L^1(Q)$. As a consequence, we can take $L = L^1(Q)$ in the fixed-point argument of Section 3.

Let us define the mapping $\psi_1 : L^1(Q) \longrightarrow L^1(Q),\ \theta \longrightarrow \psi_1(\theta) = \hat{\theta}$. The mapping ψ_1 is well defined. In the sequel, we will show that ψ_1 is compact and continuous and that there exists a ball B of $L^1(Q)$ such that $\psi_1(B) \subset B$.

*i-$\boldsymbol{\psi_1}$ **is compact**.* Let us consider a sequence θ_n, which is bounded in $L^1(Q)$, and define the sequence $\hat{\theta}_n$ by $\psi_1(\theta_n) = \hat{\theta}_n$. For a fixed $n \geq 1$, since F is bounded, there exists a unique solution $u_n \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_\sigma(\Omega))$ of the following problem (see, e.g. [14], [25], [26])

$$u_{nt} + (u_n \cdot \nabla)u_n - 2 \text{ div } (\mu(\theta_n) D u_n) = F(\theta_n) \text{ in } (H^1_\sigma)'(\Omega), \tag{5.1}$$

for almost every t $\in (0, T)$,

$$\text{div } u_n = 0 \text{ in } Q, \tag{5.2}$$

$$u_n = 0 \text{ on } \Sigma_T, \tag{5.3}$$

$$u_n(t = 0) = u_0 \text{ in } \Omega. \tag{5.4}$$

Then $\mu(\theta_n)|Du_n|^2 \in L^1(Q)$. For a fixed $n \geq 1$, $\hat{\theta}_n$ is the unique renormalized solution of the following problem (see Lemma 4.1):

$$\frac{\partial b(\hat{\theta}_n)}{\partial t} - \text{ div } (a(x, \hat{\theta}_n, \nabla \hat{\theta}_n)) + \text{ div } (\Phi(\hat{\theta}_n)) = 2\mu(\theta_n)|Du_n|^2 \text{ in } Q, \tag{5.5}$$

$$\hat{\theta}_n = 0 \text{ on } \Sigma_T, \tag{5.6}$$

$$b(\hat{\theta}_n)(t=0) = b_0 \text{ in } \Omega. \tag{5.7}$$

Due to the bounded character of F ($\alpha = 0$), we deduce that $F(\theta_n)$ is bounded in $L^\infty(Q)$. Then (see again [14], [25], [26])

$$u_n \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)), \tag{5.8}$$
$$u_{nt} \text{ is bounded in } L^2(0,T;(H^1_\sigma)'(\Omega)).$$

We can, then, extract a subsequence such that

$$u_n \rightharpoonup w \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)), \tag{5.9}$$
$$u_n \to w \text{ strongly in } L^2_\sigma(Q), \tag{5.10}$$
$$u_{nt} \rightharpoonup w_t \text{ weakly in } L^2(0,T;(H^1_\sigma)'(\Omega)), \tag{5.11}$$

as n tends to $+\infty$, where w is a function of $L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega))$.

This implies that

$$\mu(\theta_n)|Du_n|^2 \text{ is bounded in } L^1(Q). \tag{5.12}$$

In view of (5.12) and Lemma 4.2, we obtain

$$\hat{\theta}_n \text{ is bounded in } L^p(Q) \qquad \forall p \in [1,2). \tag{5.13}$$

We are now in a position to show that the sequence $\hat{\theta}_n$ converges (for a subsequence) almost everywhere in Q through the use of (5.12). However, we cannot use the method of [6] because we don't have the strong convergence of the dissipation energy $\mu(\theta_n)|Du_n|^2$ in $L^1(Q)$. Without loss of generality, we assume that the function b has only one point of discontinuity. Let us take for example $b(0^+) = 1$ and $b(0^-) = 0$. The proof which follows remains valid for any function b which satisfies the assumptions of Theorem 5.1. We define a new smooth function f (at least twice differentiable) such that $f(x) = x^3$ on (a,b) ($a < 0 < b$) and $b - a$ is small enough. The function f is linear out of (a,b). We set

$$\gamma_1 = f^+ \text{ and } \gamma_2 = -f^-.$$

Note that $\gamma_1(b(r) - \varepsilon - 1)$ and $\gamma_2(b(r) + \varepsilon)$ are C^1-functions. By an Aubin's type argument (see, e.g. [24], corollary 4), we will show that $\gamma_1(b(\hat{\theta}_n) - \varepsilon - 1)$ and $\gamma_2(b(\hat{\theta}_n) + \varepsilon)$ converge almost everywhere in Q. Thus, we deduce the pointwise convergence of $\hat{\theta}_n$.

We first introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_M such that, for any $M \geq 1$, $S_M \in W^{2,\infty}(\mathbb{R})$, S'_M has a compact support ($\mathrm{supp}(S'_M) \subset [-M, M]$), $S'_M = 1$ in $[-\frac{M}{2}, \frac{M}{2}]$ and $S_M(0) = 0$.

For a fixed $\varepsilon > 0$ and $M \geq 1$, we show that

$$S_M\Big(\gamma_1(b(\hat{\theta}_n) - \varepsilon - 1)\Big) \text{ is bounded in } L^2(0,T;H^1_0(\Omega)).$$

Indeed, we have

$$DS_M\Big(\gamma_1(b(\hat\theta_n)-\varepsilon-1)\Big)=$$
$$S_M^{'}(\gamma_1(b(\hat\theta_n)-\varepsilon-1))\gamma_1'(b(\hat\theta_n)-\varepsilon-1))b'(\hat\theta_n)\chi_{\{b(\hat\theta_n)>1+\varepsilon\}}DT_{K_{M_\varepsilon}}(\hat\theta_n),$$

almost everywhere in Q, where K_{M_ε} is a positive real number depending on M and ε. Since b is a C^1-function avoiding zero and S_M' has a compact support, we have

$$S_M^{'}(\gamma_1(b(\hat\theta_n)-\varepsilon-1))\gamma_1'(b(\hat\theta_n)-\varepsilon-1))b'(\hat\theta_n)\chi_{\{b(\hat\theta_n)>1+\varepsilon\}} \text{ is bounded in } L^\infty(Q).$$

The bounded character of $\mu(\theta_n)|Du_n|^2$ in $L^1(Q)$ allows us to conclude as in [6] that $DT_{K_{M_\varepsilon}}(\hat\theta_n)$ is bounded in $L^2(Q)$. Thus

$$S_M\Big(\gamma_1(b(\hat\theta_n)-\varepsilon-1)\Big) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{5.14}$$

for a fixed $\varepsilon>0$ and for any $M\geq 1$.

