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STRONG SOLUTIONS FOR A FLUID STRUCTURE INTERACTION SYSTEM

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Abstract. We consider the structure-interaction model, introduced by J.-L. Lions, describing the interaction of an elastic body and an incompressible fluid. Recently, many works have addressed well posedness for this model. In this paper, we prove local existence of strong solutions under the initial condition $(u_0, w_0, w_1) \in H^1 \times H^{3/2+k} \times H^{1/2+k}$ for every $k > 0$ sufficiently small and where u_0, w_0, and w_1 are the initial velocity of the fluid, initial displacement of the body, and the initial velocity of the body respectively. We also propose new alternative matching stress boundary conditions for this model.

1. Introduction

We consider a fluid structure interaction model describing the interaction of an elastic body with the surrounding viscous fluid. The model is a coupled PDE system consisting of the incompressible Navier-Stokes equation coupled with an elastic equation via velocity and stress matching boundary conditions. The history of the model goes back to 1969, when it appeared in a book by Lions [17]. Recently, this model has received a lot of attention in its linear and nonlinear formulations [12, 9, 4, 3, 6, 7, 16].

The existence of solutions for this problem was addressed in [6, 7]; in particular, the authors proved local existence of strong solutions for initial data $(u_0, w_0, w_1) \in H^2 \times H^2 \times H^1$ (here u_0, w_0, and w_1 denote the initial fluid velocity, elastic displacement and elastic velocity respectively). In our previous paper [16], we have lowered the requirement for the initial velocity u_0 to H^1 in the flat case. However, unlike [6, 7], the approach in [16] does not extend to the variable domain case since the spaces of existence require different regularities for the tangential and normal derivatives.

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The purpose of the present paper is two-fold. First, we lower the requirement on the initial data w_0, w_1 for the wave equation from $H^2 \times H^1$ as in [6, 7, 16] to $H^{3/2+k} \times H^{1/2+k}$, for any $k > 0$ which is sufficiently small, while maintaining the requirement $u_0 \in H^1$ obtained in [16]. By lowering the regularity on the initial data we achieve the second goal on establishing existence in compatible spaces for the tangential and normal derivatives of the solutions to the wave equation. We believe that this is an essential step for extending the method to the variable case, which is more involved and will be addressed elsewhere.

The main mathematical difficulties in the analysis pertain to the mismatch of regularity between the hyperbolic and parabolic components of the system with respect to the type of initial data considered. In particular, while H^1 datum in the Navier-Stokes equation is known to guarantee existence of solutions belonging to the space H^1 (cf. [16]), this is not sufficient to obtain H^2 regularity of the solution to the elastic equation. This is explained by the fact that the regularity of the boundary data is largely driven by the smoothing effect coming from the Navier-Stokes equation, thus determining the eventual regularity of the solutions to the elastic equation. However, the traces of solutions to the Navier-Stokes equation driven by H^1 type datum lack the time regularity necessary to guarantee the existence of solutions to the elastic equation in the space $H^2 \times H^1$.

To establish solutions in the desired range of spaces, we carry out estimates on fractional tangential derivatives, which we carefully couple with the estimates obtained on the time derivatives, full gradient, and the pressure. The appearance of Neumann type boundary terms in the time derivative estimates require using the Hidden trace regularity results pertaining to solutions of the wave equation (and its generalizations to second-order hyperbolic equations). Finally, the nonlinear term poses an additional complication which can be tackled by using a combination of fractional Stokes estimates and a new form of anisotropic estimates driven by the divergence-free conditions.

2. The Notation and the Main Result

Fix $0 < h_1 < h_2 < h_3$. The fluid region is defined by

$$\Omega_f = \big\{(x_1, x_2, x_3) : 0 < x_3 < h_1 \text{ or } h_2 < x_3 < h_3\big\},$$

while the elastic body occupies the domain

$$\Omega_e = \big\{(x_1, x_2, x_3) : h_1 < x_3 < h_2\big\}.$$

We seek functions (u, w, p) satisfying the system of equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \text{ in } \Omega_f \times (0, T) \tag{2.1}$$

$$\operatorname{div} u = 0 \text{ in } \Omega_f \times (0, T) \tag{2.2}$$

$$w_{tt} - \Delta w = 0 \text{ in } \Omega_e \times (0, T). \tag{2.3}$$

The flat boundary $\Gamma_c = \big\{(x_1, x_2, x_3) : x_3 = h_1 \text{ or } x_3 = h_2\big\}$ in-between the two domains is where the interaction between the structure and the fluid takes place as captured by the velocity and stress matching conditions in (2.4)–(2.6) below. The function $u = (u_1, u_2, u_3)$ represents the velocity of the fluid in the domain Ω_f, and $w = (w_1, w_2, w_3)$ the displacement function on the domain Ω_e, while p is the pressure defined on the domain Ω_f. We denote by ν the unit outward normal vector on Γ_c with respect to the region Ω_e. All functions are assumed to be periodic in x_1 and x_2 with period 2π.

In particular, we require continuity of the velocities and the stresses on Γ_c and a no slip boundary condition on the exterior boundary $\Gamma_f = \big\{(x_1, x_2, x_3) : x_3 = 0 \text{ or } x_3 = h_3\big\}$; i.e.,

$$u = w_t \qquad \text{on } \Gamma_c \times (0, T) \tag{2.4}$$

$$u = 0 \qquad \text{on } \Gamma_f \times (0, T) \tag{2.5}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} - p\nu - \frac{1}{2}(u \cdot \nu)u \qquad \text{on } \Gamma_c \times (0, T). \tag{2.6}$$

In addition, u and w satisfy the initial conditions

$$u(\cdot, 0) = u_0 \text{ in } \Omega_f \tag{2.7}$$

$$w(\cdot, 0) = w_0 \text{ in } \Omega_e \tag{2.8}$$

$$w_t(\cdot, 0) = w_1 \text{ in } \Omega_e. \tag{2.9}$$

Throughout the paper, we denote

$$H = \{u \in L^2(\Omega_f) : \operatorname{div} u = 0,\ u \cdot \nu|_{\Gamma_f} = 0\},$$

and

$$V = \{v \in H^1(\Omega_f) : \operatorname{div} v = 0,\ u|_{\Gamma_f} = 0\}.$$

The following is the main result of the paper.

Theorem 2.1. *Let $0 < k < (\sqrt{2} - 1)/2$, and assume that $u_0 \in V$, $w_0 \in H^{3/2+k}(\Omega_e)$, and $w_1 \in H^{1/2+k}(\Omega_e)$ with $w_1|_{\Gamma_c} = u_0|_{\Gamma_c}$. Then there exists a unique local in time solution*

$$u \in L^2([0, T]; H^{3/2+k}(\Omega_f)) \cap L^\infty([0, T]; V) \tag{2.10}$$

$$u_t \in L^2([0, T]; L^2(\Omega_f)) \tag{2.11}$$

$$w \in L^\infty([0,T]; H^{3/2+k}(\Omega_e)) \tag{2.12}$$

$$w_t \in L^\infty([0,T]; H^{1/2+k}(\Omega_e)) \tag{2.13}$$

$$p \in L^2([0,T]; H^{1/2+k}(\Omega_f)) \tag{2.14}$$

of the system (2.1)–(2.3) with boundary conditions (2.4)–(2.6) and initial conditions (2.7)–(2.9) for a time T depending on the initial data.

3. Proof of Main Theorem 2.1

3.1. **Past and preliminary results on existence of solutions and hidden regularity.** First, we recall the result from [6] on the existence of weak solutions to the system (2.1)–(2.3) with boundary conditions (2.4)–(2.6) starting with the definition of a weak solution.

Definition 3.1. *Assume that $(u_0, w_0, w_1) \in H \times H^1(\Omega_e) \times L^2(\Omega_e)$ and $T > 0$. We say that a triple $(u, w, w_t) \in C_w([0,T]; H \times H^1(\Omega_e) \times L^2(\Omega_e))$ is a weak solution of the system* (2.1)–(2.3) *with boundary conditions* (2.4)–(2.6) *if the following conditions hold:*

(*a*) *The time derivatives u_t and w_{tt} belong to the spaces $L^{4/3}([0,T]; V')$ and $L^2([0,T]; H^{-1}(\Omega_e))$ respectively.*
(*b*) *For the trace, we have $w_t|_{\Gamma_c} = u|_{\Gamma_c} \in L^2([0,T]; H^{1/2}(\Gamma_c))$.*
(*c*) *For the normal derivative, we have $\partial w/\partial \nu \in L^2([0,T]; H^{-1/2}(\Gamma_c))$.*
(*d*) *For all test functions $\phi \in V$ and $\psi \in H^1(\Omega_e)$, the functions u and w satisfy the variational equations*

$$(u_t, \phi)_f + (\nabla u, \nabla \phi)_f + b(u, u, \phi) + \left\langle \frac{\partial w}{\partial \nu}, \phi \right\rangle = 0 \tag{3.1}$$

$$(w_{tt}, \psi)_e + (\nabla w, \nabla \psi)_e - \left\langle \frac{\partial w}{\partial \nu}, \psi \right\rangle = 0, \tag{3.2}$$

where

$$b(u, v, w) \equiv ((u \cdot \nabla) v, w)_f + \frac{1}{2}\langle (u \cdot \nu) v, w \rangle,$$

for almost every $t \in (0,T)$.

