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CRITICAL FUJITA EXPONENTS FOR A COUPLED NON-NEWTONIAN FILTRATION SYSTEM

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Abstract. In 1966, Fujita started the close investigation of the blow-up phenomena arising in some semilinear parabolic problems and found the so-called Fujita exponent. Later, Galaktionov and Levine researched nonlinear parabolic equations involving p-Laplace operators with nonlinear boundary conditions. In the present paper we extend their results to a non-Newtonian filtration system coupled via nonlinear boundary conditions. In terms of a characteristic algebraic system introduced for the problem, we obtain a clear and simple representation of both the critical Fujita exponent and the blow-up rate for this complicated coupled system containing six nonlinear exponent parameters. The proof for establishing the critical exponents is based on careful constructions of the comparison functions, especially for the blow-up case, by using the Barlenbratt-type solutions. The analysis of the blow-up rate relies on the appropriate scale transformation of the independent and the dependent variables with some differential inequalities satisfied by the L^∞-norm of blowing up solutions. This paper provides a complete result for such a coupled degenerate parabolic system.

1. Introduction

In this paper, we investigate the critical Fujita exponents for the coupled non-Newtonian filtration equations

$$u_t = (|u_x|^{m-1}u_x)_x,\ v_t = (|v_x|^{n-1}v_x)_x, \qquad (x,t) \in \mathbb{R}^+ \times (0,T), \tag{1.1}$$

$$-(|u_x|^{m-1}u_x)(0,t) = (u^\alpha v^p)(0,t), \qquad t \in (0,T), \tag{1.2}$$

$$-(|v_x|^{n-1}v_x)(0,t) = (u^q v^\beta)(0,t), \qquad t \in (0,T), \tag{1.3}$$

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$$u(x,0) = u_0(x),\ v(x,0) = v_0(x), \qquad x \in \mathbb{R}^+, \tag{1.4}$$

where u_0, v_0 are continuous, nonnegative and compactly supported in $\mathbb{R}^+ \cup \{0\}$, satisfying the compatibility conditions at $x = 0$, the parameters $m, n > 1$, $p, q > 0, \alpha, \beta \geq 0$.

The study of critical exponents was begun after Fujita's paper [3] (1966), where it was found that the initial-value problem of the semilinear equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+$$

does not have nontrivial, nonnegative global solutions if $1 < p < p_c = 1 + 2/N$, whereas there exist both global (with small data) and non-global (with large initial data) solutions if $p > p_c$. Later, it was proved by Hayakawa [6], Kobayashi *et al* [9] and Weissler [16] that the critical situation of $p = p_c$ belongs to the blow-up case. The elegant work of Fujita revealed a new phenomenon of nonlinear partial differential equations, and stimulated the study of similar features for various problems of various nonlinear evolution equations. Since then, numerous interesting results have been obtained (see, e.g., the survey papers [1, 11] and the references therein).

In [5], Galaktionov and Levine studied the scalar problem

$$u_t = (|u_x|^{m-1}u_x)_x \quad \text{in } \mathbb{R}^+ \times (0,T), \tag{1.5}$$

$$-(|u_x|^{m-1}u_x)(0,t) = u^p(0,t) \quad \text{in } (0,T), \tag{1.6}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^+, \tag{1.7}$$

where $m > 1$. They proved for (1.5)–(1.7) that the critical global existence exponent was $p_0 = \frac{2m}{m+1}$, while for the critical Fujita exponent $p_c = 2m$, the following holds: (i) If $0 < p \leq \frac{2m}{m+1}$, the solutions of (1.5)–(1.7) are global; (ii) If $\frac{2m}{m+1} < p \leq 2m$, the solutions of (1.5)–(1.7) blow up in finite time for all nonnegative, nontrivial u_0; (iii) If $p > 2m$, the solutions of (1.5)–(1.7) are global for small u_0 and blow up in finite time for large u_0.

As for coupled systems, the critical Fujita exponents for the Newtonian filtration equations coupled via variational nonlinear boundary flux of the form

$$u_t = (u^m)_{xx},\ v_t = (v^n)_{xx}, \quad (x,t) \in \mathbb{R}^+ \times (0,T), \tag{1.8}$$

$$-(u^m)_x(0,t) = (u^\alpha v^p)(0,t),\ -(v^n)_x(0,t) = (u^q v^\beta)(0,t),\ t \in (0,T), \tag{1.9}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad x \in \mathbb{R}^+ \tag{1.10}$$

were studied by by Quirós and Rossi [15] (the case of $\alpha = \beta = 0$) and Zheng *et al* [20]. The blow-up rate for the semilinear case of (1.1)–(1.3) with $m = n = 1$ was considered by Wang *et al* in [17].

The purpose of this paper is to extend the above results to the coupled system (1.1)–(1.4). To state our results, we introduce parameters k_i $(i = 1, 2)$ satisfying

$$\begin{pmatrix} \alpha - \frac{2m}{m+1} & p \\ q & \beta - \frac{2n}{n+1} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \frac{m}{m+1} \\ \frac{n}{n+1} \end{pmatrix}, \tag{1.11}$$

namely

$$k_1 = \frac{\frac{m}{m+1}(\lambda_0 p + \frac{2n}{n+1} - \beta)}{pq - (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})},\ k_2 = \frac{\frac{n}{n+1}(\lambda_0^{-1} q + \frac{2m}{m+1} - \alpha)}{pq - (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})} \tag{1.12}$$

with $\lambda_0 = \frac{n}{n+1} \cdot \frac{m+1}{m}$.

Remark 1. We will use the signs and values of $1/k_1, 1/k_2$ to describe the critical behaviors of solutions, where $(1/k_1, 1/k_2)$=(0,0) is defined by the limit of $(1/k_1, 1/k_2)$ as $pq - (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1}) \to 0$ with $\lambda_0 p \neq \beta - \frac{2n}{n+1}$, $\lambda_0^{-1} q \neq \alpha - \frac{2m}{m+1}$. Clearly, if $\lambda_0^{-1} q = \alpha - \frac{2m}{m+1}$, $\lambda_0 p \neq \beta - \frac{2n}{n+1}$, then $k_1 = \frac{n}{q(n+1)}$, $k_2 = 0$. For $\lambda_0 p = \beta - \frac{2n}{n+1}$, $\lambda_0^{-1} q = \alpha - \frac{2m}{m+1}$, define, e.g., $k_1 = \frac{n}{2q(n+1)}$, $k_2 = \frac{m}{2p(m+1)}$. We call linear algebraic systems like (1.11) *characteristic algebraic systems* [10, 19, 20].

Now we state the main results of this paper.

Theorem 1. *If* $1/k_1, 1/k_2 < 0$*, the solutions of* (1.1)–(1.4) *are global.*

Theorem 2. *If* $0 < 1/k_1 < 2m$ *or* $0 < 1/k_2 < 2n$*, the solutions of* (1.1)–(1.4) *will blow up in a finite time for every nonnegative, nontrivial initial data.*

Theorem 3. *Assume either* $1/k_1 = 2m$ *with* $1/k_2 > 0$*, or* $1/k_2 = 2n$ *with* $1/k_1 > 0$ *holds. Then the solutions of* (1.1)–(1.4) *will blow up in a finite time for every nonnegative, nontrivial initial data.*

Theorem 4. *Assume either* $1/k_1 > 2m$ *with* $1/k_2 \notin [0, 2n]$*, or* $1/k_2 > 2n$ *with* $1/k_1 \notin [0, 2m]$ *holds. Then the solutions of* (1.1)–(1.4) *are global for small initial data and will blow up in a finite time with large initial data.*

Theorem 5. *Assume* $(1/k_1, 1/k_2) = (0, 0)$*.*

(i) *If* $\alpha < \frac{2m}{m+1}$, $\beta < \frac{2n}{n+1}$*, the solutions of* (1.1)–(1.4) *are global;*

(ii) *If* $\alpha > \frac{2m}{m+1}$, $\beta > \frac{2n}{n+1}$ *and* $\big(\lambda_0 p - (\beta - \frac{2n}{n+1})\big)\big(\lambda_0^{-1} q - (\alpha - \frac{2m}{m+1})\big) \neq 0$*, the solutions of* (1.1)–(1.4) *blow up in a finite time for every nontrivial initial data;*

