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# CONSTRUCTION OF COMPLETE EMBEDDED SELF-SIMILAR SURFACES UNDER MEAN CURVATURE FLOW. PART II

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**Abstract.** We study the Dirichlet problem associated to the equation for self-similar surfaces under mean curvature flow for graphs over the Euclidean plane with a disk removed. We show the existence of a solution provided the boundary conditions on the boundary circle are small enough and satisfy some symmetries.

## 1. Introduction

Let $X(\cdot,t) : M^2 \to \mathbf{R}^3$ be a one-parameter family of immersions of 2-dimensional smooth hypersurfaces in $\mathbf{R}^3$. We say that $M_t = X(M^2,t)$ is a solution to the mean curvature flow if

$$\frac{d}{dt}X(p,t) = \mathbf{H}(p,t), \quad p \in M, t > 0 \tag{1.1}$$
$$X(p,0) = X_0(p)$$

is satisfied for some initial data $X_0$. Here, $\mathbf{H}(p,t)$ is the mean curvature vector of the hypersurface $M_t$ at $X(p,t)$. The mean curvature flow is the gradient flow of the surface area. Huisken [5] has shown that, under certain circumstances, and after an appropriate rescaling, the family of surfaces moving under mean curvature flow tends to a surface satisfying the equation

$$H + X \cdot \nu = 0, \tag{1.2}$$

where $H$ is the mean curvature and $\nu$ is the normal vector so that the mean curvature vector is $\mathbf{H} = H\nu$ and $X$ is the position vector. The sign of $H$ is chosen so that the mean curvature of a convex surface is positive. Surfaces satisfying (1.2) are called shrinking self-similar surfaces or self-shrinkers because the mean curvature flow does not change their shape and merely contracts them. In other words, Huisken showed that self-similar surfaces model the behavior of the flow near singularities, under certain conditions.

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In this paper, we show that, for small enough boundary conditions with some symmetries on the circle $\{\xi \in \mathbf{R}^2 : |\xi| = R\}$, there exists a function $u$ matching the boundary conditions such that the graph of $u$ outside of the disk of radius $R$ is a self-shrinker. This problem is formulated as a Dirichlet problem for a quasilinear elliptic equation on an unbounded domain. The proof uses the geometric nature of the problem as well as standard tools from the theory of differential equations. It is an interesting example of the interaction of techniques from both fields. Our solution $u$ will grow at most linearly at infinity and the self-shrinker will be asymptotic to a cone at infinity. The result provides more evidence to support the existence of possible self-similar surfaces found by Chopp [1] and Ilmanen [6] using computational methods.

1.1. **Main result.** First, note that the graph of a function $u$ over a domain $\Omega$ in the Euclidean plane $\mathbf{R}^2$ satisfies the self-shrinker equation (1.2) if and only if $u$ satisfies the equation

$$\mathcal{E}(u) = g^{ij}(Du(\xi))D_{ij}u(\xi) - \xi \cdot Du(\xi) + u(\xi) = 0, \quad \xi \in \Omega, \tag{1.3}$$

where $g^{ij}(Du) = \delta^{ij} - \frac{D_i u D_j u}{1+|Du|^2}$. The domain we consider throughout this paper is the Euclidean plane with a disk removed $\Omega = \{\xi \in \mathbf{R}^2 : |\xi| > R\}$.

Let $2 > R > \sqrt{3}/2$ and $N \geq 5$ and consider a function $f \in C^4([0, 2\pi])$ satisfying the symmetries

$$f(\theta) = -f(-\theta) = f(\pi/N - \theta). \tag{1.4}$$

**Theorem 1.** *There is an $\varepsilon_0 > 0$ depending on $R$ and $N$ such that, for any $f : [0, 2\pi] \to \mathbf{R}$ with $\|f\|_{C^4[0,2\pi]} = \varepsilon \leq \varepsilon_0$ and $f$ satisfying the symmetries above, there exists a function $u$ on $\Omega = \mathbf{R}^2 \setminus \bar{B}_R$ such that*

$$\mathcal{E}(u) = g^{ij}(Du(\xi))D_{ij}u(\xi) - \xi \cdot Du(\xi) + u(\xi) = 0, \quad \xi \in \Omega, \tag{1.5}$$

$$u = f \text{ on } \partial B_R, \tag{1.6}$$

$$u(r,\theta) = -u(r,-\theta) = u(r, \pi/N - \theta) \text{ for } r > R, \theta \in [0, 2\pi). \tag{1.7}$$

*In addition, we can choose the constant $\varepsilon_0$ uniformly for all $R \in (\sqrt{3}/2, 2)$.*

Here, $u = f$ on $\partial B_R$ means that $u(R,\theta) = f(\theta)$ in polar coordinates. This is a slight abuse of notation which does not induce confusion.

In the proof, we first study the properties of the linear operator $\mathcal{L}$ associated to $\mathcal{E}$. From solutions to the linear problem, we then construct a sub and supersolution to the quasilinear equation $\mathcal{E}u = 0$.

The solution to the elliptic equation (1.3) is found as a limit for time $\tau$ going to infinity of a solution $u$ to the parabolic equation $\partial_\tau u = \mathcal{E}u$.

For initial conditions with bounded gradient, standard theory on parabolic equations assures the existence of a solution to the initial-value problem on a short time interval $[0, \tau_0)$. The constructed sub and supersolution serve as barriers for solutions of the parabolic equation $\partial_\tau u = \mathcal{E}u$ and allow us to control the growth of $u$ at infinity in the space variable for all time. The close relation between the parabolic equation $\partial_\tau u = \mathcal{E}u$ and the mean curvature flow and known results for the mean curvature flow give us interior estimates for the derivatives of $u$. We then have control over the derivatives outside of a bounded annulus around $\partial\Omega$. To bridge the gap, we invoke standard parabolic theory on bounded sets and obtain uniform estimates for derivatives of $u$ for all $\tau \in [0, \tau_0)$. This implies that the solution has to exist for all time $\tau \in [0, \infty)$. We conclude that a subsequence of $u(\cdot, \tau)$ tends to a solution of $\mathcal{E}u = 0$ as $\tau$ goes to infinity by proving a monotonicity formula.

