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# SEMICLASSICAL EVOLUTION OF TWO ROTATING SOLITONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH ELECTRIC POTENTIAL

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**Abstract.** In this paper we study the dynamics in the semiclassical limit of some classes of solutions for an energy-subcritical focusing nonlinear Schrödinger equation in the presence of an electric potential. We prove the existence of a symmetric configuration of two solitons rotating around a common pole. The existence of such solutions derives mainly from the balance between the attracting force of the two solitons and the centrifugal force due to the rotation and not from the trapping effect of the potential.

## 1. Introduction

We consider the following nonlinear Schrödinger equation (NLS):

$$(\mathrm{NLS}_\varepsilon) \quad i\varepsilon\phi(t,x) = -\varepsilon^2\Delta\phi(t,x) + E(x)\phi(t,x) - |\phi(t,x)|^{p-1}\phi(t,x), \quad (1.1)$$

where $(t,x) \in \mathbf{R} \times \mathbf{R}^2$, $p > 1$, $\varepsilon \geq 0$ and where the electric potential is defined as follows:

$$E(x) = \begin{cases} 1, & |x| \leq 1 \\ |x|^2, & |x| \geq 1+\eta \\ C^\infty(\mathbf{R}^n), & \text{otherwise} \end{cases} \quad (1.2)$$

with $\eta \ll 1$. The equation ($\mathrm{NLS}_\varepsilon$) is a model for the motion of a quantum particle in the presence of an external electric potential and appears in several applications such as nonlinear optics, water waves and Bose-Einstein condensates (see [27] and references therein for details). In the present paper we are interested in the existence and qualitative behavior of some particular classes of solutions in the *semiclassical limit*, namely when the Planck constant $\varepsilon := \hbar$ tends to zero. This limit translates in mathematical terms the *correspondence principle* between classical mechanics and quantum mechanics. It asserts that classical laws can be recovered from quantum laws, in the

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transition from a microscopic scale ($\hbar \simeq 1$) to a macroscopic scale ($\hbar \ll 1$). This issue has been widely investigated in the past thirty years. In particular many mathematicians studied special periodic-in-time solutions called *standing waves*, namely solutions of the form $\phi^\varepsilon(t,x) = U(x)e^{i\mu t\varepsilon^{-1}}$ with $\mu > 0$ and $U(x)$ satisfying (just plug $\phi_\varepsilon$ in ($\mathrm{NLS}_\varepsilon$)) the following equation:

$$-\varepsilon^2\Delta U + \mu U - |U|^{p-1}U = 0. \tag{1.3}$$

These solutions model particles at rest, since they keep their modulus constant for all $t \in \mathbf{R}$. In the case $\varepsilon = 1$, $1 < p < 1 + \frac{4}{n-2}$ if $n \geq 3$ and $p > 1$ otherwise, Strauss [26] and Berestycki-Lions [5] showed that (1.3) possesses real positive, smooth, exponentially decaying solutions of minimal energy $u_\xi := u(x-\xi)$ for $\xi \in \mathbf{R}^n$ (radial for $\xi = 0$), called *ground states*. On the other hand, the *Pohozaev's identity* implies nonexistence of nontrivial solutions in the complementary case $p \geq \frac{n+2}{n-2}$ (see [5] and [23]). In the pioneering work [12] Floer and Weinstein proved the existence of standing waves for the NLS in the presence of an electric potential $V$:

$$i\varepsilon\phi_t(t,x) = -\varepsilon^2\Delta\phi(t,x) + V(x)\phi - |\phi(t,x)|^{p-1}\phi(t,x) \tag{1.4}$$

which concentrate at nondegenerate critical points of $V$, when $0 < \varepsilon \ll 1$. These solutions $\phi^\varepsilon$ behave like $\phi^\varepsilon \simeq u_\mu(\frac{x-X_0}{\varepsilon})e^{i\mu t\varepsilon^{-1}}$, where $u_{V(X_0)}$ is a ground state solution of (1.3) with $\mu = V(X_0)$ and $X_0$ a critical point of $V$. Since then, a lot of works have appeared concerning the *semiclassical limit* of the stationary problem (1.3). For our purposes it is important to mention the works of Gui [20] and Oh [22] for solutions with many peaks and the work of Cingolani and Secchi [10] and Arioli and Szulkin [3] in the presence of a magnetic potential. Bartsch, Dancer and Peng proved in [4] the existence of multibump semiclassical states, for the nonlinear Schrödinger equation with electric and magnetic potentials, concentrating simultaneously at different local minima of the electric potential. On the other hand Wei and Xiaosong proved in [29] the existence of interacting multibump solutions concentrating near a local maximum of the electric potential. We refer the interested reader to [1] and the references therein for a more complete list of works.

For what concerns the study of the *semiclassical limit* in the dynamic case there are very few papers. Up to now only the Cauchy problem in the energy space $H^1(\mathbf{R}^n)$ has been studied (it is globally well posed for mass-subcritical nonlinearities $1 < p < 1 + \frac{4}{n}$, see for example the classical papers by Ginibre-Velo [14] and Kato [15] or the surveys by Cazenave [9], Sulem-Sulem [27] and Tao [28]). As far as we know the first result in this field is due to Bronski and Jerrard [8]. They proved that, in the semiclassical regime solutions of the Cauchy problem, corresponding to initial data of the form $\Phi^\varepsilon(x) \simeq u(\frac{x-X_0}{\varepsilon})e^{i\frac{V_0\cdot x}{\epsilon}}$, travel according to the Newton law $(\ddot{X} = -\frac{1}{2}DV(X), X(0) = X_0, \dot{X}(0) = \dot{X}_0)$ and keep the asymptotic profile $\phi^\varepsilon(t,x) \simeq u(\frac{x-X(t)}{\varepsilon})e^{i\frac{\dot{X}(t)\cdot x}{\epsilon}}$ for finite time intervals $[0,T]$ with $T > 0$. The precise description of the limit profile has been obtained by Keraani in [16] and [17]. An extension of [8] in the case of slowly varying potentials is due to Fröhlich, Gustafson, Jonsson and Sigal [13]. The first result in the presence of a magnetic field is due to the author [24], extended to the precise description of the profile by Squassina [25]. A consequence of these works is that, in the absence of any potential, the quantum particle is at rest or moves linearly ($\ddot{X} = 0$). Differently from these papers, we deal with special solutions, for which the general theory ensures just local well posedness (namely $1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$). The novelty of our result is that, even in the region of $\mathbf{R}^n$ where the electric potential $E(x)$ is constant (and so does not "virtually" affect the dynamics), there exist nonstationary solutions of (NL $\mathrm{S}_\varepsilon$). In particular we exhibit a symmetric configuration of two solitons rotating around a common pole. Moreover, differently from the papers mentioned above, the estimates for the concentration are uniform globally in time (so independent of $t$ for $t \in \mathbf{R}$). In a recent paper [18] Krieger-Martel-Raphael in the case of a Hartree type nonlinearity and in the absence of any external potential proved the existence of nonperiodic two solitons solutions whose trajectories reproduce the nontrapped dynamic of the two body problem. Moreover they conjectured the nonexistence of periodic two-body type solutions for their equation. In some sense in that paper they treat two cases complementary to our case since we think that our argument can be applied with few changes also to the Hartree equation (thanks to the stability result of Lenzmann [21]), but adding a trapping potential.

We now state our main result. To enlighten the notation we define the Banach space

$$X := L^\infty(\mathbf{R}, H^1(\mathbf{R}^n)),$$

with its natural norm

$$||u||_X := \sup_{t\in\mathbf{R}} \Big( \int_{\mathbf{R}^n} |u(t,x)|^2 dx + \int_{\mathbf{R}^n} |\nabla u(t,x)|^2 dx \Big)^{\frac{1}{2}},$$

and the matrix of rotations

$$R(\tau) := \begin{pmatrix} \cos(\tau) & \sin(\tau) & \mathbf{0} \\ -\sin(\tau) & \cos(\tau) & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{Id} \end{pmatrix}$$

for $\tau \in [0, 2\pi]$, $\mathbf{0}$ the transpose of the array of $n-2$ zeros $\mathbf{0} := (0, \dots, 0)$, $\mathbf{0}^T$ and $\mathbf{Id}$ the identity matrix on $\mathbf{R}^{n-2}$. With the notation $x \simeq y$, we will mean that $\frac{x}{y} \to 0$ as $\varepsilon \to 0$. The main result of this paper is the following.

**Theorem 1.1.** *Set $K := p\frac{\sqrt{2}}{2}$ and $V(L) = \omega + E(L) - \frac{\alpha^2 L^2}{4}$, with $E(x)$ as in* (1.2) *and $L > 0$. Then there exists $\varepsilon_0 > 0$, some constants $\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon} \in \mathbf{R}$ dependent on $\varepsilon$, and $L$ and $\overline{\alpha} \in [\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}]$ of order $\overline{\alpha}^2 \simeq e^{-V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon}}/\varepsilon L$, such that, for all $\varepsilon > 0$ with $0 < \varepsilon < \varepsilon_0$, there exist solutions $\phi^\varepsilon(t,x)$ of (NLS$_\varepsilon$), such that*

$$\begin{aligned} \Big\| \phi^\varepsilon(\varepsilon t, \varepsilon x) - u_L\Big(R(\overline{\alpha}t)x - \frac{Le_1}{\varepsilon}\Big) e^{i(\alpha L y_2 + \omega)} \\ - u_L\Big(-R(\overline{\alpha}t)x - \frac{Le_1}{\varepsilon}\Big) e^{i(-\alpha L y_2 + \omega)} \Big\|_X \to 0. \end{aligned} \tag{1.5}$$

*Here $\omega \in \mathbf{R}$, $y_2 := R_{2,1}(\overline{\alpha}t)x_1 + R_{2,2}(\overline{\alpha}t)x_2$ and $u\big(\frac{x}{\varepsilon}\big)$ is the only radial ground state of* (1.14) *below. Moreover, $\phi^\varepsilon(t,x)$ has the following symmetries: $\phi^\varepsilon(t,x) = \phi^\varepsilon(t,-x)$ and $\overline{\phi^\varepsilon(t,x_1,-x_2)} = \phi^\varepsilon(t,x)$.*

The proof of this theorem relies on a kind of Lyapunov-Schmidt reduction (see [1] for a general presentation) to study perturbative elliptic problems with lack of compactness. Our strategy proceeds as follows. We first make a change of variable $(t,x) \mapsto (\varepsilon t, \varepsilon x)$ and arrive at the equation

$$i\psi(t,x) = -\Delta\psi(t,x) + E(\varepsilon x)\psi(t,x) - |\psi(t,x)|^{p-1}\psi(t,x), \tag{1.6}$$

where $\psi(t,x) := \phi(\varepsilon t, \varepsilon x)$. Next, we make the ansatz

$$\psi(t,x) = u(R(\alpha\varepsilon t)x)e^{i\omega t}. \tag{1.7}$$

Here $\alpha > 0$ represents the angular velocity, $\omega > 0$ the phase, while the matrix of rotation $R$ is defined as in Theorem 1.1. This ansatz incorporates the qualitative behavior of the solutions of (1.6) we want to find; indeed it allows us to work in a system of coordinates rotating around the origin, in which solitons are fixed. By substituting (1.7) into (1.6) we remain with

$$\begin{aligned} &-\Delta u(R(\varepsilon\alpha\varepsilon t)x) + \omega u(R(\varepsilon\alpha\varepsilon t)x) - i\varepsilon\alpha J R(\varepsilon\alpha\varepsilon t)x\nabla u(R(\varepsilon\alpha\varepsilon t)x) \\ &+ E(\varepsilon x)u(R(\varepsilon\alpha\varepsilon t)x) - |u(R(\varepsilon\alpha\varepsilon t)x)|^{p-1}u(R(\varepsilon\alpha\varepsilon t)x) = 0, \end{aligned} \tag{1.8}$$

where $J$ is the symplectic matrix

$$J := \begin{pmatrix} 0 & 1 & \mathbf{0} \\ -1 & 0 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{Id} \end{pmatrix}.$$

Making the change of variables $x \mapsto R(\varepsilon\alpha\varepsilon t)x$, (1.8) becomes

$$-\Delta u(x) + \omega u(x) - i\varepsilon\alpha Jx\nabla u(x) + E(\varepsilon x)u(x) - |u(x)|^{p-1}u(x) = 0 \tag{1.9}$$

by invariance of the electric potential $E$ with respect to rotations. Equation (1.9) is the Euler-Lagrange equation of the functional

$$\begin{aligned} I_\varepsilon(u) : = &\frac{1}{2}\int_{\mathbf{R}^n} |\nabla u|^2 + \frac{1}{2}\int_{\mathbf{R}^n} \Big(\omega + E(\varepsilon x)\Big)|u|^2 \\ &- \frac{1}{2}\mathrm{Re}\int_{\mathbf{R}^n} i\varepsilon\alpha Jx\nabla u(x)\overline{u} - \frac{1}{p+1}\int_{\mathbf{R}^n} |u|^{p+1}. \end{aligned}$$

However, at this stage it is not so clear which is the suitable functional setting and if the quadratic part of the functional $I_\varepsilon(u)$ is coercive or not. To give a positive answer to these questions we use the following trick: we add and remove the quantity $\frac{\varepsilon^2\alpha^2}{4}|x|^2u(x)$ to equation (1.9) and this leads us to the following equation:

$$\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)^2 u(x) + V(\varepsilon x)u(x) - |u(x)|^{p-1}u(x) = 0, \tag{1.10}$$

where $A(s) = \frac{-\alpha Js}{2}$, $V(s) = \omega + E(s) - \frac{\alpha^2}{4}|s|^2$ and

$$\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)^2 u(x) := -\Delta u - 2iA(\varepsilon x)\nabla u + |A(\varepsilon x)|^2 u.$$

This allows us to write $I_\varepsilon(u)$ in the following way:

