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NONUNIQUENESS OF SOLUTIONS FOR DIRICHLET PROBLEMS RELATED TO FULLY NONLINEAR SINGULAR OR DEGENERATE OPERATORS

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Abstract. We study the Dirichlet problem for fully nonlinear elliptic operators: $G(D^2u, \nabla u, u, x) = f(x)$ in Ω, where Ω is a bounded regular domain, and f is continuous. We prove the existence, the nonexistence and the multiplicity of solutions for some particular right-hand side f when G has its two principal eigenvalues of different sign.

1. Introduction

This paper is devoted to the existence, nonexistence and nonuniqueness of solutions for the Dirichlet boundary-value problem

$$\begin{cases} G(D^2u, \nabla u, u, x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where G satisfies some homogeneity and continuity conditions and is fully nonlinear elliptic.

A particular type of operator for which our results can be applied are the following:

$$G(N, p, u, x) = |p|^\alpha \mathcal{M}^{\pm}_{a,A}(N) + c_1(x)(u^+)^{1+\alpha} + c_2(x)(u^-)^{1+\alpha}$$

where c_1 and c_2 are some continuous bounded functions, u^+ and u^- are the positive and negative parts of u, $\mathcal{M}^{+}_{a,A}(M) = Atr(M^+) - atr(M^-)$, and $\mathcal{M}^{-}_{a,A}(M) = -\mathcal{M}^{+}_{a,A}(-M)$ are the well-known Pucci's operators.

One can also consider the operator constructed with the $(\alpha + 2)$-Laplace operator

$$G(N, p, u, x) = |p|^{\alpha-2} \left(|p|^2 tr(M) + \alpha(Mp : p)\right) + c_1(x)(u^+)^{1+\alpha} + c_2(x)(u^-)^{1+\alpha}$$

The existence and uniqueness viewpoint, for this type of problem, has already been studied in the following cases:

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- When $\alpha = 0$, for the operator

$$G(D^2u, Du, x) + \lambda u$$

where G is homogeneous of degree 1 continuous, fully nonlinear and convex, and λ lies under the two principal eigenvalues of the operator $G(D^2u, Du, x)$. This is done in [24].

- When $\alpha > -1$, for operators on the model of

$$G(M, p, u, x) = |p|^\alpha \mathcal{M}^{\pm}_{a,A}(M) + b(x) \cdot p|p|^\alpha + (c(x) + \lambda)|u|^\alpha u,$$

when λ is strictly less than the two principal eigenvalues of the operator $|p|^\alpha \mathcal{M}^{\pm}_{a,A}(M) + b(x) \cdot p|p|^\alpha + c(x)|u|^\alpha u$. The results are valid also for related operators, as is made precise in the definitions given in Section 2. (See [8]).

In the case $\alpha = 0$, some nonexistence and multiplicity results have been obtained under some hypotheses concerning the sign of the principal eigenvalues of the operator considered. This is done in [25]. In that paper the author considers an operator H which is positively homogeneous of degree 1 in (D^2u, Du, u), satisfies some Lipschitz properties and is convex in (D^2u, Du, u). Moreover, he assumes that H has its two principal eigenvalues of different sign. Under these assumptions he establishes the following existence, nonexistence and multiplicity of solutions result.

Theorem 1.1. *Suppose that Ω is a bounded regular domain of $\mathbb{R}^N$ and ϕ is an eigenfunction for the first eigenvalue λ_0^+ of the operator $H(D^2u, Du, 0, x)$.*

We assume that $\lambda^+(H) < 0 < \lambda^-(H)$, and that $f = -t\phi + h$, h being bounded, $t \in \mathbb{R}$. We consider the Dirichlet problem:

$$\begin{cases} H(D^2u, Du, u, x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, there exists $t^\star(h)$ such that for $t < t^\star(h)$ there exist at least two solutions, for $t = t^\star(h)$ there exists at least one solution, for $t > t^\star(h)$ there is no solution.

The conclusion is similar to some results concerning the Ambroseti-Prodi type problem, obtained in [2], [1], [17].

The present work establishes analogous results to the one obtained in [25], in the case where the operator is homogeneous of degree $1 + \alpha$ with $\alpha > -1$.

We suppose here that

$$G(M, p, u, x) = F(M, p, x) + c_1(x)(u^+)^{1+\alpha} + c_2(x)(u^-)^{1+\alpha}$$

where F satisfies some continuity, homogeneity and ellipticity conditions which will be detailed later.

In [5], the authors prove that under these assumptions, and on the model of [4], F has two principal eigenvalue λ_o^+ and λ_o^- which are > 0. For each of them, there exists a positive (respectively negative) "eigenfunction."

In this note, we shall establish the following existence, nonexistence, and multiplicity result.

Theorem 1.2. *Let Ω be a bounded regular domain and let F satisfy the assumptions in Section 2. Let λ_o^+ be its "principal positive eigenvalue." Suppose that ϕ is a positive eigenfunction for λ_o^+, and that $G(M,p,u,x) = F(M,p,x) + c_1(u^+)^{1+\alpha} + c_2(u^-)^{1+\alpha}$ is such that*

$$\lambda^+(G) < 0 < \lambda^-(G).$$

Suppose that $h \in \mathcal{C}(\overline{\Omega})$, and $t \in \mathbb{R}$. Let us consider the Dirichlet problem $\mathcal{P}_t$:

$$\begin{cases} G(D^2u, \nabla u, u, x) = -t\phi^{1+\alpha} + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

1) *There exists $t^\star$ such that for $t \le t^\star$ there exists at least one solution.*

2) *Suppose that $\alpha \le 0$ and $G(M,p,x) = |p|^\alpha \mathcal{M}^\pm_{a,A}(M) + b(x) \cdot |p|^\alpha p + c_1(u^+)^{1+\alpha} + c_2(u^-)^{1+\alpha}$; then, there exists $t_o \le t^\star$ such that for $t < t_o$ there exist at least two solutions.*

3) *For $t > t^\star$ there is no solution.*

Let us observe that the main difficulty in obtaining these results is linked to the high nonlinearity of the operator and to the fact that it is not defined on points where the "gradient" is zero, and then one cannot test functions on such points.

In the same order of ideas, one cannot use the difference of solutions as being the solution for another operator which is proper, a key ingredient in the proofs of [25]. Moreover the multiplicity of solutions requires some further regularity results on the solutions, that we have obtained only in the case $-1 < \alpha \le 0$, [9].

2. ASSSUMPTIONS ON F AND PREVIOUS RESULTS

Let Ω be a bounded regular domain of $\mathbb{R}^N$, α be some real number > -1, and $\mathcal{S}$ be the space of symmetric matrices on $\mathbb{R}^N$. We suppose that F_o, defined on $\mathcal{S} \times \mathbb{R}^N \setminus \{0\} \times \Omega$, satisfies the following:

(H1) F_o is continuous on $\mathcal{S} \times \mathbb{R}^N \setminus \{0\} \times \Omega$ and satisfies, for all $t \in \mathbb{R}^\star$, $\mu \ge 0$, and $(x,p,X) \in \Omega \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}$, $F_o(\mu X, tp, x) = |t|^\alpha \mu F_o(X,p,x)$.

