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REFLECTION PRINCIPLES AND KERNELS IN $\mathbb{R}^n_+$ FOR THE BIHARMONIC AND STOKES OPERATORS. SOLUTIONS IN A LARGE CLASS OF WEIGHTED SOBOLEV SPACES

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Abstract. In this paper, we study the Stokes system in the half-space $\mathbb{R}^n_+$, with $n \geqslant 2$. We consider data and give solutions which live in weighted Sobolev spaces, for a whole scale of weights. We start to study the kernels of the biharmonic and Stokes operators. After the central case of the generalized solutions, we are interested in strong solutions and symmetrically in very weak solutions by means of a duality argument.

1. Introduction

The purpose of this paper is the resolution of the Stokes system

$$(S^+)\quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi &= \boldsymbol{f} \quad \text{in } \mathbb{R}^n_+, \\ \operatorname{div} \boldsymbol{u} &= h \quad \text{in } \mathbb{R}^n_+, \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \Gamma \equiv \mathbb{R}^{n-1}. \end{cases}$$

Weighted Sobolev spaces provide a functional framework quite suitable to express the regularity and the behavior at infinity of data and solutions. This paper is the continuation of a previous work in which we only dealt with the basic weights (see [8]). Here, we are interested in a large class of weights. This leads us to deal with the kernel of the operator associated to this problem and symmetrically with the compatibility condition for the data. So, an important part of this work is devoted to the study of the reflection principles for the biharmonic and Stokes operators. We give weak formulations of these principles with the aim of getting the kernels in some distribution spaces (see Section 2). The main results of [8] will be naturally included in this paper, but we will not discuss again these particular cases.

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We will also base our work on the previously established results on the harmonic and biharmonic operators (see [4], [5], [6], [7]).

Among the first works on the Stokes problem in the half-space, we can cite Cattabriga. In [10], he appeals to potential theory to explicitly get the velocity and pressure fields. For the homogeneous problem ($\boldsymbol{f} = \boldsymbol{0}$ and $h = 0$), for instance, he shows that, if $\boldsymbol{g} \in \boldsymbol{L}^p(\Gamma)$ and the semi-norm $|\boldsymbol{g}|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)} < \infty$, then $\nabla \boldsymbol{u} \in \boldsymbol{L}^p(\mathbb{R}^n_+)$ and $\pi \in L^p(\mathbb{R}^n_+)$.

We can find similar results in Farwig-Sohr (see [15]) and Galdi (see [16]), who also have chosen the setting of homogeneous Sobolev spaces. On the other hand, Maz'ya-Plamenevskiĭ-Stupyalis (see [18]) work within the suitable setting of weighted Sobolev spaces and consider different sorts of boundary conditions. However, their results are limited to the dimension 3, to the weight zero and to the Hilbertian framework, in which they give generalized and strong solutions. This is also the case with Boulmezaoud (see [9]), who only gives strong solutions; however, he suggests an interesting characterization of the kernel that we will get here in another way. Otherwise, always in dimension 3, by Fourier analysis techniques, we can find in Tanaka the case of very regular data, corresponding to velocities which belong to $\boldsymbol{W}_2^{m+3,\,2}(\mathbb{R}^3_+)$, with $m \geqslant 0$ (see [19]).

For any integer $n \geqslant 2$, writing a typical point $\in \mathbb{R}^n$ as $x = (x', x_n)$, we denote by $\mathbb{R}^n_+$ the upper half-space of $\mathbb{R}^n$ and Γ its boundary. We shall use the two basic weights $\varrho = (1 + |x|^2)^{1/2}$ and $\lg \varrho = \ln(2 + |x|^2)$, where $|x|$ is the Euclidean norm of x. For any integer q, $\mathcal{P}_q$ stands for the space of polynomials of degree smaller than or equal to q; $\mathcal{P}_q^{\Delta}$ (respectively $\mathcal{P}_q^{\Delta^2}$) is the subspace of harmonic (respectively biharmonic) polynomials of $\mathcal{P}_q$; $\mathcal{A}_q^{\Delta}$ (respectively $\mathcal{N}_q^{\Delta}$) is the subspace of polynomials of $\mathcal{P}_q^{\Delta}$, odd (respectively even) with respect to x_n, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$ (respectively $\partial_n \varphi(x', 0) = 0$), with the convention that these spaces are reduced to $\{0\}$ if $q < 0$. For any real number s, we denote by $[s]$ the integer part of s. Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X. For any $k \in \mathbb{Z}$, we shall denote by $\{1, \ldots, k\}$ the set of the first k positive integers, with the convention that this set is empty if k is nonpositive. In the whole text, bold characters are used for the vector and matrix fields.

Let Ω be an open set of $\mathbb{R}^n$. For any $m \in \mathbb{N}$, $p \in\ (1, \infty)$, $(\alpha,\, \beta) \in \mathbb{R}^2$, we define the following space:

$$W^{m,\,p}_{\alpha,\,\beta}(\Omega) = \Big\{ u \in \mathcal{D}'(\Omega) :\ 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha - m + |\lambda|}\, (\lg \varrho)^{\beta - 1}\, \partial^{\lambda} u \in L^p(\Omega);$$

$$k+1 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha-m+|\lambda|}\,(\lg\varrho)^{\beta}\,\partial^{\lambda}u \in L^p(\Omega)\Big\}, \tag{1.1}$$

where $k = m - n/p - \alpha$ if $n/p + \alpha \in \{1, \ldots, m\}$, and $k = -1$ otherwise. In the case $\beta = 0$, we simply denote the space by $W^{m,\,p}_{\alpha}(\Omega)$. Note that $W^{m,\,p}_{\alpha,\,\beta}(\Omega)$ is a reflexive Banach space equipped with the graph-norm. Now, we define the space $\overset{\circ}{W}{}^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+) = \overline{\mathcal{D}(\mathbb{R}^n_+)}^{\|\cdot\|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+)}}$, and its dual is denoted by $W^{-m,\,p'}_{-\alpha,\,-\beta}(\mathbb{R}^n_+)$. In order to define the traces of functions of $W^{m,\,p}_{\alpha}(\mathbb{R}^n_+)$ (here we don't consider the case $\beta \neq 0$), for any $\sigma \in\ (0,1)$, we introduce the space

$$W^{\sigma,\,p}_{\alpha}(\mathbb{R}^n) = \Big\{u \in \mathcal{D}'(\mathbb{R}^n) : w^{\alpha-\sigma}u \in L^p(\mathbb{R}^n),$$
$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\varrho^{\alpha}(x)\,u(x) - \varrho^{\alpha}(y)\,u(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < \infty\Big\},$$

where $w = \varrho$ if $n/p + \alpha \neq \sigma$ and $w = \varrho\,(\lg\varrho)^{1/(\sigma-\alpha)}$ if $n/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$W^{s,\,p}_{\alpha}(\mathbb{R}^n) = \Big\{u \in \mathcal{D}'(\mathbb{R}^n) : 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-s+|\lambda|}\,(\lg\varrho)^{-1}\,\partial^{\lambda}u \in L^p(\mathbb{R}^n);$$
$$k+1 \leqslant |\lambda| \leqslant [s]-1,\ \varrho^{\alpha-s+|\lambda|}\,\partial^{\lambda}u \in L^p(\mathbb{R}^n);\ \partial^{[s]}u \in W^{\sigma,\,p}_{\alpha}(\mathbb{R}^n)\Big\},$$

where $k = s - n/p - \alpha$ if $n/p + \alpha \in \{\sigma, \ldots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. In the same way, we also define, for any real number β, the space $W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^n) = \big\{v \in \mathcal{D}'(\mathbb{R}^n) : (\lg\varrho)^{\beta}\,v \in W^{s,\,p}_{\alpha}(\mathbb{R}^n)\big\}$.

Let us recall, for any integer $m \geqslant 1$ and any real number α, the following trace lemma.

Lemma 1.1. *For any integer $m \geqslant 1$ and real number α, we have the linear continuous mapping*

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) :\ W^{m,\,p}_{\alpha}(\mathbb{R}^n_+) \longrightarrow \prod_{j=0}^{m-1} W^{m-j-1/p,\,p}_{\alpha}(\mathbb{R}^{n-1}).$$

Moreover, γ is surjective and $\mathrm{Ker}\gamma = \overset{\circ}{W}{}^{m,\,p}_{\alpha}(\mathbb{R}^n_+)$.

On the Stokes problem in $\mathbb{R}^n$

$$(S):\qquad -\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}\ \text{ and }\ \operatorname{div}\boldsymbol{u} = h\ \text{ in } \mathbb{R}^n,$$

let us recall the fundamental results on which we are based in the sequel. First, for any $k \in \mathbb{Z}$, we introduce the space

$$\mathcal{S}_k = \big\{(\boldsymbol{\lambda},\,\mu) \in \boldsymbol{\mathcal{P}}_k \times \mathcal{P}^{\Delta}_{k-1} : \operatorname{div}\boldsymbol{\lambda} = 0,\ -\Delta\boldsymbol{\lambda} + \nabla\mu = \boldsymbol{0}\big\}\,.$$

Theorem 1.2 (Alliot-Amrouche [2]). *Let $\ell \in \mathbb{Z}$ and assume that $n/p' \notin \{1, \dots, \ell\}$ and $n/p \notin \{1, \dots, -\ell\}$. For any $(\boldsymbol{f}, h) \in \big(\boldsymbol{W}^{-1,p}_{\ell}(\mathbb{R}^n) \times W^{0,p}_{\ell}(\mathbb{R}^n)\big) \perp \mathcal{S}_{[1+\ell-n/p']}$, problem (S) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^n) \times W^{0,p}_{\ell}(\mathbb{R}^n)$, unique up to an element of $\mathcal{S}_{[1-\ell-n/p]}$, with the estimate*

$$\inf_{(\boldsymbol{\lambda}, \mu) \in \mathcal{S}_{[1-\ell-n/p]}} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^n)} + \|\pi + \mu\|_{W^{0,p}_{\ell}(\mathbb{R}^n)} \Big) \leqslant C \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p}_{\ell}(\mathbb{R}^n)} + \|h\|_{W^{0,p}_{\ell}(\mathbb{R}^n)} \Big).$$

Theorem 1.3 (Alliot-Amrouche [2]). *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$ be two integers and assume that $n/p' \notin \{1, \dots, \ell + 1\}$ and $n/p \notin \{1, \dots, -\ell - m\}$. For any $(\boldsymbol{f}, h) \in \big(\boldsymbol{W}^{m-1,p}_{m+\ell}(\mathbb{R}^n) \times W^{m,p}_{m+\ell}(\mathbb{R}^n)\big) \perp \mathcal{S}_{[1+\ell-n/p']}$, problem (S) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbb{R}^n) \times W^{m,p}_{m+\ell}(\mathbb{R}^n)$, unique up to an element of $\mathcal{S}_{[1-\ell-n/p]}$, with the estimate*

$$\inf_{(\boldsymbol{\lambda}, \mu) \in \mathcal{S}_{[1-\ell-n/p]}} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbb{R}^n)} + \|\pi + \mu\|_{W^{m,p}_{m+\ell}(\mathbb{R}^n)} \Big) \leqslant C \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{m-1,p}_{m+\ell}(\mathbb{R}^n)} + \|h\|_{W^{m,p}_{m+\ell}(\mathbb{R}^n)} \Big).$$

2. Reflection principles and kernels in $\mathbb{R}^n_+$

The aim of this section is to characterize the kernel of the Stokes operator with Dirichlet boundary conditions in the half-space. In this geometry, the natural way is to use a reflection principle similar to the well-known Schwarz reflection principle for harmonic functions. Since, in the Stokes system, the velocity field is biharmonic and the pressure is harmonic, it is reasonable to start with the reflection principle for the biharmonic functions. Let us notice that R. Farwig gives these continuation formulae in [14], referring to elliptic regularity theory (see Agmon, Douglis and Nirenberg, [1]). Let us especially quote R. J. Duffin, who first established in [13] the continuation formula of biharmonic functions in the three-dimensional case and then analogous formulae for the Stokes flow equations. Next, A. Huber extended in [17] this principle to polyharmonic functions. From the classical point of view, the only serious difficulty is the argument at the boundary.

Starting with the biharmonic operator, we will give a weak formulation of the continuation formula, which will allow us to characterize the kernel of this operator, even for very weak solutions.

At first, let us introduce a useful notation. For any function φ defined on an open set Ω of $\mathbb{R}^n$, we will denote by φ^*, the composite function $\varphi^* = \varphi \circ r$

defined on $\Omega^* = r(\Omega)$, of φ with the $\mathcal{C}^\infty$-diffeomorphism

$$r : \Omega \longrightarrow \Omega^*, \quad x = (x',\, x_n) \longmapsto x^* = (x',\, -x_n).$$

Thus, if $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$, then $\varphi^* \in \mathcal{D}(\mathbb{R}^n_-)$ and conversely. In the same way, if $u \in \mathcal{D}'(\Omega)$, we will denote by u^* the distribution in $\mathcal{D}'(\Omega^*)$, defined for any $\varphi \in \mathcal{D}(\Omega^*)$ by $\langle u^*,\, \varphi\rangle_{\mathcal{D}'(\Omega^*)\times\mathcal{D}(\Omega^*)} = \langle u,\, \varphi^*\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}$. Thus, if $u \in \mathcal{D}'(\mathbb{R}^n_+)$, then $u^* \in \mathcal{D}'(\mathbb{R}^n_-)$ and conversely.

Now, for the convenience of the reader, let us recall the essential tool — *i.e.,* the Green formula — in the study of singular boundary conditions for the biharmonic problem (see [7]). For any $\ell \in \mathbb{Z}$, we introduce the space

$$Y^p_{\ell,\,1}(\mathbb{R}^n_+) = \Big\{ v \in W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+) : \Delta^2 v \in W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^n_+) \Big\},$$

which is a reflexive Banach space equipped with its natural norm

$$\|v\|_{Y^p_{\ell,\,1}(\mathbb{R}^n_+)} = \|v\|_{W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)} + \|\Delta^2 v\|_{W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^n_+)}.$$

Then we proved in [7], Lemma 4.1, the following result.