Above and in the sequel, we use the following notation for the L^2 inner products:

$$(u, v)_f = \int_{\Omega_f} u \cdot v \, dx, \qquad (u, v)_e = \int_{\Omega_e} u \cdot v \, dx, \qquad \langle u, v \rangle = \int_{\Gamma_c} u \cdot v \, d\sigma(x).$$

Note that the weak formulation can be obtained formally by taking the L^2 inner product of the Navier-Stokes equation (2.1) with $\phi \in V$ and the wave equation with $\psi \in H^1(\Omega_e)$ and integrating by parts. The cancellation of the terms involving the pressure results from the boundary condition (2.6) and the divergence-free condition (2.2).

The existence of weak solutions as defined above is given in the next theorem.

Theorem 3.1. [6] *There exists a global weak solution* (u, w, w_t) *for the system* (2.1)–(2.3) *with boundary conditions* (2.4)–(2.6) *and initial conditions* $u_0 \in H$, $w_0 \in H^1(\Omega_e)$, *and* $w_1 \in L^2(\Omega_e)$. *In addition, this weak solution satisfies the energy inequality*

$$\|u(t)\|^2_{L^2(\Omega_f)} + \|w_t(t)\|^2_{L^2(\Omega_e)} + \|\nabla w(t)\|^2_{L^2(\Omega_e)} + 2\int_0^t \|\nabla u\|^2_{L^2(\Omega_f)}\,ds \\ \leq \|u_0\|^2_{L^2(\Omega_f)} + \|w_0\|^2_{H^1(\Omega_e)} + \|w_1\|^2_{L^2(\Omega_e)} = E(0), \tag{3.3}$$

for every $t \geq 0$. *Moreover, this weak solution is unique in the class specified in Definition* 3.1 *when the space dimension is two.*

Our strategy in proving Theorem 2.1 is to show that weak solutions are smooth when we impose additional regularity assumptions on the initial data. To this end, we use a hidden regularity result for the wave equation that enables us to estimate the normal derivative of w_t on the boundary in terms of the Dirichlet data and the initial conditions.

Theorem 3.2. [18] *Let* w *be a solution of the wave equation*

$$w_{tt} - \Delta w = 0 \text{ in } \Omega_e \times (0,T), \tag{3.4}$$

with the boundary condition

$$w = f \text{ on } \Gamma_c \times (0,T), \tag{3.5}$$

where $f \in L^2([0,T]; H^\alpha(\Gamma_c)) \cap H^\alpha([0,T]; L^2(\Gamma_c))$ *and* $\alpha \in [0,1]$, *while the initial conditions* w_0 *and* w_1 *belong to* $H^\alpha(\Omega_e)$ *and* $H^{\alpha-1}(\Omega_e)$ *respectively. Then there exists a solution* w *such that*

$$w \in C([0,T]; H^\alpha(\Omega_e)) \tag{3.6}$$

$$w_t \in C([0,T]; H^{\alpha-1}(\Omega_e)). \tag{3.7}$$

In addition, the normal derivative $\partial w/\partial \nu$ *on the boundary* Γ_c *satisfies*

$$\frac{\partial w}{\partial \nu} \in H^{\alpha-1}(\Gamma_c \times [0,T]), \tag{3.8}$$

with

$$\left\|\frac{\partial w}{\partial \nu}\right\|^2_{H^{\alpha-1}(\Gamma_c\times[0,T])} + \|w(t)\|^2_{H^\alpha(\Omega_e)} + \|w_t(t)\|^2_{H^{\alpha-1}(\Omega_e)} \tag{3.9}$$

$$\leq C(\|w_0\|^2_{H^\alpha(\Omega_e)} + \|w_1\|^2_{H^{\alpha-1}(\Omega_e)} + \|f\|^2_{L^2([0,T];H^\alpha(\Gamma_c))} + \|f\|^2_{H^\alpha([0,T];L^2(\Gamma_c))})$$

for all $t \in [0,T]$.

3.2. **Regularity of the tangential derivatives.** First, we obtain a priori estimates on the tangential derivatives of the weak solutions (u, w, w_t) with respect to the common flat boundary Γ_c, i.e., the fractional derivatives $|\partial_1|^{1/2+k}$ and $|\partial_2|^{1/2+k}$.

For a fixed $k \in (0, (\sqrt{2}-1)/2)$, assume $(u_0, w_0, w_1) \in V \times H^{3/2+k}(\Omega_e) \times H^{1/2+k}(\Omega_e)$. Then, by Theorem 3.1, there exists a weak solution (u, w, w_t) satisfying the conditions in Definition 3.1 and in particular the variational formulation (3.1)–(3.2).

Denoting the tangential directions with respect to Γ_c by x_j with $j = 1$ or 2, we define the fractional derivative S_j via the Fourier transform on the torus by

$$S_j u(x_j, \cdot) = \sum_{n=-\infty}^{\infty} |n|^{1/2+k} \hat{u}(\xi_j = n, \cdot) e^{i x_j n}.$$

Denote by $T \in (0, \infty)$ an a priori time of existence of strong solutions.

Lemma 3.3. *The quantity*

$$y(t) = 1 + \sum_{j=1}^{2} \Big(\|S_j u(t)\|^2_{L^2(\Omega_f)} + \|S_j w_t(t)\|^2_{L^2(\Omega_e)} + \|\nabla S_j w\|^2_{L^2(\Omega_e)} \Big),$$

satisfies an a priori estimate

$$y'(t) + \sum_{j=1}^{2} \|\nabla S_j u(t)\|^2_{L^2(\Omega_f)} \leq C y(t) \, \|\nabla u\|^4_{L^2(\Omega_f)} \tag{3.10}$$

on $[0, T)$.

Above and in the sequel, the symbol C denotes a generic positive constant which is allowed to depend on k, h_1, h_2, and h_3.

Proof. First, we replace ϕ by $S_j \phi$ for a fixed $j \in \{1,2\}$ in (3.1) and ψ by $S_j \psi$ (3.2), so that the equations (3.1) and (3.2) become

$$\begin{aligned}(S_j u_t, \phi)_f + (\nabla S_j u, \nabla \phi)_f + (S_j (u \cdot \nabla) u, \phi)_f + \frac{1}{2} \big\langle S_j (u \cdot \nu) u, \phi \big\rangle \\ + \Big\langle \frac{\partial (S_j w)}{\partial \nu}, \phi \Big\rangle = 0,\end{aligned} \tag{3.11}$$

and

$$(S_j w_{tt}, \psi)_e + (\nabla S_j w, \nabla \psi)_e - \Big\langle \frac{\partial (S_j w)}{\partial \nu}, \psi \Big\rangle = 0. \tag{3.12}$$

In addition, the boundary condition in (2.4) reads

$$S_j u = S_j w_t, \qquad \Gamma_c \times (0,T). \tag{3.13}$$

Setting $\phi = S_j u$ and $\psi = S_j w_t$ in (3.11) and (3.12) respectively, then adding the two equations while observing that the term $\langle \partial(S_j w)/\partial\nu, S_j u\rangle$ cancels with $\langle \partial(S_j w)/\partial\nu, S_j w_t\rangle$ by (3.13), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|S_j u\|^2_{L^2(\Omega_f)} + \|S_j w_t\|^2_{L^2(\Omega_e)} + \|S_j \nabla w\|^2_{L^2(\Omega_e)}\Big) + \|\nabla S_j u\|^2_{L^2(\Omega_f)} \\ \le |(S_j((u\cdot\nabla)u), S_j u)_f| + \frac{1}{2}\left|\langle S_j((u\cdot\nu)u), S_j u\rangle\right|. \end{aligned}\tag{3.14}$$

From [15], we now recall the inequality

$$\begin{aligned}&\|(-\partial_{xx})^{\alpha/2}(fg)\|_{L^p(\mathbb{R})} \\ &\le C\big(\|(-\partial_{xx})^{\alpha/2} f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})} + \|f\|_{L^{q_1}(\mathbb{R})}\|(-\partial_{xx})^{\alpha/2} g\|_{L^{q_2}(\mathbb{R})}\big),\end{aligned}\tag{3.15}$$

for $\alpha \ge 0$ and all $p, p_1, p_2, q_1, q_2 \in (1,\infty)$ such that $1/p = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$, which easily implies

$$\|S_j(fg)\|_{L^p(\Omega_f)} \le C\big(\|S_j f\|_{L^{p_1}(\Omega_f)}\|g\|_{L^{p_2}(\Omega_f)} + \|f\|_{L^{q_1}(\Omega_f)}\|S_j g\|_{L^{q_2}(\Omega_f)}\big), \tag{3.16}$$

for the same range of exponents. Using (3.16) to bound the first term on the right side of (3.14), we obtain

$$\begin{aligned}&|(S_j\big((u\cdot\nabla)u\big), S_j u)_f| \le C\|S_j(u\cdot\nabla)u\|_{L^{3/2}(\Omega_f)}\|S_j u\|_{L^3(\Omega_f)} \\ &\le C\big(\|S_j u\|_{L^6(\Omega_f)}\|\nabla u\|_{L^2(\Omega_f)} + \|u\|_{L^6(\Omega_f)}\|\nabla S_j u\|_{L^2(\Omega_f)}\big)\|S_j u\|_{L^3(\Omega_f)} \\ &\le C\|\nabla u\|_{L^2(\Omega_f)}\|S_j u\|^{1/2}_{L^2(\Omega_f)}\|\nabla S_j u\|^{3/2}_{L^2(\Omega_f)},\end{aligned}\tag{3.17}$$

where we used Hölder's inequality and the Sobolev embedding inequalities. Similarly, we have

