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LIMIT BEHAVIORS OF SOME BOUNDARY-VALUE PROBLEMS WITH HIGH AND/OR LOW VALUED PARAMETERS

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Abstract. The first aim of this paper is to give a general variational framework for bilinear forms depending on two parameters tending to zero and to infinity respectively, allowing us to analyze the three limit problems. Secondly, we give different illustrative applications for transmission problems involving some elasticity systems, diffusion problems, and Maxwell systems where one parameter tends to infinity and/or a part of the domain squeezes to a smooth surface. These limit procedures lead to new transmission problems, like a coupling between the Lamé system and the Stokes system.

1. Introduction

Some partial differential equations are characterized by the fact that their coefficients are very different in some subpart of the domain where they are set in such a way that their ratio becomes very large. As an example, we can cite the case of the diffusion problem [18]:

$$-\text{div}\,(a\nabla u) = f \quad \text{in } D,$$

where $a = 1$ in a fixed part of the domain D and a goes to infinity in the remainder D_a of the domain D. In that case, it is not difficult to obtain the limit problem, see for instance [18]. In a similar manner, it is not difficult to obtain the limit problem of the above problem if a remains fixed but the sub-domain squeezes to a smooth hypersurface of codimension 1, see [18, 22]. For diffusion problems or the Lamé system, asymptotic expansion of the solution when the sub-domain squeezes to a smooth hypersurface of codimension 1 can be found in [2, 3] for instance. But the situation becomes more difficult if both the parameter a goes to infinity and the domain D_a

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becomes small. Such a situation was studied for instance in [18] for diffusion problems in its full generality.

To our knowledge a systematic analysis of general systems of partial differential equations, where one parameter tends to infinity in a sub-domain that squeezes to a smooth hypersurface of codimension 1 has not been performed. In order to treat such problems, for a family of bilinear forms depending on two parameters ϵ and σ, the first one being devoted to tending to zero and the second one to infinity, we give a general framework as large as possible allowing us to characterize the limit problems and show strong convergence results. We also deduce a necessary and sufficient condition that guarantees that the two limits commute. We then apply this framework for different transmission problems for the elasticity systems, the anisotropic diffusion problems and Maxwell systems where one parameter tends to infinity and/or a part of the domain squeezes to a smooth surface. In some cases these limited procedures lead to new transmission problems.

The schedule of the paper is as follows. In Section 2, we present the abstract framework and prove convergence results. Section 3 is devoted to three examples. The first one concerns the isotropic elasticity system (Lamé) with the Lamé coefficients λ tending to infinity in sub-domains Ω_ϵ that approach a smooth surface S. The three limit problems are analyzed; in particular we show that, when λ tends to infinity, the limit problem is a coupling between the Lamé system and the Stokes system. The limit of this last problem when Ω_ϵ approaches a smooth surface S involves a closed subspace $W(S)$ of $H_0^1(D)^3$, that can be described in some particular situations (see Section 4). We perform a similar analysis for general anisotropic elasticity systems and diffusion problems. In some particular geometric situations the spaces $W(S)$ can be characterized; this is done in Section 4. Finally, Section 5 is devoted to an application to the Maxwell systems; here only the limit when the conductivity σ goes to infinity is possible.

2. The abstract framework

Let us consider two real positive parameters σ and ϵ where σ is destined to tend to infinity and ϵ to zero. Let V be a Hilbert space equipped with the norm $\|.\|_V$. By $< \cdot, \cdot >$ we denote the duality pairing between V and its topological dual space V'.

Inspired by concrete examples (see [18] or the examples treated below for instance), we make the following assumptions: We assume we are given a family of bilinear forms a_ϵ^σ that satisfy the following assumptions:

(H1) a_ϵ^σ depends on two positive parameters σ and ϵ and is a linear combination of two bilinear forms $a_\epsilon^{(1)}$ and $a_\epsilon^{(2)}$ independent of σ:

$$a_\epsilon^\sigma(u,\ v) = a_\epsilon^{(1)}(u,\ v) + \sigma a_\epsilon^{(2)}(u,\ v)\ ,\quad \forall u,\ v \in V.$$

(H2) $a_\epsilon^{(1)}$ and $a_\epsilon^{(2)}$ are nonnegative; i.e.,

$$a_\epsilon^{(j)}(u,\ u) \geq 0,\ \forall u \in V,\ j = 1,\ 2,$$

and uniformly continuous in ϵ in $V \times V$. Furthermore, $a_\epsilon^{(2)}$ is supposed to be symmetric.

(H3) The sum $a_\epsilon^{(1)} + a_\epsilon^{(2)}$ is uniformly coercive in ϵ; i.e., there exists a positive constant α independent of ϵ such that

$$(a_\epsilon^{(1)} + a_\epsilon^{(2)})(u,\ u) \geq \alpha \|u\|_V^2,\ \forall u \in V. \tag{2.1}$$

Hence, the square root of this left-hand side defines a norm equivalent to the usual norm of the space V.

For the last assumption, we need to introduce a new notion of convergence of bilinear forms (see section VIII.3.1 of [19] and Remark 2.3 below).

Definition 2.1. *A family of bilinear forms b_ϵ converges strongly to a bilinear form b in V as ϵ goes to zero (in short $b_\epsilon \longrightarrow b$ strongly) if*

- *there exists a family of Hilbert spaces U_ϵ containing V and such that for any $u \in V$ $\|u\|_{U_\epsilon} \longrightarrow 0$ as $\epsilon \longrightarrow 0$,*
- *the remainder $r_\epsilon = b_\epsilon - b$ satisfies*
  a) *There exists a positive constant C such that*

  $$|r_\epsilon(u,\ v)| \leq C\|u\|_{U_\epsilon}\|v\|_V,\ \forall u, v \in V,$$

  b) *for any sequence $w_\epsilon \in V$ weakly converging to zero in V, we have: $|r_\epsilon(w_\epsilon,\ v)| \longrightarrow 0$ when $\epsilon \longrightarrow 0$, for all $v \in V$.*

Remark 2.2. For symmetric bilinear forms, condition b) is a consequence of a).

Remark 2.3. If $b_\epsilon \longrightarrow b$ strongly, then b_ϵ converges to b in the sense of Kato, see section VIII.3.1 of [19]. The converse implication is false in general. Let us give the following counterexample: Consider the Hilbert space $H_0^1(0,\ 1)$ equipped with the standard inner product

$$(u,\ v)_{H_0^1(0,\ 1)} := \int_0^1 u'v'dx,$$

and associated norm $|u|_{H_0^1(0,\ 1)} = \|u'\|_{L^2(0,\ 1)}$.

It is easy to verify that the bilinear form r_ϵ defined by

$$r_\epsilon(u,\ v) = \frac{1}{\sqrt{\epsilon}}\ u(\epsilon) \int_0^1 v(x)\ dx,\quad \forall u,\ v \in H_0^1(0,\ 1),$$

converges to zero in the sense of Kato: indeed for any $u,\ v \in H_0^1(0,\ 1)$ we have

$$\begin{aligned} |r_\epsilon(u,\ v)| &= \frac{1}{\sqrt{\epsilon}}|u(\epsilon)|\Big|\int_0^1 v(x)\ dx\Big| \leq \frac{1}{\sqrt{\epsilon}}\Big|\int_0^\epsilon u'(x)\ dx\Big|\Big|\int_0^1 v(x)\ dx\Big| \\ &\leq \Big(\int_0^\epsilon |u'(x)|^2\ dx\Big)^{\frac{1}{2}}\Big(\int_0^1 |v(x)|^2\ dx\Big)^{\frac{1}{2}} \quad \longrightarrow 0 \quad \text{when} \quad \epsilon \longrightarrow 0. \end{aligned}$$

But, its convergence in the sense of Definition 2.1 does not occur because the property b) is not satisfied. Indeed, take the sequence w_ϵ defined in $H_0^1(0,\ 1)$ by

$$w_\epsilon(x) = \begin{cases} \frac{\sqrt{2}}{2\sqrt{\epsilon}}x, & \text{if } 0 \leq x < \epsilon. \\ -\frac{\sqrt{2}}{2\sqrt{\epsilon}}x + \sqrt{2\epsilon}, & \text{if } \epsilon \leq x < 2\epsilon. \\ 0, & \text{if } 2\epsilon \leq x \leq 1. \end{cases}$$

Direct calculations yield $|w_\epsilon|^2_{H_0^1(0,\ 1)} = 1$. On the other hand this sequence converges to zero weakly in $H_0^1(0,\ 1)$, since for all $v \in H_0^1(0,\ 1)$ we have

$$\begin{aligned} (w_\epsilon,\ v)_{H_0^1(0,1)} &= \int_0^1 w_\epsilon'(x)v'(x)dx \leq \Big(\int_0^{2\epsilon} |w_\epsilon'(x)|^2\ dx\Big)^{\frac{1}{2}}\Big(\int_0^{2\epsilon} |v'(x)|^2\ dx\Big)^{\frac{1}{2}} \\ &\leq \Big(\int_0^{2\epsilon} |v'(x)|^2\ dx\Big)^{1/2} \longrightarrow 0 \quad \text{when} \quad \epsilon \longrightarrow 0. \end{aligned}$$

Now, for $v \in H_0^1(0,\ 1)$, we have

$$r_\epsilon(w_\epsilon,\ v) = \frac{1}{\sqrt{\epsilon}}w_\epsilon(\epsilon)\int_0^1 v(x)\ dx = \frac{\sqrt{2}}{2}\int_0^1 v(x)\ dx.$$

This shows that the assumption b) of Definition 2.1 does not hold as soon as v has a nonzero mean value.

Now, we can state our last assumption:

(H4) The family of bilinear forms $a_\epsilon^{(2)}$ converges strongly to zero, while the family $a_\epsilon^{(1)}$ converges strongly to a bilinear form a, which is continuous and coercive on the whole space V.

We first notice the equivalence between the assumption (H3) with the coerciveness of a_ϵ^σ.

Proposition 2.4. *The following two properties are equivalent.*

i) *The sum $a_\epsilon^{(1)} + a_\epsilon^{(2)}$ is coercive on V uniformly in ϵ.*
ii) *The bilinear form a_ϵ^σ is coercive on V uniformly in ϵ and in $\sigma \geq 1$.*

Proof. This is based on the inequalities

$$\max(1,\ \sigma)(a_\epsilon^{(1)} + a_\epsilon^{(2)})(u,\ u) \geq a_\epsilon^\sigma(u,\ u) \geq \min(1,\ \sigma)(a_\epsilon^{(1)} + a_\epsilon^{(2)})(u,\ u),$$

valid for all $\sigma > 0$ and all $u \in V$. □

For further purposes, we need the following.

Definition 2.5. *For all $\epsilon > 0$, we denote by V_ϵ the space of elements v of V satisfying $a_\epsilon^{(2)}(v,\ v) = 0$. Let us further denote by W the closure in V of the union of V_ϵ for all ϵ; i.e., $W = \overline{\bigcup_{\epsilon>0} V_\epsilon}^V$.*

Finally, the spaces V_ϵ are supposed to be nested in the following sense:

$$V_{\epsilon'} \subset V_\epsilon, \quad \forall \epsilon,\ \epsilon' \ s.t \ \ \epsilon' > \epsilon. \tag{2.2}$$

This assumption is again inspired from the examples and some proofs below do not hold without it.

Given $f \in V'$, we now consider the following different problems:

Find $u_\epsilon^\sigma \in V$ such that

$$a_\epsilon^\sigma(u_\epsilon^\sigma,\ v) =< f,\ v > , \quad \forall v \in V. \tag{2.3}$$

Find $u_\epsilon \in V_\epsilon$ such that

$$a_\epsilon^{(1)}(u_\epsilon,\ v) =< f,\ v > , \quad \forall v \in V_\epsilon. \tag{2.4}$$

Find $u \in W$ such that

$$a(u,\ v) =< f,\ v > , \quad \forall v \in W. \tag{2.5}$$

Find $u^* \in V$ such that

$$a(u^*,\ v) =< f,\ v > , \quad \forall v \in V. \tag{2.6}$$

Proposition 2.6. *Under the hypotheses* (H1), (H2), (H3), *and* (H4), *problems* (2.3), (2.4), (2.5), *and* (2.6) *admit unique solutions in V, V_ϵ, W, and V respectively.*

Proof. The existence and uniqueness of the solutions of problems (2.3), (2.4), (2.5), and (2.6) in the corresponding spaces are a consequence of the Lax-Milgram lemma [8]: Indeed, the right-hand sides of the four problems are bounded linear forms on their respective spaces under the hypothesis $f \in V'$. Due to the assumption (H4), a is continuous and coercive on V.

Then, it keeps the same properties on any closed subspace. So, problems (2.5) and (2.6) admit unique solutions in W and V respectively.

The continuity of $a_\epsilon^{(1)}$, a_ϵ^σ on $V_\epsilon \times V_\epsilon$ and $V \times V$ respectively comes from the assumption (H2).

On the other hand, due to Proposition 2.4, a_ϵ^σ is coercive on V.

Finally, for $u \in V_\epsilon \subset V$, $a_\epsilon^{(2)}(u,\ u) = 0$ and therefore

$$a_\epsilon^{(1)}(u,\ u) = a_\epsilon^{(1)}(u,\ u) + a_\epsilon^{(2)}(u,\ u) \geq \alpha \|u\|_V^2,$$

due to the assumption (H3). This proves the coerciveness of $a_\epsilon^{(1)}$ on V_ϵ. Hence, the conclusion for the two problems (2.3) and (2.4) follows. □

Our main result is the following theorem that gives the relationship between these problems as $\sigma \longrightarrow \infty$ and/or $\epsilon \longrightarrow 0$.

