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# NONUNIQUENESS OF THE HEAT FLOW OF DIRECTOR FIELDS

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**Abstract.** We give a new example of nonuniqueness of weak solutions of the initial-boundary-value problem for the heat flow of director fields in an infinitely long cylinder in $\mathbb{R}^3$. The example confirms the connection between nonuniqueness of axially symmetric solutions for the harmonic map heat flow and the occurrence of point singularities in the solutions. The result is compared with earlier nonuniqueness results. Traveling wave solutions are used as barrier functions.

## 1. Introduction

In this paper we consider an initial-boundary-value problem for the heat flow of harmonic maps from an infinitely long vertical cylinder,

$$\Omega = \{(x_1, x_2, z) : x_1^2 + x_2^2 < 1, z \in \mathbb{R}\} \subset \mathbb{R}^3,$$

to the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$:

$$\begin{cases} u_t - \Delta u \;=\; |\nabla u|^2 u & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,0) \;=\; u_0(x) & \text{if } x \in \Omega \\ u(x,t) \;=\; u_0(x) & \text{if } x \in \partial\Omega, t \in \mathbb{R}^+. \end{cases} \tag{1.1}$$

We will assume that the initial-boundary data $u_0 : \;\Omega \longrightarrow \mathbb{S}^2$ is smooth and axially symmetric ($r := \sqrt{x_1^2 + x_2^2}$):

$$u_0(x_1, x_2, z) \;=\; \Big(\frac{x_1}{r} \sin h_0(r,z), \frac{x_2}{r} \sin h_0(r,z), \cos h_0(r,z)\Big). \tag{1.2}$$

It is well known (see [10] for instance) that if $u_0$ is sufficiently smooth on $\overline{\Omega}$, then problem (1.1) has a unique classical solution $u$ in a maximal time

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interval $[0,T)$, $T \in (0,\infty]$, and $u$ can be written in the form

$$u(x_1,x_2,z,t) = \left(\frac{x_1}{r}\sin h, \frac{x_2}{r}\sin h, \cos h\right)$$

for a suitable *angle function* $h = h(r,z,t)$, which solves the scalar problem

$$\begin{cases} h_t = h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & \text{for } 0<r<1, z\in\mathbb{R}, t\in\mathbb{R}^+ \\ h(r,z,0) = h_0(r,z) & \text{for } 0<r<1, z\in\mathbb{R} \\ h(1,z,t) = h_0(1,z) & \text{for } z\in\mathbb{R}, t\geq 0. \end{cases} \tag{1.3}$$

Grotowski proved in [7] that, if $\Omega = \mathbb{B}^3 \subset \mathbb{R}^3$, the $L^\infty$-norm of $\nabla u$ can blow-up in finite time even if $u_0 \in C^\infty(\bar{\Omega})$: $T < \infty$ and $\|\nabla u(\cdot,t)\|_\infty \to \infty$ as $t \to T^-$. For $t \geq T$ the solution may be extended as a weak (variational) solution, but weak solutions are not unique (see [10]): if $\Omega = \mathbb{B}^3$ and $u_0 \in C^\infty(\bar{\Omega})$, Coron ([5]) proved that problem (1.1) can have infinitely many weak solutions. These results continue to hold in the class of axially symmetric solutions of (1.1) (see [6]).

In [1] and [9], a very special case ($\Omega = \mathbb{B}^3$ and $u_0(x) = x/|x|$) has been studied in detail to get a better understanding of the nonuniqueness question: for every function $\zeta_0(t) : [0,\infty) \longrightarrow (-1,1)$ there exists an axially symmetric solution of (1.1) which is regular in $\Omega$ except for the set $\{(x_1,x_2,z,t) = (0,0,\zeta_0(t),t), t \geq 0\}$. The result is very specific but it suggests that part of the nonuniqueness of weak solutions is related to a considerable amount of freedom to prescribe the position of point singularities. For example, for every $P = (0,0,z) \in \Omega \setminus \{0,0,0\}$ there is a solution which instantaneously moves its singular point, initially located at the origin, to the point $P$. The construction in [1] and [9] is based on a quite delicate construction of barrier functions for $h$, which cannot be easily extended to more general situations. In particular it turns out to be difficult to relax the condition that $0 \leq h_0 \leq \pi$, which is restrictive since it excludes the important case in which initially the solution is smooth and $T < \infty$.

It is the aim of the present paper to construct an example of nonuniqueness in the spirit of the papers [1] and [9] for a class of initial functions $h_0$ which are smooth and bounded ($0 \leq h_0 \leq 2\pi$) in the strip $[0,1] \times \mathbb{R}$, vanish for $r = 0$, and for which $T < \infty$. They will be subsolutions of the equation

$$h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} = 0 \quad \text{for } 0<r<1, z\in\mathbb{R}. \tag{1.4}$$

For this choice of $h_0$ we first construct a "minimal" solution, $h_{\min}$, of problem (1.3) which is smooth until the blow-up time $T \in (0, \infty)$:

$$h_{\min}(0, z, t) \equiv 0 \quad \text{if } t < T, \qquad \limsup_{t \to T^-} \|\nabla h_{\min}(\cdot, \cdot, t)\|_\infty = \infty,$$

$$\text{and} \qquad h_{\min}(0, z, t) = \begin{cases} 0 & \text{if } |z| > \zeta(t) \\ \pi & \text{if } |z| < \zeta(t) \end{cases} \quad \text{if } t \geq T,$$

where $\zeta : [T, \infty) \to [0, \infty)$ is an increasing function. Then, for every $M \geq 0$, we construct a different nonnegative weak solution $h_M \geq h_{\min}$ such that

$$h_M(0, z, t) = \begin{cases} 0 & \text{if } |z| > \zeta_M(t) \\ \pi & \text{if } |z| < \zeta_M(t) \end{cases} \quad \text{if } t > 0, \tag{1.5}$$

where $\zeta_M : \mathbb{R}^+ \to [0, \infty)$ is an increasing function which satisfies, for a suitable constant $S > 0$ independent of $M$,

$$M \leq \zeta_M(t) \leq S + M + t \qquad \text{for } t > 0.$$

We observe that with a bit more work, we could have required that $h_M = h_{\min}$ for $t < T$ and $h_M$ satisfies (1.5) only for $t > T$. By the minimality of $h_{\min}$ we have that $\zeta_M(t) \geq \zeta(t)$ for $t > T$.

Comparing these results with the ones in [1] and [9] we notice significant differences: we are not able to prescribe a priori the function $\zeta_M$ but only certain intervals to which it belongs at time $t$, $\zeta_M$ is always nondecreasing and it is a priori bounded from below by $\zeta(t)$. Therefore it is natural to ask whether in our case there is less freedom to prescribe the position of the point singularities of the solutions than what was suggested in [1] and [9]. We suspect that the answer to this question is negative. Returning to the original problem for the vector field $u$, the interfaces which we consider in this paper separate segments on the vertical axis of the cylinder at which $u = \mathbf{N} = (0, 0, 1)$, the northpole, from those where $u = \mathbf{S} = (0, 0, -1)$, the south pole. Our results show that we can make the segment where $u = \mathbf{S}$ arbitrarily large, but at first sight they suggest that it will never shrink. But then we forget that also the value $h = 2\pi$ corresponds to $u = \mathbf{N}$, and in the present paper we simply did not take into account solutions $h$ which attain also the value $2\pi$. For example, following the idea of the construction of $h_{\min}$, it is easy to show that there also exists a maximal solution, $h_{\max}$, which attains only the values $\pi$ and $2\pi$ at $r = 0$. We conjecture that, in the axially symmetric setting, it is possible to prescribe the sets on the vertical axis where $u = \mathbf{N}$ and those where $u = \mathbf{S}$, but, to our best knowledge, this problem is completely open. To solve it, even in relatively simple cases where these sets are segments, for sure one has to be able to handle situations where $h$ has jumps from 0 to $2\pi$ and vice versa (in terms of $u$ these are singularities of topological degree 0 and by no means can they be controlled by barrier arguments).

For a review of the development of singularities in solutions of the harmonic map heat flow we refer to [10], where also the case of a director field defined on a two-dimensional compact Riemannian manifold, e.g. the unit disc of $\mathbb{R}^2$, is considered. The literature about the latter case is very large, see for instance [2] or [4], and includes also studies of the asymptotic behavior of solutions just before they become singular, see for instance [11] and [13].

The paper starts by specifying how the subsolution $h_0 \geq 0$ must be chosen. Then, it continues with the construction of $h_{\min}$ and of the family of solutions $\{h_M\}_{M\geq 0}$. At the end there is an Appendix which contains some technical results used throughout the paper.

## 2. Construction of $h_{\min}$

Let $\alpha \in (\sqrt{2}, 2), B \in (0, \pi/2), \mathcal{T} > 0, \mathcal{Z} \geq 0$ be four constants arbitrarily chosen and let $K \in [8/(\pi - 2B), \infty)$. If

$$\tilde{\xi}(r, z, t) := 2 \arctan \left( \frac{r}{K} \mathrm{e}^{\frac{Q}{\mathcal{T} - t + \mu_{\mathcal{Z}}(z)}} \right) + \mathcal{B} r^{\alpha},$$

where $Q > 0$,

$$\mathcal{B} := \arctan \left( \frac{\mathrm{e}^{\frac{Q}{\mathcal{T}}}}{K} \right) + \frac{B - \pi}{2},$$

and

$$\mu_{\mathcal{Z}}(z) = \begin{cases} |z - \mathcal{Z}|^4 & \text{if } z > \mathcal{Z} \\ 0 & \text{if } z \in [-\mathcal{Z}, \mathcal{Z}] \\ |z + \mathcal{Z}|^4 & \text{if } z < -\mathcal{Z}, \end{cases}$$

then by Proposition A.6 we know that, if $Q \geq \bar{Q}(\alpha, K, B, \mathcal{T}) > 0$, $\mathcal{B} > 0$ and the function $\tilde{\xi}$ is a subsolution of the equation

$$h_t \ = \ h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} \tag{2.1}$$

satisfying the conditions

$$\tilde{\xi}(0, z, t) = 0 \ \ \forall t \in [0, \mathcal{T}), z \in \mathbb{R}, \lim_{r \to 0^+} \left( \lim_{t \to \mathcal{T}^-} \tilde{\xi}(r, z, t) \right) = \begin{cases} \pi & \text{if } |z| \leq \mathcal{Z} \\ 0 & \text{if } |z| > \mathcal{Z}. \end{cases} \tag{2.2}$$

For fixed $Q \geq \bar{Q}$, by Theorem B.3 there exists a function $h_0 \in C^\infty((0,1] \times \mathbb{R})$ which is Lipschitz continuous in $[0,1] \times \mathbb{R}$ and satisfies the following properties:

(P1) $\quad (h_0)_{rr} + (h_0)_{zz} + \dfrac{(h_0)_r}{r} - \dfrac{\sin(2h_0)}{2r^2} \geq 0 \qquad \text{in } (0,1) \times \mathbb{R},$

(P2) $h_0(0,z) = 0 \quad \forall z \in \mathbb{R}$,

(P3) $2\arctan(br) \leq h_0(r,z) \leq \pi + 2\arctan(ar)$ for some $a > 0$, $b \in (0,1)$, and there exists $\bar{z} > 0$ such that $h_0(r,z) \equiv 2\arctan(br)$ if $|z| \geq \bar{z}$,

(P4) $h_0(r,-z) \equiv h_0(r,z)$, $(h_0)_z(r,z) \leq 0$ for $r \in [0,1], z \geq 0$, and

(P5) $h_0(r,z) \geq \tilde{\xi}(r,z,0)$, $h_0(1,z) \geq \tilde{\xi}(1,z,t)$ for all $r \in [0,1]$, $z \in \mathbb{R}$ and $t \in [0,\mathcal{T})$.

In order to construct the weak solution $h_{\min}$ of (1.3), for every $n \in \mathbb{N}, n \geq 2$, we consider the domain $A_n = (1/n,\, 1) \times (-n,n)$ and the differential problem

$$(P_n) \qquad \begin{cases} h_t = h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & (r,z) \in A_n,\ t > 0 \\ h(r,z,0) = h_0(r,z) & (r,z) \in \bar{A}_n \\ h(1,z,t) = h_0(1,z) & z \in [-n,n],\ t > 0 \\ h(1/n,z,t) = h_0(1/n,z) & z \in [-n,n],\ t > 0 \\ h(r,\pm n,t) = h_0(r,\pm n) & r \in [1/n,1],\ t > 0. \end{cases}$$

By standard solvability, comparison and regularity results for parabolic problems (see [8]), by properties (P1), (P3), (P4) and since the functions $\theta_b(r) = 2\arctan(br)$, $\theta_a(r) := \pi + 2\arctan(ar)$ both solve (2.1), we can say that, for every $n \in \mathbb{N}$, Problem $(P_n)$ has a unique classical solution $h_n \in C^\infty(\bar{A}_n \times \mathbb{R}^+) \cap C^0(\bar{A}_n \times [0,\infty))$ and, for $(r,z) \in \bar{A}_n$, $t' \geq t \geq 0$,

$$\begin{aligned} 2\arctan(br) \leq h_n(r,z,t) &\leq \pi + 2\arctan(ar), \\ h_0(r,z) \leq h_n(r,z,t) &\leq h_n(r,z,t'), \\ h_n(r,-z,t) &= h_n(r,z,t), \quad (h_n)_z(r,z,t) \leq 0 \quad \text{if } z \geq 0, \\ (h_n)_r, (h_n)_z \text{ are Hölder continuous in } (r,z,t), &\quad h_{n+1}(r,z,t) \geq h_n(r,z,t). \end{aligned} \tag{2.3}$$

Moreover, if we define

$$E_n(\cdot) := \iint_{A_n} \frac{r}{2}\Big((\cdot)_r^2 + (\cdot)_z^2 + \frac{\sin^2(\cdot)}{r^2}\Big) \mathrm{d}r\mathrm{d}z,$$

we have the following.