From now on, we show that for fixed ε and M

$$\frac{\partial S_M\Big(\gamma_1(b(\hat\theta_n)-\varepsilon-1)\Big)}{\partial t} \text{ is bounded in } L^1(Q)+L^2(0,T;H^{-1}(\Omega)).$$

For a fixed $n\geq 1$, $\hat\theta_n$ is a renormalized solution of the heat equation, thus

$$\begin{aligned}&\int_0^T\!\!\int_\Omega \varphi_t\int_0^{\beta_{\hat\theta_n}} S'_{M\varepsilon}(b^{-1}(r))dr\,dx\,dt \qquad (5.15)\\ &=\int_0^T\!\!\int_\Omega a(x,\hat\theta_n,\nabla\hat\theta_n)\nabla\varphi S'_{M\varepsilon}(\hat\theta_n)dx\,dt+\int_0^T\!\!\int_\Omega a(x,\hat\theta_n,\nabla\hat\theta_n)\nabla\hat\theta_n S''_{M\varepsilon}(\hat\theta_n)\,\varphi dx\,dt\\ &-\int_0^T\!\!\int_\Omega\nabla\varphi\Big(\int_0^{\hat\theta_n}\Phi'(\xi)S'_{M\varepsilon}(\xi)d\xi\Big)dx\,dt-\int_0^T\!\!\int_\Omega 2\mu(\theta_n)|Du_n|^2\,S'_{M\varepsilon}(\hat\theta_n)\varphi dx\,dt,\end{aligned}$$

with $\varphi\in C_0^\infty(Q)$ and $S_{M\varepsilon}(r)=\int_0^r S_M'(\gamma_1(b(r)-\varepsilon-1))\,\gamma_1'(b(r)-\varepsilon-1)\,dr$. We thus obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(\int_0^{\beta_{\hat\theta_n}} S'_{M\varepsilon}(b^{-1}(r))dr\Big)=\text{div }(a(x,\hat\theta_n,\nabla\hat\theta_n)S'_{M\varepsilon}(\hat\theta_n)) \qquad (5.16)\\ &-a(x,\hat\theta_n,\nabla\hat\theta_n)\nabla\hat\theta_n S''_{M\varepsilon}(\hat\theta_n)-\text{div }(\Phi(\hat\theta_n))S'_{M\varepsilon}(\hat\theta_n)+2\mu(\theta_n)|Du_n|^2\,S'_{M\varepsilon}(\hat\theta_n),\end{aligned}$$

in the distribution sense. Since $S'_{M\varepsilon}$ and $S^{''}_{M\varepsilon}$ have a compact support (supp $(S'_{M\varepsilon})\subset[-L,L]$, ε and M being fixed), we can replace $\hat\theta_n$ by its truncation $T_L(\hat\theta_n)$ in the above equality. Moreover, the growth of the operator a allows us to check that each term in the right-hand side of (5.16) is bounded in either $L^2(0,T;H^{-1}(\Omega))$ or in $L^1(Q)$. We deduce that

$$\frac{\partial}{\partial t}\Big(\int_0^{\beta_{\hat{\theta}_n}} S'_{M\varepsilon}(b^{-1}(r))dr\Big) \text{ is bounded in } L^1(Q)+L^2(0,T;H^{-1}(\Omega)),$$

for a fixed $\varepsilon > 0$ and for any $M \geq 1$.

In addition, since $S'_{M\varepsilon}(r) = 0$, in particular if $b(r) < 1+\varepsilon$, we have

$$\int_0^{\beta_{\hat{\theta}_n}} S'_{M\varepsilon}(b^{-1}(r))dr = \int_0^{b(\hat{\theta}_n)} S'_M(\gamma_1(r-\varepsilon-1))\,\gamma'_1(r-\varepsilon-1)\,dr$$
$$= S_M\Big(\gamma_1(b(\hat{\theta}_n)-\varepsilon-1)\Big). \quad (5.17)$$

We conclude that

$$\frac{\partial S_M\Big(\gamma_1(b(\hat{\theta}_n)-\varepsilon-1)\Big)}{\partial t} \text{ is bounded in } L^1(Q)+L^2(0,T;H^{-1}(\Omega)), \quad (5.18)$$

for a fixed $\varepsilon > 0$ and for any $M \geq 1$.

In view of (5.14) and (5.18), we apply an Aubin's type lemma (Simon [24]). By a construction process of diagonal sequences, there exists a subsequence, still indexed by n, $\hat{\theta}_n$, such that for any integer m, we have

$$\gamma_1(b(\hat{\theta}_n) - \frac{1}{m} - 1) \longrightarrow \phi_m, \quad (5.19)$$

and

$$\gamma_2(b(\hat{\theta}_n) + \frac{1}{m}) \longrightarrow \psi_m, \quad (5.20)$$

almost everywhere in Q, as n tends to infinity.

It remains to prove the pointwise convergence of $\hat{\theta}_n$ (for a subsequence) by using (5.19) and (5.20). The sequence ϕ_m is a positive and decreasing sequence with respect to m, while ψ_m is a negative and increasing sequence with respect to m, thus

$$\phi_m \longrightarrow \gamma^+, \quad (5.21)$$

and

$$\psi_m \longrightarrow \gamma^-, \quad (5.22)$$

almost everywhere in Q, as m tends to infinity, where γ^+ is a positive measurable function and γ^- is a negative measurable function.

It is advisable to notice that

$$\gamma_1(b(\hat{\theta}_n) - \frac{1}{m} - 1) = \gamma_1(b(\hat{\theta}_n^+) - \frac{1}{m} - 1) \quad (5.23)$$

almost everywhere in Q. We have

$$\lim_{m\to\infty}\lim_{n\to\infty}\gamma_1(b(\hat\theta_n)-\frac{1}{m}-1)=\gamma^+.$$

We also have

$$\lim_{m\to\infty}\gamma_1(b(\hat\theta_n)-\frac{1}{m}-1)=\gamma_1(b(\hat\theta_n^+)-1),$$

uniformly with respect to n, because γ_1 is Lipschitz continuous. We thus get

$$\lim_{n\to\infty}\gamma_1(b(\hat\theta_n^+)-1)=\gamma^+. \tag{5.24}$$

Let $(x,t)\in Q$ such that $\gamma^+(x,t)>0$; in view of (5.24) and since γ_1^{-1} is continuous on $[0,+\infty)$, we obtain $b(\hat\theta_n^+)(x,t)\longrightarrow\gamma_1^{-1}(\gamma^+(x,t))+1$, as n tends to infinity. The continuity of b^{-1} allows us to conclude that

$$\hat\theta_n^+(x,t)\longrightarrow b^{-1}\Big(\gamma_1^{-1}(\gamma^+(x,t))+1\Big),$$

as n tends to infinity.

Let $(x,t)\in Q$ such that $\gamma^+(x,t)=0$; by the definition of γ_1 and according to (5.24), it is clear that $\hat\theta_n^+(x,t)\longrightarrow 0$, as n tends to infinity.

We deduce that

$$\hat\theta_n^+\longrightarrow\sigma^+, \tag{5.25}$$

almost everywhere in Q, as n tends to infinity, where σ^+ is a function defined by

$$\sigma^+=\begin{cases} b^{-1}\big(\gamma_1^{-1}(\gamma^+)+1\big) & \text{in } \{(x,t)\in Q:\gamma^+(x,t)>0\},\\ 0 & \text{in } \{(x,t)\in Q:\gamma^+(x,t)=0\}.\end{cases}$$

The same proof works to show that

$$\hat\theta_n^-\longrightarrow\sigma^-, \tag{5.26}$$

almost everywhere in Q, as n tends to infinity, where σ^- is a function defined by

$$\sigma^-=\begin{cases} -b^{-1}(\gamma_2^{-1}(\gamma^-)) & \text{in } \{(x,t)\in Q:\gamma^-(x,t)<0\},\\ 0 & \text{in } \{(x,t)\in Q:\gamma^-(x,t)=0\}.\end{cases}$$