(iii) *Assume* $\big(\lambda_0 p - (\beta - \frac{2n}{n+1})\big)\big(\lambda_0^{-1} q - (\alpha - \frac{2m}{m+1})\big) = 0$. *If* $\max\{2mk_1 - 1, 2nk_2 - 1\} \geq 0$, *every nontrivial solution to* (1.1)–(1.4) *is non-global. If* $\max\{2mk_1 - 1, 2nk_2 - 1\} < 0$, *there exist both global and non-global solutions.*

Theorem 6. *Assume* $k_1, k_2 > 0$ *with* $(|u_{0x}|^{m-1}u_{0x})_x, (|v_{0x}|^{n-1}v_{0x})_x \geq 0$. *Let* (u, v) *be a solution of* (1.1)–(1.4) *with blow-up time* T. *Then there exist positive constants* c_i $(i = 1, 2, 3, 4)$ *such that*

$$c_3(T-t)^{-k_1} \leq \|u(\cdot,t)\|_\infty \leq c_1(T-t)^{-k_1},$$
$$c_4(T-t)^{-k_2} \leq \|v(\cdot,t)\|_\infty \leq c_2(T-t)^{-k_2}.$$

In summary, the critical Fujita exponents of (1.1)–(1.4) are determined by Theorems 1–5, which can be shown via the following figure:

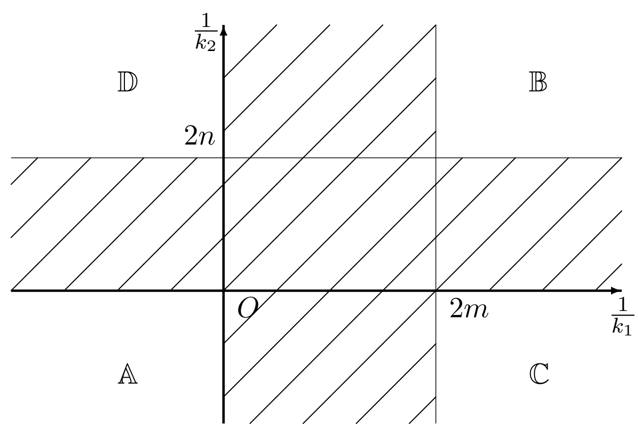

Figure 1. Critical Fujita exponents for (1.1)–(1.4)

Region $\mathbb{A}$: Global solutions
Shadow region: Non-global solutions for all nontrivial initial data
Regions $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$: Coexistence of global and non-global solutions

Remark 2. Theorem 3 says that the critical situation of $1/k_1 = 2m$ or $1/k_2 = 2n$ belongs to the blow-up case under any nontrivial initial data, just as in the scalar problem (1.5)–(1.7) [5].

Remark 3. The scalar problem (1.5)–(1.7) is deduced by letting $m = n$, $p = q$, $\alpha = \beta = 0$ and $u_0 = v_0$ in (1.1)–(1.4). Since $1/k_1 = 1/k_2 = 0$ is equivalent to $p = \frac{2m}{m+1}$ for the special case, we know that the critical Fujita exponents stated via Theorems 2–5 for (1.1)–(1.4) do agree with that for the scalar problem (1.5)–(1.7) [5]. In addition, it is easy to find that the blow-up rate for the semilinear case established in [17] can be obtained by taking $m = n = 1$ in Theorem 6.

2. Critical Fujita exponents

In this section, we will establish the critical Fujita exponents for the problem (1.1)–(1.4). By the theory of non-Newtonian filtration equations (see, e.g., [2, 8, 18]), we can establish the existence and uniqueness of local nonnegative solutions as well as the comparison principle to the problem (1.1)–(1.4) with continuous initial data u_0, v_0 with compact supports. At first consider the monotonicity of the solution (u, v) with respect to time t.

Lemma 2.1. (i) *Let* (u, v) *be a solution of* (1.1)–(1.4). *Then*

$$\frac{d}{dt}\int_0^{+\infty} u dx,\ \frac{d}{dt}\int_0^{+\infty} v dx \geq 0 \quad \text{for } t \in [0, T).$$

(ii) *If*

$$(|u_{0x}|^{m-1}u_{0x})_x,\ (|v_{0x}|^{n-1}v_{0x})_x \geq 0,$$

then $u_t, v_t \geq 0$ *for* $(x, t) \in \mathbb{R}^+ \times [0, T)$.

Proof. (i) It follows from (1.1), (1.3) that, e.g.,

$$\frac{d}{dt}\int_0^{+\infty} v dx = \int_0^{+\infty} (|v_x|^{n-1}v_x)_x dx = -(|v_x|^{n-1}v_x)(0, t) = (u^q v^\beta)(0, t) \geq 0.$$

(ii) Just as in the proof of Proposition 3.1 in [5], denote $w = u_t, z = v_t$, and then get from (1.1)–(1.4) that

$$\begin{aligned}
&w_t = (m|u_x|^{m-1}w_x)_x,\ z_t = (n|v_x|^{n-1}z_x)_x, \quad (x, t) \in \mathbb{R}^+ \times (0, T),\\
&-(m|u_x|^{m-1}w_x)(0, t) = \alpha(u^{\alpha-1}v^p w)(0, t) + p(v^{p-1}u^\alpha z)(0, t),\ t \in (0, T),\\
&-(n|v_x|^{n-1}z_x)(0, t) = q(u^{q-1}v^\beta w)(0, t) + \beta(u^q v^{\beta-1}z)(0, t),\ t \in (0, T),\\
&w(x, 0) = (|u_{0x}|^{m-1}u_{0x})_x,\ z(x, 0) = (|v_{0x}|^{n-1}v_{0x})_x, \quad x \in \mathbb{R}^+.
\end{aligned}$$

By a standard regularization procedure with the comparison principle, we can get that $w, z \geq 0$ for $(x, t) \in \mathbb{R}^+ \times (0, T)$. Refer to [5, 8] for the details. □

Lemma 2.2. *The assumption* $1/k_1, 1/k_2 < 0$ *is equivalent to* $\alpha < \frac{2m}{m+1}$, $\beta < \frac{2n}{n+1}$ *with* $pq < (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$.

Proof. It is trivial that $\alpha < \frac{2m}{m+1}$, $\beta < \frac{2n}{n+1}$ with $pq < (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$ imply $1/k_1, 1/k_2 < 0$.

On the other hand, by (1.12), the assumption $1/k_1, 1/k_2 < 0$ requires that $(\lambda_0 p + \frac{2n}{n+1} - \beta)(\lambda_0^{-1} q + \frac{2m}{m+1} - \alpha) > 0$.

If $\lambda_0 p < \frac{2n}{n+1} - \beta$ and $\lambda_0^{-1} q < \frac{2m}{m+1} - \alpha$, then $pq < (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$, and so $1/k_1, 1/k_2 > 0$, a contradiction.

It has to hold that $\lambda_0 p > \frac{2n}{n+1} - \beta$ and $\lambda_0^{-1} q > \frac{2m}{m+1} - \alpha$ with $pq < (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$ due to $k_1, k_2 < 0$. However, we would have $pq > (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$ if $\alpha \geq \frac{2m}{m+1}$ or $\beta \geq \frac{2n}{n+1}$. We conclude that $\alpha < \frac{2m}{m+1}$, $\beta < \frac{2n}{n+1}$ with $pq < (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$. □

By Lemma 2.2, the conclusion of Theorem 1 will be covered by the following lemma.

Lemma 2.3. *If $\alpha < \frac{2m}{m+1}$, $\beta < \frac{2n}{n+1}$ and $pq \leq (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$, then the solutions of* (1.1)–(1.4) *are global.*

Proof. Construct

$$\bar{u}(x,t) = \mathrm{e}^{kt}\Big(M + \mathrm{e}^{-L_1 x \mathrm{e}^{\frac{k(1-m)t}{m+1}}}\Big), \quad \bar{v}(x,t) = \mathrm{e}^{lt}\Big(M + \mathrm{e}^{-L_2 x \mathrm{e}^{\frac{l(1-n)t}{n+1}}}\Big)$$

with

$$\begin{aligned} l &= \frac{[2m - \alpha(m+1)]k}{p(m+1)}, \\ M &= \max(\|u_0\|_\infty, \|v_0\|_\infty, 1), \quad L_1 = (M+1)^{\frac{\alpha+p}{m}}, \\ L_2 &= (M+1)^{\frac{q+\beta}{n}}, \quad k = \max\Big(\frac{mL_1^{m+1}}{M}, \frac{p(m+1)L_2^{n+1}}{M[2m - \alpha(m+1)]}\Big). \end{aligned}$$

A simple computation shows

$$\begin{aligned} \bar{u}_t &= k\mathrm{e}^{kt}\Big(M + \mathrm{e}^{-L_1 x \mathrm{e}^{\frac{k(1-m)t}{m+1}}}\Big) + \frac{kL_1(m-1)x}{m+1}\mathrm{e}^{\frac{2kt}{m+1} - L_1 x \mathrm{e}^{\frac{k(1-m)t}{m+1}}} \geq kM\mathrm{e}^{kt}, \\ |\bar{u}_x|^{m-1}\bar{u}_x &= -L_1^m \mathrm{e}^{\frac{2mkt}{m+1} - L_1 m x \mathrm{e}^{\frac{k(1-m)t}{m+1}}}, \\ (|\bar{u}_x|^{m-1}\bar{u}_x)_x &= L_1^{m+1} m \mathrm{e}^{kt - L_1 m x \mathrm{e}^{\frac{k(1-m)t}{m+1}}} \leq L_1^{m+1} m \mathrm{e}^{kt}, \end{aligned}$$

and similarly

$$\bar{v}_t \geq lM\mathrm{e}^{lt}, \quad (|\bar{v}_x|^{n-1}\bar{v}_x)_x \leq L_2^{n+1} n \mathrm{e}^{lt}$$

in $\mathbb{R}^+ \times \mathbb{R}^+$. Moreover, we have on the boundary that

$$-(|\bar{u}_x|^{m-1}\bar{u}_x)(0,t) = L_1^m \mathrm{e}^{\frac{2mkt}{m+1}}, \quad -(|\bar{v}_x|^{n-1}\bar{v}_x)(0,t) = L_2^n \mathrm{e}^{\frac{2nlt}{n+1}},$$

$$(\bar{u}^\alpha\bar{v}^p)(0,t)=\mathrm{e}^{(k\alpha+lp)t}(M+1)^{\alpha+p},\ (\bar{u}^q\bar{v}^\beta)(0,t)=\mathrm{e}^{(kq+l\beta)t}(M+1)^{q+\beta}.$$

By the definitions of k, M, L_1, L_2 and the assumption $pq\le(\alpha-\frac{2m}{m+1})(\beta-\frac{2n}{n+1})$, we know that

$$\begin{aligned}&\bar{u}_t\ge(|\bar{u}_x|^{m-1}\bar{u}_x)_x,\ \bar{v}_t\ge(|\bar{v}_x|^{n-1}\bar{v}_x)_x,\quad (x,t)\in\mathbb{R}^+\times\mathbb{R}^+,\\ &-(|\bar{u}_x|^{m-1}\bar{u}_x)(0,t)\ge(\bar{u}^\alpha\bar{v}^p)(0,t),-(|\bar{v}_x|^{n-1}\bar{v}_x)(0,t)\ge(\bar{u}^q\bar{v}^\beta)(0,t),\ t\in\mathbb{R}^+,\\ &\bar{u}(x,0)\ge u_0(x),\ \bar{v}(x,0)\ge v_0(x),\quad x\in\mathbb{R}^+,\end{aligned}$$

namely, $(\bar{u},\bar{v})$ is a global super-solution of (1.1)–(1.4). □

We need a series of lemmas to prove the blow-up theorems of Fujita type. Introduce parameters l_1, l_2 defined as

$$l_1=\frac{1-k_1(m-1)}{m+1},\quad l_2=\frac{1-k_2(n-1)}{n+1}. \tag{2.1}$$

It is easy to see from Lemma 2.2 that the assumption $1/k_1$ or $1/k_2>0$ comes from the following three cases only:

(a) $pq>(\alpha-\frac{2m}{m+1})(\beta-\frac{2n}{n+1})$ with $\alpha\le\frac{2m}{m+1}$, $\beta\le\frac{2n}{n+1}$;
(b) $\alpha>\frac{2m}{m+1}$, $\beta\le\frac{2n}{n+1}$ (or $\alpha\le\frac{2m}{m+1}$, $\beta>\frac{2n}{n+1}$);
(c) $\alpha>\frac{2m}{m+1}$, $\beta>\frac{2n}{n+1}$.

Lemma 2.4. *Under one of the conditions* **(a)**, **(b)** *and* **(c)**, *the solutions of* (1.1)–(1.4) *will blow up in finite time with large initial data* u_0, v_0.

Proof. Construct

$$\underline{u}(x,t)=(T-t)^{-k_1}f_1(\xi),\ \xi=x(T-t)^{-l_1}, \tag{2.2}$$

$$\underline{v}(x,t)=(T-t)^{-k_2}f_2(\eta),\ \eta=x(T-t)^{-l_2}, \tag{2.3}$$

where T is a positive constant, f_1, f_2 are compactly supported functions to be determined.

It is easy to know from (1.11), (1.12) and (2.1) that

$$k_1+1=k_1m+(m+1)l_1,\ k_2+1=k_2n+(n+1)l_2, \tag{2.4}$$

$$(k_1+l_1)m=k_1\alpha+k_2p,\ (k_2+l_2)n=k_1q+k_2\beta. \tag{2.5}$$

A direct computation shows

$$\begin{aligned}&\underline{u}_t=(T-t)^{-(k_1+1)}[k_1f_1(\xi)+l_1\xi f_1'(\xi)],\\ &\underline{v}_t=(T-t)^{-(k_2+1)}[k_2f_2(\eta)+l_2\eta f_2'(\eta)],\\ &(|\underline{u}_x|^{m-1}\underline{u}_x)_x=(T-t)^{-[k_1m+(m+1)l_1]}(|f_1'|^{m-1}f_1')'(\xi),\end{aligned}$$

$$(|\underline{v}_x|^{n-1}\underline{v}_x)_x = (T-t)^{-[k_2n+(n+1)l_2]}(|f_2'|^{n-1}f_2')'(\eta)$$

for $(x,t)\in\mathbb{R}^+\times(0,T)$, and

$$\begin{aligned}
&(|\underline{u}_x|^{m-1}\underline{u}_x)(0,t) = (T-t)^{-(k_1+l_1)m}|f_1'(0)|^{m-1}f_1'(0),\\
&(|\underline{v}_x|^{n-1}\underline{v}_x)(0,t) = (T-t)^{-(k_2+l_2)n}|f_2'(0)|^{n-1}f_2'(0),\\
&(\underline{u}^\alpha\underline{v}^p)(0,t) = (T-t)^{-(k_1\alpha+k_2p)}f_1^\alpha(0)f_2^p(0),\\
&(\underline{u}^q\underline{v}^\beta)(0,t) = (T-t)^{-(k_1q+k_2\beta)}f_1^q(0)f_2^\beta(0)
\end{aligned}$$

for $t\in(0,T)$. Thus, if f_1 and f_2 satisfy

$$\begin{aligned}
&(|f_1'|^{m-1}f_1')'(\xi) \ge k_1f_1(\xi)+l_1\xi f_1'(\xi),\\
&(|f_2'|^{n-1}f_2')'(\eta) \ge k_2f_2(\eta)+l_2\eta f_2'(\eta), \qquad (2.6)\\
&-|f_1'(0)|^{m-1}f_1'(0) \le f_1^\alpha(0)f_2^p(0),\\
&-|f_2'(0)|^{n-1}f_2'(0) \le f_1^q(0)f_2^\beta(0), \qquad (2.7)
\end{aligned}$$

we will obtain with (2.4) and (2.5) that

$$\begin{aligned}
&\underline{u}_t \le (|\underline{u}_x|^{m-1}\underline{u}_x)_x,\ \underline{v}_t \le (|\underline{v}_x|^{n-1}\underline{v}_x)_x,\ (x,t)\in\mathbb{R}^+\times(0,T),\\
&-(|\underline{u}_x|^{m-1}\underline{u}_x)(0,t) \le (\underline{u}^\alpha\underline{v}^p)(0,t), \quad t\in(0,T),\\
&-(|\underline{v}_x|^{n-1}\underline{v}_x)(0,t) \le (\underline{u}^q\underline{v}^\beta)(0,t), \quad t\in(0,T).
\end{aligned}$$

Set

$$f_1 = A_1(c_1-\xi)_+^{\frac{m}{m-1}}, \quad f_2 = A_2(c_2-\eta)_+^{\frac{n}{n-1}}, \qquad (2.8)$$

where

$$\begin{aligned}
c_1 &= \frac{1}{k_1}\Big(\frac{m}{m-1}\Big)^{m+1}A_1^{m-1},\ c_2 = \frac{1}{k_2}\Big(\frac{n}{n-1}\Big)^{n+1}A_2^{n-1} \quad \text{if } k_1>0, k_2>0;\\
c_1 &= \frac{1}{k_1}\Big(\frac{m}{m-1}\Big)^{m+1}A_1^{m-1},\ c_2 = \frac{1}{k_1}\Big(\frac{n}{n-1}\Big)^{n+1}A_2^{n-1} \quad \text{if } k_1>0, k_2\le 0;\\
c_1 &= \frac{1}{k_2}\Big(\frac{m}{m-1}\Big)^{m+1}A_1^{m-1},\ c_2 = \frac{1}{k_2}\Big(\frac{n}{n-1}\Big)^{n+1}A_2^{n-1} \quad \text{if } k_1\le 0, k_2>0.
\end{aligned}$$

It is easy to check that (2.6) is satisfied, since

$$\begin{aligned}
&(|f_1'|^{m-1}f_1')'(\xi) - k_1f_1(\xi) - l_1\xi f_1'(\xi)\\
&= (c_1-\xi)_+^{\frac{1}{m-1}}\Big[A_1^m\big(\frac{m}{m-1}\big)^{m+1} - k_1A_1(c_1-\xi)_+ + \frac{ml_1\xi}{m-1}A_1\Big] \ge 0,
\end{aligned}$$

and similarly

$$(|f_2'|^{n-1}f_2')'(\eta) - k_2f_2(\eta) - l_2\eta f_2'(\eta) \ge 0.$$

The condition **(a)** implies $k_1, k_2 > 0$, and $\frac{p}{\frac{2m}{m+1}-\alpha} > \frac{\frac{2n}{n+1}-\beta}{q}$, or equivalently, $\frac{(n+1)p}{2m-(m+1)\alpha} > \frac{2n-(n+1)\beta}{(m+1)q}$. Thus, for any $\lambda_1, \lambda_2 > 0$, there exist $A_1, A_2 >> 1$ such that

$$\lambda_1 A_2^{\frac{2n-(n+1)\beta}{(m+1)q}} < A_1 < \lambda_2 A_2^{\frac{(n+1)p}{2m-(m+1)\alpha}},$$

and hence

$$\begin{aligned}
&\left(\frac{m}{m-1}\right)^{\frac{m[2m-(m+1)\alpha]}{m-1}} \left(\frac{1}{k_1}\right)^{\frac{m(1-\alpha)}{m-1}} A_1^{2m-(m+1)\alpha} \\
&\qquad\qquad < A_2^{(n+1)p} \left(\frac{1}{k_2}\right)^{\frac{np}{n-1}} \left(\frac{n}{n-1}\right)^{\frac{np(n+1)}{n-1}}, \\
&\left(\frac{n}{n-1}\right)^{\frac{n[2n-(n+1)\beta]}{n-1}} \left(\frac{1}{k_2}\right)^{\frac{n(1-\beta)}{n-1}} A_2^{2n-(n+1)\beta} \\
&\qquad\qquad < A_1^{(m+1)q} \left(\frac{1}{k_1}\right)^{\frac{mq}{m-1}} \left(\frac{m}{m-1}\right)^{\frac{mq(m+1)}{m-1}},
\end{aligned}$$

which fix (2.7).

While under the condition **(b)** $\alpha > \frac{2m}{m+1}$, $\beta \le \frac{2n}{n+1}$ (or $\alpha \le \frac{2m}{m+1}$, $\beta > \frac{2n}{n+1}$), or **(c)** $\alpha > \frac{2m}{m+1}$, $\beta > \frac{2n}{n+1}$, we have trivially for any positive constants μ_i $(i = 1, \dots, 4)$

$$\mu_1 A_1^{2m-(m+1)\alpha} < \mu_2 A_2^{(n+1)p}, \quad \mu_3 A_2^{2n-(n+1)\beta} < \mu_4 A_1^{(m+1)q}$$

by suitable choosing of $A_1, A_2 >> 1$. This yields (2.7) also.

From (2.2)–(2.3) and (2.8), we can take u_0, v_0 large such that $\underline{u}_0 \le u_0$, $\underline{v}_0 \le v_0$ in $\mathbb{R}^+$. Thus, the constructed non-global $(\underline{u}, \underline{v})$ is a sub-solution of (1.1)–(1.4). □

Lemma 2.5. *If $k_1 > 1/2m$ or $k_2 > 1/2n$ holds with one of conditions* **(a)**, **(b)** *and* **(c)**, *then every nonnegative, nontrivial solution of* (1.1)–(1.4) *blows up in a finite time.*

Proof. In the spirit of [5], construct

$$\begin{aligned}
&u_B(x,t) = (\tau+t)^{-\frac{1}{2m}} h_1(\xi), \quad \xi = x(\tau+t)^{-\frac{1}{2m}}, \\
&v_B(x,t) = (\tau+t)^{-\frac{1}{2n}} h_2(\eta), \quad \eta = x(\tau+t)^{-\frac{1}{2n}}, \\
&h_1(\xi) = D_m\left(d^{\frac{m+1}{m}} - \xi^{\frac{m+1}{m}}\right)_+^{\frac{m}{m-1}}, \ h_2(\eta) = D_n\left(d^{\frac{n+1}{n}} - \eta^{\frac{n+1}{n}}\right)_+^{\frac{n}{n-1}},
\end{aligned}$$

with

$$D_m=\Big[\frac{1}{2m}\big(\frac{m-1}{m+1}\big)^m\Big]^{\frac{1}{m-1}},\ D_n=\Big[\frac{1}{2n}\big(\frac{n-1}{n+1}\big)^n\Big]^{\frac{1}{n-1}}.$$

It is easy to check that

$$(|h_1'|^{m-1}h_1')'(\xi)=-\frac{1}{2m}\xi h_1'(\xi)-\frac{1}{2m}h_1(\xi),$$
$$(|h_2'|^{n-1}h_2')'(\eta)=-\frac{1}{2n}\eta h_2'(\eta)-\frac{1}{2n}h_2(\eta).$$

Moreover, $h_1'(0)=h_2'(0)=0$, which shows that the self-similar solution (u_B,v_B) satisfies $(u_B)_x(0,t)=(v_B)_x(0,t)=0$.

Since $u(0,t_0),v(0,t_0)>0$ for some $t_0\ge 0$ and $u(x,t_0),v(x,t_0)$ are continuous, there exist $\tau>0$ large and $d>0$ small such that

$$u(x,t_0)\ge u_B(x,t_0),\ v(x,t_0)\ge v_B(x,t_0)\quad \text{for } x\in\mathbb{R}^+.$$

We have shown that the self-similar (u_B,v_B) is a sub-solution to (1.1)–(1.4) in $\mathbb{R}^+\times(t_0,T)$, and hence

$$u(x,t)\ge u_B(x,t),\ v(x,t)\ge v_B(x,t)\quad \text{for } (x,t)\in\mathbb{R}^+\times(t_0,T).$$

Without loss of generality, assume $k_1>1/2m$, which implies $l_1<k_1$. Then $T^{l_1}<<T^{k_1}$ for $T>>1$. So, there exists $t^*\ge t_0$ such that

$$T^{l_1}<<(\tau+t^*)^{\frac{1}{2m}}<<T^{k_1}. \tag{2.9}$$

For $\underline{u}$ defined by (2.2) and (2.8), the inequality (2.9) implies

$$\underline{u}(x,0)\le u_B(x,t^*)\quad \text{for } x\in\mathbb{R}^+.$$

This guarantees that every nonnegative, nontrivial solution of (1.1)–(1.4) blows up in a finite time if $k_1>1/2m$ or $k_2>1/2n$. □

Lemma 2.6. *Assume $k_1<1/2m$ and $k_2<1/2n$ with one of the conditions* **(a)**, **(b)** *and* **(c)**. *Then the solutions of* (1.1)–(1.4) *are global provided the initial data u_0,v_0 are small enough.*

Proof. Construct

$$\bar u(x,t)=(\tau+t)^{-k_1}f(\xi),\quad \xi=x(\tau+t)^{-l_1}, \tag{2.10}$$
$$\bar v(x,t)=(\tau+t)^{-k_2}g(\eta),\quad \eta=x(\tau+t)^{-l_2}, \tag{2.11}$$

where

$$f(\xi)=B_1h_1(\xi+b_1),\quad g(\eta)=B_2h_2(\eta+b_2),\quad b_i\in(0,d_i) \tag{2.12}$$

with

$$h_1(\xi+b_1)=D_m\Big(d_1^{\frac{m+1}{m}}-(\xi+b_1)^{\frac{m+1}{m}}\Big)_+^{\frac{m}{m-1}}, \tag{2.13}$$

$$h_2(\eta+b_2)=D_n\Big(d_2^{\frac{n+1}{n}}-(\eta+b_2)^{\frac{n+1}{n}}\Big)_+^{\frac{n}{n-1}}. \tag{2.14}$$

We will show that $f(\xi)$, $g(\eta)$ satisfy

$$(|f'|^{m-1}f')'+l_1\xi f'+k_1 f\le 0,\ \ (|g'|^{n-1}g')'+l_2\eta g'+k_2 g\le 0, \tag{2.15}$$

$$-|f'(0)|^{m-1}f'(0)\ge f^{\alpha}(0)g^{p}(0),\ -|g'(0)|^{n-1}g'(0)\ge f^{q}(0)g^{\beta}(0) \tag{2.16}$$

with suitable constants B_i, d_i $(i=1,2)$.

It is easy to check that $h_1(\xi+b_1), h_2(\eta+b_2)$ satisfy

$$(|h_1'|^{m-1}h_1')'(\xi+b_1)=-\frac{1}{2m}(\xi+b_1)h_1'(\xi+b_1)-\frac{1}{2m}h_1(\xi+b_1), \tag{2.17}$$

$$(|h_2'|^{n-1}h_2')'(\eta+b_2)=-\frac{1}{2n}(\eta+b_2)h_2'(\eta+b_2)-\frac{1}{2n}h_2(\eta+b_2). \tag{2.18}$$

By (2.12), the inequalities in (2.15) become

$$-h_1'(\xi+b_1)\Big[\frac{B_1^{m-1}(\xi+b_1)}{2m}-l_1\xi\Big]+h_1(\xi+b_1)\Big[k_1-\frac{B_1^{m-1}}{2m}\Big]\le 0,$$

$$-h_2'(\eta+b_2)\Big[\frac{B_2^{n-1}(\eta+b_2)}{2n}-l_2\eta\Big]+h_2(\eta+b_2)\Big[k_2-\frac{B_2^{n-1}}{2n}\Big]\le 0.$$

Therefore, by the definitions (2.13)–(2.14) and the relation of l_i, k_i $(i=1,2)$ in (2.1), we get with the notation $z_1=\xi+b_1, z_2=\eta+b_2$ that (2.15) is equivalent to

$$-z_1^{\frac{m+1}{m}}\Big(\frac{1}{m-1}-\frac{B_1^{m-1}}{m-1}\Big)+\frac{1-k_1(m-1)}{m-1}b_1 z_1^{\frac{1}{m}}-\Big(\frac{B_1^{m-1}}{2m}-k_1\Big)d_1^{\frac{m+1}{m}}\le 0, \tag{2.19}$$

$$-z_2^{\frac{n+1}{n}}\Big(\frac{1}{n-1}-\frac{B_2^{n-1}}{n-1}\Big)+\frac{1-k_2(n-1)}{n-1}b_2 z_2^{\frac{1}{n}}-\Big(\frac{B_2^{n-1}}{2n}-k_2\Big)d_2^{\frac{n+1}{n}}\le 0. \tag{2.20}$$

Since $k_1<1/2m$ and $k_2<1/2n$, choose positive constants B_1, B_2 such that

$$2mk_1<B_1^{m-1}<1,\quad 2nk_2<B_2^{n-1}<1.$$

Denote $d_i=\alpha_i b_i$, $\alpha_i>1$ $(i=1,2)$, and

$$r_1=\frac{1}{m-1}-\frac{B_1^{m-1}}{m-1}>0,\quad s_1=\frac{1}{n-1}-\frac{B_2^{n-1}}{n-1}>0,$$

$$r_2 = \frac{1-k_1(m-1)}{m-1} > 0, \quad s_2 = \frac{1-k_2(n-1)}{n-1} > 0,$$
$$r_3 = \frac{B_1^{m-1}}{2m} - k_1 > 0, \quad s_3 = \frac{B_2^{n-1}}{2n} - k_2 > 0.$$

Then (2.19), (2.20) become

$$R(z_1) = -r_1 z_1^{\frac{m+1}{m}} + r_2 b_1 z_1^{\frac{1}{m}} - r_3(\alpha_1 b_1)^{\frac{m+1}{m}} \le 0, \tag{2.21}$$
$$H(z_2) = -s_1 z_2^{\frac{n+1}{n}} + s_2 b_2 z_2^{\frac{1}{n}} - s_3(\alpha_2 b_2)^{\frac{n+1}{n}} \le 0. \tag{2.22}$$

Clearly, $R(z_1)$ attains its maximum at $z_1^* = \frac{b_1 r_2}{(1+m)r_1}$, hence

$$R(z_1) \le R(z_1^*) \le 0$$

provided that

$$\alpha_1^{\frac{m+1}{m}} \ge \frac{m}{m+1}\frac{r_2}{r_3}\Big[\frac{r_2}{(m+1)r_1}\Big]^{\frac{1}{m}} \equiv \bar{\alpha}_1^{\frac{m+1}{m}}.$$

Thus, (2.21) holds if we take $\alpha_1 > \max\{1, \bar{\alpha}_1\}$. By a similar argument, (2.22) holds if we take $\alpha_2 > \max\{1, \bar{\alpha}_2\}$ with $\bar{\alpha}_2^{\frac{n+1}{n}} = \frac{n}{n+1}\frac{s_2}{s_3}\big[\frac{s_2}{(n+1)s_1}\big]^{\frac{1}{n}}$. We have proved (2.15).

Now let us check (2.16). For the case **(a)** $pq > (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$ with $\alpha \le \frac{2m}{m+1}$, $\beta \le \frac{2n}{n+1}$, we have $\frac{p(n+1)}{2m-\alpha(m+1)} > \frac{2n-\beta(n+1)}{q(m+1)}$. So, for any positive constants λ_1, λ_2, there exist positive constants b_1, b_2 small enough such that

$$\lambda_1 b_2^{\frac{p(n+1)}{2m-\alpha(m+1)}} < b_1^{\frac{n-1}{m-1}} < \lambda_2 b_2^{\frac{2n-\beta(n+1)}{q(m+1)}},$$

and therefore it is true that

$$(B_1 D_m)^{m-\alpha}\Big(\frac{m+1}{m-1}\Big)^m b_1^{\frac{2m-\alpha(m+1)}{m-1}}\Big[\alpha_1^{\frac{m+1}{m}} - 1\Big]^{\frac{m(1-\alpha)}{m-1}}$$
$$\ge B_2^p D_n^p b_2^{\frac{p(n+1)}{n-1}}\Big(\alpha_2^{\frac{n+1}{n}} - 1\Big)^{\frac{np}{n-1}},$$
$$(B_2 D_n)^{n-\beta}\Big(\frac{n+1}{n-1}\Big)^n b_2^{\frac{2n-\beta(n+1)}{n-1}}\Big[\alpha_2^{\frac{n+1}{n}} - 1\Big]^{\frac{n(1-\beta)}{n-1}}$$
$$\ge B_1^q D_m^q b_1^{\frac{q(m+1)}{m-1}}\Big(\alpha_1^{\frac{m+1}{m}} - 1\Big)^{\frac{mq}{m-1}}.$$

For either cases **(b)** or **(c)**, where $\alpha > \frac{2m}{m+1}$, $\beta \le \frac{2n}{n+1}$ (or $\alpha \le \frac{2m}{m+1}$, $\beta > \frac{2n}{n+1}$), or $\alpha > \frac{2m}{m+1}$, $\beta > \frac{2n}{n+1}$, it is trivially true that

$$\mu_1 b_1^{\frac{2m-\alpha(m+1)}{m-1}} > \mu_2 b_2^{\frac{p(n+1)}{n-1}}, \quad \mu_3 b_2^{\frac{2n-\beta(n+1)}{n-1}} > \mu_4 b_1^{\frac{q(m+1)}{m-1}}$$

hold with any positive constants μ_i $(i = 1, \dots, 4)$ by suitable choosing of positive constants $b_1, b_2 << 1$.

In summary, (2.16) holds for all cases **(a)**, **(b)** and **(c)**.

Due to (2.15), (2.16), we conclude

$$\bar u_t \ge (|\bar u_x|^{m-1}\bar u_x)_x,\ \bar v_t \ge (|\bar v_x|^{n-1}\bar v_x)_x \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+,$$
$$-(|\bar u_x|^{m-1}\bar u_x)(0,t) \ge (\bar u^\alpha \bar v^p)(0,t),\ -(|\bar v_x|^{n-1}\bar v_x)(0,t) \ge (\bar u^q \bar v^\beta)(0,t) \text{ in } \mathbb{R}^+.$$

Moreover, from (2.10)–(2.14), we can take u_0, v_0 small such that $\bar u_0 \ge u_0, \bar v_0 \ge v_0$ for $x \in \mathbb{R}^+$. Thus, $(\bar u, \bar v)$ is a global super-solution of (1.1)–(1.4). □

Now, we can easily obtain Theorem 2 from Lemma 2.5. Theorem 4 follows by combining Lemmas 2.4 and 2.6.

Proof of Theorem 3. Motivated by [5, 15, 20], assume by contradiction that there exists a nonnegative, nontrivial global solution (u, v) to the problem (1.1)–(1.4). Since $u_0(0), v_0(0) > 0$, we have

$$u_0(x) = \varphi(\xi, 0) \ge h_1(\xi + b_1) = D_m\Big(d_1^{\frac{m+1}{m}} - (\xi + b_1)^{\frac{m+1}{m}}\Big)_+^{\frac{m}{m-1}} = \underline{\varphi}_0(\xi),$$
$$v_0(x) = \psi(\eta, 0) \ge h_2(\eta + b_2) = D_n\Big(d_2^{\frac{n+1}{n}} - (\eta + b_2)^{\frac{n+1}{n}}\Big)_+^{\frac{n}{n-1}} = \underline{\psi}_0(\eta)$$

in $\mathbb{R}^+$ provided $d_i > 0$ is sufficiently small, and hence $b_i \in (0, d_i)$ is sufficiently small, $i = 1, 2$, where

$$d_1^{\frac{m+1}{m}} = b_1^{\frac{m+1}{m}} + \Big[D_m^{(m-\alpha)(1-\beta)-pq}\Big(\frac{m+1}{m-1}\Big)^{m(1-\beta)} D_n^{p(n-1)}\Big(\frac{n+1}{n-1}\Big)^{np} b_1^{1-\beta} b_2^p\Big]^{\nu_1},$$
$$d_2^{\frac{n+1}{n}} = b_2^{\frac{n+1}{n}} + \Big[D_n^{(n-\beta)(1-\alpha)-pq}\Big(\frac{n+1}{n-1}\Big)^{n(1-\alpha)} D_m^{q(m-1)}\Big(\frac{m+1}{m-1}\Big)^{mq} b_1^q b_2^{1-\alpha}\Big]^{\nu_2}$$

with $\nu_1 = \frac{m-1}{m[pq-(1-\alpha)(1-\beta)]}$, $\nu_2 = \frac{n-1}{n[pq-(1-\alpha)(1-\beta)]}$. Then, the functions

$$\varphi(\xi, \tau) = (1+t)^{k_1} u(\xi(1+t)^{l_1}, t), \quad \psi(\eta, \tau) = (1+t)^{k_2} v(\eta(1+t)^{l_2}, t)$$

with the new time $\tau = \log(1+t)$ should be a global solution to the problem

$$\varphi_\tau = (|\varphi_\xi|^{m-1}\varphi_\xi)_\xi + l_1\xi\varphi_\xi + k_1\varphi,\ \psi_\tau = (|\psi_\eta|^{n-1}\psi_\eta)_\eta + l_2\eta\psi_\eta + k_2\psi, \tag{2.23}$$

$$-(|\varphi_\xi|^{m-1}\varphi_\xi)(0,\tau)=(\varphi^\alpha\psi^p)(0,\tau),\ -(|\psi_\eta|^{n-1}\psi_\eta)(0,\tau)=(\varphi^q\psi^\beta)(0,\tau), \tag{2.24}$$

$$\varphi(\xi,0)=u_0(\xi),\quad \psi(\eta,0)=v_0(\eta). \tag{2.25}$$

We will find a solution $(\underline{\varphi},\underline{\psi})$ to (2.23)–(2.24) with initial data $\underline{\varphi}_0(\xi)\le u_0(\xi),\underline{\psi}_0(\eta)\le v_0(\eta)$. Notice that $(\underline{\varphi},\underline{\psi})$, being a pair of sub-solutions to system (2.23)–(2.25), should be global also.

Without loss of generality, assume $1/k_1=2m$ with $1/k_2>0$, which implies $l_1=k_1=1/2m$ by (2.1). According to (2.17),

$$\begin{aligned}&\big(|\underline{\varphi}_{0\xi}|^{m-1}\underline{\varphi}_{0\xi}\big)_\xi+l_1\xi\underline{\varphi}_{0\xi}+k_1\underline{\varphi}_0\\&=-\frac{1}{2m}b_1h_{1\xi}(\xi+b_1)+\Big(l_1-\frac{1}{2m}\Big)\xi h_{1\xi}(\xi+b_1)+\Big(k_1-\frac{1}{2m}\Big)h_1(\xi+b_1)\\&=-\frac{1}{2m}b_1h_{1\xi}(\xi+b_1)\ge 0\end{aligned}$$

since $h_{1\xi}\le 0$. Moreover, it is easy to check that the compatibility condition on the boundary is satisfied:

$$-|\underline{\varphi}_{0\xi}(0)|^{m-1}\underline{\varphi}_{0\xi}(0)=\underline{\varphi}_0^\alpha(0)\underline{\psi}_0^p(0),\ -|\underline{\psi}_{0\eta}(0)|^{n-1}\underline{\psi}_{0\eta}(0)=\underline{\varphi}_0^q(0)\underline{\psi}_0^\beta(0).$$

Since $|\underline{\varphi}_\xi|^{m-1}\underline{\varphi}_\xi\le 0$ on the boundary and $\underline{\varphi}_{0\xi}\le 0$, we know that $\underline{\varphi}(\xi,\tau)$ is non-increasing in ξ. In addition, we know $\underline{\varphi}_\tau\ge 0$ from the proof of Lemma 2.1.

Next, we claim that

$$+\infty>\lim_{\tau\to+\infty}\underline{\varphi}(\xi,\tau)=\Phi(\xi)\not\equiv 0$$

for any $\xi>0$. Otherwise,

$$\lim_{\tau\to+\infty}\underline{\varphi}(\xi,\tau)=+\infty$$

uniformly on $[0,\xi_0]$ since $\underline{\varphi}$ is non-increasing in ξ. Thus, for any $G>0$, there is a positive τ_0 such that $\underline{\varphi}(\xi,\tau_0)>G$ on $[0,\xi_0]$. In other words, at time $t_0=\mathrm{e}^{\tau_0}-1$, the profile $\underline{\varphi}(\xi,\tau)$ in the original variables satisfies $u(x,t_0)\ge(1+t_0)^{-k_1}G$ for $x\in[0,\xi_0(1+t_0)^{l_1}]$. Let $\underline{u}(x,t)$ be defined by (2.2). Observing $k_1=l_1$, we know that

$$G_1^{-1}\Big(\frac{1}{k_1}\Big)^{\frac{m}{m-1}}\Big(\frac{m}{m-1}\Big)^{\frac{m(m+1)}{m-1}}A_1^{m+1}\le T^{k_1}=T^{l_1}\le x_1\Big[\frac{1}{k_1}\Big(\frac{m}{m-1}\Big)^{m+1}A_1^{m-1}\Big]^{-1},$$

where $G_1 = (1+t_0)^{-k_1}G$, $x_1 = \xi_0(1+t_0)^{l_1}$ for suitable T provided $G > 0$ is large enough, thus

$$u(x,t_0) \geq \underline{u}(x,0) \qquad \text{for } x \in \mathbb{R}^+.$$

This implies that $u(x,t)$ will blow up in a finite time. However, u was assumed to be global. This contradiction shows that the function $\Phi(\xi)$ is well defined. In view of the regularity for the non-Newtonian filtration equation [18], by using the standard arguments [4, 5], we can pass to the limit in (2.23) and (2.24) respectively to get

$$(|\Phi_\xi|^{m-1}\Phi_\xi)_\xi + l_1\xi\Phi_\xi + k_1\Phi = 0, \tag{2.26}$$

$$-|\Phi_\xi(0)|^{m-1}\Phi_\xi(0) = \Phi^\alpha(0)\underline{\Psi}^p(0) \neq 0, \tag{2.27}$$

where $\underline{\Psi}(0) = \inf\lim_{\tau\to+\infty}\underline{\psi}(0,\tau) \neq 0$ by Lemma 2.1 (i).

Integrating (2.26) on $(0,+\infty)$ with $k_1 = l_1$, we have

$$\begin{aligned} 0 &= \int_0^{+\infty} (|\Phi_\xi|^{m-1}\Phi_\xi)_\xi + l_1\xi\Phi_\xi + k_1\Phi \\ &= \big(|\Phi_\xi|^{m-1}\Phi_\xi + l_1\xi\Phi\big)\Big|_0^{+\infty} + \int_0^{+\infty}(-l_1+k_1)\Phi = -|\Phi_\xi(0)|^{m-1}\Phi_\xi(0), \end{aligned}$$

which contradicts (2.27).

We have shown that the problem (1.1)–(1.4) does not admit nontrivial global solutions with either $1/k_1 = 2m$ with $1/k_2 > 0$, or $1/k_2 = 2n$ with $1/k_1 > 0$. □

Proof of Theorem 5. The case (i) is included in Lemma 2.3. For the case (ii), the assumption implies k_1 or $k_2 \to +\infty$, namely, $k_1 > 1/2m$ or $k_2 > 1/2n$ holds. The solutions of (1.1)–(1.4) blow up in finite time for every nontrivial initial data due to Lemma 2.5.

For the case (iii), the assumption $\big(\lambda_0 p - (\beta - \frac{2n}{n+1})\big)\big(\lambda_0^{-1}q - (\alpha - \frac{2m}{m+1})\big) = 0$ with $(1/k_1, 1/k_2) = (0,0)$ implies $\lambda_0 p = \beta - \frac{2n}{n+1}$, $\lambda_0^{-1}q = \alpha - \frac{2m}{m+1}$. The conclusions can be proved by Lemmas 2.4–2.6 and Theorem 3. □

3. Blow-up rates

In this section, we will determine the blow-up rate to prove Theorem 6. By (1.12), the assumption $k_1, k_2 > 0$ implies

$$(\lambda_0 p + \frac{2n}{n+1} - \beta)(\lambda_0^{-1}q + \frac{2m}{m+1} - \alpha) > 0,$$

where $\lambda_0 = \frac{n}{n+1} \cdot \frac{m+1}{m}$. There are only two cases: (i) $\lambda_0 p + \frac{2n}{n+1} - \beta, \lambda_0^{-1} q + \frac{2m}{m+1} - \alpha > 0$; (ii) $\lambda_0 p + \frac{2n}{n+1} - \beta, \lambda_0^{-1} q + \frac{2m}{m+1} - \alpha < 0$.

For case (i), the assumption $k_1, k_2 > 0$ with $\alpha \le \frac{2m}{m+1}$ and $\beta \le \frac{2n}{n+1}$ requires $pq > (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$. If at least one of $\frac{2n}{n+1} < \beta < \frac{2n}{n+1} + \lambda_0 p$ and $\frac{2m}{m+1} < \alpha < \frac{2m}{m+1} + \lambda_0^{-1} q$ holds, the assumption $k_1, k_2 > 0$ will be satisfied automatically.

Case (ii) implies that $\alpha > \frac{2m}{m+1}$, $\beta > \frac{2n}{n+1}$ with $pq < (\alpha - \frac{2m}{m+1})(\beta - \frac{2n}{n+1})$.

We know $u_t, v_t \ge 0$ by Lemma 2.1, and so $(|u_x|^{m-1} u_x)_x, (|v_x|^{n-1} v_x)_x \ge 0$. Thus, $u_x, v_x \le 0$. Define

$$M(t) = \|u(\cdot, t)\|_\infty = u(0, t), \quad N(t) = \|v(\cdot, t)\|_\infty = v(0, t),$$

and set

$$a = \Big(\frac{M^{m-\alpha}}{N^p}\Big)^{\frac{1}{m}}, \quad b = M^{1-m} \Big(\frac{M^{m-\alpha}}{N^p}\Big)^{\frac{m+1}{m}},$$
$$c = \Big(\frac{N^{n-\beta}}{M^q}\Big)^{\frac{1}{n}}, \quad d = M^{1-n} \Big(\frac{N^{n-\beta}}{M^q}\Big)^{\frac{n+1}{n}}.$$

Following [7, 15, 20], denote

$$\varphi_M(y, s) = \frac{1}{M(t)} u(ay, bs + t), \quad \psi_N(y, s) = \frac{1}{N(t)} v(cy, ds + t)$$

in $[0, +\infty) \times (-\frac{t}{b}, 0]$ and $[0, +\infty) \times (-\frac{t}{d}, 0]$ with $t < T$, respectively. Then, (φ_M, ψ_N) satisfies

$$0 \le \varphi_M, \psi_N \le 1, \quad \varphi_M(0,0) = \psi_N(0,0) = 1, \quad (\varphi_M)_s, (\psi_N)_s \ge 0.$$

Moreover, (φ_M, ψ_N) is a solution of

$$(\varphi_M)_s = (|\varphi_{My}|^{m-1} \varphi_{My})_y, \ (\psi_N)_s = (|\psi_{Ny}|^{n-1} \psi_{Ny})_y, \tag{3.1}$$

$$\begin{aligned} -(|\varphi_{My}|^{m-1} \varphi_{My})(0, s) &= (\varphi_M^\alpha \psi_N^p)(0, s), \\ -(|\psi_{Ny}|^{n-1} \psi_{Ny})(0, s) &= (\varphi_M^q \psi_N^\beta)(0, s). \end{aligned} \tag{3.2}$$

It follows from (3.2) that $\varphi_{My}(0,0) = \psi_{Ny}(0,0) = -1$. There exists a negative number s_* such that φ_M and ψ_N are well defined for $(y, s) \in A = \{(y, s) : y \ge 0, s_* \le s \le 0\}$ with M and N large enough. Otherwise, $-t/b \to 0$ as $t \to T$. We know that

$$\varphi_M\Big(y, -\frac{t}{b}\Big) = \frac{1}{M(t)} u_0(ay) < \frac{1}{2}$$

provided M and N are sufficiently large. This contradicts $\varphi_M(0,0) = 1$.

Lemma 3.1. *Under the assumption of Theorem 6, there exist positive constants c and C for M and N large enough such that*

$$c \le (\varphi_M)_s(0,0) \le C, \quad c \le (\psi_N)_s(0,0) \le C.$$

Proof. At first prove $(\varphi_M)_s(0,0) \le C$ and $(\psi_N)_s(0,0) \le C$. Denote $M_j = M(t_j), N_j = N(t_j)$ with $t_j \to T$. Since every sequence $(\varphi_{M_j}, \psi_{N_j})$ is uniformly bounded and equicontinuous [2, 18], passing to a subsequence, we have

$$\varphi_{M_j} \to \tilde{\varphi}, \quad \psi_{N_j} \to \tilde{\psi} \qquad \text{as } t_j \to T$$

uniformly on compact set of $A = \{(y,s) : y \ge 0, s_* \le s \le 0\}$, where moreover $(\varphi_{M_{jy}}, \psi_{N_{jy}})$ is uniformly Hölder continuous with $-1 \le \varphi_{M_{jy}}, \psi_{N_{jy}} \le 0$ [2, 18]. Passing to a subsequence again, we have

$$\varphi_{M_{jy}} \to \tilde{\varphi}_y, \quad \psi_{N_{jy}} \to \tilde{\psi}_y \qquad \text{as } t_j \to T$$

uniformly on compact subsets of A, where $\tilde{\varphi}_y, \tilde{\psi}_y$ are continuous and increasing with respect to y, and $\tilde{\varphi}_y(0,0) = \tilde{\psi}_y(0,0) = -1$. Suppose $U \subset A$ is a neighborhood of $(0,0)$ such that $\tilde{\varphi}_y, \tilde{\psi}_y < -\varepsilon_0$ in U with any $\varepsilon_0 > 0$. Then, $-1 \le \varphi_{M_jy}, \psi_{N_jy} \le -\varepsilon_0/2$ in U for j large enough. This means that φ_{M_j} and ψ_{N_j} are solutions of uniformly parabolic equations in $\bar{U}$. By the Schauder estimates [12, 13, 14],

$$\|\varphi_{M_j}\|_{c^{2+\alpha,1+\alpha/2}} \le C, \quad \|\psi_{N_j}\|_{c^{2+\alpha,1+\alpha/2}} \le C \qquad \text{in } \bar{U}, \tag{3.3}$$

and hence

$$(\varphi_M)_s(0,0) \le C, \quad (\psi_N)_s(0,0) \le C$$

for M and N large enough.

It remains to prove $(\varphi_M)_s(0,0) \ge c$ and $(\psi_N)_s(0,0) \ge c$. Otherwise, e.g., there exists a sequence $\{M_j\}$ such that $\frac{\partial \varphi_{M_j}}{\partial s}(0,0) \to 0$. Since φ_{M_j} satisfies (3.3), pass to a subsequence to get $\varphi_{M_j} \to \tilde{\varphi}$ for some positive function $\tilde{\varphi}$ in $C^{2+\beta,1+\beta/2}$ $(\beta < \alpha)$ satisfying $0 \le \tilde{\varphi} \le 1$, $\tilde{\varphi}(0,0) = 1$, $\frac{\partial \tilde{\varphi}}{\partial s} \ge 0$, which is a weak solution of

$$\tilde{\varphi}_s = (|\tilde{\varphi}_y|^{m-1}\tilde{\varphi}_y)_y,$$
$$- |\tilde{\varphi}_y(0,s)|^{m-1}\tilde{\varphi}_y(0,s) = \tilde{\varphi}^\alpha(0,s)\tilde{\psi}^p(0,s).$$

Set $\omega = \tilde{\varphi}_s$. Then ω satisfies

$$\omega_s = (m|\tilde{\varphi}_y|^{m-1}\omega_y)_y,$$
$$- (m|\tilde{\varphi}_y|^{m-1}\omega_y)(0,s) = p(\tilde{\psi}_s\tilde{\psi}^{p-1}\tilde{\varphi}^\alpha)(0,s) + \alpha(\omega\tilde{\varphi}^{\alpha-1}\tilde{\psi}^p)(0,s) \ge 0.$$

Thus, ω attains its minimum at $(0,0)$ with $\omega(0,0)=0$. By Hopf's lemma, which can be applied whenever $|\tilde\varphi_y|>0$, we know that $\omega\equiv 0$; i.e., $\tilde\varphi$ is independent of s. We obtain $0=\tilde\varphi_s=(|\tilde\varphi_y|^{m-1}\tilde\varphi_y)_y$, and hence

$$-|\tilde\varphi_y(y,0)|^{m-1}\tilde\varphi_y(y,0)=-|\tilde\varphi_y(0,0)|^{m-1}\tilde\varphi_y(0,0)=\tilde\varphi^{\alpha}(0,0)\tilde\psi^{p}(0,0)=1,$$

namely

$$\tilde\varphi_y=-1 \qquad \text{whenever } (y,0)\in\bar U. \tag{3.4}$$

If $\tilde\varphi_y=0$ at some $(y^*,s^*)\in A$, then (3.4) contradicts the continuity of $\tilde\varphi_y$. Otherwise, (3.4) implies that $\tilde\varphi$ is unbounded, also a contradiction. □

Proof of Theorem 6. Recall

$$M(t)=u(0,t)=\|u(\cdot,t)\|_\infty,\quad N(t)=v(0,t)=\|v(\cdot,t)\|_\infty,$$
$$\varphi_M(y,s)=\tfrac{1}{M(t)}u(ay,bs+t),\quad \psi_N(y,s)=\tfrac{1}{N(t)}v(cy,ds+t).$$

Lemma 3.1 says

$$c\le M^{-m}\Big(\frac{M^{m-\alpha}}{N^p}\Big)^{\frac{m+1}{m}}M'(t)\le C,\quad c\le N^{-n}\Big(\frac{N^{n-\beta}}{M^q}\Big)^{\frac{n+1}{n}}N'(t)\le C,$$

namely

$$\begin{aligned} &cN^{\frac{p(m+1)}{m}}\le M^{\frac{m-(m+1)\alpha}{m}}M'(t)\le CN^{\frac{p(m+1)}{m}},\\ &cM^{\frac{q(n+1)}{n}}\le N^{\frac{n-(n+1)\beta}{n}}N'(t)\le CM^{\frac{q(n+1)}{n}}.\end{aligned}\tag{3.5}$$

It follows from (3.5) that

$$cN^{\frac{p(m+1)}{m}+\frac{n-(n+1)\beta}{n}}N'(t)\le CM^{\frac{q(n+1)}{n}+\frac{m-(m+1)\alpha}{m}}M'(t),$$

which implies

$$N^{\frac{m(n+1)\left(\lambda_0p+\frac{2n}{n+1}-\beta\right)}{mn}}\le C_1M^{\frac{n(m+1)\left(\lambda_0^{-1}q+\frac{2m}{m+1}-\alpha\right)}{mn}}. \tag{3.6}$$

For the subcase (i), by (3.6)

$$N\le C_2M^{\left\{\frac{n}{n+1}\left(\lambda_0^{-1}q+\frac{2m}{m+1}-\alpha\right)\right\}/\left\{\frac{m}{m+1}\left(\lambda_0p+\frac{2n}{n+1}-\beta\right)\right\}}=C_2M^{\frac{k_2}{k_1}}. \tag{3.7}$$

Combine (3.5) with (3.7) to get

$$M^{\frac{m-(m+1)\alpha}{m}-\frac{pk_2(m+1)}{mk_1}}M'(t)\le C_3. \tag{3.8}$$

Since

$$1+\frac{m-(m+1)\alpha}{m}-\frac{pk_2(m+1)}{mk_1}=-\frac{1}{k_1}$$

by (1.11), we obtain by integrating (3.8) on (t,T) that

$$M(t) \geq c_3(T-t)^{-k_1}, \tag{3.9}$$

and hence

$$N(t) \leq c_2(T-t)^{-k_2} \tag{3.10}$$

due to (3.5) and (3.7). Similarly, we have from (3.5) that

$$CN^{\frac{p(m+1)}{m}+\frac{n-(n+1)\beta}{n}}N'(t) \geq cM^{\frac{q(n+1)}{n}+\frac{m-(m+1)\alpha}{m}}M'(t), \tag{3.11}$$

$$M \leq C_4 N^{\frac{k_1}{k_2}}, \quad N^{\frac{n-(n+1)\beta}{n}-\frac{qk_1(n+1)}{nk_2}}N'(t) \leq C_5 \tag{3.12}$$

with

$$1+\frac{n-(n+1)\beta}{n}-\frac{qk_1(n+1)}{nk_2} = -\frac{1}{k_2}.$$

It follows from (3.12) that

$$N(t) \geq c_4(T-t)^{-k_2}, \quad M(t) \leq c_1(T-t)^{-k_1}. \tag{3.13}$$

For the subcase (ii), (3.6) implies

$$N \geq C_2' M^{\left\{\frac{n}{n+1}\left(\lambda_0^{-1}q+\frac{2m}{m+1}-\alpha\right)\right\}/\left\{\frac{m}{m+1}\left(\lambda_0 p+\frac{2n}{n+1}-\beta\right)\right\}} = C_2' M^{\frac{k_2}{k_1}}. \tag{3.14}$$

Combining this with (3.5) and (3.14), we have

$$M^{\frac{m-(m+1)\alpha}{m}-\frac{pk_2(m+1)}{mk_1}}M'(t) \geq C_3'.$$

By using the procedure in the proof for the subcase (i), we can get the same estimates (3.13), (3.10) and (3.9) in turn for the subcase (ii) as well.

The proof is complete. □

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