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ON THE LIFE SPAN OF THE SCHRÖDINGER EQUATION WITH SUB-CRITICAL POWER NONLINEARITY

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Abstract. We discuss the life span of the Cauchy problem for the one-dimensional Schrödinger equation with a single power nonlinearity $\lambda|u|^{p-1}u$ ($\lambda \in \mathbb{C}$, $2 \le p < 3$) and initial data of the form $\varepsilon\varphi$ prescribed. Here, ε stands for the size of the data. It is not difficult to see that the life span $T(\varepsilon)$ is estimated by $C_0\varepsilon^{-2(p-1)/(3-p)}$ from below, provided ε is sufficiently small. In this paper, we consider a more precise estimate for $T(\varepsilon)$ and we prove that $\liminf_{\varepsilon\to 0} \varepsilon^{2(p-1)/(3-p)}T(\varepsilon)$ is larger than some positive constant expressed only by p, $\mathrm{Im}\lambda$ and φ.

1. Introduction

This paper is concerned with the life span of solutions to the Cauchy problem for the one-dimensional nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda|u|^{p-1}u & \text{in } [0,\infty)\times\mathbb{R}, \\ u|_{t=0} = \varepsilon\varphi & \text{on } \mathbb{R}. \end{cases} \tag{1.1}$$

Here, $u = u(t,x)$ is a complex-valued unknown function, $(t,x) \in [0,\infty)\times\mathbb{R}$, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\varepsilon > 0$, φ belongs to some suitable function space, $\lambda \in \mathbb{C}$ and $p > 1$.

In order to give the concrete definition of the life span, we recall a standard result for (1.1): If $1 < p < 5$ and $\lambda \in \mathbb{C}$, then (1.1) is locally well posed in $L^2(\mathbb{R})$ (see, e.g., Theorem 4.6.1 in [1]). That is, for any $\varepsilon > 0$ and $\varphi \in L^2(\mathbb{R})$, there exists some $T > 0$ such that (1.1) has a unique solution $u \in C([0,T);L^2(\mathbb{R}))$. Therefore, we can define *the life span* $T(\varepsilon)$ of (1.1) by

$$T(\varepsilon) = \sup\{T > 0 : \text{ (1.1) has a unique solution } u \in C([0,T);L^2(\mathbb{R}))\} \tag{1.2}$$

for any $p \in (1,5)$, $\lambda \in \mathbb{C}$, $\varphi \in L^2(\mathbb{R})$ and $\varepsilon > 0$.

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Remark 1.1. We give some equivalent definitions of $T(\varepsilon)$. Let $H^1(\mathbb{R})$ be the Sobolev space defined by $H^1(\mathbb{R}) = (1-\Delta)^{-1/2}L^2(\mathbb{R})$. For any $p > 1$, $\lambda \in \mathbb{C}$, $\varepsilon > 0$ and $\varphi \in H^1(\mathbb{R})$, (1.1) is locally well posed in $H^1(\mathbb{R})$, so that we can define a positive number $T'(\varepsilon)$ by

$$T'(\varepsilon) = \sup\{T > 0 : \text{ (1.1) has a unique solution } u \in C([0,T); H^1(\mathbb{R}))\}.$$

Furthermore, if we use function spaces Σ and $X(T)$ defined by (1.6) and (2.1) below, respectively, then we see that, for any $\varphi \in \Sigma$, we can define a positive number $T''(\varepsilon)$ by

$$T''(\varepsilon) = \sup\{T > 0 : \text{ (1.1) has a unique solution } u \in X(T)\}.$$

For any $p \in (1,5)$, $\lambda \in \mathbb{C}$ and $\varepsilon > 0$, if $\varphi \in H^1(\mathbb{R})$ (respectively $\varphi \in \Sigma$), then $T'(\varepsilon)$ (respectively $T''(\varepsilon)$) is equal to the life span $T(\varepsilon)$ (for the proof, see, e.g., Theorem 5.2.1 in [1]).

Before treating our problem, we mention some known results concerned with the life span of the Cauchy problem (1.1) in the case $1 < p < 5$ and $\lambda \in \mathbb{C}$.

We first focus on the case $1 < p < 5$ and $\operatorname{Im}\lambda \le 0$. It is well known that (1.1) is $L^2(\mathbb{R})$-sub-critical and that the time-local solution $u(t)$ to (1.1) satisfies the a priori estimate $\|u(t)\|_{L^2(\mathbb{R})} \le \|\varphi\|_{L^2(\mathbb{R})}$. We hence see that (1.1) is globally well posed in $L^2(\mathbb{R})$, so that $T(\varepsilon) = \infty$.

We assume that $\varepsilon > 0$ is sufficiently small and that φ belongs to some suitable function space. It is clear that we have $T(\varepsilon) = \infty$ whenever $3 < p < 5$. Indeed, for any $3 < p < 5$ and $\lambda \in \mathbb{R}$, it has been proved that the time-local solution $u(t)$ to (1.1) becomes time-global and goes to some free solution like $U(t)(\varepsilon\phi_+)$ as $t \to \infty$ (see, e.g., [2, 9]), where $U(t) = \exp(it\Delta/2)$ is the free Schrödinger propagator. We can directly apply such methods to the case $3 < p < 5$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

In order to consider the remaining case $1 < p \le 3$ and $\operatorname{Im}\lambda > 0$, we review some results of the asymptotic behavior for the solution to (1.1) in the case $1 < p \le 3$ and $\lambda \in \mathbb{C}$. We again assume that $\varepsilon > 0$ is sufficiently small and that φ belongs to some suitable function space. If $1 < p \le 3$, then $u(t)$ does not behave like any free solutions as $t \to \infty$. In the case $p = 3$ (respectively $1 < p < 3$), if $\lambda \in \mathbb{R}$, then Hayashi–Naumkin [11] (respectively Hayashi–Kaikina–Naumkin [10]) proved the existence of a time-global solution $u(t)$ tending to some modified free solution like $\mathcal{F}^{-1}\exp(i\Theta(t,\xi))\mathcal{F}U(t)(\varepsilon\phi_+)$ as $t \to \infty$. Here, $\mathcal{F}$ is the Fourier transform, $\Theta(t,\xi) = \lambda|\phi_+(\xi)|^{p-1}s_p(t)$ with

$s_p(t)$ given by

$$s_p(t) = \begin{cases} \varepsilon^2 \log t & \text{if } p = 3, \\ \dfrac{2\varepsilon^{p-1} t^{(3-p)/2}}{3-p} & \text{if } 1 < p < 3. \end{cases}$$

Let a complex-valued function $V(s, \xi)$ solve the Cauchy problem for a nonlinear ordinary differential equation

$$\begin{cases} i\partial_s V(s,\xi) = \lambda |V(s,\xi)|^{p-1} V(s,\xi), & (s,\xi) \in [0, B] \times \mathbb{R}, \\ V(0,\xi) = e^{-i\pi/4}\phi_+(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{1.3}$$

where B is some positive number. Then we see that the modified free solution $\mathcal{F}^{-1}\exp(i\Theta(t,\xi))\mathcal{F}U(t)(\varepsilon\phi_+)$ with $1 < p \le 3$ is nearly equal to the function $m_p(t,x) = \varepsilon t^{-1/2}\exp(ix^2/2t)V(s_p(t), x/t)$ for sufficiently large $t > 0$. Recently, the case $1 < p \le 3$ and $\mathrm{Im}\lambda < 0$ has also been studied. If $\mathrm{Im}\lambda < 0$ and $p = 3$ (respectively $\mathrm{Im}\lambda < 0$ and p is smaller than and sufficiently close to 3), Shimomura [13] (respectively Kita-Shimomura [7]) showed the time-global existence and that the solution $u(t)$ behaves like $m_p(t)$ as $t \to \infty$ (see also [8]). Furthermore, [13, 7, 8] proved the time-decay estimate

$$\|u(t)\|_\infty \le \begin{cases} C(1+t)^{-1/(p-1)} & \text{if } 1 < p < 3, \\ C(1+t)^{-1/2}(\log(2+t))^{-1/2} & \text{if } p = 3, \end{cases} \tag{1.4}$$

which shows that $u(t)$ decays more rapidly than the corresponding free solution does. The estimate (1.4) essentially comes from

$$\|V(s)\|_\infty \le C(1 - s\mathrm{Im}\lambda)^{-1/(p-1)}, \quad s \in [0, \infty).$$