$$\begin{aligned}&|\langle S_j((u\cdot\nu)u), S_j u\rangle| \le C\|S_j\big((u\cdot\nu)u\big)\|_{L^{3/2}(\Gamma_c)}\|S_j u\|_{L^3(\Gamma_c)} \\ &\le C\|u\|_{L^4(\Gamma_c)}\|S_j u\|_{L^{12/5}(\Gamma_c)}\|S_j u\|_{L^3(\Gamma_c)} \\ &\le C\|u\|_{H^{1/2}(\Gamma_c)}\|S_j u\|_{H^{1/6}(\Gamma_c)}\|S_j u\|_{H^{1/3}(\Gamma_c)} \\ &\le C\|\nabla u\|_{L^2(\Omega_f)}\|S_j u\|_{H^{2/3}(\Omega_f)}\|S_j u\|_{H^{5/6}(\Omega_f)} \\ &\le C\|\nabla u\|_{L^2(\Omega_f)}\|S_j u\|^{1/2}_{L^2(\Omega_f)}\|S_j \nabla u\|^{3/2}_{L^2(\Omega_f)},\end{aligned}\tag{3.18}$$

where we applied the two-dimensional Sobolev embedding theorem, the trace theorem, the interpolation inequalities, and (3.16). Using (3.17) and (3.18) in (3.14), we obtain

$$\frac{1}{2}y'(t) + \sum_{j=1}^{2} \|\nabla S_j u(t)\|^2_{L^2(\Omega_f)} \leq C\|\nabla u\|_{L^2(\Omega_f)} \|S_j u\|^{1/2}_{L^2(\Omega_f)} \|\nabla S_j u\|^{3/2}_{L^2(\Omega_f)}$$
$$\leq \frac{1}{2}\sum_{j=1}^{2} \|\nabla S_j u(t)\|^2_{L^2(\Omega_f)} + \|S_j u\|^2_{L^2(\Omega_f)} \|\nabla u\|^4_{L^2(\Omega_f)}.$$

Finally, we absorb the first term on the right into the left side and bound the term $\|S_j u\|^2_{L^2(\Omega_f)}$ by $y(t)$ to get (3.10). □

3.3. **Estimates on the nonlinear term.** In the next lemma, we establish a crucial anisotropic estimate on the nonlinear interior term, which we shall use in the next section to derive an a priori estimate for the time derivative u_t.

Lemma 3.4. *The nonlinear term $(u \cdot \nabla)u$ satisfies*

$$\int_0^t \|(u\cdot\nabla)u\|^2_{H^{-1/2+k}(\Omega_f)}\, ds$$
$$\leq C \sum_{j=1}^{2} \int_0^t \left(\|\nabla S_j u\|^2_{L^2(\Omega_f)} + \|\nabla u\|^4_{L^2(\Omega_f)} \|S_j u\|^2_{L^2(\Omega_f)} \right) ds, \tag{3.19}$$

provided the right side is finite.

Proof. Let ϕ be a function in $L^2([0,T]; H^{1/2-k}(\Omega_f))$. Then

$$\int_0^t ((u\cdot\nabla)u, \phi)_f \, ds$$
$$= \sum_{j=1}^{2}\sum_{l=1}^{3} \int_0^t \int_{\Omega_f} u_j \partial_j u_l \phi_l \, dx\, ds + \sum_{l=1}^{3} \int_0^t \int_{\Omega_f} u_3 \partial_3 u_l \phi_l \, dx\, ds$$
$$\leq \sum_{j=1}^{2} \int_0^t \|u\|_{L^6(\Omega_f)} \|\partial_j u\|_{L^{6/(3-2k)}(\Omega_f)} \|\phi\|_{L^{3/(1+k)}(\Omega_f)}\, ds$$
$$+ \int_0^t \|u_3\|_{L^{6/(1-2k)}} \|\nabla u\|_{L^2(\Omega_f)} \|\phi\|_{L^{3/(1+k)}(\Omega_f)}\, ds.$$

Applying Sobolev embedding theorems in 2 and 3 space dimensions and interpolating, we get

$$\left| \int_0^t ((u\cdot\nabla)u, \phi)_f \, ds \right| \leq C \sum_{j=1}^{2} \int_0^t \|\nabla u\|_{L^2(\Omega_f)} \|S_j u\|_{H^{1/2}(\Omega_f)} \|\phi\|_{H^{1/2-k}(\Omega_f)}\, ds$$

$$+ C \int_0^t \|u_3\|_{H^{1+k}(\Omega_f)} \|\nabla u\|_{L^2(\Omega_f)} \|\phi\|_{H^{1/2-k}(\Omega_f)} \, ds. \tag{3.20}$$

We now use the divergence-free condition $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$ to estimate

$$\begin{aligned} \|u_3\|_{H^{1+k}(\Omega_f)} &\leq C \sum_{j=1}^{2} \|\partial_j u_3\|_{H^k(\Omega_f)} + C\|\partial_3 u_3\|_{H^k(\Omega_f)} \\ &\leq C \sum_{j=1}^{2} \|\partial_j u\|_{H^k(\Omega_f)} \leq C \sum_{j=1}^{2} \|S_j u\|_{H^{1/2}(\Omega_f)}. \end{aligned}$$

Therefore, the interpolation inequalities and Hölder's inequality give

$$\begin{aligned} &\Big| \int_0^t ((u \cdot \nabla) u, \phi)_f \, ds \Big| \leq C \sum_{j=1}^{2} \int_0^t \|\nabla u\|_{L^2(\Omega_f)} \|S_j u\|_{H^{1/2}(\Omega_f)} \|\phi\|_{H^{1/2-k}(\Omega_f)} \, ds \\ &\leq C \Big(\sum_{j=1}^{2} \int_0^t \|\nabla u\|_{L^2(\Omega_f)}^2 \|S_j u\|_{L^2(\Omega_f)} \|\nabla S_j u\|_{L^2(\Omega_f)} \, ds \Big)^{1/2} \\ &\qquad \times \Big(\int_0^t \|\phi\|_{H^{1/2-k}(\Omega_f)}^2 \, ds \Big)^{1/2} \\ &\leq C \Big(\sum_{j=1}^{2} \int_0^t \Big(\|\nabla S_j u\|_{L^2(\Omega_f)}^2 + \|\nabla u\|_{L^2(\Omega_f)}^4 \|S_j u\|_{L^2(\Omega_f)}^2 \Big) \, ds \Big)^{1/2} \\ &\qquad \times \Big(\int_0^t \|\phi\|_{H^{1/2-k}(\Omega_f)}^2 \, ds \Big)^{1/2}. \end{aligned} \tag{3.21}$$

The inequality (3.19) then follows by taking the supremum over all ϕ of unit norm in $L^2([0,T]; H^{1/2-k}(\Omega_f))$. □

3.4. **Existence of pressure.** Recall that u satisfies the variational equation

$$(u_t, \phi)_f + (\nabla u, \nabla \phi)_f + ((u \cdot \nabla) u, \phi)_f + \Big\langle \frac{\partial w}{\partial \nu}, \phi \Big\rangle + \frac{1}{2} \langle (u \cdot \nu) u, \phi \rangle = 0, \tag{3.22}$$

for every $\phi \in V$. Therefore,

$$(u_t, \phi)_f + (\nabla u, \nabla \phi)_f + ((u \cdot \nabla) u, \phi)_f = 0, \tag{3.23}$$

for every $\phi \in \widetilde{V} = H_0^1(\Omega_f) \cap V$. By De Rham's theorem, for every fixed $t \in (0, \widetilde{T})$ there exists a distribution $p \in \mathcal{D}'(\Omega_f)$ such that

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0 \text{ in } \Omega_f, \tag{3.24}$$

in $\mathcal{D}'(\Omega_f)$. Therefore, u and p can be viewed as solutions of the Stokes system

$$-\Delta u + \nabla p = -u_t - (u\cdot\nabla)u \text{ in } \Omega_f \times (0,T) \tag{3.25}$$

$$\operatorname{div} u = 0 \text{ in } \Omega_f \times (0,T) \tag{3.26}$$

$$u = w_t \text{ in } \Gamma_c \times (0,T) \tag{3.27}$$

$$u = 0 \text{ in } \Gamma_f \times (0,T), \tag{3.28}$$

almost everywhere in time.

Lemma 3.5. *The functions u and p satisfy an a priori estimate*

$$\begin{aligned}
&\int_0^T \|u\|^2_{H^{3/2+k}(\Omega_f)}\,ds + \int_0^T \|\nabla p\|^2_{H^{-1/2+k}(\Omega_f)}\,ds \\
&\le C\Bigl(\int_0^T \|(u\cdot\nabla)u\|^2_{H^{-1/2+k}(\Omega_f)}\,ds + \int_0^T \|u_t\|^2_{L^2(\Omega_f)}\,ds \\
&\qquad + \sum_{j=1}^2 \int_0^T \|\nabla S_j u\|^2_{L^2(\Omega_f)}\,ds\Bigr).
\end{aligned} \tag{3.29}$$

Proof. By [26, Proposition 2.3 and Remark 2.6], the left side of (3.29) is less than or equal to

$$C\Bigl(\int_0^T \|(u\cdot\nabla)u\|^2_{H^{-1/2+k}(\Omega_f)}\,ds + \int_0^T \|u_t\|^2_{L^2(\Omega_f)}\,ds + \int_0^T \|u\|^2_{H^{1+k}(\Gamma_c)}\,ds\Bigr),$$

and the inequality (3.29) follows. □

Combining (3.19) and (3.29), we obtain

$$\begin{aligned}
&\int_0^T \|u\|^2_{H^{3/2+k}(\Omega_f)}\,ds + \int_0^T \|\nabla p\|^2_{H^{-1/2+k}(\Omega_f)}\,ds \\
&\le C\Bigl(\int_0^T \|u_t\|^2_{L^2(\Omega_f)}\,ds + \sum_{j=1}^2 \int_0^T \|\nabla S_j u\|^2_{L^2(\Omega_f)}\,ds \\
&\qquad + \sum_{j=1}^2 \int_0^T \|\nabla u\|^4_{L^2(\Omega_f)} \|S_j u\|^2_{L^2(\Omega_f)}\,ds\Bigr).
\end{aligned} \tag{3.30}$$

3.5. **Regularity of the time derivative u_t from a priori estimates.** Our strategy is to obtain a priori estimates on the $L^2([0,T];L^2(\Omega_f))$ norm of u_t and the $L^\infty([0,T];L^2(\Omega_f))$ norm of ∇u simultaneously, and then couple the estimates with the inequalities for the tangential derivatives. First, we state and prove the following three auxiliary results.

