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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A SEMILINEAR HEAT EQUATION WITH LOCALIZED REACTION

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Abstract. We consider non-negative solutions to the Dirichlet problem of a semilinear heat equation with localized reaction in Ω: $u_t = \Delta u + f(u(x_0(t),t))$, where Ω is a smooth bounded domain, $x_0(t)$ is a locally Hölder continuous function from $[0,\infty)$ into Ω and f satisfies $f(0) = f'(0) = 0$ and some blow-up condition. We show that, if $x_0(t)$ remains in some compact subset of Ω as $t \to \infty$, then all global solutions are bounded in $\Omega \times (0,\infty)$ and, if $x_0(t)$ approaches the boundary of Ω as $t \to \infty$, then some unbounded global solution (infinite time blow-up solution) exists. These results are parts of our main results on the classification of all solutions.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^{\mathbf{N}}$ with smooth boundary $\partial\Omega$. In this paper, we shall consider the Dirichlet problem of a semilinear heat equation with localized reaction $f(u(x_0(t),t))$,

$$u_t = \Delta u + f(u(x_0(t),t)), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $u_t = \partial u/\partial t$, Δ is the $N-$dimensional Laplacian, $f(\xi)$ with $\xi \geq 0$ and $u_0(x)$ with $x \in \mathbf{R}^N$ are non-negative functions, and $x_0(t)$ is a function from $[0,\infty)$ into Ω which represents the location of the point driving the heat source. We call $x_0(t)$ a localized reaction point function.

We only consider non-negative solutions.

Throughout this paper, we assume the following hypotheses:

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(A1) $f \in C^1([0,\infty))$ and $f(\xi) > 0$ in $\xi > 0$.

(A2) $x_0(t)$ is a locally Hölder continuous function in $[0,\infty)$.

(A3) $u_0 \in C(\bar{\Omega})$, $u_0(x) \geq 0$ in Ω and $u_0(x) = 0$ on $\partial\Omega$.

Under the assumptions (A1)-(A3), a unique non-negative classical solution $u \in C(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\bar{\Omega} \times (0,T))$ of (1.1)-(1.3) exists locally in time (see Example A.2 in [28], [3], [4] and Proposition 7.1). If we add the condition $f'(\xi) \geq 0$ for $\xi \geq 0$, then the comparison principle holds (See Theorem A.10 in [28] and Proposition 7.1).

Moreover, we assume the following two conditions :

(A4)

$$f'(\xi) \geq 0 \quad \text{for } \xi \geq 0, \tag{1.4}$$

and

$$\lim_{\xi\to\infty} \frac{f(\xi)}{\xi} = \infty \quad \text{and} \quad \int_\xi^\infty \frac{1}{f(\eta)}\, d\eta < \infty \quad \text{for } \xi > 0. \tag{1.5}$$

(A5) $\quad f(0) = 0 \quad$ and $\quad f'(0) = 0.$

(A4) and (A5) are conditions on $f(\xi)$ near $\xi = \infty$ and $\xi = 0$, respectively. The first condition (A4) is a blow-up condition. Under the condition (A4), the solution u of (1.1)-(1.3) blows up in finite time if $u_0(x)$ is large enough (see [3], [28] and [9]). Namely, for some $T \in (0,\infty)$,

$$\lim_{t\uparrow T} ||u(\cdot,t)||_{L^\infty(\Omega)} = \infty. \tag{1.6}$$

Under the condition (A5), the solution of (1.1)-(1.3) exists globally in time and is bounded in $\Omega \times (0,\infty)$ provided $u_0(x)$ is small enough. Moreover, the solution tends to zero as $t \to \infty$ (see Theorem 3.1).

Remark 1.1. The blow-up condition (A4) is given in [28]. We note that the condition $\lim_{\xi\to\infty} f(\xi)/\xi = \infty$ can be removed from (A4), since it can be gotten from the other conditions of (A4). Namely, we can see that the conditions (1.4) and $\int_\xi^\infty 1/f(\eta)\, d\eta < \infty$ $(\xi > 0)$ imply $\lim_{\xi\to\infty} f(\xi)/\xi = \infty$. This fact was indicated and shown by Y. Seki in our private communication. The proof will be given in Appendix II.

We are now interested in the problem whether or not the following third case exists:

(B) $u(x,t)$ exists globally but is not bounded in $\Omega \times (0,\infty)$.

This problem has been studied by a few authors and their answers were different in the cases where $x_0(t)$ remains in some compact subset K of Ω

as $t \to \infty$ and where $x_0(t)$ approaches the boundary of Ω as $t \to \infty$, which are expressed by the next conditions:

(A6) $\inf_{t \in (0,\infty)} \operatorname{dist}(x_0(t), \partial\Omega) > 0.$

(A7) $\operatorname{dist}(x_0(t), \partial\Omega) \to 0$ as $t \to \infty$.

Here, we define for $y \in \mathbf{R}^N$ and $A \subset \mathbf{R}^N$,

$$\operatorname{dist}(y, A) = \inf_{x \in A} |y - x|.$$

For this problem, Rouchon [27] answered that case (B) never happens when $N = 1$ and $x_0(t)$ remains in some compact subset K of Ω as $t \to \infty$ and when $N > 1$ and $x_0(t) \equiv x^* \in \Omega$ (namely, $x_0(t)$ satisfies (A6)). However, he did not show the boundedness of global solutions for moving $x_0(t)$ in the case of $N > 1$. On the other hand, when $N = 1$ and $\Omega = (0, 1)$, Souplet [30] answered that case (B) does occur for some $x_0(t)$ tending to zero as $t \to \infty$ (namely, $x_0(t)$ satisfies (A7)) under a weaker condition on $f(\xi)$ than (A4). Namely, for any order of decay rate of $x_0(t)$ which is not faster than exponential, he constructed an initial data $u_0(x)$ and an $x_0(t)$ (having such a decay rate) so that the solution exists globally in time and goes to infinity as $t \to \infty$. Moreover, he obtained the precise behavior of the solution as $t \to \infty$. However, he did not show that (B) occurs for any localized reaction point function $x_0(t)$ converging to 0 as $t \to \infty$. He had no results for the case of $N > 1$. We note that, in their results, the assumption (A5) is not required.

One of the aims of the present paper is to solve these open problems. Our results are as follows.

We use the notation $\|\cdot\|_\infty$ as the sup norm in Ω.

Theorem 1.2. *Assume* (A1)-(A4)(A6). *Then, there exists a constant* $C = C(\|u_0\|_\infty) > 0$ *such that any nonnegative global solution* u *of* (1.1)-(1.3) *satisfies*

$$\|u(\cdot, t)\|_\infty \le C(\|u_0\|_\infty) \quad \text{for } t \ge 0. \tag{1.7}$$

Theorem 1.3. *Assume* (A1)-(A5)(A7). *Let* $u_0 \not\equiv 0$. *Then, there exists a real number* $\tau > 0$ *such that the solution* u *of* (1.1)-(1.3) *with the initial data* $\tau u_0(x)$ *exists globally in time and satisfies*

$$\sup_{t \in (0,\infty)} ||u(\cdot, t)||_\infty = \infty. \tag{1.8}$$

Remark 1.4. Theorem 1.2 improves the result of Rouchon [27]. Theorem 1.3 improves the result of Souplet [30], since Theorem 1.3 holds for any space dimension N and any localized reaction point function $x_0(t)$ approaching the boundary $\partial\Omega$. However, we must require the assumption (A5) which is not required in [30]. We have no results on the behavior of infinite time blow-up solutions (unbounded global solutions), while Souplet [30] obtained the precise behavior of some infinite time blow-up solutions.

Remark 1.5. The corresponding problem with local reaction term

$$u_t = \Delta u + u^p, \tag{1.9}$$

with zero Dirichlet boundary condition, has been investigated by many authors. See [22, 2, 11, 6, 10, 24, 7, 25, 26] and the references therein. Roughly speaking, they showed that when $N \leq 2$ or $1 < p < (N+2)/(N-2)$ and $N \geq 3$, all global solutions are bounded, whereas when $p = (N+2)/(N-2)$ $(N \geq 3)$ and Ω is a ball, unbounded global solutions (infinite time blow-up solutions) exist.

As in [27] and [30], our methods are based on the comparison principle and the integral expression of a solution by the heat kernel. But, the key point of our proof, which is to use some lower estimate for solutions (Proposition 2.4), is different from that of [27] and that of [30]. In [27], for the proof of his result in the case $N > 1$, Rouchon used the fact that under condition (A6) an unbounded global solution becomes non-decreasing with respect to t after some time $t_0 > 0$, which was not obtained in [27] for moving $x_0(t)$. In our proof of Theorem 1.2, we combine the integral expression of a solution by the heat kernel and some upper estimate for global solutions (Lemma 4.2) which follows from the lower estimate for solutions mentioned above, and hence we get the result for moving $x_0(t)$. In [30], as above-mentioned, Souplet constructed an initial data u_0 and a localized reaction point function $x_0(t)$ exactly to obtain an unbounded global solution, whereas we get such a solution from the fact that a class of unbounded global solutions forms a separatrix between blow-up solutions and bounded global solutions tending to zero as $t \to \infty$, which is a part of our main results on the classification of all solutions of (1.1)-(1.3) (see Theorem 1.6). In the proof of our results on the classification of solutions, Lemma 4.2 mentioned above also plays a crucial role. From this method of construction of unbounded global solutions, we can not get more information about those solutions, however we get unbounded global solutions for any localized reaction point function $x_0(t)$ which approaches the boundary $\partial\Omega$ as $t \to \infty$. Also, for the same reason, we must require assumption (A5) which is not required in [30].

The main purpose of this paper is to classify all solutions of (1.1)-(1.3). Our main results are stated in the next Theorem 1.6. We note that Theorem 1.2 above is used for the proof of (II) of Theorem 1.6 and Theorem 1.3 above is included in (II) of Theorem 1.6.

