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ON RECONSTRUCTION OF BOUNDARY CONTROLS IN A PARABOLIC EQUATION

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Abstract. A problem of dynamical reconstruction of boundary controls in a nonlinear parabolic equation is considered. In the case when a control is concentrated in the Neumann boundary conditions, a solving algorithm which is stable with respect to informational noises and computational errors is described. The algorithm is based on the ideas of the theory of feedback control.

1. Introduction

Problems of determining unknown parameters through measurements of some characteristics of current phase states of a dynamical system are often called reconstruction (identification) problems. These problems are imbedded into the class of inverse problems of dynamics (and, in a more general sense, into the class of ill-posed problems (see [1–5])). The input information is assumed to appear in real time. As to unknown parameters, they should be reconstructed in real time too. One of the approaches to solving similar problems was developed in [6] (see also monographs [7–9] as well as review papers [10, 11]). This approach based on the ideas of the theory of ill-posed problems [12, 13] consists in reduction of an identification problem to a control problem for a special auxiliary dynamical system (a model) [14]. Regularization of the problem under consideration is locally realized in the process of choosing a positional control in the model. The approach mentioned above was applied to a number of problems described by some classes of both ordinary and distributed differential equations. Different, varying

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in time, characteristics of a system were under reconstruction (for example, unknown discontinuous inputs, initial and boundary data, distributed disturbances, coefficients of an elliptic operator and so on). In the present paper, the method of dynamical regularization is applied to solving a problem of reconstruction of a boundary control concentrated in the Neumann boundary conditions. In the context of the approach above, problems of reconstruction of boundary controls were investigated in [15, 9, 16, 11, 17]. It is assumed in all these papers (with the exception of [15]) that controls to be reconstructed are constrained by a priori known "instantaneous" restrictions. Namely, at every current time moment these controls belong to a given bounded and closed set (from an appropriate space). In the present paper, restrictions of a similar kind are lacking. Thus, an unknown control may be "unbounded." This renders direct application of the algorithms described in [15, 9, 16, 11, 17] impossible. For parabolic equations, an algorithm of dynamical reconstruction of "unbounded" boundary controls concentrated in the Neumann boundary conditions was proposed in [15] and substantiated in [9]. The algorithm is based on the functionally analytic representation of a boundary problem solution [18, 19] and on the well-known in the theory of ill-posed problems discrepancy method [12, 13]. In this paper, we design another solving algorithm based on an appropriate modification of the dynamical version of the smoothing functional method. Toward this end, we use the semigroup representation of solutions of boundary problems [20–25].

A parabolic equation of the form

$$x_t(t,\eta) - \Delta_L x(t,\eta) = \Phi(x(t,\eta)) \quad \text{in} \quad T \times \Omega = Q, \quad T = [0,\vartheta] \tag{1.1}$$

with initial

$$x(0,\eta) = x_0(\eta) \quad \text{in} \quad \Omega \tag{1.2}$$

and boundary

$$\left.\frac{\partial x}{\partial n}\right|_{\Sigma} = Bu \quad \text{in} \quad (0,\vartheta] \times \Gamma = \Sigma \tag{1.3}$$

conditions is under consideration. Here $\Omega \subset \mathbb{R}^n$ is an open bounded domain with a sufficiently smooth boundary Γ, Δ_L is the Laplace operator; i.e.,

$$\Delta_L x(\eta) = \sum_{j=1}^{n} \frac{\partial^2 x(\eta)}{\partial \eta_j^2}, \quad \eta = (\eta_1, \ldots, \eta_n), \quad x_0(\eta) \in L_2(\Omega),$$

$\Phi(\cdot)$ is a Lipschitz function, $u(\cdot) \in L_2(T;U)$ is a control, U is a Hilbert space (the space of controls), $B : U \to L_2(\Gamma)$ is a linear, continuous operator (i.e., $B \in \mathcal{L}(U; L_2(\Gamma))$), the symbol $\partial x/\partial n$ stands for the derivative with respect to the outward normal. The control $u(\cdot)$ is unknown. At discrete (frequent enough) time moments $\tau_i \in T$, $\tau_i = \tau_{i+1} + \delta$, $i \in [1 : m-1]$, $\tau_0 = 0$, $\tau_m = \vartheta$, phase states of system (1.1)–(1.3), $x(\tau_i, \eta) = x(\tau_i; 0, x_0, u(\cdot)) \in H = L_2(\Omega)$, are inaccurately measured. Hereinafter, the symbol $x(\cdot; 0, x_0, u^*(\cdot)) \in C(T; H)$ denotes the solution of equation (1.1) with the initial (1.2) and boundary (1.3) conditions for a control $u(\cdot) = u^*(\cdot)$. The precise definition of this solution is given below. Results of measurements, elements $\xi_i^h = \xi^h(\tau_i) \in H$, satisfy the inequalities

$$|\xi_i^h - x(\tau_i)|_H \le h, \quad i \in [0 : m-1], \tag{1.4}$$

where h is a parameter of the measurement accuracy, H is some normalized space (the space of measurements). It is required to design an algorithm of reconstruction of an unknown input $u^*(\cdot) \in U_T = L_2(T; U)$ generating an unknown output $x(\cdot)$; i.e., we should find $u^*(\cdot)$ such that $x(\cdot; 0, x_0, u^*(\cdot)) = x(\cdot)$. This is the meaningful statement of the problem in question.

2. The solving method

Let us describe the basic constructions of the solving method used in the present paper.

Let $U(x(\cdot))$ be the set of all inputs $u(\cdot) \in L_2(T; U)$ compatible with $x(\cdot)$; i.e.,

$$U(x(\cdot)) = \{u(\cdot) \in U_T : x(\cdot; 0, x_0, u(\cdot)) = x(\cdot)\}; \tag{2.1}$$

Ξ_T be the set of measurements, i.e., the set of all piecewise constant functions $\xi(\cdot) : T \to H$; $\Xi(x(\cdot), h)$ be the set of all h-accurate results of measurements, i.e., the set of all functions $\xi^h(\cdot) \in \Xi_T$ satisfying (1.4).