Theorem 2.7. *Under the assumptions* (H1), (H2), (H3), *and* (H4), *the respective solutions* u_ϵ^σ, u_ϵ, u *and* u^* *of problems* (2.3), (2.4), (2.5), *and* (2.6) *satisfy the following limit properties:*

$$\lim_{\sigma \longrightarrow \infty} \|u_\epsilon^\sigma - u_\epsilon\|_V = 0, \tag{2.7}$$

$$\lim_{\epsilon \longrightarrow 0} \|u_\epsilon - u\|_V = 0, \tag{2.8}$$

$$\lim_{\epsilon \longrightarrow 0} \|u_\epsilon^\sigma - u^*\|_V = 0. \tag{2.9}$$

Proof. Let us start with the proof of the first limit. Let ϵ be fixed. For the test function $v = u_\epsilon^\sigma$ in (2.3), and by using the fact that $f \in V'$, we have

$$a_\epsilon^\sigma(u_\epsilon^\sigma,\ u_\epsilon^\sigma) = a_\epsilon^{(1)}(u_\epsilon^\sigma,\ u_\epsilon^\sigma) + \sigma a_\epsilon^{(2)}(u_\epsilon^\sigma,\ u_\epsilon^\sigma) =< f,\ u_\epsilon^\sigma >\leq \|f\|_{V'} \|u_\epsilon^\sigma\|_V. \tag{2.10}$$

From this last inequality and Proposition 2.4, we obtain

$$\|u_\epsilon^\sigma\|_V \leq C, \text{ for } \sigma \geq 1, \tag{2.11}$$

and

$$a_\epsilon^{(2)}(u_\epsilon^\sigma,\ u_\epsilon^\sigma) \leq \frac{C}{\sigma} \longrightarrow 0 \text{ as } \sigma \longrightarrow \infty, \tag{2.12}$$

where C means different positive constants independent of ϵ and σ.

From (2.11), we deduce that there exist a subsequence, still denoted by $(u_\epsilon^\sigma)_\sigma$ and $\widetilde{u}_\epsilon \in V$ such that

$$u_\epsilon^\sigma \longrightarrow \widetilde{u}_\epsilon \text{ weakly in } V \text{ as } \sigma \longrightarrow \infty. \tag{2.13}$$

From (2.12), the symmetric property of $a_\epsilon^{(2)}$, and the convergence (2.13), we obtain

$$a_\epsilon^{(2)}(\widetilde{u}_\epsilon,\ \widetilde{u}_\epsilon) = 0. \tag{2.14}$$

This shows that $\widetilde{u}_\epsilon$ belongs to V_ϵ.

Since $a_\epsilon^{(2)}$ is symmetric and nonnegative, it satisfies Cauchy-Schwarz's inequality hence, for $v \in V_\epsilon$, we deduce $a_\epsilon^{(2)}(u_\epsilon^\sigma,\ v) = 0$. As a result, (2.3) implies that

$$a_\epsilon^{(1)}(u_\epsilon^\sigma,\ v) =< f,\ v >,\ \forall v \in V_\epsilon.$$

Taking into account (2.13), we obtain that $\widetilde{u}_\epsilon$ is the unique solution of problem (2.4). Thus $\widetilde{u}_\epsilon = u_\epsilon$. Hence, the result (2.7) is satisfied in a weak sense.

For the proof of the strong convergence, we apply a_ϵ^σ at the difference $u_\epsilon^\sigma - u_\epsilon$ which is an element of V. Since, for σ large, a_ϵ^σ is uniformly coercive, we obtain

$$\begin{aligned} C\|u_\epsilon^\sigma - u_\epsilon\|_V^2 &\le a_\epsilon^\sigma(u_\epsilon^\sigma,\ u_\epsilon^\sigma - u_\epsilon) - a_\epsilon^\sigma(u_\epsilon,\ u_\epsilon^\sigma - u_\epsilon) \\ &\le< f,\ u_\epsilon^\sigma - u_\epsilon > -a_\epsilon^{(1)}(u_\epsilon,\ u_\epsilon^\sigma - u_\epsilon). \end{aligned}$$

As a result of the weak convergence (2.13) and the bilinearity of $a_\epsilon^{(1)}(\cdot,\cdot)$, we have

$$< f,\ u_\epsilon^\sigma - u_\epsilon >\longrightarrow 0 \text{ as } \sigma \longrightarrow +\infty, \tag{2.15}$$

and

$$\begin{aligned} a_\epsilon^{(1)}(u_\epsilon,\ u_\epsilon^\sigma - u_\epsilon) = a_\epsilon^{(1)}(u_\epsilon, u_\epsilon^\sigma) - a_\epsilon^{(1)}(u_\epsilon, u_\epsilon) = a_\epsilon^{(1)}(u_\epsilon, u_\epsilon^\sigma) - < f, u_\epsilon > \\ \longrightarrow < f,\ u_\epsilon > - < f,\ u_\epsilon >= 0. \end{aligned} \tag{2.16}$$

This proves that the norm of $u_\epsilon^\sigma - u_\epsilon$ on V goes to zero, when $\sigma \longrightarrow \infty$.

Let us prove the limit (2.8). Taking $v = u_\epsilon$ as a test function in (2.4), and taking into account the property $f \in V'$ and the V-ellipticity of the bilinear form $a_\epsilon^{(1)}$ on V_ϵ, we see that the sequence (u_ϵ) is bounded in V. Therefore, there exists a subsequence that we still denote by (u_ϵ) and $\widetilde{u} \in V$ such that

$$u_\epsilon \longrightarrow \widetilde{u} \ \text{ weakly in } V \text{ as } \epsilon \longrightarrow 0. \tag{2.17}$$

From the definition of space W, we conclude that $\widetilde{u} \in W$. By the assumption (H4) made on the bilinear form $a_\epsilon^{(1)}$ and the nested property of the space V_ϵ, we have

$$a_\epsilon^{(1)}(u_\epsilon,\ v) = a(u_\epsilon,\ v) + r_\epsilon^{(1)}(u_\epsilon,\ v) =< f,\ v >,\quad \forall v \in V_{\epsilon'} \quad \epsilon' \ge \epsilon.$$

Taking $w_\epsilon = u_\epsilon - \widetilde{u}$,

$$r_\epsilon^{(1)}(u_\epsilon,\ v) = r_\epsilon^{(1)}(w_\epsilon,\ v) + r_\epsilon^{(1)}(\widetilde{u},\ v).$$

Letting ϵ tend to zero, and thanks to the convergence (2.17), we obtain that $\widetilde{u}$ satisfies (2.5) for all v in W, since $a(u_\epsilon,\ v) \longrightarrow a(\widetilde{u},\ v)$, $|r_\epsilon^{(1)}(w_\epsilon,\ v)| \longrightarrow 0$, and $|r_\epsilon^{(1)}(\widetilde{u},\ v)| \leq C\|\widetilde{u}\|_{U_\epsilon}\|v\|_V \longrightarrow 0$. Therefore, $\widetilde{u} \in W$ satisfies

$$a(\widetilde{u},\ v) =< f,\ v >, \quad \forall v \in V_{\epsilon'}, \epsilon' > 0.$$

Letting ϵ' go to zero, we deduce that $\widetilde{u} \in W$ is a solution of (2.5) and thus $\widetilde{u} = u$.

Now, we go on with the proof of the strong convergence. Consider a sequence $w_\epsilon \in V_\epsilon, \epsilon > 0$, which converges strongly to u in V. By the definition of W, such a sequence exists. We notice that, for all ϵ, $u_\epsilon - w_\epsilon \in V_\epsilon$. Applying (2.4) and (2.5) to $u_\epsilon - w_\epsilon$, and taking into account the uniform coerciveness of $a_\epsilon^{(1)}$ on V_ϵ, we obtain

$$\begin{aligned}
C\|u_\epsilon - w_\epsilon\|_V^2 &\leq a_\epsilon^{(1)}(u_\epsilon - w_\epsilon,\ u_\epsilon - w_\epsilon)\\
&\leq a_\epsilon^{(1)}(u_\epsilon,\ u_\epsilon - w_\epsilon) - a_\epsilon^{(1)}(w_\epsilon - u,\ u_\epsilon - w_\epsilon) - a_\epsilon^{(1)}(u,\ u_\epsilon - w_\epsilon)\\
&\leq a_\epsilon^{(1)}(u_\epsilon,\ u_\epsilon - w_\epsilon) - a_\epsilon^{(1)}(w_\epsilon - u,\ u_\epsilon - w_\epsilon)\\
&- a(u,\ u_\epsilon - w_\epsilon) - r_\epsilon^{(1)}(u,\ u_\epsilon - w_\epsilon)\\
&\leq < f,\ u_\epsilon - w_\epsilon > - a_\epsilon^{(1)}(w_\epsilon - u,\ u_\epsilon - w_\epsilon)\\
&- < f,\ u_\epsilon - w_\epsilon > - r_\epsilon^{(1)}(u,\ u_\epsilon - w_\epsilon)\\
&\leq C_1\|w_\epsilon - u\|_V\|u_\epsilon - w_\epsilon\|_V + C_2\|u\|_{U_\epsilon}\|u_\epsilon - w_\epsilon\|_V,
\end{aligned}$$

C_1 and C_2 being two positive constants independent of ϵ.

Using the convergence of w_ϵ to u and the assumption (H4), we obtain

$$\|u_\epsilon - w_\epsilon\|_V \leq C^{-1}C'\|w_\epsilon - u\|_V + C^{-1}C''\|u\|_{U_\epsilon} \longrightarrow 0 \ \text{ as } \ \epsilon \longrightarrow 0. \quad (2.18)$$

Hence, the strong convergence of u_ϵ to u in V is established.

Finally, we consider the third limit (2.9). Let $\sigma > 0$ be fixed. Since u_ϵ^σ is the solution of (2.3), then for $v = u_\epsilon^\sigma$ in (2.3) and since $f \in V'$ we obtain that $\{u_\epsilon^\sigma : \epsilon > 0\}$ remains bounded in V. As a result, there exist a subsequence still denoted by $(u_\epsilon^\sigma)_\epsilon$ and $\widetilde{u}^* \in V$ such that

$$u_\epsilon^\sigma \longrightarrow \widetilde{u}^* \text{ weakly in } V \ \text{ as } \epsilon \longrightarrow 0. \quad (2.19)$$

Letting ϵ tend to zero in (2.3), and through the convergence (2.19) and the assumption (H4), we see that $\widetilde{u}^*$ is indeed the solution of problem (2.6). Hence, $\widetilde{u}^* = u^*$.

For the strong convergence, we introduce $u_\epsilon^\sigma - u^* \in V$ into (2.3), which leads to

$$C\|u_\epsilon^\sigma - u^*\|_V^2 \leq a_\epsilon^\sigma(u_\epsilon^\sigma - u^*, u_\epsilon^\sigma - u^*) \leq a_\epsilon^\sigma(u_\epsilon^\sigma, u_\epsilon^\sigma - u^*) - a_\epsilon^\sigma(u^*, u_\epsilon^\sigma - u^*)$$
$$\leq a_\epsilon^\sigma(u_\epsilon^\sigma, u_\epsilon^\sigma - u^*) - a_\epsilon^{(1)}(u^*, u_\epsilon^\sigma - u^*) - \sigma a_\epsilon^{(2)}(u^*, u_\epsilon^\sigma - u^*)$$
$$\leq a_\epsilon^\sigma(u_\epsilon^\sigma, u_\epsilon^\sigma - u^*) - a(u^*, u_\epsilon^\sigma - u^*) - r_\epsilon^{(1)}(u^*, u_\epsilon^\sigma - u^*) - \sigma a_\epsilon^{(2)}(u^*, u_\epsilon^\sigma - u^*)$$
$$\leq < f, u_\epsilon^\sigma - u^* > - < f, u_\epsilon^\sigma - u^* > - r_\epsilon^{(1)}(u^*, u_\epsilon^\sigma - u^*) - \sigma a_\epsilon^{(2)}(u^*, u_\epsilon^\sigma - u^*)$$
$$\leq C'(1+\sigma)\|u^*\|_{U_\epsilon}\|u_\epsilon^\sigma - u^*\|_V.$$

Therefore,

$$\|u_\epsilon^\sigma - u^*\|_V \leq C''(1+\sigma)\|u^*\|_{U_\epsilon} \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0. \tag{2.20}$$

This completes the proof of Theorem 2.7. □

Remark 2.8. As u^* is independent of σ, the two limits $\sigma \longrightarrow \infty$ and $\epsilon \longrightarrow 0$ commute if and only if $u^* = u$ or equivalently if and only if $W = V$.

Remark 2.9. By dividing the equation (2.3) by σ and setting $\alpha = \sigma^{-1}$, we get the equation

$$a_\epsilon^{(2)}(u_\epsilon^\sigma, v) + \alpha a_\epsilon^{(1)}(u_\epsilon^\sigma,\ v) = \alpha < f,\ v >, \quad \forall v \in V.$$

Then we are in a setting similar to section I.2 of [20], except that the right-hand side here depends on the small parameter α. But clearly the analysis from section I.2 of [20] will allow us to recover the limit result (2.7).

3. Applications

Let $D \subset \mathbb{R}^3$ be a bounded domain containing in its interior a portion of a regular surface S and a sequence of three-dimensional sub-domains Ω_ϵ of thickness ϵ, containing S, that tends to S in the following sense:

The measure of Ω_ϵ tends to zero when $\epsilon \longrightarrow 0$, and if $x \notin S$, then $x \notin \Omega_\epsilon$ for ϵ small enough.

By $\partial\Omega_\epsilon$, we denote the boundary of Ω_ϵ and by ∂D the boundary of D. We denote by D_ϵ the domain $D\backslash\overline{\Omega}_\epsilon$ (see Figure 1).

Remark 3.1. For the sake of simplicity we restrict ourselves to the case of domains D of $\mathbb{R}^3$ and a regular surface S included in D, but all the results presented in this section are still valid for domains D of $\mathbb{R}^n$ with $n \geq 2$ and a regular hypersurface S of codimension 1 included in D.