This paper is part of a series of articles aimed towards the construction of new examples of complete embedded self-similar surfaces under mean curvature flow. Our general strategy is inspired by Kapouleas' article [7] and is described in detail in the previous installment [10]. Let us recall it briefly: to construct a new self-similar surface, we take two known examples suitably positioned and replace a neighborhood of their intersection with an appropriately bent scaled Scherk's singly periodic surface. The procedure is called desingularization. The resulting surface is a good approximate solution and the graph of a small function on it should provide us with an exact solution to (1.2). Before considering graphs of functions on the entire surface, we work locally and study the Dirichlet problems with small boundary conditions for graphs of functions on different pieces. The last step would consist of gluing all the solutions together in a smooth manner.

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## 2. Definitions and properties

2.1. **Mean curvature flow.** If the hypersurfaces $M_t$ in the definition of the mean curvature flow can be written locally as graphs of a function $v(\cdot, t)$ over a domain in the $xy$- plane, the quasilinear equation

$$\frac{d}{dt}v = g^{ij}(Dv)D_{ij}v, \quad g^{ij}(Dv) = \delta_{ij} - \frac{D_i v D_j v}{1 + |Dv|^2} \tag{2.1}$$

is equivalent to (1.1) up to tangential diffeomorphisms.

2.2. **Weighted Hilbert Sobolev spaces $H$, $V$ and $V_0$.** We consider the Hilbert space $H = L^2(\Omega, dm(\xi))$, where $\Omega$ is an open subset of $\mathbf{R}^n$ (not necessarily bounded) and $m$ is the Gaussian measure $dm(\xi) = e^{-\frac{1}{2}|\xi|^2}d\xi$. We write $\|f\|_H$ and $(f,g)_H$ for the norm and inner product of $H$. We also define the Hilbert space $V = \{f \in H : \partial_i f \in H \text{ for } i = 1,\dots,n\}$. The inner product of $V$ is

$$(f,g)_V = \int_\Omega \{fg + Df \cdot Dg\} dm(\xi).$$

Let $V_0$ be the closure of $C_c^\infty(\Omega)$ in $V$. Even when $\Omega$ is unbounded the inclusion $V_0 \subset H$ is compact.

## 3. The Linear Operator

Let us define sections $\Omega_{R,N}$ of the outer plane by

$$\Omega_{R,N} = \{(r,\theta) : R < r, -\pi/N < \theta < \pi/N\} \tag{3.1}$$

and their corresponding Hilbert spaces

$$V_{0;Sym}(\Omega_{R,N}) = \{u \in V_0(\Omega_{R,N}) : u(r,\theta) = -u(r,-\theta) = u(r,\pi/N - \theta)\}$$
$$H_{Sym}(\Omega_{R,N}) = \{u \in H(\Omega_{R,N}) : u(r,\theta) = -u(r,-\theta) = u(r,\pi/N - \theta)\}.$$

In this section, we study the properties of $\mathcal{L}$, the linear operator associated to $\mathcal{E}$ defined in (1.3),

$$\mathcal{L}(u) = \Delta u - \xi \cdot Du + u.$$

### 3.1. **Eigenvalues and eigenfunctions.** 
Consider the operator

$$\mathcal{L}_-(u) = \Delta u - \xi \cdot Du - u$$

on $\Omega_{0,N}$. The compactness of the inclusion $V_0(\Omega_{0,N}) \subset H(\Omega_{0,N})$ yields the following theorem.

**Theorem 2.** *The operators $\mathcal{L}_-$ and $\mathcal{L}$ have a countable set of eigenvalues having no limit point except possibly $\lambda = \infty$ and a corresponding $H$-orthogonal basis of eigenfunctions $\varphi_i \in V_0(\Omega_{0,N})$.*

**Proof.** Theory on compact bounded self-adjoint operators in Hilbert spaces gives us the result for $\mathcal{L}_-$. The result for $\mathcal{L}$ then follows immediately. □

Consider eigenfunctions of the form $f(r)g(\theta)$ satisfying the symmetries

$$g(\theta) = -g(-\theta) = g(\pi/N - \theta). \tag{3.2}$$

There is an eigenvalue $\lambda \in \mathbf{R}$ such that

$$\mathcal{L}(f(r)g(\theta)) = f_{rr}g + \frac{1}{r}f_r g + \frac{1}{r^2}fg_{\theta\theta} - rf_r g + fg = \lambda fg.$$

Separation of variables and the symmetry conditions (3.2) on $g$ imply that the only possibilities for $g$ are $g = C\sin(m\theta)$ with $m = kN$, $k \in \mathbf{N}$. Therefore,

$$g_{\theta\theta} = -m^2 g,$$
$$\mathcal{L}_0(f) = f_{rr} + \frac{1}{r}f_r - rf_r + (1-\lambda)f - \frac{1}{r^2}m^2 f = 0.$$

We have

$$\mathcal{L}_0(r^n) = (n^2 - m^2)r^{n-2} + (1-\lambda-n)r^n.$$

$\mathcal{L}_0$ is a linear map on the vector space $V = \{r^m F(r^2) : F \text{ is a polynomial }\}$ and it is upper diagonal in the basis $\{r^m, r^{m+2}, r^{m+4}, \ldots\}$. The operator $\mathcal{L}_0$ has a zero eigenvalue if one of the entries on the diagonal vanishes, i.e. if $1 - \lambda - (m + 2l) = 0$. We therefore have found some of the eigenvalues of our original operator $\mathcal{L}$,

$$\lambda = 1 - (kN + 2l), \quad k \in \mathbf{N}, l \in \{0\} \cup \mathbf{N}, \tag{3.3}$$

where $\mathbf{N} = \{1, 2, \ldots\}$ is the set of natural numbers. For each value of $k, l$ the corresponding eigenfunction is of the form

$$r^{kN} P_{k,l}(r^2)\sin(kN\theta) \tag{3.4}$$

where $P_{k,l}$ is a polynomial of degree $l$.

We now show that the eigenvalues given by (3.3) are in fact all the possible eigenvalues of $\mathcal{L}$. This is done by proving the following theorem.

