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GROW-UP RATE OF A RADIAL SOLUTION FOR A PARABOLIC-ELLIPTIC SYSTEM IN $\mathbf{R}^2$

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Abstract. We consider radial and positive solutions to a parabolic-elliptic system in $\mathbf{R}^2$. This system was introduced as a simplified version of the Keller-Segel model. The system has the critical value of the total mass. If the total mass of a solution is more than the critical value, the solution blows up in finite time. If the total mass of a solution is less than the critical value, the solution exists globally in time. Recently, some properties of solutions whose total mass is equal to the critical value have been investigated. In this paper, we construct a grow-up solution whose total mass is equal to the critical value. Furthermore, we show that the grow-up rate of the solution is equal to $O((\log t)^2)$.

1. Introduction

In this paper, we treat a parabolic-elliptic system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbf{R}^2 \times (0,\infty), \\ 0 = \Delta v + u & \text{in } \mathbf{R}^2 \times (0,\infty). \end{cases} \tag{1.1}$$

This system (1.1) is a simplified version of the so-called Keller-Segel system. Our system and the Keller-Segel system are introduced to describe the aggregation of cellular slime molds. Here, $u(x,t)$ and $v(x,t)$ represent the density of cells and the concentration of the chemical substance, respectively. The chemical substance triggers the movement of the cells. We consider that chemotactic aggregation corresponds to the blow-up and grow-up of solutions to (1.1).

Here, we say that the the solution (u,v) blows up if there exist a time $T \in (0,\infty)$ and a sequence $\{t_n\}_{n\geq 1} \subset (0,T)$ satisfying $\lim_{n\to\infty} t_n = T$ and $\lim_{n\to\infty} \sup_{x\in\mathbf{R}^2} |u(x,t_n)| = \infty$. If there exists a sequence $\{t_n\}_{n\geq 1} \subset (0,\infty)$ satisfying $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \sup_{x\in\mathbf{R}^2} |u(x,t_n)| = \infty$, we say that the solution grows up.

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The total mass of the solution (u, v) to (1.1) does not change with respect to time t if u rapidly decays at $|x| = \infty$. In other words, if u satisfies

$$(1 + |x|^2)u(x, 0) \in L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2), \tag{1.2}$$

$$u(x, t) \geq 0 \quad \text{in } \mathbf{R}^2, \tag{1.3}$$

and

$$\|u(\cdot, t)\|_\infty = \sup_{x \in \mathbf{R}^2} |u(x, t)| < \infty \quad \text{for } t \in (0, T_{\max}), \tag{1.4}$$

the solution (u, v) to (1.1) satisfies

$$\lambda \equiv \int_{\mathbf{R}^2} u(x, t)dx = \int_{\mathbf{R}^2} u(x, 0)dx \quad \text{for } t \in (0, T_{\max}),$$

where T_{max} is the maximal existence time of the classical solution. In this paper, we consider only classical solutions to (1.1) in $\mathbf{R}^2 \times (0, T_{max})$ satisfying (1.2), (1.3), and (1.4)

When $\lambda > 8\pi$, one can show that the solutions blow up in finite time.

On the other hand, when $\lambda < 8\pi$, J. Dolbeault, and B. Perthame [6] and A. Blanchet, J. Dolbeault and B. Perthame [5] have shown that the solutions exist globally in time. In other words, $\lambda = 8\pi$ is the critical case. In the critical case, the following results are shown.

P. Biler, G. Karch, Ph. Laurençot, and T. Nadzieja [2, 3] have shown the existence of global radially symmetric solutions to (1.1).

A. Blanchet, J. Carrillo, and N. Masmoudi [4] have shown the existence of nonnegative and grow-up solutions to (1.1) without the assumption of symmetry if the initial data $u(\cdot, 0)$ satisfies (1.2) and $u(\cdot, 0) \log u(\cdot, 0) \in L^1(\mathbf{R}^2)$. However, they have not determined the grow-up rate of the solution.

In this paper, we describe the existence of a radial and nonnegative solution to (1.1) that grows up and satisfies

$$\lim_{t \to \infty} \frac{\|u(\cdot, t)\|_\infty}{(\log t)^2} = C,$$

where C is some positive constant.

We find the solution by using the arguments presented in [8, 7]. M. A. Herrero and J. J. L. Velázquez [7] have shown the existence of a radial and nonnegative solution that blows up, and they have investigated the blow-up rate and the asymptotic profile of the solution by using matched asymptotic expansions techniques. N. Mizoguchi [8] found grow-up solutions to a nonlinear heat equation. A combination of the argument presented in [7] with that in [8] implies the following results.

In order to describe Theorems 1 and 2, for $t \geq 0$, $\theta \in (0,1)$, and a positive constant K, we define $E(t) = E(t;K)$ and $R(t) = R(t;K,\theta)$ as

$$E(t) = E(t;K) = \frac{\sqrt{K}}{\log(t+1)}\sqrt{1 - \frac{4\log(\log(t+1))}{\log(t+1)}} \tag{1.5}$$

and $R(t;K,\theta) = \frac{1}{2}(t+1)^{(1-\theta)/2}E(t;K)^{\theta}$, respectively. For a set $\mathcal{O} \subset [0,\infty)$, we define a function $\chi_{\mathcal{O}}(y)$ as

$$\chi_{\mathcal{O}}(y) = \begin{cases} 1 & \text{if } y \in \mathcal{O}, \\ 0 & \text{if } y \notin \mathcal{O}. \end{cases}$$

Theorem 1. *There exists a radial solution* (u,v) *to* (1.1) *in* $\mathbf{R}^2 \times (0,\infty)$ *satisfying conditions* (i), (ii), iii), *and* (iv) *for the constants* $C > 0$, $K > 0$, $T > 0$, *and* $\theta \in (0,1)$.

(*i*) $x \cdot \nabla u(x,t) \leq 0$ *in* $\mathbf{R}^2 \times (0,\infty)$ *and* $u > 0$ *in* $\mathbf{R}^2 \times (0,\infty)$.

(*ii*) $\displaystyle\int_{\mathbf{R}^2} u(x,t)dx = 8\pi$ *for* $t \in [0,\infty)$.

(*iii*) $$u(x,t) \leq \frac{C}{(t+T+1)|x|^2}\exp\Big(-\frac{|x|^2}{8(t+T+1)}\Big)$$
for $x \in \mathbf{R}^2$ *with* $|x| \geq \sqrt{t+T+1}$ *and* $t \geq 0$.

(*iv*) $$\Big|u(x,t) - \frac{8}{E(t+T;K)^2}\Big\{1+\Big(\frac{|x|}{E(t+T;K)}\Big)^2\Big\}^{-2}\Big|$$
$$\leq \frac{C}{(t+T+1)^{-1}}\Big\{1+\Big(\frac{|x|}{E(t+T;K)}\Big)^2\Big\}^{-1}$$
$$+C\Big(\chi_{[0,R(t+T;K,\theta))}(|x|) + \frac{\chi_{[R(t+T;K,\theta),\infty)(|x|)}}{E(t+T;K)}\Big)$$
$$\cdot\Big(\frac{1}{E(t+T;K)}\Big\{1+\Big(\frac{|x|}{E(t+T;K)}\Big)^2\Big\}^{-2}$$
$$+\frac{E(t+T;K)^3}{|x|^4}\Big|\log\Big(1+\Big(\frac{|x|}{E(t+T;K)}\Big)^2\Big)\Big|\chi_{[E(t+T;K),\infty)}(|x|)\Big)$$

for $x \in \mathbf{R}^2$ *with* $|x| \leq \sqrt{t+T+1}$ *and* $t \geq 0$.

The following theorem is an immediate consequence of Theorem 1.

Theorem 2. *For any* $\varepsilon > 0$, *the solution* (u,v) *in Theorem* 1 *satisfies*

$$\lim_{t\to\infty}\frac{\|u(\cdot,t)\|_\infty}{(\log t)^2} = \lim_{t\to\infty}\frac{u(0,t)}{(\log t)^2} = \frac{8}{K},$$

$$\lim_{t\to\infty} \sup_{|x|\geq\sqrt{K}/(\log t)^{1/2-\varepsilon}} u(x,t) = 0, \quad \lim_{t\to\infty} \int_{|x|\geq K/(\log t)^{(1-\varepsilon)}} u(x,t)dx = 0,$$

and

$$u(\cdot,t) \to 8\pi\delta_0 \quad \textit{in } \mathcal{M}(\mathbf{R}^2) \quad \textit{as } t\to\infty.$$

Here, K is the constant used in Theorem 1 and δ_0 is the delta function in $\mathbf{R}^2$ whose support is the origin.

2. Formal proof of Theorem 1

2.1. **Integral transformation.** Let (u,v) be a radial solution to (1.1). We define the functions $M(r,t)$ and $N(r,t)$ as

$$M(r,t) = \frac{1}{2\pi}\int_{|x|<r} u(x,t)dx \quad \text{for } r\geq 0 \tag{2.1}$$

and

$$N(r,t) = \int_0^r M(\tilde{r},t)\tilde{r}d\tilde{r} \quad \text{for } r\geq 0, \tag{2.2}$$

respectively. Then, M and N satisfy

$$M_t = M_{rr} - \frac{1}{r}M_r + \frac{1}{r}MM_r \tag{2.3}$$

and

$$N_t = N_{rr} + \frac{1}{r}N_r + \frac{1}{2}\Big(\frac{N_r}{r}\Big)^2 - 4\Big(\frac{N_r}{r}\Big),$$

respectively. We introduce self-similar variables given by

$$n(y,s) = (t+1)^{-1}N(r,t) \quad \text{and} \quad m(y,s) = M(r,t), \tag{2.4}$$

where

$$y = \frac{r}{\sqrt{t+1}} \quad \text{and} \quad s = \log(t+1).$$

Then, n and m satisfy

$$m_s = m_{yy} - \frac{1}{y}m_y + \frac{y}{2}m_y + \frac{1}{y}mm_y, \tag{2.5}$$

$$n_s = n_{yy} + \frac{1}{y}n_y + \frac{y}{2}n_y - n + \frac{1}{2}\Big(\frac{n_y}{y}\Big)^2 - 4\Big(\frac{n_y}{y}\Big), \tag{2.6}$$

$$m(y,s) = \frac{1}{2\pi}\int_{|Y|<y} z(Y,s)dY, \quad \text{and} \quad n(y,s) = \int_0^y m(\tilde{y},s)\tilde{y}d\tilde{y},$$

where $z(Y,s) = (t+1)u(x,t)$ and $Y = x/\sqrt{t+1}\in\mathbf{R}^2$.

2.2. **Matched asymptotic expansions approach.** For the solution (u, v) to (1.1) described in Theorem 1, the function m defined in (2.1) and (2.4) satisfies

$$m(\cdot, s) \to 4 \quad \text{as } s \to \infty \quad \text{in } \mathbf{R}_+ = (0, \infty).$$

Using

$$\phi(y,s) = e^{y^2/4}(m(y,s) - 4) \quad \text{and} \quad \psi(y,s) = \int_0^y \phi(\tilde{y}, s)\tilde{y} d\tilde{y}, \tag{2.7}$$

the function ψ satisfies

$$\psi_s \quad = \quad \Big[\psi_{yy} + \Big(\frac{1}{y} - \frac{y}{2}\Big)\psi_y - \psi\Big] + \frac{1}{2}e^{-y^2/4}\Big(\frac{\psi_y}{y}\Big)^2 = \mathcal{A}\psi + F. \tag{2.8}$$

In order to find the desired solution (u, v) to (1.1), we consider solutions to (2.5) approaching a stationary solution to (2.5). Since stationary solutions $m(\cdot)$ to (2.5) with $m(0) = 0$ are given by

$$m(y) = \frac{4y^2}{y^2 + a^2},$$

where a is an arbitrary positive constant, we select a solution m to (2.5) satisfying

$$m(y,s) \sim \frac{4y^2}{y^2 + \varepsilon(s)^2} \quad \text{as } s \to \infty \quad \text{for any sufficiently small } y > 0, \tag{2.9}$$

where ε is a positive function satisfying $\lim_{s\to\infty} \varepsilon(s) = 0$. Since the function F in (2.8) satisfies

$$F(y,s) = \frac{1}{2}e^{y^2/4}(m-4)^2, \tag{2.10}$$

we expect that the function F satisfies

$$F(y,s) \sim \frac{8}{\Big((y/\varepsilon(s))^2 + 1\Big)^2} \sim 4\varepsilon(s)^2\mathcal{D}_0(y) \quad \text{for } \varepsilon(s) \ll y \ll 1,$$

where $\mathcal{D}_0$ is a measure on $[0, \infty)$ satisfying

$$\int_0^\infty \mathcal{D}_0(y) f(y) y dy = f(0)$$

for any bounded and continuous functions f on $[0, \infty)$. Thus, $\mathcal{D}_0$ corresponds to the measure $2\pi\delta_0$ on $\mathbf{R}^2$.

Here and henceforth, for positive constants C and $\tilde{C}$, $C \ll \tilde{C}$ implies that $\tilde{C}/C$ is sufficiently large.

Thus, we expect that the solution ψ to (2.8) behaves asymptotically as a solution to

$$\psi_s = \mathcal{A}\psi + 4\varepsilon(s)^2\mathcal{D}_0 \quad \text{for } \varepsilon(s) \ll y \ll 1 \text{ and } s \gg 1, \tag{2.11}$$

where $\mathcal{A}$ is the differential operator in (2.8). In order to analyze solutions to (2.8) and (2.11), some properties on the spectrum of the operator $\mathcal{A}$ are required. We use

$$L^2_w = \left\{ f \in L^2_{loc}(\mathbf{R}_+) \ : \ \|f\|^2 = \int_0^\infty |f(y)|^2 e^{-y^2/4} y dy < \infty \right\}$$

and

$$H^k_w = \left\{ f \in L^2_{loc}(\mathbf{R}_+) \ : \ f^{(j)} \in L^2_w \ \text{ for } j = 0, 1, 2, \cdots, k \right\}$$

for each $k = 1, 2, 3, \cdots$. When we define the inner product $\langle \cdot, \cdot \rangle$ of L^2_w as

$$\langle f, g \rangle = \int_0^\infty f(y) g(y) e^{-y^2/4} y dy,$$

L^2_w is a Hilbert space and the differential operator $\mathcal{A}$ is a self-adjoint operator in L^2_w with domain H^2_w. For $j = 0, 1, 2, \cdots$, let

$$L_j(y) = e^y \frac{d^j}{dy^j}(y^j e^{-y}) = j! \sum_{k=0}^{j} (-1)^k \frac{j!}{(j-k)! \cdot k!} \cdot \frac{y^k}{k!} \quad \text{for } y \geq 0. \tag{2.12}$$

The function L_j is called the j-th Laguerre polynomial (see [1]). The eigenvalues of $\mathcal{A}$ consist of the sequence

$$\lambda_j = -(j+1) \quad (j = 0, 1, 2, \cdots),$$

and

$$\varphi_j(y) = \frac{1}{\sqrt{2}(j!)} L_j(y^2/4) \tag{2.13}$$

are the corresponding eigenfunctions satisfying $\|\varphi_j\| = 1$. Since the following lemma follows from (2.12) and (2.13), we omit the proof.