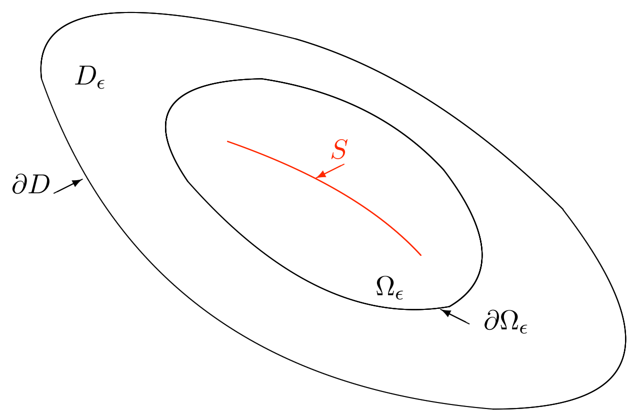

FIGURE 1

3.1. **The Lamé System.** In this subsection, we study some transmission problems for the linear isotropic elastic equations related to the previous decomposition of D. We consider an elastic medium with Lamé constants λ_0, μ_0 in D_ϵ and λ, μ in Ω_ϵ. Here, we suppose that ϵ and λ converge separately to zero and infinity respectively, and we study the limit behaviour of the boundary-value problem (described below). If D_ϵ is empty, it is well known that the limit $\lambda \longrightarrow \infty$ corresponds to the limit of incompressible elasticity and the obtained problem is the Stokes problem, see [7] and [26] for instance. In our case, D_ϵ is not empty and therefore the limit problem is the coupling between the Lamé system and the Stokes problem. This phenomenon even if expected seems to be new.

In the limit $\epsilon \longrightarrow 0$, the thin elastic subdomain becomes a surface included in a three-dimensional elastic body. The whole limit structure becomes a three-dimensional problem coupled with the bi-dimensional one. In the two-dimensional case, such a limit was considered in [3] where the aim was the asymptotic expansion of the boundary displacement field as the thickness goes to zero, while in [4] the elastic thin layer was inserted between two elastic materials.

Similar problems have been studied in [18] for the Laplacian and in [9], where the author was focused on the limit behavior when ϵ converges to zero and μ goes to infinity simultaneously.

3.1.1. *Setting of the problem.* We are concerned with the following transmission problem for the Lamé system with discontinuous coefficients:

$$\begin{cases} -\text{div }(\sigma(u)) &= & f & \text{in } D, \\ u|_{D_\epsilon} &= & u|_{\Omega_\epsilon} & \text{on } \partial\Omega_\epsilon, \\ \sigma(u|_{D_\epsilon})\cdot n &= & \sigma(u|_{\Omega_\epsilon})\cdot n & \text{on } \partial\Omega_\epsilon, \\ u &= & 0 & \text{on } \partial D. \end{cases} \tag{3.1}$$

$f \in L^2(D)^3$ is the given body force applied in D, n is the outward unit normal vector along $\partial\Omega_\epsilon$ and $\sigma(u)$ is the stress tensor given by

$$\sigma(u) = 2\widetilde{\mu}\varepsilon(u) + \widetilde{\lambda}(\text{div } u)\text{Id}, \tag{3.2}$$

where Id denotes the identity matrix and $\varepsilon(u)$ is the linearized strain tensor

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^t), \tag{3.3}$$

or equivalently

$$\varepsilon(u) = (\varepsilon_{ij}(u))_{ij} \text{ with } \varepsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i),\ 1 \le i,j \le 3. \tag{3.4}$$

The Lamé coefficients $\widetilde{\mu}$ and $\widetilde{\lambda}$ are given by

$$\widetilde{\mu} = \begin{cases} \mu_0 & \text{in } D_\epsilon \\ \mu & \text{in } \Omega_\epsilon \end{cases} \quad \text{and} \quad \widetilde{\lambda} = \begin{cases} \lambda_0 & \text{in } D_\epsilon \\ \lambda & \text{in } \Omega_\epsilon, \end{cases}$$

where $\mu_0, \mu, \lambda, \lambda_0$ are positive constants (independent of ϵ).

In this example, we want to let separately ϵ tend to zero and λ to infinity. When $\epsilon \to 0$ the thickness of the sub-domain Ω_ϵ goes to zero, and Ω_ϵ shrinks while its complement D_ϵ dilates and the corresponding material occupies the whole domain D.

The natural functional space of our problem (3.1) is given by

$$V := H_0^1(D)^3 = \{u \in H^1(D)^3 :\ u|_{\partial D} = 0\},$$

which is a Hilbert space with associated norm defined by

$$|u|_{1,D} := \|\nabla u\|_{L^2(D)^{3\times 3}}.$$

The corresponding weak formulation of (3.1) is by now standard: Find a solution $u_\epsilon^\lambda \in H_0^1(D)^3$ of

$$\int_D (2\widetilde{\mu}\varepsilon(u_\epsilon^\lambda) : \varepsilon(v) + \widetilde{\lambda} \text{ div } u_\epsilon^\lambda \text{ div } v)dx = \int_D fv\ dx, \forall v \in H_0^1(D)^3, \tag{3.5}$$

where $\tau : \theta$ defines the standard inner product between matrices $\tau : \theta = \sum_{i,j=1}^{3} \tau_{ij}\theta_{ij}$. For shortness we set

$$a_\epsilon^\lambda(u,\ v) = \int_D (2\widetilde{\mu}\varepsilon(u) : \varepsilon(v) + \widetilde{\lambda} \text{ div } u \text{ div } v)dx. \tag{3.6}$$

The splitting of the domain D and the discontinuity of the Lamé coefficients suggest that we split up the bilinear form a_ϵ^λ into the sum of two bilinear forms:

$$a_\epsilon^\lambda(u,\ v) = a_\epsilon^{(1)}(u,\ v) + \lambda a_\epsilon^{(2)}(u,\ v), \tag{3.7}$$

where

$$a_\epsilon^{(1)}(u,v) = 2\mu_0 \int_{D_\epsilon} \varepsilon(u) : \varepsilon(v)dx + 2\mu \int_{\Omega_\epsilon} \varepsilon(u) : \varepsilon(v)dx + \lambda_0 \int_{D_\epsilon} \text{div}\, u \,\text{div}\, v dx, \tag{3.8}$$

$$a_\epsilon^{(2)}(u,v) = \int_{\Omega_\epsilon} \text{div } u \text{ div } v \ dx. \tag{3.9}$$

For all $\epsilon > 0$, we denote by $V(\Omega_\epsilon)$ the closed subspace of $H_0^1(D)^3$ defined by

$$V(\Omega_\epsilon) = \{v \in H_0^1(D)^3 :\ \text{div } v = 0 \text{ in } \Omega_\epsilon\}.$$

$W(S) = \overline{\bigcup_{\epsilon>0} V(\Omega_\epsilon)}^{H_0^1(D)^3}$ and is actually the closure in $H_0^1(D)^3$ of the set of $v \in H_0^1(D)^3$ which has a divergence that vanishes in a neighborhood of the crack S.

Let us check that the bilinear forms defined by (3.6), (3.8) and (3.9) satisfy all hypotheses of the previous section.

We clearly see that the two bilinear forms $a_\epsilon^{(1)}$ and $a_\epsilon^{(2)}$ are nonnegative, symmetric, and continuous in $(H_0^1(D)^3)^2$. Consequently, their sum a_ϵ^λ is continuous too in the same space.

In order to establish the V-ellipticity of the bilinear forms, we recall the following theorem which is a consequence of *Korn*'s inequality (see [23], [27]).

Theorem 3.2 ([23], pages 50-51)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected domain with a piecewise boundary of class $\mathcal{C}^1$. Then there exists a constant $K_0 > 0$ such that*

$$\sum_{i,\ j=1}^{n} \|\varepsilon_{ij}(v)\|_{0,\Omega}^2 \geq K_0 \|v\|_{1,\Omega}^2,\ \forall v \in H_0^1(\Omega)^n. \tag{3.10}$$

This means that the mapping

$$v \longmapsto \Big(\sum_{i,\ j=1}^{n} \|\varepsilon_{ij}(v)\|_{0,\Omega}^2 \Big)^{1/2} := \|\varepsilon(v)\|_{0,\Omega},$$

defines a norm on $H_0^1(\Omega)^n$ which is equivalent to the standard norm $v \longmapsto \|v\|_{1,\Omega}$. The inequality (3.10) allows us to deduce the uniform coerciveness in ϵ of the sum $a_\epsilon^{(1)} + a_\epsilon^{(2)}$:

$$\begin{aligned}(a_\epsilon^{(1)} + a_\epsilon^{(2)})(u,u) &= \int_D 2\widetilde{\mu}|\varepsilon(u)|^2 dx + \lambda_0 \int_{D_\epsilon} |\text{div } u|^2 dx + \int_{\Omega_\epsilon} |\text{div } u|^2 dx \\ &\geq C \int_D |\varepsilon(u)|^2 dx, \quad (C = 2\min(\mu_0,\ \mu)) \geq C\|\varepsilon(u)\|_{0,D}^2 \geq CK_0\|u\|_{1,D}^2.\end{aligned}$$

This leads, for $\lambda \geq 1$, to the uniform coerciveness in ϵ of the bilinear form a_ϵ^λ, and show that (H1), (H2), and (H3) are satisfied. To verify the assumption (H4), we rewrite $a_\epsilon^{(1)}(u,v) = a(u,v) + r_\epsilon^{(1)}(u,v)$, with

$$a(u,\ v) = 2\mu_0 \int_D \varepsilon(u) : \varepsilon(v)\ dx + \lambda_0 \int_D \text{div } u \text{ div } v\ dx,$$

a symmetric continuous and coercive bilinear form in $H_0^1(D)^3$, and $r_\epsilon^{(1)}$ is given by

$$u, v \in H_0^1(D)^3 \longmapsto r_\epsilon^{(1)}(u,v) = 2(\mu - \mu_0)\int_{\Omega_\epsilon} \varepsilon(u) : \varepsilon(v) dx - \lambda_0 \int_{\Omega_\epsilon} \text{div}\, u \text{ div } v\ dx.$$

It then remains to check that $r_\epsilon^{(1)} \to 0$ strongly: For any $u, v \in H_0^1(D)^3$ thanks to Cauchy-Schwarz's inequality, we have

$$\begin{aligned}|r_\epsilon^{(1)}(u,\ v)| &\leq C_1\|\varepsilon(u)\|_{L^2(\Omega_\epsilon)}\|\varepsilon(v)\|_{L^2(\Omega_\epsilon)} + C_2\|\text{div } u\|_{L^2(\Omega_\epsilon)}\|\text{div } v\|_{L^2(\Omega_\epsilon)} \\ &\leq C_2\|u\|_{1,\Omega_\epsilon}\|v\|_{1,D}.\end{aligned}$$

This shows the first part of the requested property with $U_\epsilon = H^1(\Omega_\epsilon)^3$, since $\|u\|_{1,\Omega_\epsilon} \longrightarrow 0$ when $\epsilon \longrightarrow 0$. As $r_\epsilon^{(1)}$ is a symmetric form, we deduce the second part of the property thanks to Remark 2.2.

Similarly, we check that $a_\epsilon^{(2)} \longrightarrow 0$ strongly. This means that the assumption (H4) is verified.

Now, we can state the three limit formulations (3.11), (3.12), (3.13) in the following way:

Find $u_\epsilon \in V(\Omega_\epsilon)$ such that

$$\int_{D_\epsilon} (2\mu_0 \varepsilon(u_\epsilon) : \varepsilon(v) + \lambda_0 \text{div } u_\epsilon \text{ div } v)\ dx + 2\mu \int_{\Omega_\epsilon} \varepsilon(u_\epsilon) : \varepsilon(v)\ dx$$
$$= \int_D fv\ dx,\ \forall v \in V(\Omega_\epsilon). \tag{3.11}$$

Find $u \in W(S)$ such that

$$2\mu_0 \int_D \varepsilon(u) : \varepsilon(v)\ dx + \lambda_0 \int_D \text{div } u \text{ div } v\ dx = \int_D fv\ dx\ , \quad \forall v \in W(S). \tag{3.12}$$

Find $u^* \in H_0^1(D)^3$ such that

$$2\mu_0 \int_D \varepsilon(u^*) : \varepsilon(v)\ dx + \lambda_0 \int_D \text{div } u^* \text{ div } v\ dx = \int_D fv\ dx\ , \quad \forall v \in H_0^1(D)^3. \tag{3.13}$$

Thanks to Proposition 2.6 and Theorem 2.7, we have the following results.

Proposition 3.3. *Problems* (3.5), (3.11), (3.12), *and* (3.13) *admit unique solutions in* $H_0^1(D)^3$, $V(\Omega_\epsilon)$, $W(S)$, *and* $H_0^1(D)^3$ *respectively.*

Theorem 3.4. *The solutions* u_ϵ^λ, u_ϵ, u, *and* u^* *of the respective problems* (3.5), (3.11), (3.12), *and* (3.13) *satisfy the following limit properties:*

$$\lim_{\lambda \longrightarrow \infty} u_\epsilon^\lambda = u_\epsilon \quad \text{in } H_0^1(D)^3,$$
$$\lim_{\epsilon \longrightarrow 0} u_\epsilon = u \quad \text{in } H_0^1(D)^3,$$
$$\lim_{\epsilon \longrightarrow 0} u_\epsilon^\lambda = u^* \quad \text{in } H_0^1(D)^3.$$

3.1.2. *Strong interpretation of the limit problems.* In this section, we try to derive the corresponding strong formulations of the three weak limit formulations obtained in the above subsection.