Let λ be the first eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition and let ψ be the corresponding eigenfunction satisfying

$$\int_\Omega \psi^2(x)\,dx = 1. \tag{1.10}$$

Theorem 1.6. *Assume* (A1)-(A5). *Let* $u_0(x) \not\equiv 0$. *Then, there exist* $0 < \tau_1 \le \tau_2 < \infty$ *such that*

(I) *If* $0 < \tau < \tau_1$, *then the solution* u *of* (1.1)-(1.3) *with the initial data* τu_0 *exists globally in time and satisfies for each* $\mu \in (0, \lambda)$

$$\|e^{\mu t} u(\cdot, t)\|_\infty \to 0 \qquad \text{as } t \to \infty. \tag{1.11}$$

Furthermore, if f *satisfies the condition*

(A8) $f(\xi) = O(\xi^\alpha)$ *as* $\xi \downarrow 0$ *for some* $\alpha > 1$,

then

$$||e^{\lambda t} u(\cdot, t) - \tilde{c}\psi||_\infty \to 0 \text{ as } t \to \infty, \tag{1.12}$$

where

$$\tilde{c} = \int_\Omega u_0 \psi\,dx + \int_0^\infty f(u(x_0(t), t)) e^{\lambda t} dt \int_\Omega \psi\,dx. \tag{1.13}$$

Here, we say that $f(\xi) = O(\xi^\alpha)$ *as* $\xi \downarrow 0$ *if*

$$\limsup_{\xi \downarrow 0} |f(\xi)|/\xi^\alpha < \infty.$$

(II) *If* $\tau \in [\tau_1, \tau_2]$, *then the solution* u *of* (1.1)-(1.3) *with the initial data* τu_0 *exists globally in time and satisfies*

$$\inf_{t \in (0,\infty)} ||u(\cdot, t)||_\infty > 0. \tag{1.14}$$

Moreover, if the localized reaction point function $x_0(t)$ *satisfies* (A6), *then*

$$\sup_{t \in (0,\infty)} \|u(\cdot, t)\|_\infty < \infty, \tag{1.15}$$

and if $x_0(t)$ *satisfies* (A7), *then*

$$\sup_{t \in (0,\infty)} \|u(\cdot, t)\|_\infty = \infty. \tag{1.16}$$

(III) *If $\tau > \tau_2$, then the solution u of* (1.1)-(1.3) *with the initial data τu_0 blows up in the whole domain Ω at the blow-up time $T < \infty$ and satisfies*

$$1 \le \liminf_{t\uparrow T} \frac{\inf_{x\in K} u(x,t)}{G^{-1}(T-t)}, \tag{1.17}$$

for any compact subset K of Ω, where $\xi = G^{-1}(\eta)$ is the inverse function of $\eta = G(\xi)$ and

$$G(\xi) = \int_\xi^\infty \frac{1}{f(\xi')}\, d\xi'; \tag{1.18}$$

that is, $u = G^{-1}(T-t)$ is a solution of the ordinary differential equation $u_t = f(u)$. Moreover, if f satisfies the condition

(A9) $$\lim_{h\uparrow 1} \limsup_{\eta\downarrow 0} \frac{G^{-1}(h\eta)}{G^{-1}(\eta)} = 1,$$

then

$$\lim_{t\uparrow T} \frac{u(x,t)}{G^{-1}(T-t)} = \lim_{t\uparrow T} \frac{||u(\cdot,t)||_\infty}{G^{-1}(T-t)} = 1, \tag{1.19}$$

uniformly on compact subsets of Ω.

Remark 1.7. When $f(\xi) = \xi^p$ ($p > 1$), it satisfies (A4), (A8) and (A9). When $f(\xi) = e^\xi$, it satisfies (A4) and (A9), and, hence, the blow-up solution u satisfies (1.19). See also Theorem 5.1.

Remark 1.8. For the Dirichlet problem of (1.9), similar results were obtained in [18, 22, 2, 11] when $N \le 2$ or $1 < p < (N+2)/(N-2)$ and $N \ge 3$. Combining their results and several known results on blow-up and decay rates of solutions ([14, 12, 13, 16, 20]), we can gather these results in the following way: Let $u_0 \not\equiv 0$ and let u_τ be the solution of (1.9)(1.2)(1.3) with the initial data τu_0. Then there exists $\tau^* \in (0,\infty)$ such that (I) if $0 < \tau < \tau^*$, then u_τ exists globally in time and satisfies $\|u_\tau(\cdot,t)\|_\infty \sim e^{-\lambda t}$ as $t \to \infty$ where λ is as in Theorem 1.6, (II) if $\tau = \tau^*$, then u_τ exists globally in time and satisfies $0 < 1/C \le \|u_\tau(\cdot,t)\|_\infty \le C < \infty$ for all $t > 0$, and for some sequence $\{t_n\} \uparrow \infty$, $u_\tau(t_n)$ tends to some non-trivial stationary solution of (1.9)(1.2), (III) if $\tau > \tau^*$, then u_τ blows up at some time $T > 0$ and satisfies $\|u_\tau(\cdot,t)\|_\infty \sim \{(p-1)(T-t)\}^{-1/(p-1)}$ as $t \uparrow T$. As for the Cauchy problem to (1.9), several results concerning the above result were established in [17, 21, 31]. We note that in (II) of Theorem 1.6, we do not know whether or not $\tau_1 = \tau_2$ holds and we have no results on precise behavior of the solution as $t \to \infty$.

Remark 1.9. (1.19) was already obtained by Souplet [29] in the cases $f(\xi) = \xi^p$ and $f(\xi) = e^{\xi}$, and his method can be applied to more general $f(\xi)$ (see Remark 2.2 in [29]). Although our proof of (1.19) is similar to that of [29] and a similar estimate to our key estimate (2.12) is obtained in [29], our approach to show such an estimate is different from that of [29]. It is applicable even to the quasilinear parabolic equation

$$u_t = \Delta u^m + f(u(x_0(t), t)),$$

with $m \geq 1$, which is considered in [9]. We note that Souplet furthermore obtained more precise estimates of blow-up solutions near the boundary, but we do not get them.

Remark 1.10. For the local problem (1.9) with zero Dirichlet boundary condition, a result similar to (1.12) was obtained by Holland [14] and Kawanago [16]. However, their methods are based on the energy estimates and can not be applied to the non-local (localized) case.

The rest of the paper is organized as follows. In the next Section 2, we give preliminary lemmas and propositions. In order to prove Theorem 1.6, we need to get results about decaying solutions, global solutions and blowing-up solutions. In Section 3, we obtain the precise behavior of global solutions with small initial data. In Section 4, we get some estimates for global solutions, which depend only on the initial data. Theorem 1.2 is proved in this section. In Section 5, we study blow-up rates of blow-up solutions. Finally, combining these results we prove Theorem 1.6 in Section 6. In Appendix I, we show some L^∞-estimate for solutions, which leads to the comparison principle and will be used in Section 6. In Appendix II, we prove a lemma which is stated in Remark 1.1.

2. Preliminaries

Let Ω be a bounded domain with smooth boundary $\partial\Omega$. In this section, we state several preliminary lemmas and propositions.

We first recall the well-known results concerning the solution operator $e^{t\Delta}$ of the heat equation with zero Dirichlet boundary condition:

For any $g \in L^\infty(\Omega) \cap C(\Omega)$, let $(e^{t\Delta}g)(x) \in L^\infty(\Omega \times (0,\infty)) \cap C(\Omega \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$ denote the solution to the Dirichlet problem

$$\begin{cases} u_t = \Delta u, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = g(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Let λ be the first eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition and let ψ be a corresponding eigenfunction.

Lemma 2.1. (i) *There exists a constant* $C_1 > 1$ *such that for any nonnegative* $g \in L^\infty(\Omega) \cap C(\Omega)$

$$C_1^{-1} e^{-\lambda t} \psi(x) \inf_{x \in \Omega} g(x) \le (e^{t\Delta} g)(x) \le ||e^{t\Delta} g||_\infty \le C_1 e^{-\lambda t} ||g||_\infty \ \text{in } \Omega \times (0, \infty). \tag{2.2}$$

(ii) *Let* $g\,(\not\equiv 0) \in L^\infty(\Omega) \cap C(\Omega)$ *be nonnegative. Then, for any* $t_1 > 0$, *there exists a* $c_1 > 0$ *such that*

$$c_1 e^{-\lambda t} \psi(x) \le (e^{t\Delta} g)(x) \quad \text{in } (x, t) \in \Omega \times (t_1, \infty). \tag{2.3}$$

(iii) *Let* u *be a solution of* (1.1)-(1.3). *Then,* u *is expressed as the integral expression by* $e^{t\Delta}$ *as follows:*

$$u(x,t) = (e^{t\Delta} u_0)(x) + \int_0^t (e^{(t-s)\Delta} f(u(x_0(s), s)))(x) ds. \tag{2.4}$$

Proof. (i) Let $v = e^{t\Delta} 1 \ (\le 1)$. Since $v \in C^\infty(\bar{\Omega} \times (0, \infty))$, for $t_0 > 0$ there exists $C_1 > 0$ such that $v(x, t_0) \le C_1 \psi(x)$ in Ω. Hence, by the comparison theorem, $v(x,t) \le e^{(t-t_0)\Delta}(C_1 \psi)(x) = C_1 e^{-\lambda(t-t_0)} \psi(x)$ in $\Omega \times (t_0, \infty)$ and, hence, $v(x,t) = e^{t\Delta} 1 \le \tilde{C}_1 e^{-\lambda t}$ in $\Omega \times (0, \infty)$, where $\tilde{C}_1 = C_1 e^{\lambda t_0} \|\psi\|_\infty + e^{\lambda t_0}$. On the other hand, for some $c_1 > 0$, $c_1 \psi \le 1$ in Ω. Hence, by the comparison theorem, $c_1 e^{-\lambda t} \psi = e^{t\Delta}(c_1 \psi) \le v = e^{t\Delta} 1$ in $\Omega \times (0, \infty)$. Thus, also by the comparison theorem we have

$$\begin{aligned} c_1 e^{-\lambda t} \psi \inf_{x \in \Omega} |g(x)| &\le \inf_{x \in \Omega} |g(x)| e^{t\Delta} 1 = e^{t\Delta} (\inf_{x \in \Omega} |g(x)|) \\ &\le e^{t\Delta} g \le \|g\|_\infty e^{t\Delta} 1 \le \tilde{C}_1 e^{-\lambda t} \|g\|_\infty. \end{aligned}$$

(ii) Let $g\,(\not\equiv 0) \in L^\infty(\Omega)$ be non-negative and let $u(x,t) = (e^{t\Delta} g)(x)$. Then, the maximum principle implies that for $t_1 > 0$ $u(x, t_1) > 0$ in Ω and $\frac{\partial}{\partial n} u(x, t_1) < 0$ on $\partial\Omega$. On the other hand, since $\frac{\partial}{\partial n} \psi(x) < 0$ on $\partial\Omega$, we can choose $\mu > 0$ small enough to satisfy

$$\mu \min_{x \in \partial\Omega} \frac{\partial}{\partial n} \psi(x) > \max_{x \in \partial\Omega} \frac{\partial}{\partial n} u(x, t_1).$$

Hence, if we furthermore choose $\mu > 0$ small, then $u(x, t_1) \ge \mu \psi(x)$ in Ω. Applying the comparison theorem, we get

$$u(x,t) \ge e^{(t-t_1)\Delta} (\mu \psi)(x) = \mu e^{-\lambda(t-t_1)} \psi(x),$$

in $\Omega \times (t_1, \infty)$ to obtain (2.3).

(iii) See [8]. □

We will need the next regularity theorem (see [19]).