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbf{R}^n} \Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)u\Big|^2 + V(\varepsilon x)|u|^2 - \frac{1}{p+1}\int_{\mathbf{R}^n} |u|^{p+1}$$

and equation (1.10) as $I'_\varepsilon(u) = 0$. Therefore, the natural space associated to the functional $I_\varepsilon(u)$ is the Hilbert space

$$H_{V,A}(\mathbf{R}^n) := \overline{\Big\{u \in C_0^\infty(\mathbf{R}^n) : \|u\|_{V,A} < +\infty\Big\}}^{\|.\|_{V,A}}, \tag{1.11}$$

with

$$\|u\|_{V,A} := \Big(\int_{\mathbf{R}^n} \Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)u\Big|^2 + V(\varepsilon x)|u|^2\Big)^{\frac{1}{2}}. \tag{1.12}$$

We can easily understand that the functional $I_\varepsilon(u)$ is well defined, coercive in the space $H_{V,A}$ and is of class $C^2(H_{V,A};\mathbf{R})$ (see Lemma 2.1). We notice that equation (1.10) is the stationary nonlinear Schrödinger equation in the presence of electric and magnetic fields (see for example [3] or [10]). Once the functional setting is clarified, we can now start with the proof. Since equation (1.10) is perturbative, the strategy is to find first some approximate solutions and then an exact solution close to them in a suitable sense. We define the 1-dimensional manifold

$$\mathcal{Z}_\varepsilon := \Big\{ z_L^\varepsilon(x) = u_L\big(x - \frac{Le_1}{\varepsilon}\big)e^{i\alpha L x_2} + u_L\big(-x - \frac{Le_1}{\varepsilon}\big)e^{-i\alpha L x_2} : L > 0 \Big\}, \quad (1.13)$$

where $u_L$ is a solution of the problem

$$-\Delta u_L + V(L)u_L - u_L^p = 0 \quad (1.14)$$

and depends on $L$ like

$$u_L(x) = V(L)^{\frac{1}{p-1}} u\big(V(L)^{\frac{1}{2}} x\big) \quad (1.15)$$

($u$ is the ground state of (1.3), with $\mu = 1$). The approximate solutions $z_L^\varepsilon$ are formed by two symmetric solitons, rotating at a fixed angular velocity $\alpha > 0$ on a circle of radius $L > 0$. These functions represent the main limit profile of the exact solutions of (1.10) (see Section 3). By a Lyapunov-Schmidt reduction (Section 4) we find a manifold $\tilde{\mathcal{Z}}_\varepsilon$ close to $\mathcal{Z}_\varepsilon$ which is a *natural constraint* for $I_\varepsilon(u)$ (see Subsection 4.3). In this way we have reduced the problem of finding critical points of $I_\varepsilon(u)$ to the study of the 1-dimensional reduced functional $\Phi(L) := I_\varepsilon(u)|_{\tilde{\mathcal{Z}}}$, which takes the following form: $\Phi(L) := \tilde{\Phi}(L) + l.o.t.$, (for the notation see Section 2) where $\tilde{\Phi}(L)$ is defined as $\tilde{\Phi}(L) := V(L)^\theta\big(C_0 - C_1 e^{-V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon}}\big)$ with $K > 0$ as in Theorem 1.1 and $\theta := \frac{p+1}{p-1} - \frac{n}{2}$. By choosing $\alpha = \overline{\alpha}$ of the appropriate order in $\varepsilon$

$$\overline{\alpha}^2 \simeq \frac{e^{-V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon}}}{\varepsilon L}$$

we find a critical point of $\Phi(L)$ for some $L > 0$ (and hence a critical point of $I_\varepsilon(u)$) in the region $|x| \leq 1$ where $E = 0$.

The paper is organized as follows: in Section 2 we fix some notation, we collect some properties of embedding of the space $H_{V,A}$ and we recall some results about the existence of standing waves for (1.1) with $E(x) = 0$; in Section 3 we construct an approximate solution; Section 4 is devoted to the Lyapunov-Schmidt reduction; in Section 5 we collect all previous results to prove Theorem 1.1.

Since the higher-dimensional case requires minor changes the proofs will be carried out in the case $n = 2$ (see also Remark 5.1 (c)).

## 2. NOTATION AND PRELIMINARY FACTS

With the notation l.o.t., we generically mean lower-order terms as $\varepsilon \to 0$. The components of the matrix $R$ will be denoted by $R_{i,j}$. We will often write $V(L)$ (or $E(L)$, $A(L)$) instead of $V(Le_1)$ (or $E(Le_1)$, $A(Le_1)$). All the scalar products will be denoted with $\langle \cdot, \cdot \rangle_Y$, when it is important to specify the functional space $Y$ or with $\langle \cdot, \cdot \rangle$, where there is no chance of misunderstanding. The same will be done for norms. To simplify the notation in some of the upcoming lemmas (especially in Lemma 4.3) we introduce the following function, equivalent to the $H^1$-norm (but not to the $H_{V,A}$ norm):

$$\|\phi\|_L := \Big( \int_{\mathbf{R}^2} \Big| \Big( \frac{\nabla}{i} - A(L) \Big) \phi \Big|^2 + V(L)|\phi|^2 \Big)^{\frac{1}{2}}$$

and the corresponding scalar product

$$\langle \phi_1, \phi_2 \rangle_L := \int_{\mathbf{R}^2} \Big( \frac{\nabla}{i} - A(L) \Big) \phi_1 \overline{\Big( \frac{\nabla}{i} - A(L) \Big) \phi_2} + V(L) \phi_1 \overline{\phi_2}.$$

These quantities are well defined in $H_{V,A}$ (see Lemma 2.1 below). We point out here that the notation $\|.\|_L$ is an abuse, since $\|.\|_L$ is not an equivalent norm on $H_{V,A}$. Moreover we introduce the functional

$$I_L(u) := \int_{\mathbf{R}^2} \Big| \Big( \frac{\nabla}{i} - A(L) \Big) \phi \Big|^2 + V(L)|\phi|^2 - \frac{1}{p+1}|u|^{p+1},$$

which is well defined (see Lemma 2.1 below). We define also the space $H^s_{V,A}$ as follows:

$$H^s_{V,A}(\mathbf{R}^n) := \Big\{ u \in H_{V,A}(\mathbf{R}^n) : u(x) = u(-x) \text{ and } \overline{u(x_1, x_2)} = u(x_1, -x_2) \Big\}.$$

2.1. **The functional setting.** We have some relationships between the space $H_{V,A}(\mathbf{R}^n)$ (see formula (1.11)) with the norm (1.12), the Sobolev space $H^1(\mathbf{R}^n)$, with its natural norm

$$\|u\|_{H^1} := \Big( \int_{\mathbf{R}^2} |\nabla u|^2 + |u|^2 \Big)^{\frac{1}{2}} \tag{2.1}$$

and the weighted Sobolev space $H^1_w(\mathbf{R}^n)$ endowed with the norm

$$\|u\|_w := \Big( \int_{\mathbf{R}^2} |\nabla u|^2 + (1 + |\varepsilon x|^2)|u|^2 \Big)^{\frac{1}{2}}. \tag{2.2}$$

In particular the $H_{V,A}$-norm and the $H_w$-norm are equivalent and so $H_{V,A}$ and $H_w$ are continuously embedded into each other, while $H^1$ is continuously embedded in both of them. So the following lemma holds.

**Lemma 2.1.** *There exists some positive constants* $C > 0, C_1 > 0, C_2 > 0$, *such that*

$$C\|u\|_{H^1} \leq C_1\|u\|_w \leq \|u\|_{V,A} \leq C_2\|u\|_w. \tag{2.3}$$

**Proof.** First of all notice that

$$\begin{aligned}\|u\|^2_{V,A} &= \int_{\mathbf{R}^2} \Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)u\Big|^2 + V(\varepsilon x)|u|^2 \\ &= \int_{\mathbf{R}^2} |\nabla u|^2 + (\omega + E(\varepsilon x))|u|^2 - Re\int_{\mathbf{R}^2} i\alpha\varepsilon Jx\nabla u\bar{u}.\end{aligned}$$

Moreover, we have $|\varepsilon x|^2 \leq E(\varepsilon x) \leq 1 + |\varepsilon x|^2$ and

$$Re\big(i\alpha\varepsilon Jx\nabla u\bar{u}\big) \leq \varepsilon\alpha|x||\nabla u||u| \leq \frac{(\varepsilon\alpha)^2}{4}|\nabla u|^2 + |\varepsilon x|^2|u|^2.$$

Hence,

$$\begin{aligned}\|u\|^2_{V,A} &\leq \int_{\mathbf{R}^2} \Big(1 + \frac{(\varepsilon\alpha)^2}{4}\Big)|\nabla u|^2 + (\omega + 1 + |\varepsilon x|^2)|u|^2 \\ &\leq \max\Big\{1 + \frac{(\varepsilon\alpha)^2}{4}, 1, \omega + 1\Big\}\|u\|^2_w.\end{aligned}$$

Then for some $\delta > 0$ we have

$$-Re\big(i\alpha\varepsilon Jx\nabla u\bar{u}\big) \geq -\varepsilon\alpha|x||\nabla u||u| \geq -\frac{\delta^2}{2}|\nabla u|^2 - \frac{(\varepsilon\alpha)^2}{2\delta^2}|x|^2|u|^2$$

$$\begin{aligned}\|u\|^2_{V,A} &= \int_{\mathbf{R}^2} \Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)u\Big|^2 + V(\varepsilon x)|u|^2 \\ &= \int_{\mathbf{R}^2} |\nabla u|^2 + (\omega + E(\varepsilon x))|u|^2 - Re\int_{\mathbf{R}^2} i\alpha\varepsilon Jx\nabla u\bar{u} \\ &\geq \int_{\mathbf{R}^2} |\nabla u|^2 + (\omega + E(\varepsilon x))|u|^2 - \frac{\delta^2}{2}|\nabla u|^2 - \frac{(\varepsilon\alpha)^2}{2\delta^2}|x|^2|u|^2 \\ &\geq \Big(1 - \frac{\delta^2}{2}\Big)\int_{\mathbf{R}^2} |\nabla u|^2 + \int_{\mathbf{R}^2} |u|^2\Big(\omega + \varepsilon^2|x|^2\Big(1 - \frac{\alpha^2}{2\delta^2}\Big)\Big) \\ &\geq \min\Big\{1 - \frac{\delta^2}{2}, \omega, 1 - \frac{\alpha^2}{2\delta^2}\Big\}\|u\|^2_w.\end{aligned}$$

If we choose $\delta < \sqrt{2}$ and $2\delta^2 > \alpha^2$ we have that $\min\big\{1 - \frac{\delta^2}{2}, \omega, 1 - \frac{\alpha^2}{2\delta^2}\big\} > 0$. Moreover, $\|u\| \leq \|u\|_w$. So, by choosing $C = C_1 = \frac{1}{\sqrt{\min\{1 - \frac{\delta^2}{2}, \omega, 1 - \frac{\alpha^2}{2\delta^2}\}}}$ and $C_2 = \sqrt{\max\{1 + \frac{(\varepsilon\alpha)^2}{4}, 1, \omega + 1\}}$, we get the conclusion:

$$C\|u\|_{H^1} \leq C_1\|u\|_w \leq \|u\|_{V,A} \leq C_2\|u\|_w. \qquad \square$$

2.2. **Standing waves and the linearized equation.** It is a classical result that the problem

$$i\phi_t = -\Delta\phi + |\phi|^{p-1}\phi \tag{2.4}$$

with $1 < p < 1 + \frac{4}{n-2}$ admits *standing waves*. Making the ansatz $\phi(t,x) = e^{i\mu t}u(x)$ ($u$ is real valued), $u$ has to solve

$$E'(u) := -\Delta u + \mu u - u^p = 0 \tag{2.5}$$

and since we look for solutions decaying to zero at infinity, we work in the space $H^1(\mathbf{R}^n)$. It is well known that, due to translations, $H^1(\mathbf{R}^n)$ is not compactly embedded into $L^{p+1}(\mathbf{R}^n)$. However, we can recover some compactness working in the subspace of radial functions. Let us set $H_r^1(\mathbf{R}^n) := \{u \in H^1(\mathbf{R}^n) : u \text{ is radial})\}$. By a minimization procedure for the energy functional of (2.5)

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^n} |\nabla u|^2 + |u|^2 - \frac{1}{p+1}\int_{\mathbf{R}^n} |u|^{p+1}, \tag{2.6}$$

restricted to $H_r^1(\mathbf{R}^n)$, one shows the existence of ground states for (2.5).

**Proposition 2.2.** [5, 19, 23, 26] *For $1 < p < 2^* - 1$,* (2.5) *has a unique classical, positive, radial solution $u \in H_r^1(\mathbf{R}^n)$, exponentially decaying at infinity.*

For the sake of proving orbital stability of the *standing wave* $\phi(t,x) = e^{i\mu t}u(x)$ for $1 < p < 1 + \frac{4}{n}$, Weinstein [30], [31] (in dimensions $n = 1, 3$), characterized the kernel of $E''(u)$:

$$E''(u)w := -\Delta w + \mu w - pu^{p-1}w,$$

with $w \in H^1(\mathbf{R}^n)$, $u$ a solution of (2.5) and proved the uniform invertibility of $E''(u)$ on its range. It is important to recall that $E''(u)$ is symmetric (it is an easy direct computation) and selfadjoint on $L^2(\mathbf{R}^n)$ (see [6], Chapter 3).

**Lemma 2.3.** *Let $Z := span_{\mathbf{R}}\{u_\xi := u(x-\xi) : \xi \in \mathbf{R}^n\}$ where $u$ solves* (2.5). *Then*

$$\ker(E''(u)) = T_{u_\xi}Z = \operatorname{span}_{\mathbf{R}}\Big\{\frac{\partial u}{\partial \xi^j}, j = 1, ..., n\Big\}.$$

*Moreover, there exists $C_1, C_2 > 0$ such that*

$\langle E''(u)w, w\rangle_{L^2} \geq C_1\|w\|_{H^1}^2$ *for any* $w \in \big(\ker(E''(u)) + \{u\}\big)^\perp$ *and*

$\langle E''(u)w, w\rangle_{L^2} \leq -C_2\|w\|_{H^1}^2$ *for any* $w \in span\{u\}$.