Lemma 2.1. *For each* $j = 0, 1, 2, \cdots$, *the eigenfunction* φ_j *satisfies* $\varphi_j(0) > 0$ *and*

$$|\varphi_j(y)| \leq \frac{1}{\sqrt{2}} \left(1 + \frac{y^2}{4}\right)^j \quad \text{for } y \geq 0.$$

For a solution ψ to (2.11), we use

$$a_0(s) = \langle \psi(\cdot, s), \varphi_0 \rangle \ \text{ and } \ \psi(y, s) = a_0(s)\varphi_0(y) + Q(y, s). \tag{2.14}$$

The Fourier coefficient a_0 and the function Q satisfy

$$a_0(s)' = -a_0(s) + 4\varepsilon(s)^2 \langle \varphi_0, \mathcal{D}_0 \rangle, \tag{2.15}$$

$$Q_s = \mathcal{A}Q + 4\varepsilon(s)^2\Big(\mathcal{D}_0 - \langle \varphi_0, \mathcal{D}_0\rangle\varphi_0\Big), \tag{2.16}$$

and

$$\langle Q(\cdot, s), \varphi_0\rangle = 0 \quad \text{ for } s \geq 0. \tag{2.17}$$

Furthermore, we expect that the function Q defined in (2.14) satisfies

$$Q(y, s) \sim 4\varepsilon(s)^2 G(y) \quad \text{ as } s \to \infty \tag{2.18}$$

for some function G. Thus, we expect that the function G satisfies

$$\mathcal{A}G + \Big(\mathcal{D}_0 - \langle \varphi_0, \mathcal{D}_0\rangle\varphi_0\Big) = 0 \tag{2.19}$$

and

$$\langle G, \varphi_0\rangle = 0. \tag{2.20}$$

Since the solution G to (2.19) with (2.20) is given by

$$G(y) = e^{y^2/4}\int_y^\infty \frac{1}{\tilde{y}}\exp\Big(-\frac{\tilde{y}^2}{4}\Big)d\tilde{y} - \frac{1}{2}, \tag{2.21}$$

we observe

$$G(y) = -\log y + B + O(y^2\log y) \quad \text{ as } y \to 0, \tag{2.22}$$

where

$$B = \int_0^1 \frac{1}{\tilde{y}}\Big\{\exp\Big(-\frac{\tilde{y}^2}{4}\Big) - 1\Big\}d\tilde{y} + \int_1^\infty \frac{1}{\tilde{y}}\exp\Big(-\frac{\tilde{y}^2}{4}\Big)d\tilde{y} - \frac{1}{2}. \tag{2.23}$$

Using

$$\begin{aligned}\mathcal{G}(y) \;\; &= \;\; e^{y^2/4}\int_y^\infty \frac{1}{\tilde{y}}\exp\Big(-\frac{\tilde{y}^2}{4}\Big)\int_0^{\tilde{y}} G(\zeta)\zeta d\zeta d\tilde{y} \\ &\quad - \int_0^\infty y\int_y^\infty \frac{1}{\tilde{y}}\exp\Big(-\frac{\tilde{y}^2}{4}\Big)\int_0^{\tilde{y}} G(\zeta)\zeta d\zeta d\tilde{y}dy,\end{aligned} \tag{2.24}$$

the function $\mathcal{G}$ satisfies

$$-\mathcal{A}\mathcal{G} = G \;\text{ in } \mathbf{R}_+, \quad \langle \mathcal{G}, \varphi_0\rangle = 0, \quad \text{ and } \quad \sup_{y\geq 0}|\mathcal{G}(y)| < \infty. \tag{2.25}$$

Since we expect

$$|\varepsilon(s)^2| \ll e^{\lambda_0 s} = e^{-s} \;\text{ as } s \to \infty \quad \text{ and } \quad \int_0^\infty \varepsilon(s)^2 e^s ds < \infty,$$

it follows from (2.15) that

$$a_0(s) \sim -4\varphi_0(0)\int_s^\infty \varepsilon(\tilde{s})^2 e^{-s+\tilde{s}}d\tilde{s} \quad \text{ as } s \to \infty.$$

From this, (2.14), (2.18), and (2.22), we have

$$\begin{aligned}\psi(y,s) &\sim a_0(s)\varphi_0(y)+4\varepsilon(s)^2G(y)\\ &\sim -4\varphi_0(0)\varphi_0(y)\int_s^\infty \varepsilon(\tilde{s})^2 e^{-s+\tilde{s}}d\tilde{s}\\ &\qquad +4\varepsilon(s)^2\Big(-\log y+B+O(y^2\log y)\Big)\end{aligned}\tag{2.26}$$

for $\varepsilon(s) \ll y \le 1$ and $s \gg 1$. On the other hand, it follows from (2.7) and (2.9) that

$$\begin{aligned}\psi(y,s) &= \int_0^y (m(\tilde{y},s)-4)\exp\Big(\frac{\tilde{y}^2}{4}\Big)\tilde{y}d\tilde{y}\\ &\sim -4\int_0^y \frac{1}{1+(\tilde{y}/\varepsilon(s))^2}\tilde{y}d\tilde{y} = -2\varepsilon(s)^2\log\Big(1+\frac{y^2}{\varepsilon(s)^2}\Big)\end{aligned}\tag{2.27}$$

for $\varepsilon(s) \le y \ll 1$ and $s \gg 1$. Comparing (2.26) with (2.27) at $y=\varepsilon(s)^{1/2}$, we observe that $\varepsilon(s)$ satisfies the following integral equation:

$$\begin{aligned}-2\varepsilon(s)^2\log\Big(1+\frac{1}{\varepsilon(s)}\Big) &= -2\int_s^\infty \varepsilon(\tilde{s})^2e^{-s+\tilde{s}}d\tilde{s}+4\varepsilon(s)^2\log\frac{1}{\varepsilon(s)^{1/2}}\\ &\qquad +4\varepsilon(s)^2B+O(\varepsilon(s)^3\log\varepsilon(s)).\end{aligned}$$

Thus, we have

$$\varepsilon(s)^2 \sim \frac{K}{s^2}\Big(1-\frac{4}{s}\log s\Big)e^{-s} \quad \text{as } s\to\infty \tag{2.28}$$

for some positive constant K.

3. Topological argument

In this section, we describe a topological approach that can be used to make the formal analysis described in Section 2 rigorous. First, we need to introduce a suitable class of functions. We define $\bar{\varepsilon}(s)$ as

$$\bar{\varepsilon}(s) = \frac{K^{1/2}}{s}e^{-s/2}\sqrt{1-\frac{4}{s}\log s}, \tag{3.1}$$

where K is the constant used in (2.28).

Let μ, s_0, and s_1 be positive constants satisfying $0 \le s_0 \le s_1 < \infty$ and $0<\mu\le 1$. Further, let M be a sufficiently large constant. We shall consider nonpositive C^1-functions $f(y,s)$ satisfying the following estimates:

$$-M\mu < f(y,s) \le 0 \quad \text{for } y \ge 0 \ \text{ and } \ s\in[s_0,s_1]. \tag{3.2}$$

Set $\xi = y/\overline{\varepsilon(s)}$. We consider functions satisfying

$$\begin{aligned}&\left|f(\overline{\varepsilon(s)}\xi, s) + \frac{4}{\xi^2+1}\right| \\ &\quad < \quad M\mu\left\{\frac{\xi^2}{(\xi^2+1)^2} + \overline{\varepsilon}(s)^2|\log\overline{\varepsilon}(s)|^2\frac{1+y^2}{y^2}\chi_{(\overline{\varepsilon}(s),\infty)}(y)\right\} \qquad (3.3)\end{aligned}$$

for $y \in [0,\infty)$ and $s \in [s_0, s_1]$. Let θ be a positive and sufficiently small constant. For each C^1-function f satisfying (3.2) and (3.3), we define $\varepsilon(s)$ as

$$4\varepsilon(s)^2 = \frac{1}{2}\int_0^{\overline{\varepsilon}(s)^{2\theta}} f(y,s)^2 y e^{-y^2/4}dy. \qquad (3.4)$$

We select a function $\varepsilon(s)$ satisfying

$$\frac{\mu}{10}\overline{\varepsilon}(s) < \varepsilon(s) < \frac{10}{\mu}\overline{\varepsilon}(s) \quad \text{ for } s \in [s_0, s_1] \qquad (3.5)$$

and

$$\sup\left\{|\varepsilon(s) - \overline{\varepsilon}(\tilde{s})| \ : \ s, \tilde{s} \in [s_0, s_1] \ \text{ and } \ |s - \tilde{s}| \le \frac{1}{s}\right\} < \frac{M\mu}{s}\overline{\varepsilon}(s). \qquad (3.6)$$

Let

$$\begin{aligned}\mathcal{A}(s_0, s_1, \mu) = \Big\{f \ : \ &f \text{ is a } C^1\text{-function satisfying (3.2) and (3.3)} \\ &\text{and the function } \varepsilon \text{ defined in (3.4) satisfies (3.5) and (3.6)}\Big\}.\end{aligned}$$

We say $f \in \overline{\mathcal{A}(s_0, s_1, \mu)}$ if the function f and the function ε defined in (3.4) satisfy (3.2), (3.3), (3.5), and (3.6) when strict inequalities are replaced by the symbol "$\le$."

To construct the desired solution, we find a suitable initial function $\phi(\cdot, s_0)$ in $\mathcal{A}(s_0, s_0, 1)$ for some $s_0 \gg 1$. Let ε_0 be a positive constant satisfying $\overline{\varepsilon}(s_0)/5 < \varepsilon_0 < 5\overline{\varepsilon}(s_0)$ and let

$$m(y, s_0) = \begin{cases} 4 - \dfrac{4\varepsilon_0^2}{\varepsilon_0^2 + y^2}e^{-y^2/4} & \text{if } 0 \le y \le 3\overline{\varepsilon}(s_0)^{2\theta}/4, \\ ay^2 + b & \text{if } 3\overline{\varepsilon}(s_0)^{2\theta}/4 \le y \le \overline{\varepsilon}(s_0)^{2\theta}, \\ 4 + \dfrac{4\varepsilon_0^2}{y}G'(y)e^{-y^2/4} & \text{if } y \ge \overline{\varepsilon}(s_0)^{2\theta}, \end{cases} \qquad (3.7)$$

where a and b are positive constants satisfying

$$ay^2 + b = \begin{cases} 4 - \dfrac{4\varepsilon_0^2}{\varepsilon_0^2 + y^2} e^{-y^2/4} & \text{at } y = 3\overline{\varepsilon}(s_0)^{2\theta}/4, \\ 4 + \dfrac{4\varepsilon_0^2}{y} G'(y) e^{-y^2/4} & \text{at } y = \overline{\varepsilon}(s_0)^{2\theta}. \end{cases} \tag{3.8}$$

Furthermore, for the function $m(\cdot, s_0)$ defined in (3.7), we define $\psi(\cdot, s_0)$ as (2.7). Then, $\psi(\cdot, s_0)$ is uniquely determined by ε_0 and s_0. We use

$$\alpha_0 = \sqrt{2}\Big(\psi(\overline{\varepsilon}(s_0)^{2\theta}, s_0) - 4\varepsilon_0^2 G(\overline{\varepsilon}(s_0)^{2\theta})\Big). \tag{3.9}$$

We set

$$\tilde{\varphi}_0(y) = \begin{cases} \dfrac{1}{\alpha_0}\left\{\psi(y, s_0) - 4\varepsilon_0^2 G(y)\right\} & \text{if } 0 \le y \le \overline{\varepsilon}(s)^{2\theta}, \\ \varphi_0(y)\Big(\equiv 1/\sqrt{2}\Big) & \text{for } y \ge \overline{\varepsilon}(s_0)^{2\theta}, \end{cases} \tag{3.10}$$

where G is given in (2.21). According to the choice of ε_0 and ψ, we observe that

$$|\alpha_0 - 2\sqrt{2}\varepsilon_0^2 \log \varepsilon_0^2| \le C\varepsilon_0^2$$

and

$$\Big(10\overline{\varepsilon}(s_0)\Big)^2 \log\Big(10\overline{\varepsilon}(s_0)\Big)^2 < \frac{\alpha_0}{2\sqrt{2}} < \left(\frac{\overline{\varepsilon}(s_0)}{10}\right)^2 \log\left(\frac{\overline{\varepsilon}(s_0)}{10}\right)^2 < 0 \tag{3.11}$$

for $s_0 \gg 1$.

Here and henceforth, C denotes a positive constant independent of ε_0, s_0, θ, and M. Then, each C may vary from line to line.

The function $m(y, s_0)$ defined in (3.7) and (3.8) increases with y, while $m_y(y, s_0)/y$ does not increase with y. In other words, the corresponding initial function $u_0(x)$ of the solution (u, v) to (1.1) is positive and does not increase with $|x|$. Since ε_0 and α_0 satisfy $\overline{\varepsilon}(s_0)/5 < \varepsilon_0 < 5\overline{\varepsilon}(s_0)$ and

$$\left|\frac{\partial \alpha_0}{\partial \varepsilon_0} - 4\sqrt{2}\varepsilon_0 \log \varepsilon_0^2\right| \le C\varepsilon_0$$

for any $s_0 \gg 1$ and any α_0 satisfying (3.11), we can select a unique positive constant ε_0 and we observe that, for each $y \ge 0$, the function $\psi(y, s_0; \alpha_0)$ increases with α_0. Then, $\langle \psi(\cdot, s_0; \alpha_0), \varphi_0 \rangle$ increases with α_0. Furthermore, it follows from (3.4), (3.7), and (3.8) that

$$|\varepsilon_0^2 - \varepsilon(s_0)^2| \le C\overline{\varepsilon}(s_0)^{2+4\theta} \tag{3.12}$$

if $0 < \theta \ll 1$. For any α_0 and $s \geq s_0$, we define

$$P(\alpha_0; s) = \langle \psi(\cdot, s; \alpha_0), \varphi_0 \rangle + 2\sqrt{2} \int_s^\infty e^{-s+\tilde{s}} \overline{\varepsilon}(\tilde{s})^2 d\tilde{s}, \tag{3.13}$$

where $\psi(\cdot, \cdot; \alpha_0)$ is the solution to (2.8) with the initial function $\psi(\cdot, s_0; \alpha_0)$ defined as above.

The following proposition is a key tool in this paper.

Proposition 3.1. *Assume that $M > 0$ in* (3.2), (3.3), *and* (3.6) *is sufficiently large, and that $\theta \in (0, 1)$ in* (3.4) *is sufficiently small. Let ψ be a solution to* (2.8) *with the initial function $\psi(\cdot, s_0) = \psi(\cdot, s_0; \alpha_0)$ described above, and let the function $\phi = \phi(\cdot, \cdot; \alpha_0)$ defined in* (2.7) *satisfy*

$$\phi \in \overline{\mathcal{A}(s_0, s_1, 1)} \tag{3.14}$$

for $s \in [s_0, s_1]$ with $s_1 \geq s_0 \gg 1$. Then, if

$$P(\alpha_0; s_1) = 0, \tag{3.15}$$

it follows that $\phi \in \mathcal{A}(s_0, s_1, 1/2)$.

4. Proof of Proposition 3.1

In this section, we describe the main arguments of the proof of Proposition 3.1. Then, we describe the proofs of some technical lemmas in the last section.

4.1. **Estimates of first Fourier coefficient.** As mentioned in the previous section, we consider the desired solution in terms of the functions ψ and ϕ. We write ψ in the form

$$\psi(y, s) = a_0(s)\varphi_0(y) + R(y, s). \tag{4.1}$$

Then, the function R satisfies

$$\langle R(\cdot, s), \varphi_0 \rangle = 0 \quad \text{for } s \geq s_0.$$

It follows from (3.15) that

$$a_0(s_1) = -2\sqrt{2} \int_{s_1}^\infty e^{-s+\tilde{s}} \overline{\varepsilon}(\tilde{s})^2 d\tilde{s}. \tag{4.2}$$

Since ψ is the solution to (2.8), we have

$$a_0(s) = -\int_s^{s_1} e^{-s+\tilde{s}} \langle F(\cdot, \tilde{s}), \varphi_0 \rangle d\tilde{s} - 2\sqrt{2} \int_{s_1}^\infty e^{-s+\tilde{s}} \overline{\varepsilon}(\tilde{s})^2 d\tilde{s} \tag{4.3}$$

for $s \in [s_0, s_1]$, by (4.2). Then, we have the following estimates.

Lemma 4.1. *Assume* (3.15) *and* $\phi \in \overline{\mathcal{A}(s_0, s_1, 1)}$ *for some* $s_1 > s_0 \gg 1$. *Then, for some* $\mu \in (0,1)$ *and a positive constant* C_M *depending on* M, a_0 *satisfies*

$$\left| a_0(s) + 2\sqrt{2}\int_s^{s_1} e^{-s+\tilde{s}}\varepsilon(\tilde{s})^2 d\tilde{s} + 2\sqrt{2}\int_{s_1}^{\infty} e^{-s+\tilde{s}}\overline{\varepsilon}(\tilde{s})^2 d\tilde{s}\right| \leq C_M(\overline{\varepsilon}(s))^{4-\mu}. \tag{4.4}$$

Furthermore, for the constant α_0 *satisfying* (3.15), *it holds that*

$$\left|\alpha_0 - 2\sqrt{2}\overline{\varepsilon}(s_0)^2 \log \overline{\varepsilon}(s_0)^2\right| \leq C_M \overline{\varepsilon}(s_0)^2 \tag{4.5}$$

and that

$$|\alpha_0 - a_0(s_0)| \leq C\overline{\varepsilon}(s_0)^{2+4\theta}|\log \overline{\varepsilon}(s_0)|^2, \tag{4.6}$$

where θ *is the constant in* (3.4).

Here and henceforth, C_M represents a positive constant depending on M. Then, each C_M may vary from line to line.