Lemma 2.1. *Let $\ell \in \mathbb{Z}$ such that*

$$\frac{n}{p'} \notin \{1, \ldots, \ell - 2\} \quad \textit{and} \quad \frac{n}{p} \notin \{1, \ldots, -\ell + 2\}; \tag{2.1}$$

then the space $\mathcal{D}(\overline{\mathbb{R}^n_+})$ is dense in $Y^p_{\ell,\,1}(\mathbb{R}^n_+)$.

Thanks to this density lemma, we proved in [7], Lemma 4.2, the following result of traces with the Green formula.

Lemma 2.2. *Let $\ell \in \mathbb{Z}$. Under hypothesis* (2.1), *the mapping* $(\gamma_0,\, \gamma_1) :$ $\mathcal{D}(\overline{\mathbb{R}^n_+}) \longrightarrow \mathcal{D}(\mathbb{R}^{n-1})^2$ *can be extended to a linear continuous mapping*

$$(\gamma_0,\, \gamma_1) : Y^p_{\ell,\,1}(\mathbb{R}^n_+) \longrightarrow W^{-1/p,\,p}_{\ell-2}(\Gamma) \times W^{-1-1/p,\,p}_{\ell-2}(\Gamma),$$

and we have the following Green formula:

$$\forall v \in Y^p_{\ell,\,1}(\mathbb{R}^n_+),\ \forall \varphi \in W^{4,\,p'}_{-\ell+2}(\mathbb{R}^n_+) \textit{ such that } \varphi = \partial_n \varphi = 0 \textit{ on } \Gamma, \tag{2.2}$$

$$\big\langle \Delta^2 v, \varphi \big\rangle_{W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^n_+)\times W^{0,\,p'}_{-\ell-2,\,-1}(\mathbb{R}^n_+)} - \big\langle v, \Delta^2 \varphi \big\rangle_{W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)\times W^{0,\,p'}_{-\ell+2}(\mathbb{R}^n_+)}$$
$$= \langle v, \partial_N \Delta\varphi\rangle_{W^{-1/p,\,p}_{\ell-2}(\Gamma)\times W^{1/p,\,p'}_{-\ell+2}(\Gamma)} - \langle \partial_n v, \Delta\varphi\rangle_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)\times W^{1+1/p,\,p'}_{-\ell+2}(\Gamma)}.$$

Now, we can establish the following result.

Lemma 2.3. *Let $\ell \in \mathbb{Z}$ with hypothesis* (2.1) *and $u \in W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)$ satisfying*

$$\Delta^2 u = 0 \ \textit{ in } \mathbb{R}^n_+, \quad u = \partial_n u = 0 \ \textit{ on } \Gamma;$$

then there exists a unique biharmonic extension $\tilde{u} \in \mathcal{D}'(\mathbb{R}^n)$ of u, which is given for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by

$$\langle \tilde{u},\, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)} = \int_{\mathbb{R}^n_+} u\, \left(\varphi - 5\,\varphi^* - 6\, x_n\, \partial_n \varphi^* - x_n^2\, \Delta\varphi^*\right) \mathrm{d}x. \tag{2.3}$$

Moreover, we have $\tilde{u} \in W^{-2,\,p}_{\ell-4}(\mathbb{R}^n)$ with the estimate

$$\|\tilde{u}\|_{W^{-2,\,p}_{\ell-4}(\mathbb{R}^n_+)} \;\leqslant\; C\; \|u\|_{W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)}. \tag{2.4}$$

Proof. (1) Let us notice an important point to start with. According to Weyl's lemma, since u is a biharmonic distribution in $\mathbb{R}^n_+$, we know that $u \in \mathcal{C}^\infty(\mathbb{R}^n_+)$ — see e.g. Dautray-Lions [12], page 327, Proposition 1. Next, let us remark that the integral in (2.3) is well defined. Indeed, $u \in W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)$ and φ — thus φ^* also — has compact support.

Now, let us show that $\tilde{u}$ belongs to $\mathcal{D}'(\mathbb{R}^n)$ and more precisely the end of our statement. From (2.3) we get the following estimate:

$$|\langle \tilde{u},\, \varphi\rangle| \leqslant C\; \|u\|_{W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)} \left(\|\varphi\|_{W^{0,\,p'}_{-\ell+2}(\mathbb{R}^n)} + \|\varphi\|_{W^{1,\,p'}_{-\ell+3}(\mathbb{R}^n)} + \|\varphi\|_{W^{2,\,p'}_{-\ell+4}(\mathbb{R}^n)} \right).$$

Since $\frac{n}{p'} \notin \{\ell-3,\, \ell-2\}$, we have $W^{2,\,p'}_{-\ell+4}(\mathbb{R}^n) \hookrightarrow W^{1,\,p'}_{-\ell+3}(\mathbb{R}^n) \hookrightarrow W^{0,\,p'}_{-\ell+2}(\mathbb{R}^n)$ and then, for any $\varphi \in W^{2,\,p'}_{-\ell+4}(\mathbb{R}^n)$,

$$|\langle \tilde{u},\, \varphi\rangle| \;\leqslant C\; \|u\|_{W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)}\; \|\varphi\|_{W^{2,\,p'}_{-\ell+4}(\mathbb{R}^n)}.$$

So, we can deduce that $\tilde{u} \in W^{-2,\,p}_{\ell-4}(\mathbb{R}^n)$ and the estimate (2.4).

(2) For the uniqueness, let us consider two biharmonic extensions $\tilde{u}_1$ and $\tilde{u}_2$ of u which belong to $\mathcal{D}'(\mathbb{R}^n)$ and set $U = \tilde{u}_2 - \tilde{u}_1$. Then we have $\Delta^2 U = 0$ in $\mathbb{R}^n$ and we can deduce that U is analytic in $\mathbb{R}^n$. Since $U = 0$ in $\mathbb{R}^n_+$, the analytic continuation principle implies that in fact $U = 0$ in $\mathbb{R}^n$.

(3) Evidently, $\tilde{u}$ is an extension of u. Indeed, let $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$ and $\tilde{\varphi}$ the zero extension of φ to $\mathbb{R}^n$, then

$$\langle \tilde{u},\, \tilde{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)} = \int_{\mathbb{R}^n_+} u\, \varphi\, \mathrm{d}x;$$

that is, $\tilde{u}|_{\mathbb{R}^n_+} = u$.

On the other hand, let $\varphi \in \mathcal{D}(\mathbb{R}^n_-)$ and $\tilde{\varphi}$ the zero extension of φ to $\mathbb{R}^n$, then we get

$$\langle \tilde{u},\, \tilde{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)} = \int_{\mathbb{R}^n_+} u\, \left(-5\,\varphi^* - 6\, x_n\, \partial_n\varphi^* - x_n^2\, \Delta\varphi^*\right) \mathrm{d}x.$$

Moreover, we can express $\langle \tilde{u},\, \tilde{\varphi}\rangle$ by means of an integral in $\mathbb{R}^n_-$:

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^n_+} u\,\varphi^*\,\mathrm{d}x = \int_{\mathbb{R}^n_-} u^*\,\varphi\,\mathrm{d}x,\\
I_2 &= \int_{\mathbb{R}^n_+} u\,x_n\,\partial_n\varphi^*\,\mathrm{d}x = \int_{\mathbb{R}^n_-} x_n\,u^*\,\partial_n\varphi\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^n_-} \partial_n(x_n\,u^*)\,\varphi\,\mathrm{d}x = -\int_{\mathbb{R}^n_-} (u^* + x_n\,\partial_n u^*)\,\varphi\,\mathrm{d}x,\\
I_3 &= \int_{\mathbb{R}^n_+} u\,x_n^2\,\Delta\varphi^*\,\mathrm{d}x = \int_{\mathbb{R}^n_-} x_n^2\,u^*\,\Delta\varphi\,\mathrm{d}x = \int_{\mathbb{R}^n_-} \Delta(x_n^2\,u^*)\,\varphi\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n_-} (2\,u^* + 4\,x_n\,\partial_n u^* + x_n^2\,\Delta u^*)\,\varphi\,\mathrm{d}x.
\end{aligned}
$$

Hence,

$$
\langle \tilde{u},\, \tilde{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)} = \int_{\mathbb{R}^n_-} \left(-u^* + 2\,x_n\,\partial_n u^* - x_n^2\,\Delta u^*\right)\,\varphi\,\mathrm{d}x;
$$

that is, $\tilde{u}|_{\mathbb{R}^n_-} = -u^* - 2\,x_n\,(\partial_n u)^* - x_n^2\,(\Delta u)^*$. So, $\tilde{u}|_{\mathbb{R}^n_-} \in \mathcal{C}^\infty(\mathbb{R}^n_-)$ and we find the classical formulation obtained by R. J. Duffin (see [13]) in the three dimensional case: for any $x \in \mathbb{R}^n_-$,

$$
\tilde{u}(x) = \left(-u - 2\,x_n\,\partial_n u - x_n^2\,\Delta u\right)(x^*). \tag{2.5}
$$

(4) It remains to show that this extension is actually biharmonic in $\mathbb{R}^n$. From the definition (2.3), we obtain the following expression: for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$
\begin{aligned}
\left\langle \Delta^2\tilde{u}, \varphi\right\rangle_{\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)} &= \left\langle \tilde{u}, \Delta^2\varphi\right\rangle_{\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)}\\
&= \int_{\mathbb{R}^n_+} u\left[\Delta^2(\varphi - 5\,\varphi^*) - 6\,x_n\,\partial_n\Delta^2\varphi^* - x_n^2\,\Delta^3\varphi^*\right]\mathrm{d}x,
\end{aligned}
$$

that we can rewrite as follows:

$$
\left\langle \Delta^2\tilde{u},\, \varphi\right\rangle = \int_{\mathbb{R}^n_+} u\,\Delta^2\Phi\,\mathrm{d}x,
$$

where $\Phi = \varphi - \varphi^* - x_n^2\,\Delta\varphi^* + 2\,x_n\,\partial_n\varphi^*$. Besides, we have

$$
\begin{cases}
\Phi = \varphi - \varphi^* = 0 \quad \text{on } \Gamma,\\
\partial_n\Phi = \partial_n\varphi + \partial_n\varphi^* = \partial_n\varphi - (\partial_n\varphi)^* = 0 \quad \text{on } \Gamma.
\end{cases}
$$

Then, according to Lemma 2.2, we get $\langle \Delta^2 \tilde{u}, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$; that is, $\Delta^2 \tilde{u} = 0$ in $\mathbb{R}^n$. □

As a main consequence of Lemma 2.3, we are going to characterize the kernel $\mathcal{K}$ of the biharmonic operator $(\Delta^2, \gamma_0, \gamma_1)$ in $W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)$ as a polynomial space. For any $q \in \mathbb{Z}$, let us introduce $\mathcal{B}_q$ as a subspace of $\mathcal{P}^{\Delta^2}_q$:

$$\mathcal{B}_q = \left\{ u \in \mathcal{P}^{\Delta^2}_q : u = \partial_n u = 0 \text{ on } \Gamma \right\}.$$

Corollary 2.4. *Let $\ell \in \mathbb{Z}$ with hypothesis (2.1), then $\mathcal{K} = \mathcal{B}_{[2-\ell-n/p]}$.*

Proof. Given $u \in \mathcal{K}$, thanks to Lemma 2.3, we know that $\tilde{u} \in W^{-2,\,p}_{\ell-4}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and $\Delta^2 \tilde{u} = 0$ in $\mathbb{R}^n$. We can deduce that $\tilde{u}$, and consequently u, is a polynomial. In addition, $u \in W^{0,\,p}_{\ell-2}(\mathbb{R}^n_+)$ implies that $u \in \mathcal{P}_{[2-\ell-n/p]}$ (see [3]). □

Remark 2.5. Coming back to Lemma 2.3, since $\tilde{u} \in \mathcal{P}_{[2-\ell-n/p]}$, we get in fact $\tilde{u} \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^n)$ for any integer $m \geqslant -4$. Indeed, under hypothesis (2.1), we have the imbedding chain $W^{m+2,\,p}_{m+\ell}(\mathbb{R}^n) \hookrightarrow \cdots \hookrightarrow W^{-2,\,p}_{\ell-4}(\mathbb{R}^n)$ and besides $\mathcal{P}_{[2-\ell-n/p]} \subset W^{m+2,\,p}_{m+\ell}(\mathbb{R}^n)$.

Better, we can see that this kernel does not really depend on the regularity according to the Sobolev imbeddings. More precisely, if we denote by $\mathcal{K}^m$ the kernel of $(\Delta^2, \gamma_0, \gamma_1)$ in $W^{m+2,\,p}_{m+\ell}(\mathbb{R}^n_+)$, identical arguments lead us to the following result.

Corollary 2.6. *Let $\ell \in \mathbb{Z}$ and $m \geqslant -2$ be two integers and assume that*

$$\frac{n}{p'} \notin \{1, \ldots, \ell + \min\{m,\, 2\}\} \quad \text{and} \quad \frac{n}{p} \notin \{1, \ldots, -\ell - m\}; \tag{2.6}$$

then $\mathcal{K}^m = \mathcal{B}_{[2-\ell-n/p]}$.