On the other hand, in the case $1 < p \le 3$ and $\mathrm{Im}\lambda > 0$, the function $V(s)$ blows up at some finite $s > 0$. Therefore, we can expect that $T(\varepsilon) < \infty$ even if ε is small. In fact, it is given by Kita [6] that some blow-up property holds if p and λ satisfy $1 < p \le 3$, $\mathrm{Im}\lambda > 0$ and other suitable conditions.

Summarizing the above known results, we find that $p = 3$ is the critical exponent with respect to the asymptotic behavior of the local solution to (1.1). Furthermore, in the critical and the sub-critical cases $1 < p \le 3$, it seems that the life span $T(\varepsilon)$ is different between the cases $\mathrm{Im}\lambda < 0$ and $\mathrm{Im}\lambda > 0$.

Let us focus on the problem (1.1) in the sub-critical case $1 < p < 3$ and $\mathrm{Im}\,\lambda > 0$. Our aim in the present paper is to study the life span $T(\varepsilon)$. In particular, we consider the dependence of $T(\varepsilon)$ upon $\mathrm{Im}\lambda$. It can be easily shown that $T(\varepsilon)$ is estimated by

$$T(\varepsilon) \ge C_0 \varepsilon^{-2(p-1)/(3-p)} \tag{1.5}$$

for some positive constant C_0. In fact, introducing the space Σ for initial data and the X-norm for solutions defined by

$$\Sigma = \left\{\varphi \in L^2(\mathbb{R}) : \|\varphi\|_\Sigma \equiv \|\varphi\|_{L^2(\mathbb{R})} + \|\partial_x\varphi\|_{L^2(\mathbb{R})} + \|x\varphi\|_{L^2(\mathbb{R})} < \infty\right\} \quad (1.6)$$

and

$$\|u(t)\|_X = \|u(t)\|_{L^2(\mathbb{R})} + \|\partial_x u(t)\|_{L^2(\mathbb{R})} + \|Ju(t)\|_{L^2(\mathbb{R})}, \quad J = x + it\nabla,$$

respectively and assuming that the time-local solution $u(t)$ satisfies

$$\|u(t)\|_X \leq 2\varepsilon\|\varphi\|_\Sigma, \quad 0 < t < T \ (T > 0),$$

we see from the standard energy inequality that

$$\begin{aligned}\|u(t)\|_X &\leq \varepsilon\|\varphi\|_\Sigma + C_1\int_0^T \|u(t)\|_X\|u(t)\|_{L^\infty(\mathbb{R})}^{p-1}dt \\ &\leq \varepsilon\|\varphi\|_\Sigma + C_1\int_0^T (1+t)^{-(p-1)/2}\|u(t)\|_X^p dt \\ &\leq \varepsilon\|\varphi\|_\Sigma + C_1\varepsilon^p T^{(3-p)/2}\|\varphi\|_\Sigma^p\end{aligned}$$

for $0 < t < T$. Here, we have used (2.3) below in the second inequality and the positive constant C_1 depends only on p and $|\lambda|$. Therefore, if C_0 satisfies $C_1 C_0^{(3-p)/2}\|\varphi\|_\Sigma^{p-1} \leq 1$, then we see that $u(t)$ with $\|u(t)\|_X \leq 2\varepsilon\|\varphi\|_\Sigma$ exists in $0 < t < T_0 := C_0\varepsilon^{-2(p-1)/(3-p)}$, which implies (1.5).

Unfortunately, we can not see the dependence of $T(\varepsilon)$ upon $\mathrm{Im}\lambda$ only by the proof of (1.5). We hence have to prove a more precise estimate of $T(\varepsilon)$ to see such dependence.

1.1. **Main result.** We remark that (1.5) is equivalent to

$$\liminf_{\varepsilon\to 0} \varepsilon^{2(p-1)/(3-p)}T(\varepsilon) > 0.$$

Our goal in this paper is to show a precise lower bound of

$$\liminf_{\varepsilon\to 0} \varepsilon^{2(p-1)/(3-p)}T(\varepsilon)$$

and to see the dependence of $T(\varepsilon)$ upon $\mathrm{Im}\lambda$. To introduce our result, we define the Fourier transform $\widehat{\phi}$ by

$$\widehat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx, \quad \xi \in \mathbb{R}.$$

We are ready to mention our main result.

Theorem 1.1. *Let $2 \le p < 3$ and $\lambda \in \mathbb{C}$. Assume that* $\operatorname{Im}\lambda > 0$ *and* $(1+x^2)\varphi \in \Sigma$. *Let $T(\varepsilon)$ be the life span of* (1.1) *defined by* (1.2). *Then we have*

$$\liminf_{\varepsilon\to 0} \varepsilon^{2(p-1)/(3-p)} T(\varepsilon) \ge \Big(\frac{3-p}{2(p-1)(\mathrm{Im}\lambda)\sup_{\xi\in\mathbb{R}} |\widehat{\varphi}(\xi)|^{p-1}}\Big)^{2/(3-p)}, \quad (1.7)$$

where $\frac{1}{0}$ is understood as $+\infty$.

Remark 1.2. We see that the above estimate (1.7) depends on $\mathrm{Im}\lambda$. If $\varphi \in \Sigma$, then $\widehat{\varphi}$ is a bounded continuous function vanishing at infinity. Therefore, $\sup_{\xi\in\mathbb{R}} |\widehat{\varphi}(\xi)|^{p-1}$ is finite.

Remark 1.3. The estimates (2.5), (2.7), (3.6) and (3.9) below are essential to obtain main results. Unfortunately, such estimates can not be used in the case $1 < p < 2$. Therefore, in the case $1 < p < 2$ and $\operatorname{Im}\lambda > 0$, it is still unknown whether (1.7) holds, or not.

In order to explain the estimate (1.7) in detail, we introduce known results for the life span of classical solutions to the quasilinear Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = F(u, \partial_x u) & \text{in } [0,\infty)\times\mathbb{R}, \\ u|_{t=0} = \varepsilon\phi & \text{on } \mathbb{R}. \end{cases} \quad (1.8)$$

Here, ϕ is sufficiently smooth and vanishes at infinity and F is a gauge-invariant, cubic polynomial with respect to u, $\overline{u}$, $\partial_x u$ and $\overline{\partial_x u}$. Let $S(\varepsilon)$ be the life span of the classical solution to (1.8). Then we see from Katayama–Tsutsumi [5] that $\liminf_{\varepsilon\to 0} \varepsilon^2 \log S(\varepsilon) > 0$. Sunagawa [14] showed the following precise lower bound of $\liminf_{\varepsilon\to 0} \varepsilon^2 \log S(\varepsilon)$:

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log S(\varepsilon) \ge \frac{1}{2\sup_{\xi\in\mathbb{R}} |\widehat{\phi}(\xi)|^2 \mathrm{Im} F(1, i\xi)}. \quad (1.9)$$

From the estimate (1.9), we can expect some properties concerned with $S(\varepsilon)$. In particular, if either $\phi \equiv 0$ or

$$\mathrm{Im} F(1, i\xi) \le 0, \quad (1.10)$$

then the right-hand side of (1.9) is positive infinity and we hence expect that $S(\varepsilon)$ is much larger than $\exp(C/\varepsilon^2)$ for any $C > 0$. In fact, Hayashi–Naumkin–Sunagawa [12] recently proved small data global existence under the condition (1.10). The estimate (1.9) is a (1.8) analogue of John and Hörmander's result concerned with quasilinear wave equations (see [4, 3]).