Proposition 2.2. *Assume* (A1)-(A3). *Let u be a solution of* (1.1)-(1.3). *Suppose for some $M > 0$*

$$u(x,t) \leq M \quad \textit{in } (x,t) \in \Omega \times (0,T). \tag{2.5}$$

Then, for any $t_0 > 0$, there exist constants $C = C(M,t_0) > 0$ and $\beta = \beta(M,t_0) \in (0,1)$ (independent of u_0) such that

$$|u(x_1,t_1) - u(x_2,t_2)| \leq C \cdot (|x_1 - x_2|^\beta + |t_1 - t_2|^{\beta/2}) \tag{2.6}$$
$$\textit{for } (x_1,t_1),\ (x_2,t_2) \in \bar{\Omega} \times (t_0,T).$$

Proof. Since $|f(u(x_0(t),t))| \leq M'$ in $(0,\infty)$ for some $M' > 0$, we get the assertion by Theorem 10.1 page 204 in [19]. □

Next, we state two important propositions concerning the upper and lower estimates for solutions of (1.1)-(1.3). In particular, the lower estimate (Proposition 2.4 below) plays a crucial role in our proofs.

The next upper estimate for solutions was already shown in [29] (see Okada-Fukuda [23]).

Proposition 2.3. *Assume* (A1)-(A3). *Let u be a solution of* (1.1)-(1.3) *in $\Omega \times (0,T)$. Then, u satisfies*

$$u(x,t) \leq \int_0^t f(u(x_0(t),t))\,dt + ||u_0||_\infty \quad \textit{in } \Omega \times (0,T). \tag{2.7}$$

Proof. Let u be a solution of (1.1)-(1.3) in $\Omega \times (0,T)$. We note that

$$v(t) = \int_0^t f(u(x_0(t),t))\,dt + ||u_0||_\infty$$

is a supersolution of (1.1) in $\Omega \times (0,T)$ ($f(u(x_0(t),t))$ is considered as a given function). Since $u(x,t) = 0 \leq v(t)$ on $\partial\Omega \times (0,T)$ and $||u_0||_\infty \leq v(0)$, the usual comparison theorem implies (2.7). □

Set

$$K_T = \{x = x_0(t) : t \in [0,T]\}, \tag{2.8}$$

$$\delta_T = \mathrm{dist}(K_T, \partial\Omega), \tag{2.9}$$

$$D_T(\delta) = \{x \in \Omega : B(x;\delta) \subset \Omega\} \text{ for } 0 < \delta < \delta_T, \tag{2.10}$$

where

$$B(x;\delta) = \{y \in \mathbf{R}^N : |x - y| < \delta\}. \tag{2.11}$$

Proposition 2.4. *Assume* (A1)-(A3). *Let u be a solution of* (1.1)-(1.3) *in* $\Omega \times (0,T)$ *and let* $0 < \delta < \delta_T$. *Then,*

$$\inf_{x \in D_T(\delta)} u(x,t) \geq \int_0^t \{f(u(x_0(t),t)) - \lambda_\delta \inf_{x \in D_T(\delta)} u(x,t)\}\, dt, \tag{2.12}$$

for $t \in (0,T)$, *where* λ_δ *is the first eigenvalue of* $-\Delta$ *in* $B(0;\delta)$ *with zero Dirichlet boundary condition.*

Proof. Let $0 < \delta < \delta_T$. Put $B_\delta = B(0;\delta)$. Let ψ_δ be the corresponding eigenfunction to the first eigenvalue λ_δ satisfying

$$\int_\Omega \psi_\delta(x)\, dx = 1. \tag{2.13}$$

For the solution u of (1.1)-(1.3), let v be a solution to the problem

$$v_t - \Delta v = f(u(x_0(t),t)) \quad \text{in } B_\delta \times (0,T), \tag{2.14}$$

$$v(x,t) = 0 \quad \text{on } \partial B_\delta \times (0,T), \tag{2.15}$$

$$v(x,0) = 0 \quad \text{in } B_\delta. \tag{2.16}$$

It is not difficult to see that $v(x,t) = v(r,t)$ $(r = |x|)$ is a radially symmetric function in $x \in B_\delta$ and a non-increasing function in $r \geq 0$. Furthermore, by the usual comparison theorem, we see that for any $x' \in D_T(\delta)$, $u(x+x',t) \geq v(x,t)$ in $(x,t) \in B_\delta \times (0,T)$. In particular,

$$\inf_{x \in D_T(\delta)} u(x,t) \geq v(0,t) \quad \text{in } (0,T). \tag{2.17}$$

Multiplying (2.14) by ψ_δ and integrating by parts over $B_\delta \times (0,\tau)$, we have

$$\begin{aligned} &\int_{B_\delta} v(x,t)\psi_\delta(x)\, dx \\ &= \int_0^t \int_{B_\delta} \{-\lambda_\delta v(x,\tau)\psi_\delta(x) + f(u(x_0(\tau),\tau)\psi_\delta(x))\}\, dx d\tau, \end{aligned} \tag{2.18}$$

and, hence,

$$v(0,t) \geq \int_0^t \{-\lambda_\delta v(0,\tau) + f(u(x_0(\tau),\tau))\}\, d\tau. \tag{2.19}$$

Thus, combining this and (2.17), we get (2.12). □

The next lemma will be used in the next section. For a domain Ω with smooth boundary, let λ_Ω be the first eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition.

Lemma 2.5. *There exists a sequence of domains with smooth boundary* $\{\Omega_n\}$ *such that* $\bar{\Omega} \subset \Omega_n$, $\lambda_n < \lambda_\Omega$ *and* $\lambda_n \uparrow \lambda_\Omega$ *as* $n \to \infty$ *where* $\lambda_n = \lambda_{\Omega_n}$.

Proof. See [5].

3. Asymptotic behavior of solutions with small initial data

In this section, we assume (A1)-(A3)(A5) and study the behavior of global solutions of (1.1)-(1.3) with small initial data u_0. Furthermore, we assume (A8) and get more precise behavior of the global solutions as $t \to \infty$. We shall show the following theorem.

Let λ be the first eigenvalue of $-\Delta$ with zero Dirichlet condition and let ψ be the corresponding eigenfunction satisfying (1.10).

Theorem 3.1. *Assume* (A1)-(A3)(A5). *Let* $u_0 \not\equiv 0$. *Let* u *be a solution of* (1.1)-(1.3). *Then, there exists* $\delta_0 > 0$ *such that, if* $||u_0||_\infty < \delta_0$, *then* u *exists globally in time and satisfies, for each* $\mu \in (0, \lambda)$,

$$\|e^{\mu t} u(\cdot, t)\|_\infty \to 0 \qquad \text{as } t \to \infty. \tag{3.1}$$

Moreover, if (A8) *holds, then*

$$||e^{\lambda t} u(\cdot, t) - \tilde{c}\psi||_\infty \to 0 \text{ as } t \to \infty, \tag{3.2}$$

where

$$\tilde{c} = \int_\Omega u_0 \psi \, dx + \int_0^\infty f(u(x_0(t), t)) e^{\lambda t} dt \int_\Omega \psi \, dx. \tag{3.3}$$

In order to prove this theorem, as in [14], we transform a solution u of (1.1)-(1.3) as

$$v_\mu(x, t) = e^{\mu t} u(x, t) \qquad (0 < \mu \le \lambda). \tag{3.4}$$

(See [14] in the case $\mu = \lambda$.) Then, v_μ is a solution of the equation

$$v_t - \Delta v = \mu v + e^{\mu t} f(e^{-\mu t} v(x_0(t), t)). \tag{3.5}$$

We need several propositions. We first show the next proposition, assuming (A5).

Proposition 3.2. *Assume* (A1)-(A3)(A5). *Let* u *be a solution of* (1.1)-(1.3). *Then, for any* $h > 0$ *and* $\mu \in (0, \lambda)$ *there exists* $\delta \in (0, h)$ *such that, if* $||u_0||_\infty < \delta$, *then* u *exists globally in time and satisfies*

$$u(x, t) \le h e^{-\mu t} \qquad \text{in } \Omega \times (0, \infty). \tag{3.6}$$

Let $g(\xi) = \sup_{0\le\eta\le\xi} |f'(\eta)|$ and $\tilde{f}(\xi) = \int_0^\xi g(\eta)\,d\eta$ for $\xi \ge 0$. If we assume (A5), then $g(0) = 0$, $g(\xi)$ and $\tilde{f}(\xi)$ are non-decreasing in $\xi \ge 0$ and

$$f(\xi) \le \tilde{f}(\xi) \le g(\xi)\xi \quad \text{for } \xi \ge 0. \tag{3.7}$$

Hence, we first prove this proposition in the special case $f(\xi) = \tilde{f}(\xi)$. We note that, in this case, the comparison principle holds since $\tilde{f} \in C^1[0,\infty)$ and $\tilde{f}'(\xi) \ge 0$ in $\xi \ge 0$ (see Theorem A.10 in [28] and Proposition 7.1). For the proof of this proposition in the case of $f(\xi) = \tilde{f}(\xi)$, we need two lemmas. Let $\mu \in (0,\lambda)$. Since λ is the first eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition, by Lemma 2.5 there exists a domain D with smooth boundary such that $\bar{\Omega} \subset D$ and $\mu < \mu_D < \lambda$, where μ_D is the first eigenvalue of $-\Delta$ in D with zero Dirichlet boundary condition. Let $\tilde{\psi}_D$ be the corresponding eigenfunction to μ_D satisfying $\sup_{x\in D} \tilde{\psi}_D = 1$. We note that $\tilde{\psi}_D > 0$ in $D\,(\supset \bar{\Omega})$ and, hence, $\inf_{x\in\Omega} \tilde{\psi}_D > 0$.

Lemma 3.3. *Suppose that* $f(\xi) = \tilde{f}(\xi)$. *Assume* (A2). *Let* $0 < \mu < \lambda$ *and let* $\tilde{\psi}_D$ *be as above. Then, there exists* $h_0 > 0$ *such that* $h\tilde{\psi}_D$ $(0 < h \le h_0)$ *is a supersolution of* (3.5) *in* $\Omega \times (0,\infty)$.