**Theorem 3.** *The set $S$ of eigenfunctions* (3.4) *corresponding to the eigenvalues* (3.3) *forms a basis for the Hilbert space $H_{Sym}(\Omega_{0,N})$.*

Before we start the proof, note that, for each fixed $k \in \mathbf{N}$ and $s \neq l$,

$$(r^{kN}P_{k,l}\sin(kN\theta), r^{kN}P_{k,s}\sin(kN\theta))_{H(\Omega_{0,N})} = 0$$

since they are eigenfunctions corresponding to different eigenvalues. Hence,

$$\int_{r=0}^{\infty} P_{k,l}(r^2)P_{k,s}(r^2)r^{2kN}e^{-r^2/2}r\ dr = 0.$$

By the change of variable $\rho = r^2$,

$$\int_{\rho=0}^{\infty} P_{k,l}(\rho)P_{k,s}(\rho)\rho^{kN}e^{-\rho/2}d\rho = 0. \tag{3.5}$$

The polynomials $P_{k,l}$, which are closely related to the general Laguerre polynomials, are therefore orthogonal in $L^2(0,\infty;\rho^{kN}e^{-\rho/2}d\rho)$, the $L^2$-space weighted by $\rho^{kN}e^{-\rho/2}d\rho$. In addition, they span the functions $\rho^n$, $n \in \mathbf{N}$.

The fact that $\{P_{k,l}\}_{l\in\{0\}\cup\mathbf{N}}$ is a maximal orthogonal set in $L^2(0,\infty;\rho^{kN}e^{-\frac{\rho}{2}}d\rho)$ follows from the theorem on page 333 in [11] which is recalled below.

**Theorem 4** (Theorem on page 333 [11])**.** *The orthonormal sequence of polynomials attached to a mass distribution function $\mu(x)$ on the interval $(a,b)$, is complete in $L^2(a,b;\mu)$ whenever there exists a number $r>0$ such that the integral $\int_a^b e^{r|x|}d\mu(x)$ exists.*

**Proof of Theorem 3.** It suffices to show that the set of all finite linear combinations of members of $S$ is dense. Let $f(r,\theta)\in H_{Sym}(\Omega_{0,N})$. Since $f$ is periodic in $\theta$ and $f(r,\cdot)\in L^2(-\pi/N,\pi/N)$ for almost every $r$, we can expand $f(r,\cdot)$ in Fourier series for almost every $r\in(0,\infty)$. The symmetries of $f$ imply $f(r,\theta)=\sum_{j\in\mathbf{N}}a_j(r)\sin(jN\theta)$. Using the change of variable $\rho=r^2$, we write

$$\tilde{f}(\rho,\theta)=f(\sqrt{\rho},\theta)=\sum_{j\in\mathbf{N}}b_j(\rho)\sin(jN\theta)\text{ for a.e. }\rho\in(0,\infty). \tag{3.6}$$

The $H$-norm of $f$ is finite, therefore

$$\begin{aligned}\int_{\Omega_{0,N}}|f(r,\theta)|^2e^{-|\xi|^2/2}d\xi &= \int_{r=0}^{\infty}\int_{\theta=-\pi/N}^{\pi/N}|f(r,\theta)|^2d\theta\ e^{-r^2/2}r\ dr\\ &=\frac{\pi}{2N}\int_{\rho=0}^{\infty}\sum_{j\in\mathbf{N}}b_j^2(\rho)e^{-\rho/2}d\rho=\frac{\pi}{2N}\sum_{j\in\mathbf{N}}\int_{\rho=0}^{\infty}b_j^2(\rho)e^{-\rho/2}d\rho<\infty.\end{aligned}$$

Hence, the functions $b_j$ are in $L^2(0,\infty;e^{-\rho/2}d\rho)$ for $j\in\mathbf{N}$. For every $\varepsilon>0$, there is a $J(\varepsilon)\in\mathbf{N}$ so that

$$\frac{\pi}{2N}\sum_{j>J(\varepsilon)}\int_{\rho=0}^{\infty}b_j^2(\rho)e^{-\rho/2}d\rho\le\varepsilon/2.$$

We can approximate $b_j\rho^{-jN/2}\in L^2(0,\infty;\rho^{jN}e^{-\rho/2}d\rho)$ by a linear combination of $P_{j,l}$, since $\{P_{j,l}\}_{l\in\{0\}\cup\mathbf{N}}$ is a maximal orthogonal set . More precisely, for every $\varepsilon>0$ and every $j$, there is a linear combination of $P_{j,l}$'s denoted by $Q_{j,\varepsilon}$ so that

$$\frac{\pi}{2N}\int_0^{\infty}|b_j\ \rho^{-jN/2}-Q_{j,\varepsilon}|^2\rho^{jN}e^{-\rho/2}\ d\rho<2^{-j}\varepsilon.$$

Define $\mathcal{Q}_\varepsilon$ to be the linear combination of elements of $S$ given by

$$\mathcal{Q}_\varepsilon=\sum_{0<j\le J(\varepsilon)}Q_{j,\varepsilon}\rho^{jN/2}\sin(jN\theta).$$

Then

$$\|f - \mathcal{Q}_\varepsilon\|^2_{H(\Omega_{0,N})} = \frac{\pi}{2N}\int_{\rho=0}^{\infty}\Big(\sum_{j>J(\varepsilon)} b_j^2(\rho)e^{-\frac{\rho}{2}} + \sum_{0<j\le J(\varepsilon)} |b_j - Q_{j,\varepsilon}\rho^{\frac{jN}{2}}|^2 e^{-\frac{\rho}{2}}\Big)d\rho$$

$$\le \varepsilon/2 + \sum_{0<j\le J(\varepsilon)} 2^{-j-1}\varepsilon \le \varepsilon.$$

This shows that every function in $H_{Sym}(\Omega_{0,N})$ can be approximated arbitrarily closely by a linear combination of functions of $S$. □

3.2. **Poincaré inequality.** As an immediate consequence of Theorem 3, we have the following inequality for $u \in C_c^\infty(\Omega_{0,N})$:

$$\|u\|_V^2 = -(\Delta u - \xi\cdot Du - u, u)_H \ge -\lambda_0\|u\|_H^2 = (N+1)\|u\|_H^2,$$

where $\lambda_0$ is the lowest eigenvalue of the operator $-\mathcal{L}_-$. This implies a Poincaré inequality

$$\|u\|_{V(\Omega')} \ge \sqrt{N+1}\;\|u\|_{H(\Omega')}, \quad u \in V_0(\Omega') \tag{3.7}$$

for every domain $\Omega' \subset \Omega_{0,N}$.