Lemma 3.6. *If $u \in H^1(\Omega_f \times [0,T])$, then $u|_{\Gamma_c \times [0,T]}$ belongs to the space $H^{1/2}(\Gamma_c \times [0,T])$ and we have*

$$\|u\|^2_{H^{1/2}(\Gamma_c\times[0,T])} \le \frac{C}{\epsilon}\|u\|^2_{L^2([0,T];H^1(\Omega_f))} + \epsilon\|u\|^2_{H^1([0,T];L^2(\Omega_f))}, \tag{3.31}$$

for any $\epsilon \in (0,1)$.

Proof. The continuity of the map from $H^1(\Omega_f \times [0,T])$ into $H^{1/2}(\Gamma_c \times [0,T])$ is a known result (cf. [22, page 9]). However, we provide a proof to demonstrate the estimate (3.31). Let $U \in H^1(\mathbb{R}^3 \times \mathbb{R})$ be a compactly supported Sobolev extension of u in space and time to $\mathbb{R}^3 \times \mathbb{R}$. Then

$$\begin{aligned} &\|u\|^2_{H^{1/2}(\Gamma_c\times[0,T])} \\ &\le C\sum_{j=1}^{2}\int_{\infty}^{\infty}\int_{\infty}^{\infty}\int_{\infty}^{\infty}(1+\xi_1^2+\xi_2^2+\tau^2)^{1/2}|\widetilde{U}(\xi_1,\xi_2,h_j,\tau)|^2\,d\tau\,d\xi_1\,d\xi_2, \end{aligned} \tag{3.32}$$

where $\widetilde{U}$ is the Fourier transform of U in the x_1, x_2, and t variables. We now express $\widetilde{U}$ in terms of the full Fourier transform $\hat{U}$ of U in all variables x_1, x_2, x_3, and t. Then, applying Hölder's inequality we obtain

$$\begin{aligned} &\|u\|^2_{H^{\frac12}(\Gamma_c\times[0,T])} \\ &\le C\int_{\mathbb{R}^3}(1+\xi_1^2+\xi_2^2+\tau^2)^{\frac12}\Big|\int_{-\infty}^{\infty}\hat{U}(\xi_1,\xi_2,\xi_3,\tau)e^{ih_j\xi_3}\,d\xi_3\Big|^2\,d\tau\,d\xi_1\,d\xi_2 \\ &\le C\int_{\mathbb{R}^3}(1+\xi_1^2+\xi_2^2+\tau^2)^{\frac12}\Big(\int_{-\infty}^{\infty}\frac{\sqrt{1+\xi_1^2+\xi_2^2+\epsilon^{-2}\xi_3^2+\tau^2}}{\sqrt{1+\xi_1^2+\xi_2^2+\epsilon^{-2}\xi_3^2+\tau^2}}|\hat{U}|d\xi_3\Big)^2 d\tau d\xi_1 d\xi_2 \\ &\le C\int_{\mathbb{R}^3}\Big(\int_{-\infty}^{\infty}\big(1+\xi_1^2+\xi_2^2+\epsilon^{-2}\xi_3^2+\tau^2\big)\,|\hat{U}|^2\,d\xi_3\Big) \\ &\qquad\times\Big(\int_{-\infty}^{\infty}\frac{(1+\xi_1^2+\xi_2^2+\tau^2)^{1/2}}{1+\xi_1^2+\xi_2^2+\epsilon^{-2}\xi_3^2+\tau^2}\,d\xi_3\Big)d\tau\,d\xi_1\,d\xi_2. \end{aligned} \tag{3.33}$$

Since the second integral in ξ_3 equals $\big[\epsilon\arctan\big(\xi_3/\epsilon\sqrt{1+\xi_1^2+\xi_2^2+\tau^2}\big)\big]_{-\infty}^{\infty}$ $= \pi\epsilon$, we have

$$\begin{aligned} \|u\|^2_{H^{1/2}(\Gamma_c\times[0,T])} &\le C\epsilon\int_{\mathbb{R}^4}\big(1+\xi_1^2+\xi_2^2+\epsilon^{-2}\xi_3^2+\tau^2\big)\,|\hat{U}|^2\,d\xi_3\,d\xi_2\,d\xi_1\,d\tau \\ &\le C\epsilon\int_{\mathbb{R}^4}\big(1+\epsilon^{-2}(\xi_1^2+\xi_2^2+\xi_3^2)+\tau^2\big)\,|\hat{U}|^2\,d\xi_3\,d\xi_2\,d\xi_1\,d\tau \\ &\le \frac{C}{\epsilon}\|U\|^2_{L^2([0,T];H^1(\Omega_f))} + C\epsilon\|U\|^2_{H^1([0,T];L^2(\Omega_f))} \end{aligned}$$

$$\leq \frac{C}{\epsilon}\|u\|^2_{L^2([0,T];H^1(\Omega_f))} + C\epsilon\|u\|^2_{H^1([0,T];L^2(\Omega_f))}, \tag{3.34}$$

where the last step follows by continuity of the extension map. Therefore, the estimate in (3.31) is verified. □

Lemma 3.7. *For any $k \in (0, (\sqrt{2}-1)/2)$, we have*

$$\|u\|^2_{H^{1/2+k}(\Gamma_c\times[0,T])} \leq C(\|u\|^2_{H^1([0,T],L^2(\Omega_f))} + \|u\|^2_{L^2([0,T];H^{3/2+k}(\Omega_f))}), \tag{3.35}$$

where the constant C depends on k.

Proof. Let $\epsilon \in (0,1)$ be arbitrary. Using the standard trace inequalities, we have

$$\begin{aligned}\|u\|^2_{H^{1/2+k}(\Gamma_c\times[0,T])} &\leq C(\|u\|^2_{L^2([0,T];H^{1/2+k}(\Gamma_c))} + \|u\|^2_{H^{1/2+k}([0,T];L^2(\Gamma_c))}) \\ &\leq C(\|u\|^2_{L^2([0,T];H^{1+k}(\Omega_f))} + \|u\|^2_{H^{1/2+k}([0,T];H^{1/2+\epsilon}(\Omega_f))}) \\ &\leq C(\|u\|^2_{L^2([0,T];H^{3/2+k}(\Omega_f))} + \|u\|^2_{H^{1/2+k}([0,T];H^{1/2+\epsilon}(\Omega_f))}).\end{aligned}$$

We again use the Sobolev extension U of u to $\mathbb{R}^4$ to estimate

$$\|u\|^2_{H^{1/2+k}([0,T];H^{1/2+\epsilon}(\Omega_f))},$$

using the Fourier transform and Hölder's inequality. We get

$$\begin{aligned}&\|U\|^2_{H^{1/2+k}([0,T];H^{1/2+\epsilon}(\Omega_f))} \\ &\leq \int_{\mathbb{R}^4} \left(\sqrt{1+|\xi|^2}\right)^{1+2\epsilon}\left(\sqrt{1+\tau^2}\right)^{1+2k}|\hat{U}|^2\,d\tau\,d\xi_1\,d\xi_2 d\xi_3 \\ &\qquad \leq C\Big(\int_{\mathbb{R}^4}(1+|\xi|^2)^{(1+2\epsilon)/(1-2k)}|\hat{U}|^2\,d\tau\,d\xi_1\,d\xi_2 d\xi_3\Big)^{1/2-k} \\ &\qquad\qquad \times \Big(\int_{\mathbb{R}^4}(1+\tau^2)|\hat{U}|^2\,d\tau\,d\xi_1\,d\xi_2 d\xi_3\Big)^{1/2+k} \\ &\qquad \leq C(\|u\|^2_{H^1([0,T];L^2(\Omega_f))} + \|u\|^2_{L^2([0,T];H^s(\Omega_f))}),\end{aligned}$$

where $s > 1/(1-2k)$ is arbitrary and C depends on k and s. Setting $s = 3/2+k$, which is possible since k satisfies the quadratic inequality $k^2+k-1/4<0$, we get (3.35). □

Lemma 3.8. *Assume that u is a divergence-free vector field defined on $\mathbb{R}^3 \times [0,1]$ which is 2π-periodic in x_1 and x_2. Then*

$$\|u_3\|_{H^k(\Gamma)} \leq C\sum_{j=1}^{2}\|S_j u\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)}, \tag{3.36}$$

where $\Omega = [0, 2\pi] \times [0, 2\pi] \times [0, 1]$ *and* $\Gamma = [0, 2\pi] \times [0, 2\pi] \times \{0\}$.