(H2) For $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}$, $M \in S$, $N \in S$, $N \geq 0$,

$$a|p|^\alpha tr(N) \leq F_o(M+N,p,x) - F_o(M,p,x) \leq A|p|^\alpha tr(N).$$

(H3) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X,Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x,y) \in \mathbb{R}^N$, $x \neq y$,

$$F_o(X, \zeta(x-y), x) - F_o(-Y, \zeta(x-y), y) \leq \omega(\zeta|x-y|^2).$$

Remark 2.1. The assumption (H2) and the fact that $F_o(0,p,x) = 0$ implies that

$$|p|^\alpha(a\text{tr}(M^+) - A\text{tr}(M^-)) \leq F_o(M,p,x) \leq |p|^\alpha(A\text{tr}(M^+) - a\text{tr}(M^-)),$$

where $M = M^+ - M^-$ is a minimal decomposition of M into positive and negative symmetric matrices.

-Condition (H3) is not necessary when F_o does not depend on x; in that case condition (H2) is sufficient to get the results recalled and in the present paper.

We now introduce the terms of order of derivation one: Let b be some continuous and bounded function in Ω. One assumes that

(H4) -Either $\alpha \leq 0$ and b is Hölderian of exponent $1+\alpha$,
- or $\alpha > 0$ and, for all x and y in Ω,

$$\langle b(x) - b(y), x-y \rangle \leq 0.$$

In the following F will denote the operator

$$F(X,p,x) = F_o(X,p,x) + b(x) \cdot p|p|^\alpha.$$

Let us recall some of the properties of the solutions of the Dirichlet problem related to the operator F which will be used in this paper. Let us note that the solutions that we consider are taken in the sense of viscosity. For the convenience of the reader we reproduce here their definition.

Definition 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, and let f be some continuous function in Ω, then v, bounded and continuous on $\overline{\Omega}$ is called a viscosity super-solution (respectively sub-solution) of $F(D^2v, \nabla v, x) = f(x,v)$ if for all $x_0 \in \Omega$,*

-either there exists an open ball $B(x_0,\delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $0 \leq f(x,c)$, for all $x \in B(x_0,\delta)$ (respectively $0 \geq f(x,c)$),

-or for all $\varphi \in \mathcal{C}^2(\Omega)$*, such that* $v - \varphi$ *has a local minimum on* x_0 (*respectively a local maximum*) *and* $\nabla\varphi(x_0) \neq 0$*, one has*

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), x_0) \leq f(x_0, v(x_0))$$

(*respectively*

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), x_0) \geq f(x_0, v(x_0))).$$

Remark 2.3. One can extend the definition to uppersemicontinuous sub-solution and lowersemicontinuous super-solution.

We proved in [5] the following comparison principle, which is a key point to proving the existence of solutions for Dirichlet problems related to F.

Proposition 2.4. *Suppose that* Ω *is a bounded regular domain and* F_o *and* b *satisfy the assumptions* (H1)–(H4)*. Suppose that* $\beta(x, .)$ *is nondecreasing and continuous in* (x, u) *and* $\beta(x, 0) = 0$*,* σ *is an upper semicontinuous sub-solution of*

$$F(D^2\sigma, \nabla\sigma, x) - \beta(x, \sigma(x)) \geq g$$

and u *is a lower semicontinuous super-solution of*

$$F(D^2u, \nabla u, x) - \beta(x, u(x)) \leq f$$

with g *and* f *continuous,* $f < g$ *in* Ω *and* $\sigma \leq u$ *on* $\partial\Omega$*. Then* $\sigma \leq u$ *in* Ω*.*

Remark 2.5. The result still holds if $\beta(x, \cdot)$ is increasing and $f \leq g$ in Ω.

We now recall some estimates on sub-solutions and super-solutions which will be used throughout this paper.

Proposition 2.6. *Suppose that* F_o *and* b *satisfy the assumptions above. We assume that* Ω *is a bounded regular domain.*

Let u *be a bounded upper semicontinuous sub-solution of*

$$\begin{cases} F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha \geq m & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for some constant m*. Then there exists some constant* $C_u > 0$ *which depends only on* $a, A, b, |m|, \Omega$ *and on the* L^∞ *norm of* u*, such that*

$$u(x) \leq C_u d(x, \partial\Omega).$$

Remark 2.7. Considering $-u$ in the previous proposition and defining G_o by $-F_o(-M, -p, x) = G_o(M, p, x)$ which satisfies analogous assumptions as F_o, if u is lower semicontinuous and satisfies for some $m \in \mathbb{R}$

$$\begin{cases} F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha \leq m & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

one gets that there exists $C_u > 0$ such that

$$-u(x) \leq C_u d(x, \partial\Omega).$$

This implies in particular that any continuous solution u of

$$\begin{cases} F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with f continuous on $\overline{\Omega}$, satisfies some estimate

$$|u(x)| \leq C_u d(x, \partial\Omega)$$

with C_u which depends on $a, A, b, |f|_\infty, \Omega$.

The comparison principle in Proposition 2.4 permits us to obtain a strict maximum principle and some Hopf principle: We recall that Ω satisfies the uniform interior sphere condition if there exists $\bar{r}$ such that for all $\bar{x} \in \partial\Omega$ there exists x_1 in Ω such that $B(x_1, \bar{r}) \subset \Omega$ and $|x - x_1| = \bar{r}$.

Of course when Ω is regular (at least $W^{2,\infty}$), it satisfies the interior sphere condition.

Theorem 2.8. *Suppose that Ω is a bounded domain which satisfies the uniform interior sphere condition. Suppose that F_o and b satisfy* (H1)–(H4) *and that c is continuous and bounded. Let u be a non-negative viscosity continuous solution of*

$$F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha + c(x)u^{1+\alpha} \leq 0.$$

Then either $u \equiv 0$ or $u > 0$ in Ω.

Moreover, there exists some constant $c_u > 0$ such that for all $x_o \in \partial\Omega$ and for h small enough > 0

$$u(x_o - h\vec{n}) - u(x_o) \geq c_u h,$$

where $\vec{n}$ denotes the unit outer normal to $\partial\Omega$ at x_o.

Remark 2.9. This theorem has the following consequence: For any supersolution $u \geq 0$ of $F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha + c(x)u^{\alpha+1} \leq 0$, there exists some constant $c_u > 0$ such that $u \geq c_u d(x, \partial\Omega)$. This can also be obtained using the comparison with a convenient sub-solution near the boundary (see [7]).