3.3. **Dirichlet problem.** Let $\Omega_{R,N}$ be defined as in (3.1).

**Lemma 5.** *For any $g \in H(\Omega_{R,N})$, the equation*

$$\mathcal{L}(u) = \Delta u - \xi\cdot Du + u = g \tag{3.8}$$

*possesses a weak solution $u \in V_0(\Omega_{R,N})$.*

**Proof.** This is a classical variational argument for the functional

$$J(u) = \int_{\Omega_{R,N}} \Big\{\frac{1}{2}|Du(\xi)|^2 - \frac{1}{2}u(\xi)^2 + g(\xi)u(\xi)\Big\}\, dm(\xi),$$

over $u \in V_0(\Omega_{R,N})$ and using the Poincaré inequality (3.7). □

3.4. **Maximum Principle.** Let $\Omega'$ be any domain that is a (not necessarily proper) subset of $\Omega_{R,N}$.

**Definition 6.** *We say that $u \in V(\Omega')$ satisfies $u \le 0$ on $\partial\Omega'$ if its positive part $u^+ = \max(u,0) \in V_0(\Omega')$.*

**Theorem 7** (Maximum Principle)**.** *Let $N \ge 5$. Suppose $u \in V(\Omega')$ satisfies*

$$\mathcal{L}u = \Delta u - \xi\cdot Du + u \ge 0 \text{ in } \Omega'$$
$$u \le 0 \text{ on } \partial\Omega';$$

*then $u \le 0$ in $\Omega'$.*

**Proof.** First note that $\psi := r\cos\theta$ is a solution of $\mathcal{L}\psi = 0$. By the hypotheses, we have

$$\int_{\Omega'} Du \cdot Dv - uv\ dm \le 0, \qquad \text{for } v \ge 0, v \in C_0^1(\Omega').$$

Define the function $\varphi$ such that $u = \psi\varphi$. Since $1/\psi$ and $|D\psi|$ are uniformly bounded, $\varphi \in V$. Moreover, $\varphi$ satisfies

$$\int_{\Omega'} \psi D\varphi \cdot Dv - vD\psi \cdot D\varphi + D\psi \cdot D(\varphi v) - \psi\varphi v\ dm \le 0.$$

Since $\mathcal{L}\psi = 0$ and $\varphi v \in V_0$, we have $\int_{\Omega'} D\psi \cdot D(\varphi v) - \psi\varphi v\ dm = 0$. Hence,

$$\int_{\Omega'} \psi D\varphi \cdot Dv - vD\varphi \cdot D\psi\ dm \le 0. \tag{3.9}$$

Take $v = \max(\varphi, 0)$ in (3.9). Note that we can do this since $u^+ \in V_0$ and $\psi \neq 0$, so $v \in V_0$. Therefore,

$$\int_{\Omega'} \psi Dv \cdot Dv\ dm \le \int_{\Omega'} vDv \cdot D\psi\ dm$$

since $Dv = D\varphi$ on the support of $v$. Using $R\cos\frac{\pi}{2N} \le \psi$ and $|D\psi| \le 1$, we get

$$R\cos\frac{\pi}{2N}\int_{\Omega'} |Dv|^2 dm \le \int_{\Omega'} |Dv| v\ dm,$$

and

$$R\cos\frac{\pi}{2N}\|Dv\|_H^2 \le \|Dv\|_H\|v\|_H$$

by the Cauchy-Schwarz inequality. Our Poincaré inequality (3.7) implies that

$$R\cos\frac{\pi}{2N}\|Dv\|_H^2 \le \|Dv\|_H \frac{1}{\sqrt{N}}\|Dv\|_H.$$

A simple computation shows that, for $N \ge 5$, $R\cos\frac{\pi}{2N} - \frac{1}{\sqrt{N}} \ge \frac{1}{\sqrt{2}}\cos\frac{\pi}{2N} - \frac{1}{\sqrt{N}} > 0$. Therefore, $\|Dv\|_H = 0$. But $v \in V_0(\Omega')$, so $v \equiv 0$; this means that $\varphi \le 0$ and $u \le 0$ in $\Omega'$. □

3.5. **Boundary-value problem.** Let $F \in C^4([0, 2\pi])$ with the symmetries (1.4): $F(\theta) = -F(-\theta) = F(\pi/N - \theta)$. We will use the notation $K_0 = \|F\|_{C^4([0,2\pi])}$ throughout Section 3.

**Definition 8.** *We say that $u \in V(\Omega)$ $(V(\Omega_{R,N}))$ is a solution to the problem*

$$\mathcal{L}u = 0,\ \textit{in } \Omega\ (\Omega_{R,N} \textit{ respectively}), \quad u = F \textit{ on } \partial\Omega\ (\partial\Omega_{R,N} \textit{ respectively})$$

*if $u$ satisfies*

$$\mathcal{L}u = 0, \text{ in } \Omega \ (\Omega_{R,N} \text{ respectively}), \quad u - \psi \in V_0(\Omega) \ (V_0(\Omega_{R,N}) \text{ respectively}),$$

*where $\psi$ is a function in $C^4(\bar{\Omega})$ ($C^4(\bar{\Omega}_{R,N})$ respectively) with $\psi(R,\theta) = F(\theta)$ and $\psi(r,\theta) = -\psi(r,-\theta) = \psi(r,\pi/N - \theta)$ for $r \geq R, \theta \in [0,2\pi)$ ($r \geq R, \theta \in [-\pi/N, \pi/N]$ respectively).*

It follows from Lemma 5 and the maximum principle (Theorem 7) that the solution $u$ to $\mathcal{L}u = 0$ in $\Omega_{R,N}$, $u = F$ on $\partial\Omega_{R,N}$ exists and is well defined; in other words it is unique and independent of the choice of $\psi$.

Denote by $U_1$ the extension of $u$ to $\Omega$ given by

$$U_1(r,\theta) = u(r,\theta) \quad \text{in } \Omega_{R,N}$$
$$U_1(r,\theta) = -U_1(r,-\theta) = U_1(r,\pi/N - \theta) \quad \text{in } \Omega.$$

The function $U_1$ is a solution to the boundary-value problem

$$\mathcal{L}U_1 = \Delta U_1 - \xi \cdot DU_1 + U_1 = 0 \text{ in } \Omega, \quad U_1 = F \text{ on } \partial\Omega. \tag{3.10}$$

Moreover, it is smooth by standard elliptic theory.

