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SOLITARY WAVES OF THE TWO-DIMENSIONAL BENJAMIN EQUATION

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Abstract. In this paper, we study the existence of solitary waves associated to the two-dimensional Benjamin equation. This equation governs the evolution of waves at the interface of a two-fluid system in which surface-tension effects cannot be ignored. We classify the existence and nonexistence cases according to the sign of the transverse dispersion coefficients. Moreover, we show that the solitary waves of the 2D Benjamin equation, when they exist, converge to those of the KPI equation as the parameter preceding the nonlocal operator $H\partial_x^2$ goes to zero. We also prove the regularity of solitary waves, as well as their symmetry with respect to the transverse variable and their algebraic decay at infinity.

1. Introduction

The Benjamin equation models the dispersive wave motion of weakly nonlinear long waves in a two fluid system where the interface is subject to capillarity and when the lower fluid is very deep. (See [4].) We consider here a fluid layer of depth h_1 of a light fluid with density ρ_1, bounded above by a rigid plane and resting upon a layer of a heavier fluid with density $\rho_2 > \rho_1$. The viscosity and the compressibility are ignored. Under these flow conditions, the 2D Benjamin equation is an extension of the Benjamin equation that allows for weak spatial variations transverse to the propagation direction, and can be formally derived by a standard weakly nonlinear long wave expansion. (See [12].) It can be written as

$$(u_t + u_{xxx} - \gamma H u_{xx} + uu_x)_x \pm u_{yy} = 0, \tag{1.1}$$

where $u = u(t,x,y)$, $(x,y) \in \mathbb{R}^2$, $\gamma > 0$ and $t \geq 0$. The minus sign corresponds to the physical case. For a detailed analysis of the circumstances

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under which this equation is likely to be physically relevant, see [2]. The 2D Benjamin equation possesses two conservations laws, the momentum one

$$\int_{\mathbb{R}^2} u^2(t,x,y)dxdy = \int_{\mathbb{R}^2} u^2(0,x,y)dxdy, \tag{1.2}$$

and the energy one

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2}\Big[u_x^2 - ||D_x|^{1/2}u|^2 \mp (\partial_x^{-1}u_y)^2 - \frac{u^3}{3}\Big](t,x,y)dxdy \\ = \frac{1}{2}\int_{\mathbb{R}^2}\Big[u_x^2 - ||D_x|^{1/2}u|^2 \mp (\partial_x^{-1}u_y)^2 - \frac{u^3}{3}\Big](0,x,y)dxdy.\end{aligned} \tag{1.3}$$

The 2D Benjamin equation (1.1) combines the KDV and the Benjamin-Ono dispersive terms with the transverse variation term of the KP equation. In the absence of the BO term, it is reduced to the KP equation that governs long dispersive weakly nonlinear waves which travel predominantly in the x-direction with weak transverse effects. On the other hand, when the KDV dispersive term is ignored, we obtain the two-dimensional Benjamin-Ono equation derived by Ablowitz and Segur, [1], for internal waves in stratified fluids of large depth.

For simplicity, we will suppose that $\gamma = 1$ unless otherwise stated. The proof of our results is still the same if $\gamma > 0$. To begin with, we write (1.1) as

$$\begin{cases} u_t + u_{xxx} - Hu_{xx} + uu_x + \epsilon v_y = 0, \\ v_x = u_y, \end{cases} \tag{1.4}$$

with $\epsilon = \pm 1$. In this paper, we are interested in the existence of the solitary waves associated to (1.4) and in their properties. There are no general results concerning this problem, except for the work of B. Kim and T. R. Akylas, [13], who study formally the bifurcation mechanism of "lumps" (localized solitary waves) and the transverse instability of solitary waves.

In order to give a precise definition, we need to introduce a few spaces. We shall denote by Y the closure of $\partial_x(C_0^\infty(\mathbb{R}^2))$ endowed with the norm

$$\|\partial_x\varphi\|_Y = (\|\nabla\varphi\|_{L^2}^2 + \|\partial_x^2\varphi\|_{L^2}^2)^{1/2}.$$

Definition 1.1. *A solitary wave of* (1.1) *is a solution of the type*$(x,y,t) \to u(x-ct,y)$*, where* $u \in Y$ *and* $c > 0$*.*

Remark 1.1. The space Y defined above is introduced in [8] for the KP equation. As noticed there, by using standard imbedding theorems, one can see that, if $u \in Y$, there exists a unique, up to a constant, $\varphi \in L^q_{loc}(\mathbb{R}^2)$,

$2 \le q < +\infty$, such that $u = \partial_x\varphi$ and $v = \partial_y\varphi$. (See [8] Remark 1.1.) We then denote $v = \partial_y\varphi$ by $D_x^{-1}u_y$.

We are, thus, looking for "localized" solutions to the system

$$\begin{cases} -cu_x + u_{xxx} - Hu_{xx} + uu_x + \epsilon v_y = 0, \\ v_x = u_y. \end{cases} \tag{1.5}$$

The structure of (1.1) is reminiscent of that of the Kadomtsev-Petviashvili equation. Thus, the results of this paper are inspired from the work of A. de Bouard and J.-C. Saut [8] and P. Gravejat [10] for the KP equation. We now describe the main results of this work. In Section 2, we use Pohojaev type identities to prove the nonexistence of solitary waves, if $\epsilon = 1$. In Section 3, we prove the existence of solitary waves in the remaining cases, by considering the minimization problem

$$I_\lambda = \inf\left\{E(u), u \in Y, \text{with} \int u^3(x,y)dxdy = \lambda\right\}, \tag{1.6}$$

where $E(u) = \|u\|_Y^2 - \||D_x|^{1/2}u\|_{L^2}^2$, with $x, y \in \mathbb{R}$, $\lambda > 0$. We take the norm in Y :

$$\|u\|_Y = \|\partial_x\varphi\|_Y = (c\|\partial_x\varphi\|_{L^2}^2 + \|\partial_y\varphi\|_{L^2}^2 + \|\partial_x^2\varphi\|_{L^2}^2)^{1/2}.$$

To this aim, we shall use the concentration-compactness principle of P.-L. Lions [14]. There are some difficulties due to the functional setting of the 2D Benjamin equation, combining the operators of the KP equation and the Benjamin-Ono equation. In particular, the minimizing sequence $(u_n)_n$ is not bounded in H^1 and we have to prove a compactness lemma in L^2_{loc} for bounded sequences in Y. For the "dichotomy" case in the concentration-compactness lemma of [14], some difficulties appear in the proof of the splitting property, due to the nonlocal operator $H\partial_x^2$. In Section 4, we establish that the solitary waves, obtained by the minimization problem, converge to those of the KPI equation as γ tends to zero. In Section 5, we show that the solitary waves of (1.4) are smooth; namely, they belong to $H^\infty(\mathbb{R}^2) = \bigcap_{m\in\mathbb{N}} H^m(\mathbb{R}^2)$ where $H^m(\mathbb{R}^2)$ is the classical Sobolev space of order m. We argue by "bootstrapping," using the imbedding theorems for anisotropic Sobolev spaces [6], and a variant due to Lizorkin [15] of the Mikhlin-Hörmander multiplier theorem.

We also prove that these solutions are radially symmetric with respect to the transverse coordinate up to a translation of the origin. To obtain this result, we adapt an argument of Lopes [16].

In Section 6, we establish some algebraic decay of the solitary waves of (1.4). Namely, we show that $r^2u \in L^\infty(\mathbb{R}^2)$, where $r^2 = x^2 + y^2$, and we compute their limit when r goes to infinity. The difficulty arising here comes from the nonlocal operator $H\partial_x^2$, since (1.4) does not satisfy a Pohojaev identity on the x-variable. Then, using the Pohojaev identity on the y-variable and the algebraic decay of the kernels, we prove that $|y|u \in L^\infty(\mathbb{R}^2)$. After that, by proceeding as in [7] for the nonlocal operator, we get some integral decay of $|x|u$. By combining all these facts, we obtain the desired estimate of r^2u. At last, we show that this decay is sharp using some properties of the kernels associated to the KPI equation, proved in [10], and the properties of the solitary waves of (1.4).

2. Nonexistence of solitary waves

In this section, we shall prove that (1.4) does not admit any non-trivial solitary wave for $\epsilon = 1$. We have the following result.

Theorem 2.1. *For $\epsilon = 1$ and $c > \gamma^2/4$, equation (1.4) does not admit any non-trivial solitary wave satisfying $u = \partial_x\varphi \in Y$, $u \in H^1(\mathbb{R}^2)\bigcap L^\infty_{loc}(\mathbb{R}^2)$, and $\partial_y^2\varphi \in L^2_{loc}(\mathbb{R}^2)$.*

Proof. As noticed before, we shall take $\gamma = 1$. The proof of this theorem is based on Pohojaev type identities. To justify them, we shall need the regularity assumptions of Theorem 2.1 and use a standard truncation argument. Let $\chi_0 \in C_0^\infty(\mathbb{R})$ be such that $0 \le \chi_0(t) \le 1$, $\chi_0(t) = 1$ if $0 \le |t| \le 1$, $\chi_0(t) = 0$ if $|t| \ge 2$. We set $\chi_j(x) = \chi(\frac{x}{j})$ and $\tilde{\chi}_j(y) = \chi(\frac{y}{j^{1/4}})$, $j = 1, 2, ...$

Unlike the KP equation, we have only the y-Pohojaev identity; we cannot obtain the x-estimate, since the linear operator involves the Hilbert transform operator. Multiplying (1.5) by $y\chi_j\tilde{\chi}_jv$ and integrating over $\mathbb{R}^2$ (note that the second integral has to be interpreted as an H^1 - H^{-1} duality), we obtain

$$
\begin{aligned}
&-c\int y\chi_j\tilde{\chi}_jvu_xdxdy + \int y\chi_j\tilde{\chi}_jvu_{xxx}dxdy - \int y\chi_j\tilde{\chi}_jvHu_{xx}dxdy \\
&+\frac{1}{2}\int y\chi_j\tilde{\chi}_jv\partial_x(u^2)dxdy + \epsilon\int y\chi_j\tilde{\chi}_jvv_ydxdy = 0.
\end{aligned}
$$

After several integrations by parts, we obtain

$$
\begin{aligned}
&-\frac{c}{2}\int y\chi_j\tilde{\chi}_ju^2dxdy - \frac{1}{2}\int \chi_j\tilde{\chi}_ju_x^2dxdy + \frac{1}{2}\int \chi_j\tilde{\chi}_j||D_x|^{1/2}u|^2dxdy \\
&+\frac{1}{6}\int \chi_j\tilde{\chi}_ju^3dxdy - \frac{\epsilon}{2}\int \chi_j\tilde{\chi}_jv^2dxdy \qquad (2.1)\\
&+\frac{c}{j}\int y\chi_0'(\frac{x}{j})\tilde{\chi}_jvu_xdxdy - \frac{1}{2j^{1/4}}\int y\chi_j\chi_0'(\frac{y}{j^{1/4}})u^2dxdy
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{j}\int y\chi_0'(\frac{x}{j})\tilde{\chi}_j v u_{xx} dxdy + \frac{1}{j}\int y\chi_0'(\frac{x}{j})\tilde{\chi}_j u_y u_x dxdy \\
&- \frac{1}{2j^{1/4}}\int y\chi_j\chi_0'(\frac{y}{j^{1/4}})u_x^2 dxdy - \frac{1}{2j}\int y\chi_0'(\frac{x}{j})\tilde{\chi}_j |D_x| uv dxdy \\
&- \int \Big[|D_x|^{1/2}(y\chi_j\tilde{\chi}_j u_y) - y\chi_j\tilde{\chi}_j |D_x|^{1/2} u_y\Big] |D_x|^{1/2} u dxdy \\
&- \frac{1}{2j}\int y\chi_0'(\frac{x}{j})\tilde{\chi}_j v u^2 dxdy + \frac{1}{6j^{1/4}}\int y\chi_j\chi_0'(\frac{y}{j^{1/4}})u^3 dxdy \\
&- \frac{\epsilon}{2j^{1/4}}\int y\chi_j\chi_0'(\frac{y}{j^{1/4}})v^2 dxdy = 0.
\end{aligned}
$$

To estimate

$$\int \Big[|D_x|^{1/2}(y\chi_j\tilde{\chi}_j u_y) - y\chi_j\tilde{\chi}_j |D_x|^{1/2} u_y\Big] |D_x|^{1/2} u dxdy,$$

we shall need the following Leibniz rule for fractional derivatives. For detailed proofs of these facts, see [11].

Lemma 2.1. *Let $\alpha \in (0,1)$, $f \in C_0^\infty(\mathbb{R}^2)$ and $g \in L^2(\mathbb{R}^2)$. Then, we have*

$$\|D_x^\alpha(fg) - fD_x^\alpha g\|_{L^2} \le \|D_x^\alpha f\|_{L^\infty}\|g\|_{L^2}. \tag{2.2}$$

Note that we have the same result if we change D_x to $|D_x|$. Thus, by applying Hölder's inequality and Lemma 2.1, we obtain

$$
\begin{aligned}
&\int \Big[|D_x|^{1/2}(y\chi_j\tilde{\chi}_j u_y) - y\chi_j\tilde{\chi}_j |D_x|^{1/2} u_y\Big] |D_x|^{1/2} u dxdy \qquad (2.3)\\
&\le \||D_x|^{1/2}(y\chi_j\tilde{\chi}_j u_y) - y\chi_j\tilde{\chi}_j |D_x|^{1/2} u_y\|_{L^2}\||D_x|^{1/2}u\|_{L^2} \\
&\le \||D_x|^{1/2}(y\chi_j\tilde{\chi}_j)\|_{L^\infty}\|u_y\|_{L^2}\||D_x|^{1/2}u\|_{L^2} \le \frac{C}{j^{1/4}}\|u\|_{H^1}^2.
\end{aligned}
$$

By Lebesgue's dominated convergence theorem and (2.3), we infer from (2.1) that

$$\int \Big[\frac{c}{2}u^2 + \frac{1}{2}u_x^2 - \frac{1}{2}||D_x|^{1/2}u|^2 - \frac{1}{6}u^3 + \frac{\epsilon}{2}v^2\Big] dxdy = 0. \tag{2.4}$$

To prove the theorem, we need a second identity. We remark that if $u \in Y \cap L^6(\mathbb{R}^2)$ satisfies (1.5) in $D'(\mathbb{R}^2)$ and if we denote by Y' the dual space of Y, we have

$$-cu + u_{xx} - Hu_x + \frac{u^2}{2} + \epsilon D_x^{-1} v_y = 0 \quad \text{ in } Y',$$

where $v = D_x^{-1}u_y \in L^2(\mathbb{R}^2)$ and $D_x^{-1}v_y \in Y'$ is defined by $\langle D_x^{-1}v_y, \psi\rangle_{Y,Y'} = (v, D_x^{-1}\psi_y)$ for any $\psi \in Y$. Taking then the Y - Y' duality product of the last equation with $u \in Y$, we obtain

$$\int \Big[-cu^2 - u_x^2 + ||D_x|^{1/2}u|^2 + \frac{u^3}{2} + \epsilon v^2\Big] dxdy = 0. \tag{2.5}$$

Multiplying (2.4) by 3 and adding it to (2.5), we get

$$\int \frac{1}{2}\Big[cu^2 + u_x^2 - ||D_x|^{1/2}u|^2 + 5\epsilon v^2\Big] dxdy = 0. \tag{2.6}$$

Since $c > 1/4$, by using the Parseval identity, it is easy to see that

$$\int_{\mathbb{R}^2} cu^2 + u_x^2 - ||D_x|^{1/2}u|^2$$

is strictly positive. Using this fact and (2.6), we rule out the existence of solitary waves solutions to (1.4) if $\epsilon = 1$. □

3. Existence of solitary waves

In this section, we prove the existence of solitary waves solutions of (1.4), for $\epsilon = -1$, by using the minimization problem I_λ. To this aim, we proceed as in [8] for the KP equation. We have the following result.

Theorem 3.1. *For $\epsilon = -1$ and $c > \gamma^2/4$, equation* (1.5) *possesses a non-trivial solution (u, v) with $u \in Y$.*

Remark 3.1. As for the KP equation, the uniqueness of solitary waves for the 2D Benjamin equation (when they exist!) is an open problem.