**Proposition 2.1.** *For every $n \geq 2$ and $T > 0$*

$$E_n(h_n(\cdot\,,\cdot\,,T)) + \iiint_{A_n\times[0,T]} r(h_n)_t^2 \mathrm{d}r\mathrm{d}z\mathrm{d}t \;=\; E_n(h_0).$$

The proof is completely standard.

Thanks to properties (2.3) we can define the function

$$h_{\min}(r,z,t) := \lim_{n\to\infty} h_n(r,z,t) = \sup_{n\geq 2} h_n(r,z,t) \quad (r,z,t) \in (0,1]\times\mathbb{R}\times[0,\infty)$$

and say that, for $(r,z) \in (0,1]\times\mathbb{R}$, $t' \geq t \geq 0$,

$$\begin{array}{ll} \theta_b(r) \leq h_{\min}(r,z,t) \leq \theta_a(r)\,, & h_0(r,z) \leq h_{\min}(r,z,t) \leq h_{\min}(r,z,t'), \\ h_{\min}(r,-z,t) = h_{\min}(r,z,t), & h_{\min}(r,z',t) \leq h_{\min}(r,z,t) \text{ if } z' \geq z \geq 0. \end{array} \tag{2.4}$$

By standard regularity results, $h_{\min} \in C^\infty((0,1]\times\mathbb{R}\times\mathbb{R}^+)\cap C^0((0,1]\times\mathbb{R}\times[0,\infty))$ and it is a classical solution of (1.3). Actually, $h_{\min}$ is a "minimal" solution for problem (1.3), in the sense specified by the next statement.

**Proposition 2.2.** *Let $h \in L^\infty((0,1)\times\mathbb{R}\times\mathbb{R}^+)$ be a weak solution of* (1.3) *with $h \geq h_0$. Then $h \geq h_{\min}$.*

**Proof.** Due to parabolic Schauder-type estimates, $h$ is smooth out of $\{r = 0\}$ and is a supersolution of problem $(P_n)$ for every $n \geq 2$. Then the thesis directly follows from the parabolic maximum principle and the definition of $h_{\min}$. □

**Proposition 2.3.** *For every $\zeta > \bar{z}$ and $T > 0$*

$$\int_{-\zeta}^{\zeta} \mathrm{d}z \int_0^1 \frac{r}{2}\Big((h_{\min})_r^2 + (h_{\min})_z^2 + \frac{\sin^2(h_{\min})}{r^2}\Big)\Big|_{t=T}\mathrm{d}r$$

$$+ \iiint_{(0,1)\times\mathbb{R}\times[0,T]} r(h_{\min})_t^2 \mathrm{d}r\mathrm{d}z\mathrm{d}t$$

$$\leq \int_{-\zeta}^{\zeta} \mathrm{d}z \int_0^1 \frac{r}{2}\Big((h_0)_r^2 + (h_0)_z^2 + \frac{\sin^2(h_0)}{r^2}\Big)\mathrm{d}r < \infty.$$

**Proof.** For every $\zeta > 0$, $n \in \mathbb{N}$ with $n \geq 2$, let $E_{n,\zeta}$ be the functional defined by

$$E_{n,\zeta}(\cdot) = \int_{-\zeta}^{\zeta} \mathrm{d}z \int_{1/n}^1 \frac{r}{2}\Big((\cdot)_r^2 + (\cdot)_z^2 + \frac{\sin^2(\cdot)}{r^2}\Big)\mathrm{d}r. \tag{2.5}$$

In particular, $E_{n,n}$ coincides with the energy functional $E_n$ previously defined and, due to Proposition 2.1, for every $n \geq 2$,

$$E_{n,n}(h_n(\cdot,\,\cdot,T)) + \iiint_{A_n\times[0,T]} r(h_n)_t^2 \mathrm{d}r\mathrm{d}z\mathrm{d}t = E_{n,n}(h_0). \tag{2.6}$$

Since $h_0(r,z) = \theta_b(r)$ if $|z| \geq \bar{z}$, if $|z| > \bar{z}$ we have that

$$\int_{1/n}^{1} \frac{r}{2}\Big((h_0)_r^2 + (h_0)_z^2 + \frac{\sin^2(h_0)}{r^2}\Big)\mathrm{d}r = \int_{1/n}^{1} \frac{r}{2}\Big((\theta_b)_r^2 + \frac{\sin^2(\theta_b)}{r^2}\Big)\mathrm{d}r.$$

Let $\zeta > \bar{z}$ and $T > 0$ be two values arbitrarily chosen. Given any $n > \zeta$, if $|z| \in [\zeta, n]$, then $h_n(1/n,z,T) = h_0(1/n,z) = \theta_b(1/n)$, $h_n(1,z,T) = h_0(1,z) = \theta_b(1)$, and by [3], Corollary 21,

$$\int_{1/n}^{1} \frac{r}{2}\Big((h_n)_r^2(r,z,T) + (h_n)_z^2(r,z,T) + \frac{\sin^2 h_n(r,z,T)}{r^2}\Big)\mathrm{d}r$$

$$\geq \int_{1/n}^{1} \frac{r}{2}\Big((\theta_b)_r^2 + \frac{\sin^2(\theta_b)}{r^2}\Big)\mathrm{d}r.$$

Therefore, for every $n \geq \zeta$,

$$E_{n,\zeta}(h_n(\cdot,\,\cdot,T)) + \iiint_{A_n\times[0,T]} r(h_n)_t^2 \mathrm{d}r\mathrm{d}z\mathrm{d}t \leq E_{n,\zeta}(h_0)$$

$$\leq \iint_{[0,1]\times[-\zeta,\zeta]} \frac{r}{2}\Big((h_0)_r^2 + (h_0)_z^2 + \frac{\sin^2(h_0)}{r^2}\Big)\mathrm{d}r\mathrm{d}z.$$

Passing to the limit as $n \to \infty$ and using Fatou's lemma we obtain the thesis. $\square$

The previous proposition allows us to say that $h_{\min}$ is a weak solution of (1.3), since $(h_{\min})_t \in L_r^2((0,1)\times\mathbb{R}\times\mathbb{R}^+)$ and $\frac{\sin h_{\min}}{r}, \nabla h_{\min} \in L^\infty(\mathbb{R}^+;$ $L_r^2((0,1)\times(-\zeta,\zeta)))$ for every $\zeta > 0$. Hence, (see [12]) for all $t \geq 0$ and for almost every $z \in \mathbb{R}$ there exists

$$h_{\min}(0,z,t) := \lim_{r\to 0^+} h_{\min}(r,z,t) = k\pi \qquad \text{for some } k = k(z,t) \in \mathbb{Z}.$$

Due to the inequality $\theta_b(r) \leq h_{\min}(r,z,t) \leq \theta_a(r)$, if $I_0(t) := \{z \in \mathbb{R} : h_{\min}(0,z,t) = 0\}$ and $I_1(t) := \{z \in \mathbb{R} : h_{\min}(0,z,t) = \pi\}$ for $t \geq 0$, then for every $t \geq 0$ $\mathbb{R} \setminus (I_0(t) \cup I_1(t))$ is a set of zero Lebesgue measure. Moreover, since $h_{\min}(r,-z,t) \equiv h_{\min}(r,z,t)$ and $h_{\min}(r,z',t) \leq h_{\min}(r,z,t)$ for $r \in (0,1], t \geq 0$ and $z' \geq z \geq 0$, for every $t \geq 0$ there exists $\zeta(t) \in [0,\infty]$ such that $I_1(t) = (-\zeta(t),\zeta(t))$, $I_0(t) = (-\infty,-\zeta(t)) \cup (\zeta(t),\infty)$, up to the values

$\pm\zeta(t)$. Finally, $\zeta(t') \geq \zeta(t)$ for every $t' \geq t \geq 0$, since $h_{\min}$ is an increasing function of $t$.

**Theorem 2.4.**

(i) *There exists a constant* $S \in \mathbb{R}^+$ *such that* $\zeta(t) \leq S + t$ *for all* $t \geq 0$.
(ii) *There exists a time* $T > 0$ *such that for every* $t \in [0, T)$ $I_0(t) = \mathbb{R} \Rightarrow \zeta(t) = 0$.
(iii) *If* $\tau > 0$ *is a value such that* $I_0(\tau) = \mathbb{R}$*, then there exist* $\rho > 0$ *and* $C > 0$ *such that* $h_{\min}(r, z, t) \leq 2\arctan(Cr)$ *for all* $r \in [0, \rho], z \in \mathbb{R}, t \in [0, \tau]$.
(iv) *If* $\tau > 0$ *is a value such that* $I_0(\tau) \neq \mathbb{R}$*, then for every* $\varepsilon > 0$ *there exist* $\rho \in (0, 1)$ *and* $C > 0$ *(both* $\rho$ *and* $C$ *depend on* $\varepsilon$*) such that*

$$h_{\min}(r, z, t) \leq 2\arctan(Cr) \qquad r \in [0, \rho], |z| \geq \zeta(\tau) + \varepsilon, t \in [0, \tau].$$

**Proof.** (i) Let $g = g(y) \in C^\infty(\mathbb{R})$ be a function such that

$$g(y) = 2\pi \;\text{ if } y < -1, \quad g(y) = 2\arctan b \;\text{ if } y > 1, \quad g' \leq 0.$$

Thanks to [3], Theorem 2, there exists $\psi = \psi(r, y) \in C^\infty([0, 1] \times \mathbb{R} \setminus \{(0, \bar{y})\})$ for some $\bar{y} \in \mathbb{R}$ such that $\psi(0, y) = 0$ if $y > \bar{y}$, $\psi(0, y) = 2\pi$ if $y < \bar{y}$, $\psi(1, y) = g(y)$ for all $y \in \mathbb{R}$ and

$$\psi_{rr} + \frac{\psi_r}{r} + \psi_{yy} + \psi_y - \frac{\sin(2\psi)}{2r^2} = 0 \quad \text{in } (0, 1) \times \mathbb{R}. \tag{2.7}$$

Moreover, $\psi$ is nonincreasing with respect to $y$ and $\psi(r, y) \to \theta_b(r)$ as $y \to \infty$, $\psi(r, y) \to 2\pi$ as $y \to -\infty$ uniformly with respect to $r \in [0, 1]$. Therefore, since $h_0 \leq \pi + 2\arctan(a) < 2\pi$ and $h_0(r, z) = \theta_b(r)$ for $z \geq \bar{z}$, there exists $\sigma > 0$ such that $\psi(r, z - \sigma) \geq h_0(r, z)$ (take $\sigma > 0$ so that $\psi(r, \bar{z} - \sigma) \geq \pi + 2\arctan(a)$). If we define $h(r, z, t) = \psi(r, z - t - \sigma)$, then $h$ is an increasing function of $t$ and, in view of Proposition 2.2, $h \geq h_{\min}$. Therefore, for every $t \geq 0$ and $z > \bar{y} + \sigma + t$,

$$0 \leq h_{\min}(0, z, t) \leq \psi(0, z - t - \sigma) = 0.$$

Hence the thesis follows by taking $S = \bar{y} + \sigma$.