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^3, [0, 1])$ be a function such that $\eta \equiv 1$ in a 1/4-neighborhood of Γ with the support included in a 1/2-neighborhood of Γ. Also, let Eu be the standard extension map such that for $x_3 < 0$

$$Eu(x_1, x_2, x_3) = \sum_{m=1}^{3} a_m u\left(x_1, x_2, -\frac{x_3}{m}\right),$$

where $(a_1, a_2, a_3) = (6, -32, 27)$ (cf. [2]). Denote $U = E(\eta u)$. Then U is compactly supported, and we may freely use the Fourier transform $\hat{U}$ in all three variables. By the trace theorem, we have

$$\|u_3\|^2_{H^k(\Gamma)} \le \|U_3\|^2_{H^k(\Gamma)} \le C\|U_3\|^2_{H^{k+1/2}(\mathbb{R}^3)} \le C \int_{\mathbb{R}^3} (|\xi|^2 + 1)^{k+1/2} |\hat{U}_3|^2 \, d\xi$$
$$\le C \int_{\mathbb{R}^3} (|\xi_1|^2 + |\xi_2|^2 + 1)^{k+1/2} |\hat{U}_3|^2 \, d\xi + C \int_{\mathbb{R}^3} |\xi_3|^{2k+1} |\hat{U}_3|^2 \, d\xi = I_1 + I_2.$$

Using the continuity of the extension map, we obtain

$$I_1 \le C \sum_{j=1}^{2} \|S_j u\|^2_{L^2(\Omega)} + C\|u\|^2_{L^2(\Omega)}.$$

Next, we treat I_2. For $x_3 > 0$, we have

$$\partial_3(E(\eta u_3)) = \partial_3(\eta u_3) = u_3 \partial_3 \eta + \eta \partial_3 u_3 = u_3 \partial_3 \eta - \eta \partial_1 u_1 - \eta \partial_2 u_2$$

while, for $x_3 < 0$,

$$\partial_3(E(\eta u_3)) = \partial_3 \eta E u_3 - \eta \sum_{m=1}^{3} \frac{a_m}{m} \partial_3 u_3 \left(x_1, x_2, -\frac{x_3}{m}\right)$$
$$= \partial_3 \eta E u_3 + \eta \sum_{j=1}^{2} \sum_{m=1}^{3} \frac{a_m}{m} \partial_j u_j \left(x_1, x_2, -\frac{x_3}{m}\right) = \partial_3 \eta E u_3 - \eta \sum_{j=1}^{2} \widetilde{E}(\partial_j u_j)$$
$$= \partial_3 \eta(x_1, x_2, x_3) E u_3 + \partial_j \eta(x_1, x_2, x_3) \sum_{j=1}^{2} \widetilde{E}(u_j) - \sum_{j=1}^{2} \partial_j \big(\eta(x_1, x_2, x_3) \widetilde{E}(u_j)\big),$$

where $\widetilde{E}u(x_1, x_2, x_3) = u(x_1, x_2, x_3)$ if $x_3 \ge 0$ and

$$\widetilde{E}u(x_1, x_2, x_3) = -\sum_{m=1}^{3} \frac{a_m}{m} u\left(x_1, x_2, -\frac{x_3}{m}\right),$$

otherwise. Therefore,

$$
\begin{aligned}
I_2 = C\int |\xi_3|^{1+2k}|\hat{U}_3|^2\,d\xi &= C\int |\xi_3\hat{U}_3|^{1+2k}|\hat{U}_3|^{1-2k}\,d\xi\\
&\le C\int \bigl|(\partial_3\eta E(u_3))\hat{}\bigr|^{1+2k}|\hat{U}_3|^{1-2k}\,d\xi + C\sum_{j=1}^{2}\int \bigl|(\partial_j\eta\widetilde{E}(u_j))\hat{}\bigr|^{1+2k}|\hat{U}_3|^{1-2k}\,d\xi\\
&\qquad + C\sum_{j=1}^{2}\int \bigl|\xi_j(\eta\widetilde{E}(u_j))\hat{}\bigr|^{1+2k}|\hat{U}_3|^{1-2k}\,d\xi.
\end{aligned}
$$

Now, it is easy to check that the first and the second term on the far right are bounded by the square of the second term on the right side of (3.36). As for the third term above, we have

$$
\begin{aligned}
&\int \bigl|\xi_j(\eta\widetilde{E}(u_j))\hat{}\bigr|^{1+2k}|U_3|^{1-2k}\,d\xi\\
&\le \int \Bigl(|\xi_j|^{(1+2k)^2/2}|(\eta\widetilde{E}(u_j))\hat{}|^{1+2k}\Bigr)\Bigl(|\xi_j|^{(1-4k^2)/2}|U_3|^{1-2k}\Bigr)\,d\xi\\
&\qquad \le \sum_{j=1}^{2}\int |\xi_j|^{1+2k}|(\eta\widetilde{E}(u_j))\hat{}|^2\,d\xi + \sum_{j=1}^{2}\int |\xi_j|^{1+2k}|U_3|^2,
\end{aligned}
$$

which is bounded by the square of the first term on the right of (3.36). □

Denote

$$
E_1(0) = \|w_0\|^2_{H^{3/2+k}(\Omega_e)} + \|w_1\|^2_{H^{1/2+k}(\Omega_e)} + \|\nabla u_0\|^2_{L^2(\Omega_f)}. \tag{3.37}
$$

In the next lemma, we state and prove the main result of this section concerning the regularity of the time derivative u_t.

Lemma 3.9. *For all $\epsilon \in (0,1]$ and $t \in [0,T)$, we have*

$$
\begin{aligned}
&\int_0^t \|u_t\|^2_{L^2(\Omega_f)}\,ds + \|\nabla u(t)\|^2_{L^2(\Omega_f)} + \|w_t(t)\|^2_{H^{1/2+k}(\Omega_e)} + \|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)}\\
&\le \epsilon(t+1)\sum_{j=1}^{2}\int_0^t \|\nabla S_j u\|^2_{L^2(\Omega_f)}\,ds + \epsilon\sum_{j=1}^{2}\int_0^t \|\nabla u\|^4_{L^2(\Omega_f)}\|S_j u\|^2_{L^2(\Omega_f)}\,ds\\
&\quad + C\int_0^t \|\nabla u\|^{(1+4k)/k}_{L^2(\Omega_f)}\,ds + C\sum_{j=1}^{2}\|S_j u(t)\|^{4/k}_{L^2(\Omega_f)} + C\|u(t)\|^4_{L^2(\Omega_f)}\\
&\quad + C\int_0^t \|\nabla u\|^2_{L^2(\Omega_f)}\,ds + C\|u(t)\|^2_{L^2(\Omega_f)} + C(E_1(0) + E_1(0)^{3/2}),
\end{aligned} \tag{3.38}
$$

where C depends on ϵ.

Proof. We set $\phi = u_t$ in (3.1) and obtain

$$\|u_t\|^2_{L^2(\Omega_f)} + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(\Omega_f)} + ((u\cdot\nabla)u, u_t)_f + \frac{1}{2}\langle(u\cdot\nu)u, u_t\rangle + \Big\langle\frac{\partial w}{\partial\nu}, u_t\Big\rangle = 0. \tag{3.39}$$

We integrate the equation in time and estimate each of the terms, starting with $\int_0^t\langle\partial w/\partial\nu, u_t\rangle\,ds$. Integrating this term by parts in time we have

$$\int_0^t\Big\langle\frac{\partial w}{\partial\nu}, u_t\Big\rangle\,ds = -\int_0^t\Big\langle\frac{\partial w_t}{\partial\nu}, u\Big\rangle\,ds + \Big\langle\frac{\partial w(t)}{\partial\nu}, u(t)\Big\rangle - \Big\langle\frac{\partial w_0}{\partial\nu}, u_0\Big\rangle. \tag{3.40}$$

Let k be as before. We next estimate the right-hand side of (3.40) to get

$$\begin{aligned}\Big|\int_0^t\Big\langle\frac{\partial w}{\partial\nu}, u_t\Big\rangle\,ds\Big| &\le \Big\|\frac{\partial w_t}{\partial\nu}\Big\|_{H^{-1/2+k}(\Gamma_c\times[0,t])}\|u\|_{H^{1/2-k}(\Gamma_c\times[0,t])}\\ &\quad + \Big\|\frac{\partial w(t)}{\partial\nu}\Big\|_{L^2(\Gamma_c)}\|u(t)\|_{L^2(\Gamma_c)} + \Big\|\frac{\partial w_0}{\partial\nu}\Big\|_{L^2(\Gamma_c)}\|u_0\|_{L^2(\Gamma_c)}.\end{aligned}$$