## 3. The pseudo-critical manifold $\mathcal{Z}$

In this section we prove that the manifold

$$\mathcal{Z} = \Big\{ z_L^\varepsilon(x) = u_L\Big(x - \frac{Le_1}{\varepsilon}\Big) e^{i\alpha L x_2} + u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big) e^{-i\alpha L x_2} : L > 0 \Big\}$$

introduced in Section 1 is *pseudo-critical*. By this we mean that $I'_\varepsilon(z_L^\varepsilon)$ is nonzero in general, but uniformly small (for $\varepsilon$ small enough) in an appropriate sense. We have the following motivation for defining this manifold. Looking at equation (1.10), we notice that the external potentials $V(\varepsilon x)$ and $A(\varepsilon x)$ vary slowly in $\varepsilon$. Moreover, the function $u(x)$, a solution of (2.5), decays exponentially from the origin (see Proposition 2.2) as

$$|u(x)| \le e^{-|x|} |x|^{-\frac{n-1}{2}}$$

and so, by an easy computation, the function $u_L(x) = V(L)^{\frac{1}{p-1}} u\big(V(L)^{\frac{1}{2}} x\big)$ which indeed solves $-\Delta u_L + V(L) u_L - u_L^p = 0$ decays exponentially from the origin with the following dependence on $L$:

$$|u_L(x)| \le \frac{C}{|V(L)^{\frac{1}{2}} x|^{\frac{n-1}{2}}} e^{-(V(L)^{\frac{1}{2}}|x|}. \tag{3.1}$$

So we can guess that a good approximate 1-bump solution is a solution to

$$\Big(\frac{\nabla}{i} - A(L)\Big)^2 u + V(L) u - |u|^{p-1} u = 0, \tag{3.2}$$

which is indeed (1.10) with frozen potentials and whose one-peak solutions are of the form

$$z_1(x) = u_L\Big(x - \frac{Le_1}{\varepsilon}\Big) e^{i\alpha L x_2}. \tag{3.3}$$

In order to have a good approximation for a two-bump symmetric solution, we have split $z_L^\varepsilon$ into the two pieces

$$z_L^\varepsilon = z_1(x) + z_1(-x). \tag{3.4}$$

The function $z_L^\varepsilon$ is not a solution of (3.2) because equation (3.2) is nonlinear, but taking advantage of the decay of $z_1(x)$ and of the *semiclassical limit*, it represents a good approximation for the exact solutions of (1.10). Hence the following lemma holds true.

**Lemma 3.1.** *Suppose $\alpha = \bar\alpha$ with $\bar\alpha^2 \simeq e^{-V(L)^{\frac{1}{2}} \frac{KL}{\varepsilon}} / \varepsilon L \in [\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}]$ with $\alpha_{1,\varepsilon}$ and $\alpha_{2,\varepsilon}$ as in the hypotheses of Theorem 1.1 and $z_L^\varepsilon(x) \in \mathcal{Z}$. Then there exists positive constants $C_0$, $C_1$, $C_2$ and $C_3$ independent on $\varepsilon$ and $L$, but dependent on $p$, such that*

$$\|I'_\varepsilon(z_L^\varepsilon)\|^2 \le C_0 \varepsilon^2 |\nabla E(Le_1)|^2 V(L)^{2q}$$

$$+ C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q} e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\Big|^{\frac{1}{2}}\Big)$$
$$+ C_1\varepsilon^4|D^2 E(Le_1)|^2 V(L)^{2q-1}\Big(1+e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\Big|^{\frac{1}{2}}\Big)\Big)$$
$$+ C_2(\varepsilon\alpha)^2\Big(V(L)^{2q}+|\alpha L|^2 V(L)^{2q-1}\Big)$$
$$+ C_3 V(L)^{2q} e^{-2KV(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\Big|^{\frac{1}{2}}\Big)$$

*with* $q := \frac{1}{p-1} - \frac{n}{4} - \frac{1}{2}$ *and* $K := p\frac{\sqrt{2}}{2}$.

**Proof.** We first rewrite equation (1.10) by splitting the operator $I'_\varepsilon(z^\varepsilon_L)$ as a sum of the operator $I'_L(z^\varepsilon_L)$ of equation (3.2) and a remainder term:

$$\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)^2 z^\varepsilon_L(x) + V(\varepsilon x) z^\varepsilon_L(x) - |z^\varepsilon_L(x)|^{p-1} z^\varepsilon_L(x)$$
$$= \Big(\frac{\nabla}{i} - A(L)\Big)^2 z_1(x) + V(L) z_1(x) - |z_1(x)|^{p-1} z_1(x)$$
$$+ \Big(\frac{\nabla}{i} - A(L)\Big)^2 z_1(-x) + V(L) z_1(-x) - |z_1(-x)|^{p-1} z_1(-x)$$
$$+ \Big\{\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)^2 z^\varepsilon_L(x) - \Big(\frac{\nabla}{i} - A(L)\Big)^2 z^\varepsilon_L(x)\Big\} + \Big\{V(\varepsilon x) z^\varepsilon_L(x) - V(L) z^\varepsilon_L(x)\Big\}$$
$$\Big\{|z_1(x)|^{p-1} z_1(x) + |z_1(-x)|^{p-1} z_1(-x) - |z^\varepsilon_L(x)|^{p-1} z^\varepsilon_L(x)\Big\}$$
$$= \Big\{\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)^2 z^\varepsilon_L(x) - \Big(\frac{\nabla}{i} - A(L)\Big)^2 z^\varepsilon_L(x)\Big\} + \Big\{V(\varepsilon x) z^\varepsilon_L(x) - V(L) z^\varepsilon_L(x)\Big\}$$
$$+ \Big\{|z_1(x)|^{p-1} z_1(x) + |z_1(-x)|^{p-1} z_1(-x) - |z^\varepsilon_L(x)|^{p-1} z^\varepsilon_L(x)\Big\}$$
$$= \Big\{\Big(E(\varepsilon x) z^\varepsilon_L(x) - E(L) z^\varepsilon_L(x)\Big)\Big\} - i\varepsilon\alpha J\Big(x - \frac{Le_1}{\varepsilon}\Big)\nabla\Big(u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)e^{i\alpha L x_2}\Big)$$
$$- i\varepsilon\alpha J\Big(-x - \frac{Le_1}{\varepsilon}\Big)\nabla\Big(u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)e^{-i\alpha L x_2}\Big)$$
$$+ \Big\{|z_1(x)|^{p-1} z_1(x) + |z_1(-x)|^{p-1} z_1(-x) - |z^\varepsilon_L(x)|^{p-1} z^\varepsilon_L(x)\Big\},$$

by the definition of the *pseudo-critical* manifold $\mathcal{Z}$. We next estimate this quantity term by term. In order to clarify the steps we introduce the following notation:

$$A := \Big(E(\varepsilon x) - E(L)\Big) z^\varepsilon_L(x),$$
$$B_1 := -i\varepsilon\alpha J\Big(x - \frac{Le_1}{\varepsilon}\Big)\nabla\Big(u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)e^{i\alpha L x_2}\Big),$$
$$B_{-1} := -i\varepsilon\alpha J\Big(-x - \frac{Le_1}{\varepsilon}\Big)\nabla\Big(u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)e^{-i\alpha L x_2}\Big)$$

and

$$D := \Big\{|z_1(x)|^{p-1}z_1(x) + |z_1(-x)|^{p-1}z_1(-x) - |z_L^{\varepsilon}(x)|^{p-1}z_L^{\varepsilon}(x)\Big\}.$$

For what concerns the term $A$, from a Taylor expansion of $E(x)$ we have

$$\begin{aligned}
&|A| \leq \int_{\mathbf{R}^2} |E(\varepsilon x) - E(L)|^2 |z_L^{\varepsilon}(x)|^2 \\
&= \int_{\mathbf{R}^2} |E(\varepsilon x) - E(L)|^2 \Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)e^{i\alpha L x_2} + u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)e^{-i\alpha L x_2}\Big|^2 \\
&= \int_{\mathbf{R}^2} |E(\varepsilon x) - E(L)|^2 \Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \\
&\int_{\mathbf{R}^2} |E(\varepsilon x) - E(L)|^2 \Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|e^{-i\alpha L x_2} + e^{i\alpha L x_2}\Big| \\
&+ \int_{\mathbf{R}^2} \Big|E(\varepsilon x) - E(L)\Big|^2 \Big|u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \\
&\leq \varepsilon^2 |\nabla E(Le_1)|^2 \int_{\mathbf{R}^2} \Big|\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \\
&+ \varepsilon^2 |\nabla E(Le_1)|^2 \int_{\mathbf{R}^2} \Big|\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \Big|u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 + 2\varepsilon^2 |\nabla E(Le_1)|^2 \\
&\times \int_{\mathbf{R}^2} \Big|\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big| \\
&+ \varepsilon^4 C |D^2 E(Le_1)|^2 \int_{\mathbf{R}^2} \Big|\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^4 \Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \\
&+ \varepsilon^4 C |D^2 E(Le_1)|^2 \int_{\mathbf{R}^2} \Big|\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|^4 \Big|u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 + 2\varepsilon^4 C |D^2 E(Le_1)|^2 \\
&\times \int_{\mathbf{R}^2} \Big|\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \Big|\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|
\end{aligned}$$

and here we have used the symmetries of the potential $E$. Now we estimate just the terms with the first derivatives of $E$, since the others can be treated in a similar way. By a simple change of variables this becomes

$$\begin{aligned}
&\varepsilon^2 |\nabla E(Le_1)|^2 \int_{\mathbf{R}^2} \Big|\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \\
&+ \varepsilon^2 |\nabla E(Le_1)|^2 \int_{\mathbf{R}^2} \Big|\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 \Big|u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 + 2\varepsilon^2 |\nabla E(Le_1)|^2 \\
&\times \int_{\mathbf{R}^2} \Big|\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|\Big|u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)\Big| \\
&= \varepsilon^2 |\nabla E(Le_1)|^2 V(L)^{2q} \int_{\mathbf{R}^2} |x|^2 |u(x)|^2 + \varepsilon^2 |\nabla E(Le_1)|^2 V(L)^{2q} \int_{\mathbf{R}^2} |x|^2 |u_(x)|^2
\end{aligned}$$

$$+\,2V(L)^{2q}\varepsilon^2|\nabla E(Le_1)|^2\int_{\mathbf{R}^2}\Big|\Big(x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\Big|\Big(-x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|$$
$$\times\Big|u\Big(x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\Big|u\Big(-x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|,$$

with $q$ as in the hypotheses. To simplify the notation, just in the following we will denote $d:=V(L)^{\frac12}\frac{L}{\varepsilon}$. Since we are working with even functions, to fix the ideas we can make the computations just in the case $x_1>0$. In this way we have

$$\Big|x+V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big|^2=|x|^2+2dx_1+|de_1|^2\geq|x|^2+|de_1|^2.$$

Hence, $|x+de_1|\geq\big(|x|^2+|de_1|^2\big)^{\frac12}$. Moreover,

$$|x|^2+|de_1|^2\geq C^2\big(|x|^2+|d|^2+2|x|d\big),$$

for any $C<\frac{\sqrt2}{2}$. Hence $|x+de_1|\geq\frac{\sqrt2}{2}(|x|+d)$. Then

$$\begin{aligned}&\int_{\mathbf{R}^2}\Big|\Big(x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\Big|\Big(-x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\\&\times\Big|u\Big(x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\Big|u\Big(-x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\\&=2\int_{\mathbf{R}}dx_2\int_{\mathbf{R}^+}dx_1\Big|\Big(x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\Big|\Big(-x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\\&\times\Big|u\Big(x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|\Big|u\Big(-x-V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|.\end{aligned}$$

By estimate (3.1), we can continue the last equality as

$$\leq\int_{\mathbf{R}}dx_2\int_{\mathbf{R}^+}dx_1|x||(x+2de_1)||u(x)|\Big|\frac{C}{|x+2de_1|^{\frac{n-1}{2}}}e^{-|x+2de_1|}\Big|\tag{3.5}$$
$$\leq C\int_{\mathbf{R}}dx_2\int_{\mathbf{R}^+}dx_1|x||(x+2de_1)||u(x)|\Big|\frac{e^{-\frac{\sqrt2}{2}(|x|+2d)}}{|x+2de_1|^{\frac{n-1}{2}}}\Big|$$
$$\leq Ce^{-\frac{\sqrt2}{2}2d}\int_{\mathbf{R}}dx_2\int_{\mathbf{R}^+}dx_1|x||(x+2de_1)|^{\frac12}|u(x)|e^{-\frac{\sqrt2}{2}|x|}$$
$$\leq Ce^{-\sqrt2 d}(1+(2d)^{\frac12}).$$

Then taking into account also the estimates concerning $D^2E(Le_1)$ (which are different just in the power of $V(L)$ and in the constant $C$) we obtain

$$|A|\leq C\varepsilon^2|\nabla E(Le_1)|^2V(L)^{2q}\Big(1+e^{-\sqrt2V(L)^{\frac12}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac12}\frac{L}{\varepsilon}\Big|^{\frac12}\Big)\Big)$$

$$+ C\varepsilon^4|D^2E(Le_1)|^2V(L)^{2q-1}\Big(1+e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\Big|^{\frac{1}{2}}\Big)\Big).$$

The terms $B_1$ and $B_{-1}$ can be estimated in a similar way as follows:

$$\begin{aligned}
|B_1| &\leq \int_{\mathbf{R}^2}\Big|i\varepsilon\alpha J\Big(x-\frac{Le_1}{\varepsilon}\Big)\nabla\Big(u_L\Big(x-\frac{Le_1}{\varepsilon}\Big)e^{i\alpha Lx_2}\Big)\Big|^2\\
&\leq (\varepsilon\alpha)^2\int_{\mathbf{R}^2}\Big|\Big(x-\frac{Le_1}{\varepsilon}\Big)\Big|^2\Big|\nabla u_L\Big(x-\frac{Le_1}{\varepsilon}\Big)+i\alpha Le_2u_L\Big(x-\frac{Le_1}{\varepsilon}\Big)\Big|^2\\
&= (\varepsilon\alpha)^2\int_{\mathbf{R}^2}\Big|\Big(x-\frac{Le_1}{\varepsilon}\Big)\Big|^2\Big(\Big|\nabla u_L\Big(x-\frac{Le_1}{\varepsilon}\Big)\Big|^2+\Big|\alpha Le_2u_L\Big(x-\frac{Le_1}{\varepsilon}\Big)\Big|^2\Big)\\
&= (\varepsilon\alpha)^2V(L)^{2q}\int_{\mathbf{R}^2}|x|^2\Big(|\nabla u(x)|^2+V(L)^{-1}|\alpha Lu(x)|^2\Big)\\
&\leq (\varepsilon\alpha)^2\Big(C_0V(L)^{2q}+C_1|\alpha L|^2V(L)^{2q-1}\Big).
\end{aligned}$$