We will show that the limit problem (3.11) is a coupling between the Lamé problem set in D_ϵ with Lamé coefficients λ_0 and μ_0, and the Stokes problem set in Ω_ϵ. For that purpose we first consider the following saddle point problem: Find $(\widehat{u}_\epsilon,\ \widehat{p}_\epsilon) \in H_0^1(D)^3 \times L^2(\Omega_\epsilon)^3$ such that

$$\left\{ \begin{array}{rcll} a_\epsilon^{(1)}(\widehat{u}_\epsilon,\ v) - b(\widehat{p}_\epsilon,\ v) & = & < f,\ v >, & \forall v \in H_0^1(D)^3, \\ b(q,\ \widehat{u}_\epsilon) & = & 0, & \forall q \in L^2(\Omega_\epsilon), \end{array} \right. \tag{3.14}$$

where the form $b(q,\ v) = \int_{\Omega_\epsilon} q \text{ div } v\ dx$. The problem (3.14) is well posed since the bilinear form $(u,\ v) \longmapsto a_\epsilon^{(1)}(u,\ v)$ is coercive on $V(\Omega_\epsilon)$ and the form $(q,\ v) \in L^2(\Omega_\epsilon) \times H_0^1(D)^3 \longmapsto b(q,\ v)$ satisfies the inf-sup condition. Indeed, for any $q \in L^2(\Omega_\epsilon)$, we set

$$\widetilde{q} = \begin{cases} q & \text{in } \Omega_\epsilon \\ \alpha & \text{in } D_\epsilon \end{cases} \quad \text{with } \alpha = \frac{-1}{|D_\epsilon|} \int_{\Omega_\epsilon} q\ dx.$$

By Cauchy-Schwarz's inequality, we get

$$\begin{aligned} \|\widetilde{q}\|^2_{L^2(D)} &= \int_{\Omega_\epsilon} |q|^2\ dx + \alpha^2 |D_\epsilon| \le \int_{\Omega_\epsilon} |q|^2\ dx + \frac{|\Omega_\epsilon|}{|D_\epsilon|} \int_{\Omega_\epsilon} |q|^2\ dx \\ &\le C_\epsilon^2 \int_{\Omega_\epsilon} |q|^2\ dx \text{ with } C_\epsilon^2 = \max\{1,\ \frac{|\Omega_\epsilon|}{|D_\epsilon|}\} \\ &\le C_\epsilon^2 \|q\|^2_{L^2(\Omega_\epsilon)}. \end{aligned}$$

Since $\widetilde{q}$ has a zero mean value in D, by Corollary I.2.4 of [16], there exist a positive constant C and a function $v \in H_0^1(D)^3$ such that

$$\text{div } v = \widetilde{q} \text{ and } \|v\|_{H_0^1(D)^3} \le C \|\widetilde{q}\|_{L_0^2(D)}. \tag{3.15}$$

Therefore,

$$\begin{aligned} b(q,\ v) &= \int_{\Omega_\epsilon} q \text{ div } v\ dx = \int_{\Omega_\epsilon} q^2 dx = \|q\|_{L^2(\Omega_\epsilon)} \|q\|_{L^2(\Omega_\epsilon)} \\ &\ge C_\epsilon \|\widetilde{q}\|_{L_0^2(D)} \|q\|_{L^2(\Omega_\epsilon)} \ge C_\epsilon C^{-1} \|v\|_{H_0^1(D)^3} \|q\|_{L^2(\Omega_\epsilon)}. \end{aligned}$$

This shows that b satisfies the inf-sup condition and the well posedness of problem (3.14).

To show that (3.5) converges to (3.14), we must prove, by reference to [26], that

(1) the solution u_ϵ^λ of (3.11) converges in norm to the solution $\widehat{u}_\epsilon$ in $H_0^1(D)^3$ of (3.14);
(2) λdiv u_ϵ^λ converges in norm to $\widehat{p}_\epsilon$ in $L^2(\Omega_\epsilon)$.

We start with the first convergence. The subtraction of the first identity of (3.14) from (3.5) gives

$$a_\epsilon^{(1)}(u_\epsilon^\lambda - \widehat{u}_\epsilon,\ v) + (\lambda \text{div } u_\epsilon^\lambda,\ \text{div } v)_{\Omega_\epsilon} = -b(\widehat{p}_\epsilon,\ v), \quad \forall v \in H_0^1(D)^3. \tag{3.16}$$

By choosing $v = u_\epsilon^\lambda - \widehat{u}_\epsilon$, and since the fluid is incompressible in Ω_ϵ, i.e., div $\widehat{u}_\epsilon = 0$ in Ω_ϵ, (3.16) becomes

$$a_\epsilon^{(1)}(u_\epsilon^\lambda - \widehat{u}_\epsilon,\ u_\epsilon^\lambda - \widehat{u}_\epsilon) + (\lambda \text{div } u_\epsilon^\lambda,\ \text{div } u_\epsilon^\lambda)_{\Omega_\epsilon} = -(\widehat{p}_\epsilon,\ \text{div } u_\epsilon^\lambda)_{\Omega_\epsilon}. \tag{3.17}$$

The coerciveness of $a_\epsilon^{(1)}$ on $V(\Omega_\epsilon)$ leads to

$$\alpha\|u_\epsilon^\lambda - \widehat{u}_\epsilon\|^2_{H_0^1(D)^3} \leq \frac{1}{2\lambda}\|\widehat{p}_\epsilon\|^2_{L^2(\Omega_\epsilon)} \longrightarrow 0 \quad \text{when} \quad \lambda \longrightarrow +\infty. \tag{3.18}$$

Here, α represents the coercivity constant of $a_\epsilon^{(1)}$ on $V(\Omega_\epsilon)$ (which is indeed independent of ϵ). Hence

$$u_\epsilon^\lambda \longrightarrow \widehat{u}_\epsilon \text{ strongly in } H_0^1(D)^3 \quad \text{when} \quad \lambda \longrightarrow +\infty. \tag{3.19}$$

To establish the second convergence, we need the lemma below. For its proof, see Lemma 6.1 in [26].

Lemma 3.5. *Let Ω a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists a constant $C = C(\Omega)$ depending only on Ω, such that*

$$\|\theta\|_{L^2(\Omega)} \leq C(\Omega)\left\{ \left|\int_\Omega \theta \mathrm{dx}\right| + \sum_{i=1}^n \left\|\frac{\partial\theta}{\partial x_i}\right\|_{H^{-1}(\Omega)} \right\}, \tag{3.20}$$

for every θ in $L^2(\Omega)$.

From the identity (3.16), we have

$$a_\epsilon^{(1)}(u_\epsilon^\lambda - \widehat{u}_\epsilon,\ v) = -(\lambda \mathrm{div}\ u_\epsilon^\lambda + \widehat{p}_\epsilon,\ \mathrm{div}\ v)_{\Omega_\epsilon}. \tag{3.21}$$

For $v \in H_0^1(\Omega_\epsilon)^3$ its extension by zero outside Ω_ϵ $\widetilde{v}$ belongs to $H_0^1(D)^3$; then taking $\widetilde{v}$ as a test function in (3.21), we get

$$\int_{\Omega_\epsilon} 2\mu\varepsilon(u_\epsilon^\lambda - \widehat{u}_\epsilon) : \varepsilon(v)\ dx = -\int_{\Omega_\epsilon} (\lambda \mathrm{div}\ u_\epsilon^\lambda + \widehat{p}_\epsilon)\mathrm{div}\ v\ dx. \tag{3.22}$$

By the convergence (3.19), the left-hand side of the identity (3.22) converges to zero. Therefore, the same is true for its right-hand side. Now the divergence theorem leads to the following identity:

$$< \nabla(\lambda \mathrm{div}\ u_\epsilon^\lambda + \widehat{p}_\epsilon), v >_{H^{-1}(\Omega_\epsilon)-H_0^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} 2\mu\varepsilon(u_\epsilon^\lambda - \widehat{u}_\epsilon) : \varepsilon(v)\, dx, \tag{3.23}$$

for all $v \in H_0^1(\Omega_\epsilon)$. Consequently,

$$\|\nabla(\lambda \mathrm{div}\, u_\epsilon^\lambda + \widehat{p}_\epsilon)\|_{H^{-1}(\Omega_\epsilon)} = \sup_{v\neq 0} \frac{< \nabla(\lambda \mathrm{div}\ u_\epsilon^\lambda + \widehat{p}_\epsilon),\ v >_{H^{-1}(\Omega_\epsilon)-H_0^1(\Omega_\epsilon)}}{\|v\|_{H_0^1(\Omega_\epsilon)}}$$

$$= \sup_{v\neq 0} \frac{\left|\int_{\Omega_\epsilon} 2\mu\varepsilon(u_\epsilon^\lambda - \widehat{u}_\epsilon) : \varepsilon(v)\ dx\right|}{\|v\|_{H_0^1(\Omega_\epsilon)}} \leq C\|u_\epsilon^\lambda - \widehat{u}_\epsilon\|_{H_0^1(\Omega_\epsilon)}$$

$$\leq C\|u_\epsilon^\lambda - \widehat{u}_\epsilon\|_{H_0^1(D)} \longrightarrow 0 \;\text{ as }\; \lambda \longrightarrow \infty,$$

and we can conclude that

$$-\nabla(\lambda \text{div } u_\epsilon^\lambda) \longrightarrow \nabla \widehat{p}_\epsilon \;\text{ strongly in } H^{-1}(\Omega_\epsilon)^3 \;\text{ when }\; \lambda \longrightarrow \infty, \tag{3.24}$$

or equivalently

$$-\frac{\partial}{\partial x_i}(\lambda \text{div } u_\epsilon^\lambda) \longrightarrow \frac{\partial}{\partial x_i}\widehat{p}_\epsilon \;\; \forall i = 1,2,3 \text{ strongly in } H^{-1}(\Omega_\epsilon) \text{ when }\; \lambda \longrightarrow \infty. \tag{3.25}$$

On the other hand, for a fixed $v \in H_0^1(D)$ such that div $v = 1$ in Ω_ϵ, using (3.16) and (3.19), we have

$$\int_{\Omega_\epsilon} (\lambda \text{div } u_\epsilon^\lambda + \widehat{p}_\epsilon)\; dx \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty. \tag{3.26}$$

Lemma 3.5 allows us to write

$$\|\lambda \text{div } u_\epsilon^\lambda + \widehat{p}_\epsilon\|_{L^2(\Omega_\epsilon)} \tag{3.27}$$

$$\leq C(\Omega_\epsilon)\Big\{\Big|\int_{\Omega_\epsilon}(\lambda \text{div } u_\epsilon^\lambda + \widehat{p}_\epsilon)\text{dx}\Big| + \sum_{i=1}^{3}\Big\|\frac{\partial}{\partial x_i}(\lambda \text{div } u_\epsilon^\lambda + \widehat{p}_\epsilon)\Big\|_{H^{-1}(\Omega_\epsilon)}\Big\},$$

and by (3.25) and (3.26), we obtain that

$$\|\lambda \text{div } u_\epsilon^\lambda + \widehat{p}_\epsilon\|_{L^2(\Omega_\epsilon)} \longrightarrow 0. \tag{3.28}$$

This proves the second convergence.

The second identity of (3.14) means that $\widehat{u}_\epsilon$ belongs to $V(\Omega_\epsilon)$. Restricting the first identity of (3.14) to $v \in V(\Omega_\epsilon)$, we deduce that $\widehat{u}_\epsilon$ is a solution of (3.11), hence $\widehat{u}_\epsilon = u_\epsilon$. Now, using standard arguments (using test functions v in the first identity of (3.14) respectively in $\mathcal{D}(D_\epsilon)$, $\mathcal{D}(\Omega_\epsilon)$ and then $\mathcal{D}(D)$), we deduce that the strong formulation of the variational problem (3.11) is

$$\left\{\begin{array}{rcll} -\text{div}(\sigma(u_\epsilon)) &=& f & \text{in } D_\epsilon \\ -2\mu\Delta u_\epsilon + \nabla\widehat{p}_\epsilon &=& f & \text{in } \Omega_\epsilon \\ \text{div } u_\epsilon &=& 0 & \text{in } \Omega_\epsilon \\ u_\epsilon|_{D_\epsilon} &=& u_\epsilon|_{\Omega_\epsilon} & \text{on } \partial\Omega_\epsilon \\ \sigma(u_{\epsilon|D_\epsilon})\, n &=& 2\mu\varepsilon(u_{\epsilon|\Omega_\epsilon})\, n - \widehat{p}_\epsilon\, n & \text{on } \partial\Omega_\epsilon \\ u_\epsilon &=& 0 & \text{on } \partial D. \end{array}\right. \tag{3.29}$$

Here, $\sigma(u) = 2\mu_0\varepsilon(u) + \lambda_0 \text{div } u$ Id.

As said before, the limit problem (3.29) of (3.14) as $\lambda \longrightarrow 0$ is a coupling between the Lamé system in D_ϵ and the Stokes system in Ω_ϵ.

We cannot give a full interpretation of problem (3.12) since we cannot characterize the Hilbert space $W(S)$. As we shall see in Section 4, this characterization is strongly linked to the geometry of the crack. However, for the moment, we can only say that the solution u satisfies

$$\left\{ \begin{array}{rcll} -\text{div}(2\mu_0\varepsilon(u) + \lambda_0 \text{div } u \text{ Id}) & = & f & \text{in } D\backslash S, \\ u & = & 0 & \text{on } \partial D. \end{array} \right. \tag{3.30}$$

The limit problem (3.13) corresponds to the standard Lamé problem with the Lamé coefficients λ_0 and μ_0 set on the whole domain D with Dirichlet boundary condition

$$\left\{ \begin{array}{rcll} -\text{div}(2\mu_0\varepsilon(u^*) + \lambda_0 \text{div } u^* \text{ Id}) & = & f & \text{in } D, \\ u^* & = & 0 & \text{on } \partial D. \end{array} \right. \tag{3.31}$$

Remark 3.6. Obviously the limit procedures commute if and only if

$$W(S) = H_0^1(D)^3.$$

3.2. **General anisotropic elasticity.** In this section, we extend the previous analysis to the anisotropic elastic equations and suppose that one component of the tensor of rigidity C^1 defined in the subdomain Ω_ϵ is going to infinity. In [15], the problem of two elastic bodies joined by a soft thin adhesive of thickness ϵ along their common surface was treated, but only the limit $\epsilon \longrightarrow 0$ has been studied by using the asymptotic expansion method.