(ii) Let $L > 0$ be a constant such that $0 \leq h_0(r, 0) \leq Lr$ for all $r \in [0, 1]$. If $\rho \in \left(0, \min\left\{1, \frac{\pi}{2L}\right\}\right)$, from the trivial inequality

$$x \leq 2\arctan\left(\frac{2}{\pi}x\right) \qquad x \in [0, \pi/2] \tag{2.8}$$

it follows that, for every $r \in [0,\rho]$, $Lr \le 2\arctan(2Lr/\pi)$ and then, in view of (P4),

$$h_0(r,z) \le 2\arctan\left(\frac{2Lr}{\pi}\right) \qquad \forall r \in [0,\rho],\ z \in \mathbb{R}. \tag{2.9}$$

Since $h_{\min}$ is continuous in $(0,1] \times \mathbb{R} \times [0,\infty)$ and $h_{\min}(\rho,0,0) = h_0(\rho,0) \le 2\arctan(2L\rho/\pi) \le \pi/2$, there must be a value $T > 0$ such that $A := h_{\min}(\rho,0,T) < \pi$. Due to (2.4), we deduce that $h_{\min}(\rho,z,t) \le A < \pi$ for all $z \in \mathbb{R}$ and $t \in [0,T]$. Thus, if we define $C$ as the maximum between $2L/\pi$ and $\tan(A/2)\rho^{-1}$, for all $n > 1/\rho$, $z \in [-n,n]$ and $t \in [0,T]$,

$$h_n(\rho,z,t) \le h_{\min}(\rho,z,t) \le 2\arctan(C\rho). \tag{2.10}$$

Since the function $2\arctan(Cr)$ is a solution of (2.1), by (2.9), (2.10) and the parabolic maximum principle we obtain that for all $n > 1/\rho$

$$h_n(r,z,t) \le 2\arctan(Cr) \qquad r \in [1/n,\rho], z \in [-n,n], t \in [0,T].$$

Passing to the limit as $n \to \infty$, it follows that for all $z \in \mathbb{R}$, $t \in [0,T]$

$$h_{\min}(r,z,t) \le 2\arctan(Cr) \quad \forall r \in (0,\rho] \quad \Rightarrow \quad h_{\min}(0,z,t) = 0.$$

(iii) As in the proof of *(ii)*, if $\rho \in \left(0, \min\left\{1, \frac{\pi}{2L}\right\}\right)$, then (2.9) is satisfied. Since $\lim_{r\to 0^+} h_{\min}(r,0,\tau) = 0$, up to redefining the constant $\rho$, we may assume that $h_{\min}(\rho,0,\tau) \le A < \pi$ and then, in view of (2.4), $h_{\min}(\rho,z,t) \le A < \pi$ for all $z \in \mathbb{R}$ and $t \in [0,\tau]$. If we define $C$ as the maximum between $2L/\pi$ and $\tan(A/2)\rho^{-1}$, we can say that for all $n > 1/\rho$, $z \in [-n,n]$ and $t \in [0,\tau]$ inequality (2.10) is satisfied. Just as in the proof of *(ii)* we then obtain

$$h_{\min}(r,z,t) \le 2\arctan(Cr) \qquad \forall r \in (0,\rho],\ z \in \mathbb{R},\ t \in [0,\tau].$$

(iv) Given $\varepsilon > 0$, let $\tilde{z} = \zeta(\tau) + \varepsilon/2$ and $\mu = \mu(z)$ be the function defined by

$$\mu(z) = \frac{z - \tilde{z}}{1 + z - \tilde{z}} \qquad z \in [\tilde{z},\infty).$$

If
$$\psi(r,z,t) := \begin{cases} 2\arctan\left(re^{\frac{Q}{t\mu(z)}}\right) - r^{3/2} & z > \tilde{z}, t > 0 \\ \pi - r^{3/2} & z = \tilde{z} \text{ or } t = 0, \end{cases}$$

where $Q$ is a value greater than or equal to the constant $\hat{Q}(\tau+1,\mu)$ of Proposition A.12, then $\psi$ is a supersolution of (2.1) in the set $(0,1)\times(\tilde{z},\infty)\times(0,\tau+1)$. Let $L > 0$ be a value such that $h_0(r,0) \le Lr$ and let $\rho \in \left(0, \min\left\{1, \frac{\pi}{2L+2}\right\}\right)$. By (2.8), for every $r \in [0,\rho]$, $z \in \mathbb{R}$

$$h_0(r,z) + r^{3/2} \le h_0(r,0) + r^{3/2} \le (L+1)r \le 2\arctan\left(\frac{2L+2}{\pi}r\right). \tag{2.11}$$

In view of (2.4) and since $\lim_{r\to 0^+} h_{\min}(r,\tilde{z},\tau) = 0$, up to redefining $\rho$ we may assume

$$h_{\min}(r,z,t) \le \pi/2 \qquad \forall r \in [0,\rho],\ z \ge \tilde{z},\ t \in [0,\tau].$$

Then $\psi(r,\tilde{z},t) = \pi - r^{3/2} \ge \pi - r > \pi/2 \ge h_{\min}(r,\tilde{z},t)$ for every $r \in [0,\rho], t \in [0,\tau]$, and, choosing $Q$ sufficiently large, we also obtain

$$\psi(\rho,z,t) \ge 2\arctan(\rho e^{\frac{Q}{\tau}}) - \rho^{3/2} > \pi/2 \ge h_{\min}(\rho,z,t) \quad \forall z \ge \tilde{z}, t \in [0,\tau],$$
$$\psi(r,z,t) \ge 2\arctan(r e^{\frac{Q}{\tau}}) - r^{3/2} \ge 2\arctan((2L+2)r/\pi) - r^{3/2} \ge h_0(r,z)$$
$$\forall r \in [0,\rho], z \ge \tilde{z}, t \in [0,\tau].$$

Since, for every integer $n \ge 2$, $h_n \le h_{\min}$ in $\bar{A}_n \times [0,\infty)$, using the parabolic maximum principle we deduce that for every $n > 1/\rho$

$$h_n(r,z,t) \le \psi(r,z,t) \qquad \text{if } r \in [1/n,\rho],\ z \in [\tilde{z},n],\ t \in [0,\tau].$$

Passing to the limit as $n \to \infty$, we find that

$$h_{\min}(r,z,t) \le \psi(r,z,t) \qquad \forall r \in (0,\rho],\ z \ge \tilde{z},\ t \in [0,\tau].$$

In particular, for every $r \in [0,\rho]$, $z \ge \zeta(\tau) + \varepsilon = \tilde{z} + \varepsilon/2$

$$h_{\min}(r,z,\tau) \le 2\arctan\left(r e^{\frac{Q(2+\varepsilon)}{\tau\varepsilon}}\right),$$

from which the thesis follows using the properties of $h_{\min}$ (see (2.4)). □

The first interesting consequence of the previous theorem, namely of the points (i) and (iv), is the following.

**Corollary 2.5.** *For every $T > 0$, $h_{\min}(r,z,t) \to \theta_b(r)$ as $|z| \to \infty$ uniformly with respect to $r \in [0,1]$ and $t \in [0,T]$.*

**Proof.** By parabolic Schauder-type estimates, there exists a function $h_\infty = h_\infty(r,t)$ such that, for every $\sigma \in (0,1)$, $\tau > 0$,

$$h_{\min}(r,z,t) \to h_\infty(r,t) \text{ as } |z| \to \infty$$

in $C^{2,1}([\sigma,1] \times [\tau,\infty))$ rather than in $C^0([\sigma,1] \times [0,\infty))$. Fixing an arbitrary $T > 0$, (i) and (iv) of Theorem 2.4 imply the existence of $\zeta \in \mathbb{R}^+$, $\rho \in (0,1)$ and $C > 0$ such that, for all $r \in [0,\rho]$, $z \in \mathbb{R} \setminus (-\zeta,\zeta)$ and $t \in [0,T]$ $0 \le h_{\min}(r,z,t) \le 2\arctan(Cr)$. Thus $h_\infty(0,t) = 0$ for all $t \in [0,T]$ and $h_{\min}(r,z,t) \to h_\infty(r,t)$, as $|z| \to \infty$, uniformly with respect to $r \in [0,1]$ and

$t \in [0,T]$. Then $h_\infty \in C^{2,1}((0,1)\times\mathbb{R}^+) \cap C^0([0,1]\times[0,\infty))$ and solves

$$\begin{cases} h_t = h_{rr} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & r \in (0,1), t > 0 \\ h(r,0) = \lim_{|z|\to\infty} h_0(r,z) = \theta_b(r) & r \in (0,1) \\ h(0,t) = 0, \quad h(1,t) = \theta_b(1) & t \geq 0. \end{cases}$$

Since the classical solution of this problem is $\theta_b(r)$, it must be $h_\infty \equiv \theta_b$. □

Let $T \in [0,\infty]$ be the *first time of blow-up* for $h_{\min}$, i.e.,

$$T := \sup\{t > 0 : I_0(t) = \mathbb{R}\} = \sup\{t > 0 : h_{\min}(0,z,t) = 0 \ \ \forall z \in \mathbb{R}\}.$$

By Theorem 2.4-(ii), $T$ is greater than zero. Moreover, in view of the point (iii) of the same theorem, if $T < \infty$, then $I_0(T) \neq \mathbb{R}$. By contradiction, if $I_0(T) = \mathbb{R}$, then there exist $\rho > 0$ and $C > 0$ such that

$$h_{\min}(r,z,t) \leq 2\arctan(Cr) \qquad \forall r \in [0,\rho], z \in \mathbb{R}, t \in [0,T].$$

Since $h_{\min}$ is smooth in $(0,1]\times\mathbb{R}\times\mathbb{R}^+$, there must be $\delta > 0$ and $A \in (0,\pi)$ such that

$$h_{\min}(\rho,z,t) \leq A \qquad \forall z \in \mathbb{R}, t \in [0,T+\delta].$$

At the same time, up to redefining $\rho$ we may assume that (2.9) is true for a suitable $L > 0$. If now we take $D = \max(2L/\pi, \tan(A/2)\rho^{-1})$, then

$$h_0(r,z) \leq 2\arctan(Dr), \quad h_{\min}(\rho,z,t) \leq 2\arctan(D\rho)$$

for all $r \in [0,\rho], z \in \mathbb{R}, t \in [0,T+\delta]$. By the same argument used to conclude the proof of Theorem 2.4-(iii) we obtain that

$$h_{\min}(r,z,t) \leq 2\arctan(Dr) \qquad r \in (0,\rho], z \in \mathbb{R}, t \in [0,T+\delta].$$

Thus for all $z \in \mathbb{R}$

$$\lim_{r\to 0^+} h_{\min}(r,z,T+\delta) = 0.$$

But this implies $I_0(T+\delta) = \mathbb{R}$, thus contradicting the definition of $T$.

Let us suppose that $T < \infty$. Since $I_0(T) \neq \mathbb{R}$, one of the two following cases occurs:

(A) $\limsup_{r\to 0^+} h_{\min}(r,0,T) > 0$ while $h_{\min}(0,z,T) = 0$ for $z \neq 0$; or

(B) there exists $\zeta \in (0,\infty)$ such that $h_{\min}(0,z,T) = \pi$ for $|z| < \zeta$ and $h_{\min}(0,z,T) = 0$ for $|z| > \zeta$.

In both cases, if $\nabla h_{\min} = ((h_{\min})_r, (h_{\min})_z)$ is the gradient of $h_{\min}$ with respect to the variables $r$ and $z$, and

$$\|\nabla h_{\min}\|_\infty(t) = \sup_{(r,z)\in(0,1]\times\mathbb{R}} |\nabla h_{\min}|(r,z,t) \qquad t \geq 0,$$

we have that $\|\nabla h_{\min}\|_\infty(T) = \infty$. On the other hand, if $\tau < T$, then $I_0(\tau) = \mathbb{R}$ and, due to the regularity of $h$ out of $\{r = 0\}$ and to Theorem 2.4-(iii),

$$\sup_{t\in[0,\tau]} \|\nabla h_{\min}\|_\infty(t) < \infty.$$

This explains why $T$ is called *first time of blow-up* for the function $h_{\min}$.

For what is already known, it could also occur that $T = \infty$. In the next section we prove that $T < \infty$ and the function

$$\begin{array}{rccc} \zeta: & [T,\infty) & \longrightarrow & [0,\infty) \\ & t & \longrightarrow & |I_1(t)|/2, \end{array} \tag{2.12}$$

where $|\cdot|$ is one-dimensional Lebesgue measure, is right continuous (in addition to being sublinear, according to Theorem 2.4-(i)).

## 3. Blow up of $h_{\min}$ in finite time

**Theorem 3.1.** *The first time of blow up of $h_{\min}$ is less than or equal to $\mathcal{T}$.*

**Proof.** $h_{\min}$ is a solution of problem (1.3) smooth in $(0,1) \times \mathbb{R} \times \mathbb{R}^+$ and continuous in $[0,1] \times \mathbb{R} \times [0,T)$, while $\tilde{\xi}$ is a subsolution of it in the time interval $[0,\mathcal{T})$ by property (P5) of $h_0$. By contradiction, let $T > \mathcal{T}$. Then, $I_0(\mathcal{T}) = \mathbb{R}$ and, by Theorem 2.4, (iii), both $h_{\min}$ and $\tilde{\xi}$ satisfy the hypothesis (h1) of Lemma C.1 with $\mathfrak{T} = \mathcal{T}$. Due to (P5) and (2.4), for every $r \in [0,1], z \in \mathbb{R}$ and $t \in [0,\tau]$, with $\tau \in (0,\mathcal{T})$, we have that

$$\begin{aligned} \tilde{\xi}(r,z,t) - h_{\min}(r,z,t) &= \tilde{\xi}(r,z,t) - \tilde{\xi}(r,z,0) + \tilde{\xi}(r,z,0) - h_{\min}(r,z,t) \\ &\le \tilde{\xi}(r,z,t) - \tilde{\xi}(r,z,0). \end{aligned}$$

Since the last function goes to 0 as $|z| \to \infty$ uniformly with respect to $r \in [0,1]$ and $t \in [0,\tau]$, $h_{\min}$ and $\tilde{\xi}$ also satisfy the hypothesis (h2) of Lemma C.1 with $\mathfrak{T} = \mathcal{T}$. Then

$$\tilde{\xi} \le h_{\min} \quad \text{in } [0,1] \times \mathbb{R} \times [0,\mathcal{T})$$

and, since $h_{\min}$ is continuous in $(0,1] \times \mathbb{R} \times [0,\infty)$, it follows that, for every $r \in (0,1]$, $z \in [-\mathcal{Z},\mathcal{Z}]$,

$$\lim_{r\to 0^+} \left( \lim_{t\to\mathcal{T}^-} \tilde{\xi}(r,z,t) \right) \le \lim_{r\to 0^+} h_{\min}(r,z,\mathcal{T}).$$

Therefore, for every $z \in [-\mathcal{Z},\mathcal{Z}], h_{\min}(0,z,\mathcal{T}) = \pi \quad \Rightarrow I_0(\mathcal{T}) \neq \mathbb{R}$, which contradicts the assumption $T > \mathcal{T}$. □

Now we have to show that the function $\zeta$ defined by (2.12) is right continuous. Since $h_{\min}$ is monotone increasing with respect to the time variable

$t$ and so is $\zeta$, to prove the right continuity of $\zeta$ is sufficient to show the following.