3.6. **Behavior of solution at infinity.** Let $u$ be the solution to the problem $\mathcal{L}u = 0$ in $\Omega_{R,N}$, $u = F$ on $\partial\Omega_{R,N}$. We can choose $\psi$ in Definition 8 satisfying $\|\psi\|_\infty \leq \|F\|_\infty$ with $\psi = 0$ in $\Omega_{R,N} \setminus B_{R+2}$. Let $a = 3\|F\|_\infty$ and consider the function $w_1 = u - ar\cos\theta$. Since $N \geq 5$, $a = 3\|F\|_\infty \geq \frac{2\|F\|_\infty}{\cos(\pi/N)}$, therefore $w_1 \leq 0$ on $\partial\Omega_{R,N}$. It follows from the maximum principle that $w_1 \leq 0$ in $\Omega_{R,N}$, hence $u(r,\theta) \leq ar\cos\theta$ in $\Omega_{R,N}$. A similar argument with $w_2 = -ar\cos\theta - u$ gives a lower bound on $u(r,\theta)$, so

$$|u(r,\theta)| \leq ar\cos(\theta) \leq 3K_0 \, r \text{ on } \Omega_{R,N}. \tag{3.11}$$

Therefore, we also have

$$|U_1(\xi)| \leq 3K_0|\xi|, \quad \xi \in \Omega. \tag{3.12}$$

3.7. **Bounds on the first and second derivatives of the solution.** Let $U_1$ be the solution to (3.10) from section 3.5.

**Lemma 9.** *There is a constant $K_1$ independent of $U_1$ and $F$ so that*

$$|DU_1| + |\xi||D^2U_1| \leq K_1\|F\|_{C^4[0,2\pi]} \tag{3.13}$$

**Proof. Step 1: Estimate away from the boundary.** The function $U_1$ is smooth in $\Omega$ by elliptic theory. A computation shows that the function

$$v(x,t) = \sqrt{1-t}\, U_1\Big(\frac{x}{\sqrt{2(1-t)}}\Big)$$

satisfies the heat equation $v_t = \Delta v$. The theory for parabolic equations gives us the following estimates on the derivatives of $v$:

$$|D^m v(p,0)| \le \frac{C(m)}{s^m} \sup_{B_s(p)\times(-s^2,0)} |v(x,t)|, \tag{3.14}$$

where $C(m)$ is a constant independent of $v$ or $s$ and as long as $v$ exists in $B_s(p) \times (-s^2, 0)$. This is true provided

$$\frac{\min_{x\in B_s(p)} |x|}{\max_{t\in(-s^2,0)} \sqrt{2(1-t)}} = \frac{|p| - s}{\sqrt{2(1+s^2)}} > R.$$

In particular, it is true if $|p| > \frac{5\sqrt{2}}{3} R$ and $s = |p|/10$. From (3.14), we get

$$|Dv(p,0)| \le \frac{10\ C(1)}{|p|} \sup_{B_s(p)\times(-s^2,0)} |v(x,t)|.$$

The last factor of the right-hand side can be estimated using equation (3.12) to obtain

$$\sup_{B_s(p)\times(-s^2,0)} |v(x,t)| \le \sqrt{2}\ K_0(|p| + s).$$

Using the notation $\xi = p/\sqrt{2}$, we have

$$|DU_1(\xi)| = \sqrt{2}|Dv(p,0)| \le 22\ C(1)K_0 \text{ for } |\xi| > 5R/3.$$

Similarly, we get an estimate on the second derivative

$$|D^2U_1(\xi)| = 2|D^2v(p,0)| \le \frac{220\ C(2)}{|\xi|} K_0 \text{ for } |\xi| > 5R/3.$$

Therefore,

$$|DU_1| + |\xi||D^2U_1| \le C\|F\|_{L^\infty} \text{ for } |\xi| > 5R/3. \tag{3.15}$$

Since $R < 2$, the estimate (3.15) is valid for $|\xi| \ge 4$ in particular.

**Step 2: Estimates up to the boundary.** Consider the equation $\mathcal{L}U_1 = 0$ in the annulus $A_{R,4} = B_4 \setminus \bar{B}_R$. The domain $A_{R,4}$ is bounded so global regularity results for elliptic equations from [4] imply that

$$\|U_1\|_{W^{4,2}(A_{R,4})} \le C(\|U_1\|_{L^2(A_{R,4})} + \|\psi\|_{W^{4,2}(A_{R,4})}), \tag{3.16}$$

where $\psi$ is a function in $W^{4,2}(A_{R,4})$ such that $\psi - U_1 \in W_0^{1,2}(A_{R,4})$.

Note that we can bound the first four derivatives of $U_1$ for $|\xi| = 4$ by an argument similar to the one in step 1,

$$|D^j U_1|(\xi) \le C\|F\|_{L^\infty} \text{ for } i = 1,\dots,4 \text{ and } \xi \in \partial B_4.$$

It is then clear that we can find a constant $C$ independent of $F$ and $U_1$, and a function $\psi$, for which

$$\psi|_{\partial B_R} = F, \qquad \psi|_{\partial B_4} = U_1|_{\partial B_4}, \quad \|\psi\|_{C^4(A_{R,4})} \leq C\|F\|_{C^4[0,2\pi]}. \tag{3.17}$$

With this choice of $\psi$, equation (3.16) gives us

$$\|U_1\|_{W^{4,2}(A_{R,4})} \leq C(\|\psi\|_{W^{4,2}(A_{R,4})} + \|U_1\|_{L^2(A_{R,4})}). \tag{3.18}$$