To solve the problem, we introduce an auxiliary system M (a model) described by a linear parabolic equation. A trajectory of the model depends on a control $v^h(\cdot)$ to be formed and on a measurement $\xi^h(\cdot) \in \Xi_T$. This trajectory is denoted by the symbol

$$w^h(t) = w^h(t; 0, w_0^h, \xi_{0,t}^h(\cdot), v_{0,t}^h(\cdot)) \in C(T; H). \tag{2.2}$$

Here $v_{a,b}(\cdot)$ means the restriction of a function $v(\cdot)$ to a half-interval $[a, b)$. The initial state of the model w_0^h is chosen with the use of the result of initial measurement according to the rule $\mathcal{W}_h$ a priori fixed:

$$w_0^h = \mathcal{W}_h(\xi_0^h) \in X_0 \subset H. \tag{2.3}$$

Here X_0 is the set of initial states of the model (assumed to be known). In particular, if the initial state x_0 (see (1.2)) is known, then $X_0 = \{x_0\}$.

The rule of choosing the model control $v^h(\cdot)$ (for every $h \in (0,1)$) is identified with a pair $S_h = (\Delta_h, \mathcal{U}_h)$, where

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h} \tag{2.4}$$

is a partition of the interval T into half-intervals $[\tau_{h,i}, \tau_{h,i+1})$, $\tau_{h,i+1} = \tau_{h,i} + \delta$, $\delta = \delta(h)$, $\tau_{h,0} = 0$, $\tau_{h,m_h} = \vartheta$, $\mathcal{U}_h$ is a mapping giving a correspondence between every triple $(\tau_i, \xi_i^h, w^h(\tau_i))$, $i \in [0 : m_h - 1]$ and a function

$$v^h_{\tau_i,\tau_{i+1}}(\cdot) = \mathcal{U}_h(\tau_i, \xi_i^h, w^h(\tau_i)) \in L_2([\tau_i, \tau_{i+1}]; U). \tag{2.5}$$

Here $\tau_i = \tau_{h,i}$, $w^h(\tau_i) = w^h(\tau_i; 0, w_0^h, \xi^h_{0,\tau_i}(\cdot), v^h_{0,\tau_i}(\cdot))$, $\xi_i^h = \xi^h(\tau_i)$, $\xi^h(\cdot) \in \Xi(x(\cdot), h)$. This pair $S_h = (\Delta_h, \mathcal{U}_h)$ is called a feedback. Thus, a quadruple $(M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h)$ determines some algorithm D_h defined on the space of measurements ($D_h : \Xi_T \mapsto U_T$). This algorithm forms the output $v^h(\cdot) = D_h\xi(\cdot)$ according to the feedback principle (2.2)–(2.5) and is identified with the quadruple $(M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h)$.

Let the following condition be fulfilled.

Condition 1. The set $U_*(x(\cdot))$ of all inputs of minimal $L_2(T;U)$-norm generating the solution $x(\cdot)$ is a singleton; i.e., $U_*(x(\cdot)) = \{u_*(\cdot; x(\cdot))\}$.

Thus,

$$u_*(\cdot; x(\cdot)) = \arg\min\{|u(\cdot)|_{L_2(T;U)} : u(\cdot) \in U(x(\cdot))\}.$$

A family of operators D_h, $h \in (0,1)$, mapping the set of measurements Ξ_T into the space of controls U_T is called *regularizing*, if it possesses the property

$$\lim_{h\to 0} \sup\{|D_h\xi^h(\cdot) - u_*(\cdot; x(\cdot))|_{L_2(T;U)} : \xi^h(\cdot) \in \Xi(x(\cdot), h)\} = 0.$$

Our aim (the reconstruction problem) consists in designing a regularizing family of algorithms

$$D_h = (M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h), \qquad h \in (0,1) \tag{2.6}$$

of the form (2.2)–(2.5). Such a family is also called *a family of positional modeling algorithms* [7–9].

After choosing model (2.2) and its initial state (2.3), the work of the algorithm D_h is realized in the following way. Before the initial moment $t_0 = 0$, some accuracy h and feedback $S_h = (\Delta_h, \mathcal{U}_h)$ (2.4), (2.5) are fixed. At the ith step carried out on the time interval $[\tau_i, \tau_{i+1})$, $(\tau_i = \tau_{h,i})$, the following operations are fulfilled. First (at the moment τ_i), the phase state $x(\tau_i)$ is inaccurately measured; i.e., an element $\xi_i^h \in H$ with the property (1.4) is found. Then, according to the rule (2.5), the function $v^h(t)$ is calculated.

This function is fed (as the control in the interval $[\tau_i, \tau_{i+1})$) onto the input of the model. Thus, along with the trajectory $w^h(t)$, $t \in [t_0, \tau_i]$ computed until the moment τ_i, the new phase trajectory $w^h(t)$, $t \in (\tau_i, \tau_{i+1}]$ is formed (i.e., some kind of memory correction is realized).

The rule of constructing the regularizing family of algorithms D_h is based on the theorem presented below. Let a functional $\Lambda^0(\cdot,\cdot)$ be given on the Cartesian product $C(T;H) \times C(T;H)$.

Definition 1. [8, 9] *A family D_h, $h \in (0,1)$, (2.6) of positional modeling algorithms is called Λ^0-stable if there exist functions $k_1(\cdot)$, $k_2(\cdot)$, $k_3(\cdot)$: $[0,+\infty) \to [0,+\infty)$ such that $k_1(h) \to 1$, $k_2(h) \to 0$, $k_3(h) \to 0$ as $h \to 0$, and for any measurement $\xi^h(\cdot) \in \Xi(x(\cdot), h)$ the inequalities*

$$|v^h(\cdot)|_{L_2(T;U)} \le k_1(h)|u_*(\cdot; x(\cdot))|_{L_2(T;U)} + k_2(h), \tag{2.7}$$

$$\Lambda^0(x(\cdot), w^h(\cdot)) \le k_3(h) \tag{2.8}$$

hold.

Here $v^h(\cdot) = D_h\xi^h(\cdot)$ is the output of the algorithm, $w^h(\cdot)$ is the model trajectory generated by the algorithm D_h and the measurement $\xi^h(\cdot)$ (more precisely, by the control $v^h(\cdot)$ given by (2.5)).

Theorem 1. [8, 9] *Let a family D_h, $h \in (0,1)$, of positional modeling algorithms of the form* (2.2)–(2.5) *possess the following properties:*

a) it is Λ^0-stable;

b) for any $h_k > 0$ ($h_k \to 0+$ as $k \to +\infty$), $\xi^{h_k}(\cdot) \in \Xi(x(\cdot), h_k)$, $w^{h_k}(\cdot) = w^{h_k}(\cdot; 0, w_0^{h_k}, \xi^{h_k}(\cdot), v^{h_k}(\cdot))$, $v^{h_k}(\cdot) = D_{h_k}\xi^{h_k}(\cdot)$, the conditions

$$v^{h_k}(\cdot) \to v(\cdot) \quad \text{weakly in} \quad L_2(T;U), \quad \Lambda^0(x(\cdot), w^{h_k}(\cdot)) \to 0 \quad \text{as} \quad k \to \infty$$

imply the inclusion $v(\cdot) \in U(x(\cdot))$. Then the family D_h, $h \in (0,1)$ is regularizing.