Let us come back to the Cauchy problem (1.1). If $p = 3$ and $\mathrm{Im}\lambda > 0$, the result of [14] can be directly applicable to the (1.1) cases. That is, it follows that

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log T(\varepsilon) \geq \frac{1}{2(\mathrm{Im}\lambda)\sup_{\xi\in\mathbb{R}} |\widehat{\varphi}(\xi)|^2}. \tag{1.11}$$

For any $2 \leq p < 3$, the estimate (1.7) can be understood as the (1.1) version of (1.11). In fact, (1.11) and (1.7) are rewritten in the following form:

$$\liminf_{\varepsilon\to 0} \int_1^{T(\varepsilon)} \left(\frac{\varepsilon}{\sqrt{\tau}}\right)^{p-1} d\tau \geq \frac{1}{(p-1)(\mathrm{Im}\lambda)\sup_{\xi\in\mathbb{R}} |\widehat{\varphi}(\xi)|^{p-1}}.$$

We state our strategy for proving our main result. The estimate (1.7) formally follows from the method of [14] (see also [3], [4], etc.). As the first step, we construct a suitable approximate solution $u_a(t, x)$ which is nearly equal to the modified free solution $m_p(t, x) = \varepsilon t^{-1/2} \exp(ix^2/2t)V(s_p(t), x/t)$, where $V(s, \xi)$ solves the ordinary differential equation (1.3) with $\phi_+ = \widehat{\varphi}$. The function $m_p(t)$ is composed of the term $|\widehat{\varphi}|^{p-1}$ and the life span of $m_p(t)$ satisfies (1.7). As the second step, we show an a priori estimate for the difference between $u(t)$ and $u_a(t)$, which enables us to see that $m_p(t)$ is close to $u(t)$ in some suitable sense. However, in the sub-critical case $1 < p < 3$, a technical difficulty appears. In fact, although we have to treat higher-order derivatives of $u_a(t)$ in the above second step, the term $|\varphi|^{p-1}$ contained in $m_p(t, x)$ is not sufficiently smooth. In order to overcome such difficulty, we modify the first step. In more detail, we modify the original $u_a(t)$ by mollifying the power term $|\widehat{\varphi}|^{p-1}$. That is, we replace $|\widehat{\varphi}|^{p-1}$ by the mollified term $\rho_\delta * |\widehat{\varphi}|^{p-1}$, where $*$ is the convolution in $\mathbb{R}$ and ρ_δ $(\delta > 0)$ is some mollifier. Then we need to show that the modified u_a is close to the original u_a in some sense. For this purpose, we prove that the difference between $\rho_\delta * |\widehat{\varphi}|^{p-1}$ and $|\widehat{\varphi}|^{p-1}$ is estimated by

$$\|\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1}\|_{H^1(\mathbb{R})} \leq \mathcal{O}(\delta), \tag{1.12}$$

where $\mathcal{O}$ is a non-negative increasing function tending to 0 as $\delta \to 0$. If we suitably take δ depending on ε, then we complete the modification of the above second step and hence the proof of (1.7).

Listing the contents of this paper, we close this section. In Section 2, we state some preliminaries which will be useful to prove Theorem 1.1. In particular, the estimate (1.12) above is given. In Section 3, we next construct the modified $u_a(t, x)$ and prove some inequalities for the difference between

$u(t)$ and the modified $u_a(t)$. In Section 4, we finally show an a priori estimate which immediately implies Theorem 1.1.

2. PRELIMINARIES

In this section, we show some preliminary properties for proving Theorem 1.1. For this purpose, we state some notation. To consider derivatives of $|\widehat{\varphi}|^{p-1}$, we put a mollifier $\rho_\delta(x) = \delta^{-1}\rho(\delta^{-1}x)$ for $\delta > 0$. Here, ρ is a smooth function on $\mathbb{R}$ satisfying $0 \le \rho \le 1$, supp$\rho \subset (-1,1)$ and $\int_{\mathbb{R}} \rho(x)dx = 1$. For $1 \le q \le \infty$, we denote the $L^q(\mathbb{R})$-norm by $\|\cdot\|_q$. Recall the space Σ, the operator J and the X-norm. For $T > 0$, we define a set $X(T)$ by

$$X(T) = \left\{ w \in C([0,T); H^1(\mathbb{R})) : Jw \in C([0,T); L^2(\mathbb{R})), \sup_{t\in[0,T)} \|w(t)\|_X < \infty \right\}. \tag{2.1}$$

For a multi-index $\alpha = (\alpha_1, \alpha_2) \in (\{0,1\})^2$, we set $Z^\alpha = \partial_x^{\alpha_1} J^{\alpha_2}$. Let $M(t)$ be a multiplication operator defined by

$$M(t) = \exp\left(\frac{x^2}{2t}\right).$$

Then we have the identity

$$J = M(t)(it\partial_x)M(-t). \tag{2.2}$$

The nonlinearity $\lambda|w|^{p-1}w$ is denoted by $\mathcal{N}(w)$. For non-negative functions f_1 and f_2, we define $f_1 \lesssim f_2$ if there exists some positive constant C independent of t, x, ε and δ such that $f_1 \le Cf_2$.

The first proposition is proved by the standard argument (see, e.g., (2.5) in [14]).

Proposition 2.1. *For any $w \in X(T)$, we have*

$$\|w(t)\|_\infty \lesssim (1+t)^{-1/2}\|w(t)\|_X, \quad t \in [0,T). \tag{2.3}$$

In Section 3 below, we deal with an approximate solution containing the mollified term $\rho_\delta * |\widehat{\varphi}|^{p-1}$. Then the following estimate is essential to treating it.

Proposition 2.2. *Let $2 \le p < 3$ and $\varphi \in \Sigma$. There exists a non-negative, increasing function $\mathcal{O}$ on $(0,\infty)$ such that*

$$\mathcal{O}(\delta) \to 0 \quad \text{as } \delta \to 0 \tag{2.4}$$

and

$$\left\|(\rho_\delta * |\widehat{\varphi}|^{p-1}) - |\widehat{\varphi}|^{p-1}\right\|_{H^1(\mathbb{R})} \le \mathcal{O}(\delta). \tag{2.5}$$

Proof. If $2 \le p < 3$, then the weak derivative of $|\widehat{\varphi}|^{p-1}$ is expressed by

$$\partial_x |\widehat{\varphi}|^{p-1} = \frac{p-1}{2} |\widehat{\varphi}|^{p-3} \mathrm{Re}\left(\widehat{\varphi}\ \overline{\partial_x \widehat{\varphi}}\right). \tag{2.6}$$

Thus, we see that

$$\|\partial_x |\widehat{\varphi}|^{p-1}\|_2 \lesssim \||\widehat{\varphi}|^{p-2}\|_\infty \|\partial_x \widehat{\varphi}\|_2 \lesssim \|\widehat{\varphi}\|_\infty^{p-2} \|\partial_x \widehat{\varphi}\|_2 \lesssim \|\varphi\|_\Sigma^{p-1},$$

where we have used the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, the identity $\partial_x \widehat{\varphi} = -i\widehat{x\varphi}$ and the Plancherel theorem in the last inequality. Hence it follows that $\partial_x |\widehat{\varphi}|^{p-1} \in L^2(\mathbb{R})$ and

$$\|\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1}\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } \delta \to 0.$$

If we put, for any δ,

$$\mathcal{O}(\delta) = \min\left\{2\||\widehat{\varphi}|^{p-1}\|_{H^1(\mathbb{R})}, \sup_{0<\eta\le\delta} \|\rho_\eta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1}\|_{H^1(\mathbb{R})}\right\},$$

then $\mathcal{O}$ is a non-negative, increasing function satisfying (2.4) and (2.5). □

In Sections 3 and 4 below, we treat the X-norm of the difference of two nonlinearities $\mathcal{N}(w_1) - \mathcal{N}(w_2)$. Then the following estimate is useful.