Using Sobolev's inequality, (3.17), (3.18) and (3.12), we get

$$\|U_1\|_{C^2(A_{R,4})} \leq C\|F\|_{C^4[0,2\pi]}. \tag{3.19}$$

Combining (3.15) and (3.19), we obtain the desired result. □

## 4. Finding sub and supersolutions

This section is devoted to the construction of a subsolution $u_-$ and a supersolution $u_+$ to the problem (1.3)-(1.6), or equivalently, to

$$\mathcal{L}u = \Delta u - \xi \cdot Du + u = \frac{D_i u D_j u D_{ij} u}{1 + |Du|^2}, \qquad u|_{\partial\Omega} = \varepsilon F \tag{4.1}$$

with small $\varepsilon$. Let us discuss the strategy for finding a supersolution first. Since the boundary data is of order $\varepsilon$, we write $u = \varepsilon U$,

$$\mathcal{L}U = \varepsilon^2 \frac{D_i U D_j U D_{ij} U}{1 + \varepsilon^2 |DU|^2}, \qquad U|_{\partial\Omega} = F$$

and decompose $U$ into the two terms $U = U_1 + \varepsilon^2 U_2$, where

$$\mathcal{L}U_1 = 0, \qquad\qquad U_1|_{\partial\Omega} = F, \tag{4.2}$$

$$\mathcal{L}U_2 = \frac{D_i U D_j U D_{ij} U}{1 + \varepsilon^2 |DU|^2}, \qquad\qquad U_2|_{\partial\Omega} = 0. \tag{4.3}$$

We discussed the existence of such a function $U_1$ in Section 3.6. Choosing an appropriate function $U_2$ is done below.

4.1. **Preliminary computations for finding a supersolution.** In order to get a supersolution, we want $U_2$ to satisfy $\mathcal{L}U_2 - \frac{D_i U D_j U D_{ij} U}{1+\varepsilon^2 |DU|^2} \leq 0$. Roughly estimating $U$ by $U_1$ and bounding the derivatives of $U_1$ using Lemma 9, we are looking for a $U_2$ so that

$$\mathcal{L}U_2 - \frac{D_i U D_j U D_{ij} U}{1 + \varepsilon^2 |DU|^2} \leq \mathcal{L}U_2 + C_1^3/r \leq 0,$$

where $C_1 = K_1\|F\|_{C^4[0,2\pi]}$. Define $v_a = a(r - \frac{R^2}{r})$. A simple computation gives

$$\mathcal{L}v_a = -\frac{aR^2}{r^3} + \frac{a}{r}(1-2R^2), \qquad v_a = 0 \text{ on } \partial\Omega.$$

This makes $v_a$ a perfect candidate for $U_2$.

**Remark 10.** The term $a(1-2R^2)/r$ has to counterbalance the contribution $C_1^3/r$ from the nonlinear term. Therefore, we need $(1-2R^2)$ to be negative. This justifies the imposed lower bound $R > 1/\sqrt{2}$.

4.2. **Existence of a supersolution and a subsolution.** Let $\varepsilon > 0$ be a small constant. Consider the function $u_+ = \varepsilon(U_1 + \varepsilon^2 v_a)$, where $a$ is a constant to be chosen later.

$$\mathcal{E}u_+ = -\varepsilon^3\mathcal{L}v_a - \varepsilon^3\frac{D_i(U_1+\varepsilon^2 v_a)D_j(U_1+\varepsilon^2 v_a)D_{ij}(U_1+\varepsilon^2 v_a)}{1+\varepsilon^2|D(U_1+\varepsilon^2 v_a)|^2}. \tag{4.4}$$

We have $|D_i v_a| \le 2a$ and $|D_{ij}v_a| \le \frac{6a}{r}$, which, combined with (3.13), give us

$$\begin{aligned}\frac{1}{\varepsilon^3}\mathcal{E}u_+ &\le -\frac{aR^2}{r^3} + \frac{a}{r}(1-2R^2) + (C_1+\varepsilon^2 2a)^2\Big(\frac{C_1}{r} + \varepsilon^2\frac{6a}{r}\Big)\\ &= \frac{1}{r}\Big(-\frac{aR^2}{r^2} - a(2R^2-1) + C_1^3 + 10a\varepsilon^2C_1^2 + 28a^2\varepsilon^4C_1 + 24a^3\varepsilon^6\Big).\end{aligned}$$

Here we used the notation $C_1 = K_1\|F\|_{C^4}$. We replace $a$ by $a = kC_1^3$, with $k$ a constant to be chosen later, to get

$$\begin{aligned}\frac{r}{\varepsilon^3}\mathcal{E}u_+ \le{}& C_1^3\Big(\frac{-kR^2}{r^2} - k(2R^2-1)+1\Big)\\ &+ C_1^3\left(10\varepsilon^2kC_1^2 + 28(\varepsilon^2kC_1^2)^2 + 24(\varepsilon^2kC_1^2)^3\right).\end{aligned} \tag{4.5}$$

If $\varepsilon$ is small enough so that $\eta := \varepsilon^2kC_1^2 \le 1$, the equation (4.5) becomes

$$\frac{r}{\varepsilon^3}\mathcal{E}u_+ \le C_1^3\Big(\frac{-kR^2}{r^2} - k(2R^2-1) + 1 + 62\eta\Big).$$

Hence, to have a supersolution, it suffices to find $k$ and $\varepsilon$ that satisfy

$$\eta := \varepsilon^2kC_1^2 \le 1 \text{ and } -k(2R^2-1) + 1 + 62\eta \le 0.$$

If we take $R_0$ so that $R \ge R_0 > 1/\sqrt{2}$, both inequalities are true for $k = \frac{2}{2R_0^2-1}$ and $\varepsilon^2C_1^2 \le \frac{1}{62k} = \frac{2R_0^2-1}{124}$. Recall that $C_1 = K_1\|F\|_{C^4}$, therefore, for $\varepsilon$ and $k$ so that

$$\varepsilon\|F\|_{C^4} \le \frac{1}{K_1}\sqrt{\frac{2R_0^2-1}{124}}, \quad k = \frac{2}{2R_0^2-1}, \text{ and } R \ge R_0 > 1/\sqrt{2},$$

the function $u_+ = \varepsilon(U_1 + \varepsilon^2 k C_1^3 (r - \frac{R^2}{r}))$ is a supersolution $\mathcal{E}u_+ \leq 0$ with boundary condition $u_+|_{\partial\Omega} = \varepsilon F$. A similar argument shows that $u_- = \varepsilon(U_1 - \varepsilon^2 k C_1^3 (r - \frac{R^2}{r}))$ is a subsolution with boundary condition $u_-|_{\partial\Omega} = \varepsilon F$. If we denote by $f = \varepsilon F$ and $u_1 = \varepsilon U_1$, we have just proved the following result.