The general anisotropic elasticity problem can be written as: Find a function $u : D \longmapsto \mathbb{R}^3$ satisfying

$$\left\{ \begin{array}{rcll} -\text{div}(\sigma(u)) & = & f & \text{in } D_\epsilon \cup \Omega_\epsilon \\ u|_{D_\epsilon} & = & u|_{\Omega_\epsilon} & \text{on } \partial\Omega_\epsilon \\ \sigma(u|_{D_\epsilon}) \cdot n & = & \sigma(u|_{\Omega_\epsilon}) \cdot n & \text{on } \partial\Omega_\epsilon \\ u & = & 0 & \text{on } \partial D, \end{array} \right. \tag{3.32}$$

where u, f, and $\varepsilon(u)$ are exactly as in the case of Lamé's system, while the stress field $\sigma(u)$ is given in its general form

$$\sigma(u) = C\varepsilon(u), \tag{3.33}$$

or componentwise

$$\sigma_{ij}(u) = C_{ijkl}\varepsilon_{kl}(u), \tag{3.34}$$

where the tensor of rigidity C is supposed to be piecewise constant; namely it is given by

$$C = (c_{ijkl})_{1\leq i,j,k,l\leq 3} = \left\{ \begin{array}{ll} c^0_{ijkl} & \text{in } D_\epsilon, \\ c^1_{ijkl} & \text{in } \Omega_\epsilon; \end{array} \right.$$

and to satisfy

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \tag{3.35}$$

$$\exists M_0 > 0,\ \forall \xi \in \mathbb{R}^{3\times 3}\ :\ \sum_{i,j,k,l=1}^{3} c_{ijkl}\xi_{kl}\xi_{ij} \geq M_0 \sum_{i,j=1}^{3} |\xi_{ij}|^2. \tag{3.36}$$

As an illustrative example, we assume that the tensor of rigidity C^1 defined in Ω_ϵ is split up as follows: $C^1 = \underline{C}^1 + \sigma B$, with a tensor B that still satisfies (3.35) and σ is a positive real parameter destined to tend to infinity. For instance we may take

$B = (b_{ijkl})$ with $b_{1111} = 1$ and $b_{ijkl} = 0$ for the other indices or
$B = (\delta_{ij}\delta_{kl})$,

corresponding respectively to the constraint (see below) $\partial_1 u_1 = 0$ or div $u = 0$ in Ω_ϵ.

3.2.1. *Variational formulation.* The natural functional space of problem (3.32) is the standard Hilbert space $H_0^1(D)^3$ equipped with its natural norm. The variational formulation of our problem consists in looking for a solution $u_\epsilon^\sigma \in H_0^1(D)^3$ of

$$\forall v \in H_0^1(D)^3,\ \ a_\epsilon^\sigma(u,\ v) = \int_D fv\ dx, \tag{3.37}$$

where we have set

$$a_\epsilon^\sigma(u,v) = \int_{D_\epsilon} \sum_{i,j,k,l=1}^{3} c^0_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)\,dx + \int_{\Omega_\epsilon} \sum_{i,j,k,l=1}^{3} c^1_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)\,dx. \tag{3.38}$$

This form is bilinear and continuous in $H_0^1(D)^3$. Since the tensor C satisfies (3.35), the form a_ϵ^σ is symmetric too. The coerciveness a_ϵ^σ results from the property (3.36). So, problem (3.37) admits a unique solution in $H_0^1(D)^3$.

For the same reasons as in the above example, the bilinear form a_ϵ^σ is split up into the sum of two bilinear forms $a_\epsilon^{(1)}$ and $a_\epsilon^{(2)}$ which depend only on ϵ:

$$a_\epsilon^\sigma(u,\ v) = a_\epsilon^{(1)}(u,\ v) + \sigma a_\epsilon^{(2)}(u,\ v), \tag{3.39}$$

where

$$a_\epsilon^{(1)}(u,\ v) = \int_{D_\epsilon} C^0\varepsilon(u) : \varepsilon(v)\ dx + \int_{\Omega_\epsilon} \underline{C}^1\varepsilon(u) : \varepsilon(v)\ dx, \tag{3.40}$$

$$a_\epsilon^{(2)}(u,\ v) = \int_{\Omega_\epsilon} B\varepsilon(u) : \varepsilon(v)\ dx. \tag{3.41}$$

The two bilinear forms $a_\epsilon^{(1)}$, $a_\epsilon^{(2)}$ are positive, symmetric and continuous. Their sum is coercive on $H_0^1(D)^3$, as a consequence of (3.36). When the thickness ϵ of Ω_ϵ tends to zero, we easily show that $a_\epsilon^{(2)} \longrightarrow 0$ strongly and $a_\epsilon^{(1)} \longrightarrow a$ strongly, where the bilinear form a is defined by

$$a(u,\ v) = \int_D C^0\varepsilon(u) : \varepsilon(v)\ dx, \forall u,\ v \in H_0^1(D)^3.$$

Finally, the bilinear form a is positive, symmetric, continuous and coercive due to (3.36) and Korn's inequality (3.10).

Now we can give the three limit formulations which arise through the convergence of ϵ to zero and/or σ to infinity, separately:
Find $u_\epsilon \in V(\Omega_\epsilon)$ such that

$$\int_{D_\epsilon} C^0\varepsilon(u_\epsilon) : \varepsilon(v)dx + \int_{\Omega_\epsilon} \underline{C}^1\varepsilon(u_\epsilon) : \varepsilon(v)\,dx = \int_D fv\,dx, \quad \forall v \in V(\Omega_\epsilon), \tag{3.42}$$

where $V(\Omega_\epsilon) = \{v = (v_1,\ v_2,\ v_3) \in H_0^1(D)^3 :\ B\varepsilon(v) = 0 \text{ in } \Omega_\epsilon\}$ is a closed subspace of $H_0^1(D)^3$.
Find $u \in W(S)$ such that

$$\int_D C^0\varepsilon(u) : \varepsilon(v)\ dx = \int_D fv\ dx, \quad \forall v \in W(S), \tag{3.43}$$

where $W(S)$ is the closure in $H_0^1(D)^3$ of the space $\{v \in H_0^1(D)^3 :\ B\varepsilon(v) = 0 \text{ in a neighborhood of } S\}$.
Find $u^* \in H_0^1(D)^3$ such that

$$\int_D C^0\varepsilon(u^*) : \varepsilon(v)\ dx = \int_D fv\ dx, \quad \forall v \in H_0^1(D)^3. \tag{3.44}$$

Since we have checked that the different bilinear forms satisfy all the assumptions (H1), (H2), (H3), and (H4), Proposition 2.6 and Theorem 2.7 are applicable. Hence, we get the following results.

Proposition 3.7. *The problems* (3.37), (3.42), (3.43), *and* (3.44) *admit unique solutions in $H_0^1(D)^3$, $V(\Omega_\epsilon)$, $W(S)$, and $H_0^1(D)^3$ respectively.*

Theorem 3.8. *Let u_ϵ^σ, u_ϵ, u, and u^* be the unique solutions of the respective problems (*3.37), (3.42), (3.43), *and* (3.44). *Then, these solutions satisfy the following limit properties:*

$$\lim_{\sigma\longrightarrow\infty} u_\epsilon^\sigma = u_\epsilon \quad \text{in } H_0^1(D)^3,$$

$$\lim_{\epsilon \longrightarrow 0} u_\epsilon = u \quad \text{in } H_0^1(D)^3,$$
$$\lim_{\epsilon \longrightarrow 0} u_\epsilon^\sigma = u^* \quad \text{in } H_0^1(D)^3.$$

3.2.2. *Strong interpretation of the limit problems.* The problem (3.42) is a nonstandard transmission problem between D_ϵ and Ω_ϵ. Indeed, since $V(\Omega_\epsilon)$ is defined by the constraint $B\varepsilon(v) = 0$ in Ω_ϵ, we can introduce the Lagrange multiplier $p_\epsilon \in P_\epsilon \subset L^2(\Omega_\epsilon)^{3\times 3}$, where P_ϵ is defined as the range of the mapping

$$\mathcal{B}_\epsilon : H_0^1(D)^3 \longrightarrow L^2(\Omega_\epsilon)^{3\times 3} : v \longrightarrow B\varepsilon(v)_{|\Omega_\epsilon}.$$

Since $\mathcal{B}_\epsilon$ is linear, continuous, and surjective on P_ϵ, it becomes bijective by restricting ourselves to the quotient space with the kernel K of $\mathcal{B}_\epsilon$, which is nothing else than $V(\Omega_\epsilon) = K = \ker \mathcal{B}_\epsilon$. Hence, by the closed graph theorem the inverse of

$$\mathcal{B}_\epsilon : H_0^1(D)^3/K \longrightarrow P_\epsilon : v \longrightarrow B\varepsilon(v)_{|\Omega_\epsilon}$$

is also continuous. In other words, there exists a positive constant $C(\epsilon)$ (depending on ϵ) such that for any $p \in P_\epsilon$, there exists a unique $\dot{u} = (u + k)_{k\in K} \in H_0^1(D)^3/K$ such that

$$B\varepsilon(u)_{|\Omega_\epsilon} = p \text{ and } \|\dot{u}\|_{H_0^1(D)^3/K} \leq C(\epsilon)\|p\|_{L^2(\Omega_\epsilon)^{3\times 3}}.$$

Since K is a closed subspace of $H_0^1(D)^3$, we have

$$\|\dot{u}\|_{H_0^1(D)^3/K} = \inf_{k\in K} \|u + k\|_{H_0^1(D)^3} = \|u - \Pi_K u\|_{H_0^1(D)^3},$$

where $\Pi_K u$ is the orthogonal projection of u on K. Consequently the function $v(p) = u - \Pi_K u$ satisfies

$$B\varepsilon(v(p))_{|\Omega_\epsilon} = p \text{ and } \|v(p)\|_{H_0^1(D)^3} \leq C(\epsilon)\|p\|_{L^2(\Omega_\epsilon)^{3\times 3}}. \tag{3.45}$$

Now, we introduce the bilinear form on $P_\epsilon \times H_0^1(D)^3$ by

$$b(q,\ v) = \int_{\Omega_\epsilon} q : B\varepsilon(v)\ dx,$$

and check that b satisfies the inf-sup condition. Indeed, for any $p \in P_\epsilon$, let $v(p)$ be the element in $H_0^1(D)^3$ satisfying (3.45). With this choice we may write

$$\frac{b(p,\ v(p))}{\|v(p)\|_{H_0^1(D)^3}} = \frac{\|p\|^2_{L^2(\Omega_\epsilon)^{3\times 3}}}{\|v(p)\|_{H_0^1(D)^3}} \geq \frac{1}{C(\epsilon)}\|p\|_{L^2(\Omega_\epsilon)^{3\times 3}}.$$

Then, we can consider the saddle point problem: Find $(\hat{u}_\epsilon,\ \hat{p}_\epsilon) \in H_0^1(D)^3 \times P_\epsilon$ such that

$$\left\{ \begin{array}{rcll} a_\epsilon^{(1)}(\widehat{u}_\epsilon,\ v) - b(\widehat{p}_\epsilon,\ v) & = & < f,\ v >, & \forall v \in H_0^1(D)^3, \\ b(q,\ \widehat{u}_\epsilon) & = & 0, & \forall q \in P_\epsilon. \end{array} \right. \tag{3.46}$$

This problem admits a unique solution in $H_0^1(D)^3 \times P_\epsilon$ because $a_\epsilon^{(1)}$ is coercive on $V(\Omega_\epsilon)$ and since b satisfies the inf-sup condition.

By restricting the first identity of (3.46) to test functions in $V(\Omega_\epsilon)$, we see that $\hat{u}_\epsilon \in V(\Omega_\epsilon)$ is a solution of problem (3.42) and therefore $\hat{u}_\epsilon = u_\epsilon$. Now, the procedure described in Subsection 3.1.2 allows us to give the strong formulation to problem (3.42):

$$\left\{ \begin{array}{rcll} -\mathrm{div}(C^0\varepsilon(u_\epsilon)) & = & f & \text{in } D_\epsilon \\ -\mathrm{div}(\underline{C}^1\varepsilon(u_\epsilon)) + \mathrm{div}(B\widehat{p}_\epsilon) & = & f & \text{in } \Omega_\epsilon \\ B\varepsilon(u_\epsilon) & = & 0 & \text{in } \Omega_\epsilon \\ u_\epsilon|_{D_\epsilon} & = & u_\epsilon|_{\Omega_\epsilon} & \text{on } \partial\Omega_\epsilon \\ C^0\sigma(u_\epsilon|_{D_\epsilon})\, n & = & \underline{C}^1\varepsilon(u_\epsilon|_{\Omega_\epsilon})\, n - B\widehat{p}_\epsilon\, n & \text{on } \partial\Omega_\epsilon \\ u_\epsilon & = & 0 & \text{on } \partial D. \end{array} \right.$$

This system is a coupling between the Lamé system in D_ϵ with a Stokes like system in Ω_ϵ. Again, to our knowledge, this phenomenon is new. Note further that the main difference with the previous example is that the space P_ϵ is not known explicitly.