A proof of this theorem can be found in [7], see also [10] for a more complete strict maximum principle, analogous to the Vasquez principle.

We now recall the definition of the principal (one says also "first") eigenvalues, on the model of [4], [24], [11]:

$$\lambda^+ = \sup\Big\{\lambda : \exists\ \varphi > 0 \text{ in } \Omega,\\ F_o(D^2\varphi, \nabla\varphi, x) + b(x)\cdot\nabla\varphi|\nabla\varphi|^\alpha + (c(x)+\lambda)\varphi^{1+\alpha} \le 0 \text{ in } \Omega\Big\}$$

and symmetrically

$$\lambda^- = \sup\Big\{\lambda : \exists\ \varphi < 0 \text{ in } \Omega,\\ F_o(D^2\varphi, \nabla\varphi, x) + b(x)\cdot\nabla\varphi|\nabla\varphi|^\alpha + (c(x)+\lambda)|\varphi|^\alpha\varphi \ge 0 \text{ in } \Omega\Big\}.$$

For each of these "eigenvalues" there exists some eigenfunction [6], [7].

Theorem 2.10. *Suppose that Ω is a bounded regular domain, that F_o and b satisfy the assumptions* (H1)–(H4), *and that c is continuous and bounded on Ω. There exists $\phi > 0$ continuous in Ω such that*

$$\begin{cases} F_o(D^2\phi, \nabla\phi, x) + b(x)\cdot\nabla\phi|\nabla\phi|^\alpha + (c(x)+\lambda^+)\phi^{1+\alpha} = 0 & \text{in } \Omega\\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

and there exists $\psi < 0$ continuous such that

$$\begin{cases} F_o(D^2\psi, \nabla\psi, x) + b(x)\cdot\nabla\psi|\nabla\psi|^\alpha + (c(x)+\lambda^-)|\psi|^\alpha\psi = 0 & \text{in } \Omega\\ \phi = 0 & \text{on } \partial\Omega \end{cases}.$$

The two principal eigenvalues have the maximum and minimum property, [6], [7].

Theorem 2.11. *Let Ω be a bounded regular domain of $\mathbb{R}^N$. Suppose that F_o and b satisfy the assumptions* (H1)–(H4), *and c is continuous and bounded. Suppose that $\tau < \lambda^+$ and that u is a viscosity uppersemicontinuous sub-solution of*

$$F_o(D^2u, \nabla u, x) + b(x)\cdot\nabla u|\nabla u|^\alpha + (c(x)+\tau)|u|^\alpha u \ge 0 \ \text{ in } \ \Omega$$

with $u \le 0$ on the boundary of Ω, then $u \le 0$ in Ω.

In the same manner if $\tau < \lambda^-$ and if u is a viscosity lowersemicontinuous super-solution of

$$F_o(D^2u, \nabla u, x) + b(x)\cdot\nabla u|\nabla u|^\alpha + (c(x)+\tau)|u|^\alpha u \le 0 \ \text{ in } \ \Omega$$

with $u \ge 0$ on the boundary of Ω, then $u \ge 0$ in Ω.

We now recall some weak comparison principles which will be employed in the paper [7].

Theorem 2.12. *Suppose that Ω is a bounded domain. Suppose that F_o satisfies the assumptions above, that b and c are continuous and bounded and b satisfies $(H4)$. Suppose that f and g are continuous and bounded, that $f < 0$ on $\overline{\Omega}$, $g - f \geq m > 0$ on $\overline{\Omega}$, and that $v > 0$ and u are respectively lower semicontinuous and upper semicontinuous and satisfy*

$$F_o(D^2v, \nabla v, x) + b(x).\nabla v|\nabla v|^\alpha + c(x)v^{1+\alpha} \leq f$$

$$F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha + c(x)|u|^\alpha u \geq g.$$

Suppose that $u \leq v$ on the boundary. Then $u \leq v$ in Ω.

Remark 2.13. The existence of $v > 0$ implies that $\lambda^+ > 0$.

Remark 2.14. We shall use here this result under its symmetric form, i.e., the following.

Suppose that $f > 0$ on $\overline{\Omega}$, that $f - g > m$ on $\overline{\Omega}$ and that v is a negative upper continuous sub-solution of

$$F_o(D^2v, \nabla v, x) + b(x).\nabla v|\nabla v|^\alpha + c(x)|v|^\alpha v \geq f$$

u is a lower semicontinuous super-solution of

$$F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha + c(x)|u|^\alpha u \leq g$$

with $u \geq v$ on the boundary. Then $u \geq v$ in Ω.

We now recall some regularity and compactness results for solutions of Dirichlet problems.

Proposition 2.15. *Suppose that Ω is a bounded regular domain. Suppose that F_o satisfies the previous assumptions. Let f be some continuous function on $\overline{\Omega}$. Let u be a continuous and bounded viscosity solution of*

$$\begin{cases} F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Then if b is continuous on $\overline{\Omega}$, for any $\gamma < 1$ there exists some positive constant C which depends only on $|f|_\infty$, γ, $|b|_\infty$ and $|u|_\infty$, such that for any $(x, y) \in \bar{\Omega}^2$

$$|u(x) - u(y)| \leq C|x - y|^\gamma.$$

Corollary 2.16. *Suppose that Ω is a bounded regular domain. Suppose that F_o satisfies the previous assumptions and that b is continuous on $\overline{\Omega}$. Suppose that (f_n) is a sequence of continuous and bounded functions, uniformly bounded, and (u_n) is a sequence of continuous and bounded viscosity* *solutions of*

$$\begin{cases} F_o(D^2u_n, \nabla u_n, x) + b(x).\nabla u_n|\nabla u_n|^\alpha = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the sequence (u_n) *is relatively compact in* $\mathcal{C}(\overline{\Omega})$. *Moreover, if* f_n *converges - even simply - to some continuous and bounded function, there exists a subsequence of* $(u_n)_n$ *which converges to a solution of*

$$\begin{cases} F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For the sake of completeness we give a Lipschitz regularity result which will be required for further regularity results in Section 4. This result is obtained under an additional hypothesis on F that we recall here and denote by (H5): There exists $\nu > 0$ and $\kappa \in]1/2, 1]$ such that for all $|p| = 1$, $|q| \le \frac{1}{2}$, $B \in \mathcal{S}$

$$|F_o(B, p+q, x) - F_o(B, p, x)| \le \nu |q|^\kappa |B|$$

which implies by homogeneity that for all $p \neq 0$, $|q| \le \frac{|p|}{2}$, $B \in \mathcal{S}$

$$|F_o(B, p+q, x) - F_o(B, p, x)| \le \nu |q|^\kappa |p|^{\alpha-\kappa} |B|.$$

Theorem 2.17. *Suppose that* Ω *is a bounded regular domain. If* F_o *satisfies* (H1),(H2), (H3), *and* (H5) *and if* b *is continuous on* $\overline{\Omega}$, *then the continuous and bounded viscosity solutions of*

$$\begin{cases} F_o(D^2u, \nabla u, x) + b(x).\nabla u|\nabla u|^\alpha = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

are Lipschitz continuous inside Ω. *Moreover, if* (u_n), (f_n) *satisfy the assumptions in Corollary 2.16, the sequence* (u_n) *is uniformly Lipschitzian.*

3. EXISTENCE AND NONEXISTENCE RESULTS

3.1. **Example of an operator for which $\lambda^+ < 0$, $\lambda^- > 0$.** We now define the operator $F(M, p, x) = F_o(M, p, x) + b(x) \cdot p|p|^\alpha$.