As is known, a solution of a parabolic equation of the form (1.1), (1.2) with a boundary condition of the form (1.3) may be defined in different ways. It is possible to consider the classical solution [26] (its existence requires fulfillment of rather severe constraints on a domain Ω, an initial state x_0, and a control u) or the generalized solution used, as a rule, in control theory. The latter, in its turn, may be introduced not uniquely (see, for example, [18, 19]). Most often the semigroup approach is involved to define the generalized solution (see [20–25]). In the present paper, we use the solution defined according to this approach.

3. The solving algorithm

Before passing to the description of the algorithm for solving the problem in question, we give the strict definition of a solution of equation (1.1). Introduce the Neumann mapping $\mathcal{N}$

$$\mathcal{N}u = h \Longleftrightarrow \begin{cases} \Delta_L h - h = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial h}{\partial n}\Big|_{\Gamma} = u. \end{cases} \tag{3.1}$$

In other words, $\mathcal{N}u$ is a generalized solution of elliptic equation (3.1), i.e., a function with the following properties:

$$\mathcal{N}u \in L_2(\Omega), \quad \frac{\partial(\mathcal{N}u)}{\partial n} \in L_2(\Gamma),$$

$$\int_{\Omega} (\mathcal{N}u)(\eta)\{\Delta_L \psi(\eta) + \psi(\eta)\}\, d\eta = \int_{\Gamma} \psi(\sigma) \frac{\partial(\mathcal{N}u)(\sigma)}{\partial n}\, d\sigma \quad \forall \psi \in H_2(\Omega).$$

Consider the mapping $t \to p(t;\cdot,\cdot,\cdot) : H \times L_2(T;U) \times C(T;H) \to C(T;H)$,

$$p(t;x_0,u(\cdot),z(\cdot)) = S(t)x_0 + A\int_0^t S(t-\tau)\mathcal{N}Bu(\tau)d\tau + \int_0^t S(t-\tau)\Phi_1(z(\tau))\, d\tau,$$

$t \in T$. Here

$$Ah = \Delta_L h - h, \qquad h \in \mathcal{D}(A) = \Big\{ z \in H_2(\Omega) : \frac{\partial z}{\partial n}\Big|_{\Gamma} = 0 \Big\}$$

is the infinitesimal generator of an analytic contraction semigroup of linear continuous operators $\{S(t) : t \ge 0\}$, $H = L_2(\Omega)$, $\Phi_1(x) = \Phi(x) + x$. Hereinafter, we, as is customary [20–23], identify A with its isomorphic extension $A : H \to \mathcal{D}^*(A)$. It is known that all eigenvalues of the operator A belong to the interval $(0, +\infty)$. This fact provides the bounded invertibility of A, i.e., $A^{-1} \in \mathcal{L}(H;H)$ [22, page 291; 23, page 54]. It follows from the results [23, page 54] that $\mathcal{N}$ is a continuous mapping of the space $L_2(\Gamma)$ into the space $H^{3/2}(\Omega) \subset H^{3/2-2\varepsilon}(\Omega) = \mathcal{D}(A^{3/4-\varepsilon})$ $\forall \varepsilon > 0$. In addition [21, page 303; 24, page 292], the inclusions

$$\mathcal{N} \in \mathcal{L}(H^s(\Gamma); H^{s+3/2}(\Omega)) \quad \text{for any } s \in \mathbb{R}, \quad A^{-l} \in \mathcal{L}(H^{2\alpha}(\Omega); H^{2\alpha+2l}(\Omega))$$

are valid for any nonnegative integer l and any $\alpha \in \mathbb{R}^+ = [0, +\infty)$. The symbols $H^\alpha(\Omega)$ and $H^s(\Gamma)$ stand for the standard Sobolev spaces [see, for example, 27, pages 90–91]. Following [20; 28, remark 1], a solution of equation

(1.1)–(1.3) corresponding to a control $u(\cdot) \in L_2(T; U)$ is a unique function $x(\cdot) = x(\cdot; 0, x_0, u(\cdot)) \in C(T; H)$ satisfying the equality

$$z(t) = p(t; x_0, u(\cdot), z(\cdot)) \quad \forall\, t \in T.$$

Note that $U(x(\cdot))$, the set of all controls compatible with an output $x(\cdot)$ (see (2.1)), is of the form

$$U(x(\cdot)) = \Big\{ u(\cdot) \in U_T : x(t) - S(t)x_0 - \int_0^t S(t-\tau)\Phi_1(x(\tau))\, d\tau = \\ = A \int_0^t S(t-\tau)\mathcal{N} B u(\tau)\, d\tau \quad \forall t \in T \Big\}.$$

It is easily seen that this set is convex, bounded, and closed in the space $L_2(T; U)$. Therefore, it contains a unique element $u_*(\cdot) = u_*(\cdot; x(\cdot))$ of minimal $L_2(T; U)$-norm.

Let us describe the algorithm of reconstruction of $u_*(\cdot; x(\cdot))$. We fix a bounded set $X_0 \subset H$ (the set of initial states) and we choose a family $\{\Delta_h\}$ of partitions of the interval T of the form (2.4) and a function $\beta(h) : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

$$\delta(h) \to 0, \quad \beta(h) \to +\infty, \tag{3.2}$$
$$(h + \delta(h))\beta^{2+\varepsilon_0}(h) \to 0, \quad \varphi_x(\delta(h))\beta^{1+\varepsilon_0}(h) \to 0 \quad \text{as } h \to 0+.$$

Here $\varepsilon_0 > 0$ is a constant, $\varphi_x(\cdot)$ is the continuity modulo of the function $t \to \Phi_1(x(t)) \in H$ in T; i.e.,

$$\varphi_x(\delta) = \sup \Big\{ |\Phi_1(x(t_1)) - \Phi_1(x(t_2))|_H : t_1,\, t_2 \in T,\, |t_1 - t_2| < \delta \Big\}.$$

Below, without loss of generality, we assume that h, δ, α, $\varphi_x(\delta) \in (0, 1)$.