Since $\hat\theta_n=\hat\theta_n^+-\hat\theta_n^-$, we thus get

$$\hat\theta_n\longrightarrow\sigma, \tag{5.27}$$

almost everywhere in Q, as n tends to infinity, where σ is a function defined by $\sigma=\sigma^+-\sigma^-$. According to (5.13) and (5.27), we conclude that

$$\hat\theta_n \text{ remains in a compact subset of } L^p(Q), \tag{5.28}$$

for all p such that $1 \leq p < 2$; in particular,

$$\hat{\theta}_n \text{ remains in a compact set of } L^1(Q). \tag{5.29}$$

ii-$\boldsymbol{\psi_1}$ ***is continuous***. Let us consider a sequence θ_n, which belongs to $L^1(Q)$ such that

$$\theta_n \to \theta, \tag{5.30}$$

strongly in $L^1(Q)$ as n tends to $+\infty$, where θ is a function of $L^1(Q)$. Let $\hat{\theta}_n$ and $\hat{\theta}$ be defined by

$$\psi_1(\theta_n) = \hat{\theta}_n \quad \text{and} \quad \psi_1(\theta) = \hat{\theta}.$$

In what follows, the sequence u_n and the function w are defined as in the step (i). Since F and μ are bounded and in view of (5.9), (5.10), (5.11) and (5.30), passing to the limit as n tends to infinity in (5.1) is an easy task. So, w is nothing but the unique weak solution of (3.1) – (3.4); i.e., $w = u$ almost everywhere in Q.

$$\forall n \geq 1, u_n \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega)) \subset L^4(Q), \text{ for } N = 2.$$

This implies that the sequence $(u_n \cdot \nabla)u_n$ can be multiplied by u_n. As a consequence, we can choose u_n as a test function in (5.1). Due to (5.30), we have, in particular, $F(\theta_n) \to F(\theta)$ in $L^2(Q)$, as n tends to $+\infty$. Moreover, the sequence u_n is compact in $L^2(Q)$. Then, we can pass to the limit as n tends to $+\infty$ in (5.1) multiplied by u_n. While making a distinction between the obtained equation and the equation of (3.1) multiplied by the test function u (which is legal), we deduce, taking into account the weak convergence (5.9) that

$$u_n \to u \text{ strongly in } L^2(0,T; H^1_\sigma(\Omega)), \tag{5.31}$$

as n tends to $+\infty$. Since the function μ is continuous and bounded, we obtain

$$\mu(\theta_n)|Du_n|^2 \to \mu(\theta|Du|^2 \text{ strongly in } L^1(Q), \tag{5.32}$$

as n tends to $+\infty$. According to (5.32), Theorem 2.2 implies that $\beta_{\hat{\theta}_n}$ is a Cauchy sequence in $L^\infty(0,T; L^1(\Omega))$. In particular, there exist two functions ϑ and $\beta_\vartheta \in b(\vartheta)$ such that $\beta_{\hat{\theta}_n} \to \beta_\vartheta$ in $L^\infty(0,T; L^1(\Omega))$ and almost everywhere in Q, for a subsequence, as n tends to $+\infty$. Since b^{-1} is continuous, then $\hat{\theta}_n \to \vartheta$ almost everywhere in Q, as n tends to infinity. As a consequence $T_K(\hat{\theta}_n) \rightharpoonup T_K(\vartheta)$ in $L^2(0,T; H^1_0(\Omega))$, for any $K \geq 0$, as n tends to $+\infty$.

Upon applying Theorem 4.1 of [6], we deduce that ϑ is the unique renormalized solution of (3.5) – (3.7). Thanks to Lemma 4.1, we conclude that $\vartheta = \hat{\theta}$ almost everywhere in Q, and due to (5.28), we thus get

$$\hat{\theta}_n \to \hat{\theta}, \tag{5.33}$$

strongly in $L^p(Q)$, for all p such that $1 \leq p < 2$, as n tends to $+\infty$.

As a consequence

$$\hat{\theta}_n \to \hat{\theta}, \tag{5.34}$$

strongly in $L^1(Q)$, as n tends to $+\infty$.

iii- ***There exists a ball B of $L^1(Q)$ such that $\psi_1(B) \subset B$***. Let R be a positive real number. We will show that there exists $R_0 > 0$ such that

$$\psi_1(B_{L^1(Q)}(0, R_0)) \subset B_{L^1(Q)}(0, R_0).$$

In the following, the constant C denotes a generic constant which depends on Ω, T, m_0 and m_1. We assume that θ belongs to $B_{L^1(Q)}(0, R)$.

We recall that u, which belongs to $L^2(0, T; H^1_\sigma(\Omega)) \cap L^\infty(0, T; L^2_\sigma(\Omega))$, is the unique solution of the problem (3.1) – (3.4), then we use u as a test function in (3.1), to obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{1}{2}\int_0^T\!\!\int_\Omega \mu(\theta)|Du|^2\,dx\,dt \\
&\quad = \int_0^T\!\!\int_\Omega F(\theta)\cdot u\,dx\,dt + \frac{1}{2}\int_\Omega |u_0|^2\,dx,
\end{aligned}$$

thus

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{m_0}{2}\int_0^T\!\!\int_\Omega |Du|^2\,dx\,dt \\
&\quad \leq \int_0^T\!\!\int_\Omega F(\theta)\cdot u\,dx\,dt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)},
\end{aligned}$$

which implies that

$$m_0\int_0^T\!\!\int_\Omega |Du|^2\,dx\,dt \leq 2\int_0^T \|F(\theta)\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\,dt + \|u_0\|^2_{L^2(\Omega)}. \tag{5.35}$$

Estimate (5.35) and Poincaré's inequality lead to

$$m_0\int_0^T\!\!\int_\Omega |Du|^2\,dx\,dt \leq C\int_0^T \|F(\theta)\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\,dt + \|u_0\|^2_{L^2(\Omega)}. \tag{5.36}$$

Young's inequality, (5.36) and Korn's inequality allow us to deduce that

$$\int_0^T\!\!\int_\Omega |Du|^2\,dx\,dt \leq C\|F(\theta)\|^2_{L^2(Q)} + C\|u_0\|^2_{L^2(\Omega)}. \tag{5.37}$$

As a consequence $\mu(\theta)|Du|^2$ is bounded in $L^1(Q)$. In view of Lemma 4.2, we deduce that $\hat{\theta}$ is bounded $L^1(Q)$, and there exists a constant $C_1 > 0$, independent of θ such that $\|\hat{\theta}\|_{L^1(Q)} \leq C_1$. To conclude, it is enough to take $R_0 \geq C_1$, which implies $\psi_1(B_{L^1(Q)}(0, R_0)) \subset B_{L^1(Q)}(0, R_0)$.

Schauder's fixed-point theorem and the definition of ψ_1 allow us to conclude that there exists a weak-renormalized solution (θ, u) of $(2.14) - (2.18)$ which is the desired conclusion.

Case 2: $0 < 2\alpha \leq 1$. Let us proceed by approximation and passage to the limit. We start by replacing the function F by $F^\varepsilon = F \circ T_{\frac{1}{\varepsilon}}$, for all $\varepsilon > 0$, and we consider the following approximate system:

$$P^\varepsilon \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla)u^\varepsilon - 2 \text{ div } (\mu(\theta^\varepsilon)Du^\varepsilon) = F^\varepsilon(\theta^\varepsilon) & \text{in } (H^1_\sigma)'(\Omega), \\ \text{for almost every t} \in (0,T), & \\ \dfrac{\partial b(\theta^\varepsilon)}{\partial t} - \text{ div } (a(x, \theta^\varepsilon, \nabla\theta^\varepsilon)) + \text{ div } (\Phi(\theta^\varepsilon)) = 2\mu(\theta^\varepsilon)|Du^\varepsilon|^2 & \text{in } Q, \\ \text{div } u^\varepsilon = 0 & \text{in } Q, \\ u^\varepsilon = 0 \text{ and } \theta^\varepsilon = 0 & \text{on } \Sigma_T, \\ u^\varepsilon(t=0) = u_0 \text{ and } b(\theta^\varepsilon)(t=0) = b_0 & \text{in } \Omega. \end{cases}$$