Proof. Let μ_D be as above. For $h > 0$, put $\tilde{w}_h = h\tilde{\psi}_D$. We choose $h > 0$ small enough to satisfy

$$g\Big(h \sup_{x\in\Omega} \tilde{\psi}_D\Big) \le \frac{(\mu_D - \mu)\inf_{x\in\Omega} \tilde{\psi}_D}{\sup_{x\in\Omega} \tilde{\psi}_D}. \tag{3.8}$$

Then, we have

$$\begin{aligned} -\Delta \tilde{w}_h &= h\mu_D\tilde{\psi}_D = \mu\tilde{w}_h + (\mu_D - \mu)h\tilde{\psi}_D \\ &\ge \mu\tilde{w}_h + h\sup_{x\in\Omega}\tilde{\psi}_D \times \frac{(\mu_D - \mu)\inf_{x\in\Omega}\tilde{\psi}_D}{\sup_{x\in\Omega}\tilde{\psi}_D} \\ &\ge \mu\tilde{w}_h + h\sup_{x\in\Omega}\tilde{\psi}_D \times g\Big(h\sup_{x\in\Omega}\tilde{\psi}_D\Big) \ge \mu\tilde{w}_h + e^{\mu t}\tilde{f}(e^{-\mu t}\tilde{w}_h(x_0(t))). \end{aligned} \tag{3.9}$$

Thus, we see that $\tilde{w}_h$ is a supersolution of (3.5) with $f = \tilde{f}$ in $\Omega \times (0,\infty)$. The proof is complete. □

Lemma 3.4. *Suppose that* $f(\xi) = \tilde{f}(\xi)$. *Assume* (A2)–(A3). *Let* $0 < \mu < \lambda$ *and let* $\tilde{\psi}_D$ *and* $h_0 > 0$ *be as in Lemma* 3.3. *Let* u *be a solution of* (1.1)-(1.3). *If*

$$u_0(x) \le h\tilde{\psi}_D(x) \quad \text{in } \Omega, \tag{3.10}$$

for some $h \in (0, h_0]$, *then* $u(x,t)$ *exists globally in time and*

$$u(x,t) \le he^{-\mu t}\tilde{\psi}_D(x) \quad \text{in } (x,t) \in \Omega \times (0,\infty). \tag{3.11}$$

Proof. Let u be a solution of (1.1)-(1.3) with $f(\xi) = \tilde{f}(\xi)$ and let $v_\mu(x,t) = e^{\mu t}u(x,t)$. Since v_μ is a solution of (3.5), (1.2), (1.3) with $f(\xi) = \tilde{f}(\xi)$ and $h\tilde{\psi}_D$ $(0 < h < h_0)$ is a supersolution of (3.5) with $f(\xi) = \tilde{f}(\xi)$ by Lemma 3.3, if we assume (3.10), then we get by the comparison theorem (see [28]),

$$v_\mu(x,t) \le h\tilde{\psi}_D \quad \text{in } \Omega \times (0,T), \tag{3.12}$$

which leads to (3.12) with $T = \infty$ and, hence, (3.11). □

Proof of Proposition 3.2. Assume (A5). Then, f satisfies (3.7). Let u be a solution of (1.1)-(1.3) and let v be a solution of (1.1)-(1.3) with $f(\xi)$ replaced by $\tilde{f}(\xi)$. Let $h > 0$ and $\mu \in (0,\lambda)$, and let $\delta_h = h \inf_{x\in\Omega} \tilde{\psi}_D$ $(0 < h < h_0)$ where $\tilde{\psi}_D$ and h_0 are as in Lemma 3.4. Assume $\|u_0\|_\infty < \delta_h$ $(h < h_0)$. Then, by Lemma 3.4, we see that v exists globally in time and satisfies

$$v(x,t) \le he^{-\mu t} \quad \text{in } \Omega \times (0,\infty). \tag{3.13}$$

Note that u is a subsolution of (1.1) with $f(\xi)$ replaced by $\tilde{f}(\xi)$. Hence, applying the comparison theorem (see [28] and Proposition 7.1), we have

$$u(x,t) \le v(x,t) \le he^{-\mu t} \quad \text{in } \Omega \times (0,T), \tag{3.14}$$

which leads to (3.14) with $T = \infty$. The proof is complete. □

Next, we furthermore assume condition (A8). We note that condition (A8) leads to condition (A5). Then, there exist constants $h_1 > 0$ and $C_2 > 0$ such that

$$f(\xi) \le C_2\xi^\alpha \quad \text{for } 0 < \xi < h_1. \tag{3.15}$$

Proposition 3.5. *Assume* (A1)-(A3)(A8). *Let* u *be a solution of* (1.1)-(1.3). *Then, for any* $h \in (0, h_1)$, *there exists* $\delta \in (0,h)$ *such that if* $\|u_0\|_\infty < \delta$, *then* u *exists globally in time and satisfies*

$$u(x,t) \le he^{-\lambda t} \quad \text{in } \Omega \times (0,\infty). \tag{3.16}$$

Proof. Let u be a solution of (1.1)-(1.3). We choose $\mu \in (0,\lambda)$ close enough to λ to satisfy $\lambda/\alpha < \mu < \lambda$. Let $0 < h < h_1$. Then, from Proposition 3.2, there exists $\delta \in (0,h)$ such that, if $\|u_0\|_\infty < \delta$, then

$$u(x,t) \le he^{-\mu t} \quad \text{in } \Omega \times (0,\infty). \tag{3.17}$$

We use the integral expression (2.4) of the solution u:

$$u(x,t) = (e^{t\Delta}u_0)(x) + \int_0^t \left(e^{(t-s)\Delta} f(u(x_0(s),s))\right)(x)ds. \tag{3.18}$$

Then, because of (i) of Lemma 2.1, Lemma 3.4 and the fact that $\lambda < \alpha\mu$, we get

$$\begin{aligned} u(x,t) &\le C_1 e^{-\lambda t}||u_0||_\infty + C_1 C_2 h^\alpha \int_0^t e^{-\lambda(t-s)} e^{-\mu\alpha s}\, ds \qquad (3.19)\\ &\le C_1\Big(\delta + C_2 h^\alpha \int_0^\infty e^{-(\mu\alpha-\lambda)s}\, ds\Big)e^{-\lambda t} \le \tilde{C} h e^{-\lambda t} \quad \text{ in } \Omega\times(0,\infty), \end{aligned}$$

where $\tilde{C} = C_1\{1 + C_2 h_1^{\alpha-1}/(\mu\alpha-\lambda)\}$, which leads to the assertion of Proposition 3.5. The proof is complete. □

Hence, we shall get the next proposition.

Proposition 3.6. *Let u be as in Proposition 3.5 with $u_0 \not\equiv 0$. Assume that u satisfies (3.16) for some $h \in (0,h_1)$, where $h_1 > 0$ is as in (3.15). Let $v = e^{\lambda t}u$. Then, v satisfies*

$$\|v(\cdot,t) - \tilde{c}\psi\|_\infty \to 0 \quad \text{as } t\to\infty, \tag{3.20}$$

where $\tilde{c}$ and ψ are as in Theorem 3.1.

The method of proof of this proposition is similar to that of [18, 1, 22]. As in them, we need the next proposition.

Lemma 3.7. *Let u be as in Proposition 3.5. Assume that u satisfies (3.16) for some $h \in (0,h_1)$, where $h_1 > 0$ is as in (3.15). Let $v = e^{\lambda t}u$. Then, v satisfies*

$$\int_1^\infty \int_\Omega v_t^2\, dxdt < \infty. \tag{3.21}$$

Proof. Let $v = e^{\lambda t}u$. Then, v is a solution of the equation

$$v_t - \Delta v = \lambda v + e^{\lambda t} f(u(x_0(t),t)). \tag{3.22}$$

We note by Proposition 3.5 that

$$v(x,t) \le h \quad \text{ in } \Omega\times(0,\infty), \tag{3.23}$$

and

$$e^{\lambda t} f(u(x_0(t),t)) \le C_2 h^\alpha e^{-\lambda(\alpha-1)t} \quad \text{ in } (0,\infty). \tag{3.24}$$

Multiplying (3.22) by v_t and integrating by parts over $\Omega \times (1,T)$ $(1 < T)$, we have

$$\int_1^T \int_\Omega v_t^2\, dxdt + \frac{1}{2}\int_\Omega |\nabla v|^2\, dx \Big|_{t=1}^T \tag{3.25}$$
$$= \frac{\lambda}{2}\int_\Omega v^2\, dx \Big|_{t=1}^T + \int_1^T \int_\Omega e^{\lambda t} f(u(x_0(t),t)) v_t\, dxdt,$$

and, hence, by the inequality $ab \le (a^2+b^2)/2$, (3.23) and (3.24), we get

$$\int_1^T \int_\Omega v_t^2\, dxdt \le \int_\Omega |\nabla v|^2(x,1)\, dx + \lambda h^2|\Omega| + C_2^2 h^{2\alpha}|\Omega| \int_1^T e^{-2\lambda(\alpha-1)t}\, dt. \tag{3.26}$$

Thus, letting $T \to \infty$ in (3.26) we have (3.21), since $\alpha > 1$. □

Proof of Proposition 3.6. Let $u_0 \not\equiv 0$. Let v be as in Lemma 3.7. Then, we note that v also satisfies (2.6) with u replaced by v, for some $C > 0$ and $\beta \in (0,1)$, since $v \le h$ and $\lambda v + e^{\lambda t} f(u(x_0(t),t)) \le \lambda h + C_2 h^\alpha$ in $\Omega \times (0,\infty)$. Because of $u_0 \not\equiv 0$, we furthermore note, by (ii) of Lemma 2.1, that

$$v(x,t) = e^{\lambda t} u(x,t) \ge c_1 \psi(x) \quad \text{in } \Omega \times (1,\infty), \tag{3.27}$$

for some $c_1 > 0$, where ψ is as in Theorem 3.1.

Put $v_s(x,\tau) = v(x,s+\tau)$ in $(x,\tau) \in \Omega \times [0,1]$ $(s \ge 1)$. Then, since the family of functions $\{v_s(x,\tau)\}_{s\ge 1}$ is uniformly bounded in $\bar\Omega \times [0,1]$ and equicontinuous in $\bar\Omega \times [0,1]$, it is possible to find a subsequence $\{s_n\} \uparrow \infty$ as $n \to \infty$ and a non-negative function $w(x,t) \in C(\bar\Omega \times [0,1])$ such that

$$v_{s_n} \to w \quad \text{as } n \to \infty, \tag{3.28}$$

uniformly in $\bar\Omega \times [0,1]$. By (3.27),

$$w(x,t) \ge c_1 \psi(x) \quad \text{in } \bar\Omega \times [0,1]. \tag{3.29}$$

We note that $v_{s_n}(x,\tau)$ is a solution of the equation

$$\partial_\tau v - \Delta v = \lambda v + e^{\lambda(s_n+\tau)} f(u(x_0(s_n+\tau), s_n+\tau)) \quad \text{in } (x,\tau) \in \Omega \times [0,1]. \tag{3.30}$$

Put $\varphi(x,\tau) = \eta(\tau)\xi(x)$ where $\eta(\tau) \in C_0^\infty((0,1))$ and $\xi(x) \in C_0^\infty(\Omega)$. Multiplying (3.30) by φ and integrating by parts over $\Omega \times (0,1)$, we have

$$\int_0^1 \eta(\tau) \int_\Omega \{v_{s_n}\Delta\xi(x) + \lambda v_{s_n}\xi(x)\}\, dxd\tau \tag{3.31}$$
$$= -\int_0^1 \int_\Omega \eta\xi e^{\lambda(s_n+\tau)} f(u(x_0(s_n+\tau), s_n+\tau)\, dxd\tau$$

$$- \int_\Omega \xi(x) \int_0^1 \eta'(\tau) v_{s_n}(x,\tau)\, dx d\tau.$$

Hence, if $n \to \infty$ in (3.31), then by (3.24)

$$\int_0^1 \eta(\tau) \int_\Omega \{w\Delta\xi(x) + \lambda w\xi(x)\}\, dx d\tau \tag{3.32}$$
$$= - \int_\Omega \xi(x) \int_0^1 \eta'(\tau) w(x,\tau)\, dx d\tau.$$