**Theorem 3.2.** *For every $\tau \in [T, \infty)$, $\limsup_{t\to\tau^+} \zeta(t) \le \zeta(\tau)$.*

**Proof.** For fixed arbitrary $\tau \ge T$, in view of Theorem 2.4 (iv), for every $\varepsilon > 0$ there exist $\rho \in (0,1)$ and $C > 0$ (both $\rho$ and $C$ depend on $\varepsilon$) such that

$$h_{\min}(r,z,t) \le 2\arctan(Cr) \qquad r \in [0,\rho], |z| \ge \zeta(\tau)+\varepsilon, t \in [0,\tau].$$

Given any $\varepsilon > 0$, if $R = \min(\rho, b/(4C))$, then $R \in (0,\rho]$ and

$$h_{\min}(r,z,\tau) \le \arctan\left(\frac{br}{R}\right) \qquad \forall r \in [0,R],\, \forall z \ge \zeta(\tau)+\varepsilon. \tag{3.1}$$

If $\psi$ is the function of Lemma D.2, then in view of D.2,(i), D.2,(ii), D.2,(iv) and the inequality $h_{\min}(r,z,t) \le \theta_a(r)$, there exists $\mathcal{R} \in (0,R]$ such that

$$\psi(r,y) \ge \frac{3\pi}{2} \ge h_{\min}(r,z,t) + \frac{\pi}{4} \;\; \forall r \in [0,\mathcal{R}],\, y \le -\varepsilon \text{ and } \forall (z,t) \in \mathbb{R}\times\mathbb{R}^+. \tag{3.2}$$

At the same time, it follows from (3.1) and Lemma D.2,(ii)-(iii) that for every $r \in [0,\mathcal{R}]$, $z \ge \zeta(\tau)+\varepsilon$, and $y \in \mathbb{R}$,

$$h_{\min}(r,z,\tau) + \arctan(br/R) \le 2\arctan(br/R) \le \psi(r,y).$$

Thus, if $C = \zeta(\tau) + 2\varepsilon - \tau$ and $w$ is the function

$$w(r,z,t) = \psi(r, z-t-C) \qquad r \in [0,R],\, z \in \mathbb{R}, t \ge 0,$$

then $w$ is a solution of equation (2.1) and for all $r \in [0,\mathcal{R}], z \ge \zeta(\tau)+\varepsilon$,

$$h_{\min}(r,z,\tau) + \arctan(br/R) \le w(r,z,\tau).$$

In addition, for every $r \in [0,\mathcal{R}]$ and $z \le \zeta(\tau)+\varepsilon$,

$$\begin{aligned} w(r,z,\tau) &= \psi(r, z-\tau-C) \ge \psi(r, \zeta(\tau)+\varepsilon-\tau-C) \\ &= \psi(r,-\varepsilon) \ge h_{\min}(r,z,\tau) + \pi/4 \end{aligned}$$

by the monotonicity of $\psi$ and (3.2). Therefore, for all $r \in [0,\mathcal{R}]$ and $z \in \mathbb{R}$

$$\begin{aligned} & w(r,z,\tau) - h_{\min}(r,z,\tau) \ge \min(\pi/4,\, \arctan(br/R)) = \arctan(br/R) \\ \Rightarrow \quad & w(\mathcal{R},z,\tau) - h_{\min}(\mathcal{R},z,\tau) \ge \arctan(b\mathcal{R}/R) > 0 \qquad \forall z \in \mathbb{R}. \end{aligned}$$

Then, since $w$ is increasing with respect to $t$ and, by parabolic Schauder-type estimates, $h_{\min}(\mathcal{R},z,t)$ is continuous in $t$ uniformly with respect to $z \in \mathbb{R}$ for $t$ belonging to a neighborhood of $\tau$, there exists $\delta > 0$ such that

$$w(\mathcal{R},z,t) \ge h_{\min}(\mathcal{R},z,t) \qquad \forall z \in \mathbb{R},\, t \in [\tau, \tau+\delta].$$

Because of the inequalities

$$h_0(r,z) \le h_n(r,z,t) \le h_{\min}(r,z,t) \qquad \forall n \in \mathbb{N} \text{ with } n \ge 2,\ (r,z) \in \bar{A}_n,\ t \ge 0$$

and the monotonicity of $w$ with respect to $t$, we deduce that for every $n \in \mathbb{N}$ with $n > 1/\mathcal{R}$ the function $w$ is a supersolution of the differential problem

$$\begin{cases} h_t \;=\; h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & \text{in } (1/n,\mathcal{R}) \times (-n,n) \times (\tau,\infty) \\ h(r,z,\tau) \;=\; h_n(r,z,\tau) & \text{for } 1/n < r < \mathcal{R}, z \in (-n,n) \\ h(r,\pm n,t) = h_0(r,\pm n) & \text{for } r \in [1/n,\mathcal{R}],\ t \ge \tau \\ h(1/n,z,t) \;=\; h_0(1/n,z) & \text{for } z \in [-n,n], t \ge \tau \\ h(\mathcal{R},z,t) \;=\; h_n(\mathcal{R},z,t) & \text{for } z \in [-n,n], t \ge \tau \end{cases}$$

in the time interval $[\tau,\tau+\delta]$.

Since $h_n$ is the solution of this problem,

$$w(r,z,t) \ge h_n(r,z,t) \qquad \forall r \in [1/n,\mathcal{R}],\ z \in [-n,n],\ t \in [\tau,\tau+\delta],$$

and, passing to the limit as $n \to \infty$,

$$w(r,z,t) \ge h_{\min}(r,z,t) \qquad \forall r \in (0,\mathcal{R}],\ z \in \mathbb{R},\ t \in [\tau,\tau+\delta].$$

Then for every $t \in [\tau,\tau+\delta]$ and for all $z > t + \zeta(\tau) + 2\varepsilon - \tau$

$$0 \le \lim_{r\to 0^+} h_{\min}(r,z,t) \le \psi(0,\ z - t - \zeta(\tau) - 2\varepsilon + \tau) = 0,$$

which implies that

$$\zeta(t) = \inf\{z \in \mathbb{R}^+ : h_{\min}(0,z,t) = 0\} \le t + \zeta(\tau) + 2\varepsilon - \tau \qquad \forall t \in [\tau,\tau+\delta]$$

and

$$\limsup_{t\to\tau^+} \zeta(t) \le \zeta(\tau) + 2\varepsilon.$$

The thesis follows from the arbitrariness of $\varepsilon > 0$. □

## 4. Construction of $h_M$

This last section is devoted to the construction of a family $\{h_M\}_{M>0}$ of weak solutions of problem (1.3) such that $h_M \ge h_{\min}$ and, for every $t > 0$,

$$\lim_{r\to 0^+} h_M(r,z,t) \;=\; \pi \;\text{ if } |z| \le M, \quad \lim_{r\to 0^+} h_M(r,z,t) \;=\; 0 \;\text{ if } |z| > S + M + t,$$

where $S > 0$ is a constant independent of $M$. For every $n \in \mathbb{N}, n \geq 2$, let $\omega_n$ be the function defined as

$$\omega_n(r,z) = \begin{cases} \pi & r = 0,\ |z| < M+2 \\ 0 & r = 0,\ |z| \geq M+2 \\ 2\arctan\left(\frac{1+b}{1-b}\frac{\gamma_{M+1}(z)}{nr^2}\right) & r \in (0,1],\ z \in \mathbb{R}, \end{cases}$$

where $\gamma_{M+1}$ belongs to the family of functions given in (A.17). Then the following assertion is true.

**Proposition 4.1.** *There exists a constant $C > 0$ such that*

$$\int_0^1 \frac{r}{2}\Big((\omega_n)_r^2 + (\omega_n)_z^2 + \frac{\sin^2(\omega_n)}{r^2}\Big)\mathrm{d}r \,\leq\, C$$

*for every $n \in \mathbb{N}, n \geq 2$ and for every $z \in \mathbb{R}$.*

**Proof.** For the sake of simplicity we fix $n \in \mathbb{N}$, $n \geq 2$, and we denote by $\omega$ the function $\omega_n$ and by $\gamma$ the function $\frac{1+b}{1-b}\frac{\gamma_{M+1}}{n}$, so that

$$\omega(r,z) = 2\arctan(\gamma(z)/r^2), \qquad \frac{\sin\omega}{r} = \frac{2\gamma(z)r}{r^4+\gamma^2(z)}$$

and

$$\omega_r = \frac{-4\gamma(z)r}{r^4+\gamma^2(z)}, \quad \omega_z = \frac{2\gamma'(z)r^2}{r^4+\gamma^2(z)}.$$

Therefore, for every fixed $z \in \mathbb{R}$,

$$\int_0^1 \frac{r}{2}\Big(\omega_r^2 + \frac{\sin^2\omega}{r^2}\Big)\mathrm{d}r = \int_0^1 \frac{10\gamma^2 r^3}{(r^4+\gamma^2)^2}\mathrm{d}r = \frac{5}{2}\gamma^2\int_{\gamma^2}^{1+\gamma^2}\frac{\mathrm{d}s}{s^2} \leq 5/2.$$

If $|z| \leq M+1$ or $|z| \geq M+2$, then

$$\int_0^1 \frac{r}{2}\omega_z^2\mathrm{d}r = 0,$$

otherwise

$$\int_0^1 \frac{r}{2}\omega_z^2\mathrm{d}r = \frac{1}{2}\left|\frac{d\gamma}{dz}\right|^2\int_0^1 \frac{4r^5}{(r^4+\gamma^2)^2}\mathrm{d}r = \frac{1}{2}\left|\frac{d\gamma}{dz}\right|^2\int_{\gamma^2}^{1+\gamma^2}\frac{\sqrt{s-\gamma^2}}{s^2}\mathrm{d}s \leq$$

$$\frac{1}{2}\left|\frac{d\gamma}{dz}\right|^2\int_{\gamma^2}^{1+\gamma^2} s^{-3/2}\mathrm{d}s \leq \frac{1}{\gamma}\left|\frac{d\gamma}{dz}\right|^2 = \frac{1+b}{1-b}\,\frac{1}{n}\,\frac{1}{\gamma_{M+1}}\left|\frac{d\gamma_{M+1}}{dz}\right|^2 \leq \frac{1+b}{1-b}N_2,$$

where $N_2$ is the constant of (A.17). □

For every $n \in \mathbb{N}, n \geq 2$ we consider the differential problem

$$(\mathcal{P}_n) \begin{cases} h_t \;=\; h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & (r,z,t) \in A_n \times \mathbb{R}^+ \\ h(r,z,0) = h_{0n}(r,z) & (r,z) \in \bar{A}_n \\ h(1/n,z,t) = h_{0n}(1/n,z) & (z,t) \in [-n,n] \times \mathbb{R}^+ \\ h(1,z,t) = h_{0n}(1,z) & (z,t) \in [-n,n] \times \mathbb{R}^+ \\ h(r,\pm n,t) = h_{0n}(r,\pm n) & (r,t) \in [1/n,1] \times \mathbb{R}^+ \end{cases}$$

with $A_n = (1/n,\, 1) \times (-n,n)$ and $h_{0n} = \max(\omega_n,\, h_0)$.

The functions $h_{0n}$ ($n \in \mathbb{N},\, n \geq 2$) satisfy the following properties:

(A1) $\theta_b(r) \leq h_{0n}(r,z) \leq \theta_a(r)$, $h_{0n}(r,z) \equiv \theta_b(r)$ if $|z| \geq \max(\bar{z}, M+2)$;
(A2) there exists a constant $C = C(h_0) > 0$ such that for every $n \in \mathbb{N}$ and every $z \in \mathbb{R}$

$$\int_0^1 \frac{r}{2}\Big((h_{0n})_r^2 + (h_{0n})_z^2 + \frac{\sin^2(h_{0n})}{r^2}\Big)\mathrm{d}r \leq C;$$

(A3) $h_{0n} \geq h_0$, $h_{0n}(r,-z) \equiv h_{0n}(r,z)$, $(h_{0n})_z(r,z) \leq 0$ for $r \in [0,1], z \geq 0$; and
(A4) for every $n \geq 2$ the function $h_{0n}$ is Lipschitz continuous in $[1/n,1]\times\mathbb{R}$.