Remark 3.2. For $c < \gamma^2/4$ ($c < 1/4$ for $\gamma = 1$), the minimization problem I_λ cannot be solved since we have $I_\lambda = -\infty$. Then, in this case, we cannot prove the existence of solitary waves by using this method. Indeed, let $u \in Y$ be such that $\int u^3 = \lambda$. Define u_ϵ by $u_\epsilon(x,y) = \epsilon^{1/3}u(x, \epsilon y)$ for $(x,y) \in \mathbb{R}^2$. Then u_ϵ satisfies

$$\int u_\epsilon^3 dxdy = \int u^3 dxdy = \lambda$$

and

$$E(u_\epsilon) = \epsilon^{-1/3}\int \Big[cu^2 - ||D_x|^{1/2}u|^2 + u_x^2\Big] dxdy + \epsilon^{4/3}\int v^2 dxdy. \tag{3.1}$$

Take a particular u defined by its Fourier transform, $\hat{u}(\xi) = \alpha[1_{I_1} - 1_{I_2}]$, where $I_1 = [\xi_1, \xi_2] \times [0,1]$, $I_2 = [-\xi_2, -\xi_1] \times [-1, 0]$, ξ_1, ξ_2 are the zeros of the function $f(\xi) = c - \xi + \xi^2$ and α is a constant to be chosen such that

$\int u^3 = \lambda$. Now, one can easily say that $u \in Y$ and that u is a real function since

$$\widehat{\mathrm{Im} u}(\xi,\eta) = \frac{\hat{u} - \overline{\hat{u}}}{2}(\xi,\eta) = \frac{\hat{u}(\xi,\eta) - \hat{u}(-\xi,-\eta)}{2} = 0.$$

Then, we have

$$\begin{aligned} &\int (cu^2 - ||D_x|^{1/2}u|^2 + u_x^2)dxdy \qquad (3.2)\\ &= \int_0^1 \int_{\xi_1}^{\xi_2} (c - \xi + \xi^2) d\xi d\eta + \int_{-1}^0 \int_{-\xi_2}^{-\xi_1} (c + \xi + \xi^2) d\xi d\eta \\ &= 2\int_0^1 \int_{\xi_1}^{\xi_2} (c - \xi + \xi^2) d\xi d\eta < 0. \end{aligned}$$

Thus, (3.1) and (3.2) implies that $E(u_\epsilon) \to -\infty$ when ϵ tends to zero.

As previously said, Theorem 3.1 will be proven by considering the minimization problem (1.6). More precisely, we will show that, under the conditions of Theorem 3.1, I_λ has a non-trivial solution $u \in Y$ by using the concentration-compactness principle ([14]). Let $u \in Y$ be a minimizer of I_λ and set $v = D_x^{-1}u_y$; then there is a Lagrange multiplier θ such that

$$-u_{xx} - |D_x|u + cu + D_x^{-1}v_y = \frac{\theta}{2}u^2 \qquad \text{in } Y'(\mathbb{R}^2), \qquad (3.3)$$

where $D_x^{-1}v_y \in Y'$ is defined by $\left\langle D_x^{-1}v_y x, \psi\right\rangle_{Y',Y} = (v, D_x^{-1}\psi_y)$, for any $\psi \in Y$. By taking the x-derivative of (3.3) in $D'(\mathbb{R}^2)$, using the definition of v and performing the change of functions $\underline{u} = \theta u$ and $\underline{v} = \theta v$, one can easily see that $(\underline{u}, \underline{v})$ satisfies (1.5) in $D'(\mathbb{R}^2)$. Therefore, it suffices to prove the existence of a minimum for I_λ to achieve the proof of Theorem 3.1.

Proof of Theorem 3.1. Observe that $I_\lambda > 0$, for $\lambda > 0$; since $c > 1/4$, by using the Fourier transform, it is easy to see that

$$M\Big(\int u_x^2 + u^2\Big) \le \int u_x^2 - ||D_x|^{1/2}u|^2 + cu^2 \le N\Big(\int u_x^2 + u^2\Big), \qquad (3.4)$$

with M, $N > 0$. Now, by applying the imbedding theorem for anisotropic Sobolev spaces ([6] page 323), we show

$$\|u\|_{L^q} \le C\|u\|_Y \qquad \forall u \in Y, \ \forall q \in [2,6].$$

Hence, we get

$$\Big|\int u^3 dxdy\Big| \le C\|u\|_Y^3 \le C(E(u))^{3/2}, \ \forall u \in Y,$$

and obtain $I_\lambda \geq \left(\frac{\lambda}{C}\right)^{2/3}$, for any $\lambda > 0$. Let $\lambda > 0$ and $(u_n)_n$ be a minimizing sequence for (1.6). Then, as was noticed in Remark 1.1, there exists a sequence of functions φ_n which belong to $L^q_{loc}(\mathbb{R}^2)$ for any positive finite q and satisfy $u_n = \partial_x \varphi_n$. Let $v_n = \partial_y \varphi_n = D_x^{-1} u_{n_y}$. By (3.4), we deduce that $(u_n)_n$ is a bounded sequence in Y. Then, by taking a subsequence, we can suppose that $\|u_n\|_Y^2$ converges to $A > 0$. Now, we apply the concentration-compactness lemma of [14] to $\rho_n = |u_n|^2 + |v_n|^2 + |\partial_x u_n|^2$ (note that $\lim_{n\to+\infty} \int \rho_n = \lim_{n\to+\infty} \|u_n\|_Y^2 = A > 0$). There are three cases to be considered.

- Suppose that "vanishing" occurs. Then, we have for $R > 0$

$$\lim_{n\to+\infty} \sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_R} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2) = 0, \tag{3.5}$$

where B_R is the ball of radius R centered at 0. Using the Sobolev inequalities for anisotropic Sobolev spaces, we get

$$\begin{aligned} &\int_{(x,y)+B_1} |\partial_x \varphi_n|^3 \leq C\Big(\int_{(x,y)+B_1} (|\partial_x\varphi_n|^2 + |\partial_y\varphi_n|^2 + |\partial_x^2\varphi_n|^2)\Big)^{3/2} \\ &\leq C\Big(\sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_1} (|\partial_x\varphi_n|^2 + |\partial_y\varphi_n|^2 + |\partial_x^2\varphi_n|^2)\Big)^{1/2} \\ &\times \int_{(x,y)+B_1} (|\partial_x\varphi_n|^2 + |\partial_y\varphi_n|^2 + |\partial_x^2\varphi_n|^2), \end{aligned}$$

for $(x,y) \in \mathbb{R}^2$. Now, covering $\mathbb{R}^2$ by balls of radius 1 such that each point of $\mathbb{R}^2$ is contained in at most 3 balls, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\partial_x\varphi_n|^3 &\leq 3C\Big(\sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_1} (|\partial_x\varphi_n|^2 + |\partial_y\varphi_n|^2 + |\partial_x^2\varphi_n|^2)\Big)^{\frac{1}{2}} \|\partial_x\varphi_n\|_Y \\ &\leq 3C\Big(\sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_1} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2)\Big)^{\frac{1}{2}} \|u_n\|_Y. \end{aligned}$$

We conclude that, under the assumption (3.5), $\lim\limits_{n\to+\infty} \|u_n\|_{L^3} = 0$, which contradicts the constraint in I_λ.

- Assume now that "dichotomy" occurs; then we have

$$\left\{ \begin{array}{l} \lim_{t\to+\infty} Q(t) = \alpha \in (0, A) \qquad \text{where for} \quad t \geq 0, \\ Q(t) = \lim\limits_{n\to+\infty} \sup\limits_{(x_0,y_0)\in\mathbb{R}^2} \int_{(x_0,y_0)+B_t} \rho_n dx dy. \end{array} \right. \tag{3.6}$$

Note that the sub-additivity condition of [14] holds here, since we have, for $\lambda > 0$, $I_\lambda = \lambda^{2/3} I_1$. To rule out this case, we shall split u_n into two parts

u_n^1 and u_n^2 with disjoint supports and we shall use assumption (3.6) to get a contradiction. In order to define u_n^1 and u_n^2 in Y, we have to localize φ_n instead of u_n; but since φ_n is not in $L^2(\mathbb{R}^2)$, the splitting property of u_n is not a direct consequence of [14]. Moreover, the operator $|D_x|^{1/2}$ imposes more difficulties in the proof of this property. For that, we shall use the argument of A. de Bouard, J.-C. Saut [8] for the KP equation and the argument of Albert, Bona and Saut [3] for the non-local operator. Then, we need the following lemmas.

Lemma 3.1. *Let $q \in [2, +\infty[$. There exists a positive constant C such that for all $f \in L^1_{loc}(\mathbb{R}^2)$ with $\nabla f \in L^2_{loc}(\mathbb{R}^2)$, for all $R > 0$ and $\mathbf{x}_0 \in \mathbb{R}^2$, we have*

$$\Big(\int_{R\le|\mathbf{x}-\mathbf{x}_0|\le 2R} |f(\mathbf{x}) - m_R(f)|^q d\mathbf{x}\Big)^{1/q} \le CR^{2/q}\Big(\int_{R\le|\mathbf{x}-\mathbf{x}_0|\le 2R} |\nabla f|^2 d\mathbf{x}\Big)^{1/2},$$

where

$$m_R(f) = \frac{1}{vol(\Omega_{\mathbf{x}_0,R})}\int_{R\le|\mathbf{x}-\mathbf{x}_0|\le 2R} f(\mathbf{x})d\mathbf{x}, \qquad \mathbf{x} = (x,y) \in \mathbb{R}^2,$$

and $\Omega_{\mathbf{x}_0,R} = \left\{\mathbf{x} \in \mathbb{R}^2 : R \le |\mathbf{x}-\mathbf{x}_0| \le 2R\right\}$.

The proof of this lemma is based on the Sobolev imbedding theorem and on the Poincaré inequality. For more details, see Lemma 3.1 of [8]. The next lemma is useful to get the bound on $|D_x|^{1/2}u_n^i$, $i = 1, 2$.

Lemma 3.2. *Let $M > 0$ be such that $\|u_n\|_Y \le M$. Then, for $\epsilon > 0$ and $a \in \mathbb{R}$, there exists $R^* = R^*(\epsilon, M)$ (depending only on ϵ and M, and independent of a and u_n) such that, for all $R \ge R^*$, we have*

$$\||D_x|^{1/2}(\phi_R\partial_x\varphi_n) - \phi_R|D_x|^{1/2}\partial_x\varphi_n\|_{L^2} \le \epsilon,$$

with $u_n = \partial_x\varphi_n$, and $\phi_R = \phi(\frac{x-a}{R})$, $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $|\phi'|_\infty \le C$.

The proof of this lemma is a simple application of the Leibniz rule. Note that Lemmas 3.1 and 3.2 will be essential in the proof of the following result.

Lemma 3.3. *Assume that (3.6) holds. Then for $\epsilon > 0$ there exists β, $B \in \mathbb{R}^+$, with $0 \le \beta \le B$, $n_1 \in \mathbb{N}$ and $\delta(\epsilon)$ (with $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$), such that we can find u_n^1 and u_n^2 in Y satisfying for $n \ge n_1$*

$$\|u_n^1 + u_n^2 - u_n\|_Y,\ \Big|\|u_n^1\|_Y^2 - \alpha\Big|,\ \Big|\|u_n^2\|_Y^2 - (A-\alpha)\Big| \le \delta(\epsilon),$$

$$\Big|\int ||D_x|^{1/2}u_n^1|^2 - \beta\Big|,\ \Big|\int ||D_x|^{1/2}u_n^2|^2 - (B-\beta)\Big| \le \delta(\epsilon),$$

$$\Big| \int ||D_x|^{1/2} u_n|^2 - ||D_x|^{1/2} u_n^1|^2 - ||D_x|^{1/2} u_n^2|^2 \Big| \leq \delta(\epsilon),$$

$$\Big| \int_{\mathbb{R}^2} [(u_n^1)^3 + (u_n^2)^3 - u_n^3] \Big| \leq \delta(\epsilon),$$

and

$$\lim_{n \to +\infty} dist(supp\ \ u_n^1, supp\ \ u_n^2) = +\infty.$$

Proof of Lemma 3.3. This proof is adapted from [8] and [14] and use Lemmas 3.1 and 3.2. Indeed, suppose that (3.6) holds, then for $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$, $R_0 > 0$, $R_n > 0$ and $\mathbf{x}_n = (x_n, y_n) \in \mathbb{R}^2$ with $R_n \nearrow +\infty$, R_0, $R_n \geq R^*$ (where $R^* = R^*(\epsilon, M)$ is the constant appearing in Lemma 3.2) such that

$$\alpha - \epsilon \leq \int_{\mathbf{x}_n + B_{R_0}} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2) \leq \alpha + \epsilon, \tag{3.7}$$

and

$$\alpha - \epsilon \leq Q_n(2R_n) \leq \alpha + \epsilon, \tag{3.8}$$

for $n \geq n_0$, where

$$Q_n(t) = \sup_{(x_0,y_0) \in R^2} \int_{(x_0,y_0)+B_t} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2).$$

By taking a subsequence if necessary, we may assume that there exists $\beta, B \in \mathbb{R}^2$, with $\beta \in [0, B]$, such that $B = \lim_{n \to +\infty} \int ||D_x|^{1/2} u_n|^2$ and $\beta = \lim_{n \to +\infty} \lim_{t \to +\infty} \sup_{\mathbf{x}_n} \int_{\mathbf{x}_n + B_t} ||D_x|^{1/2} u_n|^2$, where $(\mathbf{x}_n)$ is the sequence appearing in the former estimates. Note that we have $I_\lambda = A - B$. Now, by choosing a suitable R_0 and a subsequence of $(R_n)_{n \geq 1}$, we can suppose that there exists $n_1 \geq n_0$ such that

$$\beta - \epsilon \leq \int_{\mathbf{x}_n + B_{R_0}} ||D_x|^{1/2} u_n|^2 \leq \beta + \epsilon, \tag{3.9}$$

and

$$\beta - \epsilon \leq \sup_{\mathbf{x}_n} \int_{\mathbf{x}_n + B_{2R_n}} ||D_x|^{1/2} u_n|^2 \leq \beta + \epsilon, \tag{3.10}$$

for $n \geq n_1$. It follows that

$$\int_{R_0 \leq |\mathbf{x} - \mathbf{x}_0| \leq 2R_n} ||D_x|^{1/2} u_n|^2 \leq 2\epsilon, \tag{3.11}$$

and

$$\int_{R_0\le|\mathbf{x}-\mathbf{x}_0|\le 2R_n}(|u_n|^2+|v_n|^2+|\partial_x u_n|^2)\le 2\epsilon, \tag{3.12}$$

for $n\ge n_1$. Let ξ and $\eta\in C_0^{+\infty}$ be such that $0\le\xi\le 1$, $0\le\eta\le 1$, $\xi\equiv 1$ on B_1 with supp $\xi\subset B_2$ and $\eta\equiv 1$ on $R^2\backslash B_2$ with supp $\eta\subset\mathbb{R}^2\backslash B_1$. Set $\xi_n=\xi\big(\frac{\cdot-\mathbf{x}_n}{R_1}\big)$, $\eta_n=\eta\big(\frac{\cdot-\mathbf{x}_n}{R_n}\big)$. We define u_n^1 and u_n^2 by

$$u_n^1=\partial_x(\xi_n(\varphi_n-a_n)),\qquad u_n^2=\partial_x(\eta_n(\varphi_n-b_n)),$$

where (a_n) and (b_n) are some sequences of real numbers which will be chosen later. Set

$$v_n^1=D_x^{-1}(u_n^1)_y=\partial_y(\xi_n(\varphi_n-a_n))\text{ and }v_n^2=D_x^{-1}(u_n^2)_y=\partial_y(\eta_n(\varphi_n-b_n)).$$

Let us prove the first estimate of Lemma 3.3. We have

$$\begin{aligned}\|u_n^1+u_n^2-u_n\|_{L^2}&\le\|(\partial_x\xi_n)(\varphi_n-a_n)\|_{L^2}+\|(\partial_x\eta_n)(\varphi_n-b_n)\|_{L^2}\\&\quad+\|(1-\xi_n-\eta_n)u_n\|_{L^2}.\end{aligned}$$

Inequality (3.12) yields that

$$\|(1-\xi_n-\eta_n)u_n\|_{L^2}\le\sqrt{2\epsilon}.$$

We choose a_n and b_n in such a way that Lemma 3.1 can be applied; i.e.,

$$a_n=\frac{1}{\mathrm{vol}(\omega_{\mathbf{x}_n,R_1})}\int_{R_1\le|\mathbf{x}-\mathbf{x}_n|\le 2R_1}\varphi_n(\mathbf{x})d\mathbf{x}=m_{R_1}(\varphi_n),$$

$$b_n=\frac{1}{\mathrm{vol}(\omega_{\mathbf{x}_n,R_n})}\int_{R_n\le|\mathbf{x}-\mathbf{x}_n|\le 2R_n}\varphi_n(\mathbf{x})d\mathbf{x}=m_{R_n}(\varphi_n).$$

Therefore, Lemma 3.1 implies that

$$\begin{aligned}\|u_n^1+u_n^2-u_n\|_{L^2}&\le\|\partial_x\xi_n\|_{L^\infty}R\Big(\int_{R_1\le|\mathbf{x}-\mathbf{x}_n|2R_1}|u_n|^2+|v_n|^2\Big)^{\frac12}\\&\quad+\|\partial_x\eta_n\|_{L^\infty}R\Big(\int_{R_n\le|\mathbf{x}-\mathbf{x}_n|2R_n}|u_n|^2+|v_n|^2\Big)^{\frac12}+\sqrt{2\epsilon}\\&\le CRR^{-1}\sqrt{\epsilon}+\sqrt{2\epsilon}\le C'\sqrt{\epsilon}.\end{aligned}$$

The bound on $\|v_n^1+v_n^2-v_n\|_{L^2}$ is obtained in the same way. Now, calculate

$$\begin{aligned}\|\partial_x u_n^1+\partial_x u_n^2-\partial_x u_n\|_{L^2}&=\|\partial_x^2(\xi_n(\varphi_n-a_n))+\partial_x^2(\eta_n(\varphi_n-b_n))-\partial_x^2\varphi_n\|_{L^2}\\&\le\|\partial_x^2(\xi_n)(\varphi_n-a_n)\|_{L^2}+\|\partial_x^2(\eta_n)(\varphi_n-b_n)\|_{L^2}\\&\quad+\|(1-\xi_n-\eta_n)\partial_x u_n\|_{L^2}+2\|\partial_x\xi_n u_n\|_{L^2}+2\|\partial_x\eta_n u_n\|_{L^2}.\end{aligned}$$

The first three terms in the right-hand side of the above inequality are bounded by Lemma 3.1 and (3.12). For the last two terms, we use (3.12) to get

$$\|\partial_x \xi_n u_n\|_{L^2} \leq \|\partial_x \xi_n\|_{L^\infty} \Big(\int_{R_1 \leq |\mathbf{x}-\mathbf{x}_n| \leq 2R_1} |u_n|^2 \Big)^{1/2} \leq c\sqrt{\epsilon},$$

and similarly for $\|\partial_x \eta_n u_n\|_{L^2}$. This implies the desired bound on $\|\partial_x u_n^1 + \partial_x u_n^2 - \partial_x u_n\|_{L^2}$. Combining all these estimates, we obtain that

$$\|u_n^1 + u_n^2 - u_n\|_Y \leq \delta(\epsilon),$$

where $\lim\limits_{\epsilon \to 0} \delta(\epsilon) = 0$.