Finally, we showed in [6] that we can link this kernel to those of the Dirichlet and Neumann problems for the Laplacian. With this intention, we defined the two operators Π_D and Π_N by

$$\forall r \in \mathcal{A}^{\Delta}_k, \quad \Pi_D r = \tfrac{1}{2} \int_0^{x_n} t\, r(x', t)\, \mathrm{d}t,$$
$$\forall s \in \mathcal{N}^{\Delta}_k, \quad \Pi_N s = \tfrac{1}{2}\, x_n \int_0^{x_n} s(x', t)\, \mathrm{d}t,$$

satisfying the following properties:

$$\begin{aligned} &\forall r \in \mathcal{A}_k^{\Delta}, \qquad \Delta \Pi_D r = r \ \text{ in } \mathbb{R}^n_+, \qquad \Pi_D r = \partial_n \Pi_D r = 0 \ \text{ on } \Gamma, \\ &\forall s \in \mathcal{N}_k^{\Delta}, \qquad \Delta \Pi_N s = s \ \text{ in } \mathbb{R}^n_+, \qquad \Pi_N s = \partial_n \Pi_N s = 0 \ \text{ on } \Gamma. \end{aligned} \tag{2.7}$$

So we got a second characterization of this kernel:

$$\mathcal{B}_{[2-\ell-n/p]} = \Pi_D \mathcal{A}^{\Delta}_{[-\ell-n/p]} \oplus \Pi_N \mathcal{N}^{\Delta}_{[-\ell-n/p]}. \tag{2.8}$$

Now, we can use these results in the study of the Stokes operator. But to begin with we must establish a result equivalent to Lemma 2.2. Let us denote by

$$T: \ (\boldsymbol{u}, \pi) \longmapsto (-\Delta \boldsymbol{u} + \nabla \pi, \, -\operatorname{div} \boldsymbol{u})$$

the Stokes operator. For any $\ell \in \mathbb{Z}$, we introduce the space

$$T^p_{\ell,1}(\mathbb{R}^n_+) = \Big\{ (\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^n_+) \times W^{-1,p}_{\ell-1}(\mathbb{R}^n_+) : \\ T(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell+1,1}(\mathbb{R}^n_+) \times W^{0,p}_{\ell,1}(\mathbb{R}^n_+) \Big\},$$

which is a reflexive Banach space equipped with the graph-norm. Then we have the following density result.

Lemma 2.7. *Let $\ell \in \mathbb{Z}$ such that*

$$n/p' \notin \{1, \dots, \ell - 1\} \quad \text{and} \quad n/p \notin \{1, \dots, -\ell + 1\}; \tag{2.9}$$

then the space $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$ is dense in $T^p_{\ell,1}(\mathbb{R}^n_+)$.

Proof. For every continuous linear form $\Lambda \in (T^p_{\ell,1}(\mathbb{R}^n_+))'$, there exists a unique

$$(\boldsymbol{f}, \varphi, \boldsymbol{g}, \psi) \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \overset{\circ}{W}{}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \boldsymbol{W}^{0,p'}_{-\ell-1,-1}(\mathbb{R}^n_+) \times W^{0,p'}_{-\ell,-1}(\mathbb{R}^n_+),$$

such that, for all $(\boldsymbol{u}, \pi) \in T^p_{\ell,1}(\mathbb{R}^n_+)$,

$$\langle \Lambda, (\boldsymbol{u}, \pi) \rangle = \langle (\boldsymbol{f}, \varphi), (\boldsymbol{u}, \pi) \rangle + \langle (\boldsymbol{g}, \psi), T(\boldsymbol{u}, \pi) \rangle . \tag{2.10}$$

Thanks to the Hahn-Banach theorem, it suffices to show that any Λ which vanishes on $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$ is actually zero on $T^p_{\ell,1}(\mathbb{R}^n_+)$. Let us suppose that $\Lambda = 0$ on $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$, thus on $\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)$. Then we can deduce from (2.10) that

$$(\boldsymbol{f}, \varphi) + T(\boldsymbol{g}, \psi) = 0 \quad \text{in } \mathbb{R}^n_+,$$

hence, $T(\boldsymbol{g}, \psi) \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \overset{\circ}{W}{}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)$. Let $\tilde{\boldsymbol{f}}, \tilde{\varphi}, \tilde{\boldsymbol{g}}, \tilde{\psi}$ be respectively the zero extensions of $\boldsymbol{f}, \varphi, \boldsymbol{g}, \psi$ to $\mathbb{R}^n$. By (2.10), it is clear that we have

$(\tilde{\boldsymbol{f}},\, \tilde{\varphi}) + T(\tilde{\boldsymbol{g}},\, \tilde{\psi}) = 0$ in $\mathbb{R}^n$, and thus $T(\tilde{\boldsymbol{g}},\, \tilde{\psi}) \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n)$. According to the results in the whole space (see Theorem 1.3), we can deduce that $(\tilde{\boldsymbol{g}},\, \tilde{\psi}) \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n)$. Since $\tilde{\boldsymbol{g}}$ and $\tilde{\psi}$ are the zero extensions, it follows that $(\boldsymbol{g},\, \psi) \in \mathring{\boldsymbol{W}}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$. Then, by density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)$ in $\mathring{\boldsymbol{W}}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$, we can construct a sequence $(\boldsymbol{g}_k,\, \psi_k)_{k\in\mathbb{N}} \subset \boldsymbol{\mathcal{D}}(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)$ such that $(\boldsymbol{g}_k,\, \psi_k) \to (\boldsymbol{g},\, \psi)$ in $\mathring{\boldsymbol{W}}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$. Thus, for any $(\boldsymbol{u},\, \pi) \in T^p_{\ell,\,1}(\mathbb{R}^n_+)$, we have

$$\begin{aligned}\langle \Lambda,\, (\boldsymbol{u},\, \pi)\rangle &= -\langle T(\boldsymbol{g},\, \psi),\, (\boldsymbol{u},\, \pi)\rangle + \langle (\boldsymbol{g},\, \psi),\, T(\boldsymbol{u},\, \pi)\rangle\\ &= \lim_{k\to\infty} \{-\langle T(\boldsymbol{g}_k,\, \psi_k),\, (\boldsymbol{u},\, \pi)\rangle + \langle (\boldsymbol{g}_k,\, \psi_k),\, T(\boldsymbol{u},\, \pi)\rangle\} = 0;\end{aligned}$$

i.e., Λ is identically zero. □

Thanks to this density lemma, we can prove the following result.

Lemma 2.8. *Let* $\ell \in \mathbb{Z}$. *Under hypothesis* (2.9), *we can define the linear continuous mapping* (*the trace of the velocity field*)

$$\begin{aligned}\boldsymbol{\tau}_0 : T^p_{\ell,\,1}(\mathbb{R}^n_+) &\longrightarrow \boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma),\\ (\boldsymbol{u},\, \pi) &\longmapsto \boldsymbol{u}|_\Gamma = (\gamma_0 u_1,\, \ldots,\, \gamma_0 u_n).\end{aligned}$$

Moreover, we have the following Green formula:

$$\forall (\boldsymbol{u},\, \pi) \in T^p_{\ell,\,1}(\mathbb{R}^n_+),\ \forall (\boldsymbol{\varphi},\, \psi) \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$$
$$\textit{such that } \boldsymbol{\varphi} = \boldsymbol{0} \textit{ and } \operatorname{div} \boldsymbol{\varphi} = 0 \textit{ on } \Gamma,$$

$$\begin{aligned}&\langle T(\boldsymbol{u},\, \pi),\, (\boldsymbol{\varphi},\, \psi)\rangle_{\boldsymbol{W}^{0,\,p}_{\ell+1,\,1}(\mathbb{R}^n_+)\times W^{0,\,p}_{\ell,\,1}(\mathbb{R}^n_+),\ \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^n_+)\times W^{0,\,p'}_{-\ell,\,-1}(\mathbb{R}^n_+)}\\ &= \langle (\boldsymbol{u},\, \pi),\, T(\boldsymbol{\varphi},\, \psi)\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+),\ \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\\ &\qquad - \big\langle \boldsymbol{u},\, (\partial_n \boldsymbol{\varphi}',\, -\psi)\big\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma)}.\end{aligned} \tag{2.11}$$

Proof. Let us make three remarks to start. Firstly, the left-hand term in (2.11) is nothing but the integral $\int_{\mathbb{R}^n_+} T(\boldsymbol{u},\, \pi)\cdot(\boldsymbol{\varphi},\, \psi)\, \mathrm{d}x$. Secondly, the reason for the logarithmic factor in the definition of $T^p_{\ell,\,1}(\mathbb{R}^n_+)$ is that the imbeddings $\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \hookrightarrow \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^n_+)$ and $W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \hookrightarrow W^{0,\,p'}_{-\ell,\,-1}(\mathbb{R}^n_+)$ hold without supplementary critical values with respect to (2.9) — whereas the imbedding $\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \hookrightarrow \boldsymbol{W}^{0,\,p'}_{-\ell-1}(\mathbb{R}^n_+)$ fails if $n/p' \in \{\ell,\, \ell+1\}$. Thirdly, for any $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$, the boundary conditions $\boldsymbol{\varphi} = \boldsymbol{0}$ and $\operatorname{div} \boldsymbol{\varphi} = 0$ on Γ are equivalent to $\boldsymbol{\varphi} = \boldsymbol{0}$ and $\partial_n \varphi_n = 0$ on Γ.

So we can write the following Green formula:

$$\forall(\boldsymbol{u},\,\pi)\in\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+})\times\mathcal{D}(\overline{\mathbb{R}^n_+}),\ \forall(\boldsymbol{\varphi},\,\psi)\in\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\quad(2.12)$$
$$\text{such that } \boldsymbol{\varphi}=\boldsymbol{0} \text{ and } \operatorname{div}\boldsymbol{\varphi}=0 \text{ on } \Gamma,$$
$$\int_{\mathbb{R}^n_+}T(\boldsymbol{u},\,\pi)\cdot(\boldsymbol{\varphi},\,\psi)\,\mathrm{d}x=\int_{\mathbb{R}^n_+}(\boldsymbol{u},\,\pi)\cdot T(\boldsymbol{\varphi},\,\psi)\,\mathrm{d}x-\int_\Gamma\boldsymbol{u}\cdot(\partial_n\boldsymbol{\varphi}',\,-\psi)\,\mathrm{d}x'.$$

We can deduce the following estimate:

$$\Big|\big\langle\boldsymbol{u},(\partial_n\boldsymbol{\varphi}',\,-\psi)\big\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\times\boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma)}\Big|$$
$$\leqslant\|(\boldsymbol{u},\,\pi)\|_{T^p_{\ell,\,1}(\mathbb{R}^n_+)}\|(\boldsymbol{\varphi},\,\psi)\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}.$$

By Lemma 1.1, for any $\boldsymbol{g}\in\boldsymbol{W}^{1-1/p',\,p'}_{-\ell+1}(\Gamma)$, there exists a lifting function $(\boldsymbol{\varphi},\,\psi)\in\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$ such that $\boldsymbol{\varphi}=\boldsymbol{0},\ \partial_n\boldsymbol{\varphi}'=\boldsymbol{g}',\ \partial_n\varphi_n=0$ and $-\psi=g_n$ on Γ, satisfying

$$\|(\boldsymbol{\varphi},\,\psi)\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\leqslant C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p',\,p'}_{-\ell+1}(\Gamma)},$$

where C is a constant not depending on $(\boldsymbol{\varphi},\,\psi)$ and $\boldsymbol{g}$. Then we can deduce that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)}\ \leqslant\ C\ \|(\boldsymbol{u},\,\pi)\|_{T^p_{\ell,\,1}(\mathbb{R}^n_+)}.$$

Thus, the linear mapping $\boldsymbol{\tau}_0:(\boldsymbol{u},\,\pi)\longmapsto\boldsymbol{u}|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+})\times\mathcal{D}(\overline{\mathbb{R}^n_+})$ is continuous for the norm of $T^p_{\ell,\,1}(\mathbb{R}^n_+)$. Since $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+})\times\mathcal{D}(\overline{\mathbb{R}^n_+})$ is dense in $T^p_{\ell,\,1}(\mathbb{R}^n_+)$, $\boldsymbol{\tau}_0$ can be extended by continuity to a mapping still called $\boldsymbol{\tau}_0\in\mathcal{L}\big(T^p_{\ell,\,1}(\mathbb{R}^n_+);\ \boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\big)$. Moreover, we also can deduce the formula (2.11) from (2.12) by density of $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+})\times\mathcal{D}(\overline{\mathbb{R}^n_+})$ in $T^p_{\ell,\,1}(\mathbb{R}^n_+)$. □

We now can give the continuation result for the Stokes operator.

Lemma 2.9. *Let $\ell\in\mathbb{Z}$ with hypothesis* (2.9) *and $(\boldsymbol{u},\,\pi)\in\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)$ satisfying*

$$-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{0}\quad\textit{and}\quad\operatorname{div}\boldsymbol{u}=0\quad\textit{in }\mathbb{R}^n_+,\qquad\boldsymbol{u}=\boldsymbol{0}\quad\textit{on }\Gamma;$$

then there exists a unique extension $(\tilde{\boldsymbol{u}},\,\tilde{\pi})\in\boldsymbol{\mathcal{D}}'(\mathbb{R}^n)\times\mathcal{D}'(\mathbb{R}^n)$ of $(\boldsymbol{u},\,\pi)$ satisfying

$$-\Delta\tilde{\boldsymbol{u}}+\nabla\tilde{\pi}=\boldsymbol{0}\quad\textit{and}\quad\operatorname{div}\tilde{\boldsymbol{u}}=0\qquad\textit{in }\mathbb{R}^n,\tag{2.13}$$

which is given for all $(\boldsymbol{\varphi},\,\psi)\in\boldsymbol{\mathcal{D}}(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)$ *by*

$$\begin{aligned}\langle\tilde{\boldsymbol{u}},\,\boldsymbol{\varphi}\rangle=\int_{\mathbb{R}^n_+}&[\boldsymbol{u}\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}^*)-2\,u_n\,\varphi_n^*+2\,u_n\,x_n\,(\mathrm{div}\,\boldsymbol{\varphi})^*]\,\mathrm{d}x\\&+\left\langle\pi,\,2\,x_n\,\varphi_n^*-x_n^2\,(\mathrm{div}\,\boldsymbol{\varphi})^*\right\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)},\end{aligned}\tag{2.14}$$

and

$$\langle\tilde{\pi},\,\psi\rangle=\langle\pi,\,\psi-\psi^*-2\,x_n\,\partial_n\psi^*\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}+4\int_{\mathbb{R}^n_+}u_n\,\partial_n\psi^*\,\mathrm{d}x.\tag{2.15}$$