The function F^ε being continuous and bounded, we apply the above result (Case 1: $\alpha = 0$), we deduce that there exists a weak-renormalized solution $(\theta^\varepsilon, u^\varepsilon)$ of the approximate system P^ε. By a standard estimate on the heat equation in P^ε, we have (see [6] and recall (2.2))

$$\int_\Omega |\theta^\varepsilon|(t)dx \leq \int_0^t\int_\Omega \mu(\theta^\varepsilon)|Du^\varepsilon|^2 dx\, ds + C,$$

where C is a generic constant. Now, the following estimate holds true on the Navier-Stokes problem in P^ε

$$\int_0^t\int_\Omega \mu(\theta^\varepsilon)|Du^\varepsilon|^2 dx\, ds \leq \int_0^t\int_\Omega |F^\varepsilon(\theta^\varepsilon)|^2\, dx\, ds + C.$$

Using the growth condition (2.6) on F, we obtain

$$\int_\Omega |\theta^\varepsilon|(t)dx \leq c_1 \int_0^t\int_\Omega |\theta^\varepsilon|^{2\alpha} dx\, dt + c_2, \tag{5.38}$$

where c_1 and c_2 are constants depending on Ω, T and the data b_0 and u_0.

Since $0 < 2\alpha \leq 1$, Gronwall's lemma shows that $(\theta^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T; L^1(\Omega))$.

Since $2\alpha \leq 1$ and due to the definitions of F and F^ε, we deduce that the sequence $(F^\varepsilon(\theta^\varepsilon))_{\varepsilon>0}$ is bounded in $L^2(Q)$. Then $\mu(\theta^\varepsilon)|Du^\varepsilon|^2$ is bounded in $L^1(Q)$. However, u^ε is compact in $L^2(Q)$. We proceed as in Case 1;

then there exist two functions u and θ such that $u^\varepsilon \rightharpoonup u$ in $L^2(0,T;H^1_\sigma(\Omega))$, $\theta^\varepsilon \longrightarrow \theta$ almost everywhere in Q, as ε tends to 0, for a subsequence still indexed by ε. Since $N=2$, the same argument which gives (5.32) shows that $\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \to \mu(\theta)|Du|^2$ strongly in $L^1(Q)$, as ε tends to 0. Passing to the limit as ε tends to 0 in P^ε and using the stability result given in theorem 4.1 of [6] shows that (θ,u) is a weak-renormalized solution of (2.14) – (2.18).

Case 3: $\mathbf{1 < 2\alpha < 2}$. For a fixed $\theta \in L^{2\alpha}(Q)$, due to the growth assumption (2.6) on F, we obtain $F(\theta)\in L^2(Q)$. Then there exists a unique weak solution u of (3.1) – (3.4) in $L^2(0,T;H^1_\sigma(\Omega))\cap L^\infty(0,T;L^2_\sigma(\Omega))$ (see, e.g. [19], [26]).

Now, we use exactly the same argument as in the first case ($\alpha = 0$) to show that there exists a unique renormalized solution $\hat\theta$ (see Lemma 4.1) of (3.5) – (3.7). The regularity of $\hat\theta$ (see Lemma 4.2) implies that $\hat\theta \in L^{2\alpha}(Q)$ because $1 < 2\alpha < 2$. As a consequence, we can take $L = L^{2\alpha}(Q)$ in the fixed-point argument of Section 3.

We define the mapping

$$\psi_2 : L^{2\alpha}(Q) \longrightarrow L^{2\alpha}(Q),$$
$$\theta \longrightarrow \psi_2(\theta) = \hat\theta.$$

The mapping ψ_2 is well defined. We will show that ψ_2 is compact, continuous and that there exists a ball B of $L^{2\alpha}(Q)$ such that $\psi_2(B)\subset B$.

i-$\boldsymbol{\psi_2}$ ***is compact***. Let us consider a sequence θ_n, which is bounded in $L^{2\alpha}(Q)$, and define the sequence $\hat\theta_n$ by

$$\psi_2(\theta_n) = \hat\theta_n.$$

For a fixed $n\geq 1$, due to (2.6), there exists a unique solution $u_n \in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_\sigma(\Omega))$ of the following problem (see, e.g. [14], [25], [26]):

$$u_{nt} + (u_n\cdot\nabla)u_n - 2\ \mathrm{div}\ (\mu(\theta_n)Du_n) = F(\theta_n) \text{ in } (H^1_\sigma)'(\Omega), \tag{5.39}$$

for almost every t $\in (0,T)$,

$$\mathrm{div}\ u_n = 0 \text{ in } Q, \tag{5.40}$$

$$u_n = 0 \text{ on } \Sigma_T, \tag{5.41}$$

$$u_n(t=0) = u_0 \text{ in } \Omega. \tag{5.42}$$

Then $\mu(\theta_n)|Du_n|^2 \in L^1(Q)$. For a fixed $n\geq 1$, $\hat\theta_n$ is the unique renormalized solution of the following problem (see Lemma 4.1):

$$\frac{\partial b(\hat\theta_n)}{\partial t} - \ \mathrm{div}\ (a(x,\hat\theta_n,\nabla\hat\theta_n)) + \ \mathrm{div}\ (\Phi(\hat\theta_n)) = 2\mu(\theta_n)|Du_n|^2 \text{ in } Q, \tag{5.43}$$

$$\hat{\theta}_n = 0 \text{ on } \Sigma_T, \tag{5.44}$$

$$b(\hat{\theta}_n)(t=0) = b_0 \text{ in } \Omega. \tag{5.45}$$

Due to the growth condition on F (see (2.6)), we deduce that $F(\theta_n)$ is bounded in $L^2(Q)$. Then (see again [14], [25], [26])

$$u_n \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)), \tag{5.46}$$

$$u_{nt} \text{ is bounded in } L^2(0,T;(H^1_\sigma)'(\Omega)).$$

We can, then, extract a subsequence such that

$$u_n \rightharpoonup w \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)), \tag{5.47}$$

$$u_n \to w \text{ strongly in } L^2_\sigma(Q), \tag{5.48}$$

$$u_{nt} \rightharpoonup w_t \text{ weakly in } L^2(0,T;(H^1_\sigma)'(\Omega)), \tag{5.49}$$

as n tends to $+\infty$, where w is a function in $L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega))$.

This implies that

$$\mu(\theta_n)|Du_n|^2 \text{ is bounded in } L^1(Q). \tag{5.50}$$

In view of (5.50) and Lemma 4.2, we obtain

$$\hat{\theta}_n \text{ is bounded in } L^p(Q) \qquad \forall p \in [1,2). \tag{5.51}$$

Now, with the help of (5.50), we use the same argument as in the proof of (5.27). Then, up to a subsequence, there exists a function σ such that

$$\hat{\theta}_n \longrightarrow \sigma \text{ almost everywhere in } Q, \tag{5.52}$$

as n tends to $+\infty$. In view of (5.51) and (5.52), we conclude that

$$\hat{\theta}_n \text{ belongs to a compact subset of } L^p(Q), \tag{5.53}$$

for all p such that $1 \le p < 2$. Since $1 < 2\alpha < 2$, it follows that

$$\hat{\theta}_n \text{ belongs to a compact subset of } L^{2\alpha}(Q). \tag{5.54}$$

ii-$\boldsymbol{\psi_2}$ ***is continuous***. Let us consider a sequence θ_n, which belongs to $L^{2\alpha}(Q)$, such that

$$\theta_n \to \theta, \tag{5.55}$$

strongly in $L^{2\alpha}(Q)$ as n tends to $+\infty$, where θ is a function of $L^{2\alpha}(Q)$. Let $\hat{\theta}_n$ and $\hat{\theta}$ be defined by

$$\psi_2(\theta_n) = \hat{\theta}_n \text{ and } \psi_2(\theta) = \hat{\theta}.$$