We now prove the bound on $\big| \|u_n^1\|_Y^2 - \alpha \big|$, noting that the bound on $\big| \|u_n^2\|_Y^2 - (I_\lambda - \alpha) \big|$ can be obtained in the same way. Inequalities (3.7), (3.8) and (3.12) yield

$$\begin{aligned} -\epsilon &\leq \int_{\mathbf{x}_n + B_{R_1}} \Big(|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2 \Big) - \alpha \\ &\leq \int_{\mathbb{R}^2} (\xi_n)^2 \Big(|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2 \Big) - \alpha \\ &\leq \int_{\mathbf{x}_n + B_{2R_1}} \Big(|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2 \Big) - \alpha \leq \epsilon. \end{aligned} \tag{3.13}$$

Then, to get the desired estimate, it suffices to show that

$$\Big| \|u_n^1\|_Y^2 - \int (\xi_n)^2 \Big(|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2 \Big) \Big| \leq \delta_1(\epsilon),$$

with $\lim\limits_{\epsilon \to 0} \delta_1(\epsilon) = 0$. By using Lemma 3.1 and inequality (3.12), we obtain

$$\begin{aligned} &\Big| \|u_n^1\|_{L^2} - \|\xi_n u_n\|_{L^2} \Big| \leq \|u_n^1 - \xi_n u_n\|_{L^2} \\ &= \|\partial_x(\xi_n(\varphi_n - a_n)) - \xi_n \partial_x \varphi_n\|_{L^2} \leq \|\partial_x \xi_n (\varphi_n - a_n)\|_{L^2} \\ &\leq \|\partial_x \xi_n\|_{L^\infty} \Big(\int_{R_1 \leq |\mathbf{x}-\mathbf{x}_n| \leq 2R_1} |\varphi_n - a_n|^2 d\mathbf{x} \Big)^{\frac{1}{2}} \leq c\sqrt{\epsilon}. \end{aligned}$$

By similar calculations, we can bound $\big| \|v_n^1\|_{L^2} - \|\xi_n v_n\|_{L^2} \big|$ by $c\sqrt{\epsilon}$. Now, we have

$$\begin{aligned} &\Big| \|\partial_x u_n^1\|_{L^2} - \|\xi_n \partial_x u_n\|_{L^2} \Big| \leq \| |\partial_x u_n^1 - \xi_n \partial_x u_n\|_{L^2} \\ &= \|\partial_x^2 \xi_n (\varphi_n - a_n) + 2\partial_x \xi_n u_n + \xi_n \partial_x u_n - \xi_n \partial_x u_n\|_{L^2} \\ &\leq \|\partial_x^2 \xi_n (\varphi_n - a_n) + 2\partial_x \xi_n u_n\|_{L^2}. \end{aligned}$$

By Lemma 3.2 and inequality (3.12), we obtain

$$\Big| \|\partial_x u_n^1\|_{L^2} - \|\xi_n \partial_x u_n\|_{L^2} \Big| \le c\sqrt{\epsilon}.$$

All these estimates yield that $\big| \|u_n^1\|_Y - \alpha \big| \le \delta(\epsilon)$. Let us now prove the bound on $\big| \| |D_x|^{\frac{1}{2}} u_n\|_{L^2}^2 - \beta \big|$. By (3.9) and (3.10), we get

$$-\epsilon \le \int_{\mathbf{x}_n + B_{R_1}} ||D_x|^{\frac{1}{2}} u_n|^2 - \beta \le \int_{\mathbb{R}^2} \xi_n^2 ||D_x|^{\frac{1}{2}} u_n|^2 - \beta \le \int_{\mathbf{x}_n + B_{2R_1}} ||D_x|^{\frac{1}{2}} u_n|^2 - \beta \le \epsilon.$$

Then, it suffices to prove that

$$\Big| \| |D_x|^{\frac{1}{2}} u_n^1\|_{L^2} - \int \xi_n^2 ||D_x|^{\frac{1}{2}} u_n|^2 \Big| \le \delta(\epsilon).$$

By using Lemma 3.2, (3.9) and (3.10), we get

$$\begin{aligned}
&\Big| \| |D_x|^{\frac{1}{2}} u_n^1\|_{L^2} - \|\xi_n |D_x|^{\frac{1}{2}} u_n\|_{L^2} \Big| \le \| |D_x|^{\frac{1}{2}} u_n^1 - \xi_n |D_x|^{\frac{1}{2}} u_n\|_{L^2} \\
&= \| |D_x|^{\frac{1}{2}} \partial_x (\xi_n(\varphi_n - a_n)) - \xi_n |D_x|^{\frac{1}{2}} \partial_x(\varphi_n - a_n)\|_{L^2} \\
&\le \| |D_x|^{\frac{1}{2}} (\xi_n \partial_x(\varphi_n - a_n)) - \xi_n |D_x|^{\frac{1}{2}} \partial_x (\varphi_n - a_n)\|_{L^2} + \| |D_x|^{\frac{1}{2}} (\partial_x \xi_n (\varphi_n - a_n))\|_{L^2} \\
&\le \epsilon + \|\partial_x \xi_n (\varphi_n - a_n)\|_{L^2} + \|\partial_x(\partial_x \xi_n(\varphi_n - a_n))\|_{L^2} \le \delta_1(\epsilon).
\end{aligned}$$

The bound on $\Big| \int ||D_x|^{\frac{1}{2}} u_n^2|^2 - (B - \beta) \Big|$ is proven in the same way. These estimates also yield that

$$\Big| \int ||D_x|^{\frac{1}{2}} u_n|^2 - ||D_x|^{1/2} u_n^1|^2 - ||D_x|^{1/2} u_n^2|^2 \Big| \le \delta(\epsilon).$$

Using the fact that supp $u_n^1 \bigcap$ supp $u_n^2 = \varnothing$ and the injection $Y \subset L^3(\mathbb{R}^2)$, we deduce the last bound in Lemma 3.3 from the first one.

Note that, for $c > 1/4$, we have

$$\| |D_x|^{1/2} f\|_{L^2} < \|f\|_{L^2} + \|\partial_x f\|_{L^2}, \text{ for } f \ne 0,$$

which implies that $\beta < \alpha$. □

We now continue the proof of Theorem 3.1. Taking subsequences if necessary, we may assume that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^2} (u_n^1)^3 d\mathbf{x} = \lambda_1(\epsilon), \qquad \lim_{n \to +\infty} \int_{\mathbb{R}^2} (u_n^2)^3 d\mathbf{x} = \lambda_2(\epsilon),$$

with $|\lambda_1(\epsilon) + \lambda_2(\epsilon) - \lambda| \le \delta(\epsilon)$. We have two possibilities for $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$.

(i) If $\lim_{\epsilon \to 0} \lambda_1(\epsilon) = 0$, then by choosing ϵ sufficiently small, we have for n large enough, $\int (u_n^2)^3 d\mathbf{x} > 0$. Hence, by considering

$$\Big(\frac{\lambda_2(\epsilon)}{\int_{\mathbb{R}^2} (u_n^2)^3}\Big)^{1/3} u_n^2,$$

and using Lemma 3.3, we get

$$\begin{aligned} I_{\lambda_2(\epsilon)} &\le \liminf_{n \to +\infty} \Big(\|u_n^2\|_Y^2 - \int ||D_x|^{1/2} u_n^2|^2 \Big) \\ &\le A - \alpha - B + \beta + 2\delta(\epsilon) \le I_\lambda - (\alpha - \beta) + 2\delta(\epsilon), \end{aligned}$$

which gives a contradiction since $\lim_{\epsilon \to 0} I_{\lambda_2(\epsilon)} = I_\lambda$ and $\alpha - \beta > 0$.

(ii) Therefore, we may assume that $\lim_{\epsilon \to 0} |\lambda_1(\epsilon)| > 0$ and $\lim_{\epsilon \to 0} |\lambda_2(\epsilon)| > 0$. As before, by considering $\Big(\frac{|\lambda_1(\epsilon)|}{\int_{\mathbb{R}^2} (u_n^1)^3}\Big)^{1/3} u_n^1$ and $\Big(\frac{|\lambda_2(\epsilon)|}{\int_{\mathbb{R}^2} (u_n^2)^3}\Big)^{1/3} u_n^2$, and using Lemma 3.3, we obtain

$$\begin{aligned} &I_{|\lambda_1(\epsilon)|} + I_{|\lambda_2(\epsilon)|} \\ &\le \liminf_{\epsilon \to 0} \Big(\|u_n^1\|_Y^2 - \int ||D_x|^{1/2} u_n^1|^2 \Big) + \liminf_{\epsilon \to 0} \Big(\|u_n^2\|_Y^2 - \int ||D_x|^{1/2} u_n^2|^2 \Big) \\ &\le A - B + 4\delta(\epsilon) = I_\lambda + 4\delta(\epsilon). \end{aligned}$$

This inequality gives a contradiction, when ϵ tends to zero, by using the fact that $I_\mu = \mu^{2/3} I_1$, for any positive μ. This rules out the dichotomy case.

• The only remaining possibility is the concentration-compactness case. Then, there exists a sequence $(\mathbf{x}_n)$ with $\mathbf{x}_n \in \mathbb{R}^2$, such that, for all $\epsilon > 0$, there exists $R > 0$ and $n_0 > 0$, with

$$\int_{\mathbf{x}_n + B_R} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2) d\mathbf{x} \ge A - \epsilon. \tag{3.14}$$

Using this inequality and the fact that $\lim_{n \to +\infty} \|u_n\|_Y^2 = A$, we get

$$\int_{\mathbf{x}_n + B_R} |u_n|^2 \ge A - \int_{\mathbf{x}_n + B_R} (|v_n|^2 + |\partial_x u_n|^2) d\mathbf{x} - \epsilon \ge \int_{\mathbb{R}^2} |u_n|^2 - 2\epsilon. \tag{3.15}$$

On the other hand, since (u_n) is bounded in Y, we may assume that $u_n(\cdot - \mathbf{x}_n)$ converges weakly in Y to some $u \in Y$. Then, we have

$$\int_{\mathbb{R}^2} |u|^2 d\mathbf{x} \le \liminf_{n \to +\infty} \int_{\mathbb{R}^2} |u_n|^2 d\mathbf{x} \le \liminf_{n \to +\infty} \int_{\mathbf{x}_n + B_R} |u_n|^2 d\mathbf{x} + 2\epsilon. \tag{3.16}$$

The following lemma shows that the injection $Y \subset L^2_{loc}(\mathbb{R}^2)$ is compact.

Lemma 3.4. *Let (u_n) be a bounded sequence in Y and let $R > 0$. Then, there exists a subsequence of (u_n) which converges strongly to u in $L^2(B_R)$.*

The proof of this lemma can be found in [8], page 223. We return to the proof of Theorem 3.1. By the preceding lemma, we can suppose that $u_n(\cdot - \mathbf{x}_n)$ converges strongly in $L^2_{loc}(\mathbb{R}^2)$. Then, by using inequality (3.16), we get

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus B_R} |u|^2 d\mathbf{x} = 0. \tag{3.17}$$

We deduce from (3.17) the strong convergence of (u_n) to u in $L^2(\mathbb{R}^2)$. By interpolation, we gain the convergence of $|D_x|^{1/2}(u_n - u)$ to 0 in $L^2(\mathbb{R}^2)$. Using now the injection Y into $L^6(\mathbb{R}^2)$, and by interpolation we get the strong convergence in $L^3(\mathbb{R}^2)$. It follows that $\int u^3 = \lambda$. On the other hand, we have

$$\begin{aligned} E(u) &= \|u\|_Y^2 - \int ||D_x|^{1/2} u|^2 \leq \liminf_{n \to +\infty} \|u_n\|_Y^2 - \lim_{n \to +\infty} \int ||D_x|^{1/2} u_n^2|^2 \\ &= \liminf_{n \to +\infty} \left(\|u_n\|_Y^2 - \int ||D_x|^{1/2} u_n^2|^2 \right). \end{aligned}$$

But $(u_n)_n$ is a minimizing sequence for I_λ, so that

$$E(u) \leq \liminf_{n \to +\infty} \left(\|u_n\|_Y^2 - \int ||D_x|^{1/2} u_n^2|^2 \right) = I_\lambda.$$

This shows that u is a solution of I_λ. This completes the proof of Theorem 3.1.

4. Limit to the KPI solitary waves

In this section, we show that the solitary waves of the 2D Benjamin equation converge to those of the KPI equation as γ tends to 0, for $\epsilon = -1$.

Let $c > 0$, $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+^*$, such that $\lim\limits_{k \to +\infty} \gamma_k = 0$, and

$$I_{k\lambda} = \inf \left\{ E_k(u) = \int_{\mathbb{R}^2} cu^2 - \gamma_k ||D_x|^{1/2} u|^2 + u_x^2 + (\partial_x^{-1} u_y)^2 : \int u^3 = \lambda \right\}.$$

Since $c > 0$ and $(\gamma_{k \in \mathbb{N}})$ tends to 0, there exists $k_0 > 0$ such that $c > \frac{\gamma_k^2}{4}$, for $k \geq k_0$. Then, by Theorem 3.1, the minimizing problem $I_{k\lambda}$ has a nontrivial solution in Y for $k \geq k_0$; let us denote by u_k this minimizer. Then, we have the following result.

Theorem 4.1. *Let*

$$I_\lambda = \inf\{E_0(u) = E(u) = \int_{\mathbb{R}^2} cu^2 + u_x^2 + (\partial_x^{-1}u_y)^2 : \int u^3 = \lambda\}.$$

There exists $k_0 > 0$ *such that* $(u_k)_{k\geq k_0}$ *is a minimizing sequence of* I_λ. *Consequently, there exist a subsequence, still denoted* (u_k), *and a sequence* $(\mathbf{x}_k)_k \in \mathbb{R}^2$ *such that* $u_k(\cdot - \mathbf{x}_k)$ *converges to* v *in* Y, *where* v *is a minimizer of* I_λ. *That is, the KPI-solitary waves are the limits in* Y *of solitary waves of the 2D Benjamin equation, when the parameter* (γ_k) *tends to* 0.