Moreover, we have $(\tilde{\boldsymbol{u}},\,\tilde{\pi})\in\boldsymbol{W}^{-2,\,p}_{\ell-3}(\mathbb{R}^n)\times W^{-2,\,p}_{\ell-2}(\mathbb{R}^n)$ *with the estimate*

$$\|(\tilde{\boldsymbol{u}},\,\tilde{\pi})\|_{\boldsymbol{W}^{-2,\,p}_{\ell-3}(\mathbb{R}^n)\times W^{-2,\,p}_{\ell-2}(\mathbb{R}^n)}\ \leqslant\ C\,\|(\boldsymbol{u},\,\pi)\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)}.\tag{2.16}$$

Remark 2.10. Knowing that u_n satisfies the biharmonic problem, see (2.20), naturally we must find (2.3) from (2.14). Indeed, if we take $\boldsymbol{\varphi}'=\boldsymbol{0}$ in (2.14), we get the following: for all $\varphi_n\in\mathcal{D}(\mathbb{R}^n)$,

$$\begin{aligned}\langle\tilde{u}_n,\,\varphi_n\rangle=\int_{\mathbb{R}^n_+}&[u_n\,(\varphi_n-\varphi_n^*)-2\,u_n\,\varphi_n^*-2\,u_n\,x_n\,\partial_n\varphi_n^*]\,\mathrm{d}x\\&+\left\langle\pi,\,2\,x_n\,\varphi_n^*+x_n^2\,\partial_n\varphi_n^*\right\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}.\end{aligned}$$

Since $\Delta u_n=\partial_n\pi$ in $\mathbb{R}^n_+$, we can write

$$\begin{aligned}&\left\langle\pi,\,2\,x_n\,\varphi_n^*+x_n^2\,\partial_n\varphi_n^*\right\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\\&\quad=\left\langle\pi,\,\partial_n(x_n^2\,\varphi_n^*)\right\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\\&\quad=-\left\langle\partial_n\pi,\,x_n^2\,\varphi_n^*\right\rangle_{W^{-2,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\\&\quad=-\left\langle\Delta u_n,\,x_n^2\,\varphi_n^*\right\rangle_{W^{-2,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\\&\quad=-\left\langle u_n,\,\Delta(x_n^2\,\varphi_n^*)\right\rangle_{W^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\\&\quad=-\left\langle u_n,\,2\,\varphi_n^*+4\,x_n\,\partial_n\varphi_n^*-x_n^2\,\Delta\varphi_n^*\right\rangle_{W^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}.\end{aligned}$$

Hence,

$$\langle\tilde{u}_n,\,\varphi_n\rangle=\int_{\mathbb{R}^n_+}u_n\,\left(\varphi_n-5\,\varphi_n^*-6\,x_n\,\partial_n\varphi_n^*-x_n^2\,\Delta\varphi_n^*\right)\,\mathrm{d}x,$$

which is exactly the formula (2.3) for u_n.

Proof of Lemma 2.9. (1) As for the biharmonic operator, according to (2.14) and (2.15), we can readily check that $(\tilde{\boldsymbol{u}},\tilde{\pi})\in \boldsymbol{W}^{-2,p}_{\ell-3}(\mathbb{R}^n)\times W^{-2,p}_{\ell-2}(\mathbb{R}^n)$ with the estimate (2.16). Besides, the argument for the uniqueness of the extension also holds for the Stokes operator and it is clear that (2.14) and (2.15) define an extension of $(\boldsymbol{u},\,\pi)$ to $\mathbb{R}^n$. Indeed, we have both for all $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)$, $\langle\tilde{\boldsymbol{u}},\,\tilde{\boldsymbol{\varphi}}\rangle=\int_{\mathbb{R}^n_+}\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x$ and for all $\psi\in\mathcal{D}(\mathbb{R}^n_+)$, $\langle\tilde{\pi},\,\tilde{\psi}\rangle=\langle\pi,\,\psi\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}$, where $\tilde{\boldsymbol{\varphi}}$ and $\tilde{\psi}$ are respectively the zero extensions of $\boldsymbol{\varphi}$ and ψ to $\mathbb{R}^n$.

(2) Now, we also can give the functional writing of this extension in $\mathbb{R}^n_-$. For all $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^n_-)$, we have

$$\begin{aligned}\langle\tilde{\boldsymbol{u}},\,\tilde{\boldsymbol{\varphi}}\rangle=&\int_{\mathbb{R}^n_+}\left[-\boldsymbol{u}'\cdot\boldsymbol{\varphi}'^{*}-3\,u_n\,\varphi_n^*+2\,u_n\,x_n\,(\operatorname{div}\boldsymbol{\varphi})^*\right]\,\mathrm{d}x\\&+\left\langle\pi,\,2\,x_n\,\varphi_n^*-x_n^2\,(\operatorname{div}\boldsymbol{\varphi})^*\right\rangle_{\mathcal{D}'(\mathbb{R}^n_+)\times\mathcal{D}(\mathbb{R}^n_+)}\,.\end{aligned}$$

Breaking down this expression, we get

$$\int_{\mathbb{R}^n_+}\left(-\boldsymbol{u}'\cdot\boldsymbol{\varphi}'^{*}-3\,u_n\,\varphi_n^*\right)\,\mathrm{d}x=\int_{\mathbb{R}^n_-}\left(-\boldsymbol{u}'^{*}\cdot\boldsymbol{\varphi}'-3\,u_n^*\,\varphi_n\right)\,\mathrm{d}x,$$

$$\begin{aligned}&\int_{\mathbb{R}^n_+}2\,u_n\,x_n\,(\operatorname{div}\boldsymbol{\varphi})^*\,\mathrm{d}x=\int_{\mathbb{R}^n_-}-2\,u_n^*\,x_n\,\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x\\&=\int_{\mathbb{R}^n_-}2\,\nabla(u_n^*\,x_n)\cdot\boldsymbol{\varphi}\,\mathrm{d}x=\int_{\mathbb{R}^n_-}2\,\left(u_n^*\,\varphi_n+x_n\,\nabla u_n^*\cdot\boldsymbol{\varphi}\right)\,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\langle\pi,\,2\,x_n\,\varphi_n^*\rangle_{\mathcal{D}'(\mathbb{R}^n_+)\times\mathcal{D}(\mathbb{R}^n_+)}&=-2\,\langle\pi^*,\,x_n\,\varphi_n\rangle_{\mathcal{D}'(\mathbb{R}^n_-)\times\mathcal{D}(\mathbb{R}^n_-)}\\&=-2\,\langle x_n\,\pi^*,\,\varphi_n\rangle_{\mathcal{D}'(\mathbb{R}^n_-)\times\mathcal{D}(\mathbb{R}^n_-)}\,,\end{aligned}$$

$$\begin{aligned}\left\langle\pi,\,-x_n^2\,(\operatorname{div}\boldsymbol{\varphi})^*\right\rangle_{\mathcal{D}'(\mathbb{R}^n_+)\times\mathcal{D}(\mathbb{R}^n_+)}&=-\left\langle x_n^2\,\pi,\,(\operatorname{div}\boldsymbol{\varphi})^*\right\rangle_{\mathcal{D}'(\mathbb{R}^n_+)\times\mathcal{D}(\mathbb{R}^n_+)}\\&=-\left\langle x_n^2\,\pi^*,\,\operatorname{div}\boldsymbol{\varphi}\right\rangle_{\mathcal{D}'(\mathbb{R}^n_-)\times\mathcal{D}(\mathbb{R}^n_-)}\\&=\left\langle\nabla(x_n^2\,\pi^*),\,\boldsymbol{\varphi}\right\rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^n_-)\times\boldsymbol{\mathcal{D}}(\mathbb{R}^n_-)}\\=\langle 2\,x_n\,\pi^*,\,\varphi_n\rangle_{\mathcal{D}'(\mathbb{R}^n_-)\times\mathcal{D}(\mathbb{R}^n_-)}&+\left\langle x_n^2\,\nabla\pi^*,\,\boldsymbol{\varphi}\right\rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^n_-)\times\boldsymbol{\mathcal{D}}(\mathbb{R}^n_-)}\,.\end{aligned}$$

Hence,

$$\begin{aligned}\langle\tilde{\boldsymbol{u}},\,\tilde{\boldsymbol{\varphi}}\rangle=&\int_{\mathbb{R}^n_-}\left(-\boldsymbol{u}'^{*}\cdot\boldsymbol{\varphi}'-u_n^*\,\varphi_n+2\,x_n\,\nabla u_n^*\cdot\boldsymbol{\varphi}\right)\,\mathrm{d}x\\&+\left\langle x_n^2\,\nabla\pi^*,\,\boldsymbol{\varphi}\right\rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^n_-)\times\boldsymbol{\mathcal{D}}(\mathbb{R}^n_-)}\,.\end{aligned}$$

Let us notice that here we always can replace the duality brackets by integrals. Indeed, $\boldsymbol{\varphi}$ has a compact support in $\mathbb{R}^n_-$, in addition to $\boldsymbol{u}^*$ and π^* belonging to $\mathcal{C}^\infty(\mathbb{R}^n_-)$, thus to $L^1_{loc}(\mathbb{R}^n_-)$. So, we get

$$\langle \tilde{\boldsymbol{u}}', \tilde{\boldsymbol{\varphi}}' \rangle = \int_{\mathbb{R}^n_-} \left(-\boldsymbol{u}'^* + 2\, x_n \nabla' u_n^* + x_n^2 \nabla' \pi^* \right) \cdot \boldsymbol{\varphi}' \, \mathrm{d}x;$$

i.e.,

$$\forall x \in \mathbb{R}^n_-, \quad \tilde{\boldsymbol{u}}'(x) = \left(-\boldsymbol{u}' + 2\, x_n \nabla' u_n + x_n^2 \nabla' \pi \right)(x^*)$$

and

$$\langle \tilde{u}_n, \tilde{\varphi}_n \rangle = \int_{\mathbb{R}^n_-} \left(-u_n^* + 2\, x_n \partial_n u_n^* + x_n^2 \partial_n \pi^* \right) \varphi_n \, \mathrm{d}x;$$

i.e.,

$$\forall x \in \mathbb{R}^n_-, \quad \tilde{u}_n(x) = \left(-u_n - 2\, x_n \partial_n u_n - x_n^2 \partial_n \pi \right)(x^*).$$

Likewise, for all $\psi \in \mathcal{D}(\mathbb{R}^n_-)$, we have

$$\left\langle \tilde{\pi}, \tilde{\psi} \right\rangle = \langle \pi, -\psi^* - 2\, x_n \partial_n \psi^* \rangle_{\mathcal{D}'(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)} + 4 \int_{\mathbb{R}^n_+} u_n \partial_n \psi^* \, \mathrm{d}x.$$

Separately, we get

$$\langle \pi, -\psi^* \rangle_{\mathcal{D}'(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)} = \langle -\pi^*, \psi \rangle_{\mathcal{D}'(\mathbb{R}^n_-) \times \mathcal{D}(\mathbb{R}^n_-)},$$

$$\begin{aligned} \langle \pi, -2\, x_n \partial_n \psi^* \rangle &= -2\, \langle x_n \pi, \partial_n \psi^* \rangle_{\mathcal{D}'(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)} \\ &= -2\, \langle x_n \pi^*, \partial_n \psi \rangle_{\mathcal{D}'(\mathbb{R}^n_-) \times \mathcal{D}(\mathbb{R}^n_-)} \\ &= 2\, \langle \partial_n (x_n \pi^*), \psi \rangle_{\mathcal{D}'(\mathbb{R}^n_-) \times \mathcal{D}(\mathbb{R}^n_-)} \\ &= 2\, \langle \pi^* + x_n \partial_n \pi^*, \psi \rangle_{\mathcal{D}'(\mathbb{R}^n_-) \times \mathcal{D}(\mathbb{R}^n_-)}, \end{aligned}$$

$$\int_{\mathbb{R}^n_+} u_n \partial_n \psi^* \, \mathrm{d}x = -\int_{\mathbb{R}^n_-} u_n^* \partial_n \psi \, \mathrm{d}x = \int_{\mathbb{R}^n_-} \partial_n u_n^* \, \psi \, \mathrm{d}x.$$

Hence,

$$\left\langle \tilde{\pi}, \tilde{\psi} \right\rangle_{\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)} = \langle \pi^* + 2\, x_n \partial_n \pi^* + 4\, \partial_n u_n^*, \psi \rangle_{\mathcal{D}'(\mathbb{R}^n_-) \times \mathcal{D}(\mathbb{R}^n_-)};$$

i.e.,

$$\forall x \in \mathbb{R}^n_-, \quad \tilde{\pi}(x) = (\pi - 2\, x_n \partial_n \pi - 4\, \partial_n u_n)(x^*).$$

So, we find the classical continuation formulae: for all $x \in \mathbb{R}^n_-$,

$$\left\{ \begin{array}{rl} \tilde{\boldsymbol{u}}'(x) &= \left(-\boldsymbol{u}' + 2\, x_n \nabla' u_n + x_n^2 \nabla' \pi \right)(x^*), \\ \tilde{u}_n(x) &= \left(-u_n - 2\, x_n \partial_n u_n - x_n^2 \partial_n \pi \right)(x^*), \\ \tilde{\pi}(x) &= (\pi - 2\, x_n \partial_n \pi - 4\, \partial_n u_n)(x^*). \end{array} \right.$$

(3) Finally, it remains to show that this extension satisfies (2.13) — that is the Stokes system in the whole space. For all $(\boldsymbol{\varphi},\, \psi) \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} T(\tilde{\boldsymbol{u}},\, \tilde{\pi}) \cdot (\boldsymbol{\varphi},\, \psi)\, \mathrm{d}x = \langle (\tilde{\boldsymbol{u}},\, \tilde{\pi}),\, T(\boldsymbol{\varphi},\, \psi) \rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^n)\times\mathcal{D}'(\mathbb{R}^n),\ \boldsymbol{\mathcal{D}}(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)}$$
$$= \langle \tilde{\boldsymbol{u}},\, -\Delta\boldsymbol{\varphi} + \nabla\psi \rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^n)\times\boldsymbol{\mathcal{D}}(\mathbb{R}^n)} - \langle \tilde{\pi},\, \operatorname{div}\boldsymbol{\varphi} \rangle_{\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)}\,.$$

