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VERY SLOW CONVERGENCE TO ZERO FOR A SUPERCRITICAL SEMILINEAR PARABOLIC EQUATION

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Abstract. We study the asymptotic behavior of nonnegative solutions to the Cauchy problem for a semilinear parabolic equation with a supercritical nonlinearity. It is known that there are initial data such that the corresponding solution decays to zero with an algebraic rate. Furthermore, any algebraic rate which is slower than the self-similar rate occurs as decay rate for some solution. In this paper we prove that the convergence to zero can take place with an "arbitrarily" slow rate, if the initial data are chosen properly.

1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $p > 1$, Δ denotes the Laplacian operator with respect to x and the function u_0 is nonnegative and continuous in $\mathbb{R}^N$. In spite of its simple structure, problem (1.1) offers a rich variety of mathematical phenomena and has been studied intensively by several authors. The monograph [20] and the references given therein provide a broad overview.

Concerning the large-time behavior of solutions to (1.1), the Fujita exponent

$$p_F := \frac{N+2}{N}$$

is one of the critical exponents. It is well known that if $1 < p \le p_F$ each positive solution of (1.1) blows up in finite time, whereas there are positive global solutions when $p > p_F$ (see [12]). With regard to global solutions

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converging to zero, different decay rates have been proved by several authors. In [16] it has been proved that, for $p > p_F$ and initial data u_0 satisfying

$$k_1(1+|x|)^{-l} \le u_0(x) \le k_2(1+|x|)^{-l}, \quad x \in \mathbb{R}^N,$$

with positive and small constants k_1 and k_2 the corresponding solution u of (1.1) exists globally in time and decays to zero at the same rate as the solution of the linear heat equation with the same initial data. Namely, u fulfills

$$K_1\, g(t) \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le K_2\, g(t), \qquad \text{for } t \ge t_1 > 1,$$

where g is given by

$$g(t) := \begin{cases} t^{-\frac{N}{2}} & \text{if } l > N, \\ t^{-\frac{N}{2}} \ln t & \text{if } l = N, \\ t^{-\frac{l}{2}} & \text{if } \frac{2}{p-1} \le l < N. \end{cases}$$

Moreover, it is shown in [4] that for $l > \frac{2}{p-1}$ this behavior of u occurs for a larger class of initial data without a smallness condition on k_2. We remark that the slowest of these decay rates is $t^{-\frac{1}{p-1}}$.

Furthermore, several conditions have been found which imply that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\, t^{-\frac{1}{p-1}} \qquad \text{for } t > 0$$

holds with some positive constant C (see e.g. [13], [15], [17], [21], [22]).

Additionally, there are solutions which decay to zero at a rate which is slower than $t^{-\frac{1}{p-1}}$. In order to present these results, we define another critical exponent

$$p_c := \begin{cases} \infty & \text{for } N \le 10, \\ \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{for } N \ge 11, \end{cases}$$

which satisfies $p_c > \frac{N}{N-2} > 1$ for $N \ge 11$. Moreover, let $\varphi_\infty = \varphi_\infty(|x|)$ denote the singular steady state of (1.1), which exists for $p > \frac{N}{N-2}$, $N > 2$, and is given by

$$\varphi_\infty(|x|) := L|x|^{-m}, \quad |x| > 0,$$

where $m := \frac{2}{p-1}$ and $L := \{m(N-2-m)\}^{\frac{1}{p-1}}$. Finally, if $p > p_c$ we set

$$\lambda_1 = \lambda_1(N,p) := \frac{N-2-2m-\sqrt{(N-2-2m)^2-8(N-2-m)}}{2},$$

which is the smaller positive root of

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0,$$

while λ_2 is defined to be the larger positive root of this equation. It was proved in [15] that for $p > p_c$ with initial data u_0 fulfilling

$$0 \le u_0(x) < \varphi_\infty(|x|) \quad \text{for } |x| > 0 \tag{1.2}$$

and

$$\varphi_\infty(|x|) - \kappa_1 |x|^{-l} \le u_0(x) \le \varphi_\infty(|x|) - \kappa_2 |x|^{-l} \quad \text{for } |x| > R \tag{1.3}$$

with $l \in (m, m+\lambda_1)$ and some positive constants κ_1, κ_2 and R, the solution u of (1.1) is global in time and converges to zero in such a way that

$$t^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to \infty \quad \text{as } t \to \infty$$

is satisfied. In [9] the exact decay rate of this slow convergence to zero was determined and it was shown that

$$K_1(t+1)^{-\frac{m(m+\lambda_1-l)}{2\lambda_1}} \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le K_2(t+1)^{-\frac{m(m+\lambda_1-l)}{2\lambda_1}},\ t \ge 0, \tag{1.4}$$

holds with positive constants K_1 and K_2. Moreover, (1.4) is also valid when $p > p_c$, if the initial data satisfy (1.2) and (1.3) with some $l \in [m+\lambda_1, m+\lambda_2+2)$. This shows that solutions grow up for $l > m+\lambda_1$ while they remain bounded and bounded away from zero for $l = m+\lambda_1$. We refer to [2] and [7] for the grow-up rate when $l > m+\lambda_1$ and to [8], [14], [18] and [19] for the convergence to regular steady states if $l = m+\lambda_1$. Furthermore, the rates of convergence to singular steady states (see [6]), the convergence to self-similar solutions (see [10], [11]) and the grow-up rate in the critical case $p = p_c$ (see [3]) have been established.

We remark that any algebraic decay rate slower than the self-similar one occurs due to (1.4) for solutions converging to zero, if the initial data are chosen suitably. As the grow-up can take place with any arbitrarily slow rate and in particular with rates which are slower than any algebraic rate (see [5]), we are concerned with the question whether the convergence to zero also occurs with an arbitrarily slow rate.

To this end, we assume in this paper that $p > p_c$ with $N \ge 11$ as well as (1.2) and

$$\varphi_\infty(|x|) - c_1|x|^{-m-\lambda_1}\eta(|x|) \le u_0(x) \le \varphi_\infty(|x|) - c_2|x|^{-m-\lambda_1}\eta(|x|),\ |x| > R, \tag{1.5}$$

where c_1, c_2 and R are positive constants. Here η is supposed to increase slowly at infinity like for example $\eta(z) = (\ln(z+z_0))^n$ for $n \in \mathbb{N}$ or $\eta(z) = \ln(\ln(\ldots(\ln(z+z_0))\ldots))$. Actually, the conditions on η which are raised below are satisfied for any of these examples if z_0 is chosen large enough.

Throughout this paper, $\eta \in C^2([0,\infty))$ is supposed to fulfill

$$\eta(z) > 0, \quad \eta'(z) > 0 \quad \text{and} \quad \eta''(z) \le 0 \quad \text{for all } z \ge 0 \tag{1.6}$$

such that η increases slowly near infinity in the sense that

$$\frac{z\eta'(z)}{\eta(z)} \to 0 \quad \text{as } z \to \infty. \tag{1.7}$$

Furthermore, we assume that

$$\left|\frac{z\eta''(z)}{\eta'(z)}\right| \le C_\eta \quad \text{for all } z \ge 0 \tag{1.8}$$

holds with a positive constant C_η. Finally, we suppose that for any $\alpha > 0$ and $\gamma > 0$ there is a positive constant $c_{\alpha,\gamma}$ such that

$$\eta(\gamma z^\alpha) \le c_{\alpha,\gamma}\eta(z) \quad \text{for all } z \ge 1 \tag{1.9}$$

is fulfilled. Indeed, condition (1.9) is not a consequence of (1.6), (1.7) and (1.8) which can be seen for example with the function $\eta(z) := e^{(\ln(z+2))^\varepsilon}$, where $\varepsilon > 0$ is a small constant. Now (1.7) and (1.8) imply

$$\frac{z^2\eta''(z)}{\eta(z)} \to 0 \quad \text{as } z \to \infty. \tag{1.10}$$

Moreover, we obtain from (1.6) and (1.7) that for any $\alpha > 0$ there is a constant $C_\alpha > 0$ such that

$$\eta(z) \le C_\alpha z^\alpha \quad \text{for all } z \ge 1. \tag{1.11}$$

We are now able to state our main result which shows that the convergence to zero in (1.1) takes place with arbitrarily slow decay rates, if the initial data are chosen suitably. In particular, there are solutions converging to zero with decay rates that are slower than any algebraic rate.