Proof of Lemma 4.1. We observe that

$$\begin{aligned}
&\langle F(\cdot,s),\varphi_0\rangle = \langle F(\cdot,s)\chi_{[0,\overline{\varepsilon}(s)^{2\theta}]},\varphi_0\rangle + \langle F(\cdot,s)\chi_{[\overline{\varepsilon}(s)^{2\theta},\infty)},\varphi_0\rangle\\
&= 4\varepsilon(s)^2\varphi_0(0) + \langle F(\cdot,s)\chi_{[0,\overline{\varepsilon}(s)^{2\theta}]} - 4\varepsilon(s)^2\mathcal{D}_0,\varphi_0\rangle + \langle F(\cdot,s)\chi_{[\overline{\varepsilon}(s)^{2\theta},\infty)},\varphi_0\rangle\\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{4.7}$$

By (3.4) and $\varphi_0(y) \equiv 1/\sqrt{2}$, we observe that

$$I_2 = \langle F(\cdot,s)\chi_{[0,\overline{\varepsilon}(s)^{2\theta}]},\varphi_0 - \varphi_0(0)\rangle = 0. \tag{4.8}$$

It follows from (3.3) that

$$\begin{aligned}
|I_3| &\leq C_M\int_{\overline{\varepsilon}(s)^{2\theta}}\Big(\frac{\overline{\varepsilon}(s)^4}{\tilde{y}^4} + \overline{\varepsilon}(s)^4|\log\overline{\varepsilon}(s)|^4\frac{1+\tilde{y}^4}{\tilde{y}^4}\Big)\tilde{y}\exp\Big(-\tilde{y}^2/4\Big)d\tilde{y}\\
&\leq C_M\overline{\varepsilon}(s)^{4-4\theta}|\log\overline{\varepsilon}(s)|^4.
\end{aligned}$$

By this, (4.7), and (4.8), we have

$$\left|\langle\varphi_0, F(\cdot,s)\rangle - 2\sqrt{2}\varepsilon(s)^2\right| \leq C_M\overline{\varepsilon}(s)^{4-4\theta}|\log\overline{\varepsilon}(s)|^4 \quad \text{for } s\in[s_0,s_1]$$

if $s_0 \gg 1$. Combining this with (4.3) implies

$$\begin{aligned}
&\left|a_0(s) + 2\sqrt{2}\int_s^{s_1} e^{-s+\tilde{s}}\varepsilon(\tilde{s})^2 d\tilde{s} + 2\sqrt{2}\int_{s_1}^{\infty} e^{-s+\tilde{s}}\overline{\varepsilon}(\tilde{s})^2 d\tilde{s}\right|\\
&\leq C_M\int_s^{s_1} e^{-s+\tilde{s}}\overline{\varepsilon}(\tilde{s})^{4-4\theta}|\log\overline{\varepsilon}(\tilde{s})|^4 d\tilde{s} \leq C_M\overline{\varepsilon}(s)^{4-4\theta}|\log\overline{\varepsilon}(s)|^5.
\end{aligned}$$

Taking $0 < \theta \ll 1$ and $s_0 \gg 1$, we have (4.4). By (3.1), (3.6), and (4.4), we observe that

$$\begin{aligned}&\left|a_0(s) + 2\sqrt{2}\int_s^\infty e^{-s+\tilde{s}}\overline{\varepsilon}(\tilde{s})^2 d\tilde{s}\right|\\ &\quad\le\ C_M\overline{\varepsilon}(s)^{4-\mu} + 2\sqrt{2}\int_s^{s_1} e^{-s+\tilde{s}}\left|\overline{\varepsilon}(\tilde{s})^2 - \varepsilon(\tilde{s})^2\right| d\tilde{s} \le C_M\overline{\varepsilon}(s)^2. \quad (4.9)\end{aligned}$$

By (3.1), we observe that

$$\left|\overline{\varepsilon}(s)^2 \log\overline{\varepsilon}(s)^2 + \int_s^\infty e^{-s+\tilde{s}}\overline{\varepsilon}(\tilde{s})^2 d\tilde{s}\right| \le C\overline{\varepsilon}(s)^2. \quad (4.10)$$

It follows from (3.10) that

$$\left|\langle\tilde{\varphi}_0, \varphi_0\rangle - 1\right| \le \|\tilde{\varphi}_0\chi_{[0,\overline{\varepsilon}(s_0)^{2\theta}]}\|\|\varphi_0\chi_{[0,\overline{\varepsilon}(s_0)^{2\theta}]}\| \le C\overline{\varepsilon}(s_0)^{4\theta}|\log\overline{\varepsilon}(s_0)| \quad (4.11)$$

if $s_0 \gg 1$. By this and (3.11), we observe that

$$|\alpha_0 - a_0(s_0)| = |\alpha_0\langle\varphi_0, \varphi_0\rangle - \alpha_0\langle\tilde{\varphi}_0, \varphi_0\rangle| \le C\overline{\varepsilon}(s_0)^{2+4\theta}|\log\overline{\varepsilon}(s_0)|.$$

Then, we have (4.6).

Combining (4.11) with (3.11), (4.9), and (4.10) implies

$$\begin{aligned}&|\alpha_0 - 2\sqrt{2}\overline{\varepsilon}(s_0)^2\log\overline{\varepsilon}(s_0)^2|\\ &\le |\alpha_0 - \langle\psi(\cdot, s_0), \varphi_0\rangle| + |\langle\psi(\cdot, s_0), \varphi_0\rangle - 2\sqrt{2}\overline{\varepsilon}(s_0)^2\log\overline{\varepsilon}(s_0)^2|\\ &\le |\alpha_0 - \alpha_0\langle\tilde{\varphi}_0, \varphi_0\rangle| + |a_0(s_0) - 2\sqrt{2}\overline{\varepsilon}(s_0)^2\log\overline{\varepsilon}(s_0)^2| \le C_M\overline{\varepsilon}(s_0)^2.\end{aligned}$$

Then, we obtain (4.5). Thus, we have this lemma. □

4.2. **Estimates of remainder term in outer region.** In this subsection, we describe some estimates of the remainder term $R(y, s)$ in (4.1) on the region $y \ge \overline{\varepsilon}(s)^\theta$.

Lemma 4.2. *Let θ be a positive and sufficiently small constant. Assume* (3.15) *and $\phi \in \overline{\mathcal{A}(s_0, s_1, 1)}$ for some $s_1 > s_0 \gg 1$. Then, there exist positive constants C_M and $\mu \in (0, 1)$ satisfying*

$$|R(y, s) - Q(y, s)| \le C_M(\overline{\varepsilon}(s))^{2+\mu}(1 + y^2) \quad \textit{for } y \ge \overline{\varepsilon}(s)^\theta \quad \textit{and} \quad s \in [s_0, s_1],$$

where Q is the solution to (2.16) *and* (2.17) *with $Q(\cdot, s_0) = R(\cdot, s_0)$.*

The proof of Lemma 4.2 is given in Section 5.1.

The function Q defined in (2.16) and (2.17) satisfies the following estimate.

Lemma 4.3. *Assume* (3.15) *and* $\phi \in \overline{\mathcal{A}(s_0, s_1, 1)}$ *for some* $s_1 > s_0 \gg 1$. *There exist a positive and sufficiently small constant* θ *and a positive constant* C *such that for* $y \geq \overline{\varepsilon}(s)^\theta$ *and* $s \in [s_0, s_1]$,

$$|Q(y,s) - 4\varepsilon(s)^2 G(y)| \leq C\varepsilon(s)^2(1+y^2),$$

where Q *is the solution to* (2.16) *and* (2.17) *with* $Q(\cdot, s_0) = R(\cdot, s_0)$, *and* G *is the function defined in* (2.21).

The proof of Lemma 4.3 is given in Section 5.2.

4.3. **Estimates of solutions in overlapping region.** In Lemmas 4.1, 4.2, and 4.3, we estimate the function ψ in the region $y \geq \overline{\varepsilon}(s)^\theta$. In this subsection, we show an estimate for n defined as (2.1), (2.2), and (2.4) at $y = \overline{\varepsilon}(s)^\theta$. We use $\overline{\Omega}(s) = n(\overline{\varepsilon}(\overline{s})^\theta, s)$ for $s, \overline{s} \in [s_0, s_1]$. Since it follows from (3.3) that

$$|n(y,s) - 2y^2 - \psi(y,s)| \leq \left| \int_0^y \phi(\tilde{y}, s)(e^{\tilde{y}^2/4} - 1)\tilde{y}d\tilde{y} \right| \leq C_M \overline{\varepsilon}(s)^2 |\log \overline{\varepsilon}(s)|^2 y^2$$

for $0 \leq y \ll 1$, we have

$$|\overline{\Omega}(s) - 2\overline{\varepsilon}(\overline{s})^{2\theta} - \psi(\overline{\varepsilon}(\overline{s})^\theta, s)| \leq C_M \overline{\varepsilon}(s)^{2+2\theta} |\log \overline{\varepsilon}(s)|^2 \tag{4.12}$$

for s, $\overline{s} \in [s_0, s_1]$ with $|s - \overline{s}| \leq C/s$ if $s_0 \gg 1$.

Lemma 4.4. *Under the assumptions of Lemma 4.1 as well as (3.15), there exists a positive constant* C *such that*

$$|\overline{\Omega}(s) - \overline{\Omega}(\overline{s})| \leq C\overline{\varepsilon}(\overline{s})^2$$

for s, $\overline{s} \in [s_0, s_1]$ *with* $|s - \overline{s}| \leq 1/s$ *if* $0 < \theta \ll 1$ *and* $s_0 \gg 1$.

The proof of Lemma 4.4 is given in Section 5.3.

4.4. **Analysis of inner region.** In this subsection, we prove that if $\phi \in \mathcal{A}(s_0, s_1, 1)$ ψ rapidly approaches a suitable stationary function in the region $y \in [0, \overline{\varepsilon}(s)^\theta]$ and $s_0 \gg 1$. Let $L > 0$ be a positive and sufficiently small constant, and let

$$\omega_L(\eta) = 2\Big(\eta^2 - L^2 \log(L^2 + \eta^2) + L^2 \log L^2\Big)$$

satisfy the equation

$$\omega_L'' + \frac{\omega_L'}{\eta} + \frac{1}{2}\Big(\frac{\omega_L'}{\eta}\Big)^2 - 4\frac{\omega_L'}{\eta} = 0.$$

Furthermore, we consider a related equation

$$\omega'' + \frac{\omega'}{\eta} + \frac{1}{2}\Big(\frac{\omega'}{\eta}\Big)^2 - 4\Big(\frac{\omega'}{\eta}\Big) + \overline{\Lambda}^2\Big(\frac{\eta\omega'}{2} - \omega\Big) = 0, \tag{4.13}$$

where $\overline{\Lambda}$ is a positive and sufficiently small constant. We can find a solution $\tilde{\omega}_L$ to (4.13) satisfying

$$\tilde{\omega}_L(\eta) = \omega_L(\eta) + O(\overline{\Lambda}^2 L^4)\frac{(\eta/L)^4}{1+(\eta/L)^2}\left\{1+\log\left(1+(\eta/L)^2\right)\right\}, \tag{4.14}$$

$$\tilde{\omega}_{L\eta}(\eta) = \omega_{L\eta}(\eta) + O(\overline{\Lambda}^2 L^3)\frac{(\eta/L)^3}{1+(\eta/L)^2}\left\{1+\log\left(1+(\eta/L)^2\right)\right\} \tag{4.15}$$

as $\overline{\Lambda} \to 0$ and $L \to 0$, uniformly for $\eta \in [0,1]$, and

$$\tilde{\omega}_L(\eta) = O(\eta^4) \quad \text{as } \eta \to 0. \tag{4.16}$$

We consider the solution ω to the equation

$$\omega_s = \omega_{\eta\eta} + \frac{\omega_\eta}{\eta} + \frac{1}{2}\Big(\frac{\omega_\eta}{\eta}\Big)^2 - 4\Big(\frac{\omega_\eta}{\eta}\Big) + \overline{\Lambda}^2\Big(\frac{\eta\omega_\eta}{2} - \omega\Big) = 0 \tag{4.17}$$

for $\eta \in [0,1]$ and $s \geq 0$. For the solution $n(y,s)$ to (2.6), the function $\omega(\eta,s) = \overline{\Lambda}^{-2} n(\overline{\Lambda}\eta, \overline{\Lambda}^2 s + \overline{s})$ satisfies (4.17).

In the following lemma, we describe the existence of sub-solutions and super-solutions to (4.17) that approach $\tilde{\omega}_L$ satisfying (4.13), (4.14), (4.15), and (4.16).

Lemma 4.5. *Let $\overline{\Lambda}$ and L be positive and sufficiently small. There exist functions $Z^{\pm} \equiv Z^{\pm}(\cdot, \overline{\Lambda}, L)$ and constants $E \in (0,1)$ and $B > 1$ satisfying*

$$0 < (\omega_L(\eta) - \omega_{BL}(\eta)) \leq Z^{\pm}(\eta) \leq (\omega_{EL}(\eta) - \omega_L(\eta))$$

for $\eta \in [0,1]$, and the functions

$$\begin{aligned} \omega_+(\eta,s) &= \tilde{\omega}_L(\eta) + e^{-s/100} Z^+(\eta) \\ \omega_-(\eta,s) &= \tilde{\omega}_L(\eta) - e^{-s/100} Z^-(\eta) \end{aligned}$$

are super-solutions and sub-solutions to (4.17) in $[0,1] \times \mathbf{R}_+$, respectively.

This lemma and the estimates (4.14) and (4.15) will be proved in Section 5.4.

4.5. **Estimates of size of inner boundary layer.** In this subsection, for $\overline{s} \in [s_0, s_1)$, we consider

$$\omega(\eta,s) = \Lambda(\overline{s})^{-2} n(\Lambda(\overline{s})\eta, \overline{s} + \Lambda(\overline{s})^2 s),$$

where $\Lambda(\overline{s}) = \overline{\varepsilon}(\overline{s})^\theta$ and n is a function defined in (2.1), (2.2), and (2.4). Then, we observe that ω is a solution to (4.17) with $\overline{\Lambda} = \Lambda(\overline{s})$. It follows from Lemma 4.4 that

$$|\omega(1,s) - \omega(1,0)| \leq C\Big(\frac{\overline{\varepsilon}(\overline{s})}{\Lambda(\overline{s})}\Big)^2 \quad \text{for } s \in [0, 1/\{\Lambda(\overline{s})^2\overline{s}\}].$$

Let us take $L^\pm > 0$ such that

$$\tilde{\omega}_{L^\pm}(1) = \omega(1,0) \pm C\Big(\frac{\overline{\varepsilon}(\overline{s})}{\Lambda(\overline{s})}\Big)^2 = \frac{1}{\Lambda(\overline{s})^2}\Big(\overline{\Omega}(\overline{s}) \pm C\overline{\varepsilon}(\overline{s})^2\Big), \tag{4.18}$$

where $\tilde{\omega}_{L^\pm}$ are the functions satisfying (4.13), (4.14), (4.15), and (4.16) with $\overline{\Lambda} = \Lambda(\overline{s})$ and $L = L^\pm$. Using $\overline{L} = \overline{\varepsilon}(\overline{s})/\Lambda(\overline{s})$, we have

$$|L^+ - L^-| \leq \frac{C}{\overline{s}}\overline{L} \tag{4.19}$$

if $s_0 \gg 1$.

When $\overline{s} \geq s_0 + 1/s_0$, we assume that there exists a constant $B > 1$ in Lemma 4.5 satisfying

$$|\omega(\eta,0) - \tilde{\omega}_{L^\pm}(\eta)| \leq \omega_{L^\pm}(\eta) - \omega_{BL^\pm}(\eta) \quad \text{for } \eta \in [0,1]. \tag{4.20}$$

Combining this with Lemma 4.5 implies

$$\begin{aligned}\omega(\eta,s) \quad &\leq \quad \tilde{\omega}_{L^+}(\eta) + \exp\Big(-\frac{1}{200\overline{s}\Lambda(\overline{s})^2}\Big)\Big(\omega_{EL^+}(\eta) - \omega_{L^+}(\eta)\Big) \\ &\qquad \text{for } \eta \in [0,1] \ \text{ and } \ s \in \Big[\frac{1}{2\overline{s}\Lambda(\overline{s})^2}, \frac{1}{\overline{s}\Lambda(\overline{s})^2}\Big]\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\omega(\eta,s) \quad &\geq \quad \tilde{\omega}_{L^-}(\eta) - \exp\Big(-\frac{1}{200\overline{s}\Lambda(\overline{s})^2}\Big)\Big(\omega_{EL^-}(\eta) - \omega_{L^-}(\eta)\Big) \\ &\qquad \text{for } \eta \in [0,1] \ \text{ and } \ s \in \Big[\frac{1}{2\overline{s}\Lambda(\overline{s})^2}, \frac{1}{\overline{s}\Lambda(\overline{s})^2}\Big],\end{aligned} \tag{4.22}$$

where E is the constant in Lemma 4.5. Then, replacing $\overline{s}$ by $\overline{s}_1 = \overline{s} + 1/(2\overline{s})$, we observe (4.20) with $\overline{s} = \overline{s}_1$. By this, we have (4.21) and (4.22) with $\overline{s} = \overline{s}_1$. Repeating this argument, we obtain (4.22) and (4.21) with $\overline{s} \in [s_0, \tilde{s}_1]$, where $\tilde{s}_1 \in [s_0, s_1]$ satisfies $\tilde{s}_1 + 1/\tilde{s}_1 = s_1$.

By using these estimates and the parabolic regularity, we have the following lemmas, if ω has the above mentioned properties when $\overline{s} = s_0$.

Furthermore, when $\overline{s} = s_0$, let $L^\pm$ be constants satisfying (4.18) with $\overline{s} = s_0$. Then, ω satisfies

$$\tilde{\omega}_{L^-}(\eta) \leq \omega(\eta,0) \leq \tilde{\omega}_{L^+}(\eta) \quad \text{for } \eta \in [0,1], \tag{4.23}$$

since $m(\cdot, s_0)$ satisfies (3.7) and (3.8). Then, when $\overline{s} = s_0$, ω satisfies (4.21) and (4.22) for $\eta \in [0,1]$ and $s \in [0, 1/(\Lambda(\overline{s})^2\overline{s})]$ with $E = 1$ if $s_0 \gg 1$. Then, for $\overline{s} \in [0, s_1]$, there exist constants $L^\pm$ satisfying (4.19) and (4.23). From this, we can show the following lemmas.