On the other hand, because of (3.21),

$$\int_0^1 \int_\Omega |\{v_{s_n}\}_\tau|^2\, dx d\tau = \int_{s_n}^{s_n+1} \int_\Omega v_t^2(x,t)\, dx dt \to 0 \text{ as } n \to \infty, \tag{3.33}$$

which implies that $\{v_{s_n}\}_\tau \to w_\tau$ as $n \to \infty$ weakly in $L^2(\Omega \times (0,1))$ and

$$\int_0^1 \int_\Omega w_\tau^2\, dx d\tau = 0. \tag{3.34}$$

Thereby, since w is a continuous function in $\bar\Omega \times [0,1]$, we see that for any $x \in \bar\Omega$

$$w(x,t) = w(x,0) \equiv a(x) \quad \text{for } t \in (0,1], \tag{3.35}$$

and

$$\int_0^1 \eta(\tau)\, d\tau \int_\Omega \{a(x)\Delta\xi(x) + \lambda a(x)\xi(x)\}\, dx \tag{3.36}$$
$$= - \int_\Omega \xi(x) a(x)\, dx \int_0^1 \eta'(\tau)\, d\tau = 0. \quad \text{for all } \eta(t) \in C_0^\infty(0,1),$$

which leads to

$$\int_\Omega \{a(x)\Delta\xi(x) + \lambda a(x)\xi(x)\}\, dx = 0 \quad \text{for all } \xi \in C_0^\infty(\Omega); \tag{3.37}$$

that is,

$$\Delta a + \lambda a = 0 \quad \text{in } D'(\Omega). \tag{3.38}$$

From the standard regularity theorem, we see that $a \in C^2(\Omega) \cap C(\bar\Omega)$ and, hence, $\Delta a + \lambda a = 0$ in Ω. We note that $a(x) = w(x,t) = 0$ on $x \in \partial\Omega$ and $a(x) = w(x,t) \geq c_1\psi(x) > 0$ in $\Omega \times (0,1)$ by (3.29). Hence, we get $a(x) = k\psi(x)$ for some $k > 0$.

Thus, we see that for any sequence $\{t_n\} \uparrow \infty$ there exist a subsequence (denoted by $\{t_n\}$ again) and a $k > 0$ such that

$$||v(\cdot, t_n) - k\psi||_\infty \to \infty \quad \text{as } t_n \to \infty. \tag{3.39}$$

Multiply (3.22) by ψ and integrate by parts over $\Omega \times (0, t_n)$. Then, we get

$$\int_\Omega v\psi\,dx\Big|_0^{t_n} = \int_0^{t_n} e^{\lambda t} f(u(x_0(t),t))\,dt \int_\Omega \psi\,dx. \tag{3.40}$$

If $t_n \uparrow \infty$, then

$$k\int_\Omega \psi^2\,dx = \int_\Omega \psi u_0\,dx + \int_0^\infty e^{\lambda t} f(u(x_0(t),t))\,dt \int_\Omega \psi\,dx. \tag{3.41}$$

Hence, by condition (1.10),

$$k = \int_\Omega \psi u_0\,dx + \int_0^\infty e^{\lambda t} f(u(x_0(t),t))\,dt \int_\Omega \psi\,dx. \tag{3.42}$$

Therefore, the uniqueness of k implies that

$$||v(\cdot,t) - k\psi||_\infty \to \infty \quad \text{as } t \to \infty. \tag{3.43}$$

The proof is complete. □

Proof of Theorem 3.1. Theorem 3.1 follows from Propositions 3.2, 3.5 and 3.6. □

4. The property of global solutions

In this section, we assume (A1)-(A4) and state some results concerning global solutions. We first show some properties of solutions of (1.1)-(1.3) (Proposition 4.1), which can be applied to global solutions. This result will be used in the proof of Theorem 1.6. We next show Theorem 1.2, assuming (A6).

Proposition 4.1. *Assume (A1)-(A4). Let $0 < T < \infty$. Let u be a solution of* (1.1)-(1.3) *in $\Omega\times(0,T)$. Then, there exists a constant $C_T > 0$ independent of the initial data u_0 such that*

$$u(x,t) \le ||u_0||_\infty + C_T, \quad \textit{for } (x,t) \in \Omega \times [0, T/2]. \tag{4.1}$$

For the proof of Proposition 4.1 and also Theorem 1.2, we need the next lemma which follows from Proposition 2.4.

Lemma 4.2. *Assume* (A1)-(A4). *Let $0 < T < \infty$. Let u be a solution of* (1.1)-(1.3) *in $\Omega\times(0,T)$. Let $0 < \delta < \delta_T$, where δ_T is defined by* (2.9). *Then, there exists a constant $\tilde{C} = \tilde{C}(T,\delta) > 0$ independent of the initial data u_0 and δ_T such that*

$$\int_0^{T/2} f(u(x_0(t),t))\,dt \le \tilde{C}(T,\delta). \tag{4.2}$$

Proof. Let $0 < \delta < \delta_T$. Let u be a solution of (1.1)-(1.3) in $\Omega \times (0, T)$ and set $h(t) = u(x_0(t), t)$. Then, it follows from Proposition 2.4 that

$$h(t) \geq \int_{T/2}^{t} \{f(h(t)) - \lambda_\delta h(t)\}\, dt + \int_{0}^{T/2} \{f(h(t)) - \lambda_\delta h(t)\}\, dt \quad \text{for } t \in (T/2, T), \tag{4.3}$$

since $x_0(t) \in D_T(\delta)$ for $t \in (0, T)$, where $D_T(\delta)$ is defined by (2.10).

By the condition (A4), there exists $\xi_1 = \xi_1(\delta) > 0$ such that

$$f(\xi) - \lambda_\delta \xi \geq \frac{1}{2} f(\xi) > 0 \quad \text{for } \xi \geq \xi_1. \tag{4.4}$$

Set

$$I \equiv \int_{0}^{T/2} \{f(h(t)) - \lambda_\delta h(t)\}\, dt. \tag{4.5}$$

If $I > \xi_1$, then $h(t) > \xi_1$ in $t \geq T/2$ and

$$h(t) \geq \frac{1}{2} \int_{T/2}^{t} f(h(t))\, dt + I \equiv \alpha(t) \quad \text{for } t \in (T/2, T), \tag{4.6}$$

since $h(t)$ is a continuous function in $t \geq 0$. As in the proof of Theorem 1.1 in [15], noting assumption (A4), we have $f(h(t)) \geq f(\alpha(t))$ for $t \in (T/2, T)$, or equivalently,

$$1 \leq \frac{f(h(t))}{f(\alpha(t))}. \tag{4.7}$$

Integrating both sides of (4.7) over $(T/2, t)$ $(t \in (T/2, T))$ and noting $\alpha'(t) = \frac{1}{2} f(h(t))$, we obtain

$$t - T/2 \leq \int_{T/2}^{t} \frac{f(h(t))}{f(\alpha(t))}\, dt = 2 \int_{I}^{\alpha(t)} \frac{1}{f(\xi)}\, d\xi \leq 2 \int_{I}^{\infty} \frac{1}{f(\xi)}\, d\xi < \infty,\ t \in (T/2, T). \tag{4.8}$$

Setting

$$G(u) = \int_{u}^{\infty} \frac{1}{f(\xi)}\, d\xi \ (< \infty), \tag{4.9}$$

we have $T/4 \leq G(I)$ and hence

$$I = \int_{0}^{T/2} \{f(h(t)) - \lambda_\delta h(t)\}\, dt \leq G^{-1}(T/4), \tag{4.10}$$

where $G^{-1}(\eta)$ is the inverse function of $\eta = G(u)$.

Therefore, combining this case and the case $I \le \xi_1$, we get

$$\int_0^{T/2} \{f(h(t)) - \lambda_\delta h(t)\}\, dt \le G^{-1}(T/4) + \xi_1. \tag{4.11}$$

Since

$$f(\xi) - \lambda_\delta \xi \ge \frac{1}{2} f(\xi) - \lambda_\delta \xi_1 \quad \text{for } \xi \ge 0, \tag{4.12}$$

we see that

$$\int_0^{T/2} f(h(t))\, dt \le 2\{G^{-1}(T/4) + \xi_1 + \lambda_\delta \xi_1 T/2\}. \tag{4.13}$$

The proof is complete. □

Proof of Proposition 4.1. Let u be a solution of (1.1)-(1.3) in $\Omega \times (0,T)$. Combining Proposition 2.3 and Lemma 4.2, we have

$$\begin{aligned} u(x,t) &\le \int_0^t f(u(x_0(t),t))\, dt + ||u_0||_\infty \qquad (4.14)\\ &\le ||u_0||_\infty + \tilde{C}(\delta, T) \quad \text{in } \Omega \times (0, T/2), \end{aligned}$$

where $\tilde{C}(\delta, T) > 0$ is a constant independent of u_0. The proof is complete. □

Proof of Theorem 1.2. Assume (A6). Let $0 < \delta < \inf_{t\ge 0} \operatorname{dist}(x_0(t), \partial\Omega)$. Let u be a global solution of (1.1)-(1.3). We note from (A4) that, for some $C > 0$, $\xi \le f(\xi) + C$ for all $\xi \ge 0$. Hence, by Lemma 4.2, there exists $\tilde{C} = \tilde{C}(2, \delta) > 0$ such that

$$\int_n^{n+1} f(u(x_0(t),t))\, dt \le \tilde{C} \quad \text{for } n \ge 1, \tag{4.15}$$

from which, there exists $t_n \in [n, n+1]$ such that

$$f(u(x_0(t_n), t_n)) \le \tilde{C} \quad \text{for } n \ge 1, \tag{4.16}$$

and, hence,

$$u(x_0(t_n), t_n) \le f(u(x_0(t_n), t_n)) + C \le \tilde{C} + C \quad \text{for } n \ge 1. \tag{4.17}$$