Proof. Since u_k is a minimizer of $I_{k\lambda}$, we deduce the following estimates:

$$\begin{aligned} E_k(u_k) &= \int_{\mathbb{R}^2} cu_k^2 - \gamma_k ||D_x|^{1/2}u_k|^2 + u_x^2 + (\partial_x^{-1}u_{ky})^2 \\ &\leq \int_{\mathbb{R}^2} cu^2 - \gamma_k ||D_x|^{1/2}u|^2 + u_x^2 + (\partial_x^{-1}u_y)^2 \\ &\leq \int cu^2 + u_x^2 + (\partial_x^{-1}u_y)^2 = E(u), \end{aligned} \tag{4.1}$$

for $u \in Y$ such that $\int u^3 = \lambda$. By Theorem 3.1 of [8], I_λ has a nontrivial solution $u_0 \in Y$. Thus, we obtain

$$E_k(u_k) \leq E(u_0). \tag{4.2}$$

On the other hand, by using the Parseval identity, we obtain for $\epsilon > 0$ and $u \in Y$ such that $\int u^3 = \lambda$

$$\begin{aligned} &\int cu^2 - \gamma_k ||D_x|^{1/2}u|^2 + u_x^2 + (\partial_x^{-1}u_y)^2 \\ &\geq \inf\left\{(1 - \frac{\gamma_k}{2}), (1 - \frac{\gamma_k}{2c})\right\} E_0(u) \geq (1-\epsilon)E_0(u_0), \end{aligned}$$

where γ_k is sufficiently small. This implies that

$$E_k(u_k) \geq (1-\epsilon)E_0(u_0). \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$-\gamma_k \int ||D_x|^{1/2}u_k|^2 \leq \|u_0\|_Y^2 - \|u_k\|_Y^2 \leq \epsilon E(u_0) - \gamma_k \int ||D_x|^{1/2}u_k|^2. \tag{4.4}$$

Let us now find a bound for $\int ||D_x|^{1/2}u_k|^2$. By using the inequality $c + |\xi|^2 \geq 2\sqrt{c}|\xi|$, the inverse Fourier transform and (4.2), we obtain

$$\begin{aligned} E_0(u_0) \geq E_k(u_k) &\geq \int cu_k^2 - \gamma_k ||D_x|^{1/2}u_k|^2 + u_{kx}^2 + (\partial_x^{-1}u_{ky})^2 \\ &\geq 2\sqrt{c}\int ||D_x|u_k|^2 - \gamma_k \int ||D_x|^{1/2}u_k|^2 \geq (2\sqrt{c} - \epsilon)\int ||D_x|^{1/2}u_k|^2. \end{aligned} \tag{4.5}$$

This last estimate implies the desired bound. Therefore, by (4.4), we get the convergence of $\|u_k\|_Y^2$ to $\|u_0\|_Y = I_\lambda$ when γ_k tends to zero. This shows that $(u_k)_{k\geq k_0}$ is a minimizing sequence of I_λ. Then there exists $v \in Y$, such that $\int v^3 = \lambda$ and $I_\lambda = \|v\|_Y^2$, a subsequence of $(u_k)_{k\geq k_0}$ still denoted u_k and $(\mathbf{x}_k)_{k\in\mathbb{N}}$ such that $u_k(\cdot - \mathbf{x}_k)$ converges weakly to v in Y as $k \to +\infty$; see [8]. By combining this fact with the convergence of $\|u_k\|_Y^2$ to $I_\lambda = \|v\|_Y^2$, we deduce the strong convergence of $u_k(\cdot - \mathbf{x}_k)$ to v in Y. □

5. Regularity and symmetry properties of the solitary waves

In this section, we prove that any solitary wave of (1.4) is a C^∞-function. We show this regularity property for a nonlinear elliptic equation associated to (1.5), by using the bootstrapping method. We deduce from that the regularity of solitary waves. We establish also that any minimizer of I_λ is radially symmetric with respect to the y-variable. We have the following theorem.

Theorem 5.1. *Any solitary wave solution of* (1.4) *belongs to* $H^\infty(\mathbb{R}^2)$ *provided* $\epsilon = -1$ *and* $c > \gamma^2/4$. *Moreover,* $v = D_x^{-1} u_y$ *belongs to* $H^\infty(\mathbb{R}^2)$.

Proof. The proof is adapted from [8]. Let $u \in Y$ be a solitary wave of (1.4). Then u satisfies

$$-cu_x + u_{xxx} - Hu_{xx} + \left(\frac{u^2}{2}\right)_x - D_x^{-1} v_y = 0,$$

with $v_x = u_y$. By applying ∂_x to this equation, we obtain

$$-c\partial_x^2 u - \partial_y^2 u + \partial_x^2 |D_x| u + \partial_x^4 u = \partial_x^2 \frac{u^2}{2}. \tag{5.1}$$

Then, we are left to prove the regularity of the nonlinear elliptic equation (5.1). The difficulty arises from the non-isotropy of the symbol of the linear operator $-c\partial_x^2 - \partial_y^2 + \partial_x^2|D_x| + \partial_x^4$. We proceed by bootstrapping using the following variant due to Lizorkin [15] of the Hörmander-Mikhlin theorem.

Proposition 5.1. *Let* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be a* C^n*-function for* $|\xi_j| > 0$, $j = 1, ..., n$. *Assume that there exists* $M > 0$ *such that*

$$\left|\xi_1^{k_1} ... \xi_n^{k_n} \frac{\partial^k \phi}{\partial \xi_1^{k_1} ... \partial \xi_n^{k_n}}(\xi)\right| \leq M,$$

with $k_i = 0$ *or* 1, $k = k_1 + k_2 + ... + k_n = 0, 1, ..., n$. *Then* ϕ *is a Fourier multiplier on* $L^q(\mathbb{R}^n)$, *for* $q \in (1, +\infty)$.

Next, we state a new lemma which is used to prove Theorem 5.1. Setting $g = -u^2$, (5.1) yields

$$\hat{u}_{xx} = -\xi_1^2 \hat{u} = \frac{\xi_1^4 \hat{g}}{c|\xi_1|^2 + |\xi_2|^2 - |\xi_1|^3 + \xi_1^4}.$$

Lemma 5.1. *Let $u \in Y$ be a solution of* (5.1). *Then, we have*

$$u \in \Big\{ f \in L^6(\mathbb{R}^2) \cap L^3(\mathbb{R}^2) : \ \partial_x f \in L^6(\mathbb{R}^2), \partial_y f \in L^3(\mathbb{R}^2), \ \partial_x^2 f \in L^3(\mathbb{R}^2) \Big\}.$$

Proof of Lemma 5.1. By Theorem 15.7 of [6], one has $Y \subset L^6(\mathbb{R}^2)$ and therefore $u^2 \in L^3(\mathbb{R}^2)$. Define

$$\phi_1(\xi) = \frac{\xi_1^2}{c|\xi_1|^2 + |\xi_2|^2 - |\xi_1|^3 + \xi_1^4}, \quad \phi_2(\xi) = \frac{\xi_1^4}{c|\xi_1|^2 + |\xi_2|^2 - |\xi_1|^3 + \xi_1^4},$$

and

$$\phi_3(\xi) = \frac{\xi_2 \xi_1^2}{c|\xi_1|^2 + |\xi_2|^2 - |\xi_1|^3 + \xi_1^4}.$$

Since $c > 1/4$, we have $c\xi_1^2 - |\xi_1|^3 + \xi_1^4 > 0$. Therefore, one can easily see that ϕ_1, ϕ_2 and ϕ_3 satisfy the assumptions of Proposition 5.1, yielding that u, $\partial_x^2 u$, $\partial_y u \in L^3(\mathbb{R}^2)$. By interpolation between $u \in L^6(\mathbb{R}^2)$ and $\partial_x^2 u \in L^3(\mathbb{R}^2)$, we obtain that $\partial_x u \in L^6(\mathbb{R}^2)$. □

We now turn to the proof of Theorem 5.1. Lemma 5.1 implies that u, $\partial_y u$, $\partial_x^2 u \in L^3(\mathbb{R}^2)$. By Theorem 10.2 of [6], one has $u \in L^\infty(\mathbb{R}^2)$. Then, we have $f = -(u^2)_{xx} = -2uu_{xx} - 2u_x^2 \in L^3(\mathbb{R}^2)$. Define

$$\phi_4(\xi) = \frac{\xi_2^2}{c|\xi_1|^2 + |\xi_2|^2 - |\xi_1|^3 + \xi_1^4}.$$

It is easy to see that ϕ_4 satisfies the assumptions of Proposition 5.1. Applying now Lizorkin's theorem to f, we get

$$\partial_x^2 u, \partial_x^4 u, \partial_x^2 \partial_y u, \partial_y^2 u \in L^3(\mathbb{R}^2). \tag{5.2}$$

Let $v = \partial_x^2 u$, then (5.2) implies that v, $\partial_x^2 v$, $\partial_y v \in L^3(\mathbb{R}^2)$. By the aforementioned result of [6], we have $v \in L^\infty(\mathbb{R}^2)$. This implies that $f \in L^r(\mathbb{R}^2)$ for $r \in [2, +\infty[$. Applying Lizorkin's theorem again yields that

$$\partial_x^4 u, \partial_x^2 \partial_y u, \partial_y^2 u \in L^q(\mathbb{R}^2), \text{ for } q \in [2, +\infty],$$

so that $\partial_x f$, $\partial_y f \in L^q$, for $q \in [2, +\infty]$. Reiteration of this process leads to the regularity of u. The regularity of $D_x^{-1} u_y$ is obtained by using equation (1.5) and the regularity of u.

We now state and prove the theorem concerning the symmetry property of the solitary waves.

Theorem 5.2. *Let u be a solution of the minimization problem I_λ. Then, up to a translation of the origin in the $y-$direction, u is radial; that is, u only depends on x and on $|y|$.*

Proof. To show this result, we shall use an argument of Lopes ([16]). Indeed, choose $b \in \mathbb{R}$ so that if $\Delta = \{(x,y) \in \mathbb{R}^2 : y = b\}$, we have

$$\int_{\Delta^+} \rho(u) = \int_{\Delta^-} \rho(u) = \frac{1}{2}\int_{\mathbb{R}^2} \rho(u),$$

where Δ^+ and Δ^- are the half-planes delimited by Δ. Let $u^+ = u$ in Δ^+ and u^+ be symmetric with respect to Δ. Then, $u^+ \in Y$; indeed, if $\varphi \in L^2_{loc}$ is such that $\varphi_x = u$ and $\varphi_y = D_x^{-1}u_y$, and if

$$\varphi^+(x,y) = \begin{cases} \varphi(x,y) & \text{if } y > b, \\ \varphi(x, 2b-y) & \text{if } y < b, \end{cases}$$

then $\varphi^+_x = u^+$ and $\int_{\mathbb{R}^2} (\varphi^+_y)^2 = 2\int_{\Delta^+} \varphi_y^2 < +\infty$. Since there is a sequence $(\varphi_n) \in C_0^\infty(\mathbb{R}^2)$ such that $(\varphi_n)_x$ converges to $\varphi_x = u$ in Y, it follows that $D_x^{-1}u^+_y = \varphi^+_y$. Moreover, we have

$$\int_{\mathbb{R}^2} \rho(u^+) = 2\int_{\Delta^+} \rho(u) = \int_{\mathbb{R}^2} \rho(u). \tag{5.3}$$

In the same way, if we define $u^- = u$ in Δ^- and u^- is symmetric with respect to Δ, then $u^- \in Y$ and $\int_{\mathbb{R}^2} \rho(u^-) = \int_{\mathbb{R}^2} \rho(u)$. Set

$$\lambda_1 := \int_{\mathbb{R}^2} (u^+)^3 = 2\int_{\Delta^+} u^3, \quad \lambda_2 := \int_{\mathbb{R}^2} (u^-)^3 = 2\int_{\Delta^-} u^3.$$

We have that $\lambda = \frac{\lambda_1+\lambda_2}{2}$, where $\lambda = \int u^3$. We now show that $\lambda = \lambda_1 = \lambda_2$. Suppose for example that $\lambda < \lambda_1$. Since $I_\lambda = \lambda^{2/3} I_1$, for $\lambda > 0$, we deduce that $I_\lambda < I_{\lambda_1}$. Then

$$I_\lambda = E(u) = \int \rho(u) < I_{\lambda_1} \le \int \rho(u^+),$$

which contradicts (5.3). Therefore, we have $\lambda_1 = \lambda_2 = \lambda$. We deduce that u^+, u^- and u are solutions of the minimized problem I_λ. Thus, u^+, u^- and u satisfy

$$-\partial_x^4 u + c\partial_x^2 u - \partial_x^2 |D_x| u + \partial_y^2 u = \theta \partial_x^2 \frac{u^2}{2}, \text{ in } \mathbb{R}^2.$$

Finally, since $u^+ = u$ in Δ^+ and $u^+ = u$ in Δ^-, the continuation principle (see the appendix of [9], and [17]) applied to $u^+ - u$ (respectively to $u^- - u$) implies that $u^+ = u^- = u$. Thus u is symmetric with respect to Δ.

6. Asymptotic decay of the solitary waves

In this section, we study the algebraic decay of the solitary waves associated to (1.4), for $\epsilon = -1$ and $c > \gamma^2/4$. The difficulty arises from the Benjamin-Ono operator $H\partial_x^2$. In fact, we cannot proceed as for the KPI equation, since (1.4) does not satisfy the Pohajaev identity on the x-variable. We may write the solitary wave associated to (1.4), by using the convolution equation, as

$$u = -k * \frac{u^2}{2} = -ih * (u\partial_x u), \tag{6.1}$$

where k and h are defined by their Fourier transform:

$$\hat{k}(\xi_1, \xi_2) = \frac{\xi_1^2}{c|\xi_1|^2 - |\xi_1|^3 + \xi_1^4 + \xi_2^2}, \quad \hat{h}(\xi_1, \xi_2) = \frac{\xi_1}{c|\xi_1|^2 - |\xi_1|^3 + \xi_1^4 + \xi_2^2}.$$

We first prove some integral decay of u by using the Pohojaev identity. We study also the algebraic decay of the kernels k and h. By combining all these facts, we can show, as for the KPI equation, that $|y|u \in L^\infty(\mathbb{R}^2)$. After that, we obtain the desired estimate of $|(x, y)|^2 u$ by using the convolution equation, the properties of the kernels and the bound of $|y|^l u$. The first step of the proof is inspired from the work of A. de Bouard and J.-C. Saut [9] for the KPI equation and the second step from the work of J. Bona and and Y. A. Li [7].

The second goal of this section is to compute the limit when r goes to infinity of $r^2 u$ which will imply that the algebraic decay is sharp. To get this result, we write u as

$$u = -k_0 * \frac{u^2}{2} + \gamma k_0 * |D_x| u,$$

where k_0 is the kernel associated to the KPI equation defined by

$$\widehat{k_0}(\xi_0, \xi_2) = \frac{\xi_1^2}{\xi_1^2 + \xi_1^4 + \xi_2^2}.$$

We prove an algebraic decay of $|D_x|u$ and ∇u and we recall some properties of the kernel k_0 proved in [10]. By using these facts, we deduce the pointwise convergence in $\mathbb{S}^1$ by proceeding as in [10] for the KPI equation. We obtain the uniform convergence in $\mathbb{S}^1$ thanks to Ascoli-Arzola's theorem.

Note that we suppose $\gamma = 1$ in the sequel. Then, the condition $c > \gamma^2/4$ becomes $c > 1/4$. We start with a simple integral decay estimate.

Lemma 6.1. *For $\epsilon = -1$ and $c > 1/4$, any solitary wave of* (1.4) *satisfies*

$$\int_{\mathbb{R}^2} |y|^2(|\nabla u|^2 + u_{xx}^2)dxdy < +\infty. \tag{6.2}$$

Proof. Let $\chi_0 \in C_0(\mathbb{R})$ be such that $0 \le \chi_0 \le 1$, $\chi_0(t) = 1$ if $0 \le |t| \le 1$ and $\chi_0(t) = 0$ if $|t| \ge 2$. Set $\chi_j(y) = \chi_0(\frac{y^2}{j^2})$, $j = 1, 2, \ldots$. We multiply (5.1) by $y^2\chi_j u$ and we integrate over $\mathbb{R}^2$. Using several integrations by parts yields

$$\begin{aligned}
&- c\int_{\mathbb{R}^2} y^2\chi_j(y)uu_{xx}dxdy = c\int_{\mathbb{R}^2} y^2\chi_j(y)u_x^2dxdy,\\
&- \int_{\mathbb{R}^2} y^2\chi_j(y)uu_{yy} = \int_{\mathbb{R}^2} y^2\chi_j(y)u_y^2 - \int_{\mathbb{R}^2}\chi_j(y)u^2 - \int_{\mathbb{R}^2}\Big(2y\chi_j'(y) + \frac{1}{2}y^2\chi_j''(y)\Big)u^2,\\
&\int_{\mathbb{R}^2} y^2\chi_j(y)u\partial_x^2|D_x|udxdy = -\int_{\mathbb{R}^2} y^2\chi_j(y)||D_x|^{1/2}u_x|^2dxdy,\\
&\int_{\mathbb{R}^2} y^2\chi_j(y)uu_{xxxx}dxdy = \int_{\mathbb{R}^2} y^2\chi_j(y)u_{xx}^2dxdy,\\
&\frac{1}{2}\int_{\mathbb{R}^2} y^2\chi_j(y)u(u^2)_{xx} = \int y^2\chi_j(y)u(u_x^2 + uu_{xx})dxdy = -\int y^2\chi_j(y)uu_x^2dxdy.
\end{aligned}$$

Adding these equalities, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} y^2\chi_j(y)\Big(cu_x^2 + u_y^2 + u_{xx}^2 - ||D_x|^{1/2}u_x|^2\Big) \\
&= \int_{\mathbb{R}^2}\chi_j(y)u^2 + \int_{\mathbb{R}^2} y^2\chi_j(y)uu_x^2 + \int_{\mathbb{R}^2}\Big(2y\chi_j'(y) + \frac{1}{2}\chi_j''(y)\Big)u^2.
\end{aligned} \tag{6.3}$$

Let us consider the right-hand side of (6.3). Since $u \to 0$ as $|y| \to +\infty$, there exists $R > 0$ such that, for $|y| \ge R$, we have $|u| \le \epsilon$, with $0 < \epsilon \ll 1$. Thus,

$$\int_{\mathbb{R}^2} y^2\chi_j(y)uu_x^2 \le C(R) + \epsilon\int_{\mathbb{R}^2} y^2\chi_j u_x^2.$$

Choose ϵ such that $c - \epsilon > \frac{1}{4}$. Using Lebesgue's theorem and the properties of χ_j, we get

$$\int_{\mathbb{R}^2}\Big(2y\chi_j'(y) + \frac{1}{2}\chi_j''(y)\Big)u^2 \to 0, \text{ when } j \to +\infty.$$

At last, we obtain that

$$\int_{\mathbb{R}^2} y^2\chi_j(y)\Big((c-\epsilon)u_x^2 + u_y^2 + u_{xx}^2 - ||D_x|^{1/2}u_x|^2\Big)dxdy$$

is uniformly bounded in j. By Fatou's lemma, we get

$$\int_{\mathbb{R}^2} y^2\Big[(c-\epsilon)u_x^2 + u_y^2 + u_{xx}^2 - ||D_x|^{1/2}u_x|^2\Big]dxdy \le C,$$

where C is independent of j. By using the inverse Fourier transform and the condition on ϵ, we deduce that there exists $\alpha > 0$ such that

$$\int_{\mathbb{R}^2} y^2 \Big[\alpha(u_x^2 + u_{xx}^2) + u_y^2\Big] dxdy$$
$$\leq \int_{\mathbb{R}^2} y^2 \chi_j(y) \Big((c-\epsilon)u_x^2 + u_y^2 + u_{xx}^2 - ||D_x|^{1/2} u_x|^2\Big) dxdy < +\infty. \qquad \square$$

Let us now prove some properties of the kernels h and k.