Lemma 4.6. *Under the assumptions of Lemma* 4.1 *as well as* (3.15), ε *satisfies*

$$|\varepsilon(s)^2 - \varepsilon(\overline{s})^2| \le \frac{C}{\overline{s}}\overline{\varepsilon}(\overline{s})^2 \quad \text{for } \overline{s}, s \in [s_0, s_1] \quad \text{with} \quad |\overline{s} - s| \le 1/\overline{s} \tag{4.24}$$

and

$$|\varepsilon(s)^2 - \overline{\varepsilon}(s)^2| \le \frac{C}{s}\overline{\varepsilon}(s)^2 \quad \text{for } s \in [s_0, s_1] \tag{4.25}$$

if $s_0 \gg 1$.

Lemma 4.7. *Let A be a sufficiently large constant. Under the assumptions of Lemma 4.1 as well as* (3.15), ϕ *satisfies*

$$\begin{aligned} \Big|\phi(y,s) + \frac{4}{1+(y/\overline{\varepsilon}(s))^2}\Big| &\le C\Big(\frac{1}{s}\chi_{[0,\overline{\varepsilon}(s)^\theta]}(y) + \chi_{[\overline{\varepsilon}(s)^\theta,\infty)}(y)\Big) \\ &\cdot \Big(\frac{(y/\overline{\varepsilon}(s))^2}{\{1+(y/\overline{\varepsilon}(s))^2\}^2} + \frac{\overline{\varepsilon}(s)^2|\log(1+(y/\overline{\varepsilon}(s))^2)|}{y^2}(1+y^2)\chi_{[\overline{\varepsilon}(s),\infty)}(y)\Big) \end{aligned} \tag{4.26}$$

for $s \in [s_0, s_1]$ *and* $y \in [0, A]$ *if* $s_0 \gg 1$.

The proofs of Lemmas 4.6 and 4.7 are given in Section 5.5.

4.6. **Analysis of external region.** Let A be a sufficiently large constant. By (2.5) and (2.7), we observe

$$\mathcal{M}(\phi) = \phi_s - \phi_{yy} - \Big(\frac{3}{y} - \frac{y}{2}\Big)\phi_y + 2\phi - e^{-y^2/4}\phi\Big(\frac{1}{y}\phi_y - \frac{1}{2}\phi\Big) = 0. \tag{4.27}$$

Since it follows from (3.7), (4.27), and Lemma 4.7 that

$$\phi(y, s_0) \ge -C\overline{\varepsilon}(s_0)^2 \quad \text{for } y \ge A,$$

$$\phi(A, s) \ge -C\overline{\varepsilon}(s)^2|\log\overline{\varepsilon}(s)| \quad \text{for } s \in [s_0, s_1],$$

and

$$\mathcal{M}(-Ce^{-(s-s_0)}\overline{\varepsilon}(s_0)^2|\log\overline{\varepsilon}(s_0)|) \le 0,$$

we have

$$\phi(y,s) \ge -C\overline{\varepsilon}(s)^2|\log\overline{\varepsilon}(s)|^2 \quad \text{for } y \ge A \quad \text{and} \quad s \in [s_0, s_1]. \tag{4.28}$$

Combining this with $\phi \le 0$ implies this lemma.

Lemma 4.8. *Let A be a sufficiently large constant. Under the assumption of Proposition* 3.1, ϕ *satisfies*

$$|\phi(y,s)| \le C\varepsilon(s)^2|\log\varepsilon(s)|^2$$

for $y \ge A$ *and* $s \in [s_0, s_1]$ *with some positive constant C if* $s_0 \gg 1$.

4.7. **End of proof of Proposition 3.1.** By Lemmas 4.6, 4.7, and 4.8, we have (3.2), (3.3), (3.5), and (3.6) when the constants μ and M in (3.2), (3.3), (3.5), and (3.6) are replaced by 1 and some constant C independent of M. Then, taking a sufficiently large M, we obtain $\phi \in \mathcal{A}(s_0, s_1, 1/2)$. Thus, we have Proposition 3.1.

4.8. **Proof of Theorem 1.** Let s_0 and s_1 be constants satisfying $s_1 \geq s_0 \gg 1$. We define $\mathcal{U}(s_0, s_1) \subset (-\infty, 0]$ as the open set consisting of all constants α_0 such that the corresponding constant ε_0 and the corresponding function $m(y, s_0)$ satisfy (3.7), (3.8), and (3.9), and such that the corresponding function ϕ defined in (2.7) satisfies $\phi \in \mathcal{A}(s_0, s_1, 1)$.

According to the choice of $m(\cdot, s_0)$ in Section 3, we can find a unique α_0 satisfying $P(\alpha_0, s_0) = 0$ if $s_0 \gg 1$, where P is a mapping in (3.13). Then, we have

$$\deg(P(\cdot, s_0), \mathcal{U}(s_0, s_0), 0) = 1,$$

where deg is the topological degree of the mapping $P(\cdot, s_1)$ in the set $\mathcal{U}(s_0, s_1)$ at the value 0.

Assume $\mathcal{U}(s_0, s) \neq \emptyset$ for any $s \in [s_0, s_1]$. We denote the boundary of the open set $\mathcal{U}(s_0, s)$ by $\partial\mathcal{U}(s_0, s)$. If $P(\partial\mathcal{U}(s_0, s), s) \neq 0$ for $s \in [s_0, s_1]$, then

$$\deg(P(\cdot, s), \mathcal{U}(s_0, s)), 0) = \deg(P(\cdot, s_0), \mathcal{U}(s_0, s_0)), 0) = 1 \quad \text{for } s \in [s_0, s_1].$$

If, for some $\overline{s} \in (s_0, s_1]$, $\mathcal{U}(s_0, \overline{s}) \neq \emptyset$, and $P(\alpha_0, \overline{s}) = 0$ for some $\alpha_0 \in \partial\mathcal{U}(s_0, \overline{s})$, it follows from Proposition 3.1 that the corresponding function $\phi \in \mathcal{A}(s_0, \overline{s}, 1/2)$. This contradicts $\alpha_0 \in \partial\mathcal{U}(s_0, \overline{s})$. If $s^* = \sup\{s \in [s_0, s_1] : \mathcal{U}(s_0, s) \neq \emptyset\} < s_1$, there exist two sequences $\{s_k\}_{k=1}^{\infty} \subset [s_0, s^*)$ and $\{\alpha_{0k}\}_{k=1}^{\infty} \subset (-\infty, 0]$ satisfying

$$\lim_{k \to \infty} s_k = s^*, \ \ \alpha_{0k} \in \mathcal{U}(s_0, s_k) \ \text{ and } \ P(\alpha_{0k}, s_0, s_k) = 0.$$

Since $\{\alpha_{0k}\}$ is bounded in $(-\infty, 0]$, we can assume $\alpha_0^* = \lim_{k \to \infty} \alpha_{0k}$ without loss of generality. Then, we have $P(\alpha_0^*, s_0, s^*) = 0$. Combining this with Proposition 3.1 implies $\phi \in \mathcal{A}(s_0, s^*, 1/2)$. Then, for a sufficiently small $\delta > 0$, $\mathcal{U}(s_0, s) \neq \emptyset$ for $s \in [s_0, s^* + \delta]$. This contradicts the definition of s^*. Then, we obtain

$$\mathcal{U}(s_0, s) \neq \emptyset \quad \text{for } s \in [s_0, s_1] \tag{4.29}$$

and

$$P(\partial\mathcal{U}(s_0, s), s) \neq 0 \quad \text{for } s \in [s_0, s_1]. \tag{4.30}$$

For any $s_1 \geq s_0$, (4.29) and (4.30) hold. Then, there exists a sequence $\{\alpha_{0k}\}_{k=1}^{\infty} \subset (-\infty, 0]$ satisfying $P(\alpha_{0k}, k + s_0) = 0$. Combining this with Proposition 3.1 implies that the solution ϕ_k corresponding to α_{0k} satisfies $\phi_k \in \mathcal{A}(s_0, s_0 + k, 1/2)$. Since the sequence $\{\alpha_{0k}\}_{k=1}^{\infty}$ is bounded in $(-\infty, 0]$, we can assume that $\alpha_{0,\infty} = \lim_{k\to\infty} \alpha_{0k}$ without loss of generality. The function ϕ_∞ corresponding to $\alpha_{0,\infty}$ satisfies $\phi_\infty \in \mathcal{A}(s_0, s_0 + k, 1/2)$ for any $k = 1, 2, 3, \cdots$. Then, we obtain

$$\mathcal{A}(s_0, \infty, 1) \supset \bigcap_{k=1}^{\infty} \mathcal{A}(s_0, s_0 + k, 1/2) \neq \emptyset.$$

Since the function ϕ_∞ satisfies Lemmas 4.7 and 4.8 with $s_1 = \infty$, the corresponding function m satisfies

$$\Big| m(y, s) - \frac{4(y/\overline{\varepsilon}(s))^2}{1 + (y/\overline{\varepsilon}(s))^2} \Big| \leq C\overline{\varepsilon}(s)^2 \frac{(y/\overline{\varepsilon}(s))^2}{1 + (y/\overline{\varepsilon}(s))^2} + \frac{C}{s}\Big(\frac{(y/\overline{\varepsilon}(s))^2}{\{1 + (y/\overline{\varepsilon}(s))^2\}^2} + \frac{\overline{\varepsilon}(s)^2 |\log(1 + (y/\overline{\varepsilon}(s))^2|}{y^2} (1 + y^2)\chi_{[\overline{\varepsilon}(s),\infty)}(y)\Big)$$

for $\overline{s} \in [s_0, s_1]$ and $y \in [0, \overline{\varepsilon}(s)^\theta]$.

Let $\overline{\delta}(s) = e^{s/2}\overline{\varepsilon}(s)$. For $(\overline{y}, \overline{s}) \in \mathbf{R}_+ \times [s_0, \infty)$, using $\mathcal{P}(\zeta, \tau; \overline{y}) = m(\overline{y}(1 + \zeta), \overline{s} + \overline{y}^2\tau)$ and

$$\overline{\mathcal{P}}(\zeta; \overline{y}) = \frac{4\overline{y}^2(1 + \zeta)^2}{\overline{y}^2(1 + \zeta)^2 + \exp\Big(- \overline{s} - \overline{y}^2\tau\Big)\overline{\delta}(\overline{s})^2},$$

the functions $\mathcal{P}(\zeta, \tau; \overline{y})$ and $\overline{\mathcal{P}}(\zeta; \overline{y})$ satisfy

$$\mathcal{P}_\tau = \mathcal{P}_{\zeta\zeta} - \frac{1}{1 + \zeta}\mathcal{P}_\zeta + \frac{\overline{y}^2(\zeta + 1)}{2}\mathcal{P}_\zeta + \frac{1}{1 + \zeta}\mathcal{P}\mathcal{P}_\zeta.$$

By using the parabolic regularity, we have

$$|\mathcal{P}_\zeta(\zeta, \tau; \overline{y})| \leq C \max_{-2/3\leq\zeta\leq 2/3, 0\leq\tau\leq 1} |\mathcal{P}(\zeta, \tau; \overline{y})|$$

for $\zeta \in [-1/2, 1/2]$ and $\tau \in [1/3, 1]$. By these facts and the parabolic regularity, we obtain

$$\begin{aligned} &\Big| y^{-1}m_y(y, s) - \frac{8}{\varepsilon(s)^2} \frac{1}{\{1 + (y/\overline{\varepsilon}(s))\}^2} \Big| \\ &\leq \quad \frac{C}{1 + (y/\overline{\varepsilon}(s))^2} + \frac{C}{s}\Big(\frac{1}{\overline{\varepsilon}(s)^2} \frac{1}{\{1 + (y/\overline{\varepsilon}(s))^2\}^2} \\ &\qquad + \frac{\overline{\varepsilon}(s)^2 |\log(1 + (y/\overline{\varepsilon}(s))^2)|}{y^4} (1 + y^2)\chi_{[\overline{\varepsilon}(s),\infty)}(y)\Big) \end{aligned}$$

for $s \in [s_0, \infty)$ and $y \in [0, \overline{\varepsilon}(s)^\theta/2)$. This implies

$$\Big| u(x, t) - \frac{8}{E(t)^2} \frac{1}{\{1 + (|x|/E(t; K))^2\}^2} \Big|$$

$$\leq \quad C\frac{(1+t)^{-1}}{1+(|x|/E(t;K))^2} + \frac{C}{\log(1+t)}\Big(\frac{1}{E(t;K)^2}\frac{1}{\{1+(|x|/E(t;K)^2\}^2} + \frac{E(t;K)^2|\log(1+(|x|/E(t;K))^2)|}{|x|^4}\chi_{[E(t;K),\infty)}(y)\Big) \quad (4.31)$$

for $t \in [e^{s_0}-1,\infty)$ and $x \in \mathbf{R}^2$ with $|x| \leq R(t;K,\theta)$. Using $T = e^{s_0}-1$ and replacing t by $t+T$, we obtain (iv) of Theorem 1 for $x \in \mathbf{R}^2$ with $|x| \leq R(t;K,\theta)$ and $t \geq 0$. By using Lemma 4.7 and an argument similar to that used to establish (4.31), we obtain (iv) of Theorem 1 for $x \in \mathbf{R}^2$ with $R(t+T;K,\theta) \leq |x| \leq \sqrt{t+T+1}$.

Since the function ϕ_∞ satisfies Lemmas 4.7 and 4.8 with $s_1 = \infty$, the corresponding function M satisfies

$$|M(r,t)-4| \leq \frac{C}{t+T+1}\exp\Big(-\frac{r^2}{4(t+T+1)}\Big)$$

for $r \geq \sqrt{t+T+1}/2$ and $t \geq 0$. Using this and applying the parabolic regularity to (2.3), we obtain (iii) of Theorem 1.

Since $m(y,s_0)$ increases with y and $y^{-1}m_y(y,s_0)$ does not increase with y, we have (i) and (ii). Thus, we have proved Theorem 1. □

5. Some technical results

In this section, we prove some lemmas described in Section 4.

5.1. Proof of Lemma 4.2.

Using

$$Z(y,s) = \begin{cases} R(y,s) - Q(y,s) & \text{for } y \geq 0 \text{ and } s > s_0, \\ 0 & \text{for } y \geq 0 \text{ and } s = s_0, \end{cases}$$

it follows from (2.8), (2.16), (2.17), and (4.1) that

$$\begin{aligned} Z_s - \mathcal{A}Z &= \Big(F - \langle F,\varphi_0\rangle\varphi_0\Big) - 4\varepsilon(s)^2\Big(\mathcal{D}_0 - \langle\mathcal{D}_0,\varphi_0\rangle\varphi_0\Big) \\ &= \Big(F - 4\varepsilon(s)^2\mathcal{D}_0\Big) - \Big(\langle F,\varphi_0\rangle\varphi_0 - 4\varepsilon(s)^2\langle\mathcal{D}_0,\varphi_0\rangle\varphi_0\Big) = H. \end{aligned}$$

Let $S(s)$ be the semigroup corresponding to the operator $\mathcal{A}$ in (2.8). Then, for $f_0 \in L^2_w$, $S(s)f_0$ is the solution to

$$\frac{\partial S(s)f_0}{\partial s} = \mathcal{A}S(s)f_0 \quad \text{for } s > 0 \text{ and } S(0)f_0 = f_0$$

and it satisfies

$$[S(s)f_0](|Y|) = \int_{\mathbf{R}^2} K(Y,\tilde{Y},s)f_0(|\tilde{Y}|)d\tilde{Y},$$

where

$$K(Y,\tilde{Y},s)=\frac{e^{-s}}{4\pi(1-e^{-s})}\exp\Big(-\frac{|Ye^{-s/2}-\tilde{Y}|^2}{4(1-e^{-s})}\Big)$$

for $Y,\tilde{Y}\in\mathbf{R}^2$ and $s\geq 0$.