As a model M, we take the linear equation

$$w_t(t,\eta) - \Delta_L w(t,\eta) + w(t,\eta) = \Phi_1(\xi^h(t,\eta)) \quad \text{in} \quad T \times \Omega, \tag{3.3}$$
$$w(0,\eta) = w_0^h(\eta) \quad \text{in} \quad \Omega$$

with the Neumann boundary condition

$$\frac{\partial w(t)}{\partial n}\Big|_\Gamma = Bv^h(t), \quad t \in T.$$

A solution of equation (3.3) corresponding to a control $v^h(\cdot) \in L_\infty(T; U)$ and a measurement $\xi^h(\cdot) \in \Xi(x(\cdot), h)$ is a function

$$w^h(t) = w(t; 0, w_0^h, \xi^h_{t_0,t}(\cdot), v^h_{t_0,t}(\cdot)) \in C(T; H)$$

defined by the equality [21, 22]

$$w^h(t) = S(t)w_0^h + A\int_0^t S(t-\tau)\mathcal{N}Bv_1^h(\tau)\,d\tau + \int_0^t S(t-\tau)\Phi_1(\xi^h(\tau))\,d\tau,$$

$t \in T$. The rule for choosing a model initial state is given by (2.3), where we set

$$w_0^h \in \{x \in X_0 : |\xi_0^h - x|_H \le h\}. \tag{3.4}$$

Let a feedback $S_h = (\Delta_h, \mathcal{U}_h)$ be given by (2.4), (2.5), where we set

$$\mathcal{U}_h(\tau_i, \xi_i^h, w^h(\tau_i)) = v^h(t) = v_i \quad \text{for a.a.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}); \tag{3.5}$$

here

$$v_i = \arg\min\{2(s_i^*, \mathcal{N}Bv)_H + \ln^{-1}\beta(h)|v|_U^2 : v \in U\} = -\ln\beta(h)B^*\mathcal{N}^*s_i^*, \tag{3.6}$$

$$s_i^* = A^{-1}(w^h(\tau_i) - \xi_i^h),$$

$\mathcal{N}^* \in \mathcal{L}(H; L_2(\Gamma))$ and $B^* \in \mathcal{L}(L_2(\Gamma); U)$ are conjugated operators:

$$(\mathcal{N}v, y)_H = (v, \mathcal{N}^*y)_{L_2(\Gamma)} \quad \forall v \in L_2(\Gamma), \quad y \in H,$$

$$(Bv, y)_H = (v, B^*y)_U \quad \forall v \in U, \quad y \in L_2(\Gamma).$$

Theorem 2. *Let a functional Λ^0 be of the form*

$$\Lambda^0(x(\cdot), w^h(\cdot)) = \sup_{t\in T}|A^{-1}(w^h(t) - x(t))|_H.$$

Then the family of positional modeling algorithms D_h (2.6) of the form (2.3)–(2.5), (3.3)–(3.6) satisfies the conditions of Theorem 1 and is regularizing.

Thus, the convergence $v^h(\cdot) \to u_*(\cdot; x(\cdot))$ in $L_2(T;U)$ as $h \to 0$ holds. Before proving the theorem, we give a number of auxiliary statements.

Lemma 1. *There exists a number $d_* = d_*(x_0, L)$ such that the estimate*

$$|x(t; 0, x_0, u(\cdot))|_H \le d_*(x_0, L)\{1 + |u(\cdot)|_{L_2([0;t];U)}\}$$

holds for any $t \in T$ and $u(\cdot) \in L_2(T;U)$.

Here L is a Lipschitz constant of the function $\Phi(\cdot)$. The lemma follows from the Lipschitz property of the function Φ, inequality (3.22) [28, page 233]:

$$|AS(t)\mathcal{N}Bv|_H \le C(t)|v|_U, \quad t > 0, \quad v \in U, \quad C(\cdot) \in L_2(T;\mathbb{R}), \tag{3.7}$$

the contractibility of the semigroup $\{S(t) : t \ge 0\}$, and the Gronwall inequality [29, page 219].

Consider the following linear parabolic equation:

$$w_t(t,\eta) - \Delta_L w(t,\eta) + w(t,\eta) = v_0(t,\eta) \quad \text{in} \quad T \times \Omega, \tag{3.8}$$

$$w(0,\eta) = w_0(\eta) \quad \text{in} \quad \Omega$$

with the Neumann boundary condition

$$\frac{\partial w(t)}{\partial n}\bigg|_{\Gamma} = Bv_1(t), \quad t \in T$$

and controls $v_0(\cdot)$ and $v_1(\cdot)$. A solution of this equation corresponding to controls $v_1(\cdot) \in L_2(T;U)$ and $v_0(\cdot) \in L_\infty(T;H)$ is a function

$$w(\cdot) = w(\cdot;0,w_0,v(\cdot)) \in C(T;H), \quad v(\cdot) = \{v_0(\cdot), v_1(\cdot)\}$$

of the form [20, 21]

$$w(t) = S(t)w_0 + A\int_0^t S(t-\tau)\mathcal{N}Bv_1(\tau)\,d\tau + \int_0^t S(t-\tau)v_0(\tau)\,d\tau, \quad t \in T.$$

Let bounded sets $Q_0 \subset L_2(T;H)$ and $Q_1 \subset L_2(T;U)$ be fixed. By the symbol $W(X_0,Q_0,Q_1)$ we denote a bundle of solutions of equation (3.8) corresponding to all initial states $w_0 \in X_0$ and controls $v(\cdot) = \{v_0(\cdot), v_1(\cdot)\} \in Q_0 \times Q_1$; i.e., $W(x_0,Q_0,Q_1) = \{w(\cdot;0,w_0,v(\cdot)) : w_0 \in X_0, v(\cdot) \in Q_0 \times Q_1\} \subset C(T;H)$.