Lemma 6.2. *There exists a constant $C > 0$ such that*

$$|h(x,y)| \leq \frac{C}{r}, \qquad \forall (x,y) \in \mathbb{R}^2 \quad \textit{where } r = (x^2+y^2)^{1/2}.$$

Proof. We have

$$h(x,y) = \int_{\mathbb{R}^2} \frac{\xi_1}{c\xi_1^2 + \xi_2^2 - |\xi_1|^3 + \xi_1^4} e^{ix\xi_1 + iy\xi_2} d\xi_1 d\xi_2.$$

Since $c > 1/4$, we may write

$$\frac{\xi_1}{c\xi_1^2 + \xi_2^2 - |\xi_1|^3 + \xi_1^4} = \frac{1}{\xi_1^2(c - |\xi_1| + \xi_1^2)\Big(\frac{\xi_2^2}{a^2} + 1\Big)},$$

where $a^2 = c\xi_1^2 - |\xi_1|^3 + \xi_1^4$. Under the change of variable $\xi_2 = a\xi_2'$, we get

$$h(x,y) = \int_{\mathbb{R}} \frac{\mathrm{sgn}(\xi_1)}{(c - |\xi_1| + \xi_1^2)^{1/2}} \Big[\int_{\mathbb{R}} \frac{e^{iy|\xi_1|(c-|\xi_1|+\xi_1^2)\xi_2'}}{1 + \xi_2'^2} d\xi_2'\Big] e^{ix\xi_1} d\xi_1.$$

Since $\mathcal{F}(\frac{1}{1+\xi^2})(y) = e^{-|y|}$, we obtain

$$h(x,y) = \int_{\mathbb{R}} \frac{\mathrm{sgn}(\xi)}{(c - |\xi| + \xi^2)^{1/2}} e^{-|y||\xi|(c-|\xi|+\xi^2)^{1/2})} e^{ix\xi} d\xi.$$

We first consider the case when $y = 0$. In fact, we have

$$h(x,0) = \int_{-\infty}^{+\infty} \frac{\mathrm{sgn}(\xi)}{(c - |\xi| + \xi^2)^{1/2}} e^{ix\xi} d\xi = \mathcal{F}^{-1}\Big(\frac{sgn(\xi)}{(c - |\xi| + \xi^2)^{1/2}}\Big)(x).$$

Let us check that $xh(x,0)$ is a bounded function. Actually, we have

$$\frac{d}{d\xi}\Big(\frac{\mathrm{sgn}(\xi)}{(c - |\xi| + \xi^2)^{1/2}}\Big) = \frac{2}{c^{1/2}}\delta + g,$$

where δ is the Dirac mass and $g \in L^1(\mathbb{R})$. Thus, $xh(x,0) = i(\frac{2}{c^{1/2}} + \mathcal{F}^{-1}g) \in L^\infty(\mathbb{R})$.

Now, for $y \neq 0$, we set

$$h_1(x,y) = \int_0^{+\infty} \frac{1}{(c - \xi + \xi^2)^{1/2}} e^{K(\xi)} d\xi,$$

with $K(\xi) = ix\xi - |y|\xi(c - \xi + \xi^2)^{1/2}$. We consider two cases.

• If $c > \frac{9}{32}$, we get

$$K'(\xi) = \frac{ix(c-\xi+\xi^2)^{1/2} - |y|(c-\frac{3\xi}{2}+2\xi^2)}{(c-\xi+\xi^2)^{1/2}},$$

with $c-\xi+\xi^2>0$ and $c-\frac{3\xi}{2}+2\xi^2>0$. Then, we have $|K'(\xi)|>0$ as $y\neq 0$. By integrating by parts, we get

$$\begin{aligned} h_1(x,y) &= \int_0^{+\infty} \frac{1}{(c-\xi+\xi^2)^{\frac{1}{2}}K'(\xi)}\frac{d}{d\xi}[e^{K(\xi)}] \qquad (6.4)\\ &= \Big[\frac{e^{K(\xi)}}{ix(c-\xi+\xi^2)^{\frac{1}{2}} - |y|(c-\frac{3\xi}{2}+2\xi^2)}\Big]_0^{+\infty} - \int_0^{+\infty}\frac{d}{d\xi}\Big[\frac{1}{(c-\xi+\xi^2)^{\frac{1}{2}}K'(\xi)}\Big]e^{K(\xi)}d\xi\\ &= \frac{1}{c|y|-i\sqrt{c}x} - \int_0^{+\infty}\frac{ix(\xi-\frac{1}{2}) - |y|(4\xi-\frac{3}{2})(c-\xi+\xi_1^2)^{\frac{1}{2}}}{(c-\xi+\xi_1^2)^{\frac{1}{2}}[ix(c-\xi+\xi^2)^{\frac{1}{2}} - |y|(c-\frac{3\xi}{2}+2\xi^2)]^2}e^{K(\xi)}d\xi. \end{aligned}$$

Let $F(\xi) = \frac{ix(\xi-1/2)-|y|(4\xi-3/2)(c-\xi+\xi_1^2)^{1/2}}{(c-\xi+\xi_1^2)^{\frac{1}{2}}[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^2}$. F satisfies

$$\begin{aligned} |F(\xi)|^2 &= \frac{x^2(\xi-1/2)^2 + y^2(4\xi-3/2)^2(c-\xi+\xi^2)}{(c-\xi+\xi^2)\Big[x^2(c-\xi+\xi^2)+y^2(c-\frac{3\xi}{2}+2\xi^2)^2\Big]^2}\\ &\leq \frac{C}{(c-\xi+\xi^2)\Big[x^2(c-\xi+\xi^2)+y^2(c-\frac{3\xi}{2}+2\xi^2)^2\Big]} \leq \frac{C}{(c-\frac{3\xi}{2}+2\xi^2)^2(x^2+y^2)}. \end{aligned}$$

Thus, we have

$$\Big|\int_0^{+\infty} F(\xi)e^{K(\xi)}d\xi\Big| \leq \frac{C}{r}\int_0^{+\infty}\frac{d\xi}{(c-\frac{3\xi}{2}+2\xi^2)} \leq \frac{C}{r},$$

and

$$|h_1(x,y)| \leq \frac{1}{||y|-ix|} + \frac{C}{r} = \frac{C+1}{r}, \qquad \text{if} \quad y\neq 0.$$

• If $\frac{1}{4} < c \leq \frac{9}{32}$, then the function $f(\xi) = c - \frac{3\xi}{2} + 2\xi^2$ vanishes at $\xi_1 = \frac{3-\sqrt{9-32c}}{8}$ and $\xi_2 = \frac{3+\sqrt{9-32c}}{8}$. Choose δ_1, $\delta_2 \in \mathbb{R}$ be such that $0<\delta_1<\xi_1\leq\xi_2<\delta_2$. We split the integral defining h_1 as follows:

$$h_1(x,y) = \int_0^{\delta 1} ... + \int_{\delta_1}^{\delta_2} ... + \int_{\delta_2}^{+\infty} ... = I_1 + I_2 + I_3.$$

Since $\inf_{\xi\in[0,\delta_1]\cap[\delta_2,+\infty[}|K'(\xi)|>0$, we can proceed as in the first case (when $c>\frac{9}{32}$) to obtain

$$|I_1| + |I_3| \leq \frac{C}{r}, \text{ for } y\neq 0.$$

For I_2, we have

$$|I_2| = \Big|\int_{\delta_1}^{\delta_2}\frac{1}{(c-\xi+\xi^2)^{1/2}}e^{-|y|\xi(c-|\xi|+\xi^2)^{1/2})}e^{ix\xi}d\xi\Big| \leq Be^{-A|y|}, \qquad (6.5)$$

since $\inf_{\xi\in[\delta_1,\delta_2]}(c-\xi+\xi^2)>0$. Therefore, we have

$$|y|^2 I_2 \le C. \tag{6.6}$$

Now, for $x \neq 0$, we have $|K'(\xi)| > 0$. Then, integrating by parts the integral defining I_2 yields

$$\begin{aligned} I_2 &= \int_{\delta_1}^{\delta_2} \frac{1}{(c-\xi+\xi^2)^{1/2}K'(\xi)}\frac{d}{d\xi}[e^{K(\xi)}] \\ &= \Big[\frac{e^{K(\xi)}}{ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)}\Big]_{\delta_1}^{\delta_2} - \int_{\delta_1}^{\delta_2} F(\xi)e^{K(\xi)}d\xi, \end{aligned}$$

where

$$|F(\xi)|^2 = \frac{x^2(\xi-1/2)^2+y^2(4\xi-3/2)^2(c-\xi+\xi^2)}{(c-\xi+\xi^2)\Big[x^2(c-\xi+\xi^2)+y^2(c-\frac{3\xi}{2}+2\xi^2)^2\Big]^2} \le \frac{C(1+y^2)}{(c-\xi+\xi^2)^2x^2},$$

for $\xi \in [\delta_1,\delta_2]$ and $x \neq 0$. Using the fact that

$$\inf_{\xi\in[\delta_1,\delta_2]}(c-\xi+\xi^2)>0, \tag{6.7}$$

it follows that

$$|x|I_2 \le C\Big(1+\int_{\delta_1}^{\delta_2}|y|e^{-A|y|}d\xi\Big) \le C'. \tag{6.8}$$

Inequalities (6.6) and (6.8) imply that $rI_2 \le C$. Finally, we obtain

$$|h_1(x,y)| \le \frac{C}{r}, \text{ for } y \neq 0.$$

By similar arguments, one can prove that

$$h_2(x,y) = \int_{-\infty}^{0} \frac{1}{(c+\xi+\xi^2)^{1/2}} e^{|y|\xi(c+\xi+\xi^2)^{1/2}} e^{ix\xi}d\xi$$

satisfies

$$|h_2(x,y)| \le \frac{C}{r}, \text{ for } y \neq 0. \qquad \square$$

We now prove a pointwise (non-optimal) decay estimate of u.

Lemma 6.3. *We have* $|y|u \in L^\infty(\mathbb{R}^2)$.

Proof. From (6.1), we get

$$\begin{aligned} |y||u(x,y)| \le\; & C\int_{\mathbb{R}^2}|h(x-x',y-y')(y-y')||uu_x(x',y')|dx'dy' \\ &+ C\int_{\mathbb{R}^2}|h(x-x',y-y')||y'uu_x(x',y')|dx'dy'. \end{aligned}$$

By using Lemma 6.2 and the fact that $uu_x \in L^1(\mathbb{R}^2)$, we obtain

$$\int_{\mathbb{R}^2}|h(x-x',y-y')(y-y')||uu_x(x',y')|dx'dy' \le C.$$

On the other hand, it is easy to check that $\hat{h} \in L^q$, for $q \in (1,2)$. By interpolation, we deduce that $h \in L^s(\mathbb{R}^2)$, for $s \in (2,+\infty)$. Then, by applying Young's inequality, we get

$$\int_{\mathbb{R}^2} |h(x-x',y-y')||y'uu_x(x',y')|dx'dy' \le C\|yu_x\|_{L^2}\|h\|_{L^s}\|u\|_{L^{\frac{2s}{s-2}}},$$

for $s \in (2,+\infty)$. Thus, the result follows from Lemma 6.1.

We now turn to proving some properties of the kernel k.

Lemma 6.4. *The kernel k satisfies $\hat{k} \in H^s(\mathbb{R}^2)$, for $s \in [0,1)$.*

Proof. First, prove that $\hat{k} \in L^2(\mathbb{R}^2)$. Actually, we have

$$\int_{\mathbb{R}^2} \frac{\xi_1^4}{(c\xi_1^2 - |\xi_1|^3 + \xi_1^4 + \xi_2^2)^{1/2}} = \int_{\mathbb{R}} \frac{1}{(c-|\xi_1|+\xi_1^2)^2}\Big[\int_{\mathbb{R}} \frac{d\xi_2}{\Big(1+\frac{\xi_2^2}{\xi_1^2(c-|\xi_1|+\xi_1^2)}\Big)^2}\Big]d\xi_1$$
$$= \int_{\mathbb{R}} \frac{|\xi_1|d\xi_1}{(c-|\xi_1|+\xi_1^2)^{3/2}}\int_{\mathbb{R}}\frac{d\xi_2}{(1+\xi_2^2)^2} < +\infty.$$

On the other hand, it is easy to check that $|\nabla\hat{k}| \le C|\hat{h}|$. Since $\hat{h} \in L^q(\mathbb{R}^2)$, for $q \in (1,2)$, we have $\partial_{\xi_1}\hat{k}$, $\partial_{\xi_2}\hat{k} \in L^q(\mathbb{R}^2)$, for $q \in (1,2)$; that is, $\hat{k}$ belongs to the homogeneous Sobolev space $\dot{H}^1_q(\mathbb{R}^2)$. By Bergh and Löfstrom (see Theorem 6.5.1 of [5]), we have $\dot{H}^1_q(\mathbb{R}^2) \subset \dot{H}^s_2(\mathbb{R}^2)$, for $s = 1 - \frac{1}{q}$. Thus, $\hat{k} \in \dot{H}^s_2(\mathbb{R}^2)$ for $s \in [0,1)$. Finally, $\hat{k} \in H^s(\mathbb{R}^2)$, for $s \in [0,1)$. □

From this lemma, we deduce the following result.

Lemma 6.5. *For $\delta \in [0,1)$, we have $|y|^\delta u \in L^2(\mathbb{R}^2)$.*

Proof. From the convolution equation (6.1), we can deduce

$$\Big||y|^\delta u\Big| \le C\big||y|^\delta k * u^2\big| + C\big||y|^\delta u^2 * k\big|.$$

Thanks to Lemma 6.4, we have for any $\delta \in [0,1)$

$$\||y|^\delta k * u^2\|_{L^2} \le C\|\hat{k}\|_{H^\delta}\|u\|^2_{L^2} \le C.$$

Observe now that $k \in L^q(\mathbb{R}^2)$, for $q \in (1,2]$. In fact, we have

$$\|k\|_{L^q} \le \|(1+r^2)^{s/2}k\|_{L^2}\Big\|\frac{1}{(1+r^2)^{s/2}}\Big\|_{L^\alpha},$$

where α is such that $\frac{1}{q} = \frac{1}{2} + \frac{1}{\alpha}$. Then, for $q \in (1,2]$, we may choose $\alpha \in (2,+\infty)$, such that $s\alpha > 2$. This implies $\|k\|_{L^q} \le C$ for $q \in (1,2]$. Using this fact, we get

$$\||y|^\delta u^2 * k\|_{L^2} \le \||y|^\delta u^2\|_{L^\beta}\|k\|_{L^q},$$

with β satisfying $\frac{1}{q} + \frac{1}{\beta} = \frac{3}{2}$, $\beta \in [1,2)$. On the other hand, Lemma 6.3 implies that

$$\Big|\int_{\mathbb{R}^2} |y|^{\delta\beta}u^{2\beta}\Big| \le \||y|^{\delta\beta}u^{2\beta-2}\|_{L^\infty}\|u\|^2_{L^2} \le C\|u\|^2_{L^2},$$

provided $\delta\beta \leq 2\beta - 2$. Then, by choosing $\beta > \frac{2}{2-\delta}$, we get $\||y|^\delta u^2 * k\|_{L^2} \leq C$. □

Let us now prove a decay estimate of the kernel k.