Let λ_o^+ and λ_o^- be the principal eigenvalues respectively for $F(M, p, x)$; i.e.,

$$\lambda_o^+ = \sup \left\{ \lambda : \exists\ \psi > 0 \text{ in } \Omega, F(D^2\psi, \nabla\psi, x) + \lambda\psi^{\alpha+1} \le 0 \text{ in } \Omega \right\}$$

$$\lambda_o^- = \sup \left\{ \lambda : \exists\ \psi < 0 \text{ in } \Omega, F(D^2\psi, \nabla\psi, x) + \lambda|\psi|^\alpha\psi \ge 0 \text{ in } \Omega \right\}$$

It is proved in [7] that λ_o^+ and λ_o^- are > 0. Let us note that in this definition one can consider functions ψ which are lowersemicontinuous and bounded, or continuous and bounded indifferently for the first definition, and

ψ which are uppersemicontinuous and bounded, or continuous and bounded indifferently in the second one. This remark holds also for further definitions.

As we already pointed out in Theorem 2.10 there exists an eigenfunction for each of these eigenvalues, and the maximum principle holds (cf Theorem 2.11).

We now consider some continuous and bounded functions c_1 and c_2 and we define the operator

$$\begin{aligned} G(M,p,u,x) &= F(M,p,x) + c_1(x)(u^+)^{1+\alpha} + c_2(x)(u^-)^{1+\alpha} \\ &\equiv F(M,p,x) + \vec{c}\cdot u^{1+\alpha}, \end{aligned}$$

where u^+ and u^- denote respectively the positive and negative parts of u.

We assume that the following condition on $\vec{c}$ holds:

$$\min c_1 > \lambda_o^+,\ \min c_2 > -\lambda_o^-.$$

With this choice of c_1, c_2, if we denote by λ^+ and λ^- the principal eigenvalues for G; i.e.,

$$\lambda^+ = \sup\Big\{\lambda : \exists\ \psi > 0 \text{ in } \Omega,\ G(D^2\psi, \nabla\psi, \psi, x) + \lambda\psi^{\alpha+1} \le 0 \text{ in } \Omega\Big\},$$

$$\lambda^- = \sup\Big\{\lambda : \exists\ \psi < 0 \text{ in } \Omega, G(D^2\psi, \nabla\psi, \psi, x) + \lambda|\psi|^\alpha\psi \ge 0 \text{ in } \Omega\Big\},$$

then $\lambda^+ \le \lambda_o^+ - \min\, c_1 < 0$ and $\lambda^- \ge \lambda_o^- + \min\, c_2 > 0$.

Indeed suppose by contradiction that there exist $\epsilon > 0$ and $\psi > 0$ such that

$$G(D^2\psi, \nabla\psi, \psi, x) + (\lambda_o^+ - \min c_1 + \epsilon)\psi^{1+\alpha} \le 0.$$

This would imply

$$F(D^2\psi, \nabla\psi, \psi, x) + (\lambda_o^+ + c_1 - \min c_1 + \epsilon)\psi^{1+\alpha} \le 0$$

which contradicts the definition of λ_o^+.

For the second assertion, let ϕ be a negative eigenfunction for the operator F and the eigenvalue λ_o^-. Then

$$G(D^2\phi, \nabla\phi, \phi, x) + (\lambda_o^- + \min c_2)|\phi|^\alpha\phi = (c_2 - \min c_2)|\phi|^{\alpha+1} \ge 0$$

which implies that $\lambda^- \ge \lambda_o^- + \min c_2 > 0$.

In the sequel ϕ denotes a nonnegative eigenfunction for F and for the first eigenvalue $\lambda_o^+ > 0$. We suppose also that $\phi(x_o) = 1$ for some fixed point x_o in Ω.

We consider the following problem denoted by $\mathcal{P}_t$:

$$\begin{cases} G(D^2u, \nabla u, u, x) = -t\phi^{1+\alpha} + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where h is some function in $\mathcal{C}(\overline{\Omega})$ and $t \in \mathbb{R}$.

The main result of this paper is the following existence and nonexistence result, already announced in Theorem 1.2, rewritten in a more precise form here and for the convenience of the reader.

Theorem 3.1. *Suppose that Ω is a bounded regular domain, that F_o and b satisfy* (H1)–(H4), (H5)*, and that c_1 and c_2 satisfy $\min c_1 > \lambda_o^+$, $\min c_2 > -\lambda_o^-$, where λ_o^+ and λ_o^- are the principal eigenvalues of F. Then, for all $h \in \mathcal{C}(\overline{\Omega})$ there exists $t^\star(h)$ such that the following hold.*

1) *There exists $t^\star$ such that for $t \le t^\star$ there exists at least one solution.*
2) *Suppose that $\alpha \le 0$ and $F(M,p,x) = |p|^\alpha \mathcal{M}^{\pm}_{a,A}(M) + b(x) \cdot |p|^\alpha p$, then there exists $t_o \le t^\star$ such that for $t < t_o$ there exist at least two solutions.*
3) *For $t > t^\star$ there is no solution.*

3.2. **Proof of points 1 and 3 in Theorem 1.2.** We follow partly the step in [25].

1. We prove an a priori bound on t with respect to $|u|_\infty$ when u is a solution for $\mathcal{P}_t$.

2. We prove an a priori bound on $|u|_\infty$ when u solves $\mathcal{P}_t$.

3. We exhibit some strict sub-solution for all t, and prove the existence of super-solutions for t not too large. We deduce by Perron's method that solutions exist for t not too large.

4. We prove that the set of t for which $\mathcal{P}_t$ admits a solution is an interval.

The first and second steps are contained in Proposition 3.2 below.

Proposition 3.2. *For any m_o there exists a constant R_o such that for any $t \ge -m_o$ and any solution u of $\mathcal{P}_t$,*

$$|u|_\infty \le R_o \quad \text{and} \quad t \le R_o.$$

In particular there is no solution for t large.

Proof. First step: Let u be a solution for $\mathcal{P}_t$. We prove that there exists some constant C which depends on the L^∞ norm of h and of universal constants such that $t \le C(1 + |u|_\infty^{1+\alpha})$.