It easy to check that up to a subsequence $F(\theta_n) \rightharpoonup F(\theta)$ in $L^2(Q)$, as n tends to $+\infty$. Moreover, the sequence u_n given by (5.39)- (5.42) is compact in $L^2(Q)$. Then, we can repeat the previous argument (see (5.32)) to show that

$$\mu(\theta_n)|Du_n|^2 \to \mu(\theta|Du|^2 \text{ strongly in } L^1(Q), \tag{5.56}$$

as n tends to $+\infty$. According to (5.56), Theorem 2.2 implies that $\beta_{\hat{\theta}_n}$ is a Cauchy sequence in $L^\infty(0,T;L^1(\Omega))$. In particular, there exist two functions ϑ and $\beta_\vartheta \in b(\vartheta)$ such that

$$\beta_{\hat{\theta}_n} \to \beta_\vartheta \text{ in } L^\infty(0,T;L^1(\Omega)) \text{ and a.e. in } Q,$$

for a subsequence. Since b^{-1} is continuous, we obtain

$$\hat{\theta}_n \to \vartheta \text{ a.e. in } Q,$$

as n tends to infinity. As a consequence

$$T_K(\hat{\theta}_n) \rightharpoonup T_K(\vartheta) \text{ in } L^2(0,T;H_0^1(\Omega)),$$

for any $K \geq 0$, as n tends to $+\infty$.

We deduce from theorem 4.1 of [6] that ϑ is the unique renormalized solution of (3.5) – (3.7). Thanks to Lemma 4.1, we obtain

$$\vartheta = \hat{\theta} \text{ a.e. in } Q,$$

and

$$\hat{\theta}_n \text{ converges almost everywhere to } \hat{\theta} \text{ in } Q, \tag{5.57}$$

as n tends to infinity. We thus get from (5.51) and (5.57)

$$\hat{\theta}_n \to \hat{\theta} \tag{5.58}$$

strongly in $L^p(Q)$, for all p such that $1 \leq p < 2$; as a consequence

$$\hat{\theta}_n \to \hat{\theta} \tag{5.59}$$

strongly in $L^{2\alpha}(Q)$, since $1 < 2\alpha < 2$.

iii-***There exists a ball $\boldsymbol{B}$ of $\boldsymbol{L^{2\alpha}(Q)}$ such that $\boldsymbol{\psi_2(B) \subset B}$.*** Let R be a positive real number. We will show that there exists $R_0 > 0$ such that

$$\psi_2(B_{L^{2\alpha}(Q)}(0,R_0)) \subset B_{L^{2\alpha}(Q)}(0,R_0).$$

We assume that θ belongs to $B_{L^{2\alpha}(Q)}(0,R)$. In what follows, the constant C denotes a generic constant depending on Ω, T, m_0 and m_1. We recall that $u \in L^2(0,T;H^1_\sigma(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega))$ is the unique weak solution of the problem (3.1) – (3.4). Using u as a test function in (3.1) leads to

$$\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{1}{2}\int_0^T\!\!\int_\Omega \mu(\theta)|Du|^2\,dx\,dt$$

$$= \int_0^T\int_\Omega F(\theta)\cdot u\,dx\,dt + \frac{1}{2}\int_\Omega |u_0|^2\,dx,$$

thus

$$\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{m_0}{2}\int_0^T\int_\Omega |Du|^2\,dx\,dt \le \int_0^T\int_\Omega F(\theta)\cdot u\,dx\,dt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)},$$

which implies that

$$m_0\int_0^T\int_\Omega |Du|^2\,dx\,dt \le 2\int_0^T \|F(\theta)\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\,dt + \|u_0\|^2_{L^2(\Omega)}. \quad (5.60)$$

Inequality (5.60) and Poincaré's inequality lead to

$$m_0\int_0^T\int_\Omega |Du|^2\,dx\,dt \le C\int_0^T \|F(\theta)\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\,dt + \|u_0\|^2_{L^2(\Omega)}. \quad (5.61)$$

Young's inequality, (5.61) and Korn's inequality allow us to deduce that

$$\int_0^T\int_\Omega |Du|^2\,dx\,dt \le C\|F(\theta)\|^2_{L^2(Q)} + C\|u_0\|^2_{L^2(\Omega)}. \quad (5.62)$$

In view of Lemma 4.2 and (5.62), we obtain

$$\|\hat\theta\|_{L^p(Q)} \le c\left[C\|F(\theta)\|^2_{L^2(Q)} + C\|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)}\right], \quad (5.63)$$

for all p such that $1 \le p < 2$.

By the growth assumption on F, we have

$$|F(\theta)|^2 \le 2(a^2 + M^2|\theta|^{2\alpha}) \text{ a.e. in } Q,$$

thus

$$\|F(\theta)\|^2_{L^2(Q)} \le 2a^2 \text{ meas}(\Omega)T + 2M^2\|\theta\|^{2\alpha}_{L^{2\alpha}(Q)}. \quad (5.64)$$

It follows from (5.63) and (5.64) that

$$\|\hat\theta\|_{L^p(Q)} \le c\left[Ca^2 \text{ meas}(\Omega)T + CM^2\|\theta\|^{2\alpha}_{L^{2\alpha}(Q)} + C\|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)}\right],$$

for all p such that $1 \le p < 2$. Since $1 < 2\alpha < 2$, the above inequality implies that

$$\|\hat\theta\|_{L^{2\alpha}(Q)} \le c\left[Ca^2 \text{ meas}(\Omega)T + CM^2\|\theta\|^{2\alpha}_{L^{2\alpha}(Q)} + C\|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)}\right].$$

We conclude that

$$\|\hat\theta\|_{L^{2\alpha}(Q)} \le C\left[a^2 + M^2\|\theta\|^{2\alpha}_{L^{2\alpha}(Q)} + \|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)}\right],$$

with C is a constant independent of $\|\theta\|_{L^{2\alpha}(Q)}$.

Now, we choose a positive real number R such that

$$C\left(a^2 + M^2 R^{2\alpha} + \|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)}\right) \le R.$$

It is possible to choose $\eta > 0$ small enough such that

$$a^2 + \|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)} \le \eta,$$

thus there exists $R(\eta) > 0$ such that

$$C\left(a^2 + M^2 R^{2\alpha}(\eta) + \|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)}\right) \le R(\eta);$$

this implies that

$$\psi_2(B_{L^{2\alpha}(Q)}(0, R(\eta))) \subset B_{L^{2\alpha}(Q)}(0, R(\eta)).$$

Schauder's fixed-point theorem and the definition of ψ_2 allow us to conclude that, under the condition of small data, there exists a weak-renormalized solution (θ, u) of $(2.14) - (2.18)$, which is the desired conclusion.

Case 4: $\mathbf{2 \le 2\alpha < 3}$. We choose p such that $\frac{2}{3-2\alpha} < p < \infty$, then $p > 2$ and $p' = \frac{p}{p-1}$ satisfies

$$1 < p' < \frac{2}{2\alpha - 1}.$$

We set: $q = \alpha p'$ and $r = 2\alpha$ so that $1 < q < \infty$ and $1 \le r < \frac{q}{q-1}$.