The problem (3.43) corresponds to a problem defined in $D\backslash S$ and for the same reason reported in the previous example, we can not give its full interpretation. But, we can say that at least its solution u satisfies

$$\left\{ \begin{array}{rcll} -\mathrm{div}(C^0\varepsilon(u)) & = & f & \text{in } D\backslash S, \\ u & = & 0 & \text{on } \partial D. \end{array} \right. \tag{3.47}$$

The strong problem corresponding to the weak problem (3.44) is the ordinary Dirichlet problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}(C^0\varepsilon(u^*)) & = & f & \text{in } D, \\ u^* & = & 0 & \text{on } \partial D. \end{array} \right. \tag{3.48}$$

3.3. **Anisotropic diffusion problems.** In this section, we consider the following anisotropic diffusion problem: given $f \in L^2(D)$, we look for a solution $u : D \longmapsto \mathbb{R}$ of

$$\left\{\begin{array}{rcll} -\mathrm{div}(C\nabla u) & = & f & \text{in } D \\ u|_{D_\epsilon} & = & u|_{\Omega_\epsilon} & \text{on } \partial\Omega_\epsilon \\ (C^0\nabla(u|_{D_\epsilon}))\cdot n & = & (C^1\nabla(u|_{\Omega_\epsilon}))\cdot n & \text{on } \partial\Omega_\epsilon \\ u & = & 0 & \text{on } \partial D, \end{array}\right. \tag{3.49}$$

where C is a symmetric, 3×3 matrix such that

$$C = \left\{\begin{array}{ll} C^0 = (c^0_{ij})_{ij} & \text{in } \ D_\epsilon, \\ C^1 = (c^1_{ij})_{ij} & \text{in } \ \Omega_\epsilon, \end{array}\right.$$

and is supposed to satisfy

$$\exists \beta > 0, \ \forall \xi \in \mathbb{R}^3 : \ C\xi\cdot\xi \geq \beta|\xi|^2. \tag{3.50}$$

As before we assume that $C^1 = \underline{C}^1 + \sigma B$, where B is a 3×3 symmetric matrix independent of ϵ and σ. The case $C^0 = \mathrm{Id}$ and $C^1 = \sigma\mathrm{Id}$ was treated in [18]. We, here, show that our abstract framework allows us to treat general anisotropic diffusion problems.

3.3.1. *Variational formulation.* The variational space adapted to the problem (3.49) is the standard Hilbert space

$$H^1_0(D) = \{u \in H^1(D) : \ u = 0 \ \text{ on } \ \partial D\},$$

equipped with its natural norm $|\cdot|_{1,D} = \|\nabla\cdot\|_{0,D}$.

The variational formulation of problem (3.49) consists in looking for $u^\sigma_\epsilon \in H^1_0(D)$ such that

$$\int_D C\nabla u^\sigma_\epsilon \cdot \nabla v \ dx = \int_D fv \ dx, \quad \forall v \in H^1_0(D). \tag{3.51}$$

We denote by

$$a^\sigma_\epsilon(u, \ v) = \int_D C\nabla u\cdot\nabla v \ dx, \tag{3.52}$$

which is a bilinear and continuous form thanks to Cauchy-Schwarz's inequality. The property (3.50) ensures its coerciveness. So, by a Lax-Milgram argument, we deduce the existence and uniqueness of a solution of (3.51).

As in the two previous examples, we note that the bilinear form a^σ_ϵ can be split up into the sum of two bilinear forms $a^{(1)}_\epsilon$ and $a^{(2)}_\epsilon$ depending only on ϵ:

$$a^\sigma_\epsilon(u, \ v) = a^{(1)}_\epsilon(u, \ v) + \sigma a^{(2)}_\epsilon(u, \ v), \tag{3.53}$$

where

$$a_\epsilon^{(1)}(u,\ v) = \int_{D_\epsilon} C^0 \nabla u \cdot \nabla v\ dx + \int_{\Omega_\epsilon} \underline{C}^1 \nabla u \cdot \nabla v\ dx, \tag{3.54}$$

$$a_\epsilon^{(2)}(u,\ v) = \int_{\Omega_\epsilon} B \nabla u \cdot \nabla v\ dx. \tag{3.55}$$

It is easy to show that the two bilinear forms $a_\epsilon^{(1)}$, $a_\epsilon^{(2)}$ are positive, symmetric, and continuous, and that the sum is coercive on $H_0^1(D)$. The latter property is a consequence of property (3.50) satisfied by the matrix C.

When $\epsilon \longrightarrow 0$, we show as before that $a_\epsilon^{(2)} \longrightarrow 0$ strongly and that $a_\epsilon^{(1)} \longrightarrow a$ strongly, where a is defined by

$$a(u,\ v) = \int_D C^0 \nabla u \cdot \nabla v\ dx.$$

This bilinear form a is obviously continuous, symmetric, and coercive in $H_0^1(D)$.

According to our abstract framework, we can then introduce the Hilbert spaces $W(S)$, the closure in $H_0^1(D)$ of the space $\{v \in H_0^1(D) : \ B\nabla v = 0$ in a neighborhood of $S\}$, and $V(\Omega_\epsilon) = \{v \in H_0^1(D) : B\nabla v = 0 \text{ in } \Omega_\epsilon\}$.

The three weak variational formulations obtained when $\epsilon \longrightarrow 0$ and $\sigma \longrightarrow \infty$ are as follows:

Find $u_\epsilon \in V(\Omega_\epsilon)$ such that

$$\int_{D_\epsilon} C^0 \nabla u_\epsilon \cdot \nabla v\ dx + \int_{\Omega_\epsilon} \underline{C}^1 \nabla u_\epsilon \cdot \nabla v\ dx = \int_D fv\ dx, \quad \forall v \in V(\Omega_\epsilon). \tag{3.56}$$

Find $u \in W(S)$ such that

$$\int_D C^0 \nabla u \cdot \nabla v\ dx = \int_D fv\ dx, \quad \forall v \in W(S). \tag{3.57}$$

Find $u^* \in H_0^1(D)$ such that

$$\int_D C^0 \nabla u^* \cdot \nabla v\ dx = \int_D fv\ dx, \quad \forall v \in H_0^1(D). \tag{3.58}$$

All the assumptions (H1), (H2), (H3), and (H4) being satisfied, we get the following results.

Theorem 3.9. *The problems* (3.51), (3.56), (3.57), *and* (3.58) *have unique solutions u_ϵ^σ, u_ϵ, u, and u^* in $H_0^1(D)$, $V(\Omega_\epsilon)$, $W(S)$, and $H_0^1(D)$ respectively. These solutions satisfy the following limit properties:*

$$\lim_{\sigma \longrightarrow \infty} \|u_\epsilon^\sigma - u_\epsilon\|_{H_0^1(D)} = 0,$$

$$\lim_{\epsilon \longrightarrow 0} \|u_\epsilon - u\|_{H_0^1(D)} = 0,$$
$$\lim_{\epsilon \longrightarrow 0} \|u_\epsilon^\sigma - u^*\|_{H_0^1(D)} = 0.$$

3.3.2. *Strong interpretation of the limit problems.* For this example, we have strong interpretations similar to those made in the case of general elasticity.

For the first problem (3.56), we encounter the same difficulty as before due to the general constraint $B\nabla u = 0$ in Ω_ϵ. In that case, we need to introduce the space $P_\epsilon \subset L^2(\Omega_\epsilon)^3$ defined as the range of the mapping

$$\mathcal{B}_\epsilon : H_0^1(D) \longrightarrow L^2(\Omega_\epsilon)^3 : v \longrightarrow B\nabla v_{|\Omega_\epsilon}.$$

Then, we can consider the saddle point problem: Find $(\widehat{u}_\epsilon,\ \widehat{p}_\epsilon) \in H_0^1(D) \times P_\epsilon$ such that

$$\left\{ \begin{array}{rcll} a_\epsilon^{(1)}(\widehat{u}_\epsilon,\ v) - b(\widehat{p}_\epsilon,\ v) & = & < f,\ v >, & \forall v \in H_0^1(D)^3, \\ b(q,\ \widehat{u}_\epsilon) & = & 0, & \forall q \in P_\epsilon, \end{array} \right. \tag{3.59}$$

where

$$b(q,\ v) = \int_{\Omega_\epsilon} q \cdot B\nabla v \ dx.$$

The same arguments as the ones of Section 3.2 lead to a well-posed saddle point problem. Therefore, the strong formulation of (3.56) is here given by

$$\left\{ \begin{array}{rcll} -\text{div}(C^0\nabla u_\epsilon) & = & f & \text{in } \ D_\epsilon, \\ -\text{div}(C^1\nabla u_\epsilon) + \text{div}B\widehat{p}_\epsilon & = & f & \text{in } \ \Omega_\epsilon, \\ B\nabla u_\epsilon & = & 0 & \text{in } \ \Omega_\epsilon, \\ u_\epsilon|_{D_\epsilon} & = & u_\epsilon|_{\Omega_\epsilon} & \text{on } \partial\Omega_\epsilon, \\ C^0\nabla u_\epsilon \cdot n & = & (C^1\nabla u_\epsilon - B\widehat{p}_\epsilon) \cdot n & \text{on } \partial\Omega_\epsilon, \\ u & = & 0 & \text{on } \ \partial D. \end{array} \right.$$

This problem is actually a coupling between a diffusion equation in D_ϵ with a Stokes like system in Ω_ϵ; again this seems to be a new phenomenon.

At this stage, the problem (3.57) cannot be fully interpreted since the behavior of the normal trace along the crack S is not known.

Finally, the problem (3.58) is nothing else than the weak formulation of the Dirichlet problem:

$$\left\{ \begin{array}{rcll} -\text{div}(C^0\nabla u^*) & = & f & \text{in } \ D, \\ u^* & = & 0 & \text{on } \ \partial D. \end{array} \right. \tag{3.60}$$

4. CHARACTERIZATION OF $W(S)$

As we have mentioned in the previous section, the characterization of the spaces $W(S)$ is related to the geometric shape of the crack S. In the case where the crack S is a regular surface satisfying some geometric conditions described below, we can prove that the spaces $W(S)$ corresponding to some appropriate choices of B coincide with the total functional space. A counterexample is also presented.

We begin with the characterization of $W(S)$, the closure of $\{v \in H_0^1(D) : \partial_{x_1} v = 0$ in a neighborhood of $S\}$ in $H_0^1(D)$, corresponding to the space $W(S)$ from Subsection 3.3 with

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, the assumption made on S can be stated as follows:

(H5) $\exists \delta_0 : \forall |\delta| < \delta_0;\ S_\delta \cap S = \emptyset$ and $S_\delta \subset D$,

where S_δ denotes the translated of S with distance δ in the direction of x_1.

Theorem 4.1. *Under the assumption (H5), the space $W(S)$ coincides with $H_0^1(D)$.*

Proof. Since $W(S)$ is a closed subspace of $H_0^1(D)$, we just need to show that $W(S)^\perp$, the orthogonal complement of $W(S)$ in $H_0^1(D)$, is reduced to $\{0\}$.

Let $u \in H_0^1(D)$ be such that $u\perp v$ for all v in $W(S)$; i.e., $u\perp v$ for all v in $V(\Omega_\epsilon)$ with ϵ small enough. In other words, let u satisfy

$$\int_D \nabla u \cdot \nabla v \, dx = 0,\ \forall v \in V(\Omega_\epsilon),\ \forall \epsilon \ll . \tag{4.1}$$

If $v \in \mathcal{D}(D \setminus S)$, then it vanishes in a neighborhood of S, so it vanishes in Ω_ϵ for all ϵ small enough. Therefore, v is an element of the space $V(\Omega_\epsilon)$ for these ϵ.

Hence, taking $v \in \mathcal{D}(D \setminus S)$ in (4.1), we obtain that $\Delta u = 0$ in $D \setminus S$ in the distributional sense.

Let us fix now $\varphi \in \mathcal{D}(S)$. For all δ such that $|\delta| < \delta_0$, we define a new function $\widetilde{\varphi}$ in the following manner:

$$\widetilde{\varphi}(x + \delta \overrightarrow{e}_1) = \varphi(x),\ \text{for all } x \in S. \tag{4.2}$$

The relation (4.2) defines $\widetilde{\varphi}$ in θ_{δ_0}, a neighborhood of S defined by $\theta_{\delta_0} = \{x + \delta \overrightarrow{e}_1,\ x \in S$ and $|\delta| < \delta_0\}$. We extend $\widetilde{\varphi}$ to $D\backslash\theta_{\delta_0}$ by multiplying it by a cut-off function ν of class $\mathcal{C}^\infty$, equal to 1 in $\theta_{\delta_0/2}$ and vanishing on $D\backslash\theta_{\delta_0}$. We denote the new function by

$$\widetilde{\widetilde{\varphi}} = \begin{cases} \nu\widetilde{\varphi} & \text{in } \theta_{\delta_0}, \\ 0 & \text{in } D\backslash\theta_{\delta_0}. \end{cases} \tag{4.3}$$

Then, $\widetilde{\widetilde{\varphi}} \in \mathcal{D}(D)$. In addition, with the assumption (H5) and its definition, $\widetilde{\widetilde{\varphi}}$ is constant in the direction $\overrightarrow{e}_1$ on $\theta_{\frac{\delta_0}{2}}$; i.e., $\partial_{x_1}\widetilde{\widetilde{\varphi}} \equiv 0$ on $\theta_{\delta_0/2}$. This implies that $\widetilde{\widetilde{\varphi}}$ belongs to $V(\Omega_\epsilon)$, for all $\epsilon \ll$.