Let δ be such that, for $d(x, \partial\Omega) < \delta$, the distance to $\partial\Omega$ is well defined and $\mathcal{C}^2$. One can assume that $d(x_o, \partial\Omega) > \delta$. In these conditions $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$ is also a regular subset which contains x_o.

We denote by $|\vec{c}|_\infty$ the constant $|\vec{c}|_\infty = |c_1|_\infty + |c_2|_\infty$.

Let c_ϕ and C_ϕ be such that, by previous known results contained in Remark 2.9 and Remark 2.7, $C_\phi d(x, \partial\Omega) \ge \phi \ge c_\phi d(x, \partial\Omega)$.

We assume first that $t \leq \left(\frac{2(|h|_\infty+|\vec{c}|_\infty|u|_\infty^{1+\alpha})}{(c_\phi\delta)^{1+\alpha}}\right)$; in that case the result is obtained.

If $t > \left(\frac{2(|h|_\infty+|\vec{c}|_\infty|u|_\infty^{1+\alpha})}{(c_\phi\delta)^{1+\alpha}},\right)$, let us define

$$\Omega_t = \left\{x : c_\phi d(x,\partial\Omega) > \left(\frac{2(|h|_\infty+|\vec{c}|_\infty|u|_\infty^{1+\alpha})}{t}\right)^{\frac{1}{1+\alpha}}\right\}.$$

Then Ω_t is a regular subset which contains x_o. On $\partial\Omega_t$,

$$\phi \geq \left(\frac{2(|h|_\infty+|\vec{c}|_\infty|u|_\infty^{1+\alpha})}{t}\right)^{\frac{1}{1+\alpha}}$$

and then by the comparison principle in Proposition 2.4, in Ω_t, applied with the operator $F(M,p,x)$, and using the fact that for this operator the positive constants are sub-solutions,

$$\phi^{1+\alpha} t \geq 2(|h|_\infty + |\vec{c}|_\infty |u|^{1+\alpha}) \text{ in } \Omega_t.$$

Moreover, by the Hopf principle the inequality is strict inside Ω_t.

In particular, in Ω_t

$$\begin{aligned} F(D^2u,\nabla u,x) &\leq -t\phi^{1+\alpha} + h + |\vec{c}|_\infty |u|_\infty^{1+\alpha} < -\frac{t}{2}\phi^{1+\alpha} \\ &= F\Big(D^2\Big(\big(\frac{t}{2\lambda_1}\big)^{\frac{1}{1+\alpha}}\phi\Big), \nabla\Big(\big(\frac{t}{2\lambda_1}\big)^{\frac{1}{1+\alpha}}\phi\Big), x\Big). \end{aligned}$$

Let $M = (\frac{C_\phi}{c_\phi})\left(\frac{|h|_\infty+|\vec{c}|_\infty|u|_\infty^{1+\alpha}}{\lambda_1}\right)^{\frac{1}{1+\alpha}}$. Then on the boundary of Ω_t, $|u|_\infty + u + M$ is greater than $(\frac{t}{2\lambda_1})^{\frac{1}{1+\alpha}}\phi$, since on $\partial\Omega_t$

$$\left(\frac{t}{2\lambda_1}\right)^{\frac{1}{1+\alpha}}\phi \leq \left(\frac{2(|h|_\infty+|\vec{c}|_\infty|u|_\infty^{1+\alpha})}{2\lambda_1(c_\phi d(x,\partial\Omega))^{1+\alpha}}\right)^{\frac{1}{1+\alpha}} C_\phi d(x,\partial\Omega) \leq M.$$

Using the comparison principle for F in Ω_t, and the fact that $v = |u|_\infty + u + M$ is a super-solution of

$$F(D^2v,\nabla v,x) < \frac{-t}{2}\phi^{1+\alpha}$$

one then obtains that

$$\left(\frac{t}{2\lambda_1}\right)^{\frac{1}{1+\alpha}}\phi \leq u + |u|_\infty + M \leq 2|u|_\infty + M$$

which yields the desired bound on t, considering this inequality on x_o.

Second step. We prove that for all $m_o > 0$ there exists some constant R_o such that for any $t \geq -m_o$ and any solution u of $\mathcal{P}_t$ we have

$$|u|_\infty \leq R_o.$$

We argue by contradiction and assume that there exists a sequence $(t_n)_{n,n\geq 0}$, $t_n \geq -m_o$, and $(u_n)_n$ such that $|u_n|_\infty \to +\infty$ and

$$\begin{cases} G(D^2u_n, \nabla u_n, u_n, x) = -t_n\phi^{1+\alpha} + h & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega; \end{cases}$$

dividing by $|u_n|_\infty^{1+\alpha}$, using the compactness results in Corollary (2.16), extracting a subsequence, one gets that $v_n = \frac{u_n}{|u_n|_\infty}$ converges uniformly to some v, which satisfies

$$G(D^2v, \nabla v, v, x) \leq 0.$$

In particular, since the infimum of two super-solutions is a super-solution one has

$$G(-D^2v^-, -\nabla v^-, -v^-, x) \leq 0$$

which implies by the maximum principle, since $v^- = 0$ on the boundary and $\lambda^-(G) > 0$, that $-v^- \geq 0$, finally $v \geq 0$.

The function v cannot be identically zero, since $|v|_\infty = 1$, hence by the strict maximum principle in Theorem 2.8, it is > 0. Coming back to

$$G(D^2v, \nabla v, v, x) \leq 0,$$

one would get that $\lambda^+ \geq 0$, a contradiction.

This ends the proof of Proposition 3.2.

Proposition 3.3. *There exists $t_o \in \mathbb{R}$, $t_o < 0$, such that for all $t \leq t_o$, $\mathcal{P}_t$ possesses a strict super-solution $\bar{u} > 0$ which can be chosen independently on $t \leq t_o$.*

Proof. Let $\bar{u}$ be a solution for

$$\begin{cases} F(D^2\bar{u}, \nabla\bar{u}, x) = -h^- - 1 & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

$\bar{u}$ exists and is positive, using the existence results in [8]. Moreover, the estimates in Remark 2.7 applied to $\bar{u}$ and those in Remark 2.9 applied to ϕ imply that there exists $t_0 < 0$ with

$$(-t_o)\phi^{1+\alpha} \geq |\vec{c}|_\infty \bar{u}^{1+\alpha}.$$

We then have, for $t \leq t_o$,

$$G(D^2\bar{u}, \nabla\bar{u}, \bar{u}, x) \leq F(D^2\bar{u}, \nabla\bar{u}, x) + |\vec{c}|_\infty \bar{u}^{1+\alpha} < -h^- - t_o\phi^{1+\alpha} - 1 < h - t\phi^{1+\alpha}$$

and then $\bar{u}$ is a strict super-solution > 0 for $\mathcal{P}_t$. Let us note that by Hopf's principle $\partial_n\bar{u} < 0$ on the boundary.