Theorem 1.1. *Let $N \ge 11$, $p > p_c$ and assume that $u_0 \in C^0(\mathbb{R}^N)$ fulfills* (1.2) *and* (1.5), *where η meets the conditions* (1.6), (1.7), (1.8) *and* (1.9). *Then there are positive constants C_1 and C_2 such that the solution u of* (1.1) *satisfies*

$$C_1\eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}) \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C_2\eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}) \quad \textit{for all } t \ge 0.$$

We remark that there are also bounded functions η which fulfill the conditions raised above. In this case Theorem 1.1 gives another proof of (1.4) with $l = m + \lambda_1$. Although we are here only interested in the case where η is unbounded, we prove Theorem 1.1 for general functions η as we do not need the unboundedness of η in the proof.

This paper is organized in the following way. In Section 2 we briefly introduce the self-similar change of variables which transforms radially symmetric solutions of (1.1) decaying to zero with a very slow rate into solutions of another problem which grow up. In Sections 3 and 4 we prove an upper and lower bound of the corresponding grow-up rate by constructing suitable super- and subsolutions, respectively, and using comparison arguments. These super- and subsolutions for the transformed problem are more transparent than for (1.1) itself. Finally, we complete the proof of Theorem 1.1 in Section 5.

2. Self-similar change of variables

To prove Theorem 1.1 we make use of a suitable transformation which has been an important ingredient of [9]. A radially symmetric solution of (1.1) with the behavior claimed in Theorem 1.1 will be transformed into a function which grows up. We will derive estimates for this grow-up rate which will imply the claimed behavior of solutions to (1.1). Here we shortly introduce the transformation and refer to [9] for more details.

If $u = u(r,t)$, $r = |x|$, is a radially symmetric solution of (1.1), it satisfies

$$\begin{cases} u_t = u_{rr} + \frac{N-1}{r} u_r + u^p, & r > 0, t > 0, \\ u(r,0) = u_0(r), & r > 0. \end{cases} \tag{2.1}$$

The self-similar change of variables

$$v(\rho, s) = (t+1)^{\frac{1}{p-1}} u(r,t), \quad \rho = \frac{r}{\sqrt{t+1}}, \quad s = \log(t+1) \tag{2.2}$$

transforms (2.1) into the problem

$$\begin{cases} v_s = v_{\rho\rho} + \frac{N-1}{\rho} v_\rho + v^p + \frac{\rho}{2} v_\rho + \frac{m}{2} v, & \rho > 0, s > 0, \\ v(\rho,0) = v_0(\rho) \equiv u_0(\rho), & \rho > 0. \end{cases} \tag{2.3}$$

Now our aim is to show that a radially nonincreasing solution v of (2.3) grows up such that

$$v(0,s) \simeq e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}} (e^{\frac{1}{2}s}) \quad \text{for } s \geq 0 \tag{2.4}$$

is satisfied. This will imply the claimed behavior of u.

3. Upper bound

We use an idea from [9] to prove an upper estimate for the grow-up rate of solutions to (2.3) which corresponds to (2.4). For this purpose, we construct two supersolutions of (2.3), one of them in an inner region near $\rho = 0$ and the other one in a corresponding outer region which is bounded away from $\rho = 0$.

To this end, let ψ denote the classical solution of

$$\begin{cases} \psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \psi^p = 0, & \xi > 0, \\ \psi(0) = 1, \psi_\xi(0) = 0. \end{cases} \tag{3.1}$$

Then for $p > p_c$, ψ satisfies the asymptotic expansion

$$\begin{aligned} \psi(\xi) &= L\xi^{-m} - a\xi^{-m-\lambda_1} + o(\xi^{-m-\lambda_1}), \\ \psi_\xi(\xi) &= -mL\xi^{-m-1} + a(m+\lambda_1)\xi^{-m-\lambda_1-1} + o(\xi^{-m-\lambda_1-1}), \quad \xi \simeq \infty, \end{aligned} \tag{3.2}$$

where a is a positive constant (see [7], [14]). This implies

$$L\xi^{-m} - a_1\xi^{-m-\lambda_1} \le \psi(\xi) \le L\xi^{-m} - a_2\xi^{-m-\lambda_1} \quad \text{for } \xi \ge 1 \tag{3.3}$$

with some positive constants a_1 and a_2. Furthermore, ψ has the following property.

Lemma 3.1. *Suppose $N \ge 11$, $p > p_c$ and ψ is the solution of* (3.1). *Then*

$$\psi(\xi) + \frac{1}{m}\xi\,\psi_\xi(\xi) \ge 0 \quad \text{for } \xi \ge 0$$

is satisfied.

Proof. As the positive radially symmetric steady states of (1.1) are ordered when $p \ge p_c$, we obtain $\psi(\xi) < \varphi_\infty(\xi)$ for any $\xi > 0$ (see e.g. [2] and the references given there).

We fix $\xi_0 > 0$ and set $B_\xi := B_\xi(0) \subset \mathbb{R}^N$ for $\xi > 0$. Due to the fact that $N \ge 11$ and $p > p_c > \frac{N}{N-2}$, we have

$$-m - 1 = -\frac{2}{p-1} - 1 > -\frac{2}{\frac{N}{N-2} - 1} - 1 = -(N-2) - 1 = -(N-1).$$

Adapting an idea used in [14], we conclude by Green's identity (where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$) that

$$\begin{aligned} &N\omega_N\xi_0^{N-1}[\varphi_\infty(\xi_0)\psi_\xi(\xi_0) - \psi(\xi_0)\varphi'_\infty(\xi_0)] \\ &= N\omega_N\xi_0^{N-1}[\varphi_\infty(\xi_0)\psi_\xi(\xi_0) - \psi(\xi_0)\varphi'_\infty(\xi_0)] \\ &\quad - \lim_{\xi \searrow 0} N\omega_N\xi^{N-1}[\varphi_\infty(\xi)\psi_\xi(\xi) - \psi(\xi)\varphi'_\infty(\xi)] \\ &= \lim_{\xi \searrow 0} \int_{\partial(B_{\xi_0}\setminus B_\xi)} \Big(\varphi_\infty \frac{\partial\psi}{\partial\nu} - \psi\frac{\partial\varphi_\infty}{\partial\nu}\Big)\mathrm{d}S \end{aligned}$$

$$= \lim_{\xi \searrow 0} \int\limits_{B_{\xi_0} \setminus B_\xi} (\varphi_\infty \Delta\psi - \psi \Delta\varphi_\infty) \mathrm{d}x$$

$$= \lim_{\xi \searrow 0} \int\limits_{B_{\xi_0} \setminus B_\xi} (-\varphi_\infty \psi^p + \psi \varphi_\infty^p) \mathrm{d}x$$

$$= \lim_{\xi \searrow 0} \int\limits_{B_{\xi_0} \setminus B_\xi} \varphi_\infty \psi (\varphi_\infty^{p-1} - \psi^{p-1}) \mathrm{d}x \geq 0.$$

As this implies

$$\xi_0^{-m} \psi_\xi(\xi_0) + \psi(\xi_0) m \xi_0^{-m-1} \geq 0$$

and ξ_0 is positive, the claim is valid for ξ_0. Hence, the claim is proved, since $\xi_0 > 0$ is arbitrary and the claim is obvious for $\xi = 0$. □

In order to construct a supersolution of (2.3) in an inner region, we let Ψ denote the solution of

$$\begin{cases} \Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + p\psi^{p-1}\Psi = \frac{m+\lambda_1-l}{l-m}\left(\psi + \frac{1}{m}\xi\,\psi_\xi\right) + \frac{A}{1+\xi^{m+\lambda_1}}, \ \xi > 0, \\ \Psi(0) = 0, \quad \Psi_\xi(0) = 0, \end{cases} \tag{3.4}$$

where $l \in (m, m+\lambda_1)$ is fixed. Moreover, due to Lemma 3.1 in [9], we are able to choose $A > 0$ such that

$$\begin{aligned} \Psi(\xi) &= K\xi^{2-m-\lambda_1} + o(\xi^{2-m-\lambda_1}), \\ \Psi_\xi(\xi) &= -K(m+\lambda_1-2)\xi^{1-m-\lambda_1} + o(\xi^{1-m-\lambda_1}), \end{aligned} \quad \xi \simeq \infty, \tag{3.5}$$

and

$$|\Psi(\xi)| + |\xi\,\Psi_\xi(\xi)| \leq C_\Psi (1+\xi)^{2-m-\lambda_1} \quad \text{for all } \xi \geq 0 \tag{3.6}$$

is satisfied with positive constants K and C_Ψ.

Now by an adaption of the idea used to prove Lemma 3.2 of [9] we obtain a suitable supersolution in an inner region near $\rho = 0$.