On the other hand, let $D(\delta) = \{x \in \Omega : B(x;\delta) \subset \Omega\}$ where $B(x;\delta)$ is defined by (2.11). Clearly, $\overline{D(\delta)} \subset \Omega$ and $x_0(t) \in D(\delta)$ for $t \ge 0$ by (A6). Thereby, by (i) and (iii) of Lemma 2.1, we have

$$u(x_0(t),t) = (e^{t\Delta}u_0)(x_0(t)) + \int_0^t \Big(e^{(t-s)\Delta} f(u(x_0(s),s))\Big)(x_0(t))ds \tag{4.18}$$

$$\geq C_1^{-1}\int_0^t e^{-\lambda(t-s)}f(u(x_0(s),s))\psi(x_0(t))\,ds$$
$$\geq c_2\int_0^t e^{-\lambda(t-s)}f(u(x_0(s),s))\,ds,$$

where $c_2 = C_1^{-1}\inf_{x\in D(\delta)}\psi(x) > 0$. Combining this and (4.17) we obtain

$$\int_0^{t_n} e^{-\lambda(t_n-s)}f(u(x_0(s),s))\,ds \leq c_2^{-1}u(x_0(t_n),t_n) \leq c_2^{-1}(\tilde{C}+C), \tag{4.19}$$

for $n \geq 1$, and hence

$$\int_0^{t_n} e^{\lambda s}f(u(x_0(s),s))\,ds \leq c_2^{-1}(\tilde{C}+C)e^{\lambda t_n} \quad \text{for } n \geq 1. \tag{4.20}$$

Thus, by (i) and (iii) of Lemma 2.1 we see that, if $t \in [n-1,n]$ ($n \in \mathbf{N}$), then

$$\begin{aligned} u(x,t) &\leq \|u_0\|_\infty + C_1\int_0^t e^{-\lambda(t-s)}f(u(x_0(s),s))\,ds \qquad (4.21)\\ &\leq \|u_0\|_\infty + C_1 e^{-\lambda t}\int_0^{t_n} e^{\lambda s}f(u(x_0(s),s))\,ds\\ &\leq \|u_0\|_\infty + C_1c_2^{-1}(\tilde{C}+C)e^{2\lambda} \quad \text{in } \Omega\times(0,\infty). \end{aligned}$$

The proof is complete. □

5. The blow-up rates of solutions

In this section, we assume (A1)-(A4) and obtain blow-up rates of blow-up solutions of (1.1)-(1.3). We shall show the next theorem.

Theorem 5.1. *Assume* (A1)-(A4). *Let u be a blow-up solution of* (1.1)-(1.3) *in $\Omega\times(0,T)$ with the blow-up time $T<\infty$. Then,*

$$1 \leq \liminf_{t\uparrow T}\frac{\inf_{x\in K}u(x,t)}{G^{-1}(T-t)}, \tag{5.1}$$

for any compact subset K of Ω, where $\xi = G^{-1}(\eta)$ is the inverse function of $\eta = G(\xi)$ and $\eta = G(\xi)$ is defined by (1.18). *Moreover, if f satisfies condition (A9), then*

$$\lim_{t\uparrow T}\frac{u(x,t)}{G^{-1}(T-t)} = \lim_{t\uparrow T}\frac{||u(t)||_\infty}{G^{-1}(T-t)} = 1, \tag{5.2}$$

uniformly on compact subsets of Ω.

This theorem immediately follows from the next lemma.

Lemma 5.2. *Assume* (A1)-(A4). *Let K be a compact subset of Ω. Let u be a blow-up solution of* (1.1)-(1.3) *in $\Omega \times (0,T)$ with the blow-up time $T < \infty$. Then, for any $0 < \varepsilon < 1$, there exist a $t_0 \in (0,T)$ and a constant $C_\varepsilon 0$ such that*

$$(1-\varepsilon)G^{-1}(T-t) - C_\varepsilon \le u(x,t) \le ||u(t)||_\infty \qquad (5.3)$$
$$\le \frac{G^{-1}((1-\varepsilon)(T-t))}{1-\varepsilon} + C_\varepsilon \ \textit{in } K \times (t_0, T),$$

where $\xi = G^{-1}(\eta)$ is as in Theorem 5.1.

Proof. For the proof, we use Propositions 2.3 and 2.4. Let u be a blow-up solution of (1.1)-(1.3) in $\Omega \times (0,T)$ with the blow-up time $T < \infty$. Let $0 < \varepsilon < 1$ and let $K \subset \Omega$ be a compact set in Ω. We choose $\delta \in (0, \delta_T)$ to satisfy $K \subset D_T(\delta)$ where δ_T and $D_T(\delta)$ are defined by (2.9) and (2.10), respectively. We first show the first inequality of (5.3). By Proposition 2.4 we have

$$\inf_{x \in D_T(\delta)} u(x,t) \ge \int_0^t \{f(u(x_0(t),t)) - \lambda_\delta u(x_0(t),t)\}\, dt \quad \text{for } t \in (0,T), \qquad (5.4)$$

since $x_0(t) \in D_T(\delta)$ for $t \in (0,T)$. Because of condition (A4), there exists a constant $\tilde{C}_\varepsilon > 0$ such that

$$f(\xi) - \lambda_\delta \xi \ge (1-\varepsilon) f(\xi) - \tilde{C}_\varepsilon \quad \text{for } \xi \ge 0. \qquad (5.5)$$

Hence, for $t \in (0,T)$,

$$\inf_{x \in K} u(x,t) \ge \inf_{x \in D_T(\delta)} u(x,t) \ge (1-\varepsilon) \int_0^t f(u(x_0(t),t))\, dt - \tilde{C}_\varepsilon T. \qquad (5.6)$$

On the other hand, Proposition 2.3 leads to

$$u(x_0(t),t) \le \int_0^t f(u(x_0(t),t)\, dt + ||u_0||_\infty \equiv \alpha(t) \quad \text{in } (0,T). \qquad (5.7)$$

Hence, as in the proof of Theorem 1.1 in [15] (see also the proof of Lemma 4.2), we can get the first inequality of (5.3). In fact, by (A4) we have

$$f(u(x_0(t),t)) \le f(\alpha(t)) \quad \text{in } (0,T), \qquad (5.8)$$

or equivalently,

$$\frac{f(u(x_0(t),t))}{f(\alpha(t))} \le 1 \quad \text{in } (0,T). \qquad (5.9)$$

We note by Proposition 2.3 that

$$\int_0^T f(u(x_0(t),t)\,dt = \infty, \tag{5.10}$$

since u blows up at $t = T$. Hence, integrating both sides of (5.9) over (t, T), we have, for $0 < t < T$,

$$G(\alpha(t)) = \int_{\alpha(t)}^{\infty} \frac{1}{f(\xi)}\,d\xi = \int_t^T \frac{f(u(x_0(t),t))}{f(\alpha(t))}\,dt \le T - t, \tag{5.11}$$

which implies

$$\alpha(t) \ge G^{-1}(T-t) \quad \text{for } 0 < t < T. \tag{5.12}$$

Combining this and (5.6) we obtain, for $0 < t < T$,

$$\inf_{x\in K} u(x,t) \ge (1-\varepsilon)G^{-1}(T-t) - (1-\varepsilon)||u_0||_\infty - \tilde{C}_\varepsilon T. \tag{5.13}$$

This is the first inequality of (5.3).

Next, we show the third inequality of (5.3). From (5.6),

$$u(x_0(t),t) \ge (1-\varepsilon)\int_0^t f(u(x_0(t),t))\,dt - \tilde{C}_\varepsilon T \equiv \beta(t) \quad \text{for } t \in (0,T). \tag{5.14}$$

By (5.10), we choose $t_0 \in (0, T)$ to satisfy $\beta(t_0) > 0$. Similarly to the proof of the first inequality of (5.3), we have

$$\begin{aligned} \frac{1}{1-\varepsilon}G(\beta(t)) &= \frac{1}{1-\varepsilon}\int_{\beta(t)}^{\infty} \frac{1}{f(\xi)}\,d\xi \\ &= \int_t^T \frac{f(u(x_0(t),t))}{f(\beta(t))}\,dt \ge T - t \quad \text{for } t_0 < t < T, \end{aligned} \tag{5.15}$$

which leads to

$$\beta(t) \le G^{-1}((1-\varepsilon)(T-t)) \quad \text{for } t_0 < t < T. \tag{5.16}$$

Combining this and Proposition 2.3 we get, in $\Omega \times (t_0, T)$,

$$u(x,t) \le \frac{1}{1-\varepsilon}G^{-1}((1-\varepsilon)(T-t)) + \frac{\tilde{C}_\varepsilon T}{1-\varepsilon} + ||u_0||_\infty. \tag{5.17}$$

This is the third inequality of (5.3). The proof is complete. □

Proof of Theorem 5.1. Let u be a blow-up solution of (1.1)-(1.3) in $\Omega \times (0, T)$ with the blow-up time $T < \infty$. Since $\lim_{t\uparrow T} G^{-1}(T-t) = \infty$, (5.1) immediately follows from Lemma 5.2.

Next, assuming (A9) we show (5.2). Let $0 < \varepsilon < 1$. Because of Lemma 5.2, there exist a $t_0 \in (0,\infty)$ and a constant $C_\varepsilon > 0$ such that

$$||u(t)||_\infty \le \frac{G^{-1}((1-\varepsilon)(T-t))}{1-\varepsilon} + C_\varepsilon \text{ in } (t_0,T), \tag{5.18}$$

and, hence,

$$\frac{||u(t)||_\infty}{G^{-1}(T-t)} \le \frac{G^{-1}((1-\varepsilon)(T-t))}{(1-\varepsilon)G^{-1}(T-t)} + \frac{C_\varepsilon}{G^{-1}(T-t)} \text{ in } (t_0,T). \tag{5.19}$$

Thus, because of (A9), letting $t \uparrow T$ and then $\varepsilon \downarrow 0$, we get

$$\limsup_{t\uparrow T} \frac{||u(t)||_\infty}{G^{-1}(T-t)} \le 1. \tag{5.20}$$

Combining this and (5.1) we have (5.2). □

6. Proof of Theorem 1.6

In this section, we prove Theorem 1.6. The method of the proof is similar to that of [22, 17]. We first show the next preliminary lemma which follows from the comparison theorem (Proposition 7.1).

For $\tau > 0$, we denote the solution of (1.1)-(1.3) in $\Omega \times (0,T)$ with the initial data τu_0 by u_τ. We also denote the maximum existence time of u_τ by $t_b(\tau)$.

Lemma 6.1. *Assume* (A1)-(A3) *and* $f'(\xi) \ge 0$ *in* $\xi \ge 0$. *Let* $u_0 \not\equiv 0$. *Then, the following hold:*

(i) $u_\tau(x,t)$ *is nondecreasing with* $\tau > 0$*; that is, if* $0 < \tau_1 < \tau_2$*, then* $t_b(\tau_1) \ge t_b(\tau_2)$ *and* $u_{\tau_1}(x,t) \le u_{\tau_2}(x,t)$ *in* $\Omega \times (0, t_b(\tau_2))$.

(ii) Let $\tau_1 > 0$ *and* $M > 0$. *If*

$$u_\tau(x,t) \le M \quad \text{in } \Omega \times (0,T) \quad \text{for all } \tau \in (0,\tau_1), \tag{6.1}$$

then $T < t_b(\tau_1)$ *and*

$$u_{\tau_1}(x,t) = \lim_{\tau\to\tau_1} u_\tau(x,t)\,(\le M) \text{ uniformly in } \bar\Omega \times [0,T]. \tag{6.2}$$

Proof. (i) follows from the comparison theorem.