Let $\theta \in L^r(0,T;L^q(\Omega))$. Due to Hölder's inequality and Sobolev's embedding, we have for any $\varphi \in C_0^\infty(Q)$

$$\int_0^T\!\!\int_\Omega |\theta|^\alpha \varphi \, dx\, dt \le C\, \|\theta\|^\alpha_{L^r(0,T;L^q(\Omega))} \cdot \|\varphi\|_{L^2(0,T;H_0^1(\Omega))},$$

because $N = 2 \le p$, where C is a constant depending on Ω, T, α and p. Due to the growth condition (2.6) on F, we deduce that

$$\left| \int_0^T\!\!\int_\Omega F(\theta)\, \varphi \, dx\, dt \right| \le C\,(a + \|\theta\|^\alpha_{L^r(0,T;L^q(\Omega))})\|\varphi\|_{L^2(0,T;H_0^1(\Omega))}.$$

As a consequence, $F(\theta)$ can be identified with an element of $L^2(0,T;H^{-1}(\Omega))$ with

$$\|F(\theta)\|_{L^2(0,T;H^{-1}(\Omega))} \le C(a + \|\theta\|^\alpha_{L^r(0,T;L^q(\Omega))}). \tag{5.65}$$

We deduce that there exists a unique weak solution u of Problem $(3.1)-(3.4)$ in $L^2(0,T;H^1_\sigma(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega))$ (see, e.g. [19], [26]). Moreover, Lemma 4.1 allows us to conclude that there exists a unique renormalized solution $\hat\theta$ of Problem $(3.5) - (3.7)$. As a consequence, we choose $L = L^r(0,T;L^q(\Omega))$ in the fixed-point argument of Section 3.

Let us define the mapping ψ_3 by

$$\psi_3 : L^r(0,T;L^q(\Omega)) \longrightarrow L^r(0,T;L^q(\Omega)), \quad \theta \longrightarrow \psi_3(\theta) = \hat{\theta}.$$

In view of Lemma 4.2, we deduce that $\hat{\theta} \in L^r(0,T;L^q(\Omega))$. Thus, the mapping ψ_3 is well defined.

In the sequel, we will show that ψ_3 is compact and continuous and that there exists a ball B of $L^r(0,T;L^q(\Omega))$ such that $\psi_3(B) \subset B$.

i-$\boldsymbol{\psi_3}$ ***is compact.*** Let us consider a sequence θ_n which is bounded in $L^r(0,T;L^q(\Omega))$, and define the sequence $\hat{\theta}_n$ by

$$\psi_3(\theta_n) = \hat{\theta}_n.$$

For a fixed $n \geq 1$, due to (5.65), there exists a unique weak solution $u_n \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega))$ of the following problem (see, e.g. [14], [25], [26]):

$$u_{nt} + (u_n \cdot \nabla)u_n - 2 \text{ div } (\mu(\theta_n)Du_n) = F(\theta_n) \text{ in } (H^1_\sigma)'(\Omega), \tag{5.66}$$

for almost every t $\in (0,T)$,

$$\text{div } u_n = 0 \text{ in } Q, \tag{5.67}$$

$$u_n = 0 \text{ on } \Sigma_T, \tag{5.68}$$

$$u_n(t=0) = u_0 \text{ in } \Omega. \tag{5.69}$$

Thus $\mu(\theta_n)|Du_n|^2 \in L^1(Q)$. For a fixed $n \geq 1$, $\hat{\theta}_n$ is the unique renormalized solution of the following problem:

$$\frac{\partial b(\hat{\theta}_n)}{\partial t} - \text{ div } (a(x,\hat{\theta}_n,\nabla\hat{\theta}_n)) + \text{ div } (\Phi(\hat{\theta}_n)) = 2\mu(\theta_n)|Du_n|^2 \text{ in } Q, \tag{5.70}$$

$$\hat{\theta}_n = 0 \text{ on } \Sigma_T, \tag{5.71}$$

$$b(\hat{\theta}_n)(t=0) = b_0 \text{ in } \Omega. \tag{5.72}$$

Since θ_n is bounded in $L^r(0,T;L^q(\Omega))$, estimate (5.65) implies that $F(\theta_n)$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, which yields

$$u_n \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)), \tag{5.73}$$

$$u_{nt} \text{ is bounded in } L^2(0,T;(H^1_\sigma)'(\Omega)),$$

and there exists a subsequence such that

$$u_n \rightharpoonup w \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)), \tag{5.74}$$

$$u_n \to w \text{ strongly in } L^2_\sigma(Q), \tag{5.75}$$

$$u_{nt} \rightharpoonup w_t \text{ weakly in } L^2(0,T;(H^1_\sigma)'(\Omega)), \tag{5.76}$$

as n tends to $+\infty$, where w is a function of $L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_\sigma(\Omega))$ (see again [14], [25]). This implies that

$$\mu(\theta_n)|Du_n|^2 \text{ is bounded in } L^1(Q). \tag{5.77}$$

Estimate (5.77) and Lemma 4.2 imply that

$$\hat{\theta}_n \text{ is bounded in } L^{r_1}(0,T;L^{q_1}(\Omega)), \tag{5.78}$$

for any couple (q_1, r_1) such that $1 < q_1 < \infty$ and $1 \le r_1 < \frac{q_1}{q_1-1}$.

Estimate (5.77) allows us to use exactly the same argument as in the proof of (5.27) to show that for a subsequence still indexed by n

$$\hat{\theta}_n \longrightarrow \sigma, \tag{5.79}$$

almost everywhere in Q as n tends to $+\infty$, where σ is a measurable function defined on Q.

Firstly, we show that there exist q_1 and r_1 such that $q_1 > q$, $r_1 > r$, $1 < q_1 < \infty$ and $1 \le r_1 < \frac{q_1}{q_1-1}$.

Indeed, we know that $p > \frac{2}{3-2\alpha}$ thus $\frac{\alpha p}{p-1} < \frac{2\alpha}{2\alpha-1}$. We choose q_1 such that $\frac{\alpha p}{p-1} < q_1 < \frac{2\alpha}{2\alpha-1}$.

$$q_1 > \alpha p' \Rightarrow q_1 > q, \quad q_1 < \frac{2\alpha}{2\alpha-1} \Rightarrow 2\alpha < \frac{q_1}{q_1-1}.$$

We choose r_1 such that

$$2\alpha < r_1 < \frac{q_1}{q_1-1};$$

this implies that $r_1 > r$ and $r_1 < \frac{q_1}{q_1-1}$. Since $q_1 > q$ and $r_1 > r$, we deduce from (5.78) and (5.79) that

$$\hat{\theta}_n \to \sigma \text{ strongly in } L^r(0,T;L^q(\Omega)), \tag{5.80}$$

as n tends to $+\infty$.

ii-$\boldsymbol{\psi_3}$ ***is continuous***. Let us consider a sequence θ_n in $L^r(0,T;L^q(\Omega))$ such that

$$\theta_n \to \theta \text{ strongly in } L^r(0,T;L^q(\Omega)), \tag{5.81}$$

as n tends to $+\infty$, where θ is a function of $L^r(0,T;L^q(\Omega))$. Let $\hat{\theta}_n$ and $\hat{\theta}$ be defined by

$$\psi_3(\theta_n) = \hat{\theta}_n \text{ and } \psi_3(\theta) = \hat{\theta}.$$