Lemma 3.2. *Suppose $N \geq 11$ and $p > p_c$. For $\rho \geq 0$, $s \geq 0$, $M > 0$, $\beta > 0$ and $\mu > 0$ we define*

$$\sigma(s) := M e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)}), \qquad \xi(\rho, s) := \sigma^{\frac{1}{m}}(s)\,\rho$$

and

$$v_{in}(\rho, s) := \sigma\Big(\psi(\xi) - \frac{\sigma_s}{\sigma^p}\Psi(\xi)\Big).$$

Then for any $\beta > 0$ there are $\mu > 0$, $M_0 > 0$ and $\rho_0 > 0$ such that v_{in} is a supersolution of (2.3) for $0 < \rho < \rho_0$ and $s > 0$ and

$$\frac{\sigma_s}{\sigma^p}\frac{|\Psi(\xi)|}{\psi(\xi)} \le \frac{1}{2} \quad \textit{for } \rho > 0 \textit{ and } s > 0 \tag{3.7}$$

is fulfilled, whenever $M \ge M_0$ holds.

Proof. We fix $\beta > 0$. First we compute

$$\sigma_s(s) = \frac{m}{2} M e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)})\Big(1 - \frac{2\beta}{\lambda_1} \cdot \frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})}\Big) \quad \text{for } s \ge 0$$

and define

$$\varepsilon := \frac{m + \lambda_1 - l}{l - m} > 0,$$

where l is chosen in (3.4). Due to (1.6) and (1.7) there is $\mu > 0$ such that

$$\frac{m}{2(1+\varepsilon)}\sigma(s) \le \sigma_s(s) \le \frac{m}{2}\sigma(s) \quad \text{for all } s \ge 0 \tag{3.8}$$

is satisfied for any $M > 0$. We fix $\mu > 0$ with this property.

Next, (3.3), (3.6) and $\lambda_1 > 2$ imply the existence of $\xi_0 \ge 1$ such that

$$\frac{|\Psi(\xi)|}{\psi(\xi)} \le \frac{C_\Psi}{\frac{L}{2}}\xi^{2-\lambda_1} \le \frac{2C_\Psi}{L} \quad \text{for } \xi \ge \xi_0.$$

Hence, due to the positivity of ψ there is $c_0 > 0$ such that $\frac{|\Psi(\xi)|}{\psi(\xi)} \le c_0$ holds for all $\xi > 0$. Therefore, due to (1.11) we obtain

$$\begin{aligned} \frac{\sigma_s}{\sigma^p}\frac{|\Psi(\xi)|}{\psi(\xi)} &\le c_0\frac{m}{2}\sigma^{1-p} = c_0\frac{m}{2}M^{-\frac{2}{m}}e^{-s}\eta^{\frac{2}{\lambda_1}}(e^{\beta(s+\mu)}) \\ &\le c_0\frac{m}{2}M^{-\frac{2}{m}}(C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^{\mu} \le \frac{1}{2} \quad \text{for } \rho > 0, s > 0, \end{aligned}$$

whenever

$$M \ge M_1 := \Big(c_0\, m (C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}} e^{\mu}\Big)^{\frac{m}{2}} \tag{3.9}$$

is fulfilled. Thus, (3.7) is satisfied for sufficiently large M.

As $m + \lambda_1 > \lambda_1 > 2$, we can fix $\vartheta \in (\frac{2}{m+\lambda_1}, 1)$ with $\vartheta < \frac{2}{m+\lambda_1-2}$ and obtain $c_\vartheta > 0$ such that

$$(1-z)^p \le 1 - pz + c_\vartheta |z|^{1+\vartheta} \quad \text{for } |z| \le \frac{1}{2}$$

holds. Hence, for $M \ge M_1$, we conclude

$$\Big(\psi - \frac{\sigma_s}{\sigma^p}\Psi\Big)^p \le \psi^p - \frac{\sigma_s}{\sigma^p}p\psi^{p-1}\Psi + c_\vartheta\Big(\frac{\sigma_s}{\sigma^p}\Big)^{1+\vartheta}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta} \tag{3.10}$$

for all $\rho > 0$ and $s > 0$ by (3.7).

Now, we let $\mathcal{P}$ denote the operator defined by

$$\mathcal{P}w := w_s - w_{\rho\rho} - \frac{N-1}{\rho}w_\rho - w^p - \frac{\rho}{2}w_\rho - \frac{m}{2}w.$$

Due to $\sigma\xi_s = \frac{1}{m}\sigma_s\xi$, $1 + \frac{2}{m} = p$, (3.1), (3.4) and (3.10) we compute for $M \geq M_1$

$$\begin{aligned}
Pv_{in} &= \sigma_s\psi + \sigma\psi_\xi\xi_s - \big(\frac{\sigma_s}{\sigma^{p-1}}\Psi\big)_s - \sigma^{1+\frac{2}{m}}\psi_{\xi\xi} + \frac{\sigma_s}{\sigma^{p-1-\frac{2}{m}}}\Psi_{\xi\xi} \\
&\quad - \frac{N-1}{\rho}\sigma^{1+\frac{1}{m}}\psi_\xi + \frac{N-1}{\rho}\frac{\sigma_s}{\sigma^{p-1-\frac{1}{m}}}\Psi_\xi - \sigma^p\big(\psi - \frac{\sigma_s}{\sigma^p}\Psi\big)^p - \frac{\rho}{2}\sigma^{1+\frac{1}{m}}\psi_\xi \\
&\quad + \frac{\rho}{2}\frac{\sigma_s}{\sigma^{p-1-\frac{1}{m}}}\Psi_\xi - \frac{m}{2}\sigma\psi + \frac{m}{2}\frac{\sigma_s}{\sigma^{p-1}}\Psi \\
&= \sigma_s\big(\psi + \frac{1}{m}\xi\psi_\xi\big) - \big(\frac{\sigma_s}{\sigma^{p-1}}\Psi\big)_s - \sigma^p\big(\psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi\big) + \sigma_s\big(\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi\big) \\
&\quad - \sigma^p\big(\psi - \frac{\sigma_s}{\sigma^p}\Psi\big)^p - \sigma\big(\frac{\xi}{2}\psi_\xi + \frac{m}{2}\psi\big) + \frac{\sigma_s}{\sigma^{p-1}}\big(\frac{\xi}{2}\Psi_\xi + \frac{m}{2}\Psi\big) \\
&= \sigma_s\Big(\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + \psi + \frac{1}{m}\xi\,\psi_\xi\Big) - \big(\frac{\sigma_s}{\sigma^{p-1}}\Psi\big)_s \\
&\quad + \sigma^p\psi^p - \sigma^p\big(\psi - \frac{\sigma_s}{\sigma^p}\Psi\big)^p - \sigma\big(\frac{\xi}{2}\psi_\xi + \frac{m}{2}\psi\big) + \frac{\sigma_s}{\sigma^{p-1}}\big(\frac{\xi}{2}\Psi_\xi + \frac{m}{2}\Psi\big) \\
&\geq \sigma_s\Big(\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + p\psi^{p-1}\Psi + \psi + \frac{1}{m}\xi\,\psi_\xi\Big) - \big(\frac{\sigma_s}{\sigma^{p-1}}\Psi\big)_s \\
&\quad - c_\vartheta\frac{\sigma_s^{1+\vartheta}}{\sigma^{p\vartheta}}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta} - \frac{m}{2}\sigma\big(\psi + \frac{1}{m}\xi\,\psi_\xi\big) + \frac{\sigma_s}{\sigma^{p-1}}\big(\frac{\xi}{2}\Psi_\xi + \frac{m}{2}\Psi\big) \\
&= \sigma_s\Big(\frac{A}{1+\xi^{m+\lambda_1}} + (1+\varepsilon)\big(\psi + \frac{1}{m}\xi\,\psi_\xi\big)\Big) - \big(\frac{\sigma_s}{\sigma^{p-1}}\Psi\big)_s \\
&\quad - c_\vartheta\frac{\sigma_s^{1+\vartheta}}{\sigma^{p\vartheta}}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta} - \frac{m}{2}\sigma\big(\psi + \frac{1}{m}\xi\,\psi_\xi\big) + \frac{\sigma_s}{\sigma^{p-1}}\big(\frac{\xi}{2}\Psi_\xi + \frac{m}{2}\Psi\big) \\
&\geq \frac{A\sigma_s}{1+\xi^{m+\lambda_1}} + \frac{\sigma_s}{\sigma^{p-1}}\big(\frac{\xi}{2}\Psi_\xi + \frac{m}{2}\Psi\big) - c_\vartheta\frac{\sigma_s^{1+\vartheta}}{\sigma^{p\vartheta}}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta} - \big(\frac{\sigma_s}{\sigma^{p-1}}\Psi\big)_s \\
&=: I_1 + I_2 - I_3 - I_4, \qquad (3.11)
\end{aligned}$$

for $\rho > 0$ and $s > 0$, where the last inequality is valid thanks to (3.8) and Lemma 3.1.