Introduce the notation

$$\begin{aligned} &A^{-1}W(X_0,Q_0,Q_1) \\ &\quad = \{\tilde{w}(\cdot) : \tilde{w}(t) = A^{-1}w(t),\ \forall t \in T, w(\cdot) \in W(X_0,Q_0,Q_1)\} \subset C(T;H). \end{aligned}$$

Lemma 2. *The set $A^{-1}W(X_0,Q_0,Q_1)$ is uniformly bounded and equicontinuous in the space $C(T;H)$.*

Proof. The uniform boundedness of the set $A^{-1}W(X_0,Q_0,Q_1)$ follows from the inclusions $A^{-1} \in \mathcal{L}(H;H)$, $\mathcal{N}B \in \mathcal{L}(U;H)$, the contractibility of the semigroup $\{S(t) : t \geq 0\}$ and the equality

$$\begin{aligned} &A^{-1}w^h(t;0,w_0,v_0(\cdot),v_1(\cdot)) \qquad (3.9)\\ &= S(t)A^{-1}w_0 + \int_0^t S(t-\tau)\mathcal{N}Bv_1(\tau)\,d\tau + A^{-1}\int_0^t S(t-\tau)v_0(\tau)\,d\tau. \end{aligned}$$

Let us prove the equicontinuity of the set $A^{-1}W(X_0,Q_0,Q_1)$. Let arbitrary $0 < t_1 < t_2 < \vartheta$, $v_0(\cdot) \in Q_0$, $v_1(\cdot) \in Q_1$ be chosen. It is easily seen that the equality

$$(S(t_2) - S(t_1))A^{-1}x = \int_{t_1}^{t_2} S(\tau)A(A^{-1}x)\, d\tau = \int_{t_1}^{t_2} S(\tau)x\, d\tau \quad \forall x \in H \tag{3.10}$$

is true. Using the contractibility of the semigroup $\{S(t) : t \geq 0\}$ and the boundedness of the set $X_0 \subset H$, we obtain

$$\sup_{x \subseteq X_0} |(S(t_2) - S(t_1))A^{-1}x|_H \leq c_1(t_2 - t_1). \tag{3.11}$$

Note that the following inclusion [28] is true:

$$S(t)\mathcal{N}Bu \in D(A), \quad t > 0.$$

In this case

$$\begin{aligned} &(S(t_2 - t_1) - I) \int_0^{t_1} S(t_1 - \tau)\mathcal{N}Bv_1(\tau)\, d\tau \\ &= \int_0^{t_2 - t_1} S(t)\Big(A \int_0^{t_1} S(t_1 - \tau)\mathcal{N}Bv_1(\tau)\, d\tau\Big)\, dt \\ &= \int_0^{t_2 - t_1} S(t)\Big(\int_0^{t_1} AS(t_1 - \tau)\mathcal{N}Bv_1(\tau)\, d\tau\Big)\, dt. \end{aligned} \tag{3.12}$$

However, (see (3.7)),

$$\begin{aligned} \Big| \int_0^{t_1} AS(t_1 - \tau)\mathcal{N}Bv_1(\tau)\, d\tau\Big|_H &\leq \int_0^{t_1} C(\tau)|v_1(\tau)|_U\, d\tau \\ &\leq \Big(\int_0^{t_1} |C(\tau)|^2\, d\tau \int_0^{t_1} |v_1(\tau)|_U^2\, d\tau\Big)^{1/2} \leq c_2. \end{aligned} \tag{3.13}$$

In addition,

$$\Big| \int_0^{t_2 - t_1} S(t)x\, dt\Big|_H \leq (t_2 - t_1)|x|_H. \tag{3.14}$$

It follows from (3.12)–(3.14) that

$$\Big|(S(t_2 - t_1) - I) \int_0^{t_1} S(t_1 - \tau)\mathcal{N}Bv_1(\tau)\, d\tau\Big|_H \leq (t_2 - t_1)c_2.$$

Consequently,

$$\Big| \int_0^{t_2} S(t_2 - \tau)\mathcal{N}Bv_1(\tau)\, d\tau - \int_0^{t_1} S(t_1 - \tau)\mathcal{N}Bv_1(\tau)\, d\tau\Big|_H \leq \tag{3.15}$$

$$\leq \Big| \int_{t_1}^{t_2} S(t_2 - \tau)\mathcal{N}Bv_1(\tau)\, d\tau\Big|_H + c_3(t_2 - t_1) \leq c_4(t_2 - t_1 + (t_2 - t_1)^{1/2}).$$

By analogy, one can derive the estimate

$$\Big|A^{-1}\Big(\int_0^{t_2} S(t_2-\tau)v_0(\tau)\,d\tau-\int_0^{t_1} S(t_1-\tau)v_0(\tau)\,d\tau\Big)\Big|_H \le \omega(t_2-t_1), \quad (3.16)$$

where $\omega(\delta)\to 0$ as $\delta\to 0$. The equicontinuity of the set $A^{-1}W(X_0,Q_0,Q_1)$ follows from (3.9), (3.11), (3.15), (3.16). The lemma is proved.

Lemma 3. (*the discrete Gronwall inequality*) [31]. *Let the following inequalities be true:*

$$a_1\le\varphi_0,\qquad a_{j+1}\le(1+c_0\delta)a_j+\varphi_j,\quad j\in[1:M],$$

where $c_0>0$, $\delta>0$, $a_j\ge 0$, $\varphi_j\ge 0$. *Then*

$$a_j\le\Big(\sum_{k=0}^{j}\varphi_k\Big)\exp(c_0\delta j)$$

for all $j\in[1:M]$.

In Lemmas 4–8 presented below, we fix an arbitrary measurement $\xi^h(\cdot)\in\Xi(x(\cdot),h)$, the corresponding output of the algorithm $v^h(\cdot)=D_h\xi^h(\cdot)$ and the model trajectory (2.2) generated by the algorithm D_h (2.3)–(2.5), (3.3)–(3.6) for the measurement $\xi^h(\cdot)$. By the symbol c we denote below constants not depending on $\xi^h(\cdot)$, $v^h(\cdot)$, and $w^h(\cdot)$ (these constants can be explicitly written).

Introduce the notation

$$\nu_i=|A^{-1}(w^h(\tau_i)-x(\tau_i))|_H^2,$$

$$J_{1i}=|s_i|_H^2,\qquad s_i=A^{-1}S(\delta)(w^h(\tau_i)-x(\tau_i)),$$

$$J_{2i}=2\Big(s_i,\int_0^{\delta}S(\delta-\tau)\mathcal{N}B(v^h(\tau_i+\tau)-u_*(\tau_i+\tau))\,d\tau\Big)_H.$$

Lemma 4. *The estimates*

$$\nu_{i+1}^{1/2}\le c\{h+\varphi_x(\delta)+h\ln\beta(h)+\int_0^{\tau_i}|u_*(\tau)|_U\,d\tau\}(1+\beta(h))$$

hold for all $i\in[0:m-1]$.