Proposition 2.3. *Let $2 \le p < 3$ and $\lambda \in \mathbb{C}$. Suppose that $w_j \in X(T)$, $j = 1, 2$ and $T > 0$. Then we have*

$$\begin{aligned} &\|\mathcal{N}(w_1(t)) - \mathcal{N}(w_2(t))\|_X \\ &\quad \lesssim (1+t)^{-(p-1)/2} \sup_{j=1,2} \|w_j(t)\|_X^{p-1} \|w_1(t) - w_2(t)\|_X, \quad 0 \le t < T. \end{aligned} \tag{2.7}$$

Proof. By a direct calculation, we obtain

$$\begin{aligned} &\partial_x \mathcal{N}(w_1) - \partial_x \mathcal{N}(w_2) \\ &= \frac{p+1}{2}\left(|w_1|^{p-1} - |w_2|^{p-1}\right)\partial_x w_1 + \frac{p+1}{2}|w_2|^{p-1}\left(\partial_x w_1 - \partial_x w_2\right) \\ &\quad + \frac{p-1}{2}\left(|w_1|^{p-3} w_1^2 - |w_2|^{p-3} w_2^2\right)\overline{\partial_x w_1} + \frac{p-1}{2}|w_2|^{p-3} w_2^2 \overline{(\partial_x w_1 - \partial_x w_2)}. \end{aligned}$$

Using the identity (2.2), it follows that

$$\begin{aligned} &J\mathcal{N}(w_1) - J\mathcal{N}(w_2) = M(t)(it)\partial_x\left(\mathcal{N}(M(-t)w_1(t)) - \mathcal{N}(M(-t)w_2(t))\right) \\ &= \frac{p+1}{2}\left(|w_1|^{p-1} - |w_2|^{p-1}\right) M(t)(it)\partial_x M(-t) w_1(t) \end{aligned}$$

$$
\begin{aligned}
&+\frac{p+1}{2}|w_2|^{p-1}M(t)(it)\partial_x M(-t)(w_1-w_2)\\
&+\frac{p-1}{2}M(t)(it)\left(|w_1|^{p-3}(M(-t)w_1)^2-|w_2|^{p-3}(M(-t)w_2)^2\right)\overline{\partial_x M(-t)w_1}\\
&+\frac{p-1}{2}M(t)(it)|w_2|^{p-3}(M(-t)w_2)^2\overline{\partial_x M(-t)(w_1-w_2)}\\
=&\frac{p+1}{2}\left(|w_1|^{p-1}-|w_2|^{p-1}\right)Jw_1+\frac{p+1}{2}|w_2|^{p-1}\left(Jw_1-Jw_2\right)\\
&-\frac{p-1}{2}\left(|w_1|^{p-3}w_1^2-|w_2|^{p-3}w_2^2\right)\overline{Jw_1}-\frac{p-1}{2}|w_2|^{p-3}w_2^2\overline{(Jw_1-Jw_2)}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\|\mathcal{N}(w_1)-\mathcal{N}(w_2)\|_X\\
&\lesssim \||w_1(t)|^{p-1}-|w_2(t)|^{p-1}\|_\infty\|w_1(t)\|_X+\|w_2\|_\infty^{p-1}\|w_1(t)-w_2(t)\|_X\\
&\quad+\||w_1(t)|^{p-3}w_1(t)^2-|w_2(t)|^{p-3}w_2(t)^2\|_\infty\|w_1(t)\|_X.
\end{aligned}
$$

To complete the proof of the proposition, we mention the following lemma.

Lemma 2.4. *Let $q\ge 2$. For any $\alpha_1,\alpha_2\in\mathbb{C}$, we have*

$$\left||\alpha_1|^{q-1}-|\alpha_2|^{q-1}\right|\lesssim|\alpha_1-\alpha_2|\left(|\alpha_1|+|\alpha_2|\right)^{q-2}, \tag{2.8}$$

$$\left||\alpha_1|^{q-3}\alpha_1^2-|\alpha_2|^{q-3}\alpha_2^2\right|\lesssim|\alpha_1-\alpha_2|\left(|\alpha_1|+|\alpha_2|\right)^{q-2}. \tag{2.9}$$

From Lemma 2.4 and (2.3), we have (2.7). □

We now prove Lemma 2.4 above. Let $q\ge 2$ and $j=1,2$. We set $\alpha_j=r_je^{i\theta_j}$, where $r_j>0$ and $\theta_j\in(-\pi,\pi]$. It follows from the mean value theorem that

$$|r_1^{q-1}-r_2^{q-1}|\lesssim|r_1-r_2|(r_1+r_2)^{q-2}. \tag{2.10}$$

Since

$$|r_1-r_2|\le|\alpha_1-\alpha_2|, \tag{2.11}$$

we obtain (2.8). On the other hand, we see that

$$
\begin{aligned}
&|r_1^{q-1}e^{2i\theta_1}-r_2^{q-1}e^{2i\theta_2}|\lesssim|r_1^{q-1}e^{2i\theta_1}-r_2^{q-1}e^{2i\theta_1}|+r_2^{q-1}|e^{2i\theta_1}-e^{2i\theta_2}|\\
&\lesssim|r_1^{q-1}-r_2^{q-1}|+r_2^{q-1}|e^{i\theta_1}-e^{i\theta_2}||e^{i\theta_1}+e^{i\theta_2}|\\
&\lesssim|r_1^{q-1}-r_2^{q-1}|+r_2^{q-2}|r_1e^{i\theta_1}-r_2e^{i\theta_2}|+r_2^{q-2}|r_1e^{i\theta_1}-r_2e^{i\theta_1}|\\
&\lesssim|r_1^{q-1}-r_2^{q-1}|+r_2^{q-2}|r_1e^{i\theta_1}-r_2e^{i\theta_2}|+r_2^{q-2}|r_1-r_2|.
\end{aligned}
$$

From (2.10) and (2.11), we have (2.9).

3. Approximate solution

In this section, we suppose that $2 \le p < 3$ and $\mathrm{Im}\lambda > 0$, then we define an approximate solution $u_a(t,x)$ and show some estimates in five subsections. Our goal in this section is to prove the following two inequalities which are important to show Theorem 1.1:

$$\|u_a(t)\|_X \lesssim \varepsilon, \quad 0 < t < T_B(\varepsilon) \tag{3.1}$$

and

$$\int_0^{T_B(\varepsilon)} \|R(t)\|_X dt \lesssim \varepsilon^{3/2} + \varepsilon \mathcal{O}(\varepsilon^{1/4}), \tag{3.2}$$

where

$$T_B(\varepsilon) = \Big(\frac{(3-p)B}{2\varepsilon^{p-1}}\Big)^{2/(3-p)}, \tag{3.3}$$

B is some positive number, $R = \mathcal{L}u_a - \mathcal{N}(u_a)$ and $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2$. Inequalities (3.1) and (3.2) are shown in Subsections 3.4 and 3.5, respectively.