Next, we show that I_2, I_3 and I_4 are small as compared to I_1, if $\rho \leq \rho_0$, $M \geq M_0$ and ρ_0, M_0 are chosen suitably.

For $\xi \geq 1$, we obtain by (3.6) (as $m + \lambda_1 > 2$)

$$\begin{aligned}\frac{|I_2|}{\frac{1}{3}I_1} &= \frac{3}{A}\sigma^{-\frac{2}{m}}(1+\xi^{m+\lambda_1})\Big|\frac{\xi}{2}\Psi_\xi + \frac{m}{2}\Psi\Big| \\ &\leq \frac{3}{A}\sigma^{-\frac{2}{m}}2\xi^{m+\lambda_1}\Big(\frac{1}{2}+\frac{m}{2}\Big)C_\Psi\xi^{2-m-\lambda_1} \\ &= \frac{3(m+1)C_\Psi}{A}\sigma^{-\frac{2}{m}}\xi^2 = \frac{3(m+1)C_\Psi}{A}\rho^2 \leq 1 \qquad (3.12)\end{aligned}$$

provided that

$$\rho \leq \rho_1 := \Big(\frac{3(m+1)C_\Psi}{A}\Big)^{-\frac{1}{2}}. \qquad (3.13)$$

Furthermore, if $\xi < 1$, (1.11), (3.6) and $m + \lambda_1 > 2$ imply

$$\begin{aligned}\frac{|I_2|}{\frac{1}{3}I_1} &= \frac{3}{A}\sigma^{-\frac{2}{m}}(1+\xi^{m+\lambda_1})\Big|\frac{\xi}{2}\Psi_\xi + \frac{m}{2}\Psi\Big| \\ &\leq \frac{3}{A}M^{-\frac{2}{m}}e^{-s}\eta^{\frac{2}{\lambda_1}}(e^{\beta(s+\mu)})\cdot 2\Big(\frac{1}{2}+\frac{m}{2}\Big)C_\Psi \\ &\leq \frac{3(m+1)C_\Psi(C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^\mu}{A}M^{-\frac{2}{m}} \leq 1, \qquad (3.14)\end{aligned}$$

whenever

$$M \geq M_2 := \Big(\frac{3(m+1)C_\Psi(C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}}e^\mu}{A}\Big)^{\frac{m}{2}}. \qquad (3.15)$$

Next, the choice of ϑ yields

$$\frac{2}{m+\lambda_1} \leq \vartheta \leq \frac{2}{m+\lambda_1-2} < \frac{2}{m} = p-1$$

since $\lambda_1 > 2$. As $\psi \leq 1$, we obtain

$$\frac{|I_3|}{\frac{1}{3}I_1} = \frac{3}{A}(1+\xi^{m+\lambda_1})c_\vartheta\frac{\sigma_s^\vartheta}{\sigma^{p\vartheta}}\psi^{p-1-\vartheta}|\Psi|^{1+\vartheta} \leq \frac{3c_\vartheta}{A}\Big(\frac{\sigma_s}{\sigma^p}\Big)^\vartheta(1+\xi^{m+\lambda_1})|\Psi|^{1+\vartheta}.$$

Thus, if $\xi \geq 1$ and $\rho \leq 1$, we conclude by (1.11), (3.6), (3.8) and the choice of ϑ that

$$\begin{aligned}\frac{|I_3|}{\frac{1}{3}I_1} &\leq \frac{3c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta\sigma^{-\frac{2\vartheta}{m}}2\xi^{m+\lambda_1}C_\Psi^{1+\vartheta}\xi^{(1+\vartheta)(2-m-\lambda_1)} \\ &= \frac{6c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta C_\Psi^{1+\vartheta}\sigma^{-\frac{2\vartheta}{m}}\xi^{2-(m+\lambda_1-2)\vartheta} \\ &= \frac{6c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta C_\Psi^{1+\vartheta}\sigma^{-\frac{(m+\lambda_1)\vartheta-2}{m}}\rho^{2-(m+\lambda_1-2)\vartheta}\end{aligned}$$

$$\leq \frac{6c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta C_\Psi^{1+\vartheta} M^{-\frac{(m+\lambda_1)\vartheta-2}{m}} e^{-\frac{(m+\lambda_1)\vartheta-2}{2}s} \eta^{\frac{(m+\lambda_1)\vartheta-2}{\lambda_1}}(e^{\beta(s+\mu)})$$
$$\leq \frac{6c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta C_\Psi^{1+\vartheta} (C_{\frac{\lambda_1}{2\beta}})^{\frac{(m+\lambda_1)\vartheta-2}{\lambda_1}} e^{\frac{(m+\lambda_1)\vartheta-2}{2}\mu} M^{-\frac{(m+\lambda_1)\vartheta-2}{m}}$$
$$\leq 1, \tag{3.16}$$

if

$$M \geq M_3 := \Big(\frac{6c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta C_\Psi^{1+\vartheta}(C_{\frac{\lambda_1}{2\beta}})^{\frac{(m+\lambda_1)\vartheta-2}{\lambda_1}} e^{\frac{(m+\lambda_1)\vartheta-2}{2}\mu}\Big)^{\frac{m}{(m+\lambda_1)\vartheta-2}}. \tag{3.17}$$

Similarly, if $\xi < 1$ we conclude

$$\frac{|I_3|}{\frac{1}{3}I_1} \leq \frac{3c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta \sigma^{-\frac{2\vartheta}{m}} 2C_\Psi^{1+\vartheta}$$
$$= \frac{6c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta C_\Psi^{1+\vartheta} M^{-\frac{2\vartheta}{m}} e^{-\vartheta s}\eta^{\frac{2\vartheta}{\lambda_1}}(e^{\beta(s+\mu)})$$
$$\leq \frac{6c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta C_\Psi^{1+\vartheta}(C_{\frac{\lambda_1}{2\beta}})^{\frac{2\vartheta}{\lambda_1}} e^{\vartheta\mu} M^{-\frac{2\vartheta}{m}} \leq 1, \tag{3.18}$$

provided that

$$M \geq M_4 := \Big(\frac{6c_\vartheta}{A}\Big(\frac{m}{2}\Big)^\vartheta C_\Psi^{1+\vartheta}(C_{\frac{\lambda_1}{2\beta}})^{\frac{2\vartheta}{\lambda_1}} e^{\vartheta\mu}\Big)^{\frac{m}{2\vartheta}}. \tag{3.19}$$

Concerning I_4, we compute, using $\sigma\xi_s = \frac{1}{m}\sigma_s\xi$ and (3.8),

$$\begin{aligned} |I_4| &= \Big|\Big(\frac{\sigma_s}{\sigma^{p-1}}\Psi\Big)_s\Big| = \Big|\frac{\sigma_{ss}}{\sigma^{p-1}}\Psi - (p-1)\frac{\sigma_s^2}{\sigma^p}\Psi + \frac{\sigma_s}{\sigma^{p-1}}\xi_s\Psi_\xi\Big| \\ &= \Big|\frac{\sigma_{ss}}{\sigma^{p-1}}\Psi - (p-1)\frac{\sigma_s^2}{\sigma^p}\Psi + \frac{\sigma_s^2}{m\sigma^p}\xi\,\Psi_\xi\Big| \\ &\leq \Big|\frac{\sigma_{ss}}{\sigma^{p-1}}\Psi\Big| + \frac{m(p-1)}{2}\frac{\sigma_s}{\sigma^{p-1}}|\Psi| + \frac{1}{2}\frac{\sigma_s}{\sigma^{p-1}}|\xi\,\Psi_\xi|. \end{aligned}$$

Moreover, as μ satisfies (3.8), we have

$$0 \leq \frac{2\beta}{\lambda_1}\cdot\frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} \leq \frac{\varepsilon}{1+\varepsilon} < 1 \quad \text{for all } s \geq 0.$$