For the test function $v = \widetilde{\widetilde{\varphi}}$ in (4.1), by applying Green's formula (see [21, Thm 1.4.1]), we infer that

$$< \left[\frac{\partial u}{\partial n}\right], \ \varphi >_{\widetilde{H}^{1/2}(S)' \times \widetilde{H}^{1/2}(S)} = 0, \ \forall \varphi \in \mathcal{D}(S), \tag{4.4}$$

where $[\cdot]$ means the jump through S. As $\mathcal{D}(S)$ is dense in $\widetilde{H}^{1/2}(S)$, the jump $\left[\frac{\partial u}{\partial n}\right]$ through S is equal to zero (as an element of $\widetilde{H}^{1/2}(S)'$).

Now, by Lemma 2.3 of [11], the space

$$V_0 = \{v \in H_0^1(D) : v = 0 \text{ in a neighborhood of } \partial S\}$$

is dense in $H_0^1(D)$. For $v \in V_0$, we notice that its trace on S belongs to $\widetilde{H}^{1/2}(S)$. Hence, we can apply Green's formula and, using the fact that u is harmonic in $D \setminus S$ and (4.4), we deduce that

$$\int_D \nabla u \cdot \nabla v \ dx = 0, \ \forall v \in V_0.$$

By density we have shown that u is orthogonal to the whole of $H_0^1(D)$ and is therefore equal to zero. □

Remark 4.2. Theorem 4.1 remains valid in the case of vector-valued functions. Thus, in the case of the second example of Section 3, using the same arguments as above and the generalized Green formula

$$\int_D \sigma(u) : \varepsilon(v)\, dx = -\int_D \operatorname{div}(\sigma(u)) \cdot v\, dx \\ + \int_S [\partial_n \sigma(u)].v\, ds, < [\sigma(u)\, n], v >_{(\widetilde{H}^{1/2}(S)^3)' \times \widetilde{H}^{1/2}(S)^3}, \tag{4.5}$$

we can establish the equality between $W(S)$, the closure in $H_0^1(D)^3$ of $\{v \in H_0^1(D)^3 : \partial_{x_1} v_1 = 0 \text{ in a neighborhood of } S\}$, and $H_0^1(D)^3$.

Now, we interpret the space $W(S)$ introduced in Subsection 3.1. In this example, $W(S)$ is the closure in $H_0^1(D)^3$ of $\{v \in H_0^1(D)^3 : \ \operatorname{div} v = 0$ in a neighborhood of $S\}$. Contrary to the two previous cases, the divergence of v is zero in a neighborhood of S. As a result, the assumption made on S will be transformed into

(H6) $\exists i = 1,\ 2,\ 3,\ \exists \delta_0 : \forall |\delta| < \delta_0; S_\delta^i \cap S = \emptyset$ and $S_\delta^i \subset D$;

here, S_δ^i denotes the translate of S with the distance δ in the direction $\overrightarrow{e}_i,\ i = 1,\ 2,\ 3$.

Theorem 4.3. *Under the assumption (H6), $H_0^1(D)^3$ coincides with the space $W(S)$.*

Proof. The idea of the proof is the same as the one of Theorem 4.1 except that we will build a new test function which is divergence free in a neighborhood of S.

Without loss of generality, we may suppose that the hypothesis (H6) is satisfied for $i = 1$ and we choose $\widetilde{\varphi} = (\widetilde{\varphi}_1,\ \widetilde{\varphi}_2,\ \widetilde{\varphi}_3)$ with $\widetilde{\varphi}_2$ and $\widetilde{\varphi}_3$ defined as the respective translate of φ_2 and φ_3; second and third components of $\varphi \in \mathcal{D}(D)^3$, by the distance δ in the x_1 direction (as in Theorem 4.1), while $\widetilde{\varphi}_1$ is given by

$$\widetilde{\varphi}_1(x + \delta \overrightarrow{e}_1) = \varphi_1(x) - \int_0^\delta (\partial_2 \widetilde{\varphi}_2 + \partial_3 \widetilde{\varphi}_3)(x + t\overrightarrow{e}_1)\ dt,\ \forall x \in S,\ \forall |\delta| < \delta_0. \tag{4.6}$$

We define $\widetilde{\widetilde{\varphi}}$, its extension to D, as in (4.3). Hence, by construction $\operatorname{div} \widetilde{\widetilde{\varphi}} \equiv 0$ in $\theta_{\delta_0/2}$.

By choosing such a test function and applying the generalized Green formula (4.5), we get the desired result. □

Remark 4.4. The two Theorems 4.1 and 4.3 allow us in the three examples of the previous section to identify the third limit problem with the fourth one. Thus, the commutativity of the limits is ensured.

To demonstrate the impact of the geometry of the crack on the interpretation of the space $W(S)$, we now present a counter-example. When S is a flat surface parallel to one of the three planes of the space, $W(S)$ does not coincide with the total functional space for the first two cases dealt before.

Theorem 4.5. *For a plane surface S parallel to the plane (x_1Ox_2), the space $W(S)$ introduced in Theorem* 4.1 *or in Remark* 4.2 *is strictly included in $H_0^1(D)^N$, $N = 1$ or* 3, *depending on the considered case.*

Proof. We only prove the result for the case of the space from Theorem 4.1; the other case is treated in the same way. We have supposed that S is a surface parallel to (x_1Ox_2); i.e., all points $M \in S$ have $(x_1,\ x_2,\ c)$ as coordinates, where c is a real constant. Let us set

$$W_1(S) = \{v \in H_0^1(D):\ v(x_1,\ x_2,\ c) = v(x_2,\ c),\ \forall (x_1,\ x_2,\ c) \in S\}.$$

$W_1(S)$ defines a closed subspace of $H_0^1(D)$ and is strictly included in the latter. We now prove that $W(S) \subseteq W_1(S)$. Indeed for all $\epsilon > 0$, we clearly have the inclusion $V(\Omega_\epsilon) \subset W_1(S)$ and we deduce that $W(S) \subset W_1(S)$. This proves that $W(S)$ is strictly included in $H_0^1(D)$. □

Remark 4.6. For cracks as in Theorem 4.5 a nonlocal condition formally appears on this crack for the associated problems (3.56) or (3.43). Indeed, let us take the space $W(S)$ from Remark 4.2. In this case, if the solution $u \in W(S)$ of (3.43) is regular enough, then it satisfies

$$< [\sigma(u)\, n],\ v >_{(\widetilde{H}^{1/2}(S)^3)' \times \widetilde{H}^{1/2}(S)^3} = 0, \quad \forall v \in W(S). \tag{4.7}$$

If we suppose that S coincides with the square $[0,\ 1] \times [0,\ 1] \times \{c\}$ parallel to (x_1Ox_2), then (4.7) implies that

$$< [\sigma(u)\, n],\ v >= 0, \quad \forall v \in W(S).$$

This formally implies that

$$\int_0^1 [\sigma(u)\, n](x_1,\ x_2, c)\ dx_1 = 0, \quad \text{for a.a. } x_2 \in (0,1).$$

We recognize in such an integral identity a nonlocal boundary condition on S.

5. Application to Maxwell Equations

There are several papers concerned with the asymptotic behavior of the diffracted electromagnetic fields as the conductivity $\sigma \longrightarrow +\infty$, mainly in order to derive efficient approximate models to compute the diffracted waves, let us quote [1], [17], [24] and [25]. Here, we perform a similar analysis but in a different variational framework, namely the regularized formulation, see [12] for instance, that enters into the framework of Section 2. Therefore, our convergence result is a direct consequence of Theorem 2.7.

Here, the zero-order model is obtained; higher-order models will be considered in a forthcoming work. Note that we cannot take the limit as ϵ goes to zero because the norm of the variational space depends on ϵ.

5.1. **Setting of the problem.** Let $\omega > 0$ be a fixed frequency. For an electric field E and a magnetic field H, the time harmonic Maxwell equations are given by

$$\left\{ \begin{array}{rcll} \text{curl } E + i\omega\mu \ H &=& 0 & \text{in } D, \\ \text{curl } H - (i\omega\varepsilon + \sigma) \ E &=& J_0 & \text{in } D. \end{array} \right. \tag{5.1}$$

The conductivity σ is a positive constant inside Ω_ϵ, while it vanishes outside; i.e., $\sigma = 0$ inside D_ϵ. The electric permittivity ε is supposed to be a positive constant ε_ϵ (independent of ϵ) inside Ω_ϵ and has another positive value ε_0 inside D_ϵ. Similarly, the magnetic permeability μ is equal to the constant $\mu_\epsilon > 0$ (independent of ϵ) inside Ω_ϵ and to $\mu_0 > 0$ inside D_ϵ. The source current density J_0 is supposed to be in $L^2(D)^3$, to be divergence free in D and with a support included in D_ϵ.

By eliminating the magnetic field, the system (5.1) can be reduced to a second-order equation for the electric field only which reads

$$\text{curl } \mu^{-1}\text{curl } E + (i\omega\sigma - \omega^2\varepsilon) \ E = -i\omega J_0 \quad \text{in } D. \tag{5.2}$$

We note that the magnetic flux density H is given by $H = \frac{i}{\omega\mu}$ curl E.

Taking the divergence of this last equation in D_ϵ and in Ω_ϵ, we obtain the following equation on the divergence of E :

$$\left\{ \begin{array}{rcll} \text{div } E &=& 0 & \text{in } D_\epsilon, \\ \text{div } E &=& 0 & \text{in } \Omega_\epsilon. \end{array} \right. \tag{5.3}$$

Since $\sigma = 0$ inside $D \backslash \Omega_\epsilon$, the condition of perfect conductor must be added on the exterior boundary ∂D

$$E \times n = 0 \quad \text{on } \partial D. \tag{5.4}$$

Also, due to the structure of the overall domain D, transmission conditions have to be imposed on the common surface $\partial\Omega_\epsilon$

$$\left\{ \begin{array}{rcll} E|_{D_\epsilon} \times n &=& E|_{\Omega_\epsilon} \times n & \text{on } \partial\Omega_\epsilon, \\ (\omega^2\varepsilon_0)E|_{D_\epsilon} \cdot n &=& (\omega^2\varepsilon_\epsilon - i\omega\sigma)E|_{\Omega_\epsilon} \cdot n & \text{on } \partial\Omega_\epsilon. \end{array} \right. \tag{5.5}$$

As for the previous examples, we are interested in the limit of the solution of Maxwell transmission problem (5.2), (5.3), (5.4), (5.5) as σ tends to infinity or ϵ tends to zero. Here, we suppose that we are in a low frequency regime; i.e., there exists $\omega_0 > 0$ small enough to be specified later on such that $0 < \omega < \omega_0$.

5.2. **Functional spaces and variational formulation.** Let $\mathbf{H}_0(\text{curl},\ D)$ be the standard functional space

$$\mathbf{H}_0(\text{curl},\ D) = \{E \in L^2(D)^3 : \text{curl } E \in L^2(D)^3,\ E \times n = 0 \text{ on } \partial D\}.$$

The appropriate functional space to our problem (5.2), (5.3), (5.4), (5.5) is $\mathbf{Y}_\epsilon(D)$ defined by

$$\mathbf{Y}_\epsilon(D) = \Big\{E \in \mathbf{H}_0(\text{curl}, D) :\ \text{div } E|_{\Omega_\epsilon} \in L^2(\Omega_\epsilon), \\ \text{div } E|_{D_\epsilon} \in L^2(D_\epsilon), \int_{\partial\Omega_\epsilon} E|_{D_\epsilon} \cdot n \ ds = 0\Big\},$$

endowed with the norm

$$\|E\|^2_{\mathbf{Y}_\epsilon(D)} = \|E\|^2_{0,D} + \|\text{curl } E\|^2_{0,D} + \|\text{div } E\|^2_{0,D_\epsilon} + \|\text{div } E\|^2_{0,\Omega_\epsilon}. \qquad (5.6)$$

The natural sesquilinear form of our problem is: For $E,\ H \in \mathbf{Y}_\epsilon(D)$ (in the following, H denotes a test function and no longer the magnetic field):

$$a^\sigma_\epsilon(E,\ H) = \int_D (\mu^{-1}\text{curl } E \cdot\ \text{curl } \overline{H} - \omega^2 \varepsilon E \cdot \overline{H}) dx + i\omega\sigma \int_{\Omega_\epsilon} E \cdot \overline{H} dx\ . \quad (5.7)$$

The form a^σ_ϵ is split up into the sum of two sesquilinear forms:

$$a^\sigma_\epsilon(E,\ H) = a^{(1)}_\epsilon(E,\ H) + i\omega\sigma a^{(2)}_\epsilon(E,\ H),$$

with

$$a^{(1)}_\epsilon(E,\ H) = \int_D (\mu^{-1}\text{curl } E \cdot\ \text{curl } \overline{H} - \omega^2 \varepsilon E \cdot \overline{H}) dx,$$

and

$$a^{(2)}_\epsilon(E,\ H) = \int_{\Omega_\epsilon} E \cdot \overline{H} dx.$$

The sum $a^{(1)}_\epsilon + a^{(2)}_\epsilon$ is not coercive on $\mathbf{Y}_\epsilon(D)$, so, it is necessary to regularize it by adding the two terms $(\text{div}\cdot, \text{div}\cdot)_{D_\epsilon}$ and $(\text{div}\cdot, \text{div}\cdot)_{\Omega_\epsilon}$ since the continuity of the divergence is broken through the interface $\partial\Omega_\epsilon$:

$$\begin{aligned} a^\sigma_{\epsilon,R}(E,\ H) &:= a^{(1)}_\epsilon(E,\ H) + i\omega\sigma a^{(2)}_\epsilon(E,\ H) \\ &\quad + s\int_{D_\epsilon} \text{div } E|_{D_\epsilon} \text{div } \overline{H}|_{D_\epsilon} dx + s\int_{\Omega_\epsilon} \text{div } E|_{\Omega_\epsilon} \text{div } \overline{H}|_{\Omega_\epsilon} dx \\ &= a^{(1)}_{\epsilon,R}(E,\ H) + i\omega\sigma a^{(2)}_\epsilon(E,\ H), \end{aligned}$$

where s is an arbitrarily fixed positive parameter. Then the weak formulation associated with our problem (5.2), (5.3), (5.4), (5.5) is given by: Find $E^\sigma_\epsilon \in \mathbf{Y}_\epsilon(D)$ such that

$$a^\sigma_{\epsilon,R}(E^\sigma_\epsilon,\ H) = -i\omega < J_0,\ H >,\ \forall H \in \mathbf{Y}_\epsilon(D), \tag{5.8}$$

where $a^\sigma_{\epsilon,R}$ is the regularized sesquilinear form of a^σ_ϵ defined before and $< \cdot, \cdot >$ is the $L^2(D)^3$ Hermitian inner product defined by

$$< f,\ g >:= \int_D f \cdot \overline{g}\ dx.$$

Lemma 5.1. *Let the positive constants $\varepsilon_0,\ \varepsilon_\epsilon,\ \mu_0,\ \mu_\epsilon,\ \sigma \geq 1$ be fixed. Then there exists a positive constant ω_0 (depending on ϵ, μ and s, but not on σ) such that, for all $\omega \in (0, \omega_0)$, the sesquilinear form $(1-i)a^\sigma_{\epsilon,R}$ is strongly coercive on $\boldsymbol{Y}_\epsilon(D)$; i.e., there exists a positive constant C_0 (depending on ϵ, μ, s, and ω but not on σ) such that*

$$\text{Re } \big((1-i)a^\sigma_{\epsilon,R}(E,E)\big) \geq C_0\|E\|^2_{\mathbf{Y}_\epsilon(D)}\,\forall E \in \mathbf{Y}_\epsilon(D).$$