Then, according to (2.14) and (2.15), we get

$$\int_{\mathbb{R}^n} T(\tilde{\boldsymbol{u}},\, \tilde{\pi}) \cdot (\boldsymbol{\varphi},\, \psi)\, \mathrm{d}x =$$
$$- \int_{\mathbb{R}^n_+} \left[\boldsymbol{u} \cdot \Delta(\boldsymbol{\varphi} - \boldsymbol{\varphi}^*) - 2\, u_n\, \Delta\varphi_n^* + 2\, u_n\, x_n\, (\operatorname{div}\Delta\boldsymbol{\varphi})^*\right]\, \mathrm{d}x$$
$$- \left\langle \pi,\, 2\, x_n\, \Delta\varphi_n^* - x_n^2\, (\operatorname{div}\Delta\boldsymbol{\varphi})^* \right\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}$$
$$+ \int_{\mathbb{R}^n_+} \left[\boldsymbol{u} \cdot \nabla(\psi - \psi^*) + 4\, u_n\, \partial_n\psi^* + 2\, u_n\, x_n\, \Delta\psi^*\right]\, \mathrm{d}x$$
$$+ \left\langle \pi,\, -2\, x_n\, \partial_n\psi^* - x_n^2\, \Delta\psi^* \right\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}$$
$$- \langle \pi,\, \operatorname{div}(\boldsymbol{\varphi} - \boldsymbol{\varphi}^*) + 2\, \partial_n\varphi_n^* - 2\, x_n\, \partial_n(\operatorname{div}\boldsymbol{\varphi})^* \rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}$$
$$- 4 \int_{\mathbb{R}^n_+} u_n\, \partial_n(\operatorname{div}\boldsymbol{\varphi})^*\, \mathrm{d}x.$$

With the intention of showing that $\int_{\mathbb{R}^n} T(\tilde{\boldsymbol{u}},\, \tilde{\pi}) \cdot (\boldsymbol{\varphi},\, \psi)\, \mathrm{d}x = 0$, we are going to rewrite this expression as follows:

$$\int_{\mathbb{R}^n_+} \boldsymbol{u} \cdot (-\Delta\boldsymbol{\Phi} + \nabla\Psi)\ \mathrm{d}x \ +\ \langle \pi,\, -\operatorname{div}\boldsymbol{\Phi} \rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}$$
$$= \langle (\boldsymbol{u},\, \pi),\, T(\boldsymbol{\Phi},\, \Psi) \rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+),\ \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\,,$$

where $(\boldsymbol{\Phi},\, \Psi) \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$, with $\boldsymbol{\Phi} = \boldsymbol{0}$ and $\partial_n\Phi_n = 0$ on Γ. Then, the zero of $\langle (\boldsymbol{u},\, \pi),\, T(\boldsymbol{\Phi},\, \Psi) \rangle$ will be a straightforward consequence of the Green formula (2.11).

Let us construct the functions $\boldsymbol{\Phi}$ and Ψ. We will start with the terms $\langle \pi,\, \cdot \rangle$. Noticing that $(\operatorname{div}\Delta\boldsymbol{\varphi})^* = \Delta(\operatorname{div}\boldsymbol{\varphi})^*$, we find

$$\Big\langle \pi,\, -2\, x_n\, \Delta\varphi_n^* + x_n^2\, \Delta(\operatorname{div}\boldsymbol{\varphi})^* - 2\, x_n\, \partial_n\psi^* - x_n^2\, \Delta\psi^*$$
$$- \operatorname{div}(\boldsymbol{\varphi} - \boldsymbol{\varphi}^*) - 2\, \partial_n\varphi_n^* + 2\, x_n\, \partial_n(\operatorname{div}\boldsymbol{\varphi})^* \Big\rangle$$

$$= \big\langle \pi,\, -\operatorname{div}\big[\boldsymbol{\varphi} - \boldsymbol{\varphi}^* + 2\,x_n\,\nabla\varphi_n^* + x_n^2\,\nabla(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)\big]\,\big\rangle,$$

hence we put $\boldsymbol{\Phi} = \boldsymbol{\varphi} - \boldsymbol{\varphi}^* + 2\,x_n\,\nabla\varphi_n^* + x_n^2\,\nabla(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)$.

Next, we can group together the remaining terms in $\int_{\mathbb{R}^n_+}(A+B)\,\mathrm{d}x$, where

$$A = \boldsymbol{u}\cdot[-\Delta(\boldsymbol{\varphi} - \boldsymbol{\varphi}^*) + \nabla(\psi - \psi^*)]$$
$$B = u_n\;[2\,\Delta\varphi_n^* - 2\,x_n\,\Delta(\operatorname{div}\boldsymbol{\varphi})^* + 4\,\partial_n\psi^* + 2\,x_n\,\Delta\psi^* - 4\,\partial_n(\operatorname{div}\boldsymbol{\varphi})^*]\,.$$

We can rewrite B by means of $\boldsymbol{e}_n = (0,\,\ldots,\,0,\,1)$ as follows:

$$B = \boldsymbol{u}\cdot 2\,\boldsymbol{e}_n\,\Delta\,[\varphi_n^* + x_n\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)]\,.$$

Using the identity $\boldsymbol{e}_n\,\Delta\xi = \nabla(x_n\,\Delta\xi) - \Delta(x_n\,\nabla\xi) + 2\,\nabla\partial_n\xi$, we get

$$\begin{aligned}
B \;=\; & \boldsymbol{u}\cdot\big[2\,\nabla(x_n\,\Delta(\varphi_n^* + x_n\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)))\\
&\qquad - 2\,\Delta(x_n\,\nabla(\varphi_n^* + x_n\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)))\\
&\qquad + 4\,\nabla\partial_n(\varphi_n^* + x_n\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*))\big],\\
=\; & \boldsymbol{u}\cdot\big[-\Delta(2\,x_n\,\nabla\varphi_n^* + x_n^2\,\nabla(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*))\\
&\qquad - \Delta(\nabla(x_n^2\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)))\\
&\qquad + \nabla(2\,x_n\,\Delta(\varphi_n^* + x_n\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)))\\
&\qquad + \nabla(4\,\partial_n(\varphi_n^* + x_n\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)))\big],\\
=\; & \boldsymbol{u}\cdot\big[-\Delta(2\,x_n\,\nabla\varphi_n^* + x_n^2\,\nabla(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*))\big]\\
&+ \boldsymbol{u}\cdot\nabla\big[-\Delta(x_n^2\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*))\\
&\qquad + 2\,x_n\,\Delta(\varphi_n^* + x_n\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*))\\
&\qquad + 4\,\partial_n(\varphi_n^* + x_n\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*))\big],\\
=\; & \boldsymbol{u}\cdot\big[-\Delta(2\,x_n\,\nabla\varphi_n^* + x_n^2\,\nabla(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*))\big]\\
&+ \boldsymbol{u}\cdot\nabla\big[\Delta(2\,x_n\,\varphi_n^* + x_n^2\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*))\big].
\end{aligned}$$

So, we find

$$A + B = \boldsymbol{u}\cdot(-\boldsymbol{\Delta\Phi} + \nabla\Psi)\,,$$

where

$$\begin{cases} \boldsymbol{\Phi} = \boldsymbol{\varphi} - \boldsymbol{\varphi}^* + 2\,x_n\,\nabla\varphi_n^* + x_n^2\,\nabla(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*),\\ \Psi = \psi - \psi^* + \Delta(2\,x_n\,\varphi_n^* + x_n^2\,(\psi^* - (\operatorname{div}\boldsymbol{\varphi})^*)). \end{cases}$$

We can consider $(\boldsymbol{\varphi},\,\psi) \in \boldsymbol{W}^{4,\,p'}_{-\ell+3}(\mathbb{R}^n) \times W^{3,\,p'}_{-\ell+3}(\mathbb{R}^n)$, then we can easily check that $(\boldsymbol{\Phi},\,\Psi) \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$ under hypothesis (2.9). In addition,

$$\begin{cases} \boldsymbol{\Phi} = \boldsymbol{\varphi} - \boldsymbol{\varphi}^* = \boldsymbol{0} \quad \text{on } \Gamma,\\ \partial_n\Phi_n = \partial_n\varphi_n + \partial_n\varphi_n^* = \partial_n\varphi_n - (\partial_n\varphi_n)^* = 0 \quad \text{on } \Gamma. \end{cases}$$

Then,

$$\int_{\mathbb{R}^n} T(\tilde{\boldsymbol{u}}, \tilde{\pi}) \cdot (\boldsymbol{\varphi}, \psi)\, \mathrm{d}x = 0,$$

for all $(\boldsymbol{\varphi}, \psi) \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)$; that is, $T(\tilde{\boldsymbol{u}}, \tilde{\pi}) = (\mathbf{0}, 0)$. □

Now, we can characterize the Stokes kernel. For $\ell \in \mathbb{Z}$, let us denote by $\mathcal{K}_S$ the kernel of the Stokes operator $(T, \boldsymbol{\tau}_0)$ in $\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^n_+) \times W^{-1,p}_{\ell-1}(\mathbb{R}^n_+)$ and for any $k \in \mathbb{Z}$, introduce the polynomial space

$$\mathcal{S}^+_k = \big\{(\boldsymbol{\lambda}, \mu) \in \boldsymbol{\mathcal{P}}^{\Delta^2}_k \times \mathcal{P}^{\Delta}_{k-1} : \\ -\Delta\boldsymbol{\lambda} + \nabla\mu = \mathbf{0} \text{ and } \operatorname{div}\boldsymbol{\lambda} = 0 \text{ in } \mathbb{R}^n_+,\ \boldsymbol{\lambda} = \mathbf{0} \text{ on } \Gamma\big\}.$$

Let $(\boldsymbol{u}, \pi) \in \mathcal{K}_S$. By Lemma 2.9, we can see that $\tilde{\pi}$ and $\tilde{\boldsymbol{u}}$ are respectively harmonic and biharmonic tempered distributions in $\mathbb{R}^n$, thus polynomials. Hence, we have the following result.

Corollary 2.11. *Let $\ell \in \mathbb{Z}$ with hypothesis* (2.9); *then $\mathcal{K}_S = \mathcal{S}^+_{[1-\ell-n/p]}$.*

Again, this kernel does not depend on the regularity. That is, if we denote by $\mathcal{K}^m_S$ the kernel of the Stokes operator $(T, \boldsymbol{\tau}_0)$ in $\boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbb{R}^n_+) \times W^{m,p}_{m+\ell}(\mathbb{R}^n_+)$, we have the following result.

Corollary 2.12. *Let $\ell \in \mathbb{Z}$ and $m \geqslant -1$ be two integers and assume that*

$$\frac{n}{p'} \notin \{1, \ldots, \ell + \min\{m, 1\}\} \quad \textit{and} \quad \frac{n}{p} \notin \{1, \ldots, -\ell - m\}; \tag{2.17}$$

then $\mathcal{K}^m_S = \mathcal{S}^+_{[1-\ell-n/p]}$.

We can be more specific about polynomials which build up this kernel. The idea of this characterization is due to T.Z. Boulmezaoud (see [9]). We give it with a completely different proof, based on the kernels of the Dirichlet and Neumann problems for the Laplacian and the one of the biharmonic problem with Dirichlet boundary conditions in the half-space.

Lemma 2.13. *Let $\ell \in \mathbb{Z}$. Then $(\boldsymbol{u}, \pi) \in \mathcal{S}^+_{[1-\ell-n/p]}$ if and only if there exists $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1-\ell-n/p]}$ such that*

$$\boldsymbol{u} = \boldsymbol{\varphi} - \nabla\left(\Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_n \varphi_n\right), \tag{2.18}$$

$$\pi = -\operatorname{div}\boldsymbol{\varphi}. \tag{2.19}$$

Proof. Given $(\boldsymbol{u}, \pi) \in \mathcal{S}^+_{[1-\ell-n/p]}$, then we also have $\operatorname{div}\boldsymbol{u} = 0$ on Γ and thus $\partial_n u_n = 0$ on Γ. Moreover $\Delta\pi = 0$ in $\mathbb{R}^n_+$ and thus $\Delta^2 u_n = 0$ in $\mathbb{R}^n_+$. So we get the biharmonic problem

$$\Delta^2 u_n = 0 \ \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad u_n = \partial_n u_n = 0 \ \text{ on } \Gamma. \tag{2.20}$$

Hence, $u_n \in \mathcal{B}_{[1-\ell-n/p]}$ and there exists $(r,\, s) \in \mathcal{A}^{\Delta}_{[-1-\ell-n/p]} \times \mathcal{N}^{\Delta}_{[-1-\ell-n/p]}$ such that $u_n = \Pi_D r + \Pi_N s$.