Hence, we obtain by (1.8) and (3.8)

$$|\sigma_{ss}| = \Big|\Big[\frac{m}{2}Me^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)})\Big(1-\frac{2\beta}{\lambda_1}\cdot\frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})}\Big)\Big]_s\Big|$$
$$= \Big|\Big(\frac{m}{2}\Big)^2 Me^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)})\Big(1-\frac{2\beta}{\lambda_1}\cdot\frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})}\Big)^2$$

$$\begin{aligned}
&- \frac{m}{2} M e^{\frac{m}{2}s} \eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)}) \frac{2\beta}{\lambda_1}\Big(\beta \frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} \\
&+ \beta \frac{e^{2\beta(s+\mu)}\eta''(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})} - \beta\Big(\frac{e^{\beta(s+\mu)}\eta'(e^{\beta(s+\mu)})}{\eta(e^{\beta(s+\mu)})}\Big)^2\Big)\Big| \\
&\le \frac{m}{2}\sigma_s + \frac{m}{2}\sigma\Big(\beta + \beta C_\eta + \frac{\lambda_1}{2}\Big) \le \Big(\frac{m}{2} + (1+\varepsilon)\Big(\beta(1+C_\eta) + \frac{\lambda_1}{2}\Big)\Big)\sigma_s \\
&=: \tilde{C}\sigma_s.
\end{aligned}$$

If $\xi \ge 1$, this implies due to (3.6)

$$\begin{aligned}
\frac{|I_4|}{\frac{1}{3}I_1} &\le \frac{3}{A}(1+\xi^{m+\lambda_1})\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)\sigma^{-\frac{2}{m}}(|\Psi| + |\xi\,\Psi_\xi|) \\
&\le \frac{6}{A}\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)\xi^{m+\lambda_1}\sigma^{-\frac{2}{m}}C_\Psi \xi^{2-m-\lambda_1} \\
&= \frac{6}{A}\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)C_\Psi \rho^2 \le 1, \qquad (3.20)
\end{aligned}$$

provided that

$$\rho \le \rho_2 := \Big(\frac{6}{A}\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)C_\Psi\Big)^{-\frac{1}{2}}. \qquad (3.21)$$

If $\xi < 1$, by (3.6) and (1.11) we obtain

$$\begin{aligned}
\frac{|I_4|}{\frac{1}{3}I_1} &\le \frac{3}{A}(1+\xi^{m+\lambda_1})\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)\sigma^{-\frac{2}{m}}(|\Psi| + |\xi\,\Psi_\xi|) \\
&\le \frac{6}{A}\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)\sigma^{-\frac{2}{m}}C_\Psi \\
&= \frac{6}{A}\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)C_\Psi M^{-\frac{2}{m}} e^{-s} \eta^{\frac{2}{\lambda_1}}(e^{\beta(s+\mu)}) \\
&\le \frac{6}{A}\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)C_\Psi (C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}} e^{\mu} M^{-\frac{2}{m}} \le 1 \qquad (3.22)
\end{aligned}$$

under the additional restriction

$$M \ge M_5 := \Big(\frac{6}{A}\Big(\tilde{C} + \frac{m(p-1)+1}{2}\Big)C_\Psi (C_{\frac{\lambda_1}{2\beta}})^{\frac{2}{\lambda_1}} e^{\mu}\Big)^{\frac{m}{2}}. \qquad (3.23)$$

Finally, we conclude by (3.11)-(3.23) that v_{in} is a supersolution of (2.3) for $s > 0$ and $0 < \rho \le \rho_0 := \min\{\rho_1, \rho_2, 1\}$, if

$$M \ge M_0 := \max\{M_1, M_2, M_3, M_4, M_5\}$$

is fulfilled, and hence the claim is proved. □

Next, we construct a suitable supersolution of (2.3) in a region where ρ is bounded away from zero. We let $W = W(\rho)$ denote the solution of the problem

$$\begin{cases} W_{\rho\rho} + \frac{N-1}{\rho} W_\rho + \frac{\rho}{2} W_\rho + \frac{m+\lambda_1}{2} W = 0, & \rho > 0, \\ W(0) = 1, \quad W_\rho(0) = 0. \end{cases} \tag{3.24}$$

By Lemma 3.1 in [2] this problem has a positive, decreasing solution W fulfilling

$$c_- \rho^{-m-\lambda_1} \leq W(\rho) \leq c_+ \rho^{-m-\lambda_1} \qquad \text{for } \rho \geq 1. \tag{3.25}$$

Furthermore, W is given by

$$W(\rho) = e^{-\frac{\rho^2}{4}} \mathcal{M}\Big(\frac{N-m-\lambda_1}{2}, \frac{N}{2}, \frac{\rho^2}{4}\Big), \quad \rho \geq 0,$$

where $\mathcal{M}$ denotes Kummer's function

$$\mathcal{M}(a,b,z) := 1 + \frac{az}{b} + \cdots + \frac{a(a+1)\cdots(a+n-1)z^n}{b(b+1)\cdots(b+n-1)n!} + \cdots$$

which can be found in [1]. Now we give a suitable supersolution in an outer region.

Lemma 3.3. *Let $N \geq 11$ and $p > p_c$. Then there are $\alpha > 0$ and $\beta \in (0, \frac{1}{2}]$ such that for any $\mu > 0$ there exists $b_0 > 0$ with the property that*

$$v_{out}(\rho, s) := L\rho^{-m} - b\, e^{-\frac{\lambda_1}{2}s} \eta\Big(e^{\beta(s+\mu)}\rho^\alpha\Big) W(\rho)$$

is a positive supersolution of (2.3) for $\rho > 0$ and $s > 0$, whenever $b \in (0, b_0)$ holds.

Proof. Recalling the definition of C_η in (1.8), we fix $\alpha > 0$ such that $\alpha(C_\eta - 1) \leq N-2$ is satisfied. Then, we fix $\beta \in (0, \frac{1}{2}]$ such that $\beta \leq \frac{\alpha}{N}$ holds. As $\frac{\partial}{\partial z}\mathcal{M}(a,b,z) \geq \frac{a}{b}\mathcal{M}(a,b,z)$ holds for any $z \geq 0$ when $0 < a < b$ (since $\frac{a+n-1}{b+n-1} \geq \frac{a}{b}$ for $n \in \mathbb{N}$ in this case), we have

$$\begin{aligned} W_\rho(\rho) &\geq -\frac{\rho}{2}W(\rho) + \frac{\rho}{2}\frac{N-m-\lambda_1}{N}W(\rho) = \frac{-m-\lambda_1}{N}\frac{\rho}{2}W(\rho) \\ &\geq -\frac{\frac{N-2}{2}}{N}\frac{\rho}{2}W(\rho) = -\Big(1-\frac{2}{N}\Big)\frac{\rho}{4}W(\rho) \end{aligned} \tag{3.26}$$

due to the fact that $m + \lambda_1 \leq \frac{N-2}{2}$.

Now let $\mu > 0$ be given. By (1.11) and (3.25) there is $b_0 > 0$ such that v_{out} is positive for any $\rho > 0$ and $s > 0$ in case of $b \in (0, b_0)$. As moreover

W is nonnegative, by (3.24), (3.26) and (1.8) we obtain for any $b \in (0, b_0)$ (omitting the argument $e^{\beta(s+\mu)}\rho^\alpha$ of η, η' and η'')