Then, we can represent Z in the form

$$Z(\cdot,s)=\int_{s_0}^{s}S(s-\tilde{s})H(\cdot,\tilde{s})d\tilde{s}$$

and the function F in the form

$$F(y,s)=F(y,s)\chi_{[0,\overline{\varepsilon}(s)^{2\theta}]}(y)+F(y,s)\chi_{(\overline{\varepsilon}(s)^{2\theta},\infty)}(y)=F_1(y,s)+F_2(y,s).$$

By (2.10) and (3.3), we have

$$\begin{aligned}|F_2(y,s)| &\leq C\chi_{(\overline{\varepsilon}(s)^{2\theta},\infty)}(y)e^{y^2/4}(m(y,s)-4)^2\\ &\leq C_M\chi_{(\overline{\varepsilon}(s)^{2\theta},\infty)}(y)e^{-y^2/4}\Big\{\frac{1}{(1+\xi^2)^2}+\frac{\xi^4}{(1+\xi^2)^4}\\ &\qquad +\overline{\varepsilon}(s)^4|\log\overline{\varepsilon}(s)|^4\frac{1+y^4}{y^4}\Big\},\end{aligned}$$

where $\xi=y/\overline{\varepsilon}(s)$. Using

$$\begin{aligned}H_1(y,s) &= F_1(y,s)-4\varepsilon(s)^2\mathcal{D}_0-\langle F_1(\cdot,s)-4\varepsilon(s)^2\mathcal{D}_0,\varphi_0\rangle\varphi_0\\ &= F_1(y,s)-4\varepsilon(s)^2\mathcal{D}_0\end{aligned}$$

and

$$H_2(y,s) = F_2(y,s)-\langle F_2(\cdot,s),\varphi_0\rangle\varphi_0(y),$$

we observe that

$$\begin{aligned}Z(y,s) &= \int_{s_0}^{s}\left[S(s-\tilde{s})H_1(\cdot,\tilde{s})\right](y)d\tilde{s}+\int_{s_0}^{s}\left[S(s-\tilde{s})H_2(\cdot,\tilde{s})\right](y)d\tilde{s}\\ &= Z_1(y,s)+Z_2(y,s).\end{aligned}\tag{5.1}$$

We use

$$\begin{aligned}Z_1(y,s)=&\int_{s_0}^{\max(s-1,s_0)}\left[S(s-\tilde{s})H_1(\cdot,\tilde{s})\right](y)d\tilde{s}\\ &+\int_{\max(s-1,s_0)}^{s}\left[S(s-\tilde{s})H_1(\cdot,\tilde{s})\right](y)d\tilde{s}=Z_{1,1}(y,s)+Z_{1,2}(y,s).\end{aligned}$$

Since $Z_{1,1}=0$ for $s\in[s_0,s_0+1]$, we can assume $s>s_0+1$ without loss of generality. Then, we observe that

$$Z_{1,1}(\cdot,s)=\int_{s_0}^{s-1}S\Big(s-\frac{1}{2}-\tilde{s}\Big)S\Big(\frac{1}{2}\Big)H_1(\cdot,\tilde{s})d\tilde{s}.$$

Let

$$L^2_{rad}=\Big\{f\in L^2_w\ :\ f(y)e^{y^2/4}\in L^2_w\ \Big\}$$

and let

$$\langle f,g\rangle_{L^2_{rad}}=\int_0^\infty f(y)g(y)ydy\quad\text{and}\quad\|f\|_{L^2_{rad}}=\sqrt{\langle f,f\rangle_{L^2_{rad}}}$$

for $f,\ g\in L^2_{rad}$. By (3.4), for $\varphi\in L^2_{rad}$, we observe that

$$\begin{aligned}
\langle S\Big(\frac{1}{2}\Big)&H_1(\cdot,\tilde{s}),\varphi\rangle_{L^2_{rad}}=\frac{1}{2\pi}\int_{\mathbf{R}^2}\varphi(|Y|)\left[S\Big(\frac{1}{2}\Big)H_1(\cdot,\tilde{s})\right](|Y|)dY\\
&=\frac{1}{2\pi}\int_{\mathbf{R}^2}\varphi(|Y|)\left[S\Big(\frac{1}{2}\Big)F_1(\cdot,\tilde{s})\right](|Y|)dY\\
&\qquad-\frac{1}{2\pi}\int_{\mathbf{R}^2}\varphi(|Y|)8\pi\varepsilon(\tilde{s})^2K(Y,0,1/2)dY\\
&=\frac{1}{2\pi}\iint_{\mathbf{R}^2\times\mathbf{R}^2}\varphi(|Y|)\Big(K(Y,\tilde{Y},1/2)-K(Y,0,1/2)\Big)F_1(|\tilde{Y}|,\tilde{s})dYd\tilde{Y}\\
&\quad+\frac{1}{2\pi}\iint_{\mathbf{R}^2\times\mathbf{R}^2}\varphi(|Y|)K(Y,0,1/2)F_1(|\tilde{Y}|,\tilde{s})\Big(1-e^{-|\tilde{Y}|^2/4}\Big)dYd\tilde{Y}\\
&=J_1+J_2.
\end{aligned}$$

Since

$$\begin{aligned}
|\tilde{Y}|&\le\overline{\varepsilon}(\tilde{s})^{2\theta}\le C\Big(\frac{s}{\tilde{s}}\Big)^{2\theta}e^{-2\theta(\tilde{s}-s)/2}\overline{\varepsilon}(s)^{2\theta}\\
&\le C\overline{\varepsilon}(s)^{2\theta}\le\frac{1}{10e}\overline{\varepsilon}(s)^{\theta}\le\frac{1}{10}|Ye^{-(s-\tilde{s})}|
\end{aligned}\tag{5.2}$$

for $\tilde{Y}\in\mathbf{R}^2$ with $|\tilde{Y}|\le\overline{\varepsilon}(\tilde{s})^{2\theta}$, $Y\in\mathbf{R}^2$ with $\overline{\varepsilon}(s)^{\theta}\le|Y|$, and $s,\ \tilde{s}\in[s_0,s_1]$ with $|s-\tilde{s}|\le1$, we observe that

$$\Big|e^{-|Y|^2}-e^{-|Y-\tilde{Y}|^2}\Big|\le3e^{-|Y|^2/2}|Y||\tilde{Y}|\quad\text{ if }|\tilde{Y}|\le\frac{|Y|}{10}.\tag{5.3}$$

Combining this with (3.3) implies that

$$\begin{aligned}
|J_1|=\Big|\frac{1}{2\pi}\int_{\mathbf{R}^2}\int_{|\tilde{Y}|\le\overline{\varepsilon}(\tilde{s})^{2\theta}}&\varphi(|Y|)\Big(K(Y,\tilde{Y},1/2)\\
&-K(Y,0,1/2)\Big)F_1(|\tilde{Y}|,\tilde{s})d\tilde{Y}dY\Big|
\end{aligned}$$

$$\leq C \int_{|Y| \leq \overline{\varepsilon}(\tilde{s})^{\theta}} \int_{|\tilde{Y}| \leq \overline{\varepsilon}(\tilde{s})^{2\theta}} |\varphi(|Y|)||\tilde{Y}|F_1(|\tilde{Y}|, \tilde{s}) dY d\tilde{Y}$$

$$+ C \int_{|Y| \geq \overline{\varepsilon}(\tilde{s})^{\theta}} \int_{|\tilde{Y}| \leq \overline{\varepsilon}(\tilde{s})^{2\theta}} |\varphi(|Y|)||\tilde{Y}||Y|K(Y/\sqrt{2}, 0, 1/2)F_1(|\tilde{Y}|, \tilde{s}) dY d\tilde{Y}$$

$$\leq C_M \|\varphi\|_{L^2_{rad}} \int_0^{\overline{\varepsilon}(\tilde{s})^{2\theta}} \Big\{ \frac{1}{(1+\xi^2)^2} + \frac{\xi^4}{(1+\xi^2)^4} + \overline{\varepsilon}(\tilde{s})^4 |\log \overline{\varepsilon}(\tilde{s})|^4 \frac{1+y^4}{y^4} \chi_{[\overline{\varepsilon}(\tilde{s}),\infty)}(y) \Big\} y^2 dy$$

$$\leq C_M \|\varphi\|_{L^2_{rad}} \overline{\varepsilon}(\tilde{s})^3 |\log \overline{\varepsilon}(\tilde{s})|^4. \tag{5.4}$$

By using an argument similar to that used to establish (5.4), we have

$$\begin{aligned} |J_2| &\leq C \int_{\mathbf{R}^2} \int_{|\tilde{Y}| \leq \overline{\varepsilon}(\tilde{s})^{2\theta}} |\varphi(|Y|)| K(Y/\sqrt{2}, 0, \tilde{s}) F_1(|\tilde{Y}|, \tilde{s}) |\tilde{Y}|^2 dY d\tilde{Y} \\ &\leq C_M \|\varphi\|_{L^2_{rad}} \overline{\varepsilon}(\tilde{s})^4 |\log \overline{\varepsilon}(\tilde{s})|^5. \end{aligned}$$

Then, we obtain

$$\Big| \langle S\Big(\tfrac{1}{2}\Big) H_1(\cdot, \tilde{s}), \varphi \rangle_{L^2_{rad}} \Big| \leq C_M \|\varphi\|_{L^2_{rad}} \overline{\varepsilon}(\tilde{s})^3 |\log \overline{\varepsilon}(\tilde{s})|^4.$$

This implies

$$\Big\| S\Big(\tfrac{1}{2}\Big) H_1(\cdot, \tilde{s}) \Big\|_{L^2_{rad}} \leq C_M \overline{\varepsilon}(\tilde{s})^3 |\log \overline{\varepsilon}(\tilde{s})|^4. \tag{5.5}$$

Combining this with $\langle H_1(\cdot, \tilde{s}), \varphi_0 \rangle = 0$, $\|\cdot\| \leq \|\cdot\|_{L^2_{rad}}$ and Lemma 2.1 implies

$$\begin{aligned} &\Big| S(s-\tilde{s}-1/2) S(1/2) H_1(\cdot, \tilde{s}) \Big| \\ &\leq \Big| S(s-\tilde{s}-1/2) \sum_{j=1}^{\infty} \langle S(1/2) H_1(\cdot, \tilde{s}), \varphi_j \rangle \varphi_j \Big| \\ &\leq \sum_{j=1}^{\infty} e^{-\lambda_j (s-\tilde{s}-1/2)} \|S(1/2) H_1(\cdot, \tilde{s})\| \frac{1}{\sqrt{2}} \Big(1 + \frac{y^2}{4}\Big)^j \\ &\leq C_M e^{-2(s-\tilde{s}-1/2)} \overline{\varepsilon}(\tilde{s})^3 |\log \overline{\varepsilon}(\tilde{s})|^4 \Big(1 + \frac{y^2}{4}\Big) \\ &\qquad \cdot \sum_{j=1}^{\infty} \Big\{ e^{-(s-\tilde{s}-1/2)} + \frac{1}{4} \Big(e^{-(s-\tilde{s}-1/2)/2} y \Big)^2 \Big\}^{j-1}. \end{aligned}$$

Since it follows from $s - \tilde{s} - 1/2 \geq 1/2$ that

$$e^{-(s-\tilde{s}-1/2)} + \frac{1}{4} \Big(e^{-(s-\tilde{s}-1/2)/2} y \Big)^2 \leq \frac{1}{e^{1/2}} + \frac{1}{4} < 1$$

for $y \in [0, e^{-(s-\tilde{s}-1/2)/2}]$, we have

$$\begin{aligned} &\Big|S(s-\tilde{s}-1/2)S(1/2)H_1(\cdot,\tilde{s})\Big| \\ &\le \quad C_M e^{-2(s-\tilde{s}-1/2)}\overline{\varepsilon}(\tilde{s})^3|\log\overline{\varepsilon}(\tilde{s})|^4\Big(1+\frac{y^2}{4}\Big) \end{aligned} \tag{5.6}$$

for $y \in [0, e^{-(s-\tilde{s}-1/2)/2}]$. Furthermore, it follows from (5.5) that

$$\begin{aligned} &|S(s-\tilde{s})H_1(\cdot,\tilde{s})| = |S(s-\tilde{s}-1/2)S(1/2)H_1(\cdot,\tilde{s})| \\ &\le \quad Ce^{-(s-\tilde{s}-1/2)}\left\|S(1/2)H_1(\cdot,\tilde{s})\right\|_{L^2_{rad}} \\ &\qquad \cdot\Big(\int_{\mathbf{R}^2}\exp\Big(-\frac{|Ye^{-(s-\tilde{s}-1/2)/2}-\tilde{Y}|^2}{2(1-e^{-(s-\tilde{s}-1/2)})}\Big)d\tilde{Y}\Big)^{1/2} \\ &\le \quad C_M e^{-(s-\tilde{s}-1/2)}\overline{\varepsilon}(\tilde{s})^3|\log\overline{\varepsilon}(\tilde{s})|^4 \\ &\le \quad C_M e^{-2(s-\tilde{s}-1/2)}\overline{\varepsilon}(\tilde{s})^3|\log\overline{\varepsilon}(\tilde{s})|^4\Big(1+\frac{y^2}{4}\Big) \end{aligned}$$

for $y \ge e^{-(s-\tilde{s}-1/2)}$. Combining this with (5.6) implies

$$\Big|[S(s-\tilde{s})H_1(\cdot,\tilde{s})](y)\Big| \le C_M e^{-2(s-\tilde{s})}\overline{\varepsilon}(\tilde{s})^3|\log\overline{\varepsilon}(\tilde{s})|^4\Big(1+\frac{y^2}{4}\Big)$$

for $y \ge 0$ and $\tilde{s} \in [s_0, s-1]$. Then, we have

$$\begin{aligned} |Z_{1,1}(y,s)| \quad &\le \quad C_M e^{-2s}\int_{s_0}^{s-1} e^{2\tilde{s}}\overline{\varepsilon}(\tilde{s})^3|\log\overline{\varepsilon}(\tilde{s})|^4 d\tilde{s}(1+y^2) \\ &\le \quad C_M\overline{\varepsilon}(s)^3|\log\overline{\varepsilon}(s)|^4(1+y^2). \end{aligned} \tag{5.7}$$

We shall show an estimate of $Z_{1,2}$. We observe that

$$\begin{aligned} S(s-\tilde{s})H_1(\cdot,\tilde{s}) &= \int_{|\tilde{Y}|\le\overline{\varepsilon}(\tilde{s})^{2\theta}} K(Y,\tilde{Y},s-\tilde{s})\Big(F(|\tilde{Y}|,\tilde{s})-8\pi\overline{\varepsilon}(\tilde{s})^2\delta_0(\tilde{Y})\Big)d\tilde{Y} \\ &= \int_{|\tilde{Y}|\le\overline{\varepsilon}(\tilde{s})^{2\theta}}\Big\{K(Y,\tilde{Y},s-\tilde{s})-K(Y,0,s-\tilde{s})e^{-|\tilde{Y}|^2/4}\Big\}F(|\tilde{Y}|,\tilde{s})d\tilde{Y}. \end{aligned} \tag{5.8}$$

Combining (5.3) with (5.8) implies

$$\begin{aligned} &|[S(s-\tilde{s})H_1(\cdot,\tilde{s})](|Y|)| \\ &\le \frac{Ce^{-(s-\tilde{s})}|Y|}{(1-e^{-(s-\tilde{s})})^2}\int_{|\tilde{Y}|\le\overline{\varepsilon}(\tilde{s})^{2\theta}}\exp\Big(-\frac{|Y|^2e^{-(s-\tilde{s})}}{8(1-e^{-(s-\tilde{s})})}\Big)|\tilde{Y}|F(|\tilde{Y}|,\tilde{s})d\tilde{Y} \end{aligned} \tag{5.9}$$

for $Y \in \mathbf{R}^2$ with $|Y| \geq \overline{\varepsilon}(s)^\theta$ and $\tilde{s} \in [s-1, s]$. It follows from (3.3) that

$$\int_{|\tilde{Y}| \leq \overline{\varepsilon}(\tilde{s})^{2\theta}} |\tilde{Y}| F(|\tilde{Y}|, \tilde{s}) d\tilde{Y} \leq C_M \int_0^{\overline{\varepsilon}(\tilde{s})^{2\theta}} \Big\{ \frac{\overline{\varepsilon}(\tilde{s})^4}{(\overline{\varepsilon}(\tilde{s})^2 + y^2)^2} + \frac{\overline{\varepsilon}(\tilde{s})^4 y^4}{(\overline{\varepsilon}(\tilde{s})^2 + y^2)^4}$$
$$+ \frac{\overline{\varepsilon}(\tilde{s})^4 |\log \overline{\varepsilon}(\tilde{s})|^4}{y^4} \chi_{[\overline{\varepsilon}(\tilde{s}), \infty)}(y) \Big\} y^2 dy \leq C_M \overline{\varepsilon}(\tilde{s})^3 |\log \overline{\varepsilon}(\tilde{s})|^4.$$

Combining this with (5.9) implies

$$|Z_{1,2}(|Y|, s)| \leq \int_{s-1}^s |[S(s-\tilde{s}) H_1(\cdot, \tilde{s})](|Y|)| d\tilde{s}$$
$$\leq C_M \int_{s-1}^s \frac{e^{-(s-\tilde{s})}|Y|}{4\pi(1-e^{-(s-\tilde{s})})^2} \exp\Big(- \frac{|Y|^2 e^{-(s-\tilde{s})}}{8(1-e^{-(s-\tilde{s})})} \Big) \overline{\varepsilon}(\tilde{s})^3 |\log \overline{\varepsilon}(\tilde{s})|^4 d\tilde{s}$$
$$\leq \frac{C_M}{|Y|} \overline{\varepsilon}(s)^3 |\log \overline{\varepsilon}(s)|^4 \leq C_M \overline{\varepsilon}(s)^{3-\theta} |\log \overline{\varepsilon}(s)|^4 \quad \text{for } y \geq \overline{\varepsilon}(s)^\theta.$$

By this and (5.7), we have

$$|Z_1(y, s)| \leq C_M \overline{\varepsilon}(s)^{3-\theta} |\log \overline{\varepsilon}(s)|^4 (1 + y^2) \quad \text{for } y \geq \overline{\varepsilon}(s)^\theta. \tag{5.10}$$