Proof. We use the equality

$$A^{-1}(w^h(\tau_{i+1})-x(\tau_{i+1}))=\sum_{j=1}^{3}\lambda_i^{(j)}, \quad (3.17)$$

where

$$\lambda_i^{(1)} = A^{-1}S(\delta)(w^h(\tau_i) - x(\tau_i)),$$

$$\lambda_i^{(2)} = \int_0^\delta S(\delta-\tau)\mathcal{N}B(v^h(\tau_i+\tau) - u_*(\tau_i+\tau))\,d\tau,$$

$$\lambda_i^{(3)} = A^{-1}\int_0^\delta S(\delta-\tau)[\Phi_1(\xi_i^h) - \Phi_1(x(\tau_i+\tau))]\,d\tau.$$

Since the semigroup $\{S(t) : t \geq 0\}$ is contracting, and the operator A^{-1} commutes with $S(\delta)$, the inequality

$$|\lambda_i^{(1)}|_H \leq \nu_i^{1/2} \tag{3.18}$$

holds. Then, for $\tau \in [0, \delta]$ we have (see (3.6))

$$v_1^h(\tau_i+\tau) = -\ln\beta(h)B^*\mathcal{N}^*s_i^*.$$

Thus,

$$|v_1^h(\tau_i+\tau)|_U = \ln\beta(h)|B^*\mathcal{N}^*A^{-1}(w^h(\tau_i) - \xi_i^h)|_U \tag{3.19}$$
$$\leq \{|B^*\mathcal{N}^*A^{-1}|_{\mathcal{L}(H;U)}h + |B^*\mathcal{N}^*|_{\mathcal{L}(H;U)}\nu_i^{1/2}\}\ln\beta(h), \quad \tau \in [0,\delta].$$

Estimate (3.19) implies the inequality

$$|\lambda_i^{(2)}|_H \leq \delta(ch + \pi\nu_i^{1/2})\ln\beta(h) + c\int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_U\,d\tau, \tag{3.20}$$

where $\pi = |\mathcal{N}B|_{\mathcal{L}(U;H)}|B^*\mathcal{N}^*|_{\mathcal{L}(H;U)}$. Taking into account the Lipschitz property of the function $\Phi_1(\cdot)$ and inequality (1.4), we conclude

$$|\Phi_1(\xi_i^h) - \Phi_1(x(\tau_i+\tau))|_H \leq L|\xi_i^h - x(\tau_i+\tau)|_H \leq L(h + \varphi_x(\delta)), \quad \tau \in [0,\delta]. \tag{3.21}$$

From (3.21) it follows that

$$|\lambda_i^{(3)}|_H \leq c\delta(h + \varphi_x(\delta)). \tag{3.22}$$

Then, in virtue of (3.17), we have

$$\nu_{i+1}^{1/2} \leq \sum_{j=1}^{3} |\lambda_i^{(j)}|_H.$$

Combining (3.17), (3.18), (3.20), (3.22), we obtain from the last inequality

$$\nu_{i+1}^{1/2} \leq (1 + \pi\delta\ln\beta(h))\nu_i^{1/2} + c\delta\{h + \varphi_x(\delta) + h\ln\beta(h)\} + c\int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_U\,d\tau.$$

The assertion we prove follows from the inequality above and Lemma 3. The lemma is proved.

Lemma 5. *The estimates* $J_{1,i} \le \nu_i$ *hold for all* $i \in [1 : m]$.

The proof of this lemma is evident.

Let

$$\varrho_i = |w^h(\tau_i) - x(\tau_i)|_H.$$

Lemma 6. *The inequalities*

$$\varrho_i \le c\mu_i, \quad \mu_i = 1 + \ln\beta(h)\Big(h + \delta^{1/2}\big(\sum_{j=0}^{i-1}\nu_j\big)^{1/2}\Big), \quad i \in [0 : m] \tag{3.23}$$

are true.

Proof. Taking into account the Lipschitz property of the function $\Phi_1(\cdot)$ and Lemma 1, we conclude that the following estimates are true:

$$|\Phi_1(\xi_i^h)|_H \le |\Phi_1(0)|_H + L|\xi_i^h|_H \le |\Phi_1(0)|_H + L\{|x(\tau_i)|_H + h\} \tag{3.24}$$

$$\le |\Phi_1(0)|_H + L\{1 + d_*(x_0, L)(1 + l_u)\} \le c \quad \text{for} \quad t \in \delta_i, \quad i \in [0 : m-1],$$

where

$$l_u = |u_*(\cdot)|_{L_2(T;U)}.$$

In addition, due to (3.19) we have

$$|v_1^h(t)|_U \le \ln\beta(h)c(h + \nu_i^{1/2}) \quad \text{for} \quad t \in \delta_i. \tag{3.25}$$

In this case

$$|w^h(t)|_H^2 \le c\Big(1 + \ln^2\beta(h)\big(h^2 + \delta\sum_{j=0}^{i}\nu_j\big)\Big), \quad t \in [\tau_i, \tau_{i+1}]. \tag{3.26}$$

In virtue of Lemma 1 we obtain

$$|x(t)|_H \le c \quad \forall t \in T. \tag{3.27}$$

From (3.26), (3.27) it follows that (3.23) is fulfilled. The lemma is proved.

Lemma 7. *If the semigroup* $\{S(t) : t \ge 0\}$ *is contracting, then the inequality*

$$|(S(\delta - \tau) - I)A^{-1}S(\delta)x|_H \le (\delta - \tau)|x|_H$$

is true for any $\tau \in (0, \delta)$ *and* $x \in H$.

Proof. As is known [32, page 210], if $x \in D(A)$, then the function $t \to S(t)x$ is strongly differentiable and the equality

$$\frac{d}{dt}S(t)x = S(t)Ax, \quad t > 0$$

holds. In this case

$$(S(t) - I)x = \int_0^t S(\tau)Ax\, d\tau.$$

Consequently, for $\tau \in (0, \delta)$

$$(S(\delta - \tau) - I)A^{-1}S(\delta)x = \int_0^{\delta-\tau} S(\tau)AA^{-1}S(\delta)x\, d\tau = \int_0^{\delta-\tau} S(\tau)S(\delta)x\, d\tau.$$

However, the semigroup $\{S(t) : t \geq 0\}$ is contracting. The lemma follows from the last equality. The lemma is proved.

Lemma 8. *For all $i \in [1 : m]$ the following estimates hold:*

$$J_{2i} + \ln^{-1} \beta(h) \int_{\tau_i}^{\tau_{i+1}} \{|v_1^h(s)|_U^2 - |u_*(s)|_U^2\}\, ds \leq c(\delta\mu_i + h)\kappa_i, \tag{3.28}$$

where $\kappa_i = \delta \ln \beta(h)(h + \nu_i^{1/2}) + \int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_U\, d\tau$.