$$\begin{aligned}
&\mathcal{P}v_{out} = (v_{out})_s - (v_{out})_{\rho\rho} - \frac{N-1}{\rho}(v_{out})_\rho - (v_{out})^p - \frac{\rho}{2}(v_{out})_\rho - \frac{m}{2}v_{out}\\
&= -(L\rho^{-m})_{\rho\rho} - \frac{N-1}{\rho}(L\rho^{-m})_\rho - \Big(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta W\Big)^p\\
&\quad - \frac{\rho}{2}(L\rho^{-m})_\rho - \frac{m}{2}L\rho^{-m} + b\frac{\lambda_1}{2}e^{-\frac{\lambda_1}{2}s}\eta W - b\beta e^{-\frac{\lambda_1}{2}s}e^{\beta(s+\mu)}\rho^\alpha\eta' W\\
&\quad + be^{-\frac{\lambda_1}{2}s}\eta\Big[W_{\rho\rho} + \frac{N-1}{\rho}W_\rho + \frac{\rho}{2}W_\rho + \frac{m}{2}W\Big]\\
&\quad + be^{-\frac{\lambda_1}{2}s}\Big[e^{2\beta(s+\mu)}\alpha^2\rho^{2(\alpha-1)}\eta'' W + e^{\beta(s+\mu)}\alpha(\alpha-1)\rho^{\alpha-2}\eta' W\\
&\quad + 2e^{\beta(s+\mu)}\alpha\rho^{\alpha-1}\eta' W_\rho + \Big(\frac{N-1}{\rho} + \frac{\rho}{2}\Big)e^{\beta(s+\mu)}\alpha\rho^{\alpha-1}\eta' W\Big]\\
&= (L\rho^{-m})^p - \Big(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta W\Big)^p + be^{-\frac{\lambda_1}{2}s}\eta\Big[W_{\rho\rho} + \frac{N-1}{\rho}W_\rho\\
&\quad + \frac{\rho}{2}W_\rho + \frac{m+\lambda_1}{2}W\Big] + be^{-\frac{\lambda_1}{2}s}\Big[- \beta e^{\beta(s+\mu)}\rho^\alpha\eta' W\\
&\quad + e^{2\beta(s+\mu)}\alpha^2\rho^{2(\alpha-1)}\eta'' W + e^{\beta(s+\mu)}\alpha(\alpha-1)\rho^{\alpha-2}\eta' W\\
&\quad + 2e^{\beta(s+\mu)}\alpha\rho^{\alpha-1}\eta' W_\rho + \Big(\frac{N-1}{\rho} + \frac{\rho}{2}\Big)e^{\beta(s+\mu)}\alpha\rho^{\alpha-1}\eta' W\Big]\\
&\geq be^{-\frac{\lambda_1}{2}s}e^{\beta(s+\mu)}\eta' W\Big[- \beta\rho^\alpha - e^{\beta(s+\mu)}\alpha^2\rho^{2(\alpha-1)}C_\eta e^{-\beta(s+\mu)}\rho^{-\alpha}\\
&\qquad + \alpha(\alpha-1)\rho^{\alpha-2} - 2\Big(1 - \frac{2}{N}\Big)\frac{\rho}{4}\alpha\rho^{\alpha-1} + \Big(\frac{N-1}{\rho} + \frac{\rho}{2}\Big)\alpha\rho^{\alpha-1}\Big]\\
&= be^{-\frac{\lambda_1}{2}s}e^{\beta(s+\mu)}\eta' W\Big[- \beta\rho^\alpha - C_\eta\alpha^2\rho^{\alpha-2} + \alpha(\alpha-1)\rho^{\alpha-2} + \frac{\alpha}{N}\rho^\alpha\\
&\qquad + (N-1)\alpha\rho^{\alpha-2}\Big]\\
&= be^{-\frac{\lambda_1}{2}s}e^{\beta(s+\mu)}\eta' W\Big[\Big(\frac{\alpha}{N} - \beta\Big)\rho^\alpha + \alpha(N - 2 - \alpha(C_\eta - 1))\rho^{\alpha-2}\Big]\\
&\geq 0 \qquad \text{for } \rho > 0 \text{ and } s > 0,
\end{aligned}$$

where we have used the choices of α and β. □

We now use the functions v_{in} and v_{out} to obtain a supersolution of (2.3) for $\rho > 0$ and $s > 0$ which does not grow up faster than at the rate claimed in (2.4).

Lemma 3.4. *Let $N \geq 11$, $p > p_c$ and $v_0 = v_0(\rho)$ be a nonnegative and nonincreasing continuous function of $\rho \geq 0$ fulfilling*

$$v_0(\rho) < L\rho^{-m} \quad \text{for } \rho > 0$$

and

$$v_0(\rho) \leq L\rho^{-m} - b_1\rho^{-m-\lambda_1}\eta(\rho) \quad \text{for } \rho \geq R \tag{3.27}$$

with some positive constants b_1 and R. Moreover, let $v = v(\rho, s)$ denote the nonnegative solution of (2.3)*. Then there is a positive constant c such that*

$$v(\rho, s) \leq c\, e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \tag{3.28}$$

holds for $\rho \geq 0$ and $s \geq 0$.

Proof. We fix $\alpha > 0$ and $\beta \in (0, \frac{1}{2}]$ as in Lemma 3.3. Then, we choose $\mu > 0$, $\rho_0 > 0$, $M_0 > 0$ and v_{in} as in Lemma 3.2. Furthermore, let b_0 and v_{out} be chosen as in Lemma 3.3. As v_0 is bounded, there is $\rho_1 \in (0, \rho_0)$ such that $v_0(\rho) \leq \frac{1}{4}L\rho_1^{-m}$ is satisfied for $0 \leq \rho \leq \rho_1$. Next, we set

$$M_6 := \max\Big\{M_0, \Big(\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho_1\Big)^{-m}, \Big(\frac{L\rho_1^{\lambda_1}}{2a_1\eta(e^{\beta\mu})}\Big)^{-\frac{m}{\lambda_1}}\Big\},$$

where a_1 is defined in (3.3).

Since ψ is decreasing, we thus obtain by (3.3) and Lemma 3.2

$$\begin{aligned}
v_{in}(\rho, 0) &= \sigma(0)\Big(\psi(\xi) - \frac{\sigma_s(0)}{\sigma^p(0)}\Psi(\xi)\Big) \geq \frac{1}{2}\sigma(0)\psi(\xi) \\
&= \frac{1}{2}M\eta^{-\frac{m}{\lambda_1}}(e^{\beta\mu})\,\psi\Big(M^{\frac{1}{m}}\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho\Big) \\
&\geq \frac{1}{2}M\eta^{-\frac{m}{\lambda_1}}(e^{\beta\mu})\,\psi\Big(M^{\frac{1}{m}}\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho_1\Big) \\
&\geq \frac{1}{2}M\eta^{-\frac{m}{\lambda_1}}(e^{\beta\mu})\Big(L\Big(M^{\frac{1}{m}}\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho_1\Big)^{-m} \\
&\qquad - a_1\Big(M^{\frac{1}{m}}\eta^{-\frac{1}{\lambda_1}}(e^{\beta\mu})\rho_1\Big)^{-m-\lambda_1}\Big) \\
&= \frac{1}{2}L\rho_1^{-m} - \frac{a_1}{2}M^{-\frac{\lambda_1}{m}}\eta(e^{\beta\mu})\rho_1^{-m-\lambda_1} \geq \frac{1}{4}L\rho_1^{-m} \\
&\geq v_0(\rho) \qquad \text{for } \rho \leq \rho_1,
\end{aligned} \tag{3.29}$$

provided that $M \geq M_6$ holds.

Now, we show that $v_{out} \geq v_0$ holds for $\rho > 0$, if $b > 0$ is chosen suitably small. If $\rho \geq \max\{1, R\}$, due to (3.25), (3.27) and (1.9) we conclude

$$v_{out}(\rho, 0) = L\rho^{-m} - b\,\eta(e^{\beta\mu}\rho^{\alpha})W(\rho)$$

$$\geq L\rho^{-m} - b\,(c_{\alpha,e^{\beta\mu}})\eta(\rho)c_+\rho^{-m-\lambda_1} \geq v_0(\rho),$$

if $b \leq b_2 := \frac{b_1}{(c_{\alpha,e^{\beta\mu}})c_+}$ is fulfilled. As v_0 is continuous with $v_0(\rho) < L\rho^{-m}$ for $\rho > 0$, there is $b_3 > 0$ such that $v_0(\rho) \leq L\rho^{-m} - b_3$ holds for $\rho \leq \max\{1, R\}$. Since W is nonincreasing and thus satisfies $W \leq W(0) = 1$ in $[0,\infty)$, we obtain

$$v_{out}(\rho,0) \geq L\rho^{-m} - b\,\eta(e^{\beta\mu}(R^\alpha+1)) \geq v_0(\rho) \quad \text{for } \rho \leq \max\{1,R\},$$

if $b \leq b_4 := \frac{b_3}{\eta(e^{\beta\mu}(R^\alpha+1))}$. Accordingly, we fix $b \in (0, \min\{b_0, b_2, b_4\})$ and have

$$v_{out}(\rho,0) \geq v_0(\rho) \quad \text{for all } \rho \geq 0. \tag{3.30}$$

Keeping this value of b fixed, we now claim that for any M sufficiently large

$$\rho_M(s) := \inf\left\{\rho > 0 : v_{out}(\rho,s) < v_{in}(\rho,s)\right\}$$

is well defined for all $s \geq 0$ and fulfills

$$\rho_M(s) \leq \rho_1 \quad \text{for } s \geq 0. \tag{3.31}$$

Once this has been shown, we will obtain from Lemma 3.2 and Lemma 3.3 that

$$v_{sup}(\rho,s) := \begin{cases} v_{in}(\rho,s) & \text{for } s \geq 0, \rho \leq \rho_M(s), \\ v_{out}(\rho,s) & \text{for } s \geq 0, \rho > \rho_M(s), \end{cases}$$

is a supersolution of (2.3) which moreover satisfies $v_{sup}(\rho,0) \geq v_0(\rho)$ for $\rho \geq 0$ by (3.29) and (3.30). As $v_\rho \leq 0$ holds due to the properties of v_0, the comparison principle implies

$$\begin{aligned} v(\rho,s) &\leq v(0,s) \leq v_{sup}(0,s) = v_{in}(0,s) = \sigma(s) = Me^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\beta(s+\mu)}) \\ &\leq Me^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\beta s}) \leq M(c_{\frac{1}{2\beta},1})^{\frac{m}{\lambda_1}} e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}), \quad \rho \geq 0,\ s \geq 0, \end{aligned}$$

where we have used (1.6) and (1.9). Consequently, the lemma is proved if we show (3.31).