We can deduce from (2.7) that $\partial_n \pi = \Delta u_n = r + s$ in $\mathbb{R}^n_+$ and thus π satisfies

$$\Delta\pi = 0 \;\text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \partial_n \pi = s \;\text{ on } \Gamma.$$

Then, there exists $\psi \in \mathcal{N}^{\Delta}_{[-\ell-n/p]}$ (see [6]) such that

$$\pi = \psi + Ks \;\text{ in } \mathbb{R}^n_+, \tag{2.21}$$

where

$$Ks(x',\, x_n) = \int_0^{x_n} s(x',\, t)\,\mathrm{d}t.$$

So, we have $\Delta u_n = r + s = \partial_n \pi = \partial_n \psi + s$ in $\mathbb{R}^n_+$, thus $r = \partial_n \psi$. Hence,

$$u_n = \Pi_D \partial_n \psi + \Pi_N s \;\text{ in } \mathbb{R}^n_+. \tag{2.22}$$

From (2.21), we get, for every $i \in \{1, \ldots, n-1\}$,

$$\begin{aligned}\Delta u_i = \partial_i \pi = \partial_i \psi + \partial_i K s \;\in\; \mathcal{N}^{\Delta}_{[-1-\ell-n/p]} \oplus \mathcal{A}^{\Delta}_{[-1-\ell-n/p]}\\ = \Delta \Pi_N \partial_i \psi + \Delta \Pi_D \partial_i K s.\end{aligned}$$

Then, $w_i = u_i - \Pi_N \partial_i \psi - \Pi_D \partial_i K s$ satisfies

$$\Delta w_i = 0 \;\text{ in } \mathbb{R}^n_+ \quad \text{and} \quad w_i = 0 \;\text{ on } \Gamma.$$

Hence, we have the existence of $\varphi_i \in \mathcal{A}^{\Delta}_{[1-\ell-n/p]}$ (see [4]), such that $w_i = \varphi_i$; *i.e.*,

$$u_i = \Pi_N \partial_i \psi + \Pi_D \partial_i K s + \varphi_i.$$

Thereby, writing $\boldsymbol{\varphi}' = (\varphi_1, \ldots, \varphi_{n-1})$, we get

$$\begin{aligned}\operatorname{div}' \boldsymbol{u}' &= \Pi_N \Delta' \psi + \Pi_D \Delta' K s + \operatorname{div}' \boldsymbol{\varphi}'\\ &= -\Pi_N \partial_n^2 \psi - \Pi_D \partial_n^2 K s + \operatorname{div}' \boldsymbol{\varphi}'\\ &= -\tfrac{1}{2}\, x_n\, \partial_n \psi - \tfrac{1}{2}\,(x_n\, \partial_n K s - K s) + \operatorname{div}' \boldsymbol{\varphi}'\\ &= -\tfrac{1}{2}\, x_n\, \partial_n \psi - \tfrac{1}{2}\,(x_n\, s - K s) + \operatorname{div}' \boldsymbol{\varphi}'.\end{aligned}$$

In addition, by (2.22), we have

$$\begin{aligned}\partial_n u_n &= \partial_n \Pi_D \partial_n \psi + \partial_n \Pi_N s\\ &= \tfrac{1}{2}\, x_n\, \partial_n \psi + \tfrac{1}{2}\left(x_n\, s + \int_0^{x_n} s(x',\, t)\,\mathrm{d}t\right) = \tfrac{1}{2}\, x_n\, \partial_n \psi + \tfrac{1}{2}\,\left(x_n\, s + K s\right).\end{aligned}$$

Since $\operatorname{div} \boldsymbol{u} = 0$, we can deduce that $\operatorname{div}' \boldsymbol{\varphi}' = -Ks$ and thus (2.21) can be rewritten as $\pi = \psi - \operatorname{div}' \boldsymbol{\varphi}'$. Now, if we set $\varphi_n = -\int_0^{x_n} \psi(x',\, t)\,\mathrm{d}t$, then we have $\psi = -\partial_n \varphi_n$ and $\varphi_n \in \mathcal{A}^{\Delta}_{[1-\ell-n/p]}$. So, we obtain $\pi = -\operatorname{div}\boldsymbol{\varphi}$; *i.e.,* (2.19), with $\boldsymbol{\varphi} = (\boldsymbol{\varphi}', \varphi_n) \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1-\ell-n/p]}$.

Coming back to the velocity field, we get, for every $i \in \{1, \ldots, n-1\}$,

$$u_i = \varphi_i - \partial_i \Pi_N \partial_n \varphi_n - \partial_i \Pi_D \operatorname{div}' \boldsymbol{\varphi}'. \tag{2.23}$$

Likewise, for the normal component, (2.22) yields

$$\begin{aligned} u_n &= -\Pi_D \partial_n^2 \varphi_n + \Pi_N \partial_n K s = \tfrac{1}{2}\,(\varphi_n - x_n\,\partial_n \varphi_n) + \tfrac{1}{2}\,x_n\,K s \\ &= \varphi_n - \tfrac{1}{2}\,x_n\,\partial_n\varphi_n - \tfrac{1}{2}\,\varphi_n - \tfrac{1}{2}\,x_n\,\operatorname{div}'\boldsymbol{\varphi}' \\ &= \varphi_n - \partial_n \Pi_N \partial_n \varphi_n - \partial_n \Pi_D \operatorname{div}' \boldsymbol{\varphi}'. \end{aligned}$$

So, combining this with (2.23), we get $\boldsymbol{u} = \boldsymbol{\varphi} - \nabla\left(\Pi_N \partial_n \varphi_n + \Pi_D \operatorname{div}' \boldsymbol{\varphi}'\right)$; *i.e.,* the statement (2.18).

Conversely, we can verify that such a pair $(\boldsymbol{u}, \pi)$ belongs to $\mathcal{S}^+_{[1-\ell-n/p]}$. $\square$

3. GENERALIZED SOLUTIONS TO THE STOKES SYSTEM

In this section, we will establish the central result on the generalized solutions to the Stokes system in the half-space, with Theorem 3.3. We will be interested in the existence of a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^n_+) \times W^{0,p}_{\ell}(\mathbb{R}^n_+)$ to (S^+), for data $\boldsymbol{f} \in \boldsymbol{W}^{-1,p}_{\ell}(\mathbb{R}^n_+)$, $h \in W^{0,p}_{\ell}(\mathbb{R}^n_+)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}_{\ell}(\Gamma)$. To avoid troubles with the compatibility conditions, we will start with the study of the negative weights. For this, as for the weight $\ell = 0$ in [8], we will adapt a method used by Farwig-Sohr in [15]. Then, we get back the positive weights by a duality argument, and the compatibility condition naturally comes from the kernel of the dual case.

First, we will establish the result for the homogeneous problem in the case of negative weights.

Lemma 3.1. *Let ℓ be a negative integer and assume that $n/p \notin \{1, \ldots, -\ell\}$. For any $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}_{\ell}(\Gamma)$, the homogeneous Stokes problem*

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \qquad \text{in } \mathbb{R}^n_+, \tag{3.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathbb{R}^n_+, \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma \tag{3.3}$$

has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^n_+) \times W^{0,p}_{\ell}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{S}^+_{[1-\ell-n/p]}$, with the estimate

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{S}^+_{[1-\ell-n/p]}} \left(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^n_+)} + \|\pi+\mu\|_{W^{0,p}_{\ell}(\mathbb{R}^n_+)}\right) \leqslant C\,\|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}_{\ell}(\Gamma)}.$$

Proof. The operator associated to this problem is clearly continuous, moreover its kernel is known. The last point concerns its surjectivity, then the final estimate will be a straightforward consequence of the Banach theorem. So, we only must prove the existence of a solution $(\boldsymbol{u}, \pi)$.

(1) Firstly, we will show that system (3.1)–(3.3) can be reduced to a set of three problems on the fundamental operators Δ^2 and Δ.

Applying the operator div to the first equation (3.1), we obtain

$$\Delta\pi = 0 \quad \text{in } \mathbb{R}^n_+. \tag{3.4}$$

Now, applying the operator Δ to the same equation (3.1), we deduce

$$\Delta^2\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathbb{R}^n_+. \tag{3.5}$$

From the boundary condition (3.3), we take out

$$u_n = g_n \quad \text{on } \Gamma, \tag{3.6}$$

and moreover $\operatorname{div}'\boldsymbol{u}' = \operatorname{div}'\boldsymbol{g}'$ on Γ, where $\operatorname{div}'\boldsymbol{u}' = \sum_{i=1}^{n-1}\partial_i u_i$.
Since $\operatorname{div}\boldsymbol{u} = 0$ in $\mathbb{R}^n_+$, we also have $\operatorname{div}\boldsymbol{u} = 0$ on Γ, then we can write $\partial_n u_n + \operatorname{div}'\boldsymbol{u}' = 0$ on Γ, hence

$$\partial_n u_n = -\operatorname{div}'\boldsymbol{g}' \quad \text{on } \Gamma. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we obtain the biharmonic problem

$$(P): \qquad \Delta^2 u_n = 0 \ \text{ in } \mathbb{R}^n_+, \quad u_n = g_n \ \text{ and } \ \partial_N u_n = -\operatorname{div}'\boldsymbol{g}' \ \text{ on } \Gamma.$$

Then, combining (3.4) with the trace on Γ of the n^{th} component in the equations (3.1), we obtain the Neumann problem

$$(Q): \qquad \Delta\pi = 0 \ \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \partial_n\pi = \Delta u_n \ \text{ on } \Gamma.$$

Lastly, if we consider the $n-1$ first components of the equations (3.1) and (3.3), we can write the Dirichlet problem

$$(R): \qquad \Delta\boldsymbol{u}' = \nabla'\pi \ \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \boldsymbol{u}' = \boldsymbol{g}' \ \text{ on } \Gamma.$$

(2) Next, we will solve these three problems.

`Step 1:` Problem (P). Since $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,\,p}_\ell(\Gamma)$, we have $g_n \in W^{1-1/p,\,p}_\ell(\Gamma)$ and $\operatorname{div}'\boldsymbol{g}' \in W^{-1/p,\,p}_\ell(\Gamma)$, so (P) is a homogeneous biharmonic problem with singular boundary conditions. Since $\ell < 0$, we know that problem (P) has a solution $u_n \in W^{1,\,p}_\ell(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{B}_{[1-\ell-n/p]}$ (see [7], Theorem 4.5).

`Step 2:` Problem (Q). Since $\Delta^2 u_n = 0$ in $\mathbb{R}^n_+$, according to an appropriate trace result (see [8], Lemma 3.7), we can deduce that $\Delta u_n|_\Gamma \in$

$W_\ell^{-1-1/p,\,p}(\Gamma)$. As $\ell < 0$, we know that problem (Q) has a solution $\pi \in W_\ell^{0,\,p}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{N}^{\Delta}_{[-\ell-n/p]}$ (see [5], Theorem 3.4).

`Step 3:` Problem (R). Thanks to the previous result, we can deduce that $\nabla'\pi \in \boldsymbol{W}_\ell^{-1,\,p}(\mathbb{R}^n_+)$ and moreover $\boldsymbol{g}' \in \boldsymbol{W}_\ell^{1-1/p,\,p}(\Gamma)$. Since $\ell < 0$, we know that problem (R) has a solution $\boldsymbol{u}' \in W_\ell^{1,\,p}(\mathbb{R}^n_+)$, unique up to an element of $\boldsymbol{\mathcal{A}}^{\Delta}_{[1-\ell-n/p]}$ (see [4], Theorem 3.2).

(3) In order, we have found u_n, π and $\boldsymbol{u}'$, non-unique, which satisfy (3.3) and partially satisfy (3.1), more precisely such that

$$-\Delta\boldsymbol{u}' + \nabla'\pi = \boldsymbol{0} \quad \text{in } \mathbb{R}^n_+.$$

It remains to show we can choose them satisfying (3.2) and the n^{th} component of (3.1); *i.e.,*

$$-\Delta u_n + \partial_n\pi = 0 \quad \text{in } \mathbb{R}^n_+.$$

Consider such a pair $(\boldsymbol{u},\,\pi)$ satisfying problems (P), (Q) and (R). Thanks to the first equations of (P) and (Q), we obtain

$$\Delta(\Delta u_n - \partial_n\pi) = \Delta^2 u_n = 0 \quad \text{in } \mathbb{R}^n_+.$$

Thus, with the boundary condition of (Q), we can deduce that the distribution $\Delta u_n - \partial_n\pi \in W_\ell^{-1,\,p}(\mathbb{R}^n_+)$ satisfies the Dirichlet problem

$$\Delta(\Delta u_n - \partial_n\pi) = 0 \ \text{ in } \mathbb{R}^n_+, \quad \Delta u_n - \partial_n\pi = 0 \ \text{ on } \Gamma.$$

Then, we have $\Delta u_n - \partial_n\pi = \mu \in \mathcal{A}^{\Delta}_{[-1-\ell-n/p]}$ (see [8], Theorem 3.5). Moreover, we can write $\mu = \Delta\Pi_D\mu$, with $\Pi_D\mu = q \in \mathcal{B}_{[1-\ell-n/p]}$. Setting $u_n^\dagger = u_n - q$, we now get $\Delta u_n^\dagger - \partial_n\pi = 0$ in $\mathbb{R}^n_+$, and $u_n^\dagger$ is still a solution to problem (P).

Note that π is unchanged with $u_n^\dagger$, because $\Delta q = \mu = 0$ on Γ. Thus, if we set $\boldsymbol{u}^\dagger = (\boldsymbol{u}',\,u_n^\dagger)$, the pair $(\boldsymbol{u}^\dagger,\,\pi)$ completely satisfies (3.1).

Next, as $\Delta\pi = 0$ in $\mathbb{R}^n_+$, we also have $\Delta\,\mathrm{div}\,\boldsymbol{u}^\dagger = 0$ in $\mathbb{R}^n_+$. Moreover, from the boundary condition in (R), we obtain $\mathrm{div}'\boldsymbol{u}' = \mathrm{div}'\boldsymbol{g}'$ on Γ. Then, with the boundary condition in (P), we can write

$$\mathrm{div}\,\boldsymbol{u}^\dagger = \mathrm{div}'\boldsymbol{u}' + \partial_n u_n^\dagger = \mathrm{div}'\boldsymbol{g}' - \mathrm{div}'\boldsymbol{g}' = 0 \quad \text{on } \Gamma.$$

So, we have $\mathrm{div}\,\boldsymbol{u}^\dagger \in W_\ell^{0,\,p}(\mathbb{R}^n_+)$, which satisfies the Dirichlet problem

$$\Delta\,\mathrm{div}\,\boldsymbol{u}^\dagger = 0 \ \text{ in } \mathbb{R}^n_+, \quad \mathrm{div}\,\boldsymbol{u}^\dagger = 0 \ \text{ on } \Gamma.$$

Then, we have $\mathrm{div}\,\boldsymbol{u}^\dagger = \nu \in \mathcal{A}^{\Delta}_{[-\ell-n/p]}$ (see [8], Theorem 3.8). If we take for instance $r(x) = \int_0^{x_1} \nu(t,\,x_2,\ldots,x_n)\,\mathrm{d}t$, we have $\nu = \partial_1 r$ and, thus, $\nu = \mathrm{div}\,\boldsymbol{r}$, with $\boldsymbol{r} = (r,\,0,\ldots,0)$. Setting $\boldsymbol{u}^\diamond = \boldsymbol{u}^\dagger - \boldsymbol{r}$, we get $\mathrm{div}\,\boldsymbol{u}^\diamond = 0$ in $\mathbb{R}^n_+$ and, as

$r \in \mathcal{A}^{\Delta}_{[1-\ell-n/p]}$, we still have $u_1^\diamond = u_1 - r$ a solution to the first component of the equations (3.1) and (3.3). Consequently, the pair $(\boldsymbol{u}^\diamond, \pi)$ now completely satisfies the problem (3.1)–(3.3). □

Remark 3.2. If $\boldsymbol{g}$ is sufficiently smooth; *i.e.,* $\boldsymbol{g} \in \mathcal{D}(\Gamma)$, using a potential-theoretic method, it has been shown (see [10], [11]) that there exists a unique solution of (3.1)–(3.3) with a finite Dirichlet integral. In that case, we can see that this solution is naturally coming in the functional setting of Lemma 3.1.