Analogously

$$\begin{aligned}
|B_{-1}| &\leq \int_{\mathbf{R}^2}\Big|i\varepsilon\alpha J\Big(-x-\frac{Le_1}{\varepsilon}\Big)\nabla\Big(u_L\Big(-x-\frac{Le_1}{\varepsilon}\Big)e^{-i\alpha Lx_2}\Big)\Big|^2\\
&\leq (\varepsilon\alpha)^2\Big(C_0V(L)^{2q}+C_1|\alpha L|^2V(L)^{2q-1}\Big).
\end{aligned}$$

Finally the term $D$ can be estimated in a way similar to computations in (3.5) and one gets

$$\|D\|^2 \leq CV(L)^{2q}e^{-2KV(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\Big|^{\frac{1}{2}}\Big),$$

with $K := p\frac{\sqrt{2}}{2}$. The constant $C$ depends on $p$, but is independent of $L$ and $\varepsilon$. Collecting all these estimates we get

$$\begin{aligned}
&\|I'_\varepsilon(z_L^\varepsilon)\|^2 \leq C_0\varepsilon^2|\nabla E(Le_1)|^2V(L)^{2q}\\
&+C_0\varepsilon^2|\nabla E(Le_1)|^2V(L)^{2q}e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\Big|^{\frac{1}{2}}\Big)\\
&+C_1\varepsilon^4|D^2E(Le_1)|^2V(L)^{2q-1}\Big(1+e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\Big|^{\frac{1}{2}}\Big)\Big)\\
&+C_2(\varepsilon\alpha)^2\Big(V(L)^{2q}+|\alpha L|^2V(L)^{2q-1}\Big)\\
&+C_3V(L)^{2q}e^{-2KV(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\Big|^{\frac{1}{2}}\Big),
\end{aligned}$$

with $C_0$, $C_1$, $C_2$ and $C_3$ independent of $\varepsilon$ and $L$. □

## 4. The Lyapunov-Schmidt reduction

In this section we find an exact solution $u(x) := z_L^\varepsilon(x) + w(x)$ of equation (1.10) via a Lyapunov-Schmidt reduction. An important issue is that equation (1.10) is invariant with respect to some symmetries and this allows us to work in the space $H^s_{V,A}(\mathcal{R}^n)$ (see Section 2). The choice of the approximate solutions (1.13) suggests which subspaces are more suitable for the procedure. Therefore, we split $H^s_{V,A}(\mathcal{R}^n)$ into the following two orthogonal closed subspaces $T_{z_L^\varepsilon}\mathcal{Z}^\varepsilon := \big\{\partial_L z_L^\varepsilon(x) : L \in \mathbf{R}\big\}$, where

$$\partial_L z_L^\varepsilon(x) = \partial_L\Big(u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)e^{i\alpha L x_2}\Big) + \partial_L\Big(u_L\Big(-x - \frac{Le_1}{\varepsilon}\Big)e^{-i\alpha L x_2}\Big)$$

and $W := \big(T_{z_L^\varepsilon}\mathcal{Z}^\varepsilon\big)^\perp$. Moreover, we define the associated orthogonal projections $P : H^s_{V,A} \to W$ and $Q := Id - P : H^s_{V,A} \to T_{z_L^\varepsilon}\mathcal{Z}^\varepsilon$. Thanks to this splitting equation (1.10) is equivalent to the couple of equations

$$PI'_\varepsilon(z_L^\varepsilon + w) = 0 \qquad (\text{the } \textit{range equation})$$

and

$$QI'_\varepsilon(z_L^\varepsilon + w) = 0 \qquad (\text{the } \textit{bifurcation equation}).$$

4.1. **Invertibility of $I''_\varepsilon(z_L^\varepsilon)$ in $\mathcal{Z}_\varepsilon^\perp$.** We begin with a lemma which describes the kernel of $I''_\varepsilon(z_1(x))$. We omit the proof since it is a straightforward application of Lemma 2.3 and the analysis by Cingolani and Secchi [10].

**Lemma 4.1.** *Let $z_1(x)$ be such that $\mathcal{Z}^\varepsilon \ni z_L^\varepsilon(x) = z_1(x) + z_1(-x)$. Then $\ker(I''_L(z_1)|_{H^s_{V,A}})$ is one dimensional and $\ker(I''_L(z_1)|_{H^s_{V,A}}) = T_{z_L^\varepsilon}\mathcal{Z}^\varepsilon$.*

In the following lemma we notice that $\partial_L z_L^\varepsilon(x)$ can be well approximated by $\frac{1}{\varepsilon}\partial_{x_1} z_L^\varepsilon(x)$.

**Lemma 4.2.** *One has the following estimate:*

$$\varepsilon\partial_L z_1(x) = -\partial_{x_1} z_1(x) + O(\varepsilon V(L)^\gamma \nabla V(L)) + O(\varepsilon\alpha V(L)^{\gamma+\frac{1}{2}}), \textit{ in } H_{A,V}$$

*with $\gamma := -1 - \frac{n}{2} + \frac{1}{p-1}$ and hence*

$$\varepsilon\partial_L z_L^\varepsilon(x) = -\partial_{x_1} z_L^\varepsilon(x) + O\big(\varepsilon V(L)^\gamma \nabla V(L)\big) + O(\varepsilon\alpha V(L)^{\gamma+\frac{1}{2}}), \textit{ in } H_{A,V}. \tag{4.1}$$

**Proof.** We start by computing $\partial_L z_L^\varepsilon(x)$:

$$\partial_L\Big(u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)e^{i\alpha L x_2}\Big) = \frac{1}{p-1}V(L)^{\frac{1}{p-1}-1}\nabla V(L) u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)e^{i\alpha L x_2}$$
$$+ V(L)^{\frac{1}{p-1}}\Big(i\alpha x_2 u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)e^{i\alpha L x_2} + e^{i\alpha L x_2}\nabla u \cdot \nabla\Big(V(L)^{\frac{1}{2}}\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big)\Big)$$

$$
\begin{aligned}
&= \frac{1}{p-1} V(L)^{\frac{1}{p-1}-1} \nabla V(L) u_L\Big(x - \frac{Le_1}{\varepsilon}\Big) e^{i\alpha L x_2} \\
&+ V(L)^{\frac{1}{p-1}} i\alpha x_2 u_L\Big(x - \frac{Le_1}{\varepsilon}\Big) e^{i\alpha L x_2} \\
&+ e^{i\alpha L x_2}\Big(\nabla u \cdot \Big(\frac{1}{2} V(L)^{-\frac{1}{2}} \nabla V(L)\Big(x - \frac{Le_1}{\varepsilon}\Big) - \frac{e_1}{\varepsilon} V(L)^{\frac{1}{2}}\Big)\Big).
\end{aligned}
$$

Multiplying by $\varepsilon$ we get

$$
\begin{aligned}
\varepsilon \partial_L z_1(x) &= \varepsilon \partial_L \Big( u_L\Big(x - \frac{Le_1}{\varepsilon}\Big) e^{i\alpha L x_2}\Big) \\
&= -\partial_{x_1} z_1(x) + \varepsilon \frac{1}{p-1} V(L)^{-1} \nabla V(L) z_1(x) + i\varepsilon\alpha x_2 z_1(x) \\
&+ \varepsilon e^{i\alpha L x_2} \frac{1}{2} V(L)^{-\frac{1}{2}} \nabla V(L)\Big(x - \frac{Le_1}{\varepsilon}\Big) \cdot \nabla u_L\Big(x - \frac{Le_1}{\varepsilon}\Big).
\end{aligned}
$$

Now we estimate the remainder term by term. The first part of the remainder is

$$
\begin{aligned}
&\Big(\int_{\mathbf{R}^2} |i\varepsilon\alpha x_2 z_1(x)|^2 dx\Big)^{\frac{1}{2}} = \varepsilon\alpha \Big(\int_{\mathbf{R}^2} \Big| x_2 u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 dx\Big)^{\frac{1}{2}} \\
&= \varepsilon\alpha \Big(\int_{\mathbf{R}^2} |x_2 u_L(x)|^2 dx\Big)^{\frac{1}{2}} = \varepsilon\alpha V(L)^{\frac{1}{p-1}} \Big(\int_{\mathbf{R}^2} \Big| x_2 u\big(V(L)^{\frac{1}{2}}(x)\big)\Big|^2 dx\Big)^{\frac{1}{2}} \\
&= \varepsilon\alpha V(L)^{\frac{1}{p-1} - \frac{n+1}{2}} \Big(\int_{\mathbf{R}^2} \Big| x_2 u(x)\Big|^2 dx\Big)^{\frac{1}{2}} \leq C\varepsilon\alpha V(L)^{\frac{1}{p-1} - \frac{n+1}{2}}.
\end{aligned}
$$

The second part of the remainder is

$$
\begin{aligned}
&\Big(\int_{\mathbf{R}^2} \Big| \frac{\varepsilon}{p-1} V(L)^{-1} \nabla V(L) z_1(x)\Big|^2 dx\Big)^{\frac{1}{2}} = \frac{\varepsilon V(L)^{-1}}{p-1} \nabla V(L) \\
&\times \Big(\int_{\mathbf{R}^2} \Big| u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 dx\Big)^{\frac{1}{2}} \\
&= \varepsilon \frac{V(L)^{-1+\frac{1}{p-1}}}{p-1} \nabla V(L) \Big(\int_{\mathbf{R}^2} \Big| u\Big(V(L)^{\frac{1}{2}}\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big)\Big|^2 dx\Big)^{\frac{1}{2}} \\
&= \varepsilon \frac{V(L)^{-1-\frac{n}{2}+\frac{1}{p-1}}}{p-1} \nabla V(L) \Big(\int_{\mathbf{R}^2} |u(x)|^2 dx\Big)^{\frac{1}{2}} \\
&\leq C\varepsilon V(L)^{-1-\frac{n}{2}+\frac{1}{p-1}} \nabla V(L).
\end{aligned}
$$

The third part of the remainder is

$$
\Big(\int_{\mathbf{R}^2} \Big| \varepsilon e^{i\alpha L x_2} \frac{1}{2} V(L)^{-\frac{1}{2}} \nabla V(L)\Big(x - \frac{Le_1}{\varepsilon}\Big) \cdot \nabla u_L\Big(x - \frac{Le_1}{\varepsilon}\Big)\Big|^2 dx\Big)^{\frac{1}{2}}
$$

$$\le \varepsilon\frac{1}{2}V(L)^{-\frac{1}{2}}\nabla V(L)\Big(\int_{\mathbf{R}^2}\Big|x\cdot\nabla u_L(x)\Big|^2dx\Big)^{\frac{1}{2}}$$
$$\le \varepsilon\frac{1}{2}V(L)^{-1-\frac{n}{2}+\frac{1}{p-1}}\nabla V(L)\Big(\int_{\mathbf{R}^2}\Big|x\cdot\nabla u(x)\Big|^2dx\Big)^{\frac{1}{2}}$$
$$\le C\varepsilon V(L)^{-1-\frac{n}{2}+\frac{1}{p-1}}\nabla V(L).$$

Collecting all these estimates we get

$$\varepsilon\partial_L z_1(x) = -\partial_{x_1}z_1(x) + O\big(\varepsilon V(L)^{\gamma}\nabla V(L)\big) + O\big(\varepsilon\alpha V(L)^{\gamma+\frac{1}{2}}\big), \tag{4.2}$$

in $H_{A,V}$, with $\gamma = -1-\frac{n}{2}+\frac{1}{p-1}$ and hence the thesis. □

We now prove some invertibility property of $I''_\varepsilon(z^\varepsilon_L)$ in $\mathcal{Z}^\perp_\varepsilon$. The result is obtained by a localization procedure around the two bumps. This strategy turns out to be good thanks to the decay estimates (3.1) and to the symmetries of the space $H^s_{V,A}$ . The proof of the invertibility exploits the techniques used in [1] and in [7] in some different contexts.