3.1. **Definition of** $V(t,x)$**.** Assume that $(1+x^2)\varphi \in \Sigma$. We consider an ordinary differential equation

$$\begin{cases} i\partial_s V(s,\xi) = \mathcal{N}(V(s,\xi)), & (s,\xi) \in [0,B] \times \mathbb{R}, \\ V(0,\xi) = e^{-i\pi/4}\widehat{\varphi}(\xi), & \xi \in \mathbb{R} \end{cases} \tag{3.4}$$

for some $B > 0$. We define $A \in (0,\infty]$ by

$$A^{-1} = (p-1)(\mathrm{Im}\lambda) \sup_{\xi \in \mathbb{R}} |\widehat{\varphi}(\xi)|^{p-1}. \tag{3.5}$$

Then the solution to (3.4) with $B \in (0, A)$ is expressed by

$$V(s,\xi) = W(s,\xi)^{-1/(p-1)} \exp(iG(s,\xi))\widehat{\varphi}(\xi).$$

Here,

$$W(s,\xi) = 1 - (p-1)\mathrm{Im}\lambda |\widehat{\varphi}(\xi)|^{p-1} s,$$

$$G(s,\xi) = -\mathrm{Re}\lambda |\widehat{\varphi}(\xi)|^{p-1} \int_0^s W(\sigma,\xi)^{-1} d\sigma - \frac{\pi}{4}.$$

In order to prove (3.2), we need to estimate $\partial_\xi^3 V(s,\xi)$ (see Subsection 3.2). However, $V(s,\cdot)$ generally does not belong to $C^2(\mathbb{R})$. Therefore, we have to mollify V. For $\delta > 0$, we define $V_\delta(s,\xi)$ by

$$V_\delta(s,\xi) = W_\delta(s,\xi)^{-1/(p-1)} \exp(iG_\delta(s,\xi))\widehat{\varphi}(\xi).$$

Here,

$$W_\delta(s,\xi) = 1-(p-1)\mathrm{Im}\lambda(\rho_\delta * |\widehat{\varphi}|^{p-1})(\xi)s,$$
$$G_\delta(s,\xi) = -\mathrm{Re}\lambda(\rho_\delta * |\widehat{\varphi}|^{p-1})(\xi)\int_0^s W_\delta(\sigma,\xi)^{-1}d\sigma - \frac{\pi}{4}.$$

Then we see the following property of V_δ.

Proposition 3.1. *Let $2 \le p < 3$, $\lambda \in \mathbb{C}$, $\delta > 0$ and $B \in (0, A)$. Assume that $\mathrm{Im}\,\lambda > 0$ and $(1+x^2)\varphi \in \Sigma$. Then we have $W_\delta^{-1/(p-1)}\exp(iG_\delta) \in C^\infty([0,B]\times\mathbb{R})$. Furthermore, it follows that for $l = 0, 1$ and $m = 0, 1, 2, 3$,*

$$\sup_{(s,\xi)\in[0,B]\times\mathbb{R}} \left|\partial_s^l\partial_\xi^m\left(W_\delta^{-1/(p-1)}(s,\xi)\exp(iG_\delta(s,\xi))\right)\right| \lesssim \delta^{\min\{0,1-m\}} \tag{3.6}$$

and that

$$\begin{aligned} &i\partial_s V_\delta(s,\xi) - \mathcal{N}(V_\delta(s,\xi)) \\ &= \lambda W_\delta^{-p/(p-1)}(s,\xi)\exp(iG_\delta(s,\xi))\widehat{\varphi}(\xi)\left(\rho_\delta * |\widehat{\varphi}|^{p-1}(\xi) - |\widehat{\varphi}|^{p-1}(\xi)\right). \end{aligned} \tag{3.7}$$

Proof. For $\mathrm{Im}\lambda > 0$ and $(s,\xi) \in [0,B]\times\mathbb{R}$, we see from the Hölder-Young inequality that

$$\begin{aligned} W_\delta(s,\xi) &\ge 1-(p-1)\sup_{\xi\in\mathbb{R}}\mathrm{Im}\lambda(\rho_\delta * |\widehat{\varphi}|^{p-1})(\xi)s \\ &\ge 1-(p-1)\sup_{\xi\in\mathbb{R}}\mathrm{Im}\lambda|\widehat{\varphi}|^{p-1}(\xi)s \ge 1-B/A. \end{aligned}$$

Therefore, we obtain

$$0 \le W_\delta^{-1}(s,\xi) \le \frac{A}{A-B}. \tag{3.8}$$

Thus, V_δ is well defined on $[0,B]\times\mathbb{R}$.

Using the Hölder-Young inequality and (2.6), we have for $m = 1, 2, \cdots$,

$$\begin{aligned} \|\partial_x^m(\rho_\delta * |\widehat{\varphi}|^{p-1})\|_\infty &= \|\partial_x^{m-1}\rho_\delta * \partial_x|\widehat{\varphi}|^{p-1}\|_\infty \lesssim \delta^{1-m}\|\partial_x|\widehat{\varphi}|^{p-1}\|_\infty \\ &\lesssim \delta^{1-m}\|\widehat{\varphi}\|_\infty^{p-2}\|\partial_x\widehat{\varphi}\|_\infty \lesssim \delta^{1-m}\|(1+x^2)\varphi\|_\Sigma. \end{aligned}$$

Then it follows from the definition of W_δ and G_δ that $W_\delta^{-1/(p-1)}\exp(iG) \in C^\infty([0,B]\times\mathbb{R})$ and that (3.6) holds. The identity (3.7) is given by a direct calculation. □

Remark 3.1. From the proof of the above proposition, we immediately see that

$$\sup_{(s,\xi)\in[0,B]\times\mathbb{R}} \left|\partial_\xi^m\left(W_\delta^{-p/(p-1)}(s,\xi)\exp(iG(s,\xi))\right)\right| \lesssim 1, \tag{3.9}$$

for $m = 0, 1$, which is applied later.

3.2. **Definition of $m(t,x)$ and $Q(t,x)$.** Assume that $\delta > 0$, $B \in (0, A)$ and $(t,x) \in (1, T_B(\varepsilon)] \times \mathbb{R}$. Let $m = m(t,x)$ be a function defined by

$$m(t,x) = \frac{\varepsilon M(t)}{t^{1/2}} V_\delta(s(t), \xi(t,x)),$$

where

$$s(t) = \int_0^t \left(\frac{\varepsilon}{\tau^{1/2}}\right)^{p-1} d\tau = \frac{2\varepsilon^{p-1} t^{(3-p)/2}}{3-p} \quad \text{and} \quad \xi(t,x) = \frac{x}{t}.$$