Proof. Let us set

$$a(E,H) = \int_D \mu^{-1}\text{curl } E.\text{curl } \overline{H}\,dx + i\omega\sigma\int_{\Omega_\epsilon} E\cdot\overline{H}\,dx$$
$$+ s\int_{D_\epsilon} \text{div } E\,\text{div } \overline{H}\,dx + s\int_{\Omega_\epsilon} \text{div}\,E\text{div } \overline{H}\,dx.$$

Then,

$$\text{Re } ((1-i)a(E,E))$$
$$\geq \min\{1,\omega\}C(\|\text{curl } E\|^2_{0,D} + \|E\|^2_{0,\Omega_\epsilon} + \|\text{div } E\|^2_{0,D_\epsilon} + \|\text{div } E\|^2_{0,\Omega_\epsilon}),$$

where C is a positive constant depending on μ and s. The above right-hand side is an upper bound for $\|E\|^2_{0,D_\epsilon}$ as a consequence of Lemma 2.2 of [12]. Therefore, we have

$$\text{Re } ((1-i)a(E,E)) \geq \min\{1,\ \omega\}C\|\ E\|^2_{\mathbf{Y}_\epsilon(D)}. \tag{5.9}$$

Since

$$\text{Re } \{(1-i)a^\sigma_{\epsilon,R}(E,E)\} = \text{Re } ((1-i)a(E,E)) - \omega^2\int_D \varepsilon|E|^2\ dx, \tag{5.10}$$

for $\omega^2 < C\frac{\min\{1,\ \omega\}}{\min\{\varepsilon_0,\ \varepsilon_\epsilon\}}$, we obtain

$$\text{Re } \{(1-i)a^\sigma_{\epsilon,R}(E,E)\} \geq C_0\|E\|^2_{\mathbf{Y}_\epsilon(D)},$$

where C_0 is the positive constant given by $C_0 = C\min\{1,\omega\} - \omega^2 \min\{\varepsilon_0, \varepsilon_\epsilon\}$. Hence, the conclusion follows. □

Theorem 5.2. *Under the assumptions of Lemma* 5.1, *the following holds:*

(1) *There exist a unique solution* E_ϵ^σ *to problem* (5.8).
(2) *For an appropriate choice of* s, *the solution satisfies all equations* (5.2), (5.3), (5.4), (5.5).

Proof. 1. The coerciveness of $a_{\epsilon,R}^\sigma$ on $\mathbf{Y}_\epsilon(D)$ was proved in Lemma 5.1 and we conclude by using Lax-Milgram's lemma.

2. The second result of Theorem 5.2 states that there exists an equivalence between our Maxwell transmission problem and its associated weak formulation. The arguments are quite close to the ones from Theorem 2.3 of [12]; we give the details for the sake of completeness.

First, in (5.8) we take as test function $H = \nabla\varphi$ with $\varphi \in H_0^1(D_\epsilon,\ \Delta)$ (the subspace of the $\varphi \in H_0^1(D_\epsilon)$ such that $\triangle\varphi \in L^2(D_\epsilon)$) extended to zero outside D_ϵ. This yields

$$-\omega^2\varepsilon_0 \int_{D_\epsilon} E_\epsilon^\sigma \cdot \nabla\varphi dx + \int_{D_\epsilon} s \ \mathrm{div} E_\epsilon^\sigma . \mathrm{div}\nabla\varphi dx = -i\omega \int_{D_\epsilon} J_0 \cdot \nabla\varphi dx.$$

By Green's formula and the properties of J_0, we get

$$\int_{D_\epsilon} \mathrm{div} E_\epsilon^\sigma (\Delta\varphi + \frac{\omega^2\varepsilon_0}{s}\varphi) dx = 0,\ \forall\varphi \in H_0^1(D_\epsilon,\ \Delta).$$

We note that if $\frac{-\omega^2\epsilon_0}{s}$ is not an eigenvalue of the operator $(\Delta_{Dir},\ D_\epsilon)$, e.g., the Laplacian operator in D_ϵ with Dirichlet condition on its boundary, then for all $\psi \in L^2(D_\epsilon)$ there exists a solution $\varphi \in H_0^1(D_\epsilon,\ \Delta)$ of the problem

$$\Delta\varphi + \omega^2\varepsilon_0\varphi = \psi.$$

This yields the first equation of (5.3). A similar argument in Ω_ϵ yields the second equation of (5.3) since $(\Delta\varphi + \frac{\omega^2\varepsilon_0 - i\omega\sigma}{s}\varphi)$ runs through all of $L^2(\Omega_\epsilon)$ for $\varphi \in H_0^1(D_\epsilon,\ \Delta)$.

Next, we take $H = \nabla\varphi$ with $\varphi \in H_0^1(D)$. Using this test function in (5.8), we obtain

$$\int_{D_\epsilon} \omega\varepsilon_0 E_\epsilon^\sigma \nabla\varphi \ dx + \int_{\Omega_\epsilon} (\omega\varepsilon_\epsilon - i\sigma) E_\epsilon^\sigma \nabla\varphi \ dx = i \int_D J_0 \cdot \nabla\varphi \ dx.$$

Hence, by using Green's formula, we get

$$\int_{\partial\Omega_\epsilon} \omega\varepsilon_0 (E_\epsilon^\sigma|_{D_\epsilon} \cdot n)\varphi \ dx + \int_{\partial\Omega_\epsilon} (i\sigma - \omega\varepsilon_\epsilon)(E_\epsilon^\sigma|_{\Omega_\epsilon} \cdot n)\varphi \ dx = 0.$$

From this last identity, we conclude that we have the second transmission condition of (5.5). The remaining identities of system (5.2), (5.3), (5.4), (5.5) are obtained in a standard way. □

The sesquilinear form $a_\epsilon^{(2)}$ is nonnegative. The same is true for the form $a_{\epsilon,R}^{(1)}$ since the term depending on the frequency ω is supposed to be negligible. Moreover, $a_\epsilon^{(2)}$ is Hermitian therefore it satisfies Cauchy-Schwarz's inequality.

The forms are continuous on $\mathbf{Y}_\epsilon(D)\times\mathbf{Y}_\epsilon(D)$ as a consequence of Cauchy-Schwarz's inequality:

$$\begin{aligned}|a_{\epsilon,R}^{(1)}(E,\ H)| &\le C(\mu^{-1})\|\text{rot } E\|_{0,D}\|\text{rot } H\|_{0,D}+C(\omega,\varepsilon)\|E\|_{0,D}\|H\|_{0,D}\\ &+s\|\text{div } (E|_{D_\epsilon})\|_{0,D_\epsilon}\|\text{div } (H|_{D_\epsilon})\|_{0,D_\epsilon}+s\|\text{div } (E|_{\Omega_\epsilon})\|_{0,\Omega_\epsilon}\|\text{div } (H|_{\Omega_\epsilon})\|_{0,\Omega_\epsilon}\\ &\le C'\|E\|_{\mathbf{Y}_\epsilon(D)}\|H\|_{\mathbf{Y}_\epsilon(D)},\end{aligned}$$

and

$$|a_\epsilon^{(2)}(E,H)|\le\|E\|_{0,\Omega_\epsilon}\|H\|_{0,\Omega_\epsilon}\le C\|E\|_{0,D}\|H\|_{0,D}\le C'\|E\|_{\mathbf{Y}_\epsilon(D)}\|H\|_{\mathbf{Y}_\epsilon(D)}.$$

Most of the assumptions mentioned in the abstract version are satisfied up to the following detail. If we look at the norm $\|\cdot\|_{\mathbf{Y}_\epsilon(D)}$ defined in (5.6), we clearly see that, contrary to the previous examples, it depends on the parameter ϵ. This makes the transition to the limit when the latter converges to zero impossible. Only the passage to the limit when the conductivity σ tends towards infinity is achievable.

Definition 5.3. *For all $\epsilon>0$, $\boldsymbol{Y}(\Omega_\epsilon)$ is the space of functions E in $\boldsymbol{Y}_\epsilon(D)$ such that $a_\epsilon^{(2)}(E,\ E)=0$, or equivalently*

$$\boldsymbol{Y}(\Omega_\epsilon)=\{E\in\boldsymbol{Y}_\epsilon(D):\ E=0\ \text{in}\ \Omega_\epsilon\}.$$

The limit problem as σ goes to infinity that we obtain is: Find $E_\epsilon\in\mathbf{Y}(\Omega_\epsilon)$ such that

$$a_{\epsilon,R}^{(1)}(E_\epsilon,\ H)=-i\omega<J_0,\ H>,\ \forall H\in\mathbf{Y}(\Omega_\epsilon). \tag{5.11}$$

Thus, we have a result of existence and uniqueness of the weak problem (5.8) and its limit problem (5.11) and only the first result of Theorem 2.7 is valid in this case.

Proposition 5.4. *The problems* (5.8) *and* (5.11) *admit unique solutions in $\boldsymbol{Y}_\epsilon(D)$ and $\boldsymbol{Y}(\Omega_\epsilon)$ respectively.*

Theorem 5.5. *Let E_ϵ^σ and E_ϵ be the respective solutions of problems* (5.8) *and* (5.11). *Then they satisfy the following limit property:*

$$\lim_{\sigma\longrightarrow\infty}E_\epsilon^\sigma=E_\epsilon\quad\text{in}\ \boldsymbol{Y}_\epsilon(D). \tag{5.12}$$

If s is appropriately chosen the strong formulation of problem (5.11) is

$$\left\{\begin{array}{rcll} \mu_0^{-1} \text{ curl curl } E_\epsilon - \omega^2\varepsilon_0 \; E_\epsilon & = & -i\omega J_0 & \text{in } D_\epsilon \\ \text{div } E_\epsilon & = & 0 & \text{in } D_\epsilon \\ E_\epsilon & = & 0 & \text{in } \Omega_\epsilon \\ E_\epsilon|_{D_\epsilon} \times n & = & 0 & \text{on } \partial\Omega_\epsilon \\ E_\epsilon \times n & = & 0 & \text{on } \partial D. \end{array}\right. \tag{5.13}$$

Indeed, the third and the last equations of (5.13) are satisfied since the E_ϵ solution of (5.11) belongs to $Y(\Omega_\epsilon)$. Since E_ϵ belongs to $\mathbf{H}_0(\text{curl},\ D)$, it satisfies

$$[E_\epsilon \times n]_{\partial\Omega_\epsilon} = 0.$$

Since $E_\epsilon = 0$ in Ω_ϵ, we deduce that the fifth identity of (5.13) holds.

Now, let us take in (5.11) as test function $H = \nabla\varphi$, with $\varphi \in H_0^1(D_\epsilon,\ \Delta)$ extended to zero outside D_ϵ, we find

$$\int_{D_\epsilon} s\text{div}E_\epsilon \text{ div } \Delta\varphi - \int_{D_\epsilon} \omega^2\varepsilon_0 E_\epsilon \cdot \nabla\varphi \; dx = -i\omega \int_{D_\epsilon} J_0 \cdot \nabla\varphi \; dx.$$

By Green's formula and the properties of J_0, we obtain

$$\int_{D_\epsilon} \text{div}E_\epsilon(\Delta\varphi + \frac{\omega^2\varepsilon_0}{s}\varphi)dx = 0,\ \forall\varphi \in H_0^1(D_\epsilon,\ \Delta).$$

Under the assumption that $\frac{-\omega^2\varepsilon_0}{s}$ is not an eigenvalue of the Dirichlet problem $(\Delta_{Dir},\ D_\epsilon)$, we get

$$\text{div } E_\epsilon = 0 \text{ in } D_\epsilon.$$

So, (5.11) becomes

$$\mu_0^{-1} \int_{D_\epsilon} \text{curl } E_\epsilon \cdot \text{ curl } \overline{H} - \omega^2\varepsilon_\epsilon \int_{D_\epsilon} E_\epsilon \cdot \overline{H} \; dx = -i\omega \int_{D_\epsilon} J_0 \cdot \overline{H} \; dx,\ \forall H \in \mathbf{Y}(\Omega_\epsilon). \tag{5.14}$$

By taking as test function $H \in \mathcal{D}(D_\epsilon)$ and applying Green's formula, we deduce that the first equation is satisfied. □

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