Proof. Due to (3.19) the inequality

$$\Big| \int_0^\delta \mathcal{N}B(v^h(\tau_i + \tau) - u_*(\tau_i + \tau))\, d\tau \Big|_H \leq c\kappa_i \tag{3.29}$$

is true. Taking into account the self-conjugacy of the semigroup $\{S(t) : t \geq 0\}$, Lemmas 6, 7, and inequality (3.29), we have

$$J_{2i} \leq 2\Big(s_i, \int_0^\delta \mathcal{N}B(v^h(\tau_i + \tau) - u_*(\tau_i + \tau))\, d\tau\Big)_H + c\delta\mu_i\kappa_i \tag{3.30}$$

$$= 2 \int_0^\delta \Big(A^{-1}(w^h(\tau_i) - x(\tau_i)), \mathcal{N}B(v^h(\tau_i + \tau) - u_*(\tau_i + \tau))\, d\tau\Big)_H + c\delta\mu_i\kappa_i$$

$$\leq 2 \int_0^\delta \Big(s_i^*, \mathcal{N}B(v^h(\tau_i + \tau) - u_*(\tau_i + \tau))\, d\tau\Big)_H + c(\delta\mu_i + h)\kappa_i.$$

Thus, from (3.30) for sufficiently small h ($h \in (0, h^*)$) it follows that

$$J_{2i} + \ln^{-1} \beta(h) \int_{\tau_i}^{\tau_{i+1}} \{|v^h(s)|_U^2 - |u_*(s)|_U^2\}\, ds \tag{3.31}$$

$$\leq 2 \int_0^\delta \Big(s_i^*, \mathcal{N}B(v^h(\tau_i + s) - u_*(\tau_i + s))\Big)_H\, ds$$

$$+ \ln^{-1} \beta(h) \int_{\tau_i}^{\tau_{i+1}} \{|v^h(s)|_U^2 - |u_*(s)|_U^2\}\, ds + c(\delta\mu_i + h) \leq c(\delta\mu_i + h)\kappa_i.$$

The lemma is proved. □

Proof of Theorem 2. Let us show that the family of algorithms D_h of the form (2.3)–(2.5), (3.3)–(3.6) is Λ^0-stable. Introduce the value

$$\varepsilon_h(t) = \Lambda(t, x(\cdot), w^h(\cdot)) + \ln^{-1}\beta(h)\int_0^t \{|v^h(\tau)|_U^2 - |u_*(\tau)|_U^2\}\,d\tau,$$

where

$$\Lambda(t, x(\cdot), w^h(\cdot)) = |A^{-1}(w^h(t) - x(t))|_H^2.$$

It is easily seen that the following relation is valid:

$$\begin{aligned}\varepsilon_h(\tau_{i+1}) = |A^{-1}\{S(\delta)(w^h(\tau_i) - x(\tau_i)) \qquad\qquad (3.32)\\ + A\int_0^\delta S(\delta-\tau)\mathcal{N}B(v^h(\tau_i+\tau) - u_*(\tau_i+\tau))\,d\tau\\ + \int_0^\delta S(\delta-\tau)[\Phi_1(\xi_i^h) - \Phi_1(x(\tau_i+\tau))]\,d\tau\}|_H^2\\ + \ln^{-1}\beta(h)\int_0^{\tau_{i+1}}\{|v^h(s)|_U^2 - |u_*(s)|_U^2\}\,ds\\ \le \sum_{j=1}^4 J_{ji} + \ln^{-1}\beta(h)\int_0^{\tau_{i+1}}\{|v^h(s)|_U^2 - |u_*(s)|_U^2\}\,ds,\end{aligned}$$

where the functions J_{1i}, J_{2i} are defined above,

$$J_{3i} = 2\Big(s_i, A^{-1}\int_0^\delta S(\delta-\tau)\{\Phi_1(\xi_i^h) - \Phi_1(x(\tau_i+\tau))\}\,d\tau\Big)_H,$$

$$J_{4i} = 2\{J_i^{(1)} + J_i^{(2)}\},$$

$$J_i^{(1)} = \Big|\int_0^\delta S(\delta-\tau)\mathcal{N}B(v^h(\tau_i+\tau) - u_*(\tau_i+\tau))\,d\tau\Big|_H^2,$$

$$J_i^{(2)} = \Big|A^{-1}\int_0^\delta S(\delta-\tau)\{\Phi_1(\xi_i^h) - \Phi_1(x(\tau_i+\tau))\}\,d\tau\Big|_H^2.$$

At first, we consider the value J_{3i}. Taking into account (3.21), we obtain

$$J_{3i} \le 2\delta a_1 L|A^{-1}(w^h(\tau_i) - x(\tau_i))|_H(h+\varphi_x(\delta)) \le c_3\delta(h+\varphi_x(\delta))\nu_i^{1/2}, \quad (3.33)$$

where $a_1 = |A^{-1}|_{\mathcal{L}(H;H)}$. Then, by virtue of the inclusions $A^{-1} \in \mathcal{L}(H;H)$ and $\mathcal{N} \in \mathcal{L}(L_2(\Gamma);H)$, relations (3.19) and (3.21), the inequalities

$$J_i^{(1)} \le c\delta\int_0^\delta |v_i^h - u_*(\tau_{i+1}+\tau)|_U^2 d\tau \le c\delta^2\ln^2\beta(h)(h^2+\nu_i) + c\delta\int_{\tau_i}^{\tau_{i+1}}|u_*(\tau)|_U^2 d\tau,$$

$$J_i^{(2)} \le c\delta^2(h^2 + \varphi_x^2(\delta))$$

are true. In this case

$$J_{4i} \le c\delta^2\{\varphi_x^2(\delta) + \ln^2\beta(h)(h^2+\nu_i)\} + c\delta\int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_U^2 d\tau. \tag{3.34}$$