Furthermore, we define a function $Q = Q(t,x)$ by

$$Q(t,x) = \mathcal{L}m(t,x) - \mathcal{N}(m(t,x)).$$

We see from (3.7) that Q is expressed by

$$\begin{aligned} Q(t,x) &= i\frac{\varepsilon M(t)}{t^{1/2}} \left\{ -\frac{ix^2}{2t^2} - \frac{1}{2t} - \frac{x}{t^2}\partial_\xi + \varepsilon^{p-1} t^{-(p-1)/2} \partial_s \right\} V_\delta(s(t), \xi(t,x)) \\ &\quad + \frac{1}{2}\frac{\varepsilon M(t)}{t^{1/2}} \left\{ \frac{-x^2}{t^2} + \frac{i}{t} + \frac{2ix}{t^2}\partial_\xi + \frac{1}{t^2}\partial_\xi^2 \right\} V_\delta(s(t), \xi(t,x)) \\ &= \frac{\varepsilon^p M(t)}{t^{p/2}} \{ i\partial_s V_\delta(s(t), \xi(t,x)) \} + \frac{\varepsilon M(t)}{2t^{5/2}} \partial_\xi^2 V(s(t), \xi(t,x)) \\ &= \frac{\lambda \varepsilon^p M(t)}{t^{p/2}} \left(W_\delta^{-p/(p-1)} \exp(iG_\delta) \widehat{\varphi} (\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1}) \right) (s(t), \xi(t,x)) \\ &\quad + \frac{\varepsilon M(t)}{2t^{5/2}} \partial_\xi^2 V_\delta(s(t), \xi(t,x)) \qquad (3.10) \\ &=: Q_1(t,x) + Q_2(t,x). \end{aligned}$$

3.3. **Estimates of $m(t,x)$ and $Q(t,x)$.** In this subsection, we assume that $\delta > 0$, $B \in (0, A)$ and $(t,x) \in (1, T_B(\varepsilon)] \times \mathbb{R}$, and we estimate the X-norm of $m(t)$ and $Q(t)$. The first derivative of m is given by

$$\begin{aligned} \partial_x m(t,x) &= \partial_x \left\{ \frac{\varepsilon M(t)}{t^{1/2}} \left(\left(W^{-1/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t}\right) \right) \right) \left(\widehat{\varphi}\left(\frac{x}{t}\right) \right) \right\} \\ &= \frac{\varepsilon M(t)}{t^{1/2}} \left(\left(W^{-1/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t}\right) \right) \right) \left(\frac{ix}{t} \widehat{\varphi}\left(\frac{x}{t}\right) \right) \\ &\quad + \frac{\varepsilon M(t)}{t^{1/2}} \left(\frac{1}{t}\partial_\xi \left(W^{-1/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t}\right) \right) \right) \left(\widehat{\varphi}\left(\frac{x}{t}\right) \right) \\ &\quad + \frac{\varepsilon M(t)}{t^{1/2}} \left(\left(W^{-1/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t}\right) \right) \right) \left(\frac{1}{t}\partial_\xi \widehat{\varphi}\left(\frac{x}{t}\right) \right). \end{aligned}$$

The identity (2.2) implies that

$$
\begin{aligned}
Jm(t,x) = & \frac{i\varepsilon M(t)}{t^{1/2}} \left(\partial_\xi \left(W^{-1/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t} \right) \right) \right) \left(\widehat{\varphi} \left(\frac{x}{t} \right) \right) \\
& + \frac{i\varepsilon M(t)}{t^{1/2}} \left(\left(W^{-1/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t} \right) \right) \right) \left(\partial_\xi \widehat{\varphi} \left(\frac{x}{t} \right) \right).
\end{aligned}
$$

From (3.6), we obtain

$$
\|m(t)\|_X \lesssim \frac{\varepsilon}{t^{1/2}} \left(\left\| \widehat{\varphi} \left(\frac{\cdot}{t} \right) \right\|_2 + \left\| \frac{\cdot}{t} \widehat{\varphi} \left(\frac{\cdot}{t} \right) \right\|_2 + \left\| \partial_\xi \widehat{\varphi} \left(\frac{\cdot}{t} \right) \right\|_2 \right) \lesssim \varepsilon \|\varphi\|_\Sigma \lesssim \varepsilon. \tag{3.11}
$$

We next estimate $Q_1(t,x)$. It follows that

$$
\begin{aligned}
\partial_x Q_1(t,x) = & \partial_x \Bigg\{ \frac{\lambda \varepsilon^p M(t)}{t^{p/2}} \left(\left(W^{-p/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t} \right) \right) \right) \\
& \times \left(\left(\widehat{\varphi} \left(\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1} \right) \right) \left(\frac{x}{t} \right) \right) \Bigg\} \\
= & \frac{\lambda \varepsilon^p M(t)}{t^{p/2}} \Bigg\{ \left(\left(W^{-p/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t} \right) \right) \right) \\
& \times \left(\left(\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1} \right) \left(\frac{x}{t} \right) \right) \left(\frac{ix}{t} \widehat{\varphi} \left(\frac{x}{t} \right) \right) \\
& + \left(\frac{1}{t} \partial_\xi \left(W^{-p/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t} \right) \right) \right) \\
& \times \left(\left(\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1} \right) \left(\frac{x}{t} \right) \right) \left(\widehat{\varphi} \left(\frac{x}{t} \right) \right) \\
& + \left(\left(W^{-p/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t} \right) \right) \right) \\
& \times \left(\frac{1}{t} \partial_\xi \left(\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1} \right) \left(\frac{x}{t} \right) \right) \left(\widehat{\varphi} \left(\frac{x}{t} \right) \right) \\
& + \left(\left(W^{-p/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t} \right) \right) \right) \\
& \times \left(\left(\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1} \right) \left(\frac{x}{t} \right) \right) \left(\frac{1}{t} \partial_\xi \widehat{\varphi} \left(\frac{x}{t} \right) \right) \Bigg\}
\end{aligned}
$$

and that

$$
\begin{aligned}
JQ_1(t,x) = & \frac{\varepsilon^p M(t)}{t^{p/2}} \Bigg\{ \left(\partial_\xi \left(W^{-p/(p-1)} \exp(iG_\delta) \right) \left(s(t), \left(\frac{x}{t} \right) \right) \right) \\
& \times \left(\left(\rho_\delta * |\widehat{\varphi}|^{p-1} - |\widehat{\varphi}|^{p-1} \right) \left(\frac{x}{t} \right) \right) \left(\widehat{\varphi} \left(\frac{x}{t} \right) \right)
\end{aligned}
$$

$$+\left(\left(W^{-p/(p-1)}\exp(iG_\delta)\right)\left(s(t),\left(\frac{x}{t}\right)\right)\right)$$
$$\times\left(\partial_\xi\left(\rho_\delta*|\widehat{\varphi}|^{p-1}-|\widehat{\varphi}|^{p-1}\right)\left(\frac{x}{t}\right)\right)\left(\widehat{\varphi}\left(\frac{x}{t}\right)\right)$$
$$+\left(\left(W^{-p/(p-1)}\exp(iG_\delta)\right)\left(s(t),\left(\frac{x}{t}\right)\right)\right)$$
$$\times\left(\left(\rho_\delta*|\widehat{\varphi}|^{p-1}-|\widehat{\varphi}|^{p-1}\right)\left(\frac{x}{t}\right)\right)\left(\widehat{\varphi}\left(\frac{x}{t}\right)\right)\Bigg\}.$$

We see from (3.9) and (2.5) that

$$\begin{aligned}\|Q_1(t)\|_X &\lesssim \varepsilon^p t^{-\frac{p}{2}}\mathcal{O}(\delta)\left(\left\|\widehat{\varphi}\left(\frac{\cdot}{t}\right)\right\|_\infty+\left\|\frac{\cdot}{t}\widehat{\varphi}\left(\frac{\cdot}{t}\right)\right\|_\infty+\left\|\partial_\xi\widehat{\varphi}\left(\frac{\cdot}{t}\right)\right\|_\infty\right)\\ &\lesssim \varepsilon^p t^{-(p-1)/2}\mathcal{O}(\delta)\|(1+x^2)\varphi\|_\Sigma.\end{aligned}$$