**Lemma 11.** *Assume $R \geq R_0 > 1/\sqrt{2}$ and denote by $\Omega = \mathbf{R}^2 \setminus \bar{B}_R$ the plane with a hole of radius $R$ at the origin. Let $f$ be a function in $C^4[0, 2\pi]$ with the symmetries* (1.4)*, $u_1$ a solution to the linear equation $\mathcal{L}u_1 = 0$, $u_1|_{\partial\Omega} = f$ and $K_1$ a constant* (*independent of $f$*) *so that*

$$|Du_1| + |\xi||D^2 u_1| \leq K_1 \|f\|_{C^4}.$$

*If $\varepsilon = \|f\|_{C^4[0,2\pi]}$ satisfies*

$$0 < \varepsilon = \|f\|_{C^4} \leq \frac{1}{K_1}\sqrt{\frac{2R_0^2 - 1}{124}},$$

*then the function $u_+ = u_1 + K_1^3 \varepsilon^3 k(r - \frac{R^2}{r})$, where $k = \frac{2}{2R_0^2 - 1}$, satisfies*

$$\mathcal{E}u_+ = g^{ij}(Du_+) D_{ij} u_+ - \xi \cdot Du_+ + u_+ \leq 0 \ \textit{in } \Omega$$
$$u_+ = f \ \textit{on } \partial\Omega,$$

*and the function $u_- = u_1 - K_1^3 \varepsilon^3 k(r - \frac{R^2}{r})$ satisfies*

$$\mathcal{E}u_- = g^{ij}(Du_-) D_{ij} u_- - \xi \cdot Du_- + u_- \geq 0 \ \textit{in } \Omega$$
$$u_- = f \ \textit{on } \partial\Omega.$$

From the bounds on the derivatives of $U_1$ and the definition of $u_+$ and $u_-$, we have the following.

**Corollary 12.** *Let $u_+$ and $u_-$ be defined as in the lemma above. Then there exists a constant $K_2$ depending only on $\|f\|_{C^4}$ and $R_0$ so that*

$$|\xi|^{-1}|u_\pm| + |Du_\pm| + |\xi||D^2 u_\pm| \leq K_2, \quad \xi \in \Omega.$$

## 5. THE PARABOLIC EQUATION $\partial_\tau u = \mathcal{E}u$

In this section, we prove that there exists a solution to the parabolic equation $\partial_\tau u = \mathcal{E}u$ for all time $\tau \in [0, \infty)$ for well-chosen initial and boundary conditions $u_0$ and $f$ respectively.

To avoid confusion, let us fix our notations: we denote by $D_i$ the ordinary derivative with respect to the $i$-th coordinate $x_i$ or $\xi_i$ of a spatial variable and by $Dv$ the gradient of the function $v$ with respect to the spatial variables. The notation $\partial_\tau$ or $\partial_t$ is reserved for the derivative with respect to time.

5.1. **Initial condition and short time existence.** For the initial condition to the parabolic equation, we choose a $C^3$ function $u_0$ in $\Omega$ that stays between the sub and supersolutions constructed in Lemma 11 and has bounded derivatives

$$u_- \leq u_0 \leq u_+, \qquad \|D_\xi u_0\|_{C^2(\bar{\Omega})} \leq M_0 < \infty.$$

Moreover, we assume that $u_0$ satisfies the following symmetries and compatibility conditions:

$$u_0(r,\theta) = -u_0(r,-\theta) = u_0(r,\pi/N-\theta), \tag{5.1}$$

$$u_0(R,\theta) = f(\theta), \quad \mathcal{E}(u_0)(R,\theta) = 0. \tag{5.2}$$

Let us define the domains $\mathcal{O}_t = \{x \in \mathbf{R}^2 : |x| > \sqrt{2(1-t)}\ R\}$ and $\mathcal{Q}_T = \{(x,t) : 0 < t < T, |x| > \sqrt{2(1-t)}\ R\}$. We consider the following corresponding problem for the mean curvature flow:

$$\partial_t v(x,t) = g^{ij}(Dv)(x,t)D_{ij}v(x,t), \tag{5.3a}$$

$$v(x,0) = v_0(x), x \in \mathcal{O}_0 \tag{5.3b}$$

$$v(x,t) = \sqrt{2}f(\frac{x}{\sqrt{2(1-t)}}), t \geq 0, x \in \partial\mathcal{O}_t, \tag{5.3c}$$

for some initial condition $v_0(x)$. Standard parabolic theory assures the existence of a smooth short time solution $v(x,t)$ to the boundary-value problem (5.3) with initial condition $v_0(x) = \sqrt{2}u_0(\frac{x}{\sqrt{2}})$. Moreover, if we denote the maximal time interval in which $v$ exists by $[0,t_0)$, we have

$$M_t = \sup_{\mathcal{O}_t}|Dv| + \sup_{\mathcal{O}_t}|D^2v| < \infty, \quad t \in [0,t_0). \tag{5.4}$$

For more details, we refer to [3].

In order to show that the solution $v$ exists for all time $t$, we argue by contradiction and assume that $t_0 < \infty$ then show that $M_t$ is uniformly bounded for all $t \in [0,t_0)$. Standard parabolic theory implies that the solution $v$ can then be continued past $t_0$, which contradicts the maximality of $[0,t_0)$. The rest of this section is devoted to the proof of a uniform bound on $M_t$.

The function $u$ defined by

$$v(x,t) = \sqrt{2(1-t)}\ u\big(\frac{x}{\sqrt{2(1-t)}}, -\frac{1}{2}\ln(1-t)\big) \tag{5.5}$$

satisfies

$$\partial_\tau u = g^{ij}(Du)D_{ij}u - \xi \cdot Du + u \text{ in } \Omega \times [0,\tau_0) \tag{5.6a}$$

$$u(\xi,0) = u_0(\xi), \xi \in \Omega, \qquad u(\xi,\tau) = f \text{ on } \partial\Omega \times [0,\tau_0), \tag{5.6b}$$

where we used the change of variables $\xi = \frac{x}{\sqrt{2(1-t)}}, \tau = -\frac{1}{2}\ln(1-t)$ and $\tau_0 = -\frac{1}{2}\ln(1-t_0)$.