Proposition 3.4. *For any $t \in \mathbb{R}$ there exists a strict sub-solution $\underline{u}$ for $\mathcal{P}_t$, $\underline{u} \leq 0$, which can be chosen independently on t, when t belongs to a compact set I_1 of $\mathbb{R}$. Moreover, $\underline{u}$ can be chosen such that $\underline{u} \leq u$, for all solutions u of $\mathcal{P}_t$ for $t \in I_1$.*

Proof. Let I_1 be a compact set in $\mathbb{R}$. Let us consider $\underline{u}$ a solution of the Dirichlet problem

$$\begin{cases} F(D^2\underline{u}, \nabla\underline{u}, \underline{u}, x) - c_2|\underline{u}|^\alpha \underline{u} = \sup_{t\in I_1} |-t\phi^{1+\alpha} + h|_\infty + 1 \equiv \underline{M} & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the principal eigenvalue $\lambda^-(G)$ of G is > 0, using the existence result in [8], this problem possesses a solution, which is < 0, according to the maximum principle in Theorem 2.11. It is then also a solution of

$$\begin{cases} G(D^2\underline{u}, \nabla\underline{u}, \underline{u}, x) = \sup_{t\in I_1} |-t\phi^{1+\alpha} + h|_\infty + 1 \equiv \underline{M} & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

To prove that if u is a solution of $\mathcal{P}_t$, $u \geq \underline{u}$, let us remark that 0 is a super-solution for the equation

$$G(D^2u, \nabla u, u, x) \leq \underline{M}$$

and since the infimum of two super-solutions is a super-solution, $-u^-$ is a super-solution for the same equation. One has

$$F(D^2(-u^-), \nabla(-u^-), x) + c_2(u^-)^{1+\alpha} \leq F(D^2\underline{u}, \nabla\underline{u}, x) + c_2(-\underline{u})^{1+\alpha} - 1$$

which implies by the comparison principle as it is formulated in Remark 2.14, that $-u^- \geq \underline{u}$. This ends the proof of Proposition 3.4.

We suppose that $t \leq t_o$, and assuming that I_1 contains t_o, let E be defined as

$$E = \{u : \text{sub} - \text{solution of } \mathcal{P}_t, \underline{u} \leq u \leq \overline{u}\};$$

then, using Perron's method adapted to our context, as is done in [5], one can exhibit a solution as the supremum of E. This establishes the existence of solutions for $t \leq t_0$ and finishes step 3.

We now consider $t > t_o$. Suppose that there exists a super-solution $\bar{u}_t$ for $\mathcal{P}_t$, then, arguing as before, we prove that $\bar{u}_t \geq -\bar{u}_t^- \geq \underline{u}$. Then, considering the supremum of the sub-solutions which lie between $\underline{u}$ and $\bar{u}_t$, and using once more Perron's method, one gets a solution for $\mathcal{P}_t$.

We now define $t^\star = \sup\{t \in \mathbb{R} : \mathcal{P}_t \text{ has a super} - \text{solution }\}$.

By Proposition 3.2 $t^\star$ is finite. Moreover, if $t < t^\star$, let $t_1 \in [t, t^\star[$, then a super-solution for $\mathcal{P}_{t_1}$ is a super-solution for $\mathcal{P}_t$. From this, one derives that

$\mathcal{P}_t$ has a solution, which implies that the set of $t \geq t_o$ for which there exists a solution for $\mathcal{P}_t$, is an interval.

We now prove the existence of a solution for $t = t^\star$: Let t_n be a sequence which tends to $t^\star$, $t_n < t^\star$, and (u_n) a sequence of solutions for $\mathcal{P}_{t_n}$. Since (u_n) is bounded, using the compactness result in Corollary 2.16 recalled in Section 2, it converges, up to a subsequence, towards some function u which is a solution for $\mathcal{P}_{t^\star}$.

4. Nonuniqueness results in the singular case

In this section we want to prove some results about the multiplicity of solutions in the singular case and under some additional assumptions on the operator and the parameter t.

4.1. **The operator K_t.**

We begin to introduce some operator which can be defined in the general setting employed before.

Let $t < t^\star$, let $\overline{u}$ be a super-solution for $\mathcal{P}_t$, and $\underline{u}$ be a sub-solution for $\mathcal{P}_t$; we define the closed set

$$\overline{\mathcal{O}}_{\underline{u},\overline{u}} = \{u \in \mathcal{C}(\overline{\Omega}) : \underline{u} \leq u \leq \overline{u} \text{ in } \Omega\}.$$

Let $C = 2|\vec{c}|_\infty$ and consider for $v \in \overline{\mathcal{O}}_{\underline{u},\overline{u}}$, $u = K_t(v)$ the solution of

$$\begin{cases} G(D^2u, \nabla u, v, x) - C|u|^\alpha u = -t\phi^{1+\alpha} + h - C|v|^\alpha v & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This equation may be written $F(D^2u, \nabla u, x) - C|u|^\alpha u = g$, with g continuous and bounded, consequently it possesses a unique solution, since $F(M, p, x) - C|u|^\alpha u$ admits two principal eigenvalues > 0 ([8]), using Proposition 2.4.

We check first that K_t sends $\overline{\mathcal{O}}_{\underline{u},\overline{u}}$ into itself. For the first inequality let us use $v \geq \underline{u}$, which implies that $v^- \leq -\underline{u}$. We have then

$$\begin{aligned} c_2(-\underline{u})^{1+\alpha} + C|\underline{u}|^\alpha \underline{u} &= (c_2 - C)(-\underline{u})^{1+\alpha} \leq (c_2 - C)(v^-)^{1+\alpha} \\ &\leq C|v|^\alpha v + c_1(v^+)^{1+\alpha} + c_2(v^-)^{1+\alpha}, \end{aligned}$$

and then

$$\begin{aligned} &G(D^2u, \nabla u, \underline{u}, x) = G(D^2u, \nabla u, \underline{u}, x) + C|\underline{u}|^\alpha \underline{u} - C|\underline{u}|^\alpha \underline{u} \\ &\leq G(D^2u, \nabla u, v, x) + C|v|^\alpha v - C|\underline{u}|^\alpha \underline{u} = f + C|u|^\alpha u - C|\underline{u}|^\alpha \underline{u} \\ &< G(D^2\underline{u}, \nabla \underline{u}, \underline{u}, x) + C|u|^\alpha u - C|\underline{u}|^\alpha \underline{u}. \end{aligned}$$

By the comparison principle (cf. [5]) for the operator $F(M, p, x) - C|u|^\alpha u$, one gets that $u \geq \underline{u}$.