By (3.6), the remainder $Q_2(t,x)$ is estimated by

$$\|Q_2(t)\|_X \lesssim \varepsilon t^{-2}\delta^{-2}.$$

Hence, we obtain

$$\|Q(t)\|_X \lesssim \varepsilon^p t^{-(p-1)/2}\mathcal{O}(\delta)+\varepsilon t^{-2}\delta^{-2}. \tag{3.12}$$

3.4. **Definition of $u_a(t,x)$ and $R(t,x)$.** Assume that $\delta>0$ and $B\in(0,A)$. Let χ be a smooth function on $\mathbb{R}$ satisfying $0\le\chi\le 1$, $\chi(t)=1$ if $t\le 1$ and $\chi(t)=0$ if $t\ge 2$. For $\varepsilon>0$ and $(t,x)\in(0,T_B(\varepsilon)]\times\mathbb{R}$, we put

$$u_a(t,x)=\chi(\varepsilon t)U(t)(\varepsilon\varphi(x))+(1-\chi(\varepsilon t))m(t,x),$$

where $U(t)$ is the free Schrödinger propagator. That is, $u_{0,\varepsilon}(t)=U(t)(\varepsilon\varphi)$ solves

$$\begin{cases} i\partial_t u_{0,\varepsilon}+\frac{1}{2}\partial_x^2 u_{0,\varepsilon}=0 & \text{in } [0,\infty)\times\mathbb{R},\\ u_{0,\varepsilon}|_{t=0}=\varepsilon\varphi & \text{on } \mathbb{R}.\end{cases}$$

From (3.11) and the standard equality

$$\|U(t)(\varepsilon\varphi)\|_X=\varepsilon\|\varphi\|_\Sigma, \tag{3.13}$$

we have (3.1).

Let $R=R(t,x)$ be a function defined by

$$R(t,x)=\mathcal{L}u_a(t,x)-\mathcal{N}(u_a(t,x)),\quad (t,x)\in(0,T_B(\varepsilon)]\times\mathbb{R}.$$

3.5. **Estimates of $R(t,x)$.** As the final step, we estimate R. Henceforth, we fix $\delta = \varepsilon^{1/4}$. If $t \in (0, 1/\varepsilon]$, then $R(t,x) = -\chi(\varepsilon t)\mathcal{N}(U(t)(\varepsilon\varphi(x)))$. Therefore, it follows from (2.7) and (3.13) that

$$\|R(t)\|_X \lesssim \varepsilon^p t^{-(p-1)/2}, \quad t \in (0, 1/\varepsilon]. \tag{3.14}$$

Thus, we have

$$\int_0^{1/\varepsilon} \|R(t)\|_X dt \lesssim \varepsilon^{3/2}, \tag{3.15}$$

where we have used the condition $p \geq 2$.

In the case $t \in [2/\varepsilon, T_B(\varepsilon)]$, $R(t,x)$ is equal to $Q(t,x)$. We hence obtain

$$\|R(t)\|_X \lesssim \varepsilon^p t^{-(p-1)/2}\mathcal{O}(\delta) + \varepsilon t^{-2}\delta^{-2}, \quad t \in [2/\varepsilon, T_B(\varepsilon))$$

and

$$\int_{2/\varepsilon}^{T_B(\varepsilon)} \|R(t)\|_X dt \lesssim \varepsilon\mathcal{O}(\varepsilon^{1/4}) + \varepsilon^{3/2}. \tag{3.16}$$

Let us consider the case $t \in (1/\varepsilon, 2/\varepsilon)$. Then we have

$$\begin{aligned} R(t,x) = {} & i\varepsilon\chi'(\varepsilon t)(u_{0,\varepsilon}(t,x) - m(t,x)) \\ & + (1-\chi(\varepsilon t))(\mathcal{N}(m(t,x)) - \mathcal{N}(u_a(t,x))) \\ & - \chi(\varepsilon t)\mathcal{N}(u_a(t,x)) + (1-\chi(\varepsilon t))Q(t,x). \end{aligned} \tag{3.17}$$

It is well known that the free solution $u_{0,\varepsilon}$ is expressed by

$$U(t)(\varepsilon\varphi(x)) = \frac{\varepsilon M(t)}{\sqrt{2\pi i t}} \int_{-\infty}^{\infty} \exp\left(-\frac{ixy}{t}\right) \exp\left(\frac{iy^2}{2t}\right) \varphi(y) dy.$$

Therefore, we obtain for any $t \in (1/\varepsilon, 2/\varepsilon)$,

$$\begin{aligned} U(t)(\varepsilon\varphi(x)) - m(t,x) = {} & \frac{\varepsilon M(t)}{t^{1/2}} \left\{ V_\delta\left(0, \frac{x}{t}\right) - V_\delta\left(s(t), \frac{x}{t}\right) \right\} \\ & + \frac{\varepsilon M(t)}{\sqrt{2\pi i t}} \int_{-\infty}^{\infty} \exp\left(-\frac{ixy}{t}\right) \left\{ \exp\left(\frac{iy^2}{2t}\right) - 1 \right\} \varphi(y) dy \\ = {} & \frac{2\varepsilon M(t)}{(3-p)t^{1/2}} t^{(3-p)/2} \varepsilon^{p-1} \left\{ -\int_0^1 \partial_s V_\delta\left(s(t)\theta, \frac{x}{t}\right) d\theta \right\} \\ & + \frac{\varepsilon M(t)}{\sqrt{2\pi i t}} \int_{-\infty}^{\infty} \exp\left(-\frac{ixy}{t}\right) \left\{ \exp\left(\frac{iy^2}{2t}\right) - 1 \right\} \varphi(y) dy \\ =: {} & f_1(t,x) + f_2(t,x). \end{aligned}$$

We see from (3.6) that

$$\|f_1(t)\|_X \lesssim \varepsilon^p t^{(3-p)/2}, \quad t \in (1/\varepsilon, 2/\varepsilon).$$

For the other term f_2, the following estimate was shown by [14]:

$$\|f_2(t)\|_X \lesssim \varepsilon t^{-1}, \quad t > 0.$$

Thus, we obtain for any $t \in (1/\varepsilon, 2/\varepsilon)$,

$$\|U(t)(\varepsilon\varphi) - m(t)\|_X \lesssim \varepsilon^{3/2}, \tag{3.18}$$

where we have used the relation $1/\varepsilon \leq t \leq 2/\varepsilon$ and $p \geq 2$. By (2.7) and (3.1), we have

$$\|\mathcal{N}(m(t)) - \mathcal{N}(u_a(t))\|_X \lesssim \varepsilon^{p-1} t^{-(p-1)/2} \|m(t) - u_a(t)\|_X, \quad t \in (1/\varepsilon, 2/\varepsilon).$$

Since $m - u_a = \chi(\varepsilon t)(m - u_{0,\varepsilon})$, it follows from (3.18) that

$$\|\mathcal{N}(m(t)) - \mathcal{N}(u_a(t))\|_X \lesssim \varepsilon^3, \quad t \in (1/\varepsilon, 2/\varepsilon), \tag{3.19}$$

where we have used the relation $1/\varepsilon \leq t \leq 2/\varepsilon$ and $p \geq 2$ again. By the same argument as in the proof of (3.14), we see that

$$\|\mathcal{N}(u_a(t,x))\|_X \lesssim \varepsilon^{5/2}, \quad t \in (1/\varepsilon, 2/\varepsilon). \tag{3.20}$$

Therefore, it follows from (3.17)–(3.20) and (3.12) that

$$\int_{1/\varepsilon}^{2/\varepsilon} \|R(t)\|_X dt \lesssim \varepsilon^{3/2}(1 + \mathcal{O}(\varepsilon^{1/4})). \tag{3.21}$$

We are ready to show (3.2). The estimates (3.15), (3.21) and (3.16) enable us to see that

$$\begin{aligned}\int_0^{T_B(\varepsilon)} \|R(t)\|_X dt &\leq \Big(\int_0^{1/\varepsilon} + \int_{1/\varepsilon}^{2/\varepsilon} + \int_{2/\varepsilon}^{T_B(\varepsilon)} \Big) \|R(t)\|_X dt \\ &\lesssim \varepsilon^{3/2} + \varepsilon \mathcal{O}(\varepsilon^{1/4}),\end{aligned}$$

which completes (3.2).