Moreover, up to a regularization, we can also assume that $h_{0n} \in C^\infty([1/n,1] \times\mathbb{R})$. By standard solvability, comparison and regularity results for parabolic problems (see [8]) and since the functions $\theta_b(r) = 2\arctan(br)$, $\theta_a(r) = \pi + 2\arctan(ar)$ both solve (2.1), we can say that for every $n \in \mathbb{N}$ Problem $(\mathcal{P}_n)$ has a unique classical solution $H_n \in C^\infty(\bar{A}_n \times \mathbb{R}^+) \cap C^0(\bar{A}_n \times [0,\infty))$ and

$$\begin{gathered} 2\arctan(br) \leq H_n(r,z,t) \leq \pi + 2\arctan(ar), \\ H_n(r,-z,t) = H_n(r,z,t), \\ (H_n)_r, (H_n)_z \text{ are Hölder continuous in } (r,z,t). \end{gathered} \tag{4.1}$$

In addition, due to (P1) and since $h_{0n} \geq h_0$, $h_0$ is a subsolution of Problem $(\mathcal{P}_n)$ and

$$H_n(r,z,t) \geq h_0(r,z) \qquad \forall (r,z,t) \in \bar{A}_n \times [0,\infty). \tag{4.2}$$

Due to (4.1) and to the properties of $h_{0n}$, for every $n \in \mathbb{N}$ with $n > \max(\bar{z}, M+2)$, the function $(H_n)_z$ is a subsolution of the problem

$$\begin{cases} \psi_t = \psi_{rr} + \psi_{zz} + \frac{\psi_r}{r} - \frac{\cos(2h)}{r^2}\psi & (r,z,t) \in (1/n,1) \times (0,n) \times \mathbb{R}^+ \\ \psi(r,z,0) = 0 & (r,z) \in [1/n,1] \times [0,n] \\ \psi(1/n,z,t) = 0 & (z,t) \in [0,n] \times \mathbb{R}^+ \\ \psi(1,z,t) = 0 & (z,t) \in [0,n] \times \mathbb{R}^+ \\ \psi(r,0,t) = 0 & (r,t) \in [1/n,1] \times \mathbb{R}^+ \\ \psi(r,n,t) = 0 & (r,t) \in [1/n,1] \times \mathbb{R}^+. \end{cases}$$

From the parabolic comparison principle it follows that for every $n \in \mathbb{N}$ with $n > \max(\bar{z}, M+2)$

$$(H_n)_z(r,z,t) \leq 0 \qquad \text{if } z \geq 0. \tag{4.3}$$

If we define the functional $E_n$ just as in Section 3.2, then we have the following.

**Proposition 4.2.** *For every $n \geq 2$ and $T > 0$*

$$E_n(H_n(\cdot\,,\cdot\,,T)) + \iiint_{A_n \times [0,T]} r(H_n)_t^2 \mathrm{d}r\mathrm{d}z\mathrm{d}t \;=\; E_n(h_{0n}).$$

Again the proof is completely standard.

Let $E_{n,\zeta}$ ($n \in \mathbb{N}, n \geq 2$, $\zeta > 0$) be the functional defined by (2.5). Then we have the following.

**Proposition 4.3.** *There exists a constant $C = C(h_0) > 0$ such that for every $T > 0$ and $\zeta > \max(\bar{z},\, M+2)$*

$$E_{n,\zeta}(H_n(\cdot,\,\cdot,\,T)) + \iiint_{A_n \times [0,T]} r(H_n)_t^2 \mathrm{d}r\mathrm{d}z\mathrm{d}t \leq 2C(h_0)\zeta$$

*for all $n \geq \zeta$.*

**Proof.** Let $T > 0$, $\zeta > \max(\bar{z},\, M+2)$ be arbitrarily fixed. From Proposition 4.2 it follows that for all $n \in \mathbb{N}, n \geq \zeta$,

$$E_{n,n}(H_n(\cdot\,,\cdot\,,T)) + \iiint_{A_n \times [0,T]} r(H_n)_t^2 \mathrm{d}r\mathrm{d}z\mathrm{d}t \;=\; E_{n,n}(h_{0n}). \tag{4.4}$$

If $|z| > \max(\bar{z}, M+2)$, then $h_{0n}(r,z) \equiv \theta_b(r)$ and

$$\int_{1/n}^1 \frac{r}{2}\Big((h_{0n})_r^2 + (h_{0n})_z^2 + \frac{\sin^2(h_{0n})}{r^2}\Big)\mathrm{d}r = \int_{1/n}^1 \frac{r}{2}\Big((\theta_b)_r^2 + \frac{\sin^2(\theta_b)}{r^2}\Big)\mathrm{d}r.$$

At the same time $H_n(1/n, z, T) = h_{0n}(1/n, z) = \theta_b(1/n)$, $H_n(1, z, T) = h_{0n}(1, z) = \theta_b(1)$, and by [3], Corollary 21,

$$\int_{1/n}^{1} \frac{r}{2}\Big((H_n)_r^2(r,z,T) + (H_n)_z^2(r,z,T) + \frac{\sin^2 H_n(r,z,T)}{r^2}\Big)\mathrm{d}r$$
$$\geq \int_{1/n}^{1} \frac{r}{2}\Big((\theta_b)_r^2 + \frac{\sin^2(\theta_b)}{r^2}\Big)\mathrm{d}r.$$

Therefore from (4.4) it follows that for all $n \in \mathbb{N}, n \geq \zeta$,

$$E_{n,\zeta}(H_n(\cdot,\cdot,T)) + \iiint_{A_n\times[0,T]} r(H_n)_t^2 \mathrm{d}r\mathrm{d}z\mathrm{d}t \leq E_{n,\zeta}(h_{0n})$$
$$\leq \iint_{[0,1]\times[-\zeta,\zeta]} \frac{r}{2}\Big((h_{0n})_r^2 + (h_{0n})_z^2 + \frac{\sin^2(h_{0n})}{r^2}\Big)\mathrm{d}r\mathrm{d}z \leq 2C\zeta,$$

where $C = C(h_0) > 0$ is the constant appearing in (A2). □

By parabolic Schauder type estimates (see [8]), we can say that, up to a subsequence, as $n \to \infty$,

$$H_n \to h_M \quad \text{in } C^{2,1}([\rho,1]\times\mathbb{R}\times[\sigma,\tau]) \cap C^0([\rho,1]\times\mathbb{R}\times[0,\tau])$$

for every $\rho \in (0,1), 0 < \sigma < \tau$. Thanks to Proposition 4.3 and Fatou's lemma,

$$(h_M)_t \in L_r^2((0,1)\times\mathbb{R}\times\mathbb{R}^+), \quad \frac{\sin h_M}{r}, \nabla h_M \in L^\infty(\mathbb{R}^+; L_r^2((0,1)\times(-\zeta,\zeta)))$$

for every $\zeta > 0$. Thus $h_M$ is a weak solution of problem (1.3) and, by (4.1)-(4.3),

$$\begin{gathered}\theta_b(r) \leq h_M(r,z,t) \leq \theta_a(r), \quad h_M(r,-z,t) \equiv h_M(r,z,t),\\ h_M \geq h_0, \quad h_M(r,z',t) \leq h_M(r,z,t) \quad \text{if } z' \geq z \geq 0.\end{gathered} \tag{4.5}$$

By Proposition 2.2, $h_M \geq h_{\min}$.

It follows from Proposition A.10 that the following holds.

**Theorem 4.4.** *For all $t > 0$ and $z \in [-M, M]$*

$$\lim_{r\to 0^+} h_M(r,z,t) = \pi.$$

**Proof.** Let $\mathcal{H}$ be the function defined by

$$\mathcal{H}(r,z,t) = \begin{cases} 2\arctan\left(\frac{\mathrm{e}^{-\frac{Q}{t\gamma_M(z)}}}{r}\right) + br^{3/2} & r\in(0,1],\ t\in(0,T],\ |z|<M+1\\ br^{3/2} & r\in(0,1],\ |z|\geq M+1 \text{ or } t=0,\end{cases}$$

where $\gamma_M$ belongs to the family of functions in (A.17) and $Q > 0$ is a positive constant satisfying a condition that we specify later. Since $b \in (0,1)$, for every $n \in \mathbb{N}$, $n \geq M+1$ we have that

(i) $\mathcal{H}(r,z,0) = br^{3/2} \leq br \leq 2\arctan(br) \leq h_{0n}(r,z) \quad \forall (r,z) \in (0,1]\times\mathbb{R}$,

(ii) $\mathcal{H}(r,\pm n,t) = br^{3/2} \leq br \leq 2\arctan(br) \leq h_{0n}(r,\pm n) \quad \forall (r,t) \in (0,1]\times\mathbb{R}^+$, and

(iii) $\mathcal{H}(1/n,z,t) \leq h_{0n}(1/n,z) \quad \forall z \in \mathbb{R},\ t > 0$.

Indeed, if $|z| \geq M+1$, then

$$\mathcal{H}(1/n,z,t) = b(1/n)^{3/2} \leq b/n \leq 2\arctan(b/n) \leq h_{0n}(1/n,z),$$

otherwise

$$\mathcal{H}(1/n,z,t) = 2\arctan\left(n\mathrm{e}^{-\frac{Q}{t\gamma_M(z)}}\right) + bn^{-\frac{3}{2}} < 2\arctan(n) + 2\arctan\left(\frac{b}{n}\right)$$

$$= 2\arctan\left(\frac{n+b/n}{1-b}\right) < 2\arctan\left(\frac{1+b}{1-b}\,n\right) = \omega_n(1/n,z) \leq h_{0n}(1/n,z).$$

Let $T$ be a positive value arbitrarily fixed and $t \in (0,T]$. If $|z| \geq M+1$, then $\mathcal{H}(1,z,t) = b \leq 2\arctan(b) \leq h_{0n}(1,z)$, otherwise

$$\mathcal{H}(1,z,t) = 2\arctan\left(\mathrm{e}^{-\frac{Q}{t\gamma_M(z)}}\right) + b \leq 2\arctan\left(\mathrm{e}^{-\frac{Q}{T}}\right) + b \leq h_{0n}(1,z), \tag{4.6}$$

provided that $Q \geq K$, where $K > 0$ is a suitable constant depending on $b$ and $T$. If we take $Q \geq \max(\mathcal{Q},K)$, where $\mathcal{Q}$ is the constant of Proposition A.10, whose value only depends on $b$ and $T$, then, due to (i), (ii), (iii), (4.6) and Proposition A.10, $\mathcal{H}$ is a subsolution of $(\mathcal{P}_n)$ in the time interval $[0,T)$ for every $n \in \mathbb{N}$ with $n \geq M+1$. By a parabolic comparison principle,

$$\mathcal{H} \leq H_n \ \text{ in } \bar{A}_n \times [0,T),\ \ \forall n \geq M+1 \ \Rightarrow\ \mathcal{H} \leq h_M \quad \text{in } (0,1]\times\mathbb{R}\times[0,T).$$

Together with the inequality $h_M \leq \theta_a$ this implies that for every $z \in [-M,M]$ and $t \in (0,T)$

$$h_M(r,z,t) \longrightarrow \pi \ \text{ as } \ r \to 0^+.$$

The thesis then follows from the arbitrariness of $T > 0$. □

**Theorem 4.5.** *There exists a constant $S > 0$ such that*

$$\lim_{r\to 0^+} h_M(r,z,t) = 0$$

*for every $t > 0$ and $|z| > S + M + t$.*

**Proof.** Let $\psi = \psi(r, y)$ the same function as in the proof of Theorem 2.4 (i), and let $Z > 0$ be a value such that

$$\pi + 2\arctan(a) \leq \psi(r, z) \qquad \forall r \in [0, 1],\ z \leq -Z.$$

Due to the properties of $\psi$, it must be that $-Z \leq \bar{y}$; i.e., $\bar{y} + Z \geq 0$. Since $h_{0n} \leq \pi + 2\arctan(a)$ and $h_{0n}(r, z) = \theta_b(r)$ for $z \geq \max(\bar{z}, M + 2)$, if

$$\sigma = \max(\bar{z}, M + 2) + Z,$$

then $\psi(r, \max(\bar{z}, M + 2) - \sigma) = \psi(r, -Z) \geq \pi + 2\arctan(a)$ and

$$\psi(r, z - \sigma) \geq h_{0n}(r, z) \quad \forall n \in \mathbb{N}, n \geq 2.$$

If we define

$$h(r, z, t) = \psi(r, z - t - \sigma),$$

then $h$ solves (2.1) and for every $(r, z, t) \in (0, 1] \times \mathbb{R} \times [0, \infty)$

$$h(r, z, t) \geq h(r, z, 0) = \psi(r, z - \sigma) \geq h_{0n}(r, z)$$

for all $n \in \mathbb{N}$, $n \geq 2$. By a parabolic comparison principle, $h \geq H_n$ for every $n \geq 2$. Consequently, $h \geq h_M$ and at any time $t \geq 0$

$$0 \leq h_M(0, z, t) \leq \psi(0, z - t - \sigma) = 0 \qquad \text{if } z > \bar{y} + \sigma + t.$$

Taking into account that $h_M(r, -z, t) \equiv h_M(r, z, t)$, to obtain the thesis we have just to choose

$$S = \bar{y} + \bar{z} + 2 + Z \quad \Rightarrow \quad S + M \geq \bar{y} + \sigma. \qquad \square$$

**Remark 4.6.** Let $\mathcal{M}_2 > \mathcal{M}_1 \geq 0$. From the definition of $\omega_n$ it follows that the initial and boundary data of Problem $(\mathcal{P}_n)$ for $M = \mathcal{M}_1$ are less than the data corresponding to the choice $M = \mathcal{M}_2$. By a parabolic comparison principle we then obtain $h_{\mathcal{M}_1} \leq h_{\mathcal{M}_2}$.