**Lemma 13.** *Let $u$ be a smooth solution of* (5.6a) *in* $\Omega\times[0,\tau_0)$ *with boundary condition* (5.6b)*; then*

$$\max_{\Omega\times[0,\tau_1]} |D_\xi u| + \max_{\Omega\times[0,\tau_1]} |D^2_{\xi\xi} u| < M(\tau_1) \text{ if } \tau_1 < \tau_0. \tag{5.7}$$

**Proof.** To each solution $u$ to (5.6) there corresponds a solution $v$ to the problem (5.3) via the formula (5.5). This lemma is therefore a consequence of (5.4). □

5.2. **Bounds on the function $u$.** Here, we show that a solution $u$ to the problem (5.6) stays between the subsolution $u_-$ and the supersolution $u_+$.

**Lemma 14.** *Let $u(\xi,\tau)$ be a $C^2$ solution to the problem* (5.6)*; then*

$$u_-(\xi) \le u(\xi,\tau) \le u_+(\xi), \quad |\xi| \ge R, 0 \le \tau < \tau_0. \tag{5.8}$$

**Proof.** The proof is a slight modification of the proof of a maximum principle for parabolic equations. Let $K_2$ be as in Corollary 12. For small $\varepsilon > 0$, consider the function $W = \psi(u_- - u) - \varepsilon$, where $\psi(\xi,\tau) := e^{-\varepsilon|\xi|-(1+\delta)\tau}$ and $\delta$ is to be chosen later. Note that $W$ becomes negative as $|\xi|$ tends to infinity. We have $\partial_\tau\psi = -(1+\delta)\psi, D_i\psi = -\varepsilon\frac{\xi_i}{|\xi|}\psi$, and $D_{ij}\psi = \big[-\frac{\varepsilon}{|\xi|}(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}) + \varepsilon^2\frac{\xi_i\xi_j}{|\xi|^2}\big]\psi$. A computation shows that $W$ satisfies

$$\partial_\tau W \le -\delta(W+\varepsilon) - \psi\xi\cdot DW - \varepsilon|\xi|(W+\varepsilon) + \psi\big(g^{ij}(Du_-) - g^{ij}(Du)\big)D_{ij}u_- \\ + g^{ij}(Du)\Big(D_{ij}W + 2\varepsilon\frac{\xi}{|\xi|}\cdot DW + (W+\varepsilon)\frac{D_{ij}\psi}{\psi}\Big). \tag{5.9}$$

Fix $\tau_1 \in [0,\tau_0)$. The estimates in Corollary 12 and Lemma 13 guarantee the existence of a large constant $\eta$ such that $e^{-\varepsilon|\xi|}(|u_-| + |u|) - \varepsilon \le -\varepsilon/2$ for $|\xi| \ge \eta$, $\tau \in [0,\tau_1)$. Denote by $A_{R,\eta}$ the annulus $\{R < |\xi| < \eta\}$. On the boundary of the cylinder $A_{R,\eta}\times[0,\tau_1]$, we have

$$\begin{aligned} W(\xi,\tau) &\le -\varepsilon/2 && |\xi| = \eta, \tau\in[0,\tau_1] \\ W(\xi,\tau) &\le -\varepsilon && |\xi| = R, \tau\in[0,\tau_1] \\ W(\xi,0) &\le -\varepsilon && A_{R,\eta}\times\{0\}. \end{aligned}$$

Suppose that $W > 0$ for some point in the cylinder $A_{R,\eta}\times[0,\tau_1]$. There is a first time $\tau_2 \in (0,\tau_1)$ for which there is a point $\hat\xi$ with $W(\hat\xi,\tau_2) = 0$.

At this point $(\hat{\xi}, \tau_2)$, $DW(\hat{\xi}, \tau_2) = 0, -D^2W(\hat{\xi}, \tau_2)$ is positive definite, and $\partial_\tau W(\hat{\xi}, \tau_2) \geq 0$. At $(\hat{\xi}, \tau_2)$, the equation (5.9) yields

$$0 \;\leq\; -\varepsilon\delta \;-\; \varepsilon^2|\hat{\xi}| \;+\; \psi\big(g^{ij}(Du_-) \;-\; g^{ij}(Du)\big)D_{ij}u_- \;+\; \varepsilon g^{ij}(Du)\frac{D_{ij}\psi}{\psi}.$$

The last term can be estimated by $\varepsilon g^{ij}\frac{D_{ij}\psi}{\psi} \leq \frac{\varepsilon^2}{|\xi|} + \frac{\varepsilon^2}{|\xi|} + \varepsilon^3 \leq 4\varepsilon^2$. To bound the penultimate term, note that

$$|\partial_{p_k} g^{ij}(p)| = \Big| - \frac{\delta_{ik}p_j + \delta_{jk}p_i}{1+|p|^2} + \frac{4p_ip_jp_k}{(1+|p|^2)^2}\Big| \leq 6$$
$$(Du_- - Du)(\xi_2, \tau_2) = \frac{\varepsilon D\psi}{\psi^2}(\xi_2, \tau_2),$$

therefore

$$\big|\psi\big(g^{ij}(Du_-) - g^{ij}(Du)\big)D_{ij}u_-\big| \leq 12\sqrt{2}\varepsilon^2 K_2$$

at $(\xi_2, \tau_2)$ by the mean value theorem. We obtain

$$0 \leq -\varepsilon\delta - \varepsilon^2|\xi_2| + 12\sqrt{2}\varepsilon^2 K_2 + 4\varepsilon^2 \leq \varepsilon(-\delta + 12\sqrt{2}\varepsilon K_2 + 4\varepsilon).$$

The choice of $\delta = 18\varepsilon K_2 + 4\varepsilon$ makes the last expression negative. This is not possible. We have therefore shown that $W(\xi, \tau) \leq 0$ for $R \leq |\xi| \leq \eta, \tau \in [0, \tau_1]$. The argument stays valid for any $\eta' > \eta$, thus

$$u_-(\xi, \tau) - u(\xi, \tau) \leq \varepsilon e^{\varepsilon|\xi| + (1+18K_2\varepsilon + 4\varepsilon)\tau}, \quad (\xi, \tau) \in \Omega \times [0, \tau_1].$$

Letting $\varepsilon$ tend to 0, we obtain $u_-(\xi, \tau