Due to (5.65), we have $F(\theta_n)$ bounded in $L^2(0,T;H^{-1}(\Omega))$, thus, we argue exactly as in the proof of the compactness of ψ_3 to show that, for a subsequence still indexed by n, there exists a function σ such that

$$\hat{\theta}_n \longrightarrow \sigma \text{ a.e. in } Q,$$

and

$$T_K(\hat{\theta}_n) \rightharpoonup T_K(\sigma) \text{ in } L^2(0,T;H_0^1(\Omega)),$$

as n tends to $+\infty$ for any $K > 0$. Since F is continuous and satisfies (2.6), $r = 2\alpha \geq 2$ and $q = \alpha p'$, the mapping

$$L^r(0,T;L^q(\Omega)) \longrightarrow L^2(0,T;L^{p'}(\Omega))$$

$$\theta \longrightarrow F(\theta)$$

is continuous and the embedding $L^2(0,T;L^{p'}(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$ is continuous (since $p \geq 2$). Thus, the convergence result (5.81) implies that

$$F(\theta_n) \to F(\theta) \text{ in } L^2(0,T;H^{-1}(\Omega)),$$

as n tends to $+\infty$. We can now proceed analogously to the proof of (5.32); we thus obtain

$$\mu(\theta_n)|Du_n|^2 \to \mu(\theta)|Du|^2 \text{ in } L^1(Q), \tag{5.82}$$

as n tends to $+\infty$. According to (5.82), theorem 4.1 of [6] ensures that σ is the unique renormalized solution of (3.5) – (3.7). Thanks to Lemma 4.1, we obtain

$$\sigma = \hat{\theta} \text{ a.e. in } Q,$$

and

$$\hat{\theta}_n \longrightarrow \hat{\theta} \text{ a.e. in } Q, \tag{5.83}$$

as n tends to $+\infty$. Now, it is easy to check that $\hat{\theta}_n$ is bounded in $L^{r_1}(0,T;L^{q_1}(\Omega))$ for all (q_1,r_1) such that $1 < q_1 < \infty$ and $1 \leq r_1 < \frac{q_1}{q_1-1}$. Indeed, this follows by the same method as in (5.78). We deduce (compactness of ψ_3) that $\hat{\theta}_n \to \hat{\theta}$ strongly in $L^r(0,T;L^q(\Omega))$, as n tends to $+\infty$, and thus ψ_3 is a continuous mapping.

iii-***There exists a ball B of $L^r(0,T;L^q(\Omega))$ such that $\psi_3(B) \subset B$.***

We show that, if the data are small enough, there exists a positive real number R_0 such that $\psi_3(B_{L^r(0,T;L^q(\Omega))}(0,R_0)) \subset B_{L^r(0,T;L^q(\Omega))}(0,R_0)$. Let R be a positive real number. We assume that θ belongs to $B_{L^r(0,T;L^q(\Omega))}(0,R)$.

We recall that $u \in L^2(0,T;H_\sigma^1(\Omega)) \cap L^\infty(0,T;L_\sigma^2(\Omega))$ is the unique weak solution of the problem (3.1) – (3.4). We use u as a test function in (3.1), to obtain

$$\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{1}{2}\int_0^T\int_\Omega \mu(\theta)|Du|^2\,dx\,dt = \int_0^T\int_\Omega F(\theta)\cdot u\,dx\,dt + \frac{1}{2}\int_\Omega |u_0|^2\,dx,$$

thus

$$\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{m_0}{2}\int_0^T\int_\Omega |Du|^2\,dx\,dt \leq \int_0^T \langle F(\theta), u\rangle\,dt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Due to (5.65), we have

$$m_0 \int_0^T\int_\Omega |Du|^2\,dx\,dt \\ \leq a\,\|u\|_{L^2(Q)} + M\|\theta\|^\alpha_{L^r(0,T;L^q(\Omega))}\,\|u\|_{L^2(0,T;H^1_0(\Omega))} + \|u_0\|^2_{L^2(\Omega)}.$$

Through the help of Young and Korn's inequality, we obtain

$$\int_0^T\int_\Omega |Du|^2\,dx\,dt \leq C(a^2 + M^2\|\theta\|^{2\alpha}_{L^r(0,T;L^q(\Omega))} + \|u_0\|^2_{L^2(\Omega)}).$$

In the above inequality and in what follows, C is a generic constant. In view of Lemma 4.2, we have

$$\|\hat{\theta}\|_{L^r(0,T;L^q(\Omega))} \leq C\big(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b_0\|_{L^1(\Omega)}\big),$$

and we conclude that

$$\|\hat{\theta}\|_{L^r(0,T;L^q(\Omega))} \leq C\left[a^2 + M^2\|\theta\|^{2\alpha}_{L^r(0,T;L^q(\Omega))} + \|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)}\right], \tag{5.84}$$

with C a constant independent of $\|\theta\|_{L^r(0,T;L^q(\Omega))}$, u_0, M and b_0.

Proceeding as in (iii) in the case where $1 < 2\alpha < 2$, we choose the data such that

$$a^2 + \|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)} \leq \eta,$$

where η is a small enough constant to insure that there exists a positive real number $R_0 = R(\eta)$ satisfying

$$C\left(a^2 + M^2R(\eta)^{2\alpha} + \|u_0\|^2_{L^2(\Omega)} + \|b_0\|_{L^1(\Omega)}\right) \leq R(\eta).$$

Then, inequality (5.84) shows that

$$\psi_3(B_{L^r(0,T;L^q(\Omega))}(0,R_0)) \subset B_{L^r(0,T;L^q(\Omega))}(0,R_0).$$

Schauder's fixed-point theorem and the definition of ψ_3 allow us to conclude that, under the condition of small data, there exists a weak-renormalized solution (θ,u) of $(2.14)-(2.18)$.

This allows us to end the proof of Theorem 5.1.

6. Existence of a solution for N=3

In this section, we assume that F is a continuous function from $\mathbb{R}$ into $\mathbb{R}^3$ and is bounded; i.e., $\alpha = 0$.

Theorem 6.1. *Assume that* (2.1) − (2.5), (2.7) − (2.11) *and* (2.13) *hold true. Assume that* F *is a continuous function from* $\mathbb{R}$ *into* $\mathbb{R}^3$*, and* $u_0 \in (H_0^1(\Omega))^3$ *such that* $\operatorname{div} u_0 = 0$ *in* Q *and* $u_0 \cdot n = 0$ *on* $\partial\Omega$*. There exists a real positive number* η *such that, if* $\|u_0\|_{H_0^1(\Omega)} + \|F\|_{L^\infty(\mathbb{R})} \leq \eta$*, then there exists at least a weak-renormalized solution of the system* (2.14) − (2.18) *for* $N = 3$ *(in the sense of Definition* 2.2*).*

Proof of Theorem 6.1. Since the proof relies on similar techniques to the ones developed in the previous sections, we just point out how to modify the arguments. We show this theorem through a fixed-point argument and the fixed-point space of Section 3 is $L = L^1(Q)$. For a fixed θ in $L^1(Q)$, it is known that there exists $\eta > 0$ (small enough) such that $\|u_0\|_{H_0^1(\Omega)} + \|F\|_{L^\infty(\mathbb{R})} \leq \eta$, thus the Navier-Stokes equations (3.1)-(3.4) admit a unique weak solution $u \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ (see theorem 3.11 of [26]), with $\|u\|_{L^2(0,T;H^2(\Omega))} \leq C$, where C is a constant. The unique renormalized solution $\hat{\theta}$ of (3.5)-(3.7) indeed belongs to $L^1(Q)$ (see Lemma 4.2).

We denote by ψ_4 the mapping defined by

$$\psi_4 : L^1(Q) \longrightarrow L^1(Q), \quad \theta \longrightarrow \psi_4(\theta) = \hat{\theta}.$$

By the same arguments used in the case where $\alpha = 0$, it is easily seen that ψ_4 is compact and there exists a ball B of $L^1(Q)$ such that $\psi_4(B) \subset B$ for all initial data given in Theorem 6.1. It remains to prove that ψ_4 is continuous. Since $N = 3$, we cannot use the reasoning of the preceding case ($\alpha = 0$) to have the strong convergence of u (with respect to approximations) in $L^2(0,T;H_\sigma^1(\Omega))$.