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Assume that $2 \leq p < 3$, $\operatorname{Im}\lambda > 0$, $\varepsilon > 0$, and $(1+x^2)\varphi \in \Sigma$. Suppose that $u = u(t,x)$ is the time-local solution to (1.1) satisfying $u \in X(T)$ for some $T > 0$. We immediately see the existence of such $u(t,x)$ (see Remark 1.1). Recall that positive numbers A and $T_B(\varepsilon)$ are given by

$$A^{-1} = (p-1)(\mathrm{Im}\lambda)\sup_{\xi\in\mathbb{R}}|\widehat{\varphi}(\xi)|^{p-1} \quad \text{and} \quad T_B(\varepsilon) = \left(\frac{(3-p)B}{2\varepsilon^{p-1}}\right)^{2/(3-p)},$$

respectively. Let $m(t,x)$ and $u_a(t,x)$ be functions defined in Section 3. For $\varepsilon > 0$, we fix $\delta = \varepsilon^{1/4}$. Then the inequalities (3.1) and (3.2) hold. We now prove the following lemma.

Lemma 4.1. *Let $2 \le p < 3$, $\lambda \in \mathbb{C}$ and $B \in (0, A)$. Assume that $\mathrm{Im}\,\lambda > 0$ and $(1+x^2)\varphi \in \Sigma$. Then there exists some $\varepsilon_0 > 0$ such that the following property holds for any $\varepsilon \in (0, \varepsilon_0]$: If*

$$0 < T < \min\{T(\varepsilon), T_B(\varepsilon)\} \quad \textit{and} \quad \sup_{0\le t\le T}\|u_a(t) - u(t)\|_X \le \varepsilon,$$

then we have

$$\sup_{0\le t\le T}\|u_a(t) - u(t)\|_X \le \frac{\varepsilon}{2}.$$

We first prove that we obtain Theorem 1.1 by using Lemma 4.1. We remark that the proof is very similar to that of [14]. We fix $B \in (0, A)$ and $\varepsilon \in (0, \varepsilon_0]$. Since $\|u_a(0) - u(0)\|_X = 0$ and $\|u_a(t) - u(t)\|_X$ is continuous with respect to t, there exists some $T^* > 0$ such that

$$\sup_{0\le t\le T^*}\|u_a(t) - u(t)\|_X \le \varepsilon. \tag{4.1}$$

If $T^* \ge T_B(\varepsilon)$, then it follows from (3.1) that

$$\sup_{0\le t\le T_B(\varepsilon)}\|u(t)\|_X \le \sup_{0\le t\le T_B(\varepsilon)}\|u_a(t)\|_X + \sup_{0\le t\le T^*}\|u_a(t) - u(t)\|_X \le C\varepsilon, \tag{4.2}$$

where C is a positive constant independent of ε. The a priori estimate (4.2) implies that $T(\varepsilon) > T_B(\varepsilon)$. On the other hand, assume that whenever $T^* > 0$ satisfies (4.1), T^* is smaller than $T_B(\varepsilon)$. Then we see that

$$\sup_{0\le t\le T^*}\|u(t)\|_X \le \sup_{0\le t\le T_B(\varepsilon)}\|u_a(t)\|_X + \sup_{0\le t\le T^*}\|u_a(t) - u(t)\|_X \le C\varepsilon,$$

so that $T^* < \min\{T(\varepsilon), T_B(\varepsilon)\}$. Then a positive number T^{**} defined by

$$T^{**} = \max\{T^* > 0 : T^* \text{ satisfies } (4.1)\}$$

is well defined and satisfies $0 < T^{**} < \min\{T(\varepsilon), T_B(\varepsilon)\}$ and

$$\sup_{0\le t\le T^{**}} \|u_a(t) - u(t)\|_X = \varepsilon > \frac{\varepsilon}{2}. \tag{4.3}$$

We see from Lemma 4.1 that

$$\sup_{0\le t\le T^{**}} \|u_a(t) - u(t)\|_X \le \frac{\varepsilon}{2},$$

which contradicts (4.3). We hence see that $T_B(\varepsilon) \le T(\varepsilon)$. In other words, we have

$$\varepsilon^{2(p-1)/(3-p)} T(\varepsilon) \ge \left(\frac{(3-p)B}{2}\right)^{2/(3-p)}.$$

Since $B \in (0, A)$ is arbitrary, Theorem 1.1 holds.

Let us prove Lemma 4.1. Put $v = u_a - u$. Then v solves

$$\begin{cases} \mathcal{L}Z^\alpha v = -Z^\alpha \mathcal{N}(u_a + v) + Z^\alpha \mathcal{N}(u_a) + Z^\alpha R & \text{in } [0,\infty)\times\mathbb{R}, \\ (Z^\alpha v)|_{t=0} = 0 & \text{on } \mathbb{R}. \end{cases}$$

The standard energy inequality implies that

$$\|v(t)\|_X \lesssim \int_0^t \|\mathcal{N}(u_a(\tau) + v(\tau)) - \mathcal{N}(u_a(\tau))\|_X d\tau + \int_0^t \|R(\tau)\|_X d\tau. \tag{4.4}$$

From (2.7), the assumption of Lemma 4.1, (3.1) and (3.2), we obtain

$$\|v(t)\|_X \lesssim \varepsilon^{p-1} \int_0^t (1+\tau)^{-(p-1)/2} \|v(\tau)\|_X d\tau + \varepsilon^{3/2} + \varepsilon\mathcal{O}(\varepsilon^{1/4}).$$

By Gronwall's lemma, we see that

$$\begin{aligned} \|v(t)\|_X &\le C\left(\varepsilon^{3/2} + \varepsilon\mathcal{O}(\varepsilon^{1/4})\right) \exp\left(C\varepsilon^{p-1}\int_0^{T_B(\varepsilon)} (1+\tau)^{-(p-1)/2} d\tau\right) \\ &\le C\left(\varepsilon^{3/2} + \varepsilon\mathcal{O}(\varepsilon^{1/4})\right) \exp\left(C\varepsilon^{p-1} T_B(\varepsilon)^{(3-p)/2}\right) \\ &\le C\left(\varepsilon^{3/2} + \varepsilon\mathcal{O}(\varepsilon^{1/4})\right) \exp\left(C\varepsilon^{p-1}\varepsilon^{-(p-1)}\right) \\ &\le C\left(\varepsilon^{3/2} + \varepsilon\mathcal{O}(\varepsilon^{1/4})\right) \end{aligned}$$

for some constant C independent of ε. Choosing $\varepsilon_0 > 0$ such that $C(\varepsilon_0^{1/2} + \mathcal{O}(\varepsilon_0^{1/4})) \le 1/2$, we have $\|v(t)\|_X \le \varepsilon/2$, which completes the proof of Lemma 4.1. Hence Theorem 1.1 holds.

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