(ii) We first show $t_b(\tau_1) > T$,

$$u_{\tau_1}(x,t) \le M \quad \text{in } \bar\Omega \times [0,T], \tag{6.3}$$

and

$$u_{\tau_1}(x,t) = \lim_{\tau\uparrow\tau_1} u_\tau(x,t) \text{ uniformly in } \bar\Omega \times [0,T]. \tag{6.4}$$

Assume to the contrary that $t_b(\tau_1) \le T$. Let $0 < \varepsilon < t_b(\tau_1)$. Then, by Proposition 7.1 we see, for $\tau \in (0, \tau_1)$, $t \in [0, t_b(\tau_1) - \varepsilon]$,

$$\|u_\tau(t) - u_{\tau_1}(t)\|_\infty \le C(\varepsilon)|\tau - \tau_1|\|u_0\|_\infty, \tag{6.5}$$

where $C(\varepsilon) > 0$ is a constant depending only on $\sup_{t\in(0,t_b(\tau_1)-\varepsilon)} \|u_{\tau_1}(t)\|_\infty$. Hence,

$$u_{\tau_1}(x,t) = \lim_{\tau\uparrow\tau_1} u_\tau(x,t) \quad \text{uniformly in } \bar{\Omega} \times [0, t_b(\tau_1) - \varepsilon]. \tag{6.6}$$

Since $\varepsilon \in (0, t_b(\tau_1))$ can be taken arbitrarily, we have

$$u_{\tau_1}(x,t) = \lim_{\tau\uparrow\tau_1} u_\tau(x,t) \le M \quad \text{in } \bar{\Omega} \times [0, t_b(\tau_1)), \tag{6.7}$$

and, hence, u_τ does not blow up at $t = t_b(\tau_1)$. This is a contradiction and, hence, we get $t_b(\tau_1) > T$. Taking $\varepsilon = (t_b(\tau_1) - T)/2$ in (6.6), we have

$$u_{\tau_1}(x,t) = \lim_{\tau\uparrow\tau_1} u_\tau(x,t)\, (\le M) \quad \text{uniformly in } \bar{\Omega} \times [0, T], \tag{6.8}$$

since $(t_b(\tau_1) + T)/2 > T$.

Similarly, it is not difficult to see that

$$u_{\tau_1}(x,t) = \lim_{\tau\downarrow\tau_1} u_\tau(x,t) \text{ uniformly in } \bar{\Omega} \times [0, T], \tag{6.9}$$

and, hence, we obtain (6.2). The proof is complete. □

Now, we prove Theorem 1.6. Assume (A1)-(A5) and let $u_0 \not\equiv 0$. Put

$$\begin{aligned} \Lambda_0 = \{\tau > 0 : u_\tau &\text{ exists globally in time} \qquad (6.10)\\ &\text{and satisfies } ||u_\tau(\cdot,t)||_\infty \to 0 \text{ as } t \to \infty\}, \end{aligned}$$

$$\Lambda_{GL} = \{\tau > 0 : u_\tau \text{ exists globally in time }\}, \tag{6.11}$$

and

$$\Lambda_B = \mathbf{R}^+ \backslash \Lambda_{GL} = \{\tau > 0 : u_\tau \text{ blows up in finite time }\}, \tag{6.12}$$

where $\mathbf{R}^+ = \{x \in \mathbf{R} : x > 0\}$. Clearly, $\Lambda_0 \subset \Lambda_{GL}$. Set

$$\tau_1 = \sup \Lambda_0 \qquad \text{and} \qquad \tau_2 = \sup \Lambda_{GL}. \tag{6.13}$$

We note that $0 < \tau_1 \le \tau_2 < \infty$. In fact, clearly $\tau_1 \le \tau_2$. The inequality $0 < \tau_1$ follows from Theorem 3.1 and the inequality $\tau_2 < \infty$ follows from the blow-up theorem (See Theorem B in [28] and Theorem 3.1 in [9]).

For the proof of Theorem 1.6, it is sufficient to show the next two lemmas.

Lemma 6.2. *We have*

$$\Lambda_0 = (0, \tau_1), \tag{6.14}$$

and

$$\Lambda_{GL} = (0, \tau_2]. \tag{6.15}$$

Proof. Because of (i) of Lemma 6.1, if $\tau \in \Lambda_0$ and $0 < \tau' < \tau$ then $\tau' \in \Lambda_0$. Similarly, if $\tau \in \Lambda_{GL}$ and $0 < \tau' < \tau$, then $\tau' \in \Lambda_{GL}$. Hence, we see that $(0, \tau_1) \subset \Lambda_0 \subset (0, \tau_1]$ and $(0, \tau_2) \subset \Lambda_{GL} \subset (0, \tau_2]$.

We first prove (6.14). It is sufficient to show that Λ_0 is an open set in $\mathbf{R}$. The method of the proof is the same as that of [22] and [17]. Let $\tilde{\tau} \in \Lambda_0$, and let $\delta_0 > 0$ be as in Theorem 3.1. Then, there exists $T_1 > 0$ such that $||u_{\tilde{\tau}}(\cdot, T_1)||_\infty < \delta_0/2$. By virtue of Lemma 6.1(ii), $u_\tau(x,t) \to u_{\tilde{\tau}}(x,t)$ as $\tau \downarrow \tilde{\tau}$ uniformly in $\bar{\Omega} \times [0, T_1]$. Hence, if $\tau - \tilde{\tau} > 0$ is small enough, then $||u_\tau(\cdot, T_1)||_\infty < \delta_0$. Because of Theorem 3.1, u_τ exists globally in time and $||u_\tau(\cdot, t)||_\infty \to 0$ as $t \to \infty$, and hence $\tau \in \Lambda_0$. This shows that Λ_0 is an open set.

Next, we show (6.15). Let $\tau \in (0, \tau_2) \subset \Lambda_{GL}$. Then, u_τ exists globally in time. It follows from Proposition 4.1 that for each $T > 0$

$$u_\tau(x,t) \le \tau_2 \|u_0\|_\infty + C_{2T} \quad \text{in } \Omega \times [0, T], \tag{6.16}$$

where $C_{2T} > 0$ is a constant independent of u_0 and $\tau \in (0, \tau_2)$. Thus, applying (ii) of Lemma 6.1, we see that $u_{\tau_2} = \lim_{\tau \uparrow \tau_2} u_\tau$ exists globally in time; that is, $\tau_2 \in \Lambda_{GL}$. Thus, we have (6.15). The proof is complete. $\square$

Lemma 6.3. *If* $\tau \in \Lambda_{GL} \backslash \Lambda_0 = [\tau_1, \tau_2]$*, then*

$$||u_\tau(\cdot, t)||_\infty \ge \delta_0 \quad \textit{for } t \ge 0, \tag{6.17}$$

where $\delta > 0$ *is as in Theorem* 3.1. *Moreover, if* (A6) *holds, then*

$$\sup_{t \in (0,\infty)} ||u_\tau(\cdot, t)||_\infty < \infty, \tag{6.18}$$

and if (A7) *holds, then*

$$\sup_{t \in (0,\infty)} ||u_\tau(\cdot, t)||_\infty = \infty. \tag{6.19}$$

Proof. (6.17) follows from Theorem 3.1. (6.18) is already shown in Theorem 1.2. Hence, supposing (A7), we shall show (6.19). Let $\tau \in \Lambda_{GL} \backslash \Lambda_0 = [\tau_1, \tau_2]$. Assume to the contrary that

$$\sup_{t \in (0,\infty)} ||u_\tau(\cdot, t)||_\infty \equiv M < \infty. \tag{6.20}$$

Then, by Proposition 2.2, there exist constants $C = C(M) > 0$ and $\beta = \beta(M) \in (0,1)$ such that

$$|u_\tau(x_1,t_1) - u_\tau(x_2,t_2)| \le C \cdot (|x_1 - x_2|^\beta + |t_1 - t_2|^{\beta/2}) \quad \text{for } (x_1,t_1),\ (x_2,t_2) \in \bar{\Omega} \times [1,\infty). \tag{6.21}$$

Hence, choosing $\tilde{x}(t) \in \partial\Omega$ to satisfy

$$\operatorname{dist}(x_0(t), \partial\Omega) = |x_0(t) - \tilde{x}(t)|, \tag{6.22}$$

we have

$$u_\tau(x_0(t),t) \le C|x_0(t) - \tilde{x}(t)|^\beta = C\{\operatorname{dist}(x_0(t),\partial\Omega)\}^\beta \quad \text{for } t \ge 1. \tag{6.23}$$

Thereby, by (A7)

$$u_\tau(x_0(t),t) \to 0 \quad \text{as } t \to \infty, \tag{6.24}$$

and, hence, by (A5),

$$f(u_\tau(x_0(t),t)) \to 0 \quad \text{as } t \to \infty. \tag{6.25}$$

On the other hand, by Lemma 2.1(iii), u_τ is expressed as follows:

$$u_\tau(x,t) = \tau e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} f(u_\tau(x_0(s),s))ds. \tag{6.26}$$

By (6.25), for any $\varepsilon > 0$ there exists $T > 0$ such that

$$f(u_\tau(x_0(t),t) < \varepsilon \quad \text{for } t \ge T. \tag{6.27}$$

Using Lemma 2.1(i) we get

$$\begin{aligned} u_\tau(x,t) &\le C_1 \tau e^{-\lambda t}\|u_0\|_\infty + C_1 f(M) \int_0^T e^{-\lambda(t-s)}\,ds + C_1\varepsilon \int_T^t e^{-\lambda(t-s)}\,ds \\ &\le C_1 \tau e^{-\lambda t}\|u_0\|_\infty + \frac{C_1 f(M)}{\lambda} e^{-\lambda(t-T)} + \frac{C_1\varepsilon}{\lambda} \quad \text{for } t \ge T. \end{aligned} \tag{6.28}$$

Therefore,

$$\limsup_{t\to\infty} \|u_\tau(t)\|_\infty \le \frac{C_1\varepsilon}{\lambda}. \tag{6.29}$$

Since $\varepsilon > 0$ can be taken arbitrarily,

$$\lim_{t\to\infty} \|u_\tau(t)\|_\infty = 0. \tag{6.30}$$

This is a contradiction to $\tau \notin \Lambda_0$ and hence we get (6.19). The proof is complete □

Proof of Theorem 1.6. Theorem 1.6 follows from Theorem 3.1, Theorem 5.1, Lemma 6.2 and Lemma 6.3. □

7. Appendix I

In this section, we shall show some L^∞−estimates for solutions, which leads to the uniqueness and comparison theorems. These theorems were already shown in [28] except for the L^∞-estimate, which is used in the proof of Theorem 1.6 (see the proof of Lemma 6.1). We will show the L^∞−estimate for more general equations. We note that solutions are not necessarily nonnegative.