Now, we can give the following general result.

Theorem 3.3. *Let $\ell \in \mathbb{Z}$ and assume that*

$$n/p' \notin \{1, \ldots, \ell\} \quad \text{and} \quad n/p \notin \{1, \ldots, -\ell\}. \tag{3.8}$$

For any $\boldsymbol{f} \in \boldsymbol{W}^{-1,p}_{\ell}(\mathbb{R}^n_+)$, $h \in W^{0,p}_{\ell}(\mathbb{R}^n_+)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}_{\ell}(\Gamma)$, satisfying the compatibility condition

$$\begin{aligned} \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1+\ell-n/p']}, \qquad & \langle \boldsymbol{f} - \nabla h, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,p}_{\ell}(\mathbb{R}^n_+)\times \mathring{\boldsymbol{W}}^{1,p'}_{-\ell}(\mathbb{R}^n_+)} \\ & + \left\langle \operatorname{div} \boldsymbol{f}, \Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_n \varphi_n \right\rangle_{W^{-2,p}_{\ell}(\mathbb{R}^n_+)\times \mathring{W}^{2,p'}_{-\ell}(\mathbb{R}^n_+)} \\ & + \langle \boldsymbol{g}, \partial_n \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{1-1/p,p}_{\ell}(\Gamma)\times \boldsymbol{W}^{-1/p',p'}_{-\ell}(\Gamma)} = 0, \end{aligned} \tag{3.9}$$

problem (S^+) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^n_+) \times W^{0,p}_{\ell}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{S}^+_{[1-\ell-n/p]}$, and there exists a constant C such that

$$\inf_{(\boldsymbol{\lambda},\mu)\in \mathcal{S}^+_{[1-\ell-n/p]}} \left(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^n_+)} + \|\pi + \mu\|_{W^{0,p}_{\ell}(\mathbb{R}^n_+)} \right) \leqslant C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p}_{\ell}(\mathbb{R}^n_+)} + \|h\|_{W^{0,p}_{\ell}(\mathbb{R}^n_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}_{\ell}(\Gamma)} \right).$$

Proof. (1) First, we still assume that $\ell < 0$.

We write $\boldsymbol{f} = \operatorname{div} \mathbb{F}$, where $\mathbb{F} = (\boldsymbol{F}_i)_{1\leqslant i\leqslant n} \in \boldsymbol{W}^{0,p}_{\ell}(\mathbb{R}^n_+)$, with the estimate

$$\|\mathbb{F}\|_{\boldsymbol{W}^{0,p}_{\ell}(\mathbb{R}^n_+)} \leqslant C\, \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p}_{\ell}(\mathbb{R}^n_+)}.$$

Let us respectively denote by $\tilde{\mathbb{F}} = (\tilde{\boldsymbol{F}}_i)_{1\leqslant i\leqslant n} \in \boldsymbol{W}^{0,p}_{\ell}(\mathbb{R}^n)$ and $\tilde{h} \in W^{0,p}_{\ell}(\mathbb{R}^n)$ the zero extensions of $\mathbb{F}$ and h to $\mathbb{R}^n$. By Theorem 1.2, we know that there exists a solution $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^n) \times W^{0,p}_{\ell}(\mathbb{R}^n)$ to the problem

$$(\tilde{S}): \qquad -\Delta\tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \operatorname{div}\tilde{\mathbb{F}} \ \text{ and } \ \operatorname{div}\tilde{\boldsymbol{u}} = \tilde{h} \ \text{ in } \mathbb{R}^n.$$

Consequently, we can reduce the system (S^+) to the homogeneous problem

$$(S^\sharp): \qquad -\Delta\boldsymbol{v} + \nabla\vartheta = \boldsymbol{0} \ \text{ and } \ \operatorname{div}\boldsymbol{v} = 0 \ \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{v} = \boldsymbol{g}^\sharp \ \text{ on } \Gamma,$$

where we have set $\boldsymbol{g}^\sharp = \boldsymbol{g} - \tilde{\boldsymbol{u}}|_\Gamma \in \boldsymbol{W}_\ell^{1-1/p,\,p}(\Gamma)$. Next, thanks to Lemma 3.1, we know that $(S^\sharp)$ admits a solution $(\boldsymbol{v},\,\vartheta) \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^n_+) \times W_\ell^{0,\,p}(\mathbb{R}^n_+)$. Then, $(\boldsymbol{u},\,\pi) = (\boldsymbol{v} + \tilde{\boldsymbol{u}}|_{\mathbb{R}^n_+},\,\vartheta + \tilde{\pi}|_{\mathbb{R}^n_+}) \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^n_+) \times W_\ell^{0,\,p}(\mathbb{R}^n_+)$ is a solution to (S^+).

(2) We now assume that $\ell > 0$. We will reason by duality from the case $\ell < 0$. So, we have established that, under hypothesis (3.8), the Stokes operator

$$T : \Big(\overset{\circ}{\boldsymbol{W}}{}_\ell^{1,\,p}(\mathbb{R}^n_+) \times W_\ell^{0,\,p}(\mathbb{R}^n_+)\Big)/\mathcal{S}^+_{[1-\ell-n/p]} \longrightarrow \boldsymbol{W}_\ell^{-1,\,p}(\mathbb{R}^n_+) \times W_\ell^{0,\,p}(\mathbb{R}^n_+)$$
$$(\boldsymbol{u},\,\pi) \longmapsto (-\Delta\boldsymbol{u} + \nabla\pi,\, -\operatorname{div}\boldsymbol{u})$$

is an isomorphism for any integer $\ell < 0$ and real number $p > 1$. Thus, replacing p by p' and $-\ell$ by ℓ, we deduce that its adjoint operator

$$T^* : \overset{\circ}{\boldsymbol{W}}{}_\ell^{1,\,p}(\mathbb{R}^n_+) \times W_\ell^{0,\,p}(\mathbb{R}^n_+) \longrightarrow \Big(\boldsymbol{W}_\ell^{-1,\,p}(\mathbb{R}^n_+) \times W_\ell^{0,\,p}(\mathbb{R}^n_+)\Big) \perp \mathcal{S}^+_{[1+\ell-n/p']}$$

is an isomorphism for any integer $\ell > 0$ and real number $p > 1$, always under hypothesis (3.8). Moreover, by a density argument, we can readily show that

$$T^*(\boldsymbol{v},\,\vartheta) = (-\Delta\boldsymbol{v} + \nabla\vartheta,\, -\operatorname{div}\boldsymbol{v}).$$

So, we have proved that, for any $\ell > 0$, problem (S^+) with $\boldsymbol{g} = \boldsymbol{0}$ admits a unique solution provided $(\boldsymbol{f},\,h) \perp \mathcal{S}^+_{[1+\ell-n/p']}$.

Now, it remains to show that the general problem (S^+) can be reduced to the particular case with $\boldsymbol{g} = \boldsymbol{0}$, by means of a lifting function; and then that the orthogonality condition on the lifted problem is equivalent to the compatibility condition (3.9).

First, by Lemma 1.1, there exists a lifting function $\boldsymbol{u}_g \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^n_+)$ of $\boldsymbol{g}$, *i.e.*, $\boldsymbol{u}_g = \boldsymbol{g}$ on Γ, such that

$$\|\boldsymbol{u}_g\|_{\boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^n_+)} \ \leqslant\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_\ell^{1-1/p,\,p}(\Gamma)}.$$

Set $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_g$; then problem (S^+) is equivalent to the following, with homogeneous boundary conditions:

$$(S^\star)\ \begin{cases} -\Delta\boldsymbol{v} + \nabla\pi & = \boldsymbol{f} + \Delta\boldsymbol{u}_g \quad \text{in } \mathbb{R}^n_+,\\ \operatorname{div}\boldsymbol{v} & = h - \operatorname{div}\boldsymbol{u}_g \quad \text{in } \mathbb{R}^n_+,\\ \boldsymbol{v} & = \boldsymbol{0} \quad \text{on } \Gamma. \end{cases}$$

So, provided $(\boldsymbol{f}+\Delta\boldsymbol{u}_g,\, -h+\operatorname{div}\boldsymbol{u}_g) \perp \mathcal{S}^+_{[1+\ell-n/p']}$, we know that $(S^\star)$ admits a unique solution. This condition is written in the following way:

$$\forall(\boldsymbol{\lambda},\,\mu) \in \mathcal{S}^+_{[1+\ell-n/p']}, \quad \langle\boldsymbol{f},\,\boldsymbol{\lambda}\rangle + \langle\Delta\boldsymbol{u}_g,\,\boldsymbol{\lambda}\rangle - \langle h,\,\mu\rangle + \langle\operatorname{div}\boldsymbol{u}_g,\,\mu\rangle = 0.$$

Moreover, we have the Green formula

$$\langle \Delta \boldsymbol{u}_g,\, \boldsymbol{\lambda}\rangle_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^n_+)\times \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell}(\mathbb{R}^n_+)} = \int_{\mathbb{R}^n_+} \boldsymbol{u}_g \cdot \Delta\boldsymbol{\lambda}\,\mathrm{d}x + \langle \boldsymbol{g},\, \partial_n \boldsymbol{\lambda}\rangle_{\Gamma}\,,$$
$$= \int_{\mathbb{R}^n_+} \boldsymbol{u}_g \cdot \Delta\boldsymbol{\lambda}\,\mathrm{d}x + \big\langle \boldsymbol{g}',\, \partial_n \boldsymbol{\lambda}'\big\rangle_{\Gamma}\,,$$

because $\partial_n \boldsymbol{\lambda}_n = 0$ on Γ, according to the definition of the kernel. Next, we have another Green formula

$$\langle \operatorname{div} \boldsymbol{u}_g,\, \mu\rangle_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^n_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell}(\mathbb{R}^n_+)} = -\int_{\mathbb{R}^n_+} \boldsymbol{u}_g \cdot \nabla\mu\,\mathrm{d}x - \langle g_n,\, \mu\rangle_{\Gamma}\,.$$

Finally, since $-\Delta\boldsymbol{\lambda} + \nabla\mu = \boldsymbol{0}$, we have

$$\int_{\mathbb{R}^n_+} \boldsymbol{u}_g \cdot \Delta\boldsymbol{\lambda}\,\mathrm{d}x - \int_{\mathbb{R}^n_+} \boldsymbol{u}_g \cdot \nabla\mu\,\mathrm{d}x = 0,$$

we then get a first formulation for this compatibility condition:

$$\forall(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^+_{[1+\ell-n/p']}, \quad \langle \boldsymbol{f},\, \boldsymbol{\lambda}\rangle - \langle h,\, \mu\rangle + \big\langle \boldsymbol{g}',\, \partial_n \boldsymbol{\lambda}'\big\rangle_{\Gamma} - \langle g_n,\, \mu\rangle_{\Gamma} = 0.$$

Now, according to the characterization (2.18)–(2.19), we can replace each pair $(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^+_{[1+\ell-n/p']}$ by $\big(\boldsymbol{\varphi} - \nabla(\Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_n \varphi_n),\, -\operatorname{div} \boldsymbol{\varphi}\big)$, where $\boldsymbol{\varphi}$ belongs to $\boldsymbol{\mathcal{A}}^{\Delta}_{[1+\ell-n/p']}$. Then we have

$$\langle \boldsymbol{f},\, \boldsymbol{\lambda}\rangle_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^n_+)\times \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell}(\mathbb{R}^n_+)} = \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle - \big\langle \boldsymbol{f},\, \nabla(\Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_n \varphi_n)\big\rangle\,,$$
$$= \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle + \big\langle \operatorname{div} \boldsymbol{f},\, \Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_n \varphi_n\big\rangle\,,$$

because $(\Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_n \varphi_n)|_{\Gamma} = 0$. Likewise,

$$\langle h,\, \mu\rangle_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^n_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell}(\mathbb{R}^n_+)} = \langle h,\, -\operatorname{div} \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^n_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell}(\mathbb{R}^n_+)}$$
$$= \langle \nabla h,\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^n_+)\times \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell}(\mathbb{R}^n_+)}\,.$$

Moreover, we can remark that, on the one hand , $\mu = -\partial_n \varphi_n$ on Γ, and on the other hand, according to (2.7), we have $\partial_n \boldsymbol{\lambda}' = \partial_n \boldsymbol{\varphi}'$ on Γ, hence the equivalent formulation

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1+\ell-n/p']},$$
$$\langle \boldsymbol{f} - \nabla h,\, \boldsymbol{\varphi}\rangle + \big\langle \operatorname{div} \boldsymbol{f},\, \Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_n \varphi_n\big\rangle + \langle \boldsymbol{g},\, \partial_n \boldsymbol{\varphi}\rangle_{\Gamma} = 0,$$

i.e., the compatibility condition (3.9). □

4. Strong solutions and regularity

In this section, we are interested in the existence of strong solutions, *i.e.*, of solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbb{R}^n_+) \times W^{1,p}_{\ell+1}(\mathbb{R}^n_+)$; and more generally, in the regularity of solutions to the Stokes system (S^+) according to the data.