We now observe that, since $v \le \bar u$,

$$c_1(\bar u^+)^{1+\alpha} + c_2(\bar u^-)^{1+\alpha} + C|\bar u|^\alpha \bar u \ge c_1(v^+)^{1+\alpha} + c_2(v^-)^{1+\alpha} + C|v|^\alpha v.$$

Indeed,

$$(c_1 + C)(\bar u^+)^{1+\alpha} \ge (c_1 + C)(v^+)^{1+\alpha}$$

and since $-\bar u^- \ge -v^-$

$$(c_2 - C)(v^-)^{1+\alpha} \le (c_2 - C)(\bar u^-)^{1+\alpha}.$$

From this one derives

$$\begin{aligned}
&F(D^2u, \nabla u, x) + c.\bar u^{1+\alpha} = F(D^2u, \nabla u, x) + c.\bar u^{1+\alpha} + C|\bar u|^\alpha \bar u - C|\bar u|^\alpha \bar u\\
&\ge F(D^2u, \nabla u, x) + c_1(v^+)^{1+\alpha} + c_2(v^-)^{1+\alpha} + C|v|^\alpha v - C|\bar u|^\alpha \bar u\\
&= f + C|u|^\alpha u - C|\bar u|^\alpha \bar u > F(D^2\bar u, \nabla \bar u, x) + c.\bar u^{1+\alpha} + C|u|^\alpha u - C|\bar u|^\alpha \bar u.
\end{aligned}$$

This implies using the comparison principle for the operator

$$F(M, p, x) - C|u|^\alpha u,$$

that $u \le \bar u$. We have obtained that K_t sends $\overline{\mathcal{O}}_{\underline{u},\overline{u}}$ into itself.

Let us note that the operator K_t is a compact operator. Indeed, by Corollary 2.16, K_t sends $\mathcal{C}(\overline{\Omega})$ into $\mathcal{C}(\overline{\Omega})$ and sends bounded sets into precompact sets. This result provides another proof of the existence of solutions when there exist a sub- and a super-solution ordered, using Schauder's fixed-point theorem and the fact that $K_t(\overline{\mathcal{O}}_{\underline{u},\overline{u}})$ is a compact set in $\mathcal{C}(\overline{\Omega})$. Let us also observe that all the fixed points of K_t are solutions of $\mathcal{P}_t$.

In the sequel we shall need some additional property of K_t when $t \le t_o$.

Proposition 4.1. *Suppose that $-m_o \le t < t_o$, and let $\underline{u}$ (respectively $\overline{u}$) be given by Proposition* 3.4 (*respectively* 3.3). *Then, there exists $\underline{\lambda}, \overline{\lambda} < 1$ such that*

$$\underline{\lambda}\ \underline{u} < K_t(v) \le \overline{\lambda}\ \overline{u}.$$

Proof. Indeed, let $\underline{\lambda} = \left(\sup_{x\in\bar\Omega} \frac{\underline{M} - 1 + C|\underline{u}|^{1+\alpha} - c_2(-\underline{u})^{1+\alpha}}{\underline{M} + C|\underline{u}|^{1+\alpha} - c_2(-\underline{u})^{1+\alpha}}\right)^{\frac{1}{1+\alpha}}$ where we recall that $\underline{M} = |-t\phi^{1+\alpha} + h|_\infty + 1$ has been defined in Proposition 3.4. $\underline{\lambda}$ is well defined, is positive and strictly less than 1. One has by the computations above

$$\begin{aligned}
F(D^2u, \nabla u, x) - C|u|^\alpha u &\le F(D^2\underline{u}, \nabla \underline{u}, x) - C|\underline{u}|^\alpha \underline{u} - 1\\
&\le \underline{\lambda}^{1+\alpha}\left(F(D^2\underline{u}, \nabla \underline{u}, x) - C|\underline{u}|^\alpha \underline{u}\right).
\end{aligned}$$

From this, using the homogeneity of the operator F one gets that $u \geq \underline{\lambda}\, \underline{u}$. An analogous computation permits us to prove the inequality $K_t(v) \leq \overline{\lambda}\, \overline{u}$ with $\overline{\lambda} = \left(\sup_{x\in\overline{\Omega}} \frac{-h^- - C|\overline{u}(x)|^\alpha \overline{u}(x)}{-h^- - C|\overline{u(x)}|^\alpha \overline{u}(x) - 1}\right)^{\frac{1}{1+\alpha}}$.

4.2. **Proof of point 2 in Theorem 1.2.** We now prove the multiplicity of solutions under some additional assumptions.

From now on we shall suppose that $\alpha \leq 0$ and we shall assume that the operator is of the form

$$F(x, p, M) = |p|^\alpha (\tilde{F}(M, x) + b(x) \cdot p).$$

On $\tilde{F}$ we suppose the following.

($H_1^\star$) $(M, x) \mapsto \tilde{F}(M, x)$ is continuous and for any $t \in \mathbb{R}^+$, for any $M \in S$, $\tilde{F}(tM, x) = t\tilde{F}(M, x)$.

($H_2^\star$) For any $M \in S$, and any $N \in S$, $N \geq 0$

$$atr(N) \leq \tilde{F}(M + N, x) - \tilde{F}(M, x) \leq Atr(N). \tag{4.1}$$

($H_3^\star$) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$

$$\tilde{F}(X, x) - \tilde{F}(-Y, y) \leq \omega(\zeta|x - y|^2).$$

Remark 4.2. If $\tilde{F}$ satisfies condition ($H_1^\star$), ($H_2^\star$), ($H_3^\star$), $|p|^\alpha \tilde{F}(M, x)$ satisfies (H1), (H2), (H3) (since $\alpha \leq 0$), and (H5).

The function b still satisfies (H4). Under these assumptions there exist $\mathcal{C}^{1,\beta}$ regularity results for related Dirichlet problems, [26], [14], and [12], [16]. As a consequence we proved in [9] the following.

Proposition 4.3. *Let Ω be a bounded regular domain, and $f \in \mathcal{C}(\overline{\Omega})$. Suppose that $-1 < \alpha \leq 0$, suppose that $\tilde{F}$ satisfies the conditions ($H_1^\star$), ($H_2^\star$) and ($H_3^\star$) and b satisfies (H4) Then the solutions of*

$$\begin{cases} |\nabla u|^\alpha \left(\tilde{F}(D^2 u, x) + b(x) \cdot \nabla u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

are $\mathcal{C}^{1,\beta}(\bar{\Omega})$ for all $\beta < 1$ with the estimates

$$|u|_{\mathcal{C}^{1,\beta}(\bar{\Omega})} \leq c(|f|_\infty^{\frac{1}{1+\alpha}}).$$

From this proposition one derives the following.