**Lemma 4.3.** *There exists a constant $C>0$ such that, for $\varepsilon$ small enough, one has the following estimate:*

$$< I''_\varepsilon(z^\varepsilon_L)w, w >\ \ge C\|w\|^2, \qquad w\in\Big(W^\perp, span\{z^\varepsilon_L\}\Big)^\perp. \tag{4.3}$$

**Proof.** We recall the definition of $I(u)$:

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbf{R}^2}\Big|\Big(\frac{\nabla}{i}-A(\varepsilon x)\Big)u\Big|^2 + V(\varepsilon x)|u|^2 - \frac{1}{p+1}|u|^{p+1}. \tag{4.4}$$

Since $I(u)\in\mathcal{C}^2$, we are allowed to differentiate it twice and we can easily compute $\langle I''_\varepsilon(z^\varepsilon_L)v,v\rangle$ in the following way:

$$\langle I''_\varepsilon(z^\varepsilon_L)v,v\rangle = \frac{1}{2}\int_{\mathbf{R}^2}\Big|\Big(\frac{\nabla}{i}-A(\varepsilon x)\Big)v\Big|^2 + V(\varepsilon x)|v|^2 - |z^\varepsilon_L|^{p-1}v \tag{4.5}$$
$$-(p-1)|z^\varepsilon_L|^{p-3}Re(z^\varepsilon_L)Re(v) - (p-1)|z^\varepsilon_L|^{p-3}Im(z^\varepsilon_L)Im(v).$$

In order to localize around the two bumps, we introduce the cut-off function $\chi$

$$\chi(x) = \begin{cases} 1, & B_R(0)\\ 0, & B^C_{2R}(0)\\ |\tilde{\nabla}\chi|\le\frac{2}{R}, & \text{otherwise,}\end{cases} \tag{4.6}$$

where we have used the notation $\tilde{\nabla} := \big(\frac{\nabla}{i} - A(\varepsilon x)\big)$, and split the test function $v$ as $v := v_1 + v_2 + v_3$, where

$$v_1(x) := \chi\Big(x - \frac{Le_1}{\varepsilon}\Big)v(x), \quad v_2(x) := \chi\Big(-x - \frac{Le_1}{\varepsilon}\Big)v(x)$$

and $v_3(x) := v(x) - v_1(x) - v_2(x)$. Since $v(x) = v(-x)$, then $v_1(x) = v_2(-x)$ and we also have that $supp(v_1) \subset B_{2R}\big(\frac{Le_1}{\varepsilon}\big)$, $supp(v_2) \subset B_{2R}\big(-\frac{Le_1}{\varepsilon}\big)$, $supp(v_3) \subset B^c_R\big(\frac{Le_1}{\varepsilon}\big) \cup B^c_R\big(\frac{Le_1}{\varepsilon}\big)$.

$$\begin{aligned}\|v\|^2 &= \|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2 + 2\langle v_1, v_2\rangle + 2\langle v_2, v_3\rangle + 2\langle v_3, v_1\rangle \\ &= \|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2 + 2\langle v_2, v_3\rangle + 2 < v_3, v_1 > .\end{aligned}$$

The quantity $< v_1, v_2 >$ vanishes since, as noticed above, the supports of $v_1$ and $v_2$ are disjoint. We now compute the scalar product $< v_2, v_3 >$ as follows:

$$\begin{aligned}\langle v_2, v_3\rangle &= Re \int_{\mathbf{R}^2} \tilde{\nabla} v_2 \bar{\tilde{\nabla}} \bar{v}_3 + V(\varepsilon x) v_2 \bar{v}_3 \\ &= Re \int_{\mathbf{R}^2} \tilde{\nabla} v_2 \big(\overline{\tilde{\nabla}(v - v_1 - v_2)}\big) + V(\varepsilon x) v_2 \big(\overline{v - v_1 - v_2}\big) \\ &= Re \int_{\mathbf{R}^2} \big(\tilde{\nabla} v \chi_+ + v \tilde{\nabla}\chi_+\big)\big(\overline{\tilde{\nabla} v(1 - \chi_+ - \chi_-) + v\tilde{\nabla}(1 - \chi_+ - \chi_-)}\big) \\ &\quad + Re \int_{\mathbf{R}^2} V(\varepsilon x) v \chi_+ \overline{v(1 - \chi_+ - \chi_-)} \\ &= Re \int_{\mathbf{R}^2} \big(|\tilde{\nabla} v|^2 + V(\varepsilon x)|v|^2 \chi_+ \overline{v(1 - \chi_+ - \chi_-)}\big) \\ &\quad + Re \int_{\mathbf{R}^2} v \overline{\tilde{\nabla} v} \tilde{\nabla} \chi_+ (\overline{1 - \chi_+ - \chi_-}) + \tilde{\nabla} v \bar{v} \chi_+ (\overline{\tilde{\nabla}(1 - \chi_+ - \chi_-)}) \\ &\quad + Re \int_{\mathbf{R}^2} |v|^2 \tilde{\nabla}\chi_+ (\overline{\tilde{\nabla}(1 - \chi_+ - \chi_-)}) \\ &= Re \int_{\mathbf{R}^2} \big(|\tilde{\nabla} v|^2 + V(\varepsilon x)|v|^2 \chi_+ \overline{v(1 - \chi_+)}\big) \\ &\quad + Re \int_{\mathbf{R}^2} v \overline{\tilde{\nabla} v} \tilde{\nabla} \chi_+ (\overline{1 - \chi_+}) + \tilde{\nabla} v \bar{v} \chi_+ (\overline{\tilde{\nabla}(1 - \chi_+)}) \\ &\quad + Re \int_{\mathbf{R}^2} |v|^2 \tilde{\nabla}\chi_+ (\overline{\tilde{\nabla}(1 - \chi_+)}).\end{aligned}$$

We set

$$\tau_1 := Re \int_{\mathbf{R}^2} \big(|\tilde{\nabla} v|^2 + V(\varepsilon x)|v|^2\big)\chi_+ \overline{v(1 - \chi_+)}$$

and

$$\tau_2 := Re \int_{\mathbf{R}^2} v\overline{\tilde{\nabla} v}\tilde{\nabla}\chi_+(\overline{1-\chi_+}) + \tilde{\nabla} v\bar{v}\chi_+(\overline{\tilde{\nabla}(1-\chi_+)})$$
$$+ Re \int_{\mathbf{R}^2} |v|^2\tilde{\nabla}\chi_+(\overline{\tilde{\nabla}(1-\chi_+)}).$$

We can estimate $\tau_2$ as follows:

$$\begin{aligned}
|\tau_2| &\leq \Big| \int_{\mathbf{R}^2} v\overline{\tilde{\nabla} v}\tilde{\nabla}\chi_+(\overline{1-\chi_+}) + \tilde{\nabla} v\bar{v}\chi_+(\overline{\tilde{\nabla}(1-\chi_+)})\Big| \\
&+ \Big| Re \int_{\mathbf{R}^2} |v|^2\tilde{\nabla}\chi_+(\overline{\tilde{\nabla}(1-\chi_+)})\Big| \\
&\leq \int_{\mathbf{R}^2} |v||\tilde{\nabla} v||\tilde{\nabla}\chi_+||1-\chi_+| + \int_{\mathbf{R}^2} |v||\tilde{\nabla} v||\chi_+|\big|\tilde{\nabla}(1-\chi_+)\big| \\
&+ \int_{\mathbf{R}^2} |v|^2|\tilde{\nabla}\chi_+|\big|\tilde{\nabla}(1-\chi_+)\big| \\
&\leq \frac{4}{R}\int_{R<|x|<2R} |v||\tilde{\nabla} v| + \frac{4}{R^2}\int_{R<|x|<2R} |v|^2 \\
&\leq \Big(\frac{2}{R} + \frac{4}{R^2}\Big)\int_{R<|x|<2R} \big(|\tilde{\nabla} v|^2 + |v|^2\big) \leq \frac{C}{R}\|v\|_{V,A}^2 \to 0, \text{ as } R \to +\infty.
\end{aligned}$$

This bound implies that $|\tau_2| = O_R(1)\|v\|_{V,A}^2$. We can use the same procedure for $\langle v_2, v_3\rangle = \tau_{-1} + \tau_{-2}$ with

$$\tau_{-1} := Re \int_{\mathbf{R}^2} \big(|\tilde{\nabla} v|^2 + V(\varepsilon x)|v|^2\big)\chi_-\overline{v(1-\chi_-)}$$

and

$$\tau_{-2} := Re \int_{\mathbf{R}^2} v\overline{\tilde{\nabla} v}\tilde{\nabla}\chi_-(\overline{1-\chi_-}) + \tilde{\nabla} v\bar{v}\chi_-(\overline{\tilde{\nabla}(1-\chi_-)})$$
$$+ Re \int_{\mathbf{R}^2} |v|^2\tilde{\nabla}\chi_-(\overline{\tilde{\nabla}(1-\chi_-)}).$$

Hence, $|\tau_{-2}| = O_R(1)\|v\|_{V,A}^2$. These preliminary estimates give us

$$\begin{aligned}
\|v\|^2 &= \|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2 + 2\tau_1 + 2\tau_{-1} \\
&= \|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2 + 2\langle v_2, v_3\rangle + 2\langle v_3, v_1\rangle + O_R(1)\|v\|^2.
\end{aligned}$$

We are now in position to evaluate $\langle I''_\varepsilon(z_L^\varepsilon)v, v\rangle$. By means of the cut-off function we have the following splitting:

$$\langle I''_\varepsilon(z_L^\varepsilon)v, v\rangle = \langle I''_\varepsilon(z_L^\varepsilon)v_1, v_1\rangle + \langle I''_\varepsilon(z_L^\varepsilon)v_2, v_2\rangle + \langle I''_\varepsilon(z_L^\varepsilon)v_3, v_3\rangle$$

$$+ 2\langle I''_\varepsilon(z^\varepsilon_L)v_1, v_2\rangle + 2\langle I''_\varepsilon(z^\varepsilon_L)v_2, v_3\rangle + 2\langle I''_\varepsilon(z^\varepsilon_L)v_3, v_1\rangle.$$

For simplicity we set

$$\begin{aligned}
&\sigma_1 := \langle I''_\varepsilon(z^\varepsilon_L)v_1, v_1\rangle,\ \sigma_2 := \langle I''_\varepsilon(z^\varepsilon_L)v_2, v_2\rangle,\ \sigma_3 := \langle I''_\varepsilon(z^\varepsilon_L)v_3, v_3\rangle,\\
&\sigma_{1,2} := \langle I''_\varepsilon(z^\varepsilon_L)v_1, v_2\rangle,\ \sigma_{2,3} := \langle I''_\varepsilon(z^\varepsilon_L)v_2, v_3\rangle,\\
&\sigma_{3,1} := \langle I''_\varepsilon(z^\varepsilon_L)v_3, v_1\rangle.
\end{aligned}$$

Before estimating each term, it is worthwhile to compute the second derivative of the freezed functional $I_L(u)$ (see Section 2):

$$\begin{aligned}
\langle I''_L(z^\varepsilon_L)w, w\rangle =& Re\int_{\mathbf{R}^2} \Big|\Big(\frac{\nabla}{i} - A(L)\Big)w\Big|^2 + V(L)|w|^2 - |z^\varepsilon_L|^{p-1}|w|^2 \qquad (4.7)\\
&- (p-1)|z^\varepsilon_L|^{p-3}Re(z^\varepsilon_L)Re(w)\overline{w} + -(p-1)|z^\varepsilon_L|^{p-3}Im(z^\varepsilon_L)Im(w)\overline{w}.
\end{aligned}$$

We start estimating $\sigma_1$, as follows:

$$\begin{aligned}
\sigma_1 &= \langle I''_\varepsilon(z^\varepsilon_L)v_1, v_1\rangle = \langle I''_L(z^\varepsilon_L)v_1, v_1\rangle + \int_{\mathbf{R}^2}\Big(V(\varepsilon x) - V(L)\Big)|v_1|^2\\
&+ \int_{\mathbf{R}^2}\Big(\Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)v_1\Big|^2 - \Big|\Big(\frac{\nabla}{i} - A(L)\Big)v_1\Big|^2\Big).
\end{aligned}$$

We define

$$\psi_1 := \frac{1}{\|\partial_L z_1\|^2_L}\langle v_1, \partial_L z_1\rangle_L \text{ and } \overline{v_1} := v_1 - \psi_1.$$

Then

$$\langle I''_L(z^\varepsilon_L)v_1, v_1\rangle = \langle I''_L(z^\varepsilon_L)\overline{v_1}, \overline{v_1}\rangle + 2\langle I''_L(z^\varepsilon_L)\overline{v_1}, \psi_1\rangle + \langle I''_L(z^\varepsilon_L)\psi_1, \psi_1\rangle.$$

By definition of $\psi_1$ and by the invertibility around one bump of $I''_L$ (see Lemma 4.1) we have $\langle I''_L(z^\varepsilon_L)\overline{v_1}, \overline{v_1}\rangle \geq \delta_L\|\overline{v_1}\|^2_L$. By construction we have

$$\langle v_1, \partial_L z^\varepsilon_L\rangle_L = \langle v, \partial_L z^\varepsilon_L\rangle_L - \langle v_2, \partial_L z^\varepsilon_L\rangle_L - \langle v_3, \partial_L z^\varepsilon_L\rangle_L.$$

By the symmetries of $v, v_1, v_2$ and $\partial_L z^\varepsilon_L$ we get

$$\langle v_1, \partial_L z^\varepsilon_L\rangle_L = \langle v_2, \partial_L z^\varepsilon_L\rangle_L.$$

We can verify this by direct computation:

$$\begin{aligned}
\langle \partial_L z^\varepsilon_L, v_2\rangle_L &= \int_{\mathbf{R}^2}\Big(\frac{\nabla}{i} - A(L)\Big)\partial_L z^\varepsilon_L\overline{\Big(\frac{\nabla}{i} - A(L)\Big)v_2} + V(L)\partial_L z^\varepsilon_L\overline{v_2}\\
&= \int_{\mathbf{R}^2}\Big(\overline{\nabla v_2}\nabla\partial_L z^\varepsilon_L + iA(L)\overline{v_2}\nabla\partial_L z^\varepsilon_L - iA(L)\overline{\nabla v_2}\partial_L z^\varepsilon_L + |A(L)|^2\overline{v_2}\partial_L z^\varepsilon_L\Big)\\
&+ \int_{\mathbf{R}^2}\Big(V(L)\overline{v_2}\partial_L z^\varepsilon_L\Big).
\end{aligned}$$

We make the computations just for the first two terms of the integral since the others are similar:

$$\int_{\mathbf{R}^2}\overline{\nabla v_2}\nabla\partial_L z_L^\varepsilon=\int_{\mathbf{R}^2}\overline{\nabla_x v_2(x)}\nabla_x\partial_L z_L^\varepsilon(x)=\int_{\mathbf{R}^2}\overline{\nabla_{-x}v_2(-x)}\nabla_{-x}\partial_L z_L^\varepsilon(-x)$$
$$=\int_{\mathbf{R}^2}(-1)\overline{\nabla_x v_2(-x)}(-1)\nabla_x\partial_L z_L^\varepsilon(-x)$$
$$=\int_{\mathbf{R}^2}\overline{\nabla_x v_1(x)}\nabla_x\partial_L z_L^\varepsilon(x)=\int_{\mathbf{R}^2}\overline{\nabla v_1}\nabla\partial_L z_L^\varepsilon$$

and

$$\int_{\mathbf{R}^2}iA(L)\overline{v_2}\nabla\partial_L z_L^\varepsilon=\int_{\mathbf{R}^2}iA(L)\overline{v_2(x)}\nabla_x\partial_L z_L^\varepsilon(x)$$
$$=\int_{\mathbf{R}^2}iA(L)\overline{v_2(-x)}\nabla_{-x}\partial_L z_L^\varepsilon(-x)=\int_{\mathbf{R}^2}iA(L)\overline{v_1(x)}\partial_L\nabla_{-x}z_L^\varepsilon(-x)$$
$$=\int_{\mathbf{R}^2}iA(L)\overline{v_1(x)}\partial_L\nabla_x z_L^\varepsilon(x)=\int_{\mathbf{R}^2}iA(L)\overline{v_1(x)}\nabla\partial_L z_L^\varepsilon(x).$$