It is easily seen that the relations

$$\kappa_i^2 \le c\Big\{\delta^2\ln^2\beta(h)(h^2+\nu_i) + \delta\int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_U^2 d\tau\Big\},$$

$$h\kappa_i \le h\delta + ch\Big\{\delta\ln^2\beta(h)(h^2+\nu_i) + \int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_U^2 d\tau\Big\},$$

$$\mu_i^2 \le c\Big\{1 + \ln^2\beta(h)\{h^2 + \delta\sum_{j=0}^{i-1}\nu_j\}\Big\}$$

are valid. Consequently, the following estimate of the right-hand part of inequality (3.28) holds:

$$(\delta\mu_i + h)\kappa_i \le 0,5\delta^2\mu_i^2 + 0,5\kappa_i^2 + h\kappa_i \tag{3.35}$$

$$\le c\Big\{(h+\delta)\Big(\delta+\int_{\tau_i}^{\tau_{i+1}}|u_*(\tau)|_U^2 d\tau\Big) + \delta\ln^2\beta(h)(h+\delta)(h^2+\nu_i) + \ln^2\beta(h)\delta^3\sum_{j=0}^{i-1}\nu_j\Big\}.$$

Using Lemma 4, we derive

$$\sum_{j=0}^{i-1}\nu_j \le c\delta^{-1}(1+h^2\ln^2\beta(h))(1+\beta^2(h)), \quad i\in[1:m+1].$$

Thus, for sufficiently small h ($h\in(0,h_1)$) we have

$$\nu_j \le c\beta^2(h)(1+h^2\beta^{\varepsilon_1}(h)), \quad j\in[0:m], \tag{3.36}$$

$$\sum_{j=0}^{i-1}\nu_j \le c\delta^{-1}\beta^2(h)(1+h^2\beta^{\varepsilon_1}(h)), \quad i\in[1:m+1], \quad \varepsilon_1 = \text{const} > 0.$$

Consequently,

$$\delta^3\ln^2\beta(h)\sum_{j=0}^{i-1}\nu_j \le c\delta^2\gamma(h), \tag{3.37}$$

$$\delta\ln^2\beta(h)(h+\delta)(h^2+\nu_i) \le \delta(h+\delta)(1+\gamma(h)), \tag{3.38}$$

where $\gamma(h) = \beta^{2+\varepsilon_1}(h)(1+h^2\beta^{2+\varepsilon_1}(h))$. From (3.35), (3.37)–(3.38) it follows that

$$\sum_{i=1}^{m}(\delta\mu_i + h)\kappa_i \le c(h+\delta)(1+\gamma(h)). \tag{3.39}$$

Then, taking into account (3.36), from (3.33) we derive the estimate

$$\sum_{i=1}^{m} J_{3i} \le c(h + \varphi_x(\delta))\beta(h)(1 + h\beta^{\varepsilon_1/2}(h)). \tag{3.40}$$

Inequalities (3.34), (3.36) imply

$$\sum_{i=1}^{m} J_{4i} \le \delta(1 + \gamma(h) + \varphi_x^2(\delta)). \tag{3.41}$$

From Lemmas 5, 8, estimates (3.32), (3.35), (3.39)–(3.41) it follows that there exists a number h^* such that for all $h \in (0, h^*)$ the inequalities

$$\varepsilon^h(\tau_{i+1}) \le \varepsilon^h(0) + c\bar{\gamma}(h) \quad i \in [0 : m-1], \quad m = m_h, \quad \tau_{i+1} = \tau_{h,i+1}, \tag{3.42}$$

where

$$\bar{\gamma}(h) = (h + \delta(h))\beta^{2+\varepsilon_1}(h)(1 + h^2\beta^{2+\varepsilon_1}(h))$$
$$+ (h + \varphi_x(\delta(h))\beta(h)(1 + h\beta^{\varepsilon_1/2}(h)) + \delta(h)\varphi_x^2(\delta(h)),$$

are true. Due to (3.4) we have

$$\varepsilon^h(0) \le ch. \tag{3.43}$$

Inequalities (3.42), (3.43) imply the following estimates:

$$\Lambda(\tau_{h,i}, x(\cdot), w^h(\cdot)) \le c(h + \bar{\gamma}(h) + \ln^{-1}\beta(h)), \quad i \in [0 : m_h], \tag{3.44}$$

$$|v^h(\cdot)|^2_{L_2(T;U)} \le |u_*(\cdot; x(\cdot))|^2_{L_2(T;U)} + c\ln\beta(h)(h + \bar{\gamma}(h)) \le$$
$$\le |u_*(\cdot; x(\cdot))|_{L_2(T;U)} + c\beta^{\varepsilon_1}(h)(h + \bar{\gamma}(h))$$

for sufficiently small h. Thus, inequality (2.7) holds, if

$$k_1(h) = 1, \quad k_2(h) = c\{\beta^{\varepsilon_1}(h)(h + \bar{\gamma}(h))\}^{1/2}.$$

In addition, by virtue of the relation between parameters (3.2), $k_2(h) \to 0$ as $h \to 0$ (we assume that $\varepsilon_0 = 2\varepsilon_1$), inequality (2.8) follows from (3.44) and Lemma 2. Condition b) of Theorem 1 is verified by analogy with [9] (see also [16], proof of theorem 4.1). The theorem is proved.

Remark. Let the operator B be of the form

$$Bu = \sum_{j=1}^{m} \nu_j u_j, \quad \nu_j \in L_2(\Gamma), \quad u_j \in \mathbb{R},$$

where ν_j are known functions, $u = \{u_1, \ldots, u_m\} \in \mathbb{R}^m$. Thus, $U = \mathbb{R}^m$, $B \in \mathcal{L}(\mathbb{R}^m; L_2(\Gamma))$. In this case it is natural to calculate v_i^h at every time step $\delta_i = [\tau_i, \tau_{i+1})$ by the rule

$$\begin{aligned} v_i^h &= \{v_{1i}^h, \ldots, v_{mi}^h\} \\ &= \arg\min\Big\{2\sum_{j=1}^m \gamma_{ij} v_j + \ln^{-1}\beta(h)\sum_{j=1}^m |v_j|^2 : v = \{v_1, \ldots, v_m\} \in \mathbb{R}^m\Big\}, \\ \gamma_{ij} &= (A^{-1}(w^h(\tau_i) - \xi_i^h), \mathcal{N}\nu_j)_H; \end{aligned}$$

i.e., $v_{ij}^h = -\gamma_{ij}\ln\beta(h)$. The symbol $q = A^{-1}(w^h(\tau_i) - \xi_i^h)$ means a generalized solution of the elliptic equation

$$\begin{cases} \Delta_L q - q = w^h(\tau_i) - \xi_i^h \quad \text{in} \quad \Omega, \\ \dfrac{\partial q}{\partial n}\Big|_\Gamma = 0. \end{cases}$$

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