Lemma 6.6. *We have* $r^2 k \in L^\infty(\mathbb{R}^2)$.

Proof. Using the definition of k, we may write

$$k(x,y) = \int_{\mathbb{R}^2} \frac{\xi_1^2 e^{ix\xi_1 + iy\xi_2}}{c\xi_1^2 + \xi_2^2 - |\xi_1|^3 + \xi_1^4} d\xi_1 d\xi_2.$$

As in the proof of Lemma 6.2, we find

$$k(x,y) = \int_{\mathbb{R}} \frac{|\xi|}{(c - |\xi| + \xi^2)^{1/2}} e^{-|y||\xi|(c-|\xi|+\xi^2)^{1/2}} e^{ix\xi} d\xi.$$

For $y = 0$, we have

$$k(x,0) = \int_{-\infty}^{+\infty} \frac{|\xi|}{(c - |\xi| + \xi^2)^{1/2}} e^{ix\xi} d\xi.$$

Let us show that $x^2 k(x,0)$ is bounded. Then, it suffices to prove that

$$\mathcal{F}_x^{-1}\Big(\frac{d^2}{d\xi^2} R(\xi)\Big) \in L^\infty(\mathbb{R}),$$

where $R(\xi) = \frac{|\xi|}{(c-|\xi|+\xi^2)^{1/2}}$. A simple computation shows that $R"(\xi) = \frac{2}{\sqrt{c}}\delta + g(\xi)$, where $g \in L^1(\mathbb{R})$.

For $y \neq 0$, we set

$$k_1(x,y) = \int_0^{+\infty} \frac{\xi}{(c - \xi + \xi_1^2)^{1/2}} e^{K(\xi)} d\xi,$$

with $K(\xi) = ix\xi - |y|\xi(c - \xi + \xi^2)^{1/2}$. There are two cases to be considered.

- If $c > \frac{9}{32}$, we have $|K'(\xi)| > 0$. Then, by integration by parts, we get

$$\begin{aligned} k_1(x,y) &= -\int_0^{+\infty} \frac{d}{d\xi}\Big[\frac{\xi}{(c-\xi+\xi^2)^{1/2} K'(\xi)}\Big] e^{K(\xi)} d\xi \\ &= -\int_0^{+\infty} \frac{e^{K(\xi)}}{ix(c-\xi+\xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)} d\xi \\ &+ \int_0^{+\infty} \frac{\xi[ix\xi - ix/2 - |y|(4\xi - 3/2)(c-\xi+\xi^2)^{1/2}]e^{K(\xi)}}{(c-\xi+\xi^2)^{1/2}[ix(c-\xi+\xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]^2} d\xi = I_1 + I_2. \end{aligned}$$

We integrate I_1 by parts to obtain

$$I_1 = -\Big[\frac{(c-\xi+\xi^2)^{1/2} e^{K(\xi)}}{[ix(c-\xi+\xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]^2}\Big]_0^{+\infty} + \int_0^{+\infty} \frac{d}{d\xi} H_1(\xi) e^{K(\xi)} d\xi,$$

where

$$H_1(\xi) = \frac{(c-\xi+\xi^2)^{1/2}}{[ix(c-\xi+\xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]^2}.$$

Thus, we write

$$I_1 = \frac{\sqrt{c}}{(ix\sqrt{c} - c|y|)^2} + \int_0^{+\infty} \frac{d}{d\xi} H_1(\xi) e^{K(\xi)} d\xi,$$

with

$$\begin{aligned}\frac{d}{d\xi} H_1(\xi) &= \frac{(2\xi - 1)}{2(c-\xi+\xi^2)^{1/2}} \frac{1}{[ix(c-\xi+\xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]^2} \\ &\quad - \frac{ix(2\xi-1) - 2|y|(4\xi - 3/2)(c-\xi+\xi^2)^{1/2}}{[ix(c-\xi+\xi^2)^{1/2} - |y|(c-\frac{3\xi}{2}+2\xi^2)]^3} = F_1 + F_2.\end{aligned}$$

Since $c > \frac{9}{32}$, F_1 and F_2 satisfy

$$|F_1(\xi)|^2 \le \frac{C}{(c-\xi+\xi^2)^2(x^2+y^2)^2}, \qquad |F_2(\xi)|^2 \le \frac{C}{(c-\frac{3\xi}{2}+2\xi^2)^2(x^2+y^2)^2}.$$

It follows that $I_1 \le \frac{C}{x^2+y^2}$, for $y \neq 0$.

For I_2, integrating by parts yields

$$I_2 = -\int_0^{+\infty} \frac{d}{d\xi} H_2(\xi) e^{K(\xi)} d\xi,$$

with $H_2(\xi) = \frac{\xi[ix\xi - ix/2 - |y|(4\xi-3/2)(c-\xi+\xi^2)^{1/2}]}{[ix(c-\xi+\xi^2)^{1/2} - |y|(c-\frac{3\xi}{2}+2\xi^2)]^3}$ and

$$\begin{aligned}\frac{d}{d\xi} H_2(\xi) &= \frac{ix(c-\xi+\xi^2)^{1/2}(2\xi - 1/2) - |y|(12\xi^3 - 13\xi^2 + 8c\xi + \frac{9}{4}\xi - \frac{3}{2}\xi)}{(c-\xi+\xi^2)^{1/2}[ix(c-\xi+\xi^2)^{1/2} - |y|(c-\frac{3\xi}{2}+2\xi^2)]^3} \\ &\quad - \frac{3\xi[ix\xi - ix/2 - |y|(4\xi-3/2)(c-\xi+\xi^2)^{1/2}]^2}{(c-\xi+\xi^2)^{1/2}[ix(c-\xi+\xi^2)^{1/2} - |y|(c-\frac{3\xi}{2}+2\xi^2)]^4} \\ &= G_1(\xi) + G_2(\xi).\end{aligned}$$

Since $c > \frac{9}{32}$, we can easily see that

$$|G_i(\xi)|^2 \le \frac{C}{(c-\frac{3\xi}{2}+2\xi^2)^2(x^2+y^2)^2}, \; i = 1, 2,$$

which implies that $|I_2| \le \frac{C}{x^2+y^2}$. All these estimates yield

$$|k_1(x,y)| \le \frac{C}{x^2+y^2}, \text{ for } y \neq 0.$$

• If $\frac{1}{4} < c \le \frac{9}{32}$, the function $f(\xi) = c - \frac{3\xi}{2} + 2\xi^2$ vanishes at some ξ_1, $\xi_2 > 0$. For $x \neq 0$, by integration by parts, we get

$$\begin{aligned}k_1(x,y) &= -\int_0^{+\infty} \frac{d}{d\xi}\Big[\frac{\xi}{(c-\xi+\xi^2)^{1/2} K'(\xi)}\Big] e^{K(\xi)} d\xi \\ &= -\int_0^{\delta_1} \ldots - \int_{\delta_1}^{\delta_2} \ldots - \int_{\delta_2}^{+\infty} \ldots = I_1 + I_2 + I_3,\end{aligned}$$

where $0 < \delta_1 < \xi_1 \le \xi_2 < \delta_2 < +\infty$. Using the fact that $\inf_{\xi \in [0,\delta_1]\cap[\delta_2,+\infty[} |K'(\xi)| > 0$ and reasoning as in the first case, we get

$$|I_1| + |I_3| \le \frac{C}{r^2}, \text{ for } y \ne 0.$$

For I_2, proceeding as in Lemma 6.2 (see the integral I_2, when $c \in (\frac{1}{4}, \frac{9}{32}])$, we integrate by parts and use the fact that $\inf_{\xi\in[\delta_1,\delta_2]} (c - \xi + \xi^2) > 0$ to obtain

$$(x^2 + y^2)|I_2| \le C, \text{ for } x, y \ne 0.$$

Now, for $x = 0$, split the integral defining k_1 as

$$\begin{aligned} k_1(0, y) &= \int_0^{+\infty} \frac{\xi}{(c - \xi + \xi_1^2)^{1/2}} e^{-|y|\xi(c-\xi+\xi^2)^{1/2}} d\xi \\ &= \int_0^{\delta_1} ... + \int_{\delta_1}^{\delta_2} ... + \int_{\delta_2}^{+\infty} ... = I_1' + I_2' + I_3'. \end{aligned}$$

Using the fact that $\inf_{\xi\in[0,\delta_1]\cap[\delta_2,+\infty)} |K'(\xi)| > 0$ and integrating by parts twice, we obtain $y^2\big(|I_1'| + |I_3'|\big) \le C$. For I_2', using (6.7), we get $I_2' \le Be^{-A|y|}$. Finally, we obtain

$$|k_1(x, y)| \le \frac{C}{r^2}, \text{ for } y \ne 0.$$

In a similar way, we can obtain

$$|k_2(x, y)| \le \frac{C}{r^2}, \text{ for } y \ne 0,$$

where

$$k_2(x, y) = -\int_{-\infty}^{0} \frac{\xi}{(c + \xi + \xi^2)^{1/2}} e^{K(\xi)} d\xi.$$

This completes the proof of Lemma 6.6.

Next, we show some algebraic decay of $|x|^l u$. To this aim, we shall use the preceding properties of $|y|^l u$ and k.

Lemma 6.7. *There exists $l_0 \in (0, \frac{1}{2})$ such that $|x|^{l_0} u \in L^2(\mathbb{R}^2)$.*

To prove this lemma, we proceed as in [7], Theorem 3.1.2. First, we begin by stating the next lemma proved in [7].

Lemma 6.8. *Let l and m be constants satisfying $0 < l < m - 1$. Then, there exists a constant $B > 0$ depending only on l and m such that the inequalities*

$$\int_0^{+\infty} \frac{x^l}{(1 + \epsilon x')^m (1 + |x - x'|)^m} dx' \le \frac{B|x|^l}{(1 + \epsilon|x|)^m}$$

and

$$\int_{-\infty}^{0} \frac{x^l}{(1 + \epsilon x')^m (1 + |x - x'|)^m} dx' \le \frac{B|x|^l}{(1 + \epsilon|x|)^m}$$

hold for any $\epsilon \in (0, 1]$ and $x \in \mathbb{R}$ such that $|x| > 1$.

Proof of Lemma 6.7. We define h_ϵ by

$$h_\epsilon(x,y) = \frac{|x|^l}{(1+\epsilon|x|)^s(1+|y|)^{1/2+\eta}}u(x,y),$$

where $0 < \eta \ll 1$, $1/2 < s < 1$ and l is chosen such that $0 < l < s - 1/2$. Since $u \in L^\infty(\mathbb{R}^2)$ and $\frac{|x|^l}{(1+\epsilon|x|)^s(1+|y|)^{1/2+\eta}} \in L^2(\mathbb{R}^2)$, we have $h_\epsilon \in L^2(\mathbb{R}^2)$. Now, by Lemma 6.5 and the fact that $u \in H^\infty(\mathbb{R}^2)$, we deduce that, for $\delta > 0$, there exists a constant $N \geq 1$ such that

$$\Big|(1+|y|)^{1/2+\eta}u(x,y)\Big| < \delta, \tag{6.9}$$

for $|(x,y)| \geq N$. Choose δ such that $\delta < \frac{1}{4B\|\hat{k}\|_s}$ (B is the constant appearing in Lemma 6.8) and let N be such that (6.9) holds. Then, by using the convolution equation (6.1), Hölder's inequality, and Lemma 6.4, we get

$$\begin{aligned}
&\int_{A_1} |h_\epsilon(x,y)|^2 dxdy \\
&\leq \int_{A_1} |h_\epsilon(x,y)| \frac{|x|^l}{(1+\epsilon|x|)^s(1+|y|)^{\frac{1}{2}+\eta}} \Big[\int_{\mathbb{R}^2} |k(x-x',y-y')u^2(x',y')|dx'dy'\Big]dxdy \\
&\leq \int_{A_1} |h_\epsilon(x,y)| \frac{|x|^l}{(1+\epsilon|x|)^s(1+|y|)^{\frac{1}{2}+\eta}} \\
&\Big[\Big(\int_{\mathbb{R}^2}(1+|x-x'|)^{2s}|k(x-x',y-y')|^2dx'dy'\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2}\frac{|u^2(x',y')|^2}{(1+|x-x'|)^{2s}}dx'dy'\Big)^{\frac{1}{2}}\Big]dxdy \\
&\leq \|\hat{k}\|_s\Big(\int_{A_1} |h_\epsilon(x,y)|^2 dxdy\Big)^{\frac{1}{2}}\Big(\int_{A_1}\frac{|x|^{2l}}{(1+\epsilon|x|)^{2s}(1+|y|)^{1+2\eta}} \\
&\qquad \times \Big[\int_{\mathbb{R}^2}\frac{|u^2(x',y')|2}{(1+|x-x'|)^{2s}}dx'dy'\Big]dxdy\Big)^{\frac{1}{2}},
\end{aligned}$$

where $A_1 = [N,+\infty) \times (-\infty,+\infty]$. Applying Fubini's theorem and Lemma 6.8 yields

$$\begin{aligned}
\Big(\int_{A_1} |h_\epsilon(x,y)|^2dxdy\Big)^{1/2} &\leq \|\hat{k}\|_s\Big(\int_{\mathbb{R}^2}|u^2(x',y')|^2 \\
&\quad \times \Big[\int_{A_1}\frac{x^{2l}}{(1+\epsilon x)^{2s}(1+|x-x'|)^{2s}(1+|y|)^{1+2\eta}}dxdy\Big]dx'dy'\Big)^{1/2} \\
&\leq B^{1/2}\|\hat{k}\|_s\Big(\Big(\int_{-\infty}^{-N}+\int_N^{+\infty}\Big)\int_{\mathbb{R}_{y'}}|u^2(x',y')|^2\frac{|x'|^{2l}}{(1+\epsilon|x'|)^{2s}}dy'dx'\Big)^{1/2} \\
&\quad + \|\hat{k}\|_s\Big(\int_{-N}^{N}\int_{\mathbb{R}_y}|u^2(x',y')|^2 \\
&\quad \times \Big[\int_{A_1}\frac{x^{2l}}{(1+\epsilon x)^{2s}(1+|x-x'|)^{2s}(1+|y|)^{1+2\eta}}dxdy\Big]dy'dx'\Big)^{1/2}.
\end{aligned}$$

Using (6.9), we obtain

$$\begin{aligned}
&\Big(\int_{A_1}|h_\epsilon(x,y)|^2dxdy\Big)^{1/2}\\
&\le \delta B^{1/2}\|\hat{k}\|_s\Big(\Big(\int_{-\infty}^{-N}+\int_N^{+\infty}\Big)\int_{\mathbb{R}_{y'}}|u(x',y')|^2\frac{|x'|^{2l}}{(1+\epsilon|x'|)^{2s}(1+|y|)^{1+2\eta}}dy'dx'\Big)^{1/2}\\
&+\|\hat{k}\|_s\Big(\int_{-N}^{N}\int_{\mathbb{R}_{y'}}|u^2(x',y')|^2\Big[\int_{\mathbb{R}^2}\frac{(|x|+2N)^{2l}}{(1+|x|)^{2s}(1+|y|)^{1+2\eta}}dxdy\Big]dy'dx'\Big)^{1/2}\\
&=\delta B^{1/2}\|\hat{k}\|_s\Big(\Big(\int_{-\infty}^{-N}+\int_N^{+\infty}\Big)\int_{\mathbb{R}_{y'}}|h_\epsilon(x',y')|^2dy'dx'\Big)^{1/2}\\
&+M\|\hat{k}\|_s\Big(\int_{-N}^{N}\int_{\mathbb{R}_y}|u^2(x',y')|^2dy'dx'\Big)^{1/2},
\end{aligned}$$

with $M=\int_{\mathbb{R}^2}\frac{(|x|+2N)^{2l}}{(1+|x|)^{2s}(1+|y|)^{1+2\eta}}dxdy$. A similar computation yields the inequality

$$\begin{aligned}
&\Big(\int_{A_2}|h_\epsilon(x,y)|^2dxdy\Big)^{\frac12}\le \delta B^{1/2}\|\hat{k}\|_s\Big(\Big(\int_{-\infty}^{-N}+\int_N^{+\infty}\Big)\int_{\mathbb{R}_{y'}}|h_\epsilon(x',y')|^2dy'dx'\Big)^{\frac12}\\
&+M\|\hat{k}\|_s\Big(\int_{-N}^{N}\int_{\mathbb{R}_y}|u^2(x',y')|^2dy'dx'\Big)^{\frac12},
\end{aligned}$$

where $A_2=(-\infty,-N)\times(-\infty,+\infty)$. Adding these inequalities leads to the estimate

$$\begin{aligned}
&\Big(\int_{A_2}|h_\epsilon(x,y)|^2dxdy\Big)^{1/2}+\Big(\int_{A_1}|h_\epsilon(x,y)|^2dxdy\Big)^{1/2}\\
&\le 2\delta B^{1/2}\|\hat{k}\|_s\Big(\Big(\int_{-\infty}^{-N}+\int_N^{+\infty}\Big)\int_{\mathbb{R}_{y'}}|h_\epsilon(x',y')|^2dy'dx'\Big)^{1/2}\\
&+2M\|\hat{k}\|_s\Big(\int_{-N}^{N}\int_{\mathbb{R}_y}|u^2(x',y')|^2dy'dx'\Big)^{1/2}.
\end{aligned}$$

Now, the choice of δ implies

$$\begin{aligned}
&\Big(\int_{A_2}|h_\epsilon(x,y)|^2dxdy\Big)^{1/2}+\Big(\int_{A_1}|h_\epsilon(x,y)|^2dxdy\Big)^{1/2}\\
&\le\frac{2M\|\hat{k}\|_s}{1-2\delta B^{1/2}\|\hat{k}\|_s}\Big(\int_{\mathbb{R}_{y'}}\int_{-N}^{N}|u^2(x',y')|^2dx'dy'\Big)^{1/2}.
\end{aligned}$$

Note that the constants appearing in the right-hand side of the preceding estimate are independent of ϵ. Let $\epsilon\to 0$ and apply Fatou's lemma to obtain

$$\Big(\int_{-\infty}^{-N}+\int_N^{+\infty}\Big)\int_{\mathbb{R}_y}\frac{|x|^{2l}}{(1+|y|)^{1+2\eta}}|u(x,y)|^2dydx\le C.$$

On the other hand, we have

$$\int_{-N}^{N}\int_{\mathbb{R}_y}\frac{|x|^{2l}}{(1+|y|)^{1+2\eta}}|u(x,y)|^2dydx \leq N^2\|u\|_{L^2}^2.$$

These last two estimates yield that $\frac{|x|^l}{(1+|y|)^{1/2+\eta}}u \in L^2(\mathbb{R}^2)$. By Lemma 6.5, we have $(1+|y|)^{1/2+\eta}u \in L^2(\mathbb{R}^2)$. Then, we deduce that $|x|^{l/2}u \in L^2(\mathbb{R}^2)$. Taking $l_0 = l/2$ shows the result. □

From this lemma, we deduce the next result.