**Remark 4.7.** Since for every $M \geq 0$ the function $h_M$ is a weak solution of problem (1.3) satisfying (4.5), by the same arguments used for $h_{\min}$ it is possible to prove that if

$$I_0^M(t) := \{z \in \mathbb{R} : h_M(0, z, t) = 0\} \quad \text{and} \quad I_1^M(t) := \{z \in \mathbb{R} : h_M(0, z, t) = \pi\},$$

then for every $t \geq 0$ there exists $\zeta_M(t) \in [0, \infty]$ such that

$$I_1^M(t) = (-\zeta_M(t), \zeta_M(t)), \quad I_0^M(t) = (-\infty, -\zeta_M(t)) \cup (\zeta_M(t), \infty),$$

up to the values $\pm\zeta_M(t)$. In terms of the function $\zeta_M$ the last two theorems can be reformulated by saying that

$$\exists\, S > 0 \text{ (not depending on } M) \text{ such that } M \leq \zeta_M(t) \leq S + M + t \quad \forall t > 0.$$

## APPENDIX A. SUFFICIENT CONDITIONS FOR SUB- AND SUPERSOLUTIONS

Setting

$$H_1(r,z,t) = 2\arctan\Big(\frac{r}{\lambda(z,t)}\Big), \quad H_2(r) = Br^\alpha \quad r \in [0,1], z \in \mathbb{R}, t \geq 0, \tag{A.1}$$

where $B \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and $\lambda = \lambda(z,t)$ is a positive smooth function, we look for sufficient conditions in order that $H(r,z,t) := H_1(r,z,t) + H_2(r)$ is a sub- or supersolution of the equation

$$h_t \;=\; h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2}.$$

We need the following trivial estimate.

**Lemma A.1.** *Let $B > 0$ and $\delta \in [0,1]$. If*

$$q(r) := \frac{2B^{1-\delta}r^{1+\delta}}{r^2+B^2} \quad \textit{for } r > 0,$$

*then for every $r > 0$*

$$0 \leq q(r) \leq \begin{cases} (1-\delta)^{\frac{1-\delta}{2}}(1+\delta)^{\frac{1+\delta}{2}} & \textit{if } \delta \in [0,1) \\ 2 & \textit{if } \delta = 1. \end{cases}$$

**Lemma A.2.** *If $B > 0$ and $\alpha \in (\sqrt{2}, 3]$, there exists a constant*

$$C(\alpha) \in [(\alpha^2-2)/2, \alpha^2-2]$$

*such that, if $\lambda$ satisfies the differential inequality*

$$\lambda_t - \lambda_{zz} \geq -BC\lambda^{\alpha-1} \qquad \textit{in } \mathbb{R}\times(0,T) \;\; (T \in (0,\infty]), \tag{A.2}$$

*then $H$ is a subsolution to* (2.1) *in $(0,1)\times\mathbb{R}\times(0,T)$.*

**Proof.** A straightforward computation shows that

$$\begin{aligned} \frac{(rH_r)_r}{r} - \frac{\sin(2H)}{2r^2} &= \frac{\sin(2H_1) + 2\alpha^2 H_2 - \sin(2H_1+2H_2)}{2r^2} \\ &= \frac{H_2}{r^2}\Big(\sin(2H_1)\frac{\sin^2 H_2}{H_2} + \alpha^2 - \frac{\cos(2H_1)\sin(2H_2)}{2H_2}\Big) \\ &\geq \frac{H_2}{r^2}(\alpha^2-2) = Br^{\alpha-2}(\alpha^2-2), \end{aligned} \tag{A.3}$$

where we have used the fact that $|\sin x| \leq |x|$. On the other hand

$$H_t - H_{zz} = -\frac{\sin H_1}{\lambda}\Big(\lambda_t - \lambda_{zz} + (1+\cos H_1)\frac{\lambda_z^2}{\lambda}\Big) \leq -\frac{\sin H_1}{\lambda}\,(\lambda_t - \lambda_{zz}) \tag{A.4}$$

since $\sin H_1 \geq 0$. From (A.3) and (A.4) it follows that

$$Br^{\alpha-2}(\alpha^2 - 2) \geq -\frac{\sin H_1}{\lambda}(\lambda_t - \lambda_{zz}) \tag{A.5}$$

is a sufficient condition for $H$ to be a subsolution of (2.1). Since $\sin H_1 = 2\lambda r\left(\lambda^2 + r^2\right)^{-1} > 0$, (A.5) can be written as

$$\lambda_t - \lambda_{zz} \geq -B(\alpha^2 - 2)\frac{\lambda^2 + r^2}{2r^{3-\alpha}}. \tag{A.6}$$

By Lemma A.1 there exists $C(\alpha) \in [1, 2]$ such that

$$\frac{2r^{3-\alpha}}{\lambda^2 + r^2} = \frac{2r^{3-\alpha}\lambda^{\alpha-1}}{\lambda^2 + r^2}\lambda^{1-\alpha} \leq C(\alpha)\lambda^{1-\alpha}.$$

Then

$$-B(\alpha^2 - 2)\frac{\lambda^2 + r^2}{2r^{3-\alpha}} \leq \frac{-B(\alpha^2 - 2)}{C(\alpha)}\lambda^{\alpha-1}.$$

Redefining $C(\alpha)$ as $(\alpha^2 - 2)/C(\alpha)$, we obtain from (A.6) that (A.2) is a sufficient condition for $H$ to be a subsolution of (2.1). □

Let $\zeta \in [-\infty, \infty)$, $T \in (0, \infty]$.

**Lemma A.3.** *If $B < 0$ and $\alpha \in (\sqrt{2}, 3]$, then there exists a constant $C(\alpha) \in [(\alpha^2 - 2)/2, \alpha^2 - 2]$ such that if $\lambda$ satisfies the differential inequality*

$$\lambda_t - \lambda_{zz} + 2\,\frac{\lambda_z^2}{\lambda} \leq |B|C\lambda^{\alpha-1} \qquad \text{in } (\zeta, \infty) \times (0, T), \tag{A.7}$$

*then $H$ is a supersolution to (2.1) in the open set $(0, 1) \times (\zeta, \infty) \times (0, T)$.*

**Proof.** Just as in the proof of Lemma A.2

$$H_{rr} + \frac{H_r}{r} - \frac{\sin(2H)}{2r^2} = \frac{H_2}{r^2}\Big(\sin(2H_1)\frac{\sin^2 H_2}{H_2} + \alpha^2 - \frac{\cos(2H_1)\sin(2H_2)}{2H_2}\Big).$$

Since the expression within parentheses is always greater than or equal to $\alpha^2 - 2$ and $\frac{H_2}{r^2} = Br^{\alpha-2} < 0$,

$$H_t - H_{zz} \geq (\alpha^2 - 2)Br^{\alpha-2} \tag{A.8}$$

is a sufficient condition for $H$ to be a supersolution of (2.1). But

$$H_t - H_{zz} = -\frac{\sin H_1}{\lambda}\Big(\lambda_t - \lambda_{zz} + (1 + \cos H_1)\frac{\lambda_z^2}{\lambda}\Big)$$

and (A.8) is then equivalent to

$$\lambda_t - \lambda_{zz} + (1 + \cos H_1)\frac{\lambda_z^2}{\lambda} \leq \frac{\lambda}{\sin H_1}(\alpha^2 - 2)|B|r^{\alpha-2}. \tag{A.9}$$

Since

$$\frac{\lambda}{\sin H_1}(\alpha^2-2)|B|r^{\alpha-2} = \frac{\lambda^2+r^2}{2r^{3-\alpha}}(\alpha^2-2)|B|$$

and, as in the proof of Lemma A.2,

$$\frac{2r^{3-\alpha}}{\lambda^2+r^2} \leq C(\alpha)\lambda^{1-\alpha} \qquad \text{for a suitable } C(\alpha)\in[1,2],$$

$$\lambda_t - \lambda_{zz} + (1+\cos H_1)\frac{\lambda_z^2}{\lambda} \leq \frac{\lambda^{\alpha-1}}{C(\alpha)}|B|(\alpha^2-2)$$

implies (A.9). To end the proof redefine $C(\alpha)$ as $(\alpha^2-2)/C(\alpha)$. □

**Remark A.4.** It is easy to verify that the constants $C(\alpha)$ in Lemma's A.2 and A.3 are the same.

We want to construct, for every $\sigma \geq 0$ and $T > 0$, a subsolution $\xi$ of equation (2.1) having the form

$$\xi(r,z,t) = 2\arctan\left(\frac{r}{\lambda(z,t)}\right) + Br^\alpha,$$

where $B > 0$ and $\alpha \in (\sqrt{2}, 3]$ are given constants and $\lambda: \mathbb{R}\times[0,T) \longrightarrow \mathbb{R}^+$ is a smooth function, such that

$$\xi(0,z,t) = 0 \;\; \forall t \in [0,T), z\in\mathbb{R} \quad \text{and} \tag{A.10}$$

$$\lim_{r\to 0^+}\left(\lim_{t\to T^-}\xi(r,z,t)\right) = \begin{cases}\pi & \text{if } |z|\leq\sigma\\ 0 & \text{if } |z|>\sigma.\end{cases}$$

**Lemma A.5.** *Let $B, K, Q, \sigma > 0, \alpha \in (\sqrt{2}, 2)$ and let $\lambda$ be the function*

$$\lambda(z,t) = K\mathrm{e}^{\frac{-Q}{T-t+\mu_\sigma(z)}} \qquad z\in\mathbb{R}, t\in[0,T],$$

*where*

$$\mu_\sigma(z) = \begin{cases}|z-\sigma|^4 & \text{if } z>\sigma\\ 0 & \text{if } z\in[-\sigma,\sigma]\\ |z+\sigma|^4 & \text{if } z<-\sigma.\end{cases} \tag{A.11}$$

*There exists $\bar{Q} = \bar{Q}(B,K,\alpha) > 0$ such that, if $Q \geq \bar{Q}$, then $\lambda$ satisfies (A.2) for any $\sigma$.*

**Proof.** If we define $\phi(t) = T - t$, then

$$\lambda_t - \lambda_{zz} = KQ\mathrm{e}^{\frac{-Q}{\phi(t)+\mu_\sigma(z)}}\left(\frac{\phi'(t)-\mu_\sigma''(z)}{(\phi(t)+\mu_\sigma(z))^2} + \frac{2(\mu_\sigma'(z))^2}{(\phi(t)+\mu_\sigma(z))^3} - \frac{Q(\mu_\sigma'(z))^2}{(\phi(t)+\mu_\sigma(z))^4}\right).$$

Since

$$|\mu'_\sigma(z)| \equiv 4\mu_\sigma(z)^{3/4}, \qquad \mu''_\sigma(z) \equiv 12\mu_\sigma(z)^{1/2}$$

and $\phi \geq 0$ in $[0,T]$, we deduce that

$$\lambda_t - \lambda_{zz} \geq KQ\mathrm{e}^{\frac{-Q}{\phi(t)+\mu_\sigma(z)}}\Big(\frac{-1-12\mu_\sigma(z)^{1/2}}{(\phi(t)+\mu_\sigma(z))^2} - \frac{16Q\mu_\sigma(z)^{3/2}}{(\phi(t)+\mu_\sigma(z))^4}\Big) \geq$$

$$KQ\mathrm{e}^{\frac{-Q}{\phi(t)+\mu_\sigma(z)}}\Big(-\frac{1}{(\phi(t)+\mu_\sigma(z))^2} - \frac{12}{(\phi(t)+\mu_\sigma(z))^{3/2}} - \frac{16Q}{(\phi(t)+\mu_\sigma(z))^{5/2}}\Big).$$

Then

$$Q\lambda^{2-\alpha}\Big(\frac{1}{(\phi(t)+\mu_\sigma(z))^2} + \frac{12}{(\phi(t)+\mu_\sigma(z))^{3/2}} + \frac{16Q}{(\phi(t)+\mu_\sigma(z))^{5/2}}\Big) \leq BC \tag{A.12}$$

is a sufficient condition in order that (A.2) be true.

Taking $k = Q(2-\alpha)$ and $x = (\phi(t)+\mu_\sigma(z))^{-1}$, from the elementary inequality

$$\forall k,\beta > 0 \qquad x^\beta \mathrm{e}^{-kx} \leq \Big(\frac{\beta}{k\mathrm{e}}\Big)^\beta \quad \forall x \geq 0 \tag{A.13}$$

we infer that the left-hand side of (A.12) is less than or equal to

$$K^{2-\alpha}Q\Big(\Big(\frac{2}{Q(2-\alpha)\mathrm{e}}\Big)^2 + 12\Big(\frac{3/2}{Q(2-\alpha)\mathrm{e}}\Big)^{3/2} + 16Q\Big(\frac{5/2}{Q(2-\alpha)\mathrm{e}}\Big)^{5/2}\Big)$$
$$= K^{2-\alpha}\mathcal{C}(\alpha)\Big(\frac{1}{Q} + \frac{1}{\sqrt{Q}}\Big),$$

where $\mathcal{C}(\alpha)$ is a positive constant only depending on $\alpha$. Since also the constant $C$ in (A.12) only depends on $\alpha$, the thesis follows. □

Lemmas A.2 and A.5 imply at once the following.