Next, we shall consider an estimate of Z_2. We use

$$Z_2(y, s) = \int_{s_0}^{\max(s-1, s_0)} S(s-\tilde{s}) H_2(\cdot, \tilde{s}) d\tilde{s} + \int_{\max(s-1, s_0)}^s S(s-\tilde{s}) H_2(\cdot, \tilde{s}) d\tilde{s}$$
$$= Z_{2,1}(y, s) + Z_{2,2}(y, s). \tag{5.11}$$

Since it follows from (3.3) that

$$F_2(y, \tilde{s}) \leq C_M \overline{\varepsilon}(\tilde{s})^4 |\log \overline{\varepsilon}(\tilde{s})|^4 \Big\{ \frac{1}{y^4} + 1 \Big\} \chi_{[\overline{\varepsilon}(\tilde{s})^{2\theta}, \infty)}(y) \leq C_M \overline{\varepsilon}(\tilde{s})^{4-8\theta} |\log \overline{\varepsilon}(\tilde{s})|^4, \tag{5.12}$$

we have

$$\|H_2(\cdot, \tilde{s})\| \leq \|F_2(\cdot, \tilde{s})\| + |\langle F_2(\cdot, \tilde{s}), \varphi_0 \rangle| \|\varphi_0\| \leq C_M \overline{\varepsilon}(\tilde{s})^{4-8\theta} |\log \overline{\varepsilon}(\tilde{s})|^4. \tag{5.13}$$

As mentioned in the argument used to establish the estimate of Z_1, we may assume $s \geq s_0 + 1$ without loss of generality. Then, by using an argument similar to that used to establish (5.6), it follows from (5.13) and Lemma 2.1 that

$$\Big| S(s-\tilde{s}) H_2(\cdot, \tilde{s}) \Big| \leq \sum_{j=1}^\infty \Big| \langle H_2(\cdot, \tilde{s}), \varphi_j \rangle S(s-\tilde{s}) \varphi_j \Big| \tag{5.14}$$
$$\leq C_M \sum_{j=1}^\infty e^{-\lambda_j (s-\tilde{s})} \overline{\varepsilon}(\tilde{s})^{4-6\theta} |\log \overline{\varepsilon}(\tilde{s})|^4 |\varphi_j| d\tilde{s}$$

$$\leq C_M e^{-2(s-\tilde{s})}\overline{\varepsilon}(\tilde{s})^{4-6\theta}|\log\overline{\varepsilon}(\tilde{s})|^4(1+y^2)$$

for $y \in [0, e^{(s-\tilde{s})/2}]$ and $\tilde{s} \in [s_0, s-1]$. For $y \geq e^{(s-\tilde{s})/2}$, it follows from (5.12) that

$$\begin{aligned}
&|[S(s-\tilde{s})H_2(\cdot,\tilde{s})](y)| \\
&\leq \quad |[S(s-\tilde{s})F_2(\cdot,\tilde{s})](y)| + e^{-(s-\tilde{s})}|\langle F_2(\cdot,\tilde{s}),\varphi_0\rangle||\varphi_0(y)| \\
&\leq \quad C_M e^{-(s-\tilde{s})}\overline{\varepsilon}(\tilde{s})^{4-8\theta}|\log\overline{\varepsilon}(\tilde{s})|^4 \\
&\leq \quad C_M e^{-2(s-\tilde{s})}\overline{\varepsilon}(\tilde{s})^{4-8\theta}|\log\overline{\varepsilon}(\tilde{s})|^4(1+y^2).
\end{aligned}$$

Combining this with (5.14) implies

$$\begin{aligned}
|Z_{2,1}(y,s)| &\leq \int_{s_0}^{s-1} |[S(s-\tilde{s})H_2(\cdot,\tilde{s})](y)d\tilde{s} \\
&\leq C_M\overline{\varepsilon}(s)^{4-8\theta}|\log\overline{\varepsilon}(s)|^4(1+y^2) \qquad (5.15)
\end{aligned}$$

for $y \geq 0$. By (3.3) and (5.12), $Z_{2,2}$ in (5.11) satisfies

$$\begin{aligned}
|Z_{2,2}(y,s)| &\leq \int_{s-1}^{s} \Big(|[S(s-\tilde{s})F_2(\cdot,\tilde{s})](y)| \\
&\qquad + e^{-(s-\tilde{s})}|\langle F_2(\cdot,\tilde{s}),\varphi_0\rangle||\varphi_0(y)|\Big)d\tilde{s} \\
&\leq C_M \int_{s-1}^{s} e^{-(s-\tilde{s})}\overline{\varepsilon}(\tilde{s})^{4-8\theta}|\log\overline{\varepsilon}(\tilde{s})|^4 d\tilde{s} \\
&\leq C_M\overline{\varepsilon}(s)^{4-8\theta}|\log\overline{\varepsilon}(s)|^4
\end{aligned}$$

for $y \geq 0$. Combining this with (5.1), (5.10), (5.11), and (5.15) implies

$$|Z(y,s)| \leq C_M\overline{\varepsilon}(s)^{3-\theta}|\log\overline{\varepsilon}(s)|^4(1+y^2) \qquad (5.16)$$

for $y \geq \overline{\varepsilon}(s)^\theta$ and $s \in [s_0, s_1]$. Thus, we have proved Lemma 4.2. □

5.2. **Proof of Lemma 4.3.** It follows from (2.16) and (2.17) that

$$\begin{aligned}
Q(\cdot,s) &= S(s-s_0)Q(\cdot,s_0) + 4\int_{s_0}^{s} S(s-\tilde{s})\varepsilon(\tilde{s})^2\Big(2\pi\delta_0 - \langle\mathcal{D}_0,\varphi_0\rangle\varphi_0\Big)d\tilde{s} \\
&= S(s-s_0)R(\cdot,s_0) + 8\pi\int_{s_0}^{s}\varepsilon(\tilde{s})^2\Big\{K(Y,0,s-\tilde{s}) - \frac{e^{-(s-\tilde{s})}}{4\pi}\Big\}d\tilde{s}.
\end{aligned}$$

We use $\overline{Q}(y,s) = 4\overline{\varepsilon}(s)^2G(y)$. Since G defined in (2.21) satisfies (2.19) and (2.20), it holds that

$$\overline{Q}_s = \mathcal{A}\overline{Q} + 4\overline{\varepsilon}(s)^2\Big(\mathcal{D}_0 - \langle\varphi_0,\mathcal{D}_0\rangle\varphi_0\Big) + (8\overline{\varepsilon}(s)\overline{\varepsilon}(s)')G.$$

Then, we observe that

$$\begin{aligned}\overline{Q}(\cdot,s) &= S(s-s_0)\overline{Q}(\cdot,s_0)+4\int_{s_0}^{s}S(s-\tilde{s})[\overline{\varepsilon}(\tilde{s})^2\Big(2\pi\delta_0-\langle\mathcal{D}_0,\varphi_0\rangle\varphi_0\Big)]d\tilde{s}\\ &\qquad+\int_{s_0}^{s}S(s-\tilde{s})\left[(8\overline{\varepsilon}(\tilde{s})\overline{\varepsilon}(\tilde{s})')G\right]d\tilde{s}\\ &= S(s-s_0)(4\overline{\varepsilon}(s_0)^2G)+8\pi\int_{s_0}^{s}\overline{\varepsilon}(\tilde{s})^2\Big\{K(\cdot,0,s-\tilde{s})-\frac{e^{-(s-\tilde{s})}}{4\pi}\Big\}d\tilde{s}\\ &\qquad+\int_{s_0}^{s}(8\overline{\varepsilon}(\tilde{s})\overline{\varepsilon}(\tilde{s})')S(s-\tilde{s})Gd\tilde{s}\end{aligned}$$

and

$$\begin{aligned}&Q(|Y|,s)-4\varepsilon(s)^2G(|Y|)\\ &= (Q(|Y|,s)-\overline{Q}(|Y|,s))+(\overline{Q}(|Y|,s)-4\varepsilon(s)^2G(|Y|))\\ &= 4(\overline{\varepsilon}(s)^2-\varepsilon(s)^2)G(|Y|)+S(s-s_0)\Big(R(\cdot,s_0)-4\overline{\varepsilon}(s_0)^2G)\Big)\\ &\quad+8\pi\int_{s_0}^{\max(s-1,s_0)}\Big(\varepsilon(\tilde{s})^2-\overline{\varepsilon}(\tilde{s})^2\Big)\Big\{K(Y,0,s-\tilde{s})-\frac{e^{-(s-\tilde{s})}}{4\pi}\Big\}d\tilde{s}\\ &\quad+8\pi\int_{\max(s-1,s_0)}^{s}\Big(\varepsilon(\tilde{s})^2-\overline{\varepsilon}(\tilde{s})^2\Big)\Big\{K(Y,0,s-\tilde{s})-\frac{e^{-(s-\tilde{s})}}{4\pi}\Big\}d\tilde{s}\\ &\qquad+\int_{s_0}^{\max(s-1,s_0)}(8\overline{\varepsilon}(\tilde{s})\overline{\varepsilon}(\tilde{s})')[S(s-\tilde{s})G](Y)d\tilde{s}\\ &\qquad+\int_{\max(s-1,s_0)}^{s}(8\overline{\varepsilon}(\tilde{s})\overline{\varepsilon}(\tilde{s})')[S(s-\tilde{s})G](Y)d\tilde{s}\\ &= J_3+J_4+J_5+J_6+J_7+J_8. \qquad (5.17)\end{aligned}$$

By (3.6) and (2.21), we have

$$|J_3|\le\frac{9M}{s}\overline{\varepsilon}(s)^2|G(\overline{\varepsilon}(s)^\theta)|\le 10M\theta\overline{\varepsilon}(s)^2\qquad\text{for } y\ge\overline{\varepsilon}(s)^\theta. \qquad (5.18)$$

We consider an estimate of J_4 when $s\in[s_0,s_0+1]$. By (3.10) and (4.1), we observe that

$$\begin{aligned}&R(y,s_0)-4\overline{\varepsilon}(s_0)^2G(y)=\psi(y,s_0)-a_0(s_0)\varphi_0(y)-4\overline{\varepsilon}(s_0)^2G(y)\\ &= (\alpha_0\tilde{\varphi}_0(y)-a_0(s_0)\varphi_0(y))+(4\varepsilon_0^2G(y)-4\overline{\varepsilon}(s_0)^2G(y))=J_9+J_{10}.\end{aligned}$$

We observe that

$$S(s-s_0)J_9=\int_{\mathbf{R}^2}K(\cdot,\tilde{Y},s-s_0)(\alpha_0\tilde{\varphi}_0(|\tilde{Y}|)-a_0(s_0)\varphi_0(|\tilde{Y}|))d\tilde{Y}$$

$$
\begin{aligned}
&= \int_{|\tilde{Y}|\le\overline{\varepsilon}(s_0)^{2\theta}} K(Y,\tilde{Y},s-s_0)(\alpha_0\tilde{\varphi}_0(|\tilde{Y}|)-a_0(s_0)\varphi_0(|\tilde{Y}|))d\tilde{Y} \\
&\qquad + \int_{|\tilde{Y}|\ge\overline{\varepsilon}(s_0)^{\theta}} K(Y,\tilde{Y},s-s_0)(\alpha_0\tilde{\varphi}_0(|\tilde{Y}|)-a_0(s_0)\varphi_0(|\tilde{Y}|))d\tilde{Y} \\
&= J_{4,1}+J_{4,2}. \qquad (5.19)
\end{aligned}
$$

Since $|\tilde{Y}|\le|Ye^{-(s-s_0)}|/10$ for $\tilde{Y}\in\mathbf{R}^2$ with $|\tilde{Y}|\le\overline{\varepsilon}(s)^{2\theta}$, $Y\in\mathbf{R}^2$ with $|Y|\ge\overline{\varepsilon}(s)^{\theta}$, and $s\in[s_0,s_0+1]$, it follows from (4.5) and (5.3) that

$$
\begin{aligned}
|J_{4,1}| \le & \Big|\int_{|\tilde{Y}|\le\overline{\varepsilon}(s_0)^{2\theta}} (K(Y,\tilde{Y},s-s_0)-K(Y,0,s-s_0)) \\
&\qquad \cdot(\alpha_0\tilde{\varphi}_0(|\tilde{Y}|)-a_0(s_0)\varphi_0(|\tilde{Y}|))d\tilde{Y}\Big| \\
&+\Big|\int_{|\tilde{Y}|\le\overline{\varepsilon}(s_0)^{2\theta}} K(Y,0,s-s_0)(\alpha_0\tilde{\varphi}_0(|\tilde{Y}|)-a_0(s_0)\varphi_0(|\tilde{Y}|))d\tilde{Y}\Big| \\
\le & \; C\Big|\int_{|\tilde{Y}|\le\overline{\varepsilon}(s_0)^{2\theta}} \frac{|Y||\tilde{Y}|}{1-e^{-(s-s_0)}}K(Y/\sqrt{2},0,s-s_0)(\alpha_0\tilde{\varphi}_0(|\tilde{Y}|))d\tilde{Y}\Big| \\
&+K(Y,0,s-s_0)\Big(\sqrt{2}|\langle\alpha_0\tilde{\varphi}_0-a_0(s_0)\varphi_0,\varphi_0\rangle| \\
&+\Big|\int_{|\tilde{Y}|\le\overline{\varepsilon}(s_0)^{2\theta}} (\alpha_0\tilde{\varphi}_0(|\tilde{Y}|)-a_0(s_0)\varphi_0(|\tilde{Y}|))(1-e^{-|\tilde{Y}|^2/4})d\tilde{Y}\Big|\Big) \\
\le & \; \frac{C}{|Y|}|\alpha_0|\overline{\varepsilon}(s_0)^{6\theta}|\log\overline{\varepsilon}(s_0)|+C|\alpha_0|\overline{\varepsilon}(s_0)^{8\theta}|\log\overline{\varepsilon}(s_0)| \\
\le & \; C\overline{\varepsilon}(s_0)^{2+5\theta}|\log\overline{\varepsilon}(s_0)|^2 \quad \text{for } |Y|\ge\overline{\varepsilon}(s_0)^{\theta}. \qquad (5.20)
\end{aligned}
$$

By using (3.10), (4.5), and (4.6), we obtain

$$
\begin{aligned}
|J_{4,2}| &\le \Big|\int_{|\tilde{Y}|\ge\overline{\varepsilon}(s_0)^{2\theta}} K(Y,\tilde{Y},s-s_0)(\alpha_0-a_0(s_0))\varphi_0(|\tilde{Y}|))d\tilde{Y}\Big| \qquad (5.21) \\
&\le C|\alpha_0-a_0(s_0)| \le C\overline{\varepsilon}(s_0)^{2+4\theta}|\log\overline{\varepsilon}(s_0)|.
\end{aligned}
$$

By (2.24), (2.25), and (3.12), and an argument similar to that used to establish (5.20), we have

$$
\begin{aligned}
|S(s-s_0)J_{10}| \le & \; |4\varepsilon_0^2-4\overline{\varepsilon}(s_0)^2|\int_{|\tilde{Y}|\le\overline{\varepsilon}(s)^{2\theta}} K(Y,\tilde{Y},s-s_0)G(|\tilde{Y}|)d\tilde{Y} \\
&+|4\varepsilon_0^2-4\overline{\varepsilon}(s_0)^2|\int_{|\tilde{Y}|\ge\overline{\varepsilon}(s_0)^{2\theta}} K(Y,\tilde{Y},s-s_0)G(|\tilde{Y}|)d\tilde{Y}
\end{aligned}
$$

$$\begin{aligned}
&\le |4\varepsilon_0^2 - 4\overline{\varepsilon}(s_0)^2| \int_{|\tilde{Y}|\le\overline{\varepsilon}(s_0)^{2\theta}} |K(Y,\tilde{Y},s-s_0) - K(Y,0,s-s_0)| G(|\tilde{Y}|) d\tilde{Y} \\
&\quad + |4\varepsilon_0^2 - 4\overline{\varepsilon}(s_0)^2| \int_{|\tilde{Y}|\le\overline{\varepsilon}(s_0)^{2\theta}} K(Y,0,s-s_0) G(|\tilde{Y}|) d\tilde{Y} \\
&\quad + |4\varepsilon_0^2 - 4\overline{\varepsilon}(s_0)^2| (2\theta |\log \overline{\varepsilon}(s_0)| + C) \\
&\le |4\varepsilon_0^2 - 4\overline{\varepsilon}(s_0)^2| \Big(\frac{C}{|Y|} \int_0^{\overline{\varepsilon}(s_0)^{2\theta}} G(y) y^2 dy + C + 2\theta |\log \overline{\varepsilon}(s_0)| \Big) \\
&\le \frac{9M}{s} \Big(C\varepsilon(s_0)^{5\theta} |\log \varepsilon(s_0)| + C + 2\theta |\log \varepsilon(s_0)| \Big) \varepsilon(s_0)^2.
\end{aligned}$$

By this, (5.19), (5.20), and (5.21), we obtain

$$|J_4| \le \Big(C + 18M\theta \Big) \overline{\varepsilon}(s)^2 \quad \text{for } y \ge \overline{\varepsilon}(s)^\theta \text{ and } s \in [s_0, s_0+1]. \tag{5.22}$$