By analogous computations on the other terms we have

$$\langle v_1,\partial_L z_L^\varepsilon\rangle_L=\langle v_2,\partial_L z_L^\varepsilon\rangle_L$$

as desired. Hence, $\langle v_1,\partial_L z_L^\varepsilon\rangle_L=\langle v,\partial_L z_L^\varepsilon\rangle_L-\langle v_2,\partial_L z_L^\varepsilon\rangle_L-\langle v_3,\partial_L z_L^\varepsilon\rangle_L$ implies

$$\tfrac{1}{2}\langle v_1,\partial_L z_L^\varepsilon\rangle_L=\langle v,\partial_L z_L^\varepsilon\rangle_L-\langle v_3,\partial_L z_L^\varepsilon\rangle_L.$$

Moreover,

$$\tfrac{1}{2}\langle v_1,\partial_L z_L^\varepsilon\rangle_L=\tfrac{1}{2}\langle v_1,\partial_L z_1\rangle_L+\tfrac{1}{2}\langle v_1,\partial_L z_2\rangle_L$$

and so

$$\tfrac{1}{2}\langle v_1,\partial_L z_1\rangle_L=\langle v,\partial_L z_L^\varepsilon\rangle_L-\langle v_3,\partial_L z_L^\varepsilon\rangle_L-\tfrac{1}{2}\langle v_1,\partial_L z_2\rangle_L.$$

We now estimate term by term the last quantity. Since by hypothesis $\langle v,\partial_L z_L^\varepsilon\rangle=0$, then

$$\langle v,\partial_L z_L^\varepsilon\rangle_L=\int_{\mathbf{R}^2}\Big(\frac{\nabla}{i}-A(\varepsilon x)\Big)\partial_L z_L^\varepsilon\overline{\Big(\frac{\nabla}{i}-A(\varepsilon x)\Big)v}+V(\varepsilon x)\partial_L z_L^\varepsilon\overline{v}$$
$$-\int_{\mathbf{R}^2}\Big(\frac{\nabla}{i}-A(L)\Big)\partial_L z_L^\varepsilon\overline{\Big(\frac{\nabla}{i}-A(L)\Big)v}+V(L)\partial_L z_L^\varepsilon\overline{v},$$

but these terms can be estimated in a way similar to the one used in the proof of Lemma 3.1 and so $\langle v,\partial_L z_L^\varepsilon\rangle_L=o_{R,\varepsilon}(1)\|v\|$. For what concerns $\langle v_3,\partial_L z_L^\varepsilon\rangle_L$, we can just notice that $v_3(x)$ is supported in $B_R^c(\frac{Le_1}{\varepsilon})\cup B_R^c(-\frac{Le_1}{\varepsilon})$ where $\partial_L z_L^\varepsilon$ is exponentially decaying and this leads directly to

$\langle v_3, \partial_L z_L^\varepsilon\rangle_L = o_{R,\varepsilon}(1)\|v\|$ and in a similar way $\langle v_1, \partial_L z_2\rangle_L = o_{R,\varepsilon}(1)\|v\|$. By these reasons we can conclude that $\langle v_1, \partial_L z_1\rangle_L = o_{R,\varepsilon}(1)\|v\|$ and so $\|\psi_1\| = o_{R,\varepsilon}(1)\|v\|$. Collecting all this information we have

$$\begin{aligned}\langle I_L''(z_L^\varepsilon)v_1, v_1\rangle &= \langle I_L''(z_L^\varepsilon)\overline{v_1}, \overline{v_1}\rangle + 2\langle I_L''(z_L^\varepsilon)\overline{v_1}, \psi_1\rangle + \langle I_L''(z_L^\varepsilon)\psi_1, \psi_1\rangle \\ &\geq \delta_L\|\overline{v_1}\|_L^2 + o_{R,\varepsilon}(1)\|v\|^2.\end{aligned}$$

Going back to $\sigma_1$ we have the following:

$$\begin{aligned}\sigma_1 &= \langle I_\varepsilon''(z_L^\varepsilon)v_1, v_1\rangle = \langle I_L''(z_L^\varepsilon)v_1, v_1\rangle + \int_{\mathbf{R}^2}\Big(V(\varepsilon x) - V(L)\Big)|v_1|^2 \\ &+ \int_{\mathbf{R}^2}\Big(\Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)v_1\Big|^2 - \Big|\Big(\frac{\nabla}{i} - A(L)\Big)v_1\Big|^2\Big) \\ &\geq \delta_L\|\overline{v_1}\|_L^2 + o_{R,\varepsilon}(1)\|v\|^2 + \int_{\mathbf{R}^2}\Big(V(\varepsilon x) - V(L)\Big)|v_1|^2 \\ &+ \int_{\mathbf{R}^2}\Big(\Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)v_1\Big|^2 - \Big|\Big(\frac{\nabla}{i} - A(L)\Big)v_1\Big|^2\Big) \\ &= \delta_L\|\overline{v_1}\|^2 - (1+\delta_L)\int_{\mathbf{R}^2}\Big(V(\varepsilon x) - V(L)\Big)|v_1|^2 \\ &- (1+\delta_L)\int_{\mathbf{R}^2}\Big(\Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)v_1\Big|^2 - \Big|\Big(\frac{\nabla}{i} - A(L)\Big)v_1\Big|^2\Big) + o_{R,\varepsilon}(1)\|v\|^2 \\ &\geq C_1\|v_1\|^2\end{aligned}$$

for some $C_1 > 0$, by the estimate performed in Lemma 3.1. To estimate the corresponding $\sigma_2 = \langle I_\varepsilon''(z_L^\varepsilon)v_2, v_2\rangle$, one has to make the same steps but with $\psi_2 := \frac{1}{\|\partial_L z_2\|_L^2}\langle v_2, \partial_L z_2\rangle_L$ and $\overline{v_2} := v_2 - \psi_2$ and obtain

$$\sigma_2 = \langle I_\varepsilon''(z_L^\varepsilon)v_2, v_2\rangle \geq C_2\|v_2\|^2,$$

for some $C_2 > 0$. For what concerns $\sigma_3 = \langle I_\varepsilon''(z_L^\varepsilon)v_3, v_3\rangle$ we have

$$\begin{aligned}\sigma_3 &= \langle I_L''(z_L^\varepsilon)v_3, v_3\rangle = Re\int_{\mathbf{R}^2}\Big|\Big(\frac{\nabla}{i} - A(L)\Big)v_3\Big|^2 + V(L)|v_3|^2 \\ &- |z_L^\varepsilon|^{p-1}|v_3|^2 - (p-1)|z_L^\varepsilon|^{p-3}Re(z_L^\varepsilon)Re(v_3)\overline{v_3} \\ &- (p-1)|z_L^\varepsilon|^{p-3}Im(z_L^\varepsilon)Im(v_3)\overline{v_3} = \|v_3\|^2 - |z_L^\varepsilon|^{p-1}|v_3|^2 \\ &- (p-1)|z_L^\varepsilon|^{p-3}Re(z_L^\varepsilon)Re(v_3)\overline{v_3} - (p-1)|z_L^\varepsilon|^{p-3}Im(z_L^\varepsilon)Im(v_3)\overline{v_3}.\end{aligned}$$

As before, since $v_3$ is supported where $z_L^\varepsilon$ is exponentially decaying, we have $\sigma_3 = \langle I_\varepsilon''(z_L^\varepsilon)v_3, v_3\rangle \geq C_3\|v_3\|^2$. Without repeating all the computations we have that for $\sigma_{2,3}$ and $\sigma_{3,1}$ there holds the following estimates:

$$\sigma_{2,3} = \langle I_\varepsilon''(z_L^\varepsilon)v_2, v_3\rangle \geq C_{2,3}\tau_1 + o_{R,\varepsilon}(1)$$

and

$$\sigma_{3,1} = \langle I_\varepsilon''(z_L^\varepsilon)v_3, v_1\rangle \geq C_{3,1}\tau_{-1} + o_{R,\varepsilon}(1).$$

Collecting all these estimates, we get

$$\begin{aligned}\langle I_\varepsilon''(z_L^\varepsilon)v, v\rangle &= \sigma_1 + \sigma_2 + \sigma_3 + \sigma_{1,2} + \sigma_{2,3} + \sigma_{3,1} \\ &\geq C_1\|v_1\|^2 + C_2\|v_2\|^2 + C_3\|v_3\|^2 + C_{2,3}\tau_1 + C_{3,1}\tau_{-1} + o_{R,\varepsilon}(1) \\ &\geq \min\{C_1, C_2, C_3, C_{2,3}, C_{3,1}\} \\ &\times \Big(\|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2 + \tau_1 + \tau_{-1} + o_{R,\varepsilon}(1)\Big) \geq C\|v\|^2\end{aligned}$$

for some $0 < C < \min\{C_1, C_2, C_3, C_{2,3}, C_{3,1}\}$. This concludes the proof. □

4.2. **The range equation.** We are now in position to solve the *range equation*. We rewrite $PI_\varepsilon'(z_L^\varepsilon + w) = 0$ in an equivalent way, suitable for the contraction mapping principle. First we have

$$I_\varepsilon'(z_L^\varepsilon + w) = I_\varepsilon'(z_L^\varepsilon) + I_\varepsilon''(z_L^\varepsilon)[w] + R(z_L^\varepsilon, w)$$

with $R(z_L^\varepsilon, w) := I_\varepsilon'(z_L^\varepsilon + w) - I_\varepsilon'(z_L^\varepsilon) - I_\varepsilon''(z_L^\varepsilon)[w]$. Then, taking the projection to the range and by Lemma 4.3, we arrive at

$$w = N_{\varepsilon,L}(w), \text{ where } N_{\varepsilon,L}(w) := -S_L^{-1}\big(PI_\varepsilon'(z_L^\varepsilon) + PR(z_L^\varepsilon, w)\big),$$

where $S_L := P \circ I_\varepsilon''(z_L^\varepsilon) \circ P$.

**Lemma 4.4.** *For $\varepsilon$ sufficiently small the operator $S_L$ is invertible for every $L \in \mathbf{R}$ and there exists $C > 0$ such that $\|S_L^{-1}\| \leq C$, $L \in \mathbf{R}$.*

**Proof.** The proof is a direct consequence of the computations in Lemma 4.3, Lemma 8.10 in [1] and Lemma 3.2 in [10] and so we omit the proof here. □

Our aim is now to prove that $N_{\varepsilon,L}(w)$ is a contraction in some ball of $H^s_{V,A}$.

**Proposition 4.5.** *For $\varepsilon > 0$ small enough, there exists a unique $w = w(\varepsilon, L)$ and constants $C_0$, $C_1$, $C_2$ and $C_3$ (independent of $\varepsilon$ and $L$) such that $PI_\varepsilon'(z_L^\varepsilon + w) = 0$ and*

$$\begin{aligned}\|w\|^2 &\leq C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q}\Big(1 + e^{-\sqrt{2}V(L)^{\frac12}\frac{L}{\varepsilon}}\Big(1 + \big|2V(L)^{\frac12}\frac{L}{\varepsilon}\big|^{\frac12}\Big)\Big) \\ &+ C_1\varepsilon^4|D^2E(Le_1)|^2 V(L)^{2q-1}\Big(1 + e^{-\sqrt{2}V(L)^{\frac12}\frac{L}{\varepsilon}}\Big(1 + \big|2V(L)^{\frac12}\frac{L}{\varepsilon}\big|^{\frac12}\Big)\Big) \\ &+ C_2(\varepsilon\alpha)^2\Big(V(L)^{2q} + |\alpha L|^2 V(L)^{2q-1}\Big)\end{aligned}$$

$$+ C_3 V(L)^{2q} e^{-KV(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big).$$

**Proof.** By the same estimates in [1], page 128, and the embeddings proved in Lemma 2.1 we have

$$\|R(z_L^\varepsilon, w)\| \le C\big(\|w\|^2 + \|w\|^p\big) \tag{4.8}$$

and

$$\|R(z_L^\varepsilon, w_2) - R(z_L^\varepsilon, w_1)\| \le C\big(\|w_2\|^2 + \|w_2\|^{p-1} + \|w_1\|^2 + \|w_1\|^{p-1}\big)\|w_2 - w_1\|.$$

By Lemma 3.1 we have

$$\begin{aligned}
&\|I'_\varepsilon(z_L^\varepsilon)\|^2 \le C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q}\\
&+ C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q} e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big)\\
&+ C_1\varepsilon^4|D^2E(Le_1)|^2 V(L)^{2q-1}\Big(1+e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big)\Big)\\
&+ C_2(\varepsilon\alpha)^2\Big(V(L)^{2q} + |\alpha L|^2 V(L)^{2q-1}\Big)\\
&+ C_3 V(L)^{2q} e^{-KV(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big),
\end{aligned}$$

hence we obtain

$$\begin{aligned}
&\|N_{\varepsilon,L}(w)\| \le C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q}\\
&+ C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q} e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big)\\
&+ C_1\varepsilon^4|D^2E(Le_1)|^2 V(L)^{2q-1}\Big(1+e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big)\Big)\\
&+ C_2(\varepsilon\alpha)^2\Big(V(L)^{2q} + \big|\alpha L\big|^2 V(L)^{2q-1}\Big) + C_3 V(L)^{2q} e^{-KV(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\\
&+ C\big(\|w\|^2 + \|w\|^p\big)
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&\|N_{\varepsilon,L}(w_2) - N_{\varepsilon,L}(w_1)\|\\
&\le C\Big(\|w_2\|^2 + \|w_2\|^{p-1} + \|w_1\|^2 + \|w_1\|^{p-1}\Big)\|w_2 - w_1\|.
\end{aligned} \tag{4.10}$$