**Proposition A.6.** *Let $B, T, \sigma, Q, K > 0$, $\alpha \in (\sqrt{2}, 2)$ and let $\xi$ be the function*

$$\xi(r,z,t) = 2\arctan\Big(\frac{r}{K}\mathrm{e}^{\frac{Q}{T-t+\mu_\sigma(z)}}\Big) + Br^\alpha.$$

*There exists $\bar{Q} = \bar{Q}(B,\alpha,K) > 0$ such that if $Q \geq \bar{Q}$, then $\xi$ is a subsolution of* (2.1) *in $(0,1)\times\mathbb{R}\times(0,T)$ and satisfies* (A.10).

**Remark A.7.** From the proof of Lemma A.5 it follows that

(1) $\bar{Q} \to \infty$ when $K \to \infty$,
(2) $\bar{Q}$ is a decreasing function of $B$.

The next results concern the construction of another type of subsolution for (2.1).

**Lemma A.8.** *Let $\alpha \in (\sqrt{2}, 3]$, $B \in \mathbb{R}^+$ and let $h$ be the function*

$$h(r,t) = 2\arctan\Big(\frac{\lambda(z,t)}{r}\Big) + Br^\alpha$$

*with $\lambda : (-\zeta, \zeta) \times (0,T) \longrightarrow \mathbb{R}^+$ a regular function ($\zeta, T > 0$).*
*There exists $C(\alpha) \in [(\alpha^2-2)/2, \alpha^2-2]$ such that, if $\lambda$ is a solution of*

$$\lambda_t - \lambda_{zz} + 2\frac{\lambda_z^2}{\lambda} \leq BC\lambda^{\alpha-1} \qquad in\ (-\zeta,\, \zeta) \times (0,T), \tag{A.14}$$

*then $h$ is a subsolution of (2.1) in $(0,1) \times (-\zeta,\, \zeta) \times (0,T)$.*

**Proof.** Let $H$ be the function defined by

$$H(r,z,t) = 2\arctan\Big(\frac{r}{\lambda(z,t)}\Big) - Br^\alpha$$

for $r \in [0,1]$, $z \in (-\zeta, \zeta)$ and $t \in (0,T)$. Since $\alpha \in (\sqrt{2}, 3]$ and $-B < 0$, the same arguments used in the proof of Lemma A.3 show that there exists $C(\alpha) \in [(\alpha^2-2)/2, \alpha^2-2]$ such that if $\lambda$ is a solution of (A.14), then $H$ is a supersolution of (2.1) in $(0,1) \times (-\zeta,\, \zeta) \times (0,T)$. On the other hand, for every $(r,z,t) \in (0,1) \times (-\zeta, \zeta) \times (0,T)$ we have trivially that $h(r,z,t) = \pi - H(r,z,t)$. □

**Lemma A.9.** *Let $\alpha \in (\sqrt{2}, 2)$, $B, \zeta, T > 0$ and let $\gamma : (-\zeta, \zeta) \times (0,T) \longrightarrow \mathbb{R}^+$ be a positive function of $z \in (-\zeta, \zeta)$ and $t \in (0,T)$ such that*

$$|\gamma_t| \leq M_1,\ |\gamma_z| \leq M_2,\ |\gamma_{zz}| \leq M_{22} \qquad in(-\zeta,\zeta) \times (0,T) \tag{A.15}$$

*for some constants $M_1, M_2, M_{22} > 0$. There exists $\tilde{Q} = \tilde{Q}(\alpha, B, M_1, M_2, M_{22}) > 0$ such that for every $Q \geq \tilde{Q}$ the function $\lambda := \mathrm{e}^{-\frac{Q}{\gamma}}$ satisfies (A.14).*

**Proof.** A straightforward computation shows that

$$\lambda_t - \lambda_{zz} + 2\,\frac{\lambda_z^2}{\lambda} = \frac{Q\lambda}{\gamma^2}\Big(\gamma_t - \gamma_{zz} + \frac{Q}{\gamma^2}\gamma_z^2 + \frac{2}{\gamma}\gamma_z^2\Big)$$

and then (A.14) is equivalent to

$$\frac{Q\lambda^{2-\alpha}}{\gamma^2}\Big(\gamma_t - \gamma_{zz} + \frac{Q}{\gamma^2}\gamma_z^2 + \frac{2}{\gamma}\gamma_z^2\Big) \leq BC. \tag{A.16}$$

But due to (A.15) and (A.13) we have that

$$\frac{Q\lambda^{2-\alpha}}{\gamma^2}\Big(\gamma_t - \gamma_{zz} + \frac{Q}{\gamma^2}\gamma_z^2 + \frac{2}{\gamma}\gamma_z^2\Big)$$

$$\leq \mathrm{e}^{-\frac{(2-\alpha)Q}{\gamma}}\Big(\frac{M_1 Q}{\gamma^2} + \frac{M_{22}Q}{\gamma^2} + \frac{Q^2 M_2^2}{\gamma^4} + \frac{2M_2^2 Q}{\gamma^3}\Big) \leq \mathcal{G}\Big(\frac{M_1 + M_{22}}{Q} + \frac{3M_2^2}{Q^2}\Big)$$

for a suitable constant $\mathcal{G}$ depending on $\alpha$. Here we have used the fact that $(2-\alpha)Q > 0$. Then the result follows from (A.16) since $C$ is a constant depending on $\alpha$. □

Let $\{\gamma_\zeta\}_{\zeta\geq 0} \subset C^\infty(\mathbb{R})$ be a family of functions such that $\gamma_{\zeta_1} \leq \gamma_{\zeta_2}$ if $0 \leq \zeta_1 < \zeta_2$,

$$\gamma_\zeta(z) \equiv \gamma_\zeta(-z), \quad \gamma_\zeta(z) = 1 \ \ \forall z \in [0,\zeta], \quad \gamma_\zeta(z) = 0 \ \ \forall z \geq \zeta+1,$$

$$-N_1 \leq \frac{d\gamma_\zeta}{dz} < 0, \quad \left|\frac{d\gamma_\zeta}{dz}\right|^2 \frac{1}{\gamma_\zeta} \leq N_2, \quad \text{and} \left|\frac{d^2\gamma_\zeta}{dz^2}\right| \leq N_3 \quad \forall z \in (\zeta, \zeta+1) \tag{A.17}$$

where the constants $N_1, N_2, N_3 \in \mathbb{R}^+$ do not depend on $\zeta$. We remark that for every $\zeta \geq 0 \quad \gamma_\zeta(z) \in (0,1)$ if $z \in (\zeta, \zeta+1)$.

**Proposition A.10.** *Let $\alpha \in (\sqrt{2}, 2)$, $B, T \in \mathbb{R}^+$. There exists $\mathcal{Q} = \mathcal{Q}(\alpha, B, T) > 0$ such that for every $Q \geq \mathcal{Q}$ and for every $\zeta \geq 0$ the function*

$$h(r,z,t) = \begin{cases} 2\arctan\left(\dfrac{\mathrm{e}^{-\frac{Q}{t\gamma_\zeta(z)}}}{r}\right) + Br^\alpha & r \in (0,1],\ t \in (0,T],\ |z| < \zeta+1 \\ Br^\alpha & r \in (0,1],\ |z| \geq \zeta+1 \ \textit{or}\ t = 0 \end{cases}$$

*is a subsolution of (2.1) in $(0,1) \times \mathbb{R} \times (0,T)$.*

**Proof.** We can write $h$ as

$$2\arctan\left(\frac{\lambda(z,t)}{r}\right) + Br^\alpha$$

where

$$\lambda(z,t) = \begin{cases} \mathrm{e}^{-\frac{Q}{t\gamma_\zeta(z)}} & t \in (0,T],\ |z| < \zeta+1 \\ 0 & |z| \geq \zeta+1 \text{ or } t = 0. \end{cases}$$

Since $\lambda \in C^\infty(\mathbb{R} \times (0,T))$ we have that $h \in C^\infty((0,1) \times \mathbb{R} \times (0,T))$ and so the same is true for

$$\mathcal{L}(h) := h_t - h_{rr} - h_{zz} - \frac{h_r}{r} + \frac{\sin(2h)}{2r^2}.$$

If $|z| > \zeta + 1$, then

$$\mathcal{L}(h) = \frac{\sin(2h) - \alpha^2 2h}{2r^2} \leq \frac{\sin(2h) - 2h}{2r^2} \leq 0$$

while, by A.8 and A.9, $\mathcal{L}(h) \leq 0$ in $(0,1) \times (-\zeta-1, \zeta+1) \times (0,T)$ provided that $Q \geq \mathcal{Q}$ for a suitable constant $\mathcal{Q} = \mathcal{Q}(\alpha, B, T)$. □

**Lemma A.11.** *Let $B < 0$, $\alpha \in (\sqrt{2}, 2)$ and let $\gamma = \gamma(z,t)$ be a positive function of $z \in (\zeta, \infty)$ and $t \in (0,T)$ $(\zeta \in \mathbb{R}, T > 0)$ such that*

$$|\gamma_t| \le M_1, \ \ |\gamma_z| \le M_2, \ \ |\gamma_{zz}| \le M_{22} \qquad \text{in } (\zeta,\infty) \times (0,T)$$

*for some constants $M_1$, $M_2$, $M_{22} > 0$. There exists $\hat{Q} = \hat{Q}(\alpha, |B|, M_1, M_2, M_{22}) > 0$ such that for every $Q \ge \hat{Q}$ the function $\lambda := \mathrm{e}^{-\frac{Q}{\gamma}}$ satisfies (A.7) in $(\zeta, \infty) \times (0,T)$.*

We omit the proof, which is formally identical to the one of Lemma A.9. The following result is an immediate consequence of A.3 and A.11.

**Proposition A.12.** *Let $\phi \in C^1([0,T))$ $(T \in (0,\infty))$ and $\mu \in C^2([\zeta,\infty))$ $(\zeta \in \mathbb{R})$ be two functions satisfying the conditions:*

1. *$\phi > 0$ in $(0,T)$, and $A_0 := \sup_{[0,T)} |\phi|$, $A_1 := \sup_{[0,T)} |\phi'|$ are both finite,*
2. *$\mu > 0$ in $(\zeta,\infty)$, and $M_0 := \sup_{[\zeta,\infty)} |\mu|, M_1 := \sup_{[\zeta,\infty)} |\mu'|, M_2 := \sup_{[\zeta,\infty)} |\mu''|$ are all finite.*

*If $\alpha \in (\sqrt{2}, 2)$ and $B < 0$, then there exists $\hat{Q} = \hat{Q}(\alpha, |B|, A_0, A_1, M_0, M_1, M_2) > 0$ such that for every $Q \ge \hat{Q}$ the function*

$$\psi(r,z,t) = 2\arctan\left(r\mathrm{e}^{\frac{Q}{\phi(t)\mu(z)}}\right) + Br^{\alpha}$$

*is a supersolution of (2.1) in the open set $(0,1)\times(\zeta,\infty)\times(0,T)$.*

## APPENDIX B. SPECIAL SUBSOLUTIONS WITH FINITE BLOW-UP TIME

Let $\alpha \in (\sqrt{2}, 2), B \in (0, \pi/2), \mathcal{T} > 0, \mathcal{Z} \ge 0$ be four constants arbitrarily chosen. Let $Z = \mathcal{Z} + \sqrt[4]{\mathcal{T}}$, $K \in [8/(\pi - 2B), \infty)$ and let $Q > 0$ be a constant such that

$$\mathcal{B} := \arctan\left(\frac{\mathrm{e}^{\frac{Q}{\mathcal{T}}}}{K}\right) + \frac{B-\pi}{2} \ > \ 0$$

and $Q \ge \bar{Q}(\mathcal{B}, \alpha, K) \ge \bar{Q}(B, \alpha, K)$, where $\bar{Q}$ is the same as in Proposition A.6. If $\mu_Z$ and $\mu_{\mathcal{Z}}$ are the functions defined by (A.11) for $\sigma = Z, \mathcal{Z}$ respectively and

$$\xi(r,z,t) := 2\arctan\left(\frac{r}{K}\mathrm{e}^{\frac{Q}{\mathcal{T}-t+\mu_Z(z)}}\right) + Br^{\alpha},$$

$$\tilde{\xi}(r,z,t) := 2\arctan\left(\frac{r}{K}\mathrm{e}^{\frac{Q}{\mathcal{T}-t+\mu_{\mathcal{Z}}(z)}}\right) + \mathcal{B}r^{\alpha}, \qquad \text{(B.1)}$$

then, according to Proposition A.6, $\xi, \tilde{\xi}$ are subsolutions of (2.1) satisfying conditions (A.10) for $T = \mathcal{T}$ and $\sigma = Z, \mathcal{Z}$. Since $B > 2\mathcal{B}$ and $Z \geq \mathcal{Z}$, which implies $\mu_Z \leq \mu_{\mathcal{Z}}$, it is obvious that

$$\xi(r, z, 0) \geq \tilde{\xi}(r, z, 0) + \mathcal{B}r^{\alpha} \qquad \forall (r, z) \in [0, 1] \times \mathbb{R}.$$

Moreover, we have the following.