Theorem 4.1. *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$ be two integers and assume that*

$$n/p' \notin \{1, \dots, \ell+1\} \quad \text{and} \quad n/p \notin \{1, \dots, -\ell-m\}. \tag{4.1}$$

For any $\boldsymbol{f} \in \boldsymbol{W}^{m-1,p}_{m+\ell}(\mathbb{R}^n_+)$, $h \in W^{m,p}_{m+\ell}(\mathbb{R}^n_+)$ and $\boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,p}_{m+\ell}(\Gamma)$, satisfying the compatibility condition (3.9), *problem* (S^+) *admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbb{R}^n_+) \times W^{m,p}_{m+\ell}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{S}^+_{[1-\ell-n/p]}$, and there exists a constant C such that*

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{S}^+_{[1-\ell-n/p]}} \Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbb{R}^n_+)} + \|\pi+\mu\|_{W^{m,p}_{m+\ell}(\mathbb{R}^n_+)}\Big) \leqslant C\Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{m-1,p}_{m+\ell}(\mathbb{R}^n_+)} + \|h\|_{W^{m,p}_{m+\ell}(\mathbb{R}^n_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{m+1-1/p,p}_{m+\ell}(\Gamma)}\Big).$$

We have already proved this result for $\ell = 0$ and $\ell = -1$ in our previous work (see [8], Corollaries 5.5 and 5.7). We will use similar arguments for the other negative weights, with the aim of minimizing the set of critical values, thanks to the known results on the harmonic and biharmonic operators in the half-space. Then, for the positive weights, we will use a regularity argument to avoid the compatibility conditions which would naturally appear in the auxiliary problems with the previous method.

At first, we adapt Lemma 3.1 and its proof for more regular data.

Lemma 4.2. *Let $\ell \leqslant -2$ and $m \geqslant 1$ be two integers and assume that*

$$n/p \notin \{1, \dots, -\ell-m\}. \tag{4.2}$$

For any $\boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,p}_{m+\ell}(\Gamma)$, the Stokes problem (3.1)–(3.3) *has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbb{R}^n_+) \times W^{m,p}_{m+\ell}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{S}^+_{[1-\ell-n/p]}$, with the corresponding estimate.*

Proof. Point (1) of Lemma 3.1 is clearly unchanged.

Since $\boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,p}_{m+\ell}(\Gamma)$, under hypothesis (4.2), problem (P) has a solution $u_n \in W^{m+1,p}_{m+\ell}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{B}_{[1-\ell-n/p]}$ (see [6], Lemma 4.10). Hence we have $\Delta u_n|_\Gamma \in W^{m-1-1/p,p}_{m+\ell}(\Gamma)$, and then problem (Q) has a solution $\pi \in W^{m,p}_{m+\ell}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{N}^\Delta_{[-\ell-n/p]}$ (see [8], Theorem 3.4, for $m = 1$; and [6], Theorem 2.8, for $m \geqslant 2$). Hence, $\nabla'\pi \in$

$\boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^n_+)$, and then problem (R) has a solution $\boldsymbol{u}' \in \boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^n_+)$, unique up to an element of $\boldsymbol{\mathcal{A}}^{\Delta}_{[1-\ell-n/p]}$ (see [4], Corollary 3.4). Likewise, point (3) is unchanged with respect to the proof of Lemma 3.1. □

Proof of Theorem 4.1. (1) Assume that $\ell \leqslant -2$. The proof is quite similar to the one of Theorem 3.3. Here again, the only question is the surjectivity of the Stokes operator for such data. For that, we must simply replace Theorem 1.2 by Theorem 1.3 and Lemma 3.1 by Lemma 4.2 in the proof of the existence of a solution for negative weights in Theorem 3.3.

(2) Assume that $\ell > 0$. We simply extend the regularity argument used in [8] for the cases $\ell = 0$ and $\ell = -1$. Now, hypothesis (4.1) is reduced to

$$n/p' \notin \{1, \dots, \ell + 1\}. \tag{4.3}$$

Since $n/p' \neq \ell + 1$, we have the imbedding $W^{m-1,\,p}_{m+\ell}(\mathbb{R}^n_+) \hookrightarrow W^{-1,\,p}_{\ell}(\mathbb{R}^n_+)$; moreover, $W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+) \hookrightarrow W^{0,\,p}_{\ell}(\mathbb{R}^n_+)$ and $W^{m+1-1/p,\,p}_{m+\ell}(\Gamma) \hookrightarrow W^{1-1/p,\,p}_{\ell}(\Gamma)$ hold. So, thanks to Theorem 3.3, we know that problem (S^+) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n_+) \times W^{0,\,p}_{\ell}(\mathbb{R}^n_+)$. We will show by induction that

$$\begin{aligned}(\boldsymbol{f}, h, \boldsymbol{g}) \in \boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^n_+) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+) \times \boldsymbol{W}^{m+1-1/p,\,p}_{m+\ell}(\Gamma)&\\ \Longrightarrow (\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^n_+) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+).&\end{aligned} \tag{4.4}$$

For $m = 0$, (4.4) is true. Now, assume that (4.4) is true for $0, 1, \dots, m$ and suppose that $(\boldsymbol{f}, h, \boldsymbol{g}) \in \boldsymbol{W}^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+) \times W^{m+1,\,p}_{m+\ell+1}(\mathbb{R}^n_+) \times \boldsymbol{W}^{m+2-1/p,\,p}_{m+\ell+1}(\Gamma)$. Let us prove that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+2,\,p}_{m+\ell+1}(\mathbb{R}^n_+) \times W^{m+1,\,p}_{m+\ell+1}(\mathbb{R}^n_+)$. Since we also have the imbeddings $W^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+) \hookrightarrow W^{m-1,\,p}_{m+\ell}(\mathbb{R}^n_+)$, $W^{m+1,\,p}_{m+\ell+1}(\mathbb{R}^n_+) \hookrightarrow W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+)$ and $W^{m+2-1/p,\,p}_{m+\ell+1}(\Gamma) \hookrightarrow W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$, according to the induction hypothesis, we can deduce that the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^n_+) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+)$. Now, for any $i \in \{1, \dots, n-1\}$, we have

$$-\Delta(\varrho\, \partial_i \boldsymbol{u}) + \nabla(\varrho \partial_i \pi) = \varrho\, \partial_i \boldsymbol{f} + \frac{2}{\varrho} x.\nabla \partial_i \boldsymbol{u} + \Big(\frac{n-1}{\varrho} + \frac{1}{\varrho^3}\Big)\, \partial_i \boldsymbol{u} + \frac{1}{\varrho} x \partial_i \pi.$$

Thus, $-\Delta(\varrho\, \partial_i \boldsymbol{u}) + \nabla(\varrho\, \partial_i \pi) \in \boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^n_+)$. Moreover,

$$\operatorname{div}(\varrho\, \partial_i \boldsymbol{u}) = \frac{1}{\varrho}\, x\, \partial_i \boldsymbol{u} + \varrho\, \partial_i h.$$

Thus, $\operatorname{div}(\varrho\,\partial_i \boldsymbol{u}) \in W^{m,p}_{m+\ell}(\mathbb{R}^n_+)$. We also have $\gamma_0(\varrho\,\partial_i \boldsymbol{u}) = \varrho'\,\partial_i\gamma_0\boldsymbol{u} = \varrho'\,\partial_i \boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$. So, by the induction hypothesis, we can deduce that

$$\forall i \in \{1,\dots,n-1\}, \quad (\partial_i \boldsymbol{u},\, \partial_i \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+\ell+1}(\mathbb{R}^n_+) \times W^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+).$$

It remains to prove that $(\partial_n \boldsymbol{u},\, \partial_n \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+\ell+1}(\mathbb{R}^n_+) \times W^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+)$. For that, let us observe that, for any $i \in \{1,\dots,n-1\}$, we have

$$\begin{array}{lll}
\partial_i\partial_n \boldsymbol{u} & = \partial_n\partial_i \boldsymbol{u} & \in \boldsymbol{W}^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+),\\
\partial_n^2 u_i & = -\Delta' u_i + \partial_i \pi - f_i & \in W^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+),\\
\partial_n^2 u_n & = \partial_n h - \partial_n \operatorname{div}' \boldsymbol{u}' & \in W^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+),\\
\partial_n \pi & = f_n + \Delta u_n & \in W^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+).
\end{array}$$

Hence, $\nabla(\partial_n \boldsymbol{u}) \in \boldsymbol{W}^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+)$ and, knowing that $\partial_n \boldsymbol{u} \in \boldsymbol{W}^{m,\,p}_{m+\ell}(\mathbb{R}^n_+)$, we can deduce that $\partial_n \boldsymbol{u} \in \boldsymbol{W}^{m+1,\,p}_{m+\ell+1}(\mathbb{R}^n_+)$, according to definition (1.1). Consequently, we have $\nabla \boldsymbol{u} \in \boldsymbol{W}^{m+1,\,p}_{m+\ell+1}(\mathbb{R}^n_+)$. Likewise, $\nabla \pi \in \boldsymbol{W}^{m,\,p}_{m+\ell+1}(\mathbb{R}^n_+)$ and, finally, we can conclude that $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{m+2,\,p}_{m+\ell+1}(\mathbb{R}^n_+) \times W^{m+1,\,p}_{m+\ell+1}(\mathbb{R}^n_+)$. □

5. Very weak solutions

The aim of this section is to come back to the homogeneous Stokes system (3.1)–(3.3), but with singular data on the boundary. In [8], we solved it in the cases $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)$ and $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_{0}(\Gamma)$. Here, we will extend these results to the other weights, introducing the question of the kernel and, by duality, the compatibility condition. Thanks to Lemma 2.8, the proof will be more direct.

Theorem 5.1. *Let $\ell \in \mathbb{Z}$ with hypothesis* (2.9). *For any $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)$, satisfying the compatibility condition*

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1+\ell-n/p']}, \qquad \langle \boldsymbol{g},\, \partial_n \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma)} = 0, \tag{5.1}$$

problem (3.1)–(3.3) *admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{S}^+_{[1-\ell-n/p]}$, and there exists a constant C such that*

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in \mathcal{S}^+_{[1-\ell-n/p]}} \Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)} + \|\pi+\mu\|_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)}\Big) \leqslant C\, \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)}.$$

Proof. To start with, let us observe that Lemma 2.8 gives a meaning to these boundary conditions. Besides, thanks to the Green formula (2.11), we get the equivalence between problem (3.1)–(3.3) and the variational formulation: Find $(\boldsymbol{u}, \pi) \in T^p_{\ell,1}(\mathbb{R}^n_+)$ satisfying

$$\begin{aligned} &\forall(\boldsymbol{v}, \vartheta) \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+), \\ &\qquad \text{such that } \boldsymbol{v} = \mathbf{0} \text{ and } \operatorname{div} \boldsymbol{v} = 0 \text{ on } \Gamma, \\ &\langle (\boldsymbol{u}, \pi), T(\boldsymbol{v}, \vartheta)\rangle_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{-1,p}_{\ell-1}(\mathbb{R}^n_+),\ \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)} \\ &\qquad = \langle \boldsymbol{g}, (\partial_n \boldsymbol{v}', -\vartheta)\rangle_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,p'}_{-\ell+1}(\Gamma)}. \end{aligned} \tag{5.2}$$

Now, let us solve problem (5.2). By Theorem 4.1, we know that under hypothesis (2.9), for all $(\boldsymbol{f}, h) \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+) \perp \mathcal{S}^+_{[1-\ell-n/p]}$, there exists a unique solution $(\boldsymbol{v}, \vartheta) \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)/\mathcal{S}^+_{[1+\ell-n/p']}$ to

$$-\Delta \boldsymbol{v} + \nabla\vartheta = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = h \quad \text{in } \mathbb{R}^n_+, \qquad \boldsymbol{v} = \mathbf{0} \quad \text{on } \Gamma,$$

with the estimate

$$\|(\boldsymbol{v}, \vartheta)\|_{\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)/\mathcal{S}^+_{[1+\ell-n/p']}} \leqslant C\Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)} + \|h\|_{W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)}\Big).$$

Consider the linear form $\Lambda:\ (\boldsymbol{f}, h) \longmapsto \langle \boldsymbol{g}, (\partial_n \boldsymbol{v}', -\vartheta)\rangle_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,p'}_{-\ell+1}(\Gamma)}$ defined on $\boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+) \perp \mathcal{S}^+_{[1-\ell-n/p]}$. By (5.1), we have for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1+\ell-n/p']}$, or, equivalently, for any $(\boldsymbol{\lambda}, \mu) \in \mathcal{S}^+_{[1+\ell-n/p']}$,

$$\begin{aligned} |\Lambda(\boldsymbol{f}, h)| &= \Big|\langle \boldsymbol{g}, (\partial_n \boldsymbol{v}', -\vartheta) + \partial_n\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,p'}_{-\ell+1}(\Gamma)}\Big| \\ &= \Big|\langle \boldsymbol{g}, (\partial_n[\boldsymbol{v}' + \boldsymbol{\lambda}'], -[\vartheta + \mu])\rangle_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,p'}_{-\ell+1}(\Gamma)}\Big| \\ &\leqslant C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)}\ \|(\boldsymbol{v}, \vartheta) + (\boldsymbol{\lambda}, \mu)\|_{\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)}. \end{aligned}$$

Thus,

$$\begin{aligned} |\Lambda(\boldsymbol{f}, h)| &\leqslant C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)}\ \|(\boldsymbol{v}, \vartheta)\|_{\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)/\mathcal{S}^+_{[1+\ell-n/p']}} \\ &\leqslant C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)} \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)} + \|h\|_{W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)}\Big). \end{aligned}$$

In other words, Λ is continuous on $\boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+) \perp \mathcal{S}^+_{[1-\ell-n/p]}$, and according to the Riesz representation theorem, we can deduce that there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^n_+) \times W^{-1,p}_{\ell-1}(\mathbb{R}^n_+)/\mathcal{S}^+_{[1-\ell-n/p]}$, which is the dual space of $\boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+) \perp \mathcal{S}^+_{[1-\ell-n/p]}$, such that

$$\forall(\boldsymbol{f}, h) \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+),$$
$$\Lambda(\boldsymbol{f}, h) = \langle \boldsymbol{u}, \boldsymbol{f}\rangle_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^n_+)\times \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)} + \langle \pi, -h\rangle_{W^{-1,p}_{\ell-1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)};$$

i.e., the pair $(\boldsymbol{u}, \pi)$ satisfies (5.2) and the kernel of the associated operator is $\mathcal{S}^+_{[1-\ell-n/p]}$. □

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