For $\overline{C} > 0$ we define the set $W_{\overline{C}} := \big\{w \in W : \|w\| \le \overline{C}\Lambda(\varepsilon, L)\big\}$ with

$$\begin{aligned}
&\Lambda(\varepsilon, L) := C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q}\\
&+ C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q} e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big)
\end{aligned}$$

$$+ C_1\varepsilon^4|D^2E(Le_1)|^2V(L)^{2q-1}\Big(1+e^{-\sqrt{2}V(L)^{\frac12}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac12}\frac{L}{\varepsilon}\big|^{\frac12}\Big)\Big)$$
$$+ C_2(\varepsilon\alpha)^2\Big(V(L)^{2q}+|\alpha L|^2V(L)^{2q-1}\Big)$$
$$+ C_3V(L)^{2q}e^{-KV(L)^{\frac12}\frac{L}{\varepsilon}}\Big(1+\big|2V(L)^{\frac12}\frac{L}{\varepsilon}\big|^{\frac12}\Big).$$

We show that $N_{\varepsilon,L}$ is a contraction for sufficiently large $\overline{C}$ and for sufficiently small $\varepsilon$. If $\overline{C}\geq 2C$, by (4.9) the set $W_{\overline{C}}$ is mapped into itself if $\varepsilon$ is sufficiently small. Moreover if $w_1,w_2\in W_{\overline{C}}$, by (4.10) we have

$$\begin{aligned}&\|N_{\varepsilon,L}(w_2)-N_{\varepsilon,L}(w_1)\|\\ &\leq C\big(\overline{C}^2+\overline{C}^{p-1}\big)\big(\Lambda(\varepsilon,L)^2+\Lambda(\varepsilon,L)^{p-1}\big)\|w_2-w_1\|.\end{aligned}\tag{4.11}$$

So for $\varepsilon$ sufficiently small $C(\overline{C}^2+\overline{C}^{p-1})(\Lambda(\varepsilon,L)^2+\Lambda(\varepsilon,L)^{p-1})\leq\frac12$ and hence

$$\|N_{\varepsilon,L}(w_2)-N_{\varepsilon,L}(w_1)\|\leq\tfrac12\|w_2-w_1\|,$$

so $N_{\varepsilon,L}$ is a contraction and by the contraction mapping principle there exists a unique $w\in W$ solving the range equation and satisfying

$$\|w\|^2\leq C_0\varepsilon^2|\nabla E(Le_1)|^2V(L)^{2q}\Big(1+e^{-\sqrt{2}V(L)^{\frac12}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac12}\frac{L}{\varepsilon}\Big|^{\frac12}\Big)\Big)$$
$$+ C_1\varepsilon^4|D^2E(Le_1)|^2V(L)^{2q-1}\Big(1+e^{-\sqrt{2}V(L)^{\frac12}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac12}\frac{L}{\varepsilon}\Big|^{\frac12}\Big)\Big)$$
$$+ C_2(\varepsilon\alpha)^2\Big(V(L)^{2q}+\Big|\alpha L\Big|^2V(L)^{2q-1}\Big)$$
$$+ C_3V(L)^{2q}e^{-KV(L)^{\frac12}\frac{L}{\varepsilon}}\Big(1+\Big|2V(L)^{\frac12}\frac{L}{\varepsilon}\Big|^{\frac12}\Big).$$

This concludes the proof. □

4.3. **The natural constraint.** This subsection is the key point to reducing the problem of finding critical points of $I_\varepsilon(u)$ to the study of the critical points of an appropriate 1-dimensional function. This will be done by proving that the *pseudo-critical* manifold $\mathcal{Z}$ is near a manifold $\tilde{\mathcal{Z}}$ (in a sense which will be explained below) which is a *natural constraint* (see [2]). To reduce the problem to a finite-dimensional one, first we have to prove that $w$ depends smoothly (in the $C^1$ sense) on $L$; more precisely, we have to prove that the application $L\mapsto w(L,\varepsilon)$ belongs to $C^1(\mathbf{R},H_{V,A}$. This is a consequence of the implicit function theorem.

**Lemma 4.6.** *The function $w(\varepsilon,L)$ found in Proposition 4.5 is of class $\mathcal{C}^1$ with respect to $L$ and $\frac{\partial w}{\partial L}\to 0$ as $\varepsilon\to 0$.*

**Proof.** Consider the map $H : \mathbf{R} \times H_{V,A} \times \mathbf{R} \times \mathbf{R} \longrightarrow H_{V,A} \times \mathbf{R}$ defined as follows:

$$H(L, w, \alpha, \varepsilon) := \begin{pmatrix} I'_\varepsilon(z^\varepsilon_L + w) - \alpha\partial_L z^\varepsilon_L \\ \langle w, z^\varepsilon_L\rangle \end{pmatrix}.$$

Then $w \in W$ is a solution of $PI'_\varepsilon(z^\varepsilon_L + w) = 0$ if and only if $H(L, w, \alpha, \varepsilon) = 0$. We check now that the operator $H$ satisfies the hypothesis of the implicit function theorem. For $v \in H^s_{V,A}$ and $\beta \in \mathbf{R}$, there holds

$$\frac{H(L, w, \alpha, \varepsilon)}{\partial(w, \alpha)} := \begin{pmatrix} I''_\varepsilon(z^\varepsilon_L + w)[v] - \beta\partial_L z^\varepsilon_L \\ \langle v, \partial_L z^\varepsilon_L\rangle \end{pmatrix}.$$

By Sobolev embeddings and Lemma 2.1, and by estimates similar to the estimate (4.8) we obtain

$$\frac{H(L, w, \alpha, \varepsilon)}{\partial(w, \alpha)}[v, \beta] := \begin{pmatrix} I''_\varepsilon(z^\varepsilon_L)[v] - \beta\partial_L z^\varepsilon_L \\ \langle v, \partial_L z^\varepsilon_L\rangle \end{pmatrix} + O(\|w\|^2 + \|w\|^{p-1}).$$

By Lemma 4.3, we get that $[\frac{H(L,w,\alpha,\varepsilon)}{\partial(w,\alpha)}]_{(L,0,0,\varepsilon)}$ is uniformly invertible in $l$ for $\varepsilon$ small enough and so $\frac{H(L,w,\alpha,\varepsilon)}{\partial(w,\alpha)}$ is uniformly invertible in $L$ in a neighborhood of $(L, 0, 0, \varepsilon)$ for $\varepsilon$ small enough. We now apply the implicit function theorem and we get that the map $L \mapsto w(L, \varepsilon)$ belongs to $C^1(\mathbf{R}, H^s_{V,A})$. □

Now using a geometric argument we get that the manifold

$$\tilde{\mathcal{Z}}_\varepsilon := \{u(x) := z^\varepsilon_L(x) + w(x) : L > 0\},$$

with $w$ as in Proposition 4.5, is a natural constraint for $I'_\varepsilon$.

**Proposition 4.7.** *The manifold $\tilde{\mathcal{Z}}_\varepsilon$ is a natural constraint for $I'_\varepsilon$. More precisely $I'_\varepsilon \mid_{\tilde{\mathcal{Z}}_\varepsilon} (u) = 0 \Longrightarrow I'_\varepsilon(u) = 0$.*

**Proof.** Suppose that $I'_\varepsilon|_{\tilde{\mathcal{Z}}_\varepsilon}(z^\varepsilon_L + w) = 0$ for some $z^\varepsilon_L + w \in \tilde{\mathcal{Z}}_\varepsilon$. Then $I'_\varepsilon|_{\tilde{\mathcal{Z}}_\varepsilon}$ is orthogonal to $T_{z^\varepsilon_L+w}\tilde{\mathcal{Z}}_\varepsilon$. On the other hand $I'_\varepsilon|_{\tilde{\mathcal{Z}}_\varepsilon} \in T_{z^\varepsilon_L}\mathcal{Z}_\varepsilon$ by definition of $w$ and the two tangent spaces $T_{z^\varepsilon_L+w}\tilde{\mathcal{Z}}_\varepsilon$ and $T_{z^\varepsilon_L}\mathcal{Z}_\varepsilon$ are close if $\varepsilon$ is small enough. This implies that $I'_\varepsilon(z^\varepsilon_L + w) = 0$, which concludes the proof. □

4.4. **The reduced functional.** Thanks to the discussion in Subsection 4.3 and in order to solve equation (1.10), it is enough to find critical points of the following 1-dimensional functional:

$$\Phi_\varepsilon(L) := I_\varepsilon(u) \mid_{\tilde{\mathcal{Z}}_\varepsilon} = \frac{1}{2}\left\|z^\varepsilon_L + w\right\|^2_{V,A} - \frac{1}{p+1}\int_{\mathbf{R}^2} |z^\varepsilon_L + w|^{p+1}. \qquad (4.12)$$

This functional can be rewritten in a more explicit way as in the following lemma.

**Lemma 4.8.** *The reduced functional $\Phi_\varepsilon(L)$ has the following explicit dependence on L:*

$$\Phi_\varepsilon(L) = \tilde{\Phi}_\varepsilon(L) + R(z_L^\varepsilon, w),$$

*where* $\tilde{\Phi}_\varepsilon := V(L)^\theta(\tilde{C} - Ce^{-V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon}})$, $C > 0$, $\tilde{C} > 0$ *and is independent of* $L$ *and* $\varepsilon$, $\theta := \frac{p+1}{p-1} - \frac{n}{2}$ *and*

$$\begin{aligned}
&\|R(z_L^\varepsilon, w)\|^2 \leq C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q}\\
&+C_0\varepsilon^2|\nabla E(Le_1)|^2 V(L)^{2q} e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1 + \big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big)\\
&+C_1\varepsilon^4|D^2E(Le_1)|^2 V(L)^{2q-1}\Big(1 + e^{-\sqrt{2}V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1 + \big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big)\Big)\\
&+C_2(\varepsilon\alpha)^2\Big(V(L)^{2q} + V(L)^{2q-1}\Big)\\
&+C_3V(L)^{2q}e^{-KV(L)^{\frac{1}{2}}\frac{L}{\varepsilon}}\Big(1 + \big|2V(L)^{\frac{1}{2}}\frac{L}{\varepsilon}\big|^{\frac{1}{2}}\Big),
\end{aligned}$$

*with* $C_0$, $C_1$, $C_2$ *and* $C_3$ *independent of* $L$ *and* $\varepsilon$.

**Proof.** Since

$$\|z_1\|_{V,A}^2 = \int_{\mathbf{R}^2} |z_1|^{p+1} \quad \text{and} \quad \|z_2\|_{V,A}^2 = \int_{\mathbf{R}^2} |z_2|^{p+1},$$

by a Taylor expansion, one has

$$\begin{aligned}
&\Phi_\varepsilon(L = \frac{1}{2}\|z_L^\varepsilon\|_{V,A}^2 + \langle z_L^\varepsilon, w\rangle_{V,A} + \frac{1}{2}\|w\|_{V,A}^2 - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_L^\varepsilon|^{p+1}\\
&+ \Big\{\frac{1}{p+1}\int_{\mathbf{R}^2}|z_L^\varepsilon|^{p+1} - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_L^\varepsilon + w|^{p+1}\Big\}\\
&= \frac{1}{2}\|z_L^\varepsilon\|_{V,A}^2 - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_L^\varepsilon|^{p+1} + R_1(z_L^\varepsilon, w) + R_2(z_L^\varepsilon, w)\\
&= \Big\{\frac{1}{2}\|z_1\|_L^2 - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_1|^{p+1}\Big\} + \Big\{\frac{1}{2}\|z_2\|_L^2 - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_2|^{p+1}\Big\}\\
&+ \Big\{\frac{1}{p+1}\int_{\mathbf{R}^2}|z_1|^{p+1} + \frac{1}{p+1}\int_{\mathbf{R}^2}|z_2|^{p+1} - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_1 + z_2|^{p+1}\Big\}\\
&+ \langle z_1, z_2\rangle_L + \big\{\langle z_1, z_2\rangle_{V,A} - \langle z_1, z_2\rangle_L\big\} + R_1(z_L^\varepsilon, w) + R_2(z_L^\varepsilon, w)\\
&+ \Big\{\frac{1}{2}\|z_1\|_{V,A}^2 - \frac{1}{2}\|z_1\|_L^2\Big\} + \Big\{\frac{1}{2}\|z_2\|_{V,A}^2 - \frac{1}{2}\|z_2\|_L^2\Big\}
\end{aligned}$$

$$= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathbf{R}^2}|z_1|^{p+1} + \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathbf{R}^2}|z_2|^{p+1} + \langle z_1, z_2\rangle_L$$
$$+ \Big\{\frac{1}{p+1}\int_{\mathbf{R}^2}|z_1|^{p+1} + \frac{1}{p+1}\int_{\mathbf{R}^2}|z_2|^{p+1} - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_1+z_2|^{p+1}\Big\}$$
$$+ R_1(z_L^\varepsilon, w) + R_2(z_L^\varepsilon, w) + R_3(z_L^\varepsilon, w) + R_4(z_L^\varepsilon, w)$$
$$= 2\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathbf{R}^2}|z_1|^{p+1} + Re\int_{\mathbf{R}^2}|z_1|^{p-1}z_1\overline{z_2}$$
$$+ \Big\{\frac{1}{p+1}\int_{\mathbf{R}^2}|z_1|^{p+1} + \frac{1}{p+1}\int_{\mathbf{R}^2}|z_2|^{p+1} - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_1+z_2|^{p+1}\Big\} + R(z_L^\varepsilon, w)$$
$$= C_0V(L)^\theta + V(L)^\theta\int_{\mathbf{R}^2}\Big|u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\Big|^{p-1}u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)$$
$$\times u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\cos(2\alpha Lx_2) + \frac{1}{p+1}V(L)^\theta\int_{\mathbf{R}^2}\Big|u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\Big|^{p+1}$$
$$+ \Big|u\Big(-x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\Big|^{p+1} - \frac{1}{p+1}V(L)^\theta\int_{\mathbf{R}^2}\Big|u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)e^{i\alpha Lx_2}$$
$$+ u\Big(-x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)e^{-i\alpha Lx_2}\Big|^{p+1} + R(z_L^\varepsilon, w).$$