Let us consider a sequence θ_n in $L^1(Q)$ such that

$$\theta_n \to \theta \tag{6.1}$$

strongly in $L^1(Q)$ as n tends to $+\infty$, where θ is a function of $L^1(Q)$. Let $\hat{\theta}_n$ and $\hat{\theta}$ be defined by

$$\psi_4(\theta_n) = \hat{\theta}_n \text{ and } \psi_4(\theta) = \hat{\theta}.$$

For a fixed $n \geq 1$, since $\|u_0\|_{H_0^1(\Omega)} + \|F\|_{L^\infty(\mathbb{R})} \leq \eta$, where η is a small enough positive real number, there exists a unique weak solution $u_n \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ of the following problem:

$$u_{nt} + (u_n \cdot \nabla)u_n - 2 \operatorname{div} (\mu(\theta_n)Du_n) = F(\theta_n) \text{ in } (H_\sigma^1)'(\Omega), \tag{6.2}$$

for almost every t $\in (0,T)$,

$$\operatorname{div} u_n = 0 \text{ in } Q, \tag{6.3}$$

$$u_n = 0 \text{ on } \Sigma_T, \tag{6.4}$$

$$u_n(t=0) = u_0 \text{ in } \Omega, \tag{6.5}$$

with u_n bounded in $L^2(0,T;H^2(\Omega))$, and

$$\frac{\partial u_n}{\partial t} \text{ bounded in } L^2(0,T;(H^1_\sigma)'(\Omega)).$$

Thus $\mu(\theta_n)|Du_n|^2 \in L^1(Q)$. For a fixed $n \geq 1$, $\hat{\theta}_n$ is the unique renormalized solution of the following problem:

$$\frac{\partial b(\hat{\theta}_n)}{\partial t} - \text{ div } (a(x,\hat{\theta}_n,\nabla\hat{\theta}_n)) + \text{ div } (\Phi(\hat{\theta}_n)) = 2\mu(\theta_n)|Du_n|^2 \text{ in } Q, \tag{6.6}$$

$$\hat{\theta}_n = 0 \text{ on } \Sigma_T, \tag{6.7}$$

$$b(\hat{\theta}_n)(t=0) = b_0 \text{ in } \Omega. \tag{6.8}$$

According to an Aubin's type lemma (see, e.g. [24]), we may, then, extract a subsequence such that

$$u_n \to u \text{ strongly in } L^2(0,T;H^1_\sigma(\Omega)), \tag{6.9}$$

as n tends to $+\infty$, where u is a function of $L^\infty(0,T;H^1_0(\Omega)) \cap L^2(0,T;H^2(\Omega))$. The proof is completed by using (6.9) and proceeding analogously to the proof of the continuity of ψ_1.

The conditions of Schauder's fixed-point theorem being satisfied, the definition of ψ_4 allows us to conclude that there exists a weak-renormalized solution (θ,u) of the system $(2.14) - (2.18)$.

References

[1] H.W. Alt and S. Luckhaus, *Quasilinear elliptic-parabolic differential equations,* Math. Z., 183 (1983), 311–341.

[2] A. Attaoui, *Existence de solutions pour une classe de systèmes non linéaires de Boussinesq,* C. R. Acad. Sci. Paris Ser. I, 346 (2008), 515–520.

[3] D. Blanchard and O. Guibé, *Existence of solution for a nonlinear system in thermoviscoelasticity,* Adv. Differential Equations, 5 (2000), 1221–1252.

[4] D. Blanchard and F. Murat, *Renormalized solution for nonlinear parabolic problems with L^1 data: Existence and uniqueness,* Proc. Roy. Soc. Edinburgh Sect. A., 127 (1997), 1137–1152.

[5] D. Blanchard, F. Murat, and H. Redwane, *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems,* J. Differential Equations, 177 (2001), 331–374.

[6] D. Blanchard and A. Porretta, *Stefan problems with nonlinear diffusion and convection,* J. Differential Equations, 210 (2005), 383–428.

[7] D. Blanchard and H. Redwane, *Renormalized solutions for a class of nonlinear parabolic evolution problems,* J. Math. Pures Appl., 77 (1998), 117–151.

[8] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.L. Vasquez, *An L^1-theory of existence and uniquness of solutions of nonlinear elliptic equations,* Ann. Scuola Norm. sup. Pisa, 22 (1995), 241–273.

[9] L. Boccardo, D. Giachetti, J.I. Diaz, and F. Murat, *Existence and regularity of renormalized solutions for some elliptic problems involving derivations of nonlinear terms,* J. Differential. Equations, 106 (1993), 215–237.

[10] B. Climent and E. Fernández-Cara, *Existence and uniqueness results for a coupled problem related to the stationary Navier-Stokes system,* J. Math. Pures Appl., 76 (1997), 307–319.

[11] B. Climent and E. Fernández-Cara, *Some existence and uniqueness results for a time-dependent coupled problem of the Navier-Stokes kind,* Math. Models Methods Appl. Sci., 8 (1998), 603–622.

[12] A. Dall' Aglio and L. Orsina, *Nonlinear parabolic equations with natural growth conditions and L^1 data,* Nonlinear Analysis, 27 (1986), 59–73.

[13] G. De Rham, "Varietés différentiables," Hermann, Paris, 1960.

[14] J.I. Diaz and G. Galiano, *Existence and uniqueness of solutions of the Boussinesq system with nonlinear thermal diffusion,* Topological Methods in Nonlinear Analysis, 11 (1998), 59–82.

[15] R.J. DiPerna and P.L. Lions, *On the Cauchy problem for Boltzmann equations: Global existence and weak stability,* Ann of Math., 130 (1989), 321–366.

[16] R.J. DiPerna and P.L. Lions, *Ordinary differential equations: Sobolev spaces and transport theory,* Invent. Math., 98 (1989), 511–547.

[17] N. Igbida and M. Urbano, *Uniqueness for nonlinear degenerate problems,* NoDea Nonlin. Diff. Eq. Appl., 30 (2003), 287–307.

[18] J. Leray, *Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique,* J. Math. Pures Appl., 12 (1933), 1–82.

[19] P.L. Lions, "Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible Models," Oxford Lect. Series in Math. Appl 3., Oxford, 1996.

[20] F. Murat, "Soluciones renormalizadas de EDP elipticas non lineales: Cours à l'Université de Séville, Publication R93023," Laboratoire d'Analyse Numérique, Paris VI 1993.

[21] F. Murat, *Équations elliptiques non linéaires avec second membre L^1 ou mesure,* In "Comptes Rendus du 26éme Congrès National d'Analyse Numérique, les Karellis," 1994, A12–A24.

[22] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncations,* Ann. Mat. Pura ed Applicata, 177 (1999), 143–172.

[23] S. Segura de León and J. Toledo, *Regularity for entropy solutions of parabolic p-Laplacian type equations,* Publ. Mat., 43 (1999), 665–683.

[24] J. Simon, *Compact sets in the space $L^p(0,T;B)$,* Ann. Mat. Pura. Appl., 146 (1987), 65–96.

[25] L. Tartar, "Topics in nonliear analysis," Publications Mathématiques d'Orsay, 1982.

[26] R. Temam, "Navier-Stokes Equations: Theory and Numerical Analysis, 3rd revised edition," North-Holland, Amsterdam, 1984.