Applying the standard trace inequalities to the initial conditions as well as the inequality

$$\|u\|_{L^2(\Gamma_c)} \le C\|u\|^{1/2}_{L^2(\Omega_f)}\|u\|^{1/2}_{H^1(\Omega_f)} \le C\|u\|^{1/2}_{L^2(\Omega_f)}\|\nabla u\|^{1/2}_{L^2(\Omega_f)},$$

we obtain

$$\begin{aligned}\Big|\int_0^t\Big\langle\frac{\partial w}{\partial\nu}, u_t\Big\rangle\,ds\Big| &\le \epsilon_1\Big\|\frac{\partial w_t}{\partial\nu}\Big\|^2_{H^{-1/2+k}(\Gamma_c\times[0,t])} + \frac{C}{\epsilon_1}\|u\|^2_{H^{1/2-k}(\Gamma_c\times[0,t])}\\ &\quad + C\|w(t)\|_{H^{3/2+k}(\Omega_e)}\|u(t)\|^{1/2}_{L^2(\Omega_f)}\|\nabla u(t)\|^{1/2}_{L^2(\Omega_f)}\\ &\quad + C(\|w_0\|^2_{H^{3/2+k}(\Omega_e)} + \|\nabla u_0\|^2_{L^2(\Omega_f)}),\end{aligned}$$

for some $\epsilon_1 \in (0,1]$ to be determined. Now, we appeal to the hidden regularity result in Theorem 3.2. Namely, we have

$$\begin{aligned}&\Big\|\frac{\partial w_t}{\partial\nu}\Big\|^2_{H^{-1/2+k}(\Gamma_c\times[0,T])}\\ &\le C(\|w_1\|^2_{H^{1/2+k}(\Omega_e)} + \|w_{tt}(\cdot,0)\|^2_{H^{-1/2+k}(\Omega_e)} + \|u\|^2_{H^{1/2+k}(\Gamma_c\times[0,T])})\\ &\qquad - \|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)} - \|w_t(t)\|^2_{H^{1/2+k}(\Omega_e)},\end{aligned} \tag{3.41}$$

which follows from (3.9) applied to w_t instead of w with $\alpha = 1/2 + k$, $f = w_t|_{\Gamma_c} = u|_{\Gamma_c}$ and with initial conditions $w_t(\cdot,0) = w_1 \in H^{1/2+k}(\Omega_e)$ and

$w_{tt}(\cdot,0) = \Delta w_0 \in H^{-1/2+k}(\Omega_e)$. Consequently,

$$\begin{aligned}\Big|\int_0^t \Big\langle \frac{\partial w}{\partial \nu}, u_t \Big\rangle\, ds\Big| &\le C\epsilon_1 \|u\|^2_{H^{1/2+k}(\Gamma_c\times[0,t])} + \frac{C}{\epsilon_1}\|u\|^2_{H^{1/2-k}(\Gamma_c\times[0,t])} \\ &+ \frac{1}{8}\|\nabla u(t)\|^2_{L^2(\Omega_f)} + \frac{C}{\delta^2}\|u(t)\|^2_{L^2(\Omega_f)} + \delta\|w(t)\|^2_{H^{3/2+k}(\Omega_e)} \\ &- \epsilon_1\|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)} - \epsilon_1\|w_t(t)\|^2_{H^{1/2+k}(\Omega_e)} + CE_1(0), \end{aligned} \tag{3.42}$$

where $\delta \in (0,1]$ is to be chosen. Applying (3.35) on the first term on the right of (3.42) and (3.31) on the second, we get

$$\begin{aligned}\Big|\int_0^t \Big\langle \frac{\partial w}{\partial \nu}, u_t \Big\rangle\, ds\Big| &\le C\epsilon_1 \|u_t\|^2_{L^2([0,t];L^2(\Omega_f))} + C\epsilon_1\|u\|^2_{L^2([0,t];H^{3/2+k}(\Omega_f))} \\ &+ \frac{C}{\epsilon_1^3}\|u\|^2_{L^2([0,t];H^1(\Omega_f))} + \frac{1}{8}\|\nabla u(t)\|^2_{L^2(\Omega_f)} + \frac{C}{\delta^2}\|u(t)\|^2_{L^2(\Omega_f)} \\ &+\delta\|w(t)\|^2_{H^{3/2+k}(\Omega_e)} - \epsilon_1\|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)} - \epsilon_1\|w_t(t)\|^2_{H^{1/2+k}(\Omega_e)} + CE_1(0), \end{aligned} \tag{3.43}$$

for $\epsilon_1, \delta \in (0,1]$ to be determined. In order to estimate the norm of w above, we recall that w satisfies the elliptic problem

$$\Delta w = w_{tt} \text{ in } \Omega_e \times (0,T),$$

with the boundary condition

$$w|_{\Gamma_c} = \int_0^t u\, ds + w_0 \in C([0,T]; H^{3/2+k}(\Gamma_c)),$$

for almost every $t < T$. Hence, by the available theory for elliptic boundary value problems (cf. [5, Theorem 3.8.1]), w satisfies

$$\begin{aligned}\|w(t)\|^2_{H^{3/2+k}(\Omega_e)} &\le C(\|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)} + \|w(t)\|^2_{H^{1+k}(\Gamma_c)}) \\ &\le C(\|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)} + t\|u\|^2_{L^2([0,t];H^{1+k}(\Gamma_c))} + \|w_0\|^2_{H^{3/2+k}(\Omega_e)}) \\ &\le C\|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)} + Ct\sum_{j=1}^{2}\|\nabla S_j u\|^2_{L^2([0,t];L^2(\Omega_f))} + CE_1(0), \end{aligned} \tag{3.44}$$

where the last step follows from the trace theorem. Therefore, the estimate in (3.43) becomes

$$\begin{aligned}\Big|\int_0^t \Big\langle \frac{\partial w}{\partial \nu}, u_t \Big\rangle\, ds\Big| &\le C\epsilon_1 \|u_t\|^2_{L^2([0,t];L^2(\Omega_f))} + C\epsilon_1\|u\|^2_{L^2([0,t];H^{3/2+k}(\Omega_f))} \\ &+ \frac{C}{\epsilon_1^3}\int_0^t \|\nabla u\|^2_{L^2(\Omega_f)}\, ds + \frac{1}{8}\|\nabla u(t)\|^2_{L^2(\Omega_f)} + \frac{C}{\delta^2}\|u(t)\|^2_{L^2(\Omega_f)}\end{aligned}$$

$$+ C\delta t \sum_{j=1}^{2} \int_0^t \|\nabla S_j u\|_{L^2(\Omega_f)}^2 \, ds - (\epsilon_1 - C\delta)\|w_{tt}(t)\|_{H^{-1/2+k}(\Omega_e)}^2$$
$$- \epsilon_1 \|w_t(t)\|_{H^{1/2+k}(\Omega_e)}^2 + CE_1(0). \tag{3.45}$$

We now return to (3.39) and treat the nonlinear interior term next. Namely,

$$\begin{aligned}
&\Big| \int_0^t ((u\cdot\nabla)u, u_t)_f \, ds \Big| \le C \int_0^t \|u\|_{L^6(\Omega_f)} \|\nabla u\|_{L^3(\Omega_f)} \|u_t\|_{L^2(\Omega_f)} \, ds \\
&\le \frac{C}{\epsilon_1} \int_0^t \|\nabla u\|_{L^2(\Omega_f)}^2 \|u\|_{H^{3/2}(\Omega_f)}^2 \, ds + \epsilon_1 \|u_t\|_{L^2([0,T];L^2(\Omega_f))}^2 \\
&\le \frac{C}{\epsilon_1} \int_0^t \|\nabla u\|_{L^2(\Omega_f)}^{(8k+2)/(2k+1)} \|u\|_{H^{3/2+k}(\Omega_f)}^{2/(1+2k)} \, ds + \epsilon_1 \|u_t\|_{L^2([0,T];L^2(\Omega_f))}^2 \qquad (3.46) \\
&\le C_{\epsilon_1,\delta} \int_0^t \|\nabla u\|_{L^2(\Omega_f)}^{(1+4k)/k} \, ds + \delta \int_0^t \|u\|_{H^{3/2+k}(\Omega_f)}^2 \, ds + \epsilon_1 \|u_t\|_{L^2([0,T];L^2(\Omega_f))}^2.
\end{aligned}$$

Next, we estimate the nonlinear boundary term $\int_0^t \langle (u\cdot\nu)u, u_t\rangle \, ds$ by first integrating it by parts with respect to time obtaining

$$\begin{aligned}
\int_0^t \langle (u\cdot\nu)u, u_t\rangle \, ds &= \frac{1}{2}\sum_{k=1}^{3} \int_0^t \int_{\Gamma_c} u_3 \frac{\partial u_k^2}{\partial t} \, d\sigma(x)\, ds \\
&= -\frac{1}{2}\sum_{k=1}^{3} \int_0^t \int_{\Gamma_c} \frac{\partial u_3}{\partial t} u_k^2 \, d\sigma(x)\, ds + \frac{1}{2}\sum_{k=1}^{3} \int_{\Gamma_c} u_3(x,t) u_k(x,t)^2 \, d\sigma(x) \\
&\qquad - \frac{1}{2}\sum_{k=1}^{3} \int_{\Gamma_c} u_3(x,0) u_k(x,0)^2 \, d\sigma(x).
\end{aligned}$$

Estimating each of these terms via Hölder, we get

$$\Big| \int_0^t \langle (u\cdot\nu)u, u_t\rangle \, ds \Big| \le C \int_0^t \|\partial_t u_3\|_{H^{-1/2}(\Gamma_c)} \||u|^2\|_{H^{1/2}(\Gamma_c)} \, ds \tag{3.47}$$
$$+ C\|u_3(t)\|_{L^{2/(1-k)}(\Gamma_c)} \|u(t)\|_{L^{4/(1+k)}(\Gamma_c)}^2 + C\|u_0\cdot\nu\|_{L^4(\Gamma_c)} \|u_0\|_{L^{8/3}(\Gamma_c)}^2.$$