Lemma 6.9. *We have* $|x|^l u \in L^\infty(\mathbb{R}^2)$, $\forall l,\ 0 \leq l \leq l_0$.

Proof. From (6.1), we obtain

$$\begin{aligned}\Big||x|^l u(x,y)\Big| &\leq C\int_{\mathbb{R}^2}|k(x-x',y-y')||x-x'|^l|u^2(x',y')|dx'dy' \\ &\quad + C\int_{\mathbb{R}^2}|k(x-x',y-y')|x'|^l|u^2(x',y')|dx'dy'.\end{aligned} \tag{6.10}$$

Note that we have $l_0 < 1$. Then, by using Lemma 6.4 and Hölder's inequality, the first term of the right-hand side of (6.10) belongs to $L^\infty(\mathbb{R}^2)$. For the second term, we use Hölder's inequality and the facts that $k \in L^2(\mathbb{R}^2)$, $|x|^l u \in L^2(\mathbb{R}^2)$, and $u \in L^\infty(\mathbb{R}^2)$. □

We are now ready to prove the algebraic decay of u.

Theorem 6.1. *We have* $|(x,y)|^2 u \in L^\infty(\mathbb{R}^2)$, *with* $|(x,y)| = \sqrt{x^2+y^2}$.

Proof. Let us first prove that $|(x,y)|^l u \in L^\infty$, for $l \in [0,2] \cap [0,2l_0)$. Actually, by the convolution equation (6.1), we get

$$\begin{aligned}|(x,y)|^l|u(x,y)| &\leq C\int_{\mathbb{R}^2}|k(x-x',y-y')||(x-x',y-y')|^l|u^2(x',y')|dx'dy' \\ &\quad + C\int_{\mathbb{R}^2}|k(x-x',y-y')||(x',y')|^l|u^2(x',y')|dx'dy'.\end{aligned} \tag{6.11}$$

By applying Lemma 6.6 and Hölder's inequality, we obtain that, for $l \in [0,2] \cap [0,2l_0)$,

$$\begin{aligned}&\int_{\mathbb{R}^2}|k(x-x',y-y')||(x-x',y-y')|^l|u^2(x',y')|dx'dy' \\ &\leq \int_{B((x,y),1)^c}|(x-x',y-y')|^{-2}|(x-x',y-y')|^l|u^2(x',y')|dx'dy' \\ &+ \int_{B((x,y),1)}|k(x-x',y-y')||(x-x',y-y')|^l|u^2(x',y')|dx'dy' \\ &\leq C\|u\|_{L^2}^2 + \|k\|_{L^2}\|u\|_{L^\infty}^2 \leq C.\end{aligned}$$

For the second term in the right-hand side of (6.11), we use Lemma 6.6 to get

$$\begin{aligned}&\int_{\mathbb{R}^2}|k(x-x',y-y')||(x',y')|^l|u^2(x',y')|dx'dy'\\&\le\int_{B((x,y),1)^c}|(x-x',y-y')|^{-2}|(x',y')|^l|u^2(x',y')|dx'dy'\\&+\int_{B((x,y),1)}|k(x-x',y-y')||(x',y')|^l|u^2(x',y')|dx'dy'.\end{aligned}$$

Applying Hölder's inequality and Lemmas 6.3 and 6.9, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}|k(x-x',y-y')||(x',y')|^l|u^2(x',y')|dx'dy'\\&\le\Big(\int_{B(0,1)^c}\frac{1}{|(x,y)|^{2q}}dxdy\Big)^{1/q}\Big(\int_{B((x,y),1)^c}|(x',y')|^{lq'}|u^2(x',y')|^{q'}dx'dy'\Big)^{1/q'}\\&+\|k\|_{L^2}\||(x,y)|^{l/2}u\|_{L^\infty}^2\\&\le\Big(\int_{B(0,1)^c}\frac{1}{|(x,y)|^{2q}}dxdy\Big)^{1/q}\Big(\int_{\mathbb{R}^2}|(x',y')|^{lq'}|u^2(x',y')|^{q'}dx'dy'\Big)^{1/q'}+C,\end{aligned}$$

where q, q' are such that $q>1$ and $\frac{1}{q}+\frac{1}{q'}=1$. On the other hand, by choosing a suitable q, we obtain that $|(x,y)|^lu^2\in L^{q'}$. Indeed,

$$\begin{aligned}&\int_{\mathbb{R}^2}|(x',y')|^{lq'}|u^2(x',y')|^{q'}dx'dy'\\&\le\int_{B(0,1)}|u^2(x',y')|^{lq'}dx'dy'+\int_{B(0,1)^c}|(x',y')|^{l'q'}|u^2(x',y')|^{q'}\frac{dx'dy'}{|(x',y')|^{(l-l')q'}},\end{aligned}$$

where $l'\in[0,2]\cap[0,2l_0)$ and l', q' are chosen such that $(l-l')q'>2$. Then, by Hölder's inequality and Lemmas 6.3 and 6.9, we get

$$\int_{\mathbb{R}^2}|(x',y')|^{lq'}|u^2(x',y')|^{q'}dx'dy'\le C+C\||(x,y)|^{l'/2}u\|_{L^\infty}^{2q'}\le C.$$

Finally, we obtain that $|(x,y)|^lu\in L^\infty(\mathbb{R}^2)$ for $l\in[0,2]\cap[0,2l_0)$. Note that, in the previous arguments, we have used essentially the properties of the kernel k and the fact that $|(x,y)|^lu\in L^\infty$ for $l\in[0,2]\cap[0,2l_0)$. Therefore, by a straightforward induction argument, it follows that $|(x,y)|^lu$ belongs to $L^\infty(\mathbb{R}^2)$ for $l\in[0,2]\cap[0,nl_0)$, for all $n\in\mathbb{N}$. Thus, for n sufficiently large, we obtain that $|(x,y)|^lu\in L^\infty$ for $l\in[0,2]$. □

In order to get the desired limit of r^2u when r tends to infinity, we have to prove some estimates of ∇u and $|D_x|u$. Thus, we write $|D_x|u$ and ∇u as

$$|D_x|u=-M*\frac{u^2}{2},\ \partial_xu=iHM*\frac{u^2}{2},\ \partial_yu=-iN*\frac{u^2}{2},\tag{6.12}$$

where H is the Hilbert transform operator and M, N are defined by

$$\widehat{M}(\xi_1,\xi_2)=\frac{|\xi_1|^3}{c|\xi_1|^2-|\xi_1|^3+\xi_1^4+\xi_2^2},\ \widehat{N}(\xi_1,\xi_2)=\frac{\xi_1^2\xi_2}{c|\xi_1|^2-|\xi_1|^3+\xi_1^4+\xi_2^2}.$$

We now state some properties of the kernels M and N.

Lemma 6.10. *We have r^3M, r^3HM and $r^3N\in L^\infty(B(0,1)^c)$.*

Proof. The proof of these properties is similar to that of the kernels h and k in Lemmas 6.2 and 6.6. Note that we will suppose that $c>\frac{9}{32}$ in the sequel. The proof in the other cases is analogous to that of Lemmas 6.2 and 6.6.

(i) We begin by proving that $r^3M\in L^\infty(B(0,1)^c)$. Using the inverse Fourier transform, we write

$$M(x,y)=\int_{\mathbb{R}^2}\frac{|\xi_1|^3}{c\xi_1^2+\xi_2^2-|\xi_1|^3+\xi_1^4}e^{ix\xi_1+iy\xi_2}d\xi_1d\xi_2.$$

By the same argument of Lemma 6.3, we get

$$M(x,y)=\int_{\mathbb{R}}\frac{\xi^2}{(c-|\xi|+\xi^2)^{1/2}}e^{-|y||\xi|(c-|\xi|+\xi^2)^{1/2}}e^{ix\xi}d\xi.$$

For $y=0$, we have

$$M(x,0)=\int_{-\infty}^{+\infty}\frac{\xi^2}{(c-|\xi|+\xi^2)^{1/2}}e^{ix\xi}d\xi.$$

It suffices to prove that $\mathcal{F}_x^{-1}\left(\frac{d^3}{d\xi^3}R(\xi)\right)\in L^\infty(\mathbb{R})$, where $R(\xi)=\frac{\xi^2}{(c-|\xi|+\xi^2)^{1/2}}$. Indeed, a simple computation shows that $\frac{d^3}{d\xi^3}R(\xi)\in L^1(\mathbb{R}^2)$.

For $y\neq 0$, we consider

$$M_1(x,y)=\int_0^{+\infty}\frac{\xi^2}{(c-\xi+\xi_1^2)^{1/2}}e^{K(\xi)}d\xi,$$

with $K(\xi)=ix\xi-|y|\xi(c-\xi+\xi^2)^{1/2}$. By integration by parts, we obtain

$$\begin{aligned}M(x,y)=&-\int_0^{+\infty}\frac{2\xi}{ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)}e^{K(\xi)}d\xi\\&+\int_0^{+\infty}\frac{\xi^2[ix(\xi-1/2)-|y|(4\xi-3/2)(c-\xi+\xi^2)^{1/2}]}{c-\xi+\xi^2)^{1/2}[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^2}e^{K(\xi)}d\xi\\&=I_1+I_2.\end{aligned}$$

For I_1, by integration by parts, we get

$$\begin{aligned}I_1=&\int_0^{+\infty}\frac{2(c-\xi+\xi^2)^{1/2}e^{K(\xi)}}{[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^2}d\xi\\&+\int_0^{+\infty}\frac{\xi(2\xi-1)e^{K(\xi)}}{(c-\xi+\xi^2)^{1/2}[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^2}d\xi\end{aligned}$$

$$-\int_0^{+\infty}\frac{2\xi[ix(\xi-1/2)-|y|(4\xi-3/2)(c-\xi+\xi^2)^{1/2}]e^{K(\xi)}}{[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^3}d\xi=I_{11}+I_{12}+I_{13}.$$

Integrating by parts again yields

$$I_{11}=\frac{2c}{[ix\sqrt{c}-|y|c]^3}-\int_0^{+\infty}H_1(\xi)e^{K(\xi)}d\xi,$$

where

$$H_1(\xi)=\frac{d}{d\xi}\Big[\frac{2(c-\xi+\xi^2)}{[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^3}d\xi\Big].$$

By a simple calculation, we can easily check that (for more details, see [17])

$$|H_1(\xi)|^2\leq\frac{C}{(c-\xi+\xi^2)^2(x^2+y^2)^3}.$$

It follows that $I_{11}\leq\frac{C}{r^3}$. In the same way, we can prove $|I_{12}|+|I_{13}|\leq\frac{C}{r^3}$. Finally, we deduce that $|I_1|\leq\frac{C}{r^3}$, for $y\neq 0$.

For I_2, we proceed in the same way. We integrate by parts to get

$$\begin{aligned}I_2=&-\int_0^{+\infty}\frac{2\xi[ix(\xi-1/2)-|y|(4\xi-3/2)(c-\xi+\xi^2)^{1/2}]}{[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^3}e^{K(\xi)}d\xi\\&-\int_0^{+\infty}\frac{\xi^2[ix(c-\xi+\xi^2)^{1/2}-|y|(8\xi^2-\frac{15}{2}\xi+3/4+4c)]}{(c-\xi+\xi^2)^{1/2}[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^3}e^{K(\xi)}d\xi\\&+\int_0^{+\infty}\frac{3\xi^2[ix(\xi-1/2)-|y|(4\xi-3/2)(c-\xi+\xi^2)^{1/2}]^2}{(c-\xi+\xi^2)^{1/2}[ix(c-\xi+\xi^2)^{1/2}-|y|(c-\frac{3\xi}{2}+2\xi^2)]^4}e^{K(\xi)}d\xi\\&=I_{21}+I_{22}+I_{23}.\end{aligned}$$

Integrating by parts again yields

$$I_{2i}=\int_0^{+\infty}G_i(\xi)e^{K(\xi)d\xi},\ i=1,2,3,$$

where G_1, G_2 and G_3 satisfy (for detailed computation, see [17])

$$|G_i(\xi)|^2\leq\frac{C}{(c-\xi+\xi^2)^2(x^2+y^2)^3},\ \text{i=1, 2, 3.}$$

This implies that $|I_2|\leq\frac{C}{r^3}$, for $y\neq 0$. We conclude that $|M_1(x,y)|\leq\frac{C}{r^3}$, for $y\neq 0$.

In a similar way, we can prove that

$$|M_2(x,y)|\leq\frac{C}{r^3},\ \text{for } y\neq 0,$$

where

$$M_2(x,y)=\int_{-\infty}^0\frac{\xi^2}{(c+\xi+\xi^2)^{1/2}}e^{K(\xi)}d\xi.$$

(ii) For the kernel HM, the proof is still the same since $\widehat{HM}(\xi) = i\mathrm{sgn}(\xi)\widehat{M}(\xi)$. Except for the case $y = 0$, we have $\frac{d^3}{d\xi^3}R(\xi) = i(4/\sqrt{c}\delta + g(\xi))$, with $g \in L^1$.

(iii) For the kernel N, we use the same method. We will only state the difference between the proofs. We can write N as

$$N(x,y) = \int_{\mathbb{R}^2} \frac{\xi_1^2\xi_2}{c\xi_1^2 + \xi_2^2 - |\xi_1|^3 + \xi_1^4} e^{ix\xi_1 + iy\xi_2} d\xi_1 d\xi_2.$$

By the change of variables $\xi_2 = |\xi_1|(c - |\xi_1| + \xi_1^2)\xi_2'$ (see Lemma 6.2), we get

$$N(x,y) = \int_{\mathbb{R}} \frac{\xi^2|\xi|}{\xi^2(c - |\xi| + \xi^2)^{\frac{1}{2}}} \partial_y \Big[e^{-|y||\xi|(c-|\xi|+\xi^2)^{\frac{1}{2}}} \Big] e^{ix\xi} d\xi = \mathrm{sgn}(y) \int_{\mathbb{R}} \xi^2 e^{K(\xi)} d\xi.$$

For $y = 0$, we have $N(x,0) = \int_{\mathbb{R}} \xi^2 e^{ix\xi}$. Since $\frac{d^3}{d\xi^3}(\xi^2) = 0 \in L^1(\mathbb{R})$, we deduce that $x^3|N(x,0)| \leq C$.