Proposition 4.4. *We suppose that the assumptions in Proposition* 4.3 *are satisfied. Then, for all* $\beta \in [0,1[$, *there exists some constant* c_β *such that for all* $v \in \mathcal{C}(\overline{\Omega})$

$$|K_t(v)|_{\mathcal{C}^{1,\beta}(\overline{\Omega}))} \leq c_\beta(|v|_\infty + |f|_\infty^{\frac{1}{1+\alpha}}).$$

In particular, K_t is a compact operator on $\mathcal{C}^1(\overline{\Omega})$. We now observe the following.

Proposition 4.5. *Suppose that* $t \leq t_o$, *and let* $\underline{u}$ *and* $\overline{u}$ *be given by Proposition* 3.4 *and* 3.3. *Let* v *be in* $\mathcal{C}(\overline{\Omega})$ *with* $\underline{u} \leq v \leq \overline{u}$ in Ω, *then* $K_t(v)$ *belongs to the open set* $\mathcal{O} = \{v \in \mathcal{C}_0^1(\Omega) : \ \underline{u} < v < \overline{u} \text{ in } \Omega, \partial_n \overline{u} < \partial_n u < \partial_n \underline{u} \text{ on } \partial\Omega\}$.

Proof of Proposition 4.5. It is sufficient to use the inequality

$$\underline{\lambda}\ \underline{u} \leq u \leq \overline{\lambda}\ \overline{u}$$

proved in Proposition 4.1 with some $\overline{\lambda}$, $\underline{\lambda} \in]0,1[$. Using the fact that $u = 0$ on the boundary and that for $\bar{x} \in \partial\Omega$, for $h > 0$, and $\vec{n}(\bar{x})$ the unit outer normal to $\partial\Omega$ on $\bar{x}$, one has

$$\underline{\lambda}\ \frac{\underline{u}(\bar{x} - h\vec{n}(\bar{x})) - \underline{u}(\bar{x})}{-h} \geq \frac{u(x - h\vec{n}(\bar{x})) - u(\bar{x})}{-h} \geq \overline{\lambda}\ \frac{\overline{u}(\bar{x} - h\vec{n}(\bar{x})) - \overline{u}}{-h},$$

one gets the result.

We have obtained that for $t \leq t_o$, $K_t(\overline{\mathcal{O}}) \subset \mathcal{O}$ and thus the degree $d(I - K_t, \mathcal{O}, 0)$ is well defined.

Remark 4.6. Suppose that $t' > t$, $t' < t^\star$ is such that $\mathcal{P}_{t'}$ possesses a super-solution $u_{t'}$ which satisfies the Hopf principle on the boundary. Then by a strict comparison principle [9], one gets that if $\overline{\mathcal{O}}_t = \{u : \underline{u} \leq u \leq u_{t'}\}$ and $\mathcal{O}_t = \{u \in \mathcal{C}_0^1(\Omega) : \underline{u} < u < u_{t'} \text{ in } \Omega, \partial_n \overline{u} < \partial_n u < \partial_n \underline{u} \text{ on } \partial\Omega\}$, $K_t(\overline{\mathcal{O}_t}) \subset \mathcal{O}_t$. Consequently in fact the multiplicity result which follows is still valid for $t < t^{\star\star} = \sup\{t : \exists$ some super-solution for $\mathcal{P}_t$ which satisfies Hopf on the boundary$\}$. Since this comparison principle is rather technical, we have preferred to announce the multiplicity result only in the case $t \leq t_o$ since in that case the arguments are rather elementary (see Proposition 4.5).

Proposition 4.7. *Let* $B_R = \{u \in \mathcal{C}^1(\overline{\Omega}) : ||u||_{\mathcal{C}^1(\overline{\Omega})} < R\}$. *Let* $t_1 \in (-\infty, t_o)$; *there exist* R_1 *and* $R_2 \in \mathbb{R}^+$, $R_1 < R_2$, *such that* $d(I - K_{t_1}, B_{R_2}, 0) = 0$, $d(I - K_{t_1}, \mathcal{O} \cap B_{R_1}, 0) = 1$ *and* $d(I - K_{t_1}, B_{R_2} \setminus \overline{\mathcal{O} \cap B_{R_1}}, 0) = -1$.

Proof. Let $t_1 < t_o$, $\underline{u}$ and $\overline{u}$ be respectively sub- and super-solutions for $\mathcal{P}_t$. Let $\bar{R}$ be an upper bound for the $\mathcal{C}^1(\overline{\Omega})$ norm of $K_t(v)$ when $|v|_\infty \leq R_o$, where R_o is defined in Proposition 3.2. Let $R_1 = \sup\{\bar{R} : |\underline{u}|_{\mathcal{C}^1(\overline{\Omega})}, |\overline{u}|_{\mathcal{C}^1(\overline{\Omega})}\}$ and $R_2 = \sup(\bar{R}+1, R_1+1)$.

To prove the first equality, let us note that $d(I-K_t, B_{R_2}, 0)$ is well defined. Let $t_2 = t^\star + 1$. $t \mapsto K_t$ is a compact homotopy linking K_{t_1} to K_{t_2}. We know by Proposition 3.2 that for all $u \in \partial B_{R_2}$ and $t \in [t_1, t_2]$ $(I - K_t)(u) \neq 0$, which implies that $d(I - K_{t_1}, B_{R_2}, 0) = d(I - K_{t_2}, B_{R_2}, 0) = 0$ since K_{t_2} has no fixed point in B_{R_2}.

To prove the second equality we fix $w \in \mathcal{O} \cap B_{R_1}$ and consider $H_s(v) = sK_{t_1}(v) + (1-s)w$, $v \in \mathcal{C}(\overline{\Omega})$. H_s is a compact homotopy linking K_{t_1} to the mapping H_0. Moreover, $H_s(\overline{\mathcal{O}}) \cap \partial\mathcal{O} = \emptyset$. One has $d(I - K_{t_1}, \mathcal{O} \cap B_{R_1}, 0) = d(I - H_1, \mathcal{O} \cap B_{R_1}, 0) = d(I - H_o, \mathcal{O} \cap B_{R_1}, 0) = 1$, since H_o is a constant mapping.

Finally, to complete the proof of the multiplicity result, we use the excision property (cf [17], [21]) of the degree. One has $d(I - K_{t_1}, \mathcal{O} \cap B_{R_1}, 0) = 1$ and $d(I - K_{t_1}, B_{R_2}, 0) = 0$, hence,

$$d(I - K_{t_1}, B_{R_2} \setminus \overline{\mathcal{O} \cap B_{R_1}}, 0) = -1,$$

which implies that $I - K_{t_1}$ has at least also one zero in $B_{R_2} \setminus \overline{(\mathcal{O} \cap B_{R_1})}$.

Then we conclude with the degree theory that K_{t_1} has at least two fixed points, one in $\mathcal{O} \cap B_{R_1}$, the other in $B_{R_2} \setminus \overline{\mathcal{O} \cap B_{R_1}}$.

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