To this end, we prove that

$$v_{out}(\rho_1,s) < v_{in}(\rho_1,s) \quad \text{for } s \geq 0 \tag{3.32}$$

holds if M is chosen large enough. By (3.3), (3.6) and (3.8) we have

$$\begin{aligned} v_{in}(\rho_1,s) &= \sigma\Big(\psi(\xi) - \frac{\sigma_s}{\sigma^p}\Psi(\xi)\Big) \\ &\geq \sigma\Big(L(\sigma^{\frac{1}{m}}\rho_1)^{-m} - a_1(\sigma^{\frac{1}{m}}\rho_1)^{-m-\lambda_1} - \frac{m}{2}\sigma^{-\frac{2}{m}}C_\Psi(\sigma^{\frac{1}{m}}\rho_1)^{2-m-\lambda_1}\Big) \\ &= L\rho_1^{-m} - \Big(a_1\rho_1^{-m-\lambda_1} + \frac{m}{2}C_\Psi\rho_1^{2-m-\lambda_1}\Big)\sigma^{-\frac{\lambda_1}{m}} \quad \text{for } s \geq 0 \end{aligned}$$

whenever

$$M \geq M_7 := \max\Big\{M_6, \Big((C_{\lambda_1})^{-\frac{1}{\lambda_1}} e^{-\beta\mu}\rho_1\Big)^{-m}\Big\},$$

because

$$\sigma^{\frac{1}{m}}(s)\rho_1 \geq M^{\frac{1}{m}} e^{\frac{1}{2}s}(C_{\lambda_1})^{-\frac{1}{\lambda_1}} e^{-\beta(s+\mu)}\rho_1 \geq M^{\frac{1}{m}}(C_{\lambda_1})^{-\frac{1}{\lambda_1}} e^{-\beta\mu}\rho_1$$

is satisfied due to (1.11) and $\beta \in (0, \frac{1}{2}]$. As moreover (1.9) implies

$$\begin{aligned} v_{out}(\rho_1, s) &= L\rho_1^{-m} - b\, e^{-\frac{\lambda_1}{2}s}\eta(e^{\beta(s+\mu)}\rho_1^{\alpha})W(\rho_1) \\ &\leq L\rho_1^{-m} - b\, e^{-\frac{\lambda_1}{2}s}(c_{1,\rho_1^{-\alpha}})^{-1}\eta(e^{\beta(s+\mu)})W(\rho_1) \\ &= L\rho_1^{-m} - b\,(c_{1,\rho_1^{-\alpha}})^{-1}W(\rho_1)M^{\frac{\lambda_1}{m}}\sigma^{-\frac{\lambda_1}{m}} \qquad \text{for } s \geq 0, \end{aligned}$$

(3.32) is valid for any

$$M > M_8 := \max\Big\{M_7, \Big(\frac{a_1\rho_1^{-m-\lambda_1} + \frac{m}{2}C_\Psi\rho_1^{2-m-\lambda_1}}{b\,(c_{1,\rho_1^{-\alpha}})^{-1}W(\rho_1)}\Big)^{\frac{m}{\lambda_1}}\Big\}.$$

Finally, $\rho_M(s)$ is well defined for any $s \geq 0$, since $\lim_{\rho\searrow 0} v_{in}(\rho, s) = \sigma(s) < \infty$ and $v_{out}(\rho, s) \to \infty$ as $\rho \searrow 0$ holds for any $s \geq 0$. Thus, (3.31) is fulfilled due to (3.32) if we choose $M > M_8$, and the proof is complete. □

4. Lower bound

In this section, we derive the corresponding lower bound for v and adapt an idea from [9].

The function

$$\bar{W}(\rho) := \rho^{-m-\lambda_1}, \qquad \rho > 0,$$

is a positive solution of the equation

$$\bar{W}_{\rho\rho} + \frac{N-1}{\rho}\bar{W}_\rho + \frac{\rho}{2}\bar{W}_\rho + \frac{m+\lambda_1}{2}\bar{W} + \frac{pL^{p-1}}{\rho^2}\bar{W} = 0 \qquad \text{for } \rho > 0. \tag{4.1}$$

Now we construct a suitable subsolution of (2.3).

Lemma 4.1. *Suppose $N \geq 11$ and $p > p_c$. Then for any $\beta \geq 2N$ and each $b > 0$ the function*

$$v_{sub}(\rho, s) := \max\Big\{0, L\rho^{-m} - b\, e^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2))\bar{W}(\rho)\Big\}, \quad \rho > 0, s \geq 0,$$

is a subsolution of (2.3) *for all $\rho > 0$ and $s > 0$.*

Proof. Fixing $\beta \geq 2N$ and $b > 0$, we choose $\rho > 0$ and $s > 0$ such that $v_{sub}(\rho, s)$ is positive.
As $be^{-\frac{\lambda_1}{2}}\eta(e^{\beta s}(1+\rho^2))\bar{W}(\rho)$ is positive and $p > 1$, the mean value theorem implies

$$\begin{aligned}
&(L\rho^{-m})^p - \Big(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2))\bar{W}(\rho)\Big)^p \\
&\leq p(L\rho^{-m})^{p-1}be^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2))\bar{W}(\rho) \\
&= be^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2))\frac{pL^{p-1}}{\rho^2}\bar{W}(\rho).
\end{aligned}$$

Thus, we obtain due to (1.6) and (4.1) (suppressing the argument $e^{\beta s}(1+\rho^2)$ of η, η' and η'')

$$\begin{aligned}
\mathcal{P}v_{sub} &= (v_{sub})_s - (v_{sub})_{\rho\rho} - \frac{N-1}{\rho}(v_{sub})_\rho - (v_{sub})^p - \frac{\rho}{2}(v_{sub})_\rho - \frac{m}{2}v_{sub} \\
&= -(L\rho^{-m})_{\rho\rho} - \frac{N-1}{\rho}(L\rho^{-m})_\rho - \Big(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta\bar{W}\Big)^p \\
&\quad - \frac{\rho}{2}(L\rho^{-m})_\rho - \frac{m}{2}L\rho^{-m} + b\frac{\lambda_1}{2}e^{-\frac{\lambda_1}{2}s}\eta\bar{W} - b\beta e^{-\frac{\lambda_1}{2}s}e^{\beta s}(1+\rho^2)\eta'\bar{W} \\
&\quad + be^{-\frac{\lambda_1}{2}s}\eta\Big[\bar{W}_{\rho\rho} + \frac{N-1}{\rho}\bar{W}_\rho + \frac{\rho}{2}\bar{W}_\rho + \frac{m}{2}\bar{W}\Big] + be^{-\frac{\lambda_1}{2}s}\Big[e^{2\beta s}4\rho^2\eta''\bar{W} \\
&\quad + 2e^{\beta s}\eta'\bar{W} + 2e^{\beta s}2\rho\eta'\bar{W}_\rho + \Big(\frac{N-1}{\rho} + \frac{\rho}{2}\Big)e^{\beta s}2\rho\eta'\bar{W}\Big] \\
&= (L\rho^{-m})^p - \Big(L\rho^{-m} - be^{-\frac{\lambda_1}{2}s}\eta\bar{W}\Big)^p + be^{-\frac{\lambda_1}{2}s}\eta\Big[\bar{W}_{\rho\rho} + \frac{N-1}{\rho}\bar{W}_\rho \\
&\quad + \frac{\rho}{2}\bar{W}_\rho + \frac{m+\lambda_1}{2}\bar{W}\Big] + be^{-\frac{\lambda_1}{2}s}\Big[-\beta e^{\beta s}(1+\rho^2)\eta'\bar{W} \\
&\quad + 4e^{2\beta s}\rho^2\eta''\bar{W} + 2e^{\beta s}\eta'\bar{W} + 4e^{\beta s}\rho\eta'\bar{W}_\rho + \Big(\frac{N-1}{\rho} + \frac{\rho}{2}\Big)2e^{\beta s}\rho\eta'\bar{W}\Big] \\
&\leq be^{-\frac{\lambda_1}{2}s}\eta\Big[\bar{W}_{\rho\rho} + \frac{N-1}{\rho}\bar{W}_\rho + \frac{\rho}{2}\bar{W}_\rho + \frac{m+\lambda_1}{2}\bar{W} + \frac{pL^{p-1}}{\rho^2}\bar{W}\Big] \\
&\quad + be^{-\frac{\lambda_1}{2}s}e^{\beta s}\eta'\bar{W}\Big[-\beta(1+\rho^2) + 2 + 2(N-1) + \rho^2\Big] \\
&\leq be^{-\frac{\lambda_1}{2}s}e^{\beta s}\eta'\bar{W}(-\beta + 2N)(1+\rho^2) \leq 0
\end{aligned}$$

due to the choice of β. Now the claim is proved since $v_1 \equiv 0$ as well is a subsolution of (2.3). □

Now we are in position to prove the lower bound for the grow-up rate of solutions to (2.3) which corresponds to the rate claimed in (2.4).