For $s > s_0 + 1$, we consider an estimate of J_4. Since

$$\langle R(\cdot, s_0) - 4\overline{\varepsilon}(s_0)^2 G, \varphi_0 \rangle = 0,$$

we observe that

$$\begin{aligned}
|J_4| &\le \sum_{j=1}^{\infty} e^{-\lambda_j(s-s_0)} |\langle \alpha_0 \tilde{\varphi}_0 - a_0(s_0)\varphi_0 + 4(\varepsilon_0^2 - \overline{\varepsilon}(s_0)^2) G, \varphi_j \rangle| |\varphi_j(y)| \\
&\le e^{-2(s-s_0)} (1+y^2) \Big(C\overline{\varepsilon}(s_0)^{4\theta} |\log \overline{\varepsilon}(s_0)|^2 + \frac{9M}{s_0} \|G\| \Big) \\
&\qquad \cdot \sum_{j=1}^{\infty} \Big\{ e^{-(s-s_0)} + \frac{1}{4} \Big(e^{-(s-s_0)/2} y^2 \Big) \Big\}^{j-1} \\
&\le \frac{10M}{s} \|G\| \overline{\varepsilon}(s)^2 (1+y^2) \quad \text{for } y \in [0, e^{-(s-s_0)/2}],
\end{aligned} \tag{5.23}$$

by (3.6), (3.12), (4.6), and (4.11). For $Y \in \mathbf{R}^2$ with $|Y| \ge e^{(s-s_0)/2}$, it follows from (3.6), (3.12), and the fact that $s - s_0 \ge 1$ that

$$\begin{aligned}
|J_4| &\le C \int_{|\tilde{Y}|\le\overline{\varepsilon}(s_0)^{2\theta}} K(Y,\tilde{Y},s-s_0) \Big(|\alpha_0 \tilde{\varphi}_0(|\tilde{Y}|) - a_0(s_0)\varphi_0(|\tilde{Y}|)| \\
&\qquad + 4|\varepsilon(s_0)^2 - \varepsilon_0^2| G(|\tilde{Y}|) \Big) d\tilde{Y} \\
&\le C e^{-(s-s_0)} \Big(|\alpha_0| \overline{\varepsilon}(s_0)^{4\theta} + \frac{M}{s_0} \overline{\varepsilon}(s_0)^{2+8\theta} |\log \overline{\varepsilon}(s_0)| \Big) \\
&\le C e^{-2(s-s_0)} \overline{\varepsilon}(s_0)^{2+4\theta} |\log \overline{\varepsilon}(s_0)| (1+|Y|^2) \\
&\le C \overline{\varepsilon}(s)^{2+4\theta} |\log \overline{\varepsilon}(s)| (1+|Y|^2).
\end{aligned} \tag{5.24}$$

When we consider the estimates of J_5, J_6, J_7, and J_8, we may assume $s > s_0 + 1$ without loss of generality.

For $y \in [0, e^{-(s-\tilde{s}-1/2)}]$ and $\tilde{s} \in [s_0, s-1]$, it follows from (3.6) and Lemma 2.1 that

$$\begin{aligned} & |\varepsilon(\tilde{s})^2 - \overline{\varepsilon}(\tilde{s})^2| \Big| K(Y, 0, s-\tilde{s}) - \frac{e^{-(s-\tilde{s})}}{4\pi} \Big| \\ \leq \; & \sum_{j=1}^{\infty} e^{-\lambda_j (s-\tilde{s}-1/2)} |\varepsilon(\tilde{s})^2 - \overline{\varepsilon}(\tilde{s})^2| \Big| \langle K(\cdot, 0, 1/2), \varphi_j \rangle \Big| |\varphi_j(|Y|)| \\ \leq \; & C e^{-2(s-\tilde{s}-1/2)} \frac{M}{\tilde{s}} \overline{\varepsilon}(\tilde{s})^2 (1 + y^2). \end{aligned} \tag{5.25}$$

For $y \in [e^{-(s-\tilde{s}-1/2)}, \infty)$ and $\tilde{s} \in [s_0, s-1]$, we have

$$\begin{aligned} & |\varepsilon(\tilde{s})^2 - \overline{\varepsilon}(\tilde{s})^2| \Big| K(Y, 0, s-\tilde{s}) - \frac{e^{-(s-\tilde{s})}}{4\pi} \Big| \\ \leq \; & C e^{-(s-\tilde{s})} \frac{M}{\tilde{s}} \overline{\varepsilon}(\tilde{s})^2 \leq C e^{-2(s-\tilde{s})} \frac{M}{\tilde{s}} \overline{\varepsilon}(\tilde{s})^2 (1 + |Y|^2). \end{aligned}$$

Combining this with (5.25) implies

$$|J_5| \leq C \frac{M}{s} \overline{\varepsilon}(s)^2 (1 + |Y|^2) \quad \text{ for } Y \in \mathbf{R}^2. \tag{5.26}$$

By an argument similar to that used to establish (5.26), we obtain

$$|J_7| \leq C \overline{\varepsilon}(s)^2 (1 + |Y|^2) \quad \text{ for } Y \in \mathbf{R}^2 \tag{5.27}$$

since it holds that

$$|\frac{d}{ds} \overline{\varepsilon}(s)| \leq C \overline{\varepsilon}(s). \tag{5.28}$$

Using $\eta = \eta(\tilde{s}) = y^2 e^{-(s-\tilde{s})}/[4(1 - e^{-(s-\tilde{s})})]$, it follows from (3.6) that

$$\begin{aligned} |J_6| & \leq \frac{M}{s} \overline{\varepsilon}(s)^2 \Big\{ \frac{1}{2} + \int_{|Y|^2 e^{-1}/[4(1-e^{-1})]}^{\infty} \frac{2}{\eta} e^{-\eta} d\eta \Big\} \\ & \leq \frac{M}{s} \overline{\varepsilon}(s)^2 (C - 4 \log[\min(1, |Y|)]) \leq \Big(\frac{CM}{s} + 4M\theta \Big) \overline{\varepsilon}(s)^2 \end{aligned} \tag{5.29}$$

for $Y \in \mathbf{R}^2$ with $|Y| \geq \overline{\varepsilon}(s)^{\theta}$. Since the function $\mathcal{G}$ defined in (2.24) satisfies

$$\frac{d}{ds} S(s) \mathcal{G} = \mathcal{A} S(s) \mathcal{G} = -S(s) G,$$

we observe that

$$|J_8| \leq C \Big| \overline{\varepsilon}(s)^2 \mathcal{G}(y) - \overline{\varepsilon}(s-1)^2 S(1) \mathcal{G}(y) \Big| + C \int_{s-1}^{s} \overline{\varepsilon}(\tilde{s})^2 |S(s-\tilde{s}) \mathcal{G}| d\tilde{s} \leq C \overline{\varepsilon}(s)^2$$

by (5.28) and (2.25). By this, (5.17), (5.18), (5.22), (5.23), (5.24), (5.26), (5.27), and (5.29), we have

$$|Q(y,s) - 4\varepsilon(s)^2 G(y)| \le \Big(C + \frac{CM}{s} + 32M\theta\Big)\overline{\varepsilon}(s)^2(1+y^2)$$

for $y \ge \overline{\varepsilon}(s)^\theta$ and $s \in [s_0, s_1]$. Taking $\theta = 1/M$ and $s_0 \gg 1$, we obtain the proof of this lemma. □

5.3. **Proof of Lemma 4.4.** For the sake of simplicity, we write $\Lambda(\overline{s}) = \overline{\varepsilon}(\overline{s})^\theta$. By (4.12), we observe that

$$\begin{aligned}&|\overline{\Omega}(s) - \overline{\Omega}(\overline{s})|\\ &\le |a_0(s) - a_0(\overline{s})|\frac{1}{\sqrt{2}} + |R(\Lambda(\overline{s}), s) - R(\Lambda(\overline{s}), \overline{s})| + C_M\overline{\varepsilon}(s)^{2+2\theta}|\log \overline{\varepsilon}(s)|^2\end{aligned}$$

and

$$\begin{aligned}|a_0(s) - a_0(\overline{s})| \le &\Big|a_0(s) + 2\sqrt{2}\int_s^\infty e^{-(s-\tilde{s})}\tilde{\varepsilon}(\tilde{s})^2 d\tilde{s}\Big|\\ &+\Big|2\sqrt{2}\int_s^\infty e^{-(s-\tilde{s})}\tilde{\varepsilon}(\tilde{s})^2 d\tilde{s} - 2\sqrt{2}\int_{\overline{s}}^\infty e^{-(\overline{s}-\tilde{s})}\tilde{\varepsilon}(\tilde{s})^2 d\tilde{s}\Big|\\ &+\Big|a_0(\overline{s}) + 2\sqrt{2}\int_{\overline{s}}^\infty e^{-(\overline{s}-\tilde{s})}\tilde{\varepsilon}(\tilde{s})^2 d\tilde{s}\Big| = J_{11} + J_{12} + J_{13},\end{aligned}$$

where

$$\tilde{\varepsilon}(s) = \begin{cases} \varepsilon(s) & \text{if } s \in [s_0, s_1],\\ \overline{\varepsilon}(s) & \text{if } s \ge s_1.\end{cases}$$

By Lemma 4.1, we obtain

$$J_{11} + J_{13} \le C_M\overline{\varepsilon}(s)^{4-\mu},$$

where μ is the constant in Lemma 4.1. We can assume $s > \overline{s}$ for simplicity. Since it follows from (3.6) that

$$\begin{aligned}J_{12} &\le 2\sqrt{2}\int_{\overline{s}}^s e^{-(\overline{s}-\tilde{s})}\varepsilon(\tilde{s})^2 d\tilde{s} + 2\sqrt{2}\int_{\overline{s}}^\infty |e^{-(\overline{s}-\tilde{s})} - e^{-(s-\tilde{s})}|\varepsilon(\tilde{s})^2 d\tilde{s}\\ &\le \frac{4}{s}\overline{\varepsilon}(s)^2 + \frac{4}{s}\int_{\overline{s}}^\infty e^{-(s-\tilde{s})}\varepsilon(\tilde{s})^2 d\tilde{s} \le C\overline{\varepsilon}(s)^2\end{aligned}$$

for s, $\overline{s} \in [s_0, s_1]$ with $s > \overline{s} > s - 1/s$, we have

$$|a_0(s) - a_0(\overline{s})| \le C\overline{\varepsilon}(s)^2$$

if $s_0 \gg 1$. By (3.6), and Lemmas 4.2 and 4.3, we observe that

$$\begin{aligned}|R(\Lambda(\overline{s}), \overline{s}) - R(\Lambda(\overline{s}), s)| \le &|R(\Lambda(\overline{s}), \overline{s}) - \varepsilon(\overline{s})^2 G(\Lambda(\overline{s}))|\\ &+|\varepsilon(\overline{s})^2 - \varepsilon(s)^2|G(\Lambda(\overline{s})) + |\varepsilon(s)^2 G(\Lambda(\overline{s})) - R(\Lambda(\overline{s}), s)|\end{aligned}$$

$$\leq \quad C\overline{\varepsilon}(s)^2 + \Big(\frac{3M}{s}\Big)\theta|\log\varepsilon(\overline{s})|\overline{\varepsilon}(s)^2.$$

Then, we have

$$|\overline{\Omega}(s) - \overline{\Omega}(\overline{s})| \leq C\overline{\varepsilon}(s)^2$$

for s, $\overline{s} \in [s_0, s_1]$ with $|s - \overline{s}| \leq 1/s$ if $0 < \theta \ll 1$ and $s_0 \gg 1$. Thus, we have proved this lemma. □

5.4. **Proof of Lemma 4.5.** In this subsection, we show estimates of $\tilde{\omega}_L$ and Lemma 4.5. These proofs are shown by using an argument similar to the one in [7, Section 4.5]. Then, we describe only the main concept of these proofs.

Using

$$\tilde{\omega}_L = \omega_R + \kappa \quad \text{and} \quad J(\tau) = \frac{1}{L^2}\kappa(L\tau), \tag{5.30}$$

J satisfies

$$J_{\tau\tau} - \frac{3}{\tau}J_\tau + \frac{4\tau}{\tau^2+1}J_\tau + \frac{1}{2}\Big(\frac{1}{\tau}J_\tau\Big)^2 + \overline{\Lambda}^2 L^2\Big(\frac{\tau}{2}J_\tau - J\Big) + \overline{\Lambda}^2 L^2\left\{-\frac{2\tau^2}{\tau^2+1} + 2\log\left(1+\tau^2\right)\right\} = 0.$$

Since we expect $\tilde{\omega}_L(\eta) = O(\eta^4)$ as $\eta \to 0$, we can regard the function J as a fixed point of the operator

$$[\mathcal{T}(J)](\tau) = \int_0^\tau \frac{\tilde{\tau}^3}{(1+\tilde{\tau}^2)^2}\int_0^{\tilde{\tau}} \frac{(1+\zeta^2)^2}{\zeta^3}\Big\{-\frac{1}{2}\Big(\frac{1}{\zeta}J_\zeta(\zeta)\Big)^2 - \overline{\Lambda}^2 L^2\Big(\frac{\zeta}{2}J_\zeta(\zeta) - J(\zeta)\Big) - \overline{\Lambda}^2 L^2\Big(-\frac{2\zeta^2}{\zeta^2+1} + 2\log\left(1+\zeta^2\right)\Big)\Big\}d\zeta d\tilde{\tau}.$$

When C^1-functions J_1 and J_2 satisfy $J_i(\tau) = O(\tau^4)$ and $J_{i\tau}(\tau) = O(\tau^3)$ as $\tau \to 0$ for $i = 1, 2$, we can show

$$\|\mathcal{T}(J_1) - \mathcal{T}(J_2)\|_{\infty,\omega} \leq C\Big(\overline{\Lambda}^2 + \|J_1\|_{\infty,\omega} + \|J_2\|_{\infty,\omega}\Big)\|J_1 - J_2\|_{\infty,\omega}, \tag{5.31}$$

where

$$\|J\|_{\infty,\omega} = \max_{0\leq\tau\leq 1/L}\Big|\frac{(1+\tau^2)^2}{\tau^3(1+\log(1+\tau^2))}J_\tau(\tau)\Big| + \max_{0\leq\tau\leq 1/L}\Big|\frac{(1+\tau^2)^2}{\tau^4(1+\log(1+\tau^2))}J(\tau)\Big|.$$

Furthermore, we can show

$$\|\mathcal{T}^j(0)\|_{\infty,\omega} \leq C\overline{\Lambda}^2 \quad \text{for } j = 1, 2, 3, \cdots$$

if $0 < \overline{\Lambda} \ll 1$. By this and (5.31), we note the existence of the fixed point of the operator $\mathcal{T}$, and the fixed point J satisfies

$$|J(\tau)| \le C\overline{\Lambda}^2 L^2 \frac{\tau^6}{(1+\tau^2)^2} \left\{1 + \log(1+\tau^2)\right\}$$

and

$$|J_\tau(\tau)| \le C\overline{\Lambda}^2 L^2 \frac{\tau^5}{(1+\tau^2)^2} \left\{1 + \log(1+\tau^2)\right\}.$$

Thus, we have (4.14) and (4.15).