Here we have used the following notation:

$$R(z_L^\varepsilon, w) := R_1(z_L^\varepsilon, w) + R_2(z_L^\varepsilon, w) + R_3(z_L^\varepsilon, w) + R_4(z_L^\varepsilon, w),$$

with

$$R_1(z_L^\varepsilon, w) := \frac{1}{p+1}\int_{\mathbf{R}^2}|z_L^\varepsilon|^{p+1} - \frac{1}{p+1}\int_{\mathbf{R}^2}|z_L^\varepsilon + w|^{p+1},$$
$$R_2(z_L^\varepsilon, w) := \langle z_L^\varepsilon, w\rangle_{V,A} + \tfrac{1}{2}\|w\|_{V,A}^2,$$
$$R_3(z_L^\varepsilon, w) := \tfrac{1}{2}\|z_1\|_{V,A}^2 - \tfrac{1}{2}\|z_1\|_L^2 + \tfrac{1}{2}\|z_2\|_{V,A}^2 - \tfrac{1}{2}\|z_2\|_L^2,$$
$$R_4(z_L^\varepsilon, w) := \langle z_1, z_2\rangle_{V,A} - \langle z_1, z_2\rangle_L, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2}$$

and

$$\tilde{C} := 2\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathbf{R}^2}|u(x)|^{p+1}.$$

To simplify the notation we will write

$$\rho_1 := u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big), \qquad \rho_2 := u\Big(-x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)$$

and $\beta := \alpha Lx_2$. Then

$$|\rho_1|^{p+1} + |\rho_2|^{p+1} - |\rho_1e^{i\beta} + \rho_2e^{-i\beta}|^{p+1}$$

$$= |\rho_1|^{p+1} + |\rho_2|^{p+1} - \Big(|\rho_1|^2 + |\rho_2|^2 + 2\rho_1\rho_2\cos(2\beta)\Big)^{\frac{p+1}{2}}$$
$$= |\rho_1|^{p+1} + |\rho_2|^{p+1} - \Big(|\rho_1|^2 + |\rho_2|^2\Big)^{\frac{p+1}{2}}\Big(1 + \frac{2\rho_1\rho_2}{|\rho_1|^2+|\rho_2|^2}\cos(2\beta)\Big)^{\frac{p+1}{2}}$$
$$= |\rho_1|^{p+1} + |\rho_2|^{p+1} - \Big(|\rho_1|^2 + |\rho_2|^2\Big)^{\frac{p+1}{2}}$$
$$- (p+1)\frac{\rho_1\rho_2}{|\rho_1|^2+|\rho_2|^2}\cos(2\beta)\Big(|\rho_1|^2 + |\rho_2|^2\Big)^{\frac{p+1}{2}}$$
$$= -(p+1)|\rho_1|^{p-1}|\rho_1|^2 - (p+1)|\rho_2|^{p-1}|\rho_1|^2$$
$$- (p+1)\frac{\rho_1\rho_2}{|\rho_1|^2+|\rho_2|^2}\cos(2\beta)\Big(|\rho_1|^2 + |\rho_2|^2\Big)^{\frac{p+1}{2}} + l.o.t.$$

Putting all the terms together we have

$$(p+1)|\rho_1|^p\rho_2\cos(2\beta) + |\rho_1|^{p+1} + |\rho_2|^{p+1} - |\rho_1 e^{i\beta} + \rho_2 e^{-i\beta}|^{p+1}$$
$$= -(p+1)|\rho_1|^{p-1}|\rho_1|^2 - (p+1)|\rho_2|^{p-1}|\rho_1|^2$$
$$- (p+1)\rho_1\rho_2\cos(2\beta)\Big(|\rho_1|^2 + |\rho_2|^2\Big)^{\frac{p-1}{2}} + (p+1)|\rho_1|^p\rho_2\cos(2\beta) + l.o.t.$$
$$= -(p+1)|\rho_1|^{p-1}|\rho_1|^2 - (p+1)|\rho_2|^{p-1}|\rho_1|^2 - (p+1)|\rho_2|^p\rho_1\cos(2\beta)$$
$$+ l.o.t. = -(p+1)|\rho_2|^p\rho_1\cos(2\beta) + l.o.t.$$

Coming back with the original notation we have obtained that

$$\int_{\mathbf{R}^2}\Big|u\Big(x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|^{p-1}u\Big(x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)$$
$$\times\, u\Big(-x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\cos(2\alpha L x_2)$$
$$+ \frac{1}{p+1}\int_{\mathbf{R}^2}\Big|u\Big(x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|^{p+1} + \Big|u\Big(-x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|^{p+1}$$
$$- \frac{1}{p+1}\int_{\mathbf{R}^2}\Big|u\Big(x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)e^{i\alpha L x_2} + u\Big(-x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)e^{-i\alpha L x_2}\Big|^{p+1}$$
$$= -\int_{\mathbf{R}^2}\Big|u\Big(-x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|^{p-1}u\Big(-x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)$$
$$\times\, u\Big(x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\cos(2\alpha L x_2) + l.o.t.$$

Now we make estimates on this last term:

$$\int_{\mathbf{R}^2}\Big|u\Big(-x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)\Big|^{p-1}u\Big(-x - V(L)^{\frac12}\frac{Le_1}{\varepsilon}\Big)$$

$$\times u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\cos(2\alpha L x_2) = A + B,$$

where

$$A := \int_{\mathbf{R}} dx_1 \int_{|x_2|\le \frac{M}{\alpha L}} dx_2 \Big|u\Big(-x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\Big|^{p-1}$$
$$\times u\Big(-x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\cos(2\alpha L x_2)$$

and

$$B := \int_{\mathbf{R}} dx_1 \int_{|x_2|\ge \frac{M}{\alpha L}} dx_2 \Big|u\Big(-x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\Big|^{p-1}$$
$$\times u\Big(-x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)u\Big(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon}\Big)\cos(2\alpha L x_2),$$

with $M > 0$. With computations close to the ones performed in the proof of Lemma 3.1, we can bound $A$ by

$$|A| \le Ce^{-V(L)^{\frac{1}{2}}(\frac{M}{\alpha L} + V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon})}\Big(1 + \Big(2V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon}\Big)\Big)$$

and $B$ by

$$|B| \le Ce^{-V(L)^{\frac{1}{2}}(V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon})}\Big(1 + \Big(2V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon}\Big)\Big).$$

In all these computations the constant $C$ is independent of $\varepsilon$ and $L$. This implies that $B$ is far smaller than $A$ as $\varepsilon \to 0$. Moreover this implies that the sign of $A+B$ is the same as the sign of $A$. Since both $u(-x-V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon})$ and $u(x - V(L)^{\frac{1}{2}}\frac{Le_1}{\varepsilon})$ decay exponentially from the line $x_2 = 0$, and since $\cos(2\alpha L x_2)$ is almost 1 on bounded sets as $\varepsilon \to 0$, the sign of $A$ is positive. □

Thanks to Lemma 4.8, the last thing to do is to find critical points of the reduced functional $\Phi_\varepsilon(L)$:

$$\Phi_\varepsilon(L) = V(L)^\theta(C_0 - C_1 e^{-V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon}}) + R(z_L^\varepsilon, w).$$

**Proposition 4.9.** *There exist $\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon} > 0$ and $\overline{\alpha} \in [\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}]$ such that $\Phi_\varepsilon(L)$ has a critical point in some $\bar{L}$, with $0 < \bar{L} < 1$.*

**Proof.** Since $R(z_L^\varepsilon, w)$ is an l.o.t., to find critical points of $\Phi_\varepsilon$ it is enough to study the function $\tilde{\Phi}_\varepsilon := V(L)^\theta(C_0 - C_1 e^{-V(L)^{\frac{1}{2}}\frac{KL}{\varepsilon}})$. First we derive it with respect to $L$ and we obtain

$$\begin{aligned}\tilde{\Phi}'_\varepsilon &= \theta C_0 V(L)^{\theta-1} V'(L)(C_0 - C_1 f_\varepsilon(L)) \\ &\quad + C_1 V(L)^\theta f_\varepsilon(L)\Big(\frac{1}{2} V(L)^{-\frac{1}{2}} V'(L) K \frac{L}{\varepsilon} + V(L)^{\frac{1}{2}} \frac{K}{\varepsilon}\Big)\end{aligned}$$

where we wrote $f_\varepsilon(L) := e^{-V(L)^{\frac{1}{2}} \frac{KL}{\varepsilon}}$. So critical points of $\tilde{\Phi}_\varepsilon$ must satisfy $\tilde{\Phi}'_\varepsilon = 0$:

$$\theta C_0^2 V'(L) + f_\varepsilon(L)\Big( - C_0 C_1 \theta V'(L) + \frac{C_1}{2} V(L)^{\frac{1}{2}} V'(L) K \frac{L}{\varepsilon} + V(L)^{\frac{3}{2}} \frac{K}{\varepsilon}\Big) = 0,$$

which multiplying by $\varepsilon$ becomes

$$\varepsilon\theta C_0^2 V'(L) + f_\varepsilon(L)\Big( - C_0 C_1 \varepsilon\theta V'(L) + \frac{C_1}{2} V(L)^{\frac{1}{2}} V'(L) K L + V(L)^{\frac{3}{2}} K\Big) = 0.$$

Since we want to find a critical point near the maximum of $V(L)$, by the definition of $V(L)$ we have

$$\begin{aligned}&\Big( - C_0 C_1 - \frac{\alpha^2}{2} L\varepsilon + \frac{C_1}{2}\Big(\omega + 1 - \frac{\alpha^2}{4} L^2\Big)^{\frac{1}{2}}\Big( - K\frac{\alpha^2}{2} L^2\Big) \\ &+ K\Big(\omega + 1 - \frac{\alpha^2}{4} L^2\Big)^{\frac{1}{2}}\Big) - f_\varepsilon(L)^{-1} \varepsilon\theta C_0^2 \frac{\alpha^2}{2} L = 0.\end{aligned}$$

Considering just higher-order terms we get

$$f_\varepsilon(L)(-C_0 C_1 + K(\omega+1)) - \varepsilon\theta C_0^2 \frac{\alpha^2}{2} L = 0.$$

If we choose $\omega$ big enough, we have that $-C_0C_1 + K(\omega+1) > 0$ and hence we have that $\alpha$ must satisfy

$$\overline{C} e^{-V(L)^{\frac{1}{2}} \frac{KL}{\varepsilon}} = \varepsilon\alpha^2 L \tag{4.13}$$

with $\bar{C} := \frac{-C_0C_1 + K(\omega+1)}{\theta C_0^2}$. Since equation (4.13) is linear in $\alpha^2$, there exists $\alpha_{1,\varepsilon}$ such that $\bar{C} e^{-V(L)^{\frac{1}{2}} \frac{KL}{\varepsilon}} > \varepsilon\alpha_{1,\varepsilon}^2 L$ and $\alpha_{2,\varepsilon}$ such that $\bar{C} e^{-V(L)^{\frac{1}{2}} \frac{KL}{\varepsilon}} < \varepsilon\alpha_{2,\varepsilon}^2 L$. In this way we find a critical point of $\tilde{\Phi}_\varepsilon$ and, since the remainder is l.o.t., we also find a critical point of $\Phi_\varepsilon$. □

**Remark 4.10.** We could also find a critical point of $\tilde{\Phi}_\varepsilon$ without taking the first derivative. It would be enough to notice that the value of $\tilde{\Phi}_\varepsilon$ for $\alpha = \alpha_{1,\varepsilon}$ and for $\alpha = \alpha_{2,\varepsilon}$ is less then its value in some point $\alpha = \tilde{\alpha}$ such that $\alpha_{1,\varepsilon} < \tilde{\alpha} < \alpha_{2,\varepsilon}$. This clearly leads to the existence of a critical point of $\tilde{\Phi}_\varepsilon$ for some $\alpha \in [\alpha_{1,\varepsilon}^2, \alpha_{2,\varepsilon}^2]$.

**Remark 4.11.** We notice that when $\alpha = 0$, namely in the stationary case, it is well known that equation (1.10) admits one-bump concentrating solutions for all $L > 0$ in the region in which $E(x) = 1$: $|x| \leq 1$ (see [1]).

## 5. Proof of Theorem 1.1 and Final Comments

In this section we will use all the propositions and lemmas discussed in the previous sections in order to prove Theorem 1.1.

**Proof of the main theorems.** By Section 4 there exists a solution $u(x) = z_L^\varepsilon + w(x)$ of (1.10). Coming back to the ansatz, substituting this function into it and scaling back from $\phi(t,x)$ to $\psi(t,x)$ we get the existence result for the equation (1.1). By Proposition 4.5 the variation $w(x)$ tends to zero as $\varepsilon \to 0$. Coming back to the ansatz and by the same scaling as before we get the concentration result. The symmetries of the solutions are just consequences of the fact that we solve (1.10) in the space $H^s_{V,A}$. This concludes the proof of Theorem 1.1. □

**Remark 5.1.** (a) We can prove that there exist one-bump solutions of (1.10) which concentrate on minima of $V$ (see Remark 4.11). The proof is similar to the one presented above, but simpler since the term due to the interaction between bumps disappears. Moreover the analogous result with the same proof can be proved for more general potentials $E(x)$ that are at least quadratic at infinity.

(b) We can also prove that there are other multibump solutions of (1.10), with each bump concentrating at different minima of $V$ and in this case our result extends to rotating solitons the work of Bartsch and Dancer [4] in the stationary case.

(c) Even if the proof is presented here mainly in the case of $\mathbf{R}^2$, it holds as well in higher dimensions. We recall that, for $n = 2$, we do not have the restriction $1 < p < \frac{n+2}{n-2}$ and the result holds for all $p > 1$.

**Remark 5.2.** (a) It would be interesting to understand if there are solutions of (1.1) in the case $E(x) \equiv 1$ on all $\mathbf{R}^n$. In favor of this idea there is the fact that in our case the concentration takes place in a region in which the potential $E(x)$ is constant, while the difficulty comes from the possible dispersion due to the absence of any trapping potential. The main problem for proving the result arises in the invertibility of the linearized operator $S_L$.

(b) The configuration we have studied in this paper is not the only possible. With our method we can prove, with slightly different computations, that there are also configurations of polygons and combinations of polygons and lines, such as concentric polygons.

(c) With a similar argument and dropping the potential we think it is possible to prove the existence of rotating solitons on rotating-invariant bounded domains, like disks in $\mathbf{R}^2$. The result derives from the fact that in bounded domains the $H^1$-norm controls the angular term.

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