In order to estimate the first factor $\|\partial_t u_3\|_{H^{-\frac{1}{2}}(\Gamma_c)}$, we use the trace theorem for divergence-free $L^2(\Omega_f)$ functions, and to deal with the factor $\||u|^2\|_{H^{\frac{1}{2}}(\Gamma_c)}$ we use

$$\||u|^2\|_{H^{\frac{1}{2}}(\Gamma_c)} \le C\||u|^2\|_{W^{1,4/3}(\Gamma_c)} \le C\sum_{i=1}^{2} \big\| |\partial_i u||u| \big\|_{L^{4/3}(\Gamma_c)} + C\||u|^2\|_{L^{\frac{4}{3}}(\Gamma_c)}$$

$$\leq C\|u\|_{H^1(\Gamma_c)}\|u\|_{L^4(\Gamma_c)} \leq C\|u\|_{H^1(\Gamma_c)}\|\nabla u\|_{H^{\frac{1}{2}}(\Gamma_c)} \leq C\|u\|_{H^{\frac{3}{2}}(\Omega_f)}\|\nabla u\|_{L^2(\Omega_f)}.$$

Therefore, (3.47) becomes

$$\begin{aligned}\Big|\int_0^t \langle (u\cdot\nu)u, u_t\rangle\, ds\Big| &\leq C\int_0^t \|u_t\|_{L^2(\Omega_f)}\|u\|_{H^{3/2}(\Omega_f)}\|\nabla u\|_{L^2(\Omega_f)}\, ds \\ &+ C\|u_3(t)\|_{H^k(\Gamma_c)}\|u(t)\|^2_{H^{(1-k)/2}(\Gamma_c)} + C\|u_0\|_{H^{1/2}(\Gamma_c)}\|u_0\|^2_{H^{1/4}(\Gamma_c)} \\ &\leq \epsilon_1\int_0^t \|u_t\|^2_{L^2(\Omega_f)} + \frac{C}{\epsilon_1}\int_0^t \|u\|^2_{H^{3/2}(\Omega_f)}\|\nabla u\|^2_{L^2(\Omega_f)}\, ds \\ &\quad + C\|u_3(t)\|_{H^k(\Gamma_c)}\|u(t)\|^2_{H^{1-k/2}(\Omega_f)} + C\|\nabla u_0\|^3_{L^2(\Omega_f)}.\end{aligned}$$

Using Lemma 3.8 and interpolation inequalities, we obtain

$$\begin{aligned}&\Big|\int_0^t \langle (u\cdot\nu)u, u_t\rangle\, ds\Big| \\ &\leq \epsilon_1\int_0^t \|u_t\|^2_{L^2(\Omega_f)} + \frac{C}{\epsilon_1}\int_0^t \|\nabla u\|^{(8k+2)/(2k+1)}_{L^2(\Omega_f)}\|u\|^{2/(2k+1)}_{H^{3/2+k}(\Omega_f)}\, ds \\ &\qquad + C\sum_{j=1}^2 \|S_j u(t)\|_{L^2(\Omega_f)}\|\nabla u(t)\|^{2-k}_{L^2(\Omega_f)}\|u(t)\|^k_{L^2(\Omega_f)} \\ &\qquad + C\|u(t)\|_{L^2(\Omega_f)}\|\nabla u(t)\|^{2-k}_{L^2(\Omega_f)}\|u(t)\|^k_{L^2(\Omega_f)} + C\|\nabla u_0\|^3_{L^2(\Omega_f)}.\end{aligned}$$

Now, applying Young's inequality once more we get

$$\begin{aligned}\Big|\int_0^t \langle (u\cdot\nu)u, u_t\rangle\, ds\Big| &\leq \epsilon_1\|u_t\|^2_{L^2([0,T];L^2(\Omega_f))} + \delta\int_0^t \|u\|^2_{H^{3/2+k}(\Omega_f)}\, ds \\ &\quad + C_{\epsilon_1,\delta}\int_0^t \|\nabla u\|^{(1+4k)/k}_{L^2(\Omega_f)}\, ds + C\sum_{j=1}^2 \|S_j u(t)\|^{4/k}_{L^2(\Omega_f)} \\ &\quad + \frac{1}{8}\|\nabla u(t)\|^2_{L^2(\Omega_f)} + C\|u\|^4_{L^2(\Omega_f)} + C\|\nabla u_0\|^3_{L^2(\Omega_f)}. \qquad (3.48)\end{aligned}$$

We are now ready to complete the estimates in (3.39). Indeed, integrating (3.39) in time and collecting the bounds in (3.45), (3.46), and (3.48), we get

$$\begin{aligned}&(1 - C\epsilon_1)\int_0^t \|u_t\|^2_{L^2(\Omega_f)}\, ds + \frac{1}{4}\|\nabla u(t)\|^2_{L^2(\Omega_f)} + \epsilon_1\|w_t(t)\|^2_{H^{1/2+k}(\Omega_e)} \\ &\quad + (\epsilon_1 - C\delta)\|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)} \\ &\leq C\epsilon_1\int_0^t \|u_t\|^2_{L^2(\Omega_f)}\, ds + C(\delta+\epsilon_1)\int_0^t \|u\|^2_{H^{\frac{3}{2}+k}(\Omega_f)} ds + C_{\epsilon_1}\int_0^t \|\nabla u\|^2_{L^2(\Omega_f)} ds\end{aligned}$$

$$+\frac{C}{\delta^2}\|u(t)\|^2_{L^2(\Omega_f)}+C\delta t\sum_{j=1}^{2}\int_0^t\|\nabla S_j u\|^2_{L^2(\Omega_f)}\,ds+C_{\epsilon_1,\delta}\int_0^t\|\nabla u\|^{(1+4k)/k}_{L^2(\Omega_f)}\,ds$$

$$+C\sum_{j=1}^{2}\|S_j u(t)\|^{4/k}_{L^2(\Omega_f)}+C\|u(t)\|^4_{L^2(\Omega_f)}+CE_1(0)+CE_1(0)^{3/2}. \qquad (3.49)$$

Using (3.30), we obtain

$$(1-C\epsilon_1)\int_0^t\|u_t\|^2_{L^2(\Omega_f)}\,ds+\frac{1}{4}\|\nabla u(t)\|^2_{L^2(\Omega_f)}+\epsilon_1\|w_t(t)\|^2_{H^{1/2+k}(\Omega_e)}$$
$$+(\epsilon_1-C\delta)\|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)}$$
$$\le C(\delta+\epsilon_1)\int_0^t\|u_t\|^2_{L^2(\Omega_f)}\,ds+C(\delta+\epsilon_1)\sum_{j=1}^{2}\int_0^t\|\nabla S_j u\|^2_{L^2(\Omega_f)}\,ds$$
$$+C(\delta+\epsilon_1)\sum_{j=1}^{2}\int_0^t\|\nabla u\|^4_{L^2(\Omega_f)}\|S_j u\|^2_{L^2(\Omega_f)}\,ds+C_{\epsilon_1}\int_0^t\|\nabla u\|^2_{L^2(\Omega_f)}\,ds$$
$$+\frac{C}{\delta^2}\|u(t)\|^2_{L^2(\Omega_f)}+C\delta t\sum_{j=1}^{2}\int_0^t\|\nabla S_j u\|^2_{L^2(\Omega_f)}\,ds+C_{\epsilon_1,\delta}\int_0^t\|\nabla u\|^{(1+4k)/k}_{L^2(\Omega_f)}\,ds$$
$$+C\sum_{j=1}^{2}\|S_j u(t)\|^{4/k}_{L^2(\Omega_f)}+C\|u(t)\|^4_{L^2(\Omega_f)}+CE_1(0)+CE_1(0)^{3/2}. \qquad (3.50)$$

Choosing $\epsilon_1\in(0,1]$ sufficiently small and $\delta\in(0,1]$ sufficiently small compared to ϵ_1, we get (3.38). □

3.6. **Collecting the estimates.** Now, we are in position to prove Theorem 2.1. The proof follows by coupling the estimates for time and tangential derivatives.

Proof of Theorem 2.1. For simplicity, assume $T\in(0,1]$. First, we integrate (3.10) in time and add the resulting equation to (3.38) while taking $\epsilon>0$ sufficiently small to absorb the first term on the right of (3.38). We get

$$y(t)+\sum_{j=1}^{2}\int_0^t\|\nabla S_j u\|^2_{L^2(\Omega_f)}\,ds+\int_0^t\|u_t\|^2_{L^2(\Omega_f)}\,ds$$
$$+\|\nabla u(t)\|^2_{L^2(\Omega_f)}+\|w_t(t)\|^2_{H^{1/2+k}(\Omega_e)}+\|w_{tt}(t)\|^2_{H^{-1/2+k}(\Omega_e)}$$

$$\begin{aligned}\le C\int_0^t \|\nabla u\|_{L^2(\Omega_f)}^4 y(s)\,ds + C\int_0^t \|\nabla u\|_{L^2(\Omega_f)}^{(1+4k)/k}\,ds \\ + C\sum_{j=1}^2 \|S_j u(t)\|_{L^2(\Omega_f)}^{4/k} + f(E_1(0),E(0)),\end{aligned} \tag{3.51}$$

where f denotes an explicit generic smooth function. We next estimate $\sum_{j=1}^2 \|S_j u(t)\|_{L^2(\Omega_f)}^{4/k}$ by $y^{2/k}$ which can be further bounded through multiplying (3.10) by $y^{2/k-1}$ and integrating in time to obtain

$$Cy(t)^{2/k} \le C\int_0^t y(s)^{2/k}\|\nabla u\|_{L^2(\Omega_f)}^4\,ds + y(0)^{2/k}. \tag{3.52}$$