For $y \neq 0$, let

$$N_1(x,y) = \int_0^{+\infty} \xi^2 e^{K(\xi)} d\xi.$$

After an integration by parts, we obtain

$$\begin{aligned} N_1(x,y) = &- \int_0^{+\infty} \frac{2\xi(c - \xi + \xi^2)^{1/2}}{[ix(c - \xi + \xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]} e^{K(\xi)} \\ &- \int_0^{+\infty} \frac{\xi^2(\xi - 1/2)}{[(c - \xi + \xi^2)^{1/2} ix(c - \xi + \xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]} e^{K(\xi)} d\xi \\ &+ \int_0^{+\infty} \frac{\xi^2[ix(\xi - 1/2) - |y|(4\xi - 3/2)(c - \xi + \xi^2)^{1/2}]}{[ix(c - \xi + \xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]^2} e^{K(\xi)} d\xi \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For I_2 and I_3, we integrate by parts four times to obtain $r^4(I_2 + I_3) \leq C$.

For I_1, it follows by integration by parts that

$$\begin{aligned} I_1 = &\int_0^{\infty} \frac{2(c - \xi + \xi^2)}{[ix(c - \xi + \xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]^2} e^{K(\xi)} d\xi \\ &+ \int_0^{\infty} \frac{\xi(2\xi - 1)}{[ix(c - \xi + \xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]^2} e^{K(\xi)} d\xi \\ &- \int_0^{+\infty} \frac{4\xi(c - \xi + \xi^2)^{1/2}[ix(\xi - 1/2) - |y|(4\xi - 3/2)(c - \xi + \xi^2)^{1/2}]}{[ix(c - \xi + \xi^2)^{1/2} - |y|(c - \frac{3\xi}{2} + 2\xi^2)]^3} \\ &= I_{11} + I_{12} + I_{13}. \end{aligned}$$

After several integrations by parts, we obtain $r^3|I_1| \leq C$. Finally, it follows that

$$r^3|N_1(x,y)| \leq C, \text{ for } r = (x^2 + y^2)^{1/2} > 1.$$

By similar arguments, we can prove that

$$r^3|N_2(x,y)| \leq C, \text{ for } r = (x^2 + y^2)^{1/2} > 1,$$

where

$$N_2(x,y) = \int_{-\infty}^{0} \xi^2 e^{K(\xi)} d\xi. \qquad \square$$

We now prove a decay estimate of the gradient.

Lemma 6.11. *We have $r\nabla u \in L^\infty(\mathbb{R}^2)$.*

Proof. From the convolution equation (6.1), we may write ∇u as $\nabla u = -k * \nabla \frac{u^2}{2}$. It follows that

$$\begin{aligned} |(x,y)||\nabla u(x,y)| &\le C \int_{\mathbb{R}^2} |(x,y)||k(x-x',y-y')\nabla u^2(x',y')|dx'dy' \qquad (6.13)\\ &\le C \int_{\mathbb{R}^2} |(x-x',y-y')||k(x-x',y-y')u\nabla u(x',y')|dx'dy' \\ &+ C \int_{\mathbb{R}^2} |(x',y')||k(x-x',y-y')u\nabla u(x',y')|dx'dy'. \end{aligned}$$

Applying Hölder's inequality and Lemmas 6.4 and 6.6, we get

$$\begin{aligned} &\int_{\mathbb{R}^2} |(x-x',y-y')||k(x-x',y-y')u\nabla u(x',y')|dx'dy' \\ &\le \int_{B((x,y),1)^c} |(x-x',y-y')||k(x-x',y-y')u\nabla u(x',y')|dx'dy' \\ &\int_{B((x,y),1)} |(x-x',y-y')||k(x-x',y-y')u\nabla u(x',y')|dx'dy' \\ &\le C\|u\|_{L^2}\|\partial_x u\|_{L^2} + \|k\|_{L^2}\|u\|_{L^\infty}\|\partial_x u\|_{L^2}. \end{aligned}$$

For the second term in the right-hand side of (6.13), we use Theorem 6.1 and Lemma 6.4 to obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} |(x',y')||k(x-x',y-y')u\nabla u(x',y')|dx'dy' \qquad (6.14)\\ &\le C\|ru\|_{L^\infty} \int_{\mathbb{R}^2} |k(x-x',y-y')||\nabla u(x',y')|dx'dy' \le c\|k\|_{L^2}\|\nabla u\|_{L^2}. \end{aligned}$$

This completes the proof of Lemma 6.11. $\square$

We now prove the bounds of $r^3|D_x|u$ and $r^3\nabla u$, which are useful to compute the limit of r^2u, when r goes to infinity.

Lemma 6.12. *We have $r^3|D_x|u$, $r^3\nabla u \in L^\infty(\mathbb{R}^2)$.*

Proof. Let us prove first that $r^3|D_x|u \in L^\infty(\mathbb{R}^2)$. We may write $|D_x|u$ as $|D_x|u = -Hk * \partial_x \frac{u^2}{2}$. Then, we have

$$\begin{aligned} |(x,y)|^3||D_x|u(x,y)| &\le |(x,y)|^3 \Big| \int_{\mathbb{R}^2} Hk(x-x',y-y')\partial_x \frac{u^2}{2}(x',y')dx'dy' \Big| \\ &\le |(x,y)|^3 \Big| \int_{B((x,y),1)^c} Hk(x-x',y-y')\partial_x \frac{u^2}{2}(x',y')dx'dy' \Big| \end{aligned}$$

$$+\,|(x,y)|^3\Big|\int_{B((x,y),1)} Hk(x-x',y-y')u\partial_x u(x',y')dx'dy'\Big|.$$

Using the facts that the kernels k and M are continuous on $B(0,1)^c$ and $-i\xi\widehat{Hk}=\widehat{M}$, it follows by integration by parts that

$$\begin{aligned}
&\int_{B((x,y),1)^c} Hk(x-x',y-y')\partial_x u^2(x',y')dx'dy'\\
&=-\int_{B((x,y),1)^c} M(x-x',y-y')\partial_x u^2(x',y')dx'dy'\\
&-\int_{\mathbb{S}^1} Hk(\sigma_1,\sigma_2)\sigma_1 u^2(x-\sigma_1,y-\sigma_2)d\sigma_1 d\sigma_2.
\end{aligned}$$

Thus, by Lemma 6.10, Theorem 6.1 and Hölder's inequality, we obtain

$$\begin{aligned}
&|(x,y)|^3\Big|\int_{B((x,y),1)^c} Hk(x-x',y-y')\partial_x u^2(x',y')dx'dy'\Big|\\
&\le C\Big|\int_{B((x,y),1)^c}(|(x-x',y-y')|^3+|(x',y')|^3)M(x-x',y-y')u^2(x',y')dx'dy'\Big|\\
&+C\Big|\int_{\mathbb{S}^1}(|(x-\sigma_1,y-\sigma_2)|^3+|(\sigma_1,\sigma_2)|^3)Hk(\sigma_1,\sigma_2)\sigma_1 u^2(x-\sigma_1,y-\sigma_2)d\sigma_1 d\sigma_2\Big|\\
&\le C\Big(\|u\|_{L^2}+\|r^{3/2}u\|_{L^\infty}^2\int_{\mathbb{R}^2}\frac{1}{|z|^3}dz+\|r^{1/2}u\|_{L^\infty}^2+\|u\|_{L^\infty}^2\Big)\le C.
\end{aligned}$$

Using now Lemmas 6.4 and 6.11, Theorem 6.1 and Hölder's inequality yields

$$\begin{aligned}
&|(x,y)|^3\Big|\int_{B((x,y),1)} Hk(x-x',y-y')u\partial_x u(x',y')dx'dy'\Big|\\
&\le\Big|\int_{B((x,y),1)}|(x-x',y-y')|^3 Hk(x-x',y-y')u\partial_x u(x',y')dx'dy'\Big|\\
&+C\Big|\int_{B((x,y),1)}|(x',y')|^3 Hk(x-x',y-y')u\partial_x u(x',y')dx'dy'\Big|\\
&\le\|k\|_{L^2}\|u\|_{L^\infty}\|\partial_x u\|_{L^2}+\|k\|_{L^2}\|r^2u\|_{L^\infty}\|r\partial_x u\|_{L^\infty}.
\end{aligned}$$

Combining all these estimates, we deduce that $r^3|D_x|u\in L^\infty$. By using the same technique and Lemma 6.10, one can easily see that $r^3\nabla u\in L^\infty$. □

Remark 6.1. Note that we may assume that $c=1$ by making the scale change $\tilde{u}(x,y)=c^{-1}u(c^{-1/2}x,c^{-1}y)$. Then, (1.5) may be written

$$-\tilde{u}_x+\tilde{u}_{xxx}-c^{-1/2}H\tilde{u}_{xx}+(u^2)_x-\partial_x^{-1}\tilde{u}_{yy}=0, \tag{6.15}$$

for $\epsilon=-1$. Let $\gamma=c^{-1/2}$. Then the condition $c>1/4$ becomes $0<\gamma<2$. Therefore, we shall use equation (6.15) in what follows.

To obtain the desired result, we shall use some properties of the kernels associated to the KPI equation. As noticed above, we write u as

$$u = -k_0 * \frac{u^2}{2} + \gamma k_0 * |D_x| u. \tag{6.16}$$

We next recall some properties of the kernel k_0 proved in [10].

Lemma 6.13. *(i) We have $r^2 k_0 \in L^\infty(\mathbb{R}^2)$. (ii) Let $\sigma \in \mathbb{S}^1$ and $\mathbf{x} \in \mathbb{R}^2$. Then*

$$\lim_{r \to +\infty} r^2 k_0(r\sigma - \mathbf{x}) = \frac{\Gamma(1)}{2\pi}(1 - 2\sigma_1^2), \text{ where } \Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1.$$

Using the previous lemmas, we are now able to show the convergence result.

Theorem 6.2. *Let u be a solution of equation (6.15) and $u_\infty \in C^\infty(\mathbb{S}^1)$ be defined by*

$$u_\infty(\sigma) = \frac{1}{2\pi}(1 - 2\sigma_1^2) \int_{\mathbb{R}^2} u^2(\mathbf{x}) d\mathbf{x},$$

for $\sigma = (\sigma_1, \sigma_2) \in \mathbb{S}^1$. Then, $r^2 u(r\cdot)$ converges to u_∞ in $L^\infty(\mathbb{S}^1)$.

Proof of Theorem 6.2. The proof of this theorem is analogous to that of Theorem 1 in [10]. We begin by proving that $u_r(\sigma) = r^2 u(r\sigma)$ converges to $u_\infty(\sigma)$, when r tends to infinity, for all $\sigma \in \mathbb{S}^1$. We then deduce, by using Theorem 6.1, Lemma 6.12 and Ascoli-Arzela's theorem, that this convergence is uniform on $\mathbb{S}^1$. In fact, for $\sigma \in \mathbb{S}^1$, by the convolution equation (6.16), we write for $r > 0$

$$\begin{aligned} u_r(\sigma) = &\int_{B(r\sigma, \frac{r}{2})^c} r^2 k_0(r\sigma - \mathbf{x}) \Big[-\frac{u^2}{2}(\mathbf{x}) + \gamma |D_x| u(\mathbf{x}) \Big] d\mathbf{x} \\ &\times \int_{B(r\sigma, \frac{r}{2})} r^2 k_0(r\sigma - \mathbf{x}) \Big[-\frac{u^2}{2}(\mathbf{x}) + \gamma |D_x| u(\mathbf{x}) \Big] d\mathbf{x}. \end{aligned}$$

By Lemma 6.12, we have

$$\lim_{r \to +\infty} r^2 k_0(r\sigma - \mathbf{x}) \Big[-\frac{u^2}{2}(\mathbf{x}) + \gamma |D_x| u(\mathbf{x}) \Big] = \frac{1}{2\pi}(1 - 2\sigma_1^2) \Big[-\frac{u^2}{2}(\mathbf{x}) + \gamma |D_x| u(\mathbf{x}) \Big].$$

Using now Lemmas 6.6 and 6.12 and Theorem 6.1, we get

$$\begin{aligned} &\Big| r^2 k_0(r\sigma - \mathbf{x}) \Big[-\frac{u^2}{2}(\mathbf{x}) + \gamma |D_x| u(\mathbf{x}) \Big] 1_{B(r\sigma, \frac{r}{2})^c} \Big| \\ &\leq C \frac{r^2}{(r\sigma - \mathbf{x})^2 (1 + |\mathbf{x}|^4)} + \frac{r^2}{(r\sigma - \mathbf{x})^2 (1 + |\mathbf{x}|^3)} \leq \frac{C}{(1 + |\mathbf{x}|^3)}. \end{aligned}$$

The dominated convergence theorem yields

$$\begin{aligned} &\lim_{r \to +\infty} \int_{B(r\sigma, \frac{r}{2})^c} r^2 k_0(r\sigma - \mathbf{x}) \Big[-\frac{u^2}{2}(\mathbf{x}) + \gamma |D_x| u(\mathbf{x}) \Big] d\mathbf{x} \\ &= \frac{1}{2\pi}(1 - 2\sigma_1^2) \int_{\mathbb{R}^2} \Big[-\frac{u^2}{2}(\mathbf{x}) + \gamma |D_x| u(\mathbf{x}) \Big] d\mathbf{x} = -\frac{1}{2\pi}(1 - 2\sigma_1^2) \int_{\mathbb{R}^2} \frac{u^2}{2}(\mathbf{x}) d\mathbf{x}, \end{aligned}$$

since

$$\int_{\mathbb{R}^2} |D_x|u(\mathbf{x})d\mathbf{x} = -\int_{\mathbb{R}^2} \partial_x Hu(\mathbf{x})d\mathbf{x} = 0.$$

By Theorem 6.1 and Lemma 6.12, we have

$$\begin{aligned}&\Big|\int_{B(r\sigma,\frac{r}{2})} r^2 k_0(r\sigma-\mathbf{x})\Big[-\frac{u^2}{2}(\mathbf{x})+\gamma|D_x|u(\mathbf{x})\Big]d\mathbf{x}\Big|\\&\leq C\int_{B(r\sigma,\frac{r}{2})}\Big(\frac{r^2}{|x|^4}+\frac{r^2}{|x|^3}\Big)|k_0(r\sigma-\mathbf{x})|d\mathbf{x}.\end{aligned}$$

Using Lemma 6.6 and the fact that on the ball $B(r\sigma,\frac{r}{2})$ we have $|x|>\dfrac{r}{2}$, we obtain

$$\begin{aligned}&\Big|\int_{B(r\sigma,\frac{r}{2})} r^2 k_0(r\sigma-\mathbf{x})\Big[-\frac{u^2}{2}(\mathbf{x})+\gamma|D_x|u(\mathbf{x})\Big]d\mathbf{x}\Big|\\&\leq \frac{C}{r}\Big[\int_{B(0,1)}|k_0(\mathbf{x})|d\mathbf{x}+\int_{1\leq|\mathbf{x}|\leq\frac{r}{2}}\frac{d\mathbf{x}}{|\mathbf{x}|^2}\Big]\leq\frac{C}{r}\Big(C+\ln r\Big).\end{aligned}$$

We deduce that

$$\lim_{r\to+\infty}\int_{B(r\sigma,\frac{r}{2})} r^2 k_0(r\sigma-\mathbf{x})\Big[-\frac{u^2}{2}(\mathbf{x})+\gamma|D_x|u(\mathbf{x})\Big]d\mathbf{x}=0.$$

Finally, we obtain that

$$\lim_{r\to+\infty} u_r(\sigma) = -\frac{1}{2\pi}(1-2\sigma_1^2)\int_{\mathbb{R}^2}\frac{u^2}{2}(\mathbf{x})d\mathbf{x}. \tag{6.17}$$

Let us now show the uniform convergence. Assume that, by contradiction, $(u_r)_{r>0}$ does not converge uniformly to u_∞ in $\mathbb{S}^1$. Then there exist $\epsilon>0$ and a sequence $(r_n)_{n\in\mathbb{N}}$, $r_n\to+\infty$ such that

$$\|u_{r_n}-u_\infty\|_{L^\infty(\mathbb{S}^1)}\geq\epsilon,\ \forall n\in\mathbb{N}.$$

By Theorem 6.1 and Lemma 6.12, we get

$$\|u_{r_n}\|_{L^\infty(\mathbb{S}^1)}\leq C,\ \|\nabla^{\mathbb{S}^1}u_{r_n}\|_{L^\infty(\mathbb{S}^1)}\leq Cr_n^3\|\nabla u(r_n.)\|_{L^\infty(\mathbb{S}^1)}\leq C.$$

Therefore, by applying Ascoli-Arzela's theorem, we deduce that there exists a subsequence $(r_{n_k})_{k\in\mathbb{N}}$ such that $(u_{r_{n_k}})_{k\in\mathbb{N}}$ converges in $L^\infty(\mathbb{S}^1)$. Then its limit is necessarily equal to u_∞, which leads to a contradiction. Finally, we obtain the uniform convergence to u_∞ in $L^\infty(\mathbb{S}^1)$.

□

Remark 6.2. Consider the case when the sign + precedes the nonlocal operator $H\partial_x^2$. Then, we can easily see that we obtain the same result of existence, regularity, symmetry and decay at infinity, for $c>0$, since $f(\xi)=c+|\xi|+\xi^2$ is positive. The proof is analogous to the previous case and easier, since $f(\xi)$ is positive for all $\xi\in\mathbb{R}$.

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