In order to find sub-solutions and super-solutions, it is necessary for us to select $Z^\pm$ such that

$$Z^\pm_{\eta\eta} + \frac{1}{\eta} Z^\pm_\eta + \Big(\frac{(4+\gamma(\eta))\eta^2}{\eta^2+L^2} - 4\Big)\frac{Z^\pm_\eta}{\eta} \pm \frac{1}{2}\Big(\frac{Z^\pm_\eta}{\eta}\Big)^2 e^{-s/100}$$
$$+\overline{\Lambda}^2\Big(\frac{\eta Z^\pm_\eta}{2} - Z^\pm\Big) + \frac{1}{100} Z^\pm \le 0 \quad \text{for } \eta \in [0,1] \ \text{ and } \ s \ge 0,$$

respectively, where

$$\gamma(\eta) = \frac{L^2+\eta^2}{\eta^3}\kappa_\eta(\eta)$$

and κ is the function in (5.30). Then, we can select Z^+ and Z^- as solutions to

$$\begin{aligned}\mathcal{L}_+(Z^+) \quad &= \quad Z^+_{\eta\eta} + \frac{1}{\eta} Z^+_\eta + \Big(\frac{(4+\gamma(\eta))\eta^2}{\eta^2+L^2} - 4\Big)\frac{Z^+_\eta}{\eta} + \frac{1}{2}\Big(\frac{Z^+_\eta}{\eta}\Big)^2 \\ &\qquad +\overline{\Lambda}^2\Big(\frac{\eta Z^+_\eta}{2} - Z^+\Big) + \frac{1}{100} Z^+ \le 0 \quad \text{for } \eta \in [0,1],\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_-(Z^-) \quad &= \quad Z^-_{\eta\eta} + \frac{1}{\eta} Z^-_\eta + \Big(\frac{(4+\gamma(\eta))\eta^2}{\eta^2+L^2} - 4\Big)\frac{Z^-_\eta}{\eta} \\ &\qquad +\overline{\Lambda}^2\Big(\frac{\eta Z^-_\eta}{2} - Z^-\Big) + \frac{1}{100} Z^- \le 0 \quad \text{for } \eta \in [0,1],\end{aligned}$$

respectively. If a function Z^+ satisfies $\mathcal{L}_+(Z^+) \le 0$ in $[0,1]$, it holds that $\mathcal{L}_-(Z^+) \le 0$ in $[0,1]$. Then, we consider only Z^+. For simplicity, we drop the superscript $+$ in Z^+. Let D be a positive constant satisfying $\overline{\Lambda}$, $L \ll D \ll 1$. Using

$$Z(\eta) = DL^2 \log\Big(1 + \frac{\eta^2}{L^2}\Big) - \frac{L^2}{4}\Big(\overline{\Lambda}^2 + \frac{2D}{100}\Big)\eta^2 \log\Big(1 + \frac{\eta^2}{L^2}\Big), \qquad (5.32)$$

it follows from (4.15) that

$$\mathcal{L}_+(Z) \le 0 \quad \text{for } \tau \in [0, 1/L].$$

If constants E and B satisfy

$$\overline{\Lambda}, L \ll D \ll B - 1 \ll 1 - E,$$

the function Z in (5.32) and the constants E and B satisfy Lemma 4.5. Thus, we establish Lemma 4.5. □

5.5. **Proofs of Lemmas 4.6 and 4.7.** Using $\eta \in (0,1)$, $\overline{s} \in [s_0, s_1]$, and

$$\mathcal{P}(\zeta, \tau) = \omega(\eta(2+\zeta), \overline{s} + \eta^2\tau) \quad \text{for } \zeta \in [-1, 1] \text{ and } \tau \ge 0,$$

$\mathcal{P}$ satisfies

$$\begin{aligned} \mathcal{P}_\tau &= \mathcal{P}_{\zeta\zeta} + \frac{1}{2+\zeta}\mathcal{P}_\zeta + \Lambda(\overline{s})^2\Big(\frac{\eta^2}{2}(2+\zeta)\mathcal{P}_\zeta - \eta^2\mathcal{P}\Big) \\ &\quad + \left\{\frac{1}{2}\Big(\frac{\mathcal{P}_\zeta}{2+\zeta}\Big)^2 - 4\Big(\frac{\mathcal{P}_\zeta}{2+\zeta}\Big)\right\}. \end{aligned} \tag{5.33}$$

It follows from Lemma 4.4 that

$$|\omega(1,s) - \omega(1,0)| \le C\Big(\frac{\overline{\varepsilon}(\overline{s})}{\Lambda(\overline{s})}\Big)^2 \quad \text{for } s \in [0, 1/\{\Lambda(\overline{s})^2\overline{s}\}].$$

Then, we take $L > 0$ and $L^\pm > 0$ such that

$$\tilde{\omega}_L(1) = \omega(1,0) \text{ and } \tilde{\omega}_{L^\pm}(1) = \omega(1,0) \pm C\Big(\frac{\overline{\varepsilon}(\overline{s})}{\Lambda(\overline{s})}\Big)^2,$$

respectively. Then, $L^\pm$ satisfy

$$0 < L^+ < L < L^- < C\Big(\frac{\overline{\varepsilon}(\overline{s})}{\Lambda(\overline{s})}\Big)^2 \text{ and } |L^+ - L^-| \le \frac{C}{s}L.$$

Using $\mathcal{P}_0(\zeta) = \tilde{\omega}_L(\eta(2+\zeta))$, $\mathcal{P}_0$ satisfies (5.33). Then, $P = \mathcal{P} - \mathcal{P}_0$ satisfies

$$\begin{aligned} P_\tau &= P_{\zeta\zeta} + \frac{1}{2+\zeta}P_\zeta + \Lambda(\overline{s})^2\Big(\frac{\eta^2}{2}(2+\zeta)P_\zeta - \eta^2\mathcal{P}\Big) \\ &\quad + \left\{\frac{1}{2}\Big(\frac{\omega_\eta}{\eta(2+\zeta)} + \frac{\tilde{\omega}_{L\eta}}{\eta(2+\zeta)}\Big)\frac{P_\zeta}{2+\zeta} - 4\Big(\frac{\mathcal{P}_\zeta}{2+\zeta}\Big)\right\}. \end{aligned}$$

Since $m(\cdot, s_0)$ satisfies (3.7), we observe that $0 \le m \le 4$ in $[0,\infty) \times [s_0, s_1]$. It follows from (4.15) that $|\tilde{\omega}_{L\eta}(\eta)/\eta| \le C$. Furthermore,

$$\phi(\cdot, s_0) = e^{y^2/4}(m(\cdot, s_0) - 4)$$

satisfies (3.3) with $M = C$, where C is a positive constant independent of M. Combining these with the parabolic regularity implies

$$\begin{aligned}&\Big|\eta\omega_\eta(\eta, s_0+\eta^2\tau) - \eta\tilde{\omega}_{L\eta}(\eta)\Big|\\ &\quad\le C \max_{-1\le\zeta\le 1, 0\le\tau\le 2}\Big|\omega(\eta(2+\zeta), s_0+\eta^2\tau) - \tilde{\omega}_L(\eta(2+\zeta))\Big|, \qquad (5.34)\end{aligned}$$

for $\tau \in [0, \min(1, (s_1-s_0)/\eta^2)]$, and

$$\begin{aligned}&\Big|\eta\omega_\eta(\eta, \overline{s}+\eta^2) - \eta\tilde{\omega}_{L\eta}(\eta)\Big|\\ &\quad\le C \max_{-1\le\zeta\le 1, 0\le\theta\le 2}\Big|\omega(\eta(2+\zeta), \overline{s}+\eta^2\theta) - \tilde{\omega}_L(\eta(2+\zeta))\Big| \qquad (5.35)\end{aligned}$$

for $\overline{s} \in [s_0, s_1]$.

As mentioned in Section 4.5, for each $\overline{s} \in [s_0, s_1]$, we have

$$\begin{aligned}&|\omega(\eta, s) - \tilde{\omega}_L(\eta)| \le 2|\omega_{L^+}(\eta) - \omega_{L^-}(\eta)|\\ &\le \frac{C}{\overline{s}}\Big\{\Big(\Big|\log\Big(1+\frac{\eta^2}{L^2}\Big)\Big| + \frac{\eta^2}{(\overline{\varepsilon}(\overline{s})/\Lambda(\overline{s}))^2+\eta^2}\Big)\Big(\frac{\overline{\varepsilon}(\overline{s})}{\Lambda(\overline{s})}\Big)^2\chi_{[L,1]}(\eta)\Big\}\\ &\quad + \frac{C}{\overline{s}}\frac{\eta^4}{L^2}\chi_{[0,L]}(\eta) \text{ for } \eta\in[0,1] \text{ and } s\in[0, 1/\{\Lambda(\overline{s})\overline{s}\}]. \qquad (5.36)\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|m(y,\overline{s}) - m(y,s)| &\le \frac{C}{\overline{s}}\Big\{\frac{\chi_{[\overline{\varepsilon}(\overline{s}),\overline{\varepsilon}(\overline{s})^\theta]} + \xi^2\chi_{[0,\overline{\varepsilon}(\overline{s})]}(y)}{1+\xi^2}\\ &\quad + \frac{\overline{\varepsilon}(\overline{s})^2}{y^2}\Big|\log\Big(1+\xi^2\Big)\Big|\chi_{[\overline{\varepsilon}(\overline{s}),\overline{\varepsilon}(\overline{s})^\theta])}(y)\Big\} \qquad (5.37)\end{aligned}$$

for $y \in [0, \overline{\varepsilon}(\overline{s})^\theta]$ and $s \ge \overline{s}$ with $|s-\overline{s}| \le 1/\overline{s}$, where $\xi = y/\overline{\varepsilon}(\overline{s})$. Since

$$|\eta^{-1}\tilde{\omega}_{L\eta}(\eta) - 4| \le C\frac{(\overline{\varepsilon}(\overline{s})/\Lambda(\overline{s}))^2}{(\overline{\varepsilon}(\overline{s})/\Lambda(\overline{s}))^2+\eta^2} \quad \text{for } \eta\in[0,1],$$

we have

$$\begin{aligned}&|(m(y,\overline{s})-4)^2 e^{y^2/4} - (m(y,s)-4)^2 e^{y^2/4}|\\ &\le \frac{C}{\overline{s}}\Big\{\frac{\chi_{[\overline{\varepsilon}(\overline{s}),\overline{\varepsilon}(\overline{s})^\theta]} + \xi^2\chi_{[0,\overline{\varepsilon}(\overline{s})]}(y)}{1+\xi^2} + \frac{\overline{\varepsilon}(\overline{s})^2}{y^2}\Big|\log\Big(1+\xi^2\Big)\Big|\chi_{[\overline{\varepsilon}(\overline{s}),\infty)}(y)\Big\}\\ &\quad\times\Big\{\frac{1}{1+\xi^2} + \frac{\overline{\varepsilon}(\overline{s})^2}{y^2}\Big|\log\Big(1+\xi^2\Big)\Big|\chi_{[\overline{\varepsilon}(\overline{s}),\infty)}(y)\Big\} \qquad (5.38)\end{aligned}$$

for $y \in [0, \overline{\varepsilon}(\overline{s})^\theta]$, where $\xi = y/\overline{\varepsilon}(\overline{s})$. By this and (3.4), we obtain

$$|\varepsilon(s)^2 - \varepsilon(\overline{s})^2| \le \frac{C}{\overline{s}} \overline{\varepsilon}(\overline{s})^2 \tag{5.39}$$

for s, $\overline{s} \in [s_0, s_1]$ with $|s - \overline{s}| \le 1/\overline{s}$. Thus, we have (4.24).

It follows from (4.15) that

$$|\eta^{-1}\tilde{\omega}_{L\eta}(\eta) - \frac{4\Lambda(\overline{s})^2\eta^2}{\overline{\varepsilon}(\overline{s})^2 + \Lambda(\overline{s})^2\eta^2}| \le C\overline{\varepsilon}(\overline{s})^2 \log\left(1 + \frac{\eta^2}{(\overline{\varepsilon}(\overline{s})/\Lambda(\overline{s}))^2}\right)$$

for $\eta \in [0, 1]$. Combining this with (5.34) and (5.35) implies

$$\begin{aligned} &|\phi(y, \overline{s}) + \frac{4}{1 + \xi^2}| \\ \le \quad &\frac{C}{\overline{s}}\Big\{\frac{\xi^2}{(1 + \xi^2)^2} + \frac{\overline{\varepsilon}(\overline{s})^2}{y^2}\Big|\log\left(1 + \xi^2\right)\Big|\chi_{[\overline{\varepsilon}(\overline{s}), \overline{\varepsilon}(\overline{s})^\theta])}(y)\Big\} \end{aligned} \tag{5.40}$$

for $y \in [0, \overline{\varepsilon}(\overline{s})^\theta]$ and $s \ge \overline{s} \in [s_0, s_1]$, where $\xi = y/\overline{\varepsilon}(\overline{s})$.

Let A be a sufficiently large constant. Since it follows from (5.40) that

$$|\psi(y, s)| \le C\overline{\varepsilon}(s)^2|\log \overline{\varepsilon}(s)| \quad \text{for } y \in [\overline{\varepsilon}(s)^\theta/2, \overline{\varepsilon}(s)^\theta] \text{ and } s \in [s_0, s_1],$$

we observe that

$$|\psi(y, s)| \le C\overline{\varepsilon}(s)^2|\log \overline{\varepsilon}(s)| \quad \text{for } y \in [\overline{\varepsilon}(s)^\theta/2, 2A] \text{ and } s \in [s_0, s_1],$$

if $s_0 \gg 1$, by Lemmas 4.1, 4.2, and 4.3. By this, (3.7), and a parabolic regularity argument similar to that used to establish (5.37), we have

$$|\phi(y, s)| \le \frac{C}{y^2}\overline{\varepsilon}(s)^2|\log \overline{\varepsilon}(s)| \quad \text{for } y \in [2\overline{\varepsilon}(s)^\theta/3, 3A/2] \text{ and } s \in [s_0, s_1].$$

Combining this with (5.37) implies (4.26). Thus, we have proved Lemma 4.7.

In order to prove (4.25), we shall prove

$$|n(\Lambda(s), s) - \Lambda(s)^2\omega_{L(s)}(1)| \le C\overline{\varepsilon}(s)^2, \tag{5.41}$$

where $L(s) = \varepsilon(s)/\Lambda(s)$. To prove (5.41), we argue by contradiction. Then, suppose that, for some $\overline{s} \in [s_0, s_1]$ and some $C_1 \gg 1$, we should have either

$$n(\Lambda(s), s) - \Lambda(s)^2\omega_{L(s)}(1) \ge C_1\overline{\varepsilon}(s)^2$$

or

$$n(\Lambda(s), s) - \Lambda(s)^2\omega_{L(s)}(1) \le -C_1\overline{\varepsilon}(s)^2.$$

We consider only the first inequality, since the argument for the second inequality is similar to that for the first.

Let L be the constant satisfying

$$\tilde{\omega}_L(1) = \omega_{L(\overline{s})}(1) + C_1 \frac{\overline{\varepsilon}(\overline{s})^2}{\Lambda(\overline{s})^2}.$$

By using an argument similar to that used to establish (5.39), we obtain

$$\varepsilon(\overline{s} + 1/\overline{s}) - \varepsilon(\overline{s}) \geq \frac{CC_1}{\overline{s}} \overline{\varepsilon}(\overline{s}).$$

Noting that C is independent of M and C_1, for any sufficiently large $C_1 > 0$, this contradicts (5.39). Thus, we have (5.41).

Since it follows from the fact that $0 \leq m \leq 4$, and Lemmas 4.2 and 4.3, that

$$\begin{aligned}
&|n(\Lambda(s), s) - 2\Lambda(s)^2 - \psi(\Lambda(s), s)| \\
&\leq \Big| \int_0^{\Lambda(s)} (m(\tilde{y}, s) - 4)\tilde{y} d\tilde{y} - \int_0^{\Lambda(s)} (m(\tilde{y}, s) - 4) e^{-\tilde{y}^2/4} \tilde{y} d\tilde{y} \Big| \\
&\leq \Big(1 - e^{-\Lambda(s)^2/4}\Big) \int_0^{\Lambda(s)} (4 - m(\tilde{y}, s)) \tilde{y} e^{-\tilde{y}^2/4} d\tilde{y} \\
&\leq C\Lambda(s)^2 |\psi(\Lambda(s), s)|,
\end{aligned}$$

and

$$|\psi(\Lambda(s), s) - a_0(s)\varphi_0(\Lambda(s)) - 4\varepsilon(s)^2 G(\Lambda(s))| \leq C\overline{\varepsilon}(s)^2,$$

we have

$$|\frac{a_0(s)}{\sqrt{2}} + 4\varepsilon(s)^2 G(\Lambda(s)) + 2\Lambda(s)^2 - \Lambda(s)^2 \omega_{L(s)}(1)| \leq C\varepsilon(s)^2$$

by (5.41). Combining this with

$$|4\varepsilon(s)^2 G(\Lambda(s)) + 4\theta\varepsilon(s)^2 \log \varepsilon(s)| \leq C\overline{\varepsilon}(s)^2$$

and

$$|\Lambda(s)^2 \omega_{L(s)}(1) - 2\Lambda(s)^2 - 4(1-\theta)\varepsilon(s)^2 \log \varepsilon(s)| \leq \frac{CM}{s} \overline{\varepsilon}(s)^2$$

implies

$$\Big| 2\tilde{\varepsilon}(s)^2 \log \tilde{\varepsilon}(s) + \int_s^\infty e^{-(s-\tilde{s})} \tilde{\varepsilon}(\tilde{s})^2 d\tilde{s} \Big| \leq C\tilde{\varepsilon}(s)^2, \tag{5.42}$$

where

$$\tilde{\varepsilon}(s) = \begin{cases} \overline{\varepsilon}(s) & \text{if } s \geq s_1, \\ \varepsilon(s) & \text{if } s_0 \leq s \leq s_1. \end{cases}$$

Using

$$\varepsilon(s)^2 = \Big(1 + \frac{C(s)}{s}\Big) \overline{\varepsilon}(s)^2,$$

we observe that $C(\cdot)$ is continuous in $[s_0, s_1]$ and that $|C(s)| \leq M$ for $s \in [s_0, s_1]$. Thus, there exists $s^* \in [s_0, s_1]$ satisfying

$$|C(s^*)| = \max_{s_0 \leq s \leq s_1} |C(s)|.$$

By this and (5.42), we have

$$|C(s)| \leq |C(s^*)| \leq C \quad \text{ for } s \in [s_0, s_1],$$

since $\overline{\varepsilon}$ satisfies (5.42). Thus, we obtain

$$|\tilde{\varepsilon}(s)^2 - \overline{\varepsilon}(s)^2| \leq \frac{C}{s}\overline{\varepsilon}(s)^2 \quad \text{ for } s \geq s_0.$$

In particular, we have (4.25). Thus, we have proved Lemma 4.6 □

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