Denoting

$$Y(t) = y(t) + \|\nabla u(t)\|_{L^2(\Omega_f)}^2 + \|w_t(t)\|_{H^{1/2+k}(\Omega_e)}^2 + \|w_{tt}(t)\|_{H^{-1/2+k}(\Omega_e)}^2,$$

the inequality (3.51) becomes

$$\begin{aligned}Y(t) + \sum_{j=1}^2\int_0^t \|\nabla S_j u\|_{L^2(\Omega_f)}^2\,ds + \int_0^t \|u_t\|_{L^2(\Omega_f)}^2\,ds \\ \le C\int_0^t \|\nabla u\|_{L^2(\Omega_f)}^4 y(s)\,ds + C\sum_{j=1}^2\int_0^t \|\nabla u\|_{L^2(\Omega_f)}^{(1+4k)/k}\,ds \\ + C\int_0^t y(s)^{2/k}\|\nabla u\|_{L^2(\Omega_f)}^4\,ds + f(E_1(0),E(0)).\end{aligned} \tag{3.53}$$

We finally bound the terms on the right by powers of $Y(t)$:

$$\begin{aligned}Y(t) + \sum_{j=1}^2\int_0^t \|\nabla S_j u\|_{L^2(\Omega_f)}^2\,ds + \int_0^t \|u_t\|_{L^2(\Omega_f)}^2\,ds \\ \le C\int_0^t \Big(Y(s)^{(1+4k)/2k} + Y(s)^{2+2/k} + Y(s)^3\Big)\,ds + y(0)^{2/k} + f(E_1(0),E(0)).\end{aligned} \tag{3.54}$$

The a priori bounds (2.10)–(2.14) follow by the Gronwall lemma.

Now, we briefly sketch the argument which shows how the a priori estimates lead to a solution as in the statement. For this purpose, we construct a solution of

$$u_t - \Delta u + (u_\epsilon\cdot\nabla)u + \nabla p = 0 \text{ in } \Omega_f\times(0,T) \tag{3.55}$$

$$\operatorname{div} u = 0 \text{ in } \Omega_f\times(0,T) \tag{3.56}$$

$$w_{tt} - \Delta w = 0 \text{ in } \Omega_e\times(0,T), \tag{3.57}$$

with the boundary conditions

$$u = w_t \qquad \text{on } \Gamma_c \times (0,T) \tag{3.58}$$

$$u = 0 \qquad \text{on } \Gamma_f \times (0,T) \tag{3.59}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} - p\nu - \frac{1}{2}(u_\epsilon \cdot \nu)u \qquad \text{on } \Gamma_c \times (0,T). \tag{3.60}$$

and the initial conditions

$$u(\cdot,0) = u_{0\epsilon} \text{ in } \Omega_f \tag{3.61}$$

$$w(\cdot,0) = w_{0\epsilon} \text{ in } \Omega_e \tag{3.62}$$

$$w_t(\cdot,0) = w_{1\epsilon} \text{ in } \Omega_e, \tag{3.63}$$

where $\{f_\epsilon\}_{\epsilon>0}$ is the standard mollification of f. Using the arguments in [6, 7], we may construct a local strong solution (u,p,w), depending on $\epsilon > 0$, but with the interval of existence which is ϵ independent. This solution satisfy the above a priori bounds, and we may use the standard compactness techniques (cf. [8, 25, 26]) to pass to the limit $\epsilon \to 0$ along a subsequence. We omit further details.

Now, we establish uniqueness. Let (u_1,p_1,w_1) and (u_2,p_2,w_2) be two solutions of our problem. The differences $U = u_1 - u_2$, $P = p_1 - p_2$, and $W = w_1 - w_2$ satisfy the equations

$$U_t - \Delta U + u_1 \cdot \nabla U + U \cdot \nabla u_2 + \nabla P = 0 \text{ in } \Omega_f \times (0,T) \tag{3.64}$$

$$\operatorname{div} U = 0 \text{ in } \Omega_f \times (0,T), \tag{3.65}$$

and

$$W_{tt} - \Delta W = 0 \text{ in } \Omega_e \times (0,T). \tag{3.66}$$

Multiplying (3.64) by U and (3.66) by W_t and integrating, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|U\|^2_{L^2(\Omega_f)} + \|W_t\|^2_{L^2(\Omega_e)} + \|\nabla W\|^2_{L^2(\Omega_e)}\big) + \|\nabla U\|^2_{L^2(\Omega_f)} \\
&\quad = -\int_{\Gamma_c} U \cdot \frac{\partial U}{d\nu}\, d\sigma(x) - \int_{\Omega_f} (u_1 \cdot \nabla U)\cdot U\, dx - \int_{\Omega_f} (U \cdot \nabla u_2)\cdot U\, dx \\
&\qquad + \int_{\Gamma_c} (U\cdot\nu)P\, d\sigma(x) + \int_{\Gamma_c} U \cdot \frac{\partial W}{\partial \nu}\, d\sigma(x) \\
&\quad = -\int_{\Omega_f} (U\cdot\nabla u_2)\cdot U\, dx - \frac{1}{2}\int_{\Gamma_c} (U\cdot\nu)u_2 \cdot U\, d\sigma(x) = I_1 + I_2.
\end{aligned}$$

The Hölder and Gagliardo-Nirenberg inequalities give

$$I_1 \le C\|U\|^2_{L^{6/(2+k)}(\Omega_f)} \|\nabla u_2\|_{L^{3/(1-k)}(\Omega_f)}$$

$$\le C\|U\|_{L^2(\Omega_f)}^{1+k}\|\nabla U\|_{L^2(\Omega_f)}^{1-k}\|u_2\|_{H^{3/2+k}(\Omega_f)}$$
$$\le \frac{1}{4}\|\nabla U\|_{L^2(\Omega_f)}^2 + C\|u_2\|_{H^{3/2+k}(\Omega_f)}^{2/(1+k)}\|U\|_{L^2(\Omega_f)}^2.$$

Similarly, using the Hölder and trace inequalities, we obtain

$$I_2 \le \|U\cdot\nu\|_{L^2(\Gamma_c)}\|u_2\cdot U\|_{L^2(\Gamma_c)}$$
$$\le C\|U\|_{L^2(\Omega_f)}^{1/2}\|\nabla U\|_{L^2(\Omega_f)}^{1/2}\|u\cdot U\|_{L^2(\Omega_f)}^{1/2}\|\nabla(u\cdot U)\|_{L^2(\Omega_f)}^{1/2},$$

and after a short calculation

$$I_2 \le C\|U\|_{L^2(\Omega_f)}\|\nabla U\|_{L^2(\Omega_f)}\|u_2\|_{L^2(\Omega_f)}^{2k/(3+2k)}\|u_2\|_{H^{3/2+k}(\Omega_f)}^{3/(3+2k)}$$
$$\le \frac{1}{4}\|\nabla U\|_{L^2(\Omega_f)}^2 + C\|u_2\|_{L^2(\Omega_f)}^{4k/(3+2k)}\|u_2\|_{H^{3/2+k}(\Omega_f)}^{6/(3+2k)}\|U\|_{L^2(\Omega_f)}^2.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\big(\|U\|_{L^2(\Omega_f)}^2 + \|W_t\|_{L^2(\Omega_e)}^2 + \|\nabla W\|_{L^2(\Omega_e)}^2\big)$$
$$\le C\Big(\|u_2\|_{H^{3/2+k}(\Omega_f)}^{2/(1+k)} + \|u_2\|_{L^2(\Omega_f)}^{4k/(3+2k)}\|u_2\|_{H^{3/2+k}(\Omega_f)}^{6/(3+2k)}\Big)\|U\|_{L^2(\Omega_f)}^2,$$

and the uniqueness follows. □

4. Alternatives to the Stress Matching Boundary Condition

The condition (2.6) represents the matching of stresses between the elastic body and the liquid. In view of the free boundary case considered in [9], it would seem more natural to replace (2.6) with the condition

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} - p\nu \qquad \text{on } \Gamma_c \times (0,T). \tag{4.1}$$

With this correction, we can no longer obtain the global existence of weak solutions, except in the 2D case with a small initial datum. However, we still obtain the local existence of strong solutions under the same conditions as in Theorem 2.1. The estimates (3.10) and (3.38) remain unchanged, but (3.3) is replaced by

$$\frac{d}{dt}\Big(\|u(t)\|_{L^2(\Omega_f)}^2 + \|w_t(t)\|_{L^2(\Omega_e)}^2 + \|\nabla w(t)\|_{L^2(\Omega_e)}^2\Big) + 2\|\nabla u\|_{L^2(\Omega_e)}^2 \tag{4.2}$$
$$= \int_{\Gamma_c}|u|^2 u\cdot\nu \le \|u\|_{L^3(\Gamma_c)}^3 \le C\|u\|_{H^{1/3}(\Gamma_c)}^3 \le C\|u\|_{L^2(\Omega_f)}^{1/2}\|\nabla u\|_{L^2(\Omega_f)}^{5/2}.$$

The a priori bounds then follow by the Gronwall inequality applied to (3.10), (3.38), and (4.2).

Another natural alternative to (2.6) is the condition

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} - p_B \nu \qquad \text{on } \Gamma_c \times (0,T), \tag{4.3}$$

where $p_B = p + \frac{1}{2}|u|^2$ is the Bernoulli pressure. In this case, we have (3.3), and the global weak solutions exist. The proof of existence of strong solutions is similar as with the condition (2.6)—in fact, the proof is easier since we do not integrate by parts in t as in (3.40).

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