Lemma 4.2. *Assume $N \geq 11$ and $p > p_c$. Moreover, let $v_0 = v_0(\rho)$ be radially symmetric and nonnegative satisfying*

$$v_0(\rho) \geq L\rho^{-m} - b_2\,\rho^{-m-\lambda_1}\eta(\rho) \quad \text{for } \rho > 0 \tag{4.2}$$

with some $b_2 > 0$. Then the solution v of (2.3) fulfills

$$\sup_{\rho>0} v(\rho,s) \geq c\,e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for all } s \geq 0$$

with some constant $c > 0$.

Proof. We fix $\beta \geq 2N$, define

$$b := \max\left\{b_2, \frac{L}{\eta(1)}\right\}$$

and take the function v_{sub} from Lemma 4.1. Due to (1.6) and the choice of b, we obtain

$$L\rho^{-m} - b\,\eta(1+\rho^2)\rho^{-m-\lambda_1} \leq L\rho^{-m} - b\,\eta(1)\rho^{-m-\lambda_1} \leq L\rho^{-m} - L\rho^{-m-\lambda_1} \leq 0$$

for $\rho \leq 1$. Thus, the definition of $\bar{W}$ implies

$$v_{sub}(\rho,0) = 0 \leq v_0(\rho) \quad \text{for } \rho \leq 1.$$

Furthermore, by (1.6) and (4.2) we have

$$\begin{aligned} v_{sub}(\rho,0) &= L\rho^{-m} - b\,\eta(1+\rho^2)\rho^{-m-\lambda_1} \leq L\rho^{-m} - b\,\eta(\rho)\rho^{-m-\lambda_1} \\ &\leq L\rho^{-m} - b_2\eta(\rho)\rho^{-m-\lambda_1} \leq v_0(\rho) \quad \text{for all } \rho \geq 1 \text{ where } v_{sub}(\rho,0) > 0. \end{aligned}$$

Altogether, we conclude $v_{sub}(\rho,0) \leq v_0(\rho)$ for all $\rho \geq 0$. Hence, the comparison principle yields $v \geq v_{sub}$ for all $\rho \geq 0$ and $s \geq 0$. Defining

$$\rho(s) := \left(\frac{L}{2b(c_{2\beta,2})}\right)^{-\frac{1}{\lambda_1}} e^{-\frac{1}{2}s}\eta^{\frac{1}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq 0,$$

where $c_{2\beta,2}$ is defined in (1.9), we find $s_0 \geq 0$ such that $\rho(s) \leq 1$ is satisfied for all $s \geq s_0$ by (1.11). Hence, due to (1.6) and (1.9) we obtain

$$\begin{aligned} \sup_{\rho>0} v(\rho,s) \geq v_{sub}(\rho(s),s) &= L\rho^{-m}(s) - b\,e^{-\frac{\lambda_1}{2}s}\eta(e^{\beta s}(1+\rho^2(s)))\bar{W}(\rho(s)) \\ &\geq L\rho^{-m}(s) - b\,e^{-\frac{\lambda_1}{2}s}\eta(2e^{\beta s})\rho^{-m-\lambda_1}(s) \\ &\geq L\rho^{-m}(s) - b\,e^{-\frac{\lambda_1}{2}s}(c_{2\beta,2})\eta(e^{\frac{1}{2}s})\rho^{-m-\lambda_1}(s) \end{aligned}$$

$$= \rho^{-m}(s)\Big(L - b(c_{2\beta,2})e^{-\frac{\lambda_1}{2}s}\eta(e^{\frac{1}{2}s})\rho^{-\lambda_1}(s)\Big) = \frac{L}{2}\rho^{-m}(s)$$
$$= \frac{L}{2}\Big(\frac{L}{2b(c_{2\beta,2})}\Big)^{\frac{m}{\lambda_1}} e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq s_0.$$

This implies the claim as v is continuous and

$$\sup_{\rho>0} v(\rho,s) \geq \sup_{\rho>0} v_{sub}(\rho,s) > 0 \quad \text{for any } s \in [0,s_0]$$

is fulfilled due to the choice of v_{sub}. □

5. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1 with the help of the estimates which are derived in Sections 3 and 4.

Let $u_0 \in C^0(\mathbb{R}^N)$ satisfy (1.2) and (1.5). Moreover, we let u denote the corresponding solution of (1.1) and define the radially symmetric functions

$$\underline{u}_0(r) := \min\{u_0(x) \ : \ x \in \mathbb{R}^N, |x| = r\} \quad \text{for } r \geq 0$$

and

$$\overline{u}_0(r) := \max\{u_0(x) \ : \ x \in \mathbb{R}^N, |x| \geq r\} \quad \text{for } r \geq 0.$$

Then the properties of u_0 imply that $\underline{u}_0(r)$ and $\overline{u}_0(r)$ are continuous in $r \geq 0$ and satisfy (1.5) (with a possibly larger constant R). Moreover, $\overline{u}_0(r)$ is nonincreasing for $r \geq 0$ and we have

$$0 \leq \underline{u}_0(|x|) \leq u_0(x) \leq \overline{u}_0(|x|) < \varphi_\infty(|x|) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.$$

We define $\underline{u}(r,t)$ and $\overline{u}(r,t)$ to be the solutions of (2.1) corresponding to the initial data $\underline{u}_0(r)$ and $\overline{u}_0(r)$, respectively. Both solutions exist globally in time and $\overline{u}(r,t)$ is nonincreasing in r for any $t \geq 0$. Furthermore, let $\underline{v}(\rho,s)$ and $\overline{v}(\rho,s)$ denote the solutions of (2.3) which are obtained from $\underline{u}$ and $\overline{u}$, respectively, by the self-similar change of variables defined in (2.2). As the initial data $\underline{v}_0(\rho) = \underline{u}_0(\rho)$ and $\overline{v}_0(\rho) = \overline{u}_0(\rho)$ fulfill the conditions of Lemma 4.2 and Lemma 3.4, respectively, we conclude

$$\sup_{\rho\geq 0} \underline{v}(\rho,s) \geq C_1\, e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq 0$$

by Lemma 4.2 and

$$\sup_{\rho\geq 0} \overline{v}(\rho,s) \leq C_2\, e^{\frac{m}{2}s}\eta^{-\frac{m}{\lambda_1}}(e^{\frac{1}{2}s}) \quad \text{for } s \geq 0$$

by Lemma 3.4 with some positive constants C_1 and C_2. Hence, (2.2) implies

$$\|\underline{u}(|\cdot|,t)\|_{L^\infty(\mathbb{R}^N)} = (t+1)^{-\frac{1}{p-1}} \sup_{\rho\geq 0} \underline{v}(\rho,\log(t+1)) \geq C_1\,\eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}),\ t\geq 0,$$

and

$$\|\overline{u}(|\cdot|,t)\|_{L^\infty(\mathbb{R}^N)} = (t+1)^{-\frac{1}{p-1}} \sup_{\rho\geq 0} \overline{v}(\rho,\log(t+1)) \leq C_2\,\eta^{-\frac{m}{\lambda_1}}((t+1)^{\frac{1}{2}}),\ t\geq 0.$$

As the comparison principle yields

$$\|\underline{u}(|\cdot|,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\overline{u}(|\cdot|,t)\|_{L^\infty(\mathbb{R}^N)} \quad \text{for } t\geq 0,$$

the proof of Theorem 1.1 is complete.

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