Let $g(x,t,\xi,\eta) \in C^1(\bar{\Omega}\times[0,\infty)\times\mathbf{R}\times\mathbf{R})$ and consider the equation

$$u_t = \Delta u + g(x,t,u,u(x_0(t),t)) \quad \text{in } \Omega\times(0,T). \tag{7.1}$$

A function u [or v] $\in C(\bar{\Omega}\times[0,T))\cap C^{2,1}(\Omega\times(0,T))$ is called a subsolution [or supersolution] if u [or v] satisfies the equation (7.1) in $\Omega\times(0,T)$ with equality replaced by $\le$ [or $\ge$].

Proposition 7.1. *Assume* (A2). *Suppose that*

$$-M \le u(x,t),\, v(x,t) \le M \quad \textit{in } \Omega\times(0,T), \tag{7.2}$$

for some $M>0$. Then, the following hold.

(*i*) *Let u and v be solutions of* (7.1) *in $\Omega\times(0,T)$. Then, there exists a constant $C>0$ depending only on M and T such that*

$$\begin{aligned}\|u(\cdot,t)-v(\cdot,t)\|_\infty \le C\Big\{&\|u(\cdot,0)-v(\cdot,0)\|_\infty \\ &+\sup_{(x,t)\in\partial\Omega\times(0,T)}|u(x,t)-v(x,t)|\Big\} \quad \textit{for } t\in(0,T).\end{aligned} \tag{7.3}$$

(*ii*) *Assume $(\partial g/\partial\eta)(x,t,\xi,\eta)\ge 0$ in $\bar{\Omega}\times[0,\infty)\times\mathbf{R}\times\mathbf{R}$. Let u [or v] be a subsolution [or supersolultion] of* (7.1) *in $\Omega\times(0,T)$. Then, there exists a constant $C>0$ depending only on M and T such that*

$$\begin{aligned}\|[u(\cdot,t)-v(\cdot,t)]_+\|_\infty \le C\Big\{&\|[u(\cdot,0)-v(\cdot,0)]_+\|_\infty \\ &+\sup_{(x,t)\in\partial\Omega\times(0,T)}[u(x,t)-v(x,t)]_+\Big\} \quad \textit{for } t\in(0,T),\end{aligned} \tag{7.4}$$

where $[a]_+=\max\{a,0\}$.

Proof. (i) Let $w=u-v$. Then, w satisfies the equation

$$w_t = \Delta w + h_1(x,t)w + h_2(x,t)w(x_0(t),t) \quad \text{in } \Omega\times(0,T), \tag{7.5}$$

where h_1 and h_2 are some functions satisfying

$$|h_1(x,t)| \leq C_1 \quad \text{in } \Omega \times (0,T), \tag{7.6}$$

and

$$|h_2(x,t)| \leq C_2 \quad \text{in } \Omega \times (0,T), \tag{7.7}$$

for some $C_1 > 0$ and $C_2 > 0$ (depending only on M and T).

As in the proof of Proposition 2.3, for $\tilde{C} = C_1 + C_2$,

$$g(t) = \tilde{C} \int_0^t \|w(\cdot,t)\|_\infty \, dt + \|w(\cdot,0)\|_\infty + \sup_{(x,t)\in\partial\Omega\times(0,T)} |w(x,t)|$$

is a supersolution of (7.5), where $h_1(x,t)w + h_2(x,t)w(x_0(t),t)$ is considered as a given function. Since $w(x,0) \leq g(0)$ in Ω and $w(x,t) \leq g(t)$ in $\partial\Omega \times (0,T)$, by the usual comparison theorem we have $w(x,t) \leq g(t)$ in $\Omega \times (0,T)$ and hence $[w(x,t)]_+ \leq g(t)$ in $\Omega \times (0,T)$. Similarly, $[-w(x,t)]_+ \leq g(t)$ in $\Omega \times (0,T)$ and, hence,

$$\|w(\cdot,t)\|_\infty \leq g(t) = \tilde{C} \int_0^t \|w(\cdot,t)\|_\infty \, dt + \|w(\cdot,0)\|_\infty + \sup_{(x,t)\in\partial\Omega\times(0,T)} |w(x,t)| \quad \text{in } (0,T). \tag{7.8}$$

Therefore, by Gronwall's inequality we get

$$\|w(\cdot,t)\|_\infty \leq e^{\tilde{C}t}\Big\{\|w(\cdot,0)\|_\infty + \sup_{(x,t)\in\partial\Omega\times(0,T)} |w(x,t)|\Big\} \quad \text{for } t \in (0,T); \tag{7.9}$$

that is,

$$\|u(\cdot,t) - v(\cdot,t)\|_\infty \leq e^{\tilde{C}t}\Big\{\|u(\cdot,0) - v(\cdot,0)\|_\infty + \sup_{(x,t)\in\partial\Omega\times(0,T)} |u(x,t) - v(x,t)|\Big\} \quad \text{for } t \in (0,T). \tag{7.10}$$

(ii) Assume $(\partial g/\partial\eta)(x,t,\xi,\eta) \geq 0$ in $\bar{\Omega} \times [0,\infty) \times \mathbf{R} \times \mathbf{R}$. Let u [or v] be a subsolution [or supersolultion] of (7.1) in $\Omega \times (0,T)$. Similarly to above, if we set $w = u - v$, then w satisfies

$$w_t \leq \Delta w + h_1(x,t)w + h_2(x,t)w(x_0(t),t) \quad \text{in } \Omega \times (0,T), \tag{7.11}$$

where h_1 and h_2 (≥ 0) are some functions satisfying (7.6) and (7.7) for some positive constants C_1 and C_2 (depending only on M and T). Furthermore, if we set $\tilde{w} = e^{C_1 t} w$, $\tilde{w}$ satisfies

$$\tilde{w}_t \leq \Delta\tilde{w} + (h_1(x,t) + C_1)\tilde{w} + h_2(x,t)\tilde{w}(x_0(t),t) \quad \text{in } \Omega \times (0,T). \tag{7.12}$$

Set $C_3 = 2C_1 + C_2$. Note that $h_1(x,t) + C_1 \geq 0$ and $h_2(x,t) \geq 0$ in $\Omega \times (0,T)$. Then, we see that

$$\tilde{g}(t) = C_3 \int_0^t \|[\tilde{w}(\cdot,t)]_+\|_\infty \, dt + \|[\tilde{w}(\cdot,0)]_+\|_\infty + \sup_{(x,t)\in\partial\Omega\times(0,T)} [\tilde{w}(x,t)]_+$$

is a supersolution of

$$w_t = \Delta w + (h_1(x,t) + C_1)\tilde{w} + h_2(x,t)\tilde{w}(x_0(t),t). \tag{7.13}$$

Similarly to above, we have $\tilde{w}(x,t) \leq \tilde{g}(t)$ in $\Omega \times (0,T)$ and, hence,

$$\begin{aligned}[\tilde{w}(x,t)]_+ \leq C_3 \int_0^t \|[\tilde{w}(\cdot,t)]_+\|_\infty \, dt + \|[\tilde{w}(\cdot,0)]_+\|_\infty \\ + \sup_{(x,t)\in\partial\Omega\times(0,T)} [\tilde{w}(x,t)]_+ \quad \text{in } \Omega \times (0,T),\end{aligned} \tag{7.14}$$

which leads to

$$\begin{aligned}\|[\tilde{w}(\cdot,t)]_+\|_\infty \leq & e^{C_3 t}\Big\{\|[\tilde{w}(\cdot,0)]_+\|_\infty \\ & + \sup_{(x,t)\in\partial\Omega\times(0,T)} [\tilde{w}(x,t)]_+\Big\} \quad \text{for } t \in (0,T).\end{aligned} \tag{7.15}$$

That is,

$$\begin{aligned}\|[u(\cdot,t) - v(\cdot,t)]_+\|_\infty \leq & e^{C_3 t}\Big\{\|[u(\cdot,0) - v(\cdot,0)]_+\|_\infty \\ & + \sup_{(x,t)\in\partial\Omega\times(0,T)} [u(x,t) - v(x,t)]_+\Big\} \quad \text{for } t \in (0,T).\end{aligned} \tag{7.16}$$

The proof is complete. □

8. Appendix II

In this section we prove the next lemma.

Lemma 8.1. *Assume* (1.4) *and*

$$\int_\xi^\infty \frac{1}{f(\eta)} \, d\eta < \infty \quad \textit{for } \xi > 0. \tag{8.1}$$

Then

$$\lim_{\xi\to\infty} \frac{f(\xi)}{\xi} = \infty. \tag{8.2}$$

Proof. Assume (1.4) and (8.1). Then, it is not difficult to see that

$$\limsup_{\xi\to\infty} \frac{f(\xi)}{\xi} = \infty. \tag{8.3}$$

In fact, otherwise,

$$\int_1^\infty \frac{1}{f(\eta)}\, d\eta = \infty, \tag{8.4}$$

and this is a contradiction. So, it is sufficient to show

$$\liminf_{\xi\to\infty} \frac{f(\xi)}{\xi} = \infty. \tag{8.5}$$

Assume to the contrary that

$$\liminf_{\xi\to\infty} \frac{f(\xi)}{\xi} = C < \infty. \tag{8.6}$$

By (8.3), there exists a sequence $\{\xi_n\} \uparrow \infty \subset \mathbf{R}$ such that

$$\frac{f(\xi_n)}{\xi_n} > C + 3 \quad \text{for } n \geq 1, \tag{8.7}$$

and by (8.6), there exists a sequence $\{\eta_n\} \subset \mathbf{R}$ such that

$$\eta_n > \xi_n \quad \text{and} \quad \frac{f(\eta_n)}{\eta_n} < C + 1 \quad \text{for } n \geq 1. \tag{8.8}$$

Hence, by virtue of the intermediate value theorem, there exists a sequence $\{z_n\} \subset \mathbf{R}$ such that

$$\xi_n < z_n < \eta_n \quad \text{and} \quad \frac{f(z_n)}{z_n} = C + 2 \quad \text{for } n \geq 1. \tag{8.9}$$

We note that $z_n \to \infty$ as $n \to \infty$. Thereby, we can choose a subsequence of $\{z_n\}$ (still denoted by $\{z_n\}$) to satisfy $2z_n \leq z_{n+1}$ for $n \geq 1$. Since $f(\xi)$ is nondecreasing in $\xi \geq 0$, we see that $f(\xi) \leq f(z_n) = (C+2)z_n$ in $[z_n/2, z_n]$ for $n \geq 2$. Therefore,

$$\int_{z_2/2}^\infty \frac{1}{f(\xi)}\, d\xi \geq \sum_{n=2}^\infty \int_{z_n/2}^{z_n} \frac{1}{f(\xi)}\, d\xi \geq \frac{1}{C+2} \sum_{n=2}^\infty \int_{z_n/2}^{z_n} \frac{1}{z_n}\, d\xi = \infty. \tag{8.10}$$

This is a contradiction to (8.1) and hence we get (8.5), that is, (8.2). The proof is complete. □

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