**Lemma B.1.** $\xi(1, z, 0) \geq \tilde{\xi}(1, z, t) + \mathcal{B}$, *for all* $z \in \mathbb{R}, t \in [0, \mathcal{T})$.

**Proof.** For every $z \in [-Z, Z]$

$$\xi(1, z, 0) = 2 \arctan\left(\frac{e^{\frac{Q}{\mathcal{T}}}}{K}\right) + B = \pi + 2\mathcal{B} \geq \tilde{\xi}(1, z, t) + \mathcal{B} \qquad \forall t \in [0, \mathcal{T})$$

by definition of $\mathcal{B}$ and $\tilde{\xi}$. On the other hand, if $z \geq Z$ then

$$\mu_{\mathcal{Z}}(z) = (z - \mathcal{Z})^4 = ((z - Z) + (Z - \mathcal{Z}))^4 \geq (z - Z)^4 + (Z - \mathcal{Z})^4 = \mu_Z(z) + \mathcal{T},$$

and $\mu_{\mathcal{Z}}(z) = \mu_{\mathcal{Z}}(|z|) \geq \mu_Z(|z|) + \mathcal{T} = \mu_Z(z) + \mathcal{T}$ if $z \leq -Z$. Therefore, for every $z \in \mathbb{R} \setminus [-Z, Z]$ and $t \in [0, \mathcal{T})$

$$\frac{Q}{\mathcal{T} - t + \mu_{\mathcal{Z}}(z)} \leq \frac{Q}{\mu_Z(z) + \mathcal{T}},$$

which, together with $B > 2\mathcal{B}$, implies $\tilde{\xi}(1, z, t) + \mathcal{B} \leq \xi(1, z, 0)$. □

At last, since $K \geq 8/(\pi - 2B)$, it turns out that the following holds.

**Lemma B.2.** *There exist* $\bar{z} = \bar{z}(Z, Q) > 0$ *and* $b \in (0, 1)$ *such that*

$$\xi(r, z, 0) \leq 2 \arctan(br) \qquad \forall r \in [0, 1],\ |z| \geq \bar{z}.$$

**Proof.** Let $\bar{z} = \bar{z}(Z, Q) > 0$ be a constant such that for every $z \in (-\infty, -\bar{z}] \cup [\bar{z}, \infty)$, $e^{\frac{Q}{\mu_Z(z)}} \leq 3/2$. Therefore, for every $z \in (-\infty, -\bar{z}] \cup [\bar{z}, \infty)$ and $r \in [0, 1]$,

$$\xi(r, z, 0) \leq 2 \arctan\left(\frac{3r}{2K}\right) + Br^{\alpha} \leq (3/K + B)r \leq \left(\pi/2 - \frac{\pi/2 - B}{4}\right)r.$$

Since $A := \pi/2 - (\pi/2 - B)/4 \in (0, \pi/2)$, $Ar \leq 2 \arctan(2Ar/\pi)$ for all $r \in [0, 1]$. Then the thesis follows by defining $b = \frac{2A}{\pi}$. □

**Theorem B.3.** *There exists a function* $h_0 \in C^{\infty}((0, 1] \times \mathbb{R})$*, Lipschitz continuous in* $[0, 1] \times \mathbb{R}$*, which satisfies the five properties* (P1)-(P5) *listed in the beginning of Section* 2.

**Proof.** If $q(r, z) = \xi(r, z, 0)$, then

- $$q_{rr} + q_{zz} + \frac{q_r}{r} - \frac{\sin(2q)}{2r^2} \geq \xi_t(r,z,0) = \frac{2Kr\mathrm{e}^{\frac{Q}{\mathcal{T}+\mu_Z(z)}}}{K^2 + r^2\mathrm{e}^{\frac{2Q}{\mathcal{T}+\mu_Z(z)}}} \frac{Q}{(\mathcal{T}+\mu_Z(z))^2} > 0$$
  in $(0,1] \times \mathbb{R}$,
- $q(0,z) = 0$ for all $z \in \mathbb{R}$,
- $2\arctan(r/K) \leq q(r,z) \leq \pi + 2\arctan(ar)$ for some $a > 0$, and there exist $\bar{z} > 0$, $b \in (0,1)$ such that $q(r,z) \leq 2\arctan(br)$ if $|z| \geq \bar{z}$,
- $q(r,-z) \equiv q(r,z)$, $q_z(r,z) \leq 0$ for $r \in [0,1], z \geq 0$, and
- $q(r,z) \geq \tilde{\xi}(r,z,0) + \mathcal{B}r^\alpha$, $q(1,z) \geq \tilde{\xi}(1,z,t) + \mathcal{B}$ for all $r \in [0,1]$, $z \in \mathbb{R}$ and $t \in [0,\mathcal{T})$.

Therefore, since the function $\theta_b(r) := 2\arctan(br)$ is a subsolution of (2.1) and the maximum of two subsolutions is itself a subsolution, to prove the statement it is sufficient to take, up to a small regularization, $h_0(r,z) := \max\{q(r,z), \theta_b(r)\}$. □

## APPENDIX C. A COMPARISON PRINCIPLE

In this section we prove a comparison principle which has been obtained by slightly modifying a similar result contained in [7].

**Lemma C.1.** *Let $\psi, \xi$ be respectively a regular super- and subsolution to problem* (1.3) *on a time interval* $[0, \mathcal{T})$ $(\mathcal{T} > 0)$. *If, for every $\tau \in (0, \mathcal{T})$,*

(h1)
$$\xi(r,z,t), \psi(r,z,t) \longrightarrow 0 \qquad \text{as } r \to 0^+$$
*uniformly with respect to $z \in \mathbb{R}$ and $t \in [0,\tau]$, and*

(h2)
$$\limsup_{|z|\to\infty} \sup_{r\in[0,1], t\in[0,\tau]} (\xi(r,z,t) - \psi(r,z,t)) \leq 0,$$

*then $\xi \leq \psi$ on $[0,1] \times \mathbb{R} \times [0,\mathcal{T})$.*

**Proof.** Let $\eta := \xi - \psi$. Then $\eta \leq 0$ on $[0,1] \times \mathbb{R} \times \{0\}$ and on $\{0,1\} \times \mathbb{R} \times [0,\mathcal{T})$. In addition, $\eta$ satisfies on $(0,1) \times \mathbb{R} \times (0,\mathcal{T})$ the differential inequality

$$\eta_t - \Delta\eta - \frac{\eta_r}{r} + \frac{f}{2r^2}\eta \leq 0, \tag{C.1}$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian in $(r,z)$ coordinates, and

$$f(r,z,t) := \int_0^1 \cos(2(s\xi(r,z,t) + (1-s)\psi(r,z,t)))\mathrm{d}s$$

is bounded on $[0,1]\times\mathbb{R}\times[0,\mathcal{T})$. In view of (h1) there exists $\rho\in(0,1)$ such that, for every $r\in[0,\rho]$, $z\in\mathbb{R}$ and $t\in[0,\tau]$,

$$|\xi(r,z,t)|,|\psi(r,z,t)|\le\pi/4 \tag{C.2}$$

and therefore $f\ge 0$ in $[0,\rho]\times\mathbb{R}\times[0,\tau]$. Let $M>0$ be a constant such that $|f|<M$. Multiplying (C.1) by $\mathrm{e}^{-\frac{Mt}{2\rho^2}}$ and introducing $h(r,z,t):=\mathrm{e}^{-\frac{Mt}{2\rho^2}}\eta$ we have that

$$h_t-\Delta h-\frac{h_r}{r}+h\Big(\frac{f}{2r^2}+\frac{M}{2\rho^2}\Big)\le 0. \tag{C.3}$$

Since $\mathrm{e}^{-\frac{M\tau}{2\rho^2}}\eta\le h\le\eta$ on $[0,1]\times\mathbb{R}\times[0,\tau]$, if the thesis is false, there exists a time $\tau\in(0,\mathcal{T})$ such that $s:=\sup_{[0,1]\times\mathbb{R}\times[0,\tau]}h>0$. From (h2) we deduce that

$$\limsup_{|z|\to\infty}\ \sup_{r\in[0,1],t\in[0,\tau]}h(r,z,t)\le 0. \tag{C.4}$$

Thus there must be $\zeta>0$ such that

$$\sup_{r\in[0,1],\,|z|>\zeta,\,t\in[0,\tau]}h(r,z,t)<s/2\quad\text{and thus}\quad s=\max_{[0,1]\times[-\zeta,\zeta]\times[0,\tau]}h;$$

i.e., $h$ attains a positive maximum on $[0,1]\times\mathbb{R}\times[0,\tau]$. On the other hand, just like $\eta$, $h$ is nonpositive on $\{0,1\}\times\mathbb{R}\times[0,\tau]$ and on $(0,1)\times\mathbb{R}\times\{0\}$. Therefore, the positive maximum is achieved on $(0,1)\times\mathbb{R}\times(0,\tau]$, say at $(\bar r,\bar z,\bar t)$, and by the regularity of $h$ we must have $h_t\ge 0$, $\Delta h\le 0$, $h_r=0$ at this point. Then, due to (C.3),

$$\frac{f(\bar r,\bar z,\bar t)}{2\bar r^2}+\frac{M}{2\rho^2}\le 0. \tag{C.5}$$

Since $f(\bar r,\bar z,\bar t)\ge 0$ if $\bar r\le\rho$, and $\bar r>\rho$ implies

$$\frac{f(\bar r,\bar z,\bar t)}{2\bar r^2}>-\frac{M}{2\rho^2},$$

we have obtained a contradiction. □

## Appendix D. Traveling wave solutions of (2.1)

By repeating the same construction of [3] in the strip $(0,R)\times\mathbb{R}$, $R>0$, rather than in $(0,1)\times\mathbb{R}$, one can show the following.

**Lemma D.1.** *Given $c>0$ and a function $g=g(y)$ satisfying*

$$g\in C^4(\mathbb{R}),\ g'\le 0\ \text{in}\ \mathbb{R},\ g=A\ \text{in}\ (-\infty,y_0),\ g=B\ \text{in}\ (y_1,\infty),$$

*for some* $y_0 < y_1$ *and*

$$\pi < A < 3\pi \quad \text{and} \quad 0 < B < \pi/2,$$

*there exists a function* $\psi : [0,R] \times \mathbb{R} \to \mathbb{R}$ *smooth in* $(0,R] \times \mathbb{R}$ *which satisfies equations* (2.7) *and* $\psi(R,y) \equiv g(y)$. *In addition the following properties are satisfied:*

(*i*) *there exists* $\hat{y}$ *such that* $\psi$ *is continuous in* $\{(0,y) : y \neq \hat{y}\}$, $\psi(0,y) = 0$ *if* $y > \hat{y}$ *and* $\psi(0,y) = 2\pi$ *if* $y < \hat{y}$;

(*ii*) $\psi(r,y)$ *is nonincreasing with respect to* $y$;

(*iii*) $\psi(r,y) \to 2\arctan(\beta r/R)$ *uniformly with respect to* $r \in [0,R]$ *as* $y \to \infty$, *where* $\beta$ *is defined by* $2\arctan\beta = B$;

(*iv*) $\psi(r,y) \to 2\pi + 2\arctan(\alpha r/R)$ *uniformly with respect to* $r \in [0,R]$ *as* $y \to -\infty$, *where* $\alpha$ *is defined by* $2\pi + 2\arctan\alpha = A$;

(*v*) $\psi$ *is real analytic in* $[0,R) \times \mathbb{R} \setminus \{(0,\hat{y})\}$.

We remark that, up to a translation in the $y$ variable, it is always possible to make $\hat{y} = 0$.

Let $b \in (0,1)$. The following result follows straightforwardly from the previous one.

**Lemma D.2.** *There exists a function*

$$\begin{array}{llll} \psi: & [0,R] \times \mathbb{R} \setminus \{(0,0)\} & \longrightarrow & \mathbb{R} \\ & (r,y) & \longrightarrow & \psi(r,y) \end{array}$$

*which satisfies all the following properties:*

(i) $\psi$ *is smooth* ($C^\infty$) *in its domain,*

(ii) $\psi$ *is nonincreasing with respect to* $y$,

(iii) $\psi \to 2\arctan(br/R)$ *as* $y \to \infty$, *and* $\psi \to 2\pi$ *as* $y \to -\infty$ *uniformly with respect to* $r \in [0,R]$,

(iv) $\psi$ *solves the problem*

$$\begin{cases} \psi_{yy} + \psi_y + \psi_{rr} + \frac{\psi_r}{r} - \frac{\sin(2\psi)}{2r^2} = 0 & (0,R) \times \mathbb{R} \\ \psi(0,y) = \begin{cases} 0 & \text{if } y > 0 \\ 2\pi & \text{if } y < 0 \end{cases} & \\ \psi(R,y) = g(y) & y \in \mathbb{R} \end{cases}$$

*for a suitable function* $g \in C^\infty(\mathbb{R})$ *such that* $g' \leq 0$, $g(y) = 2\arctan(b)$ *for every* $y \geq 1$ *and* $g(y) = 2\pi$ *for every* $y \leq -1$.

Of course, for every constant value $C \in \mathbb{R}$ the function $w_C(r,z,t) = \psi(r, z - t - C)$ is a solution of (2.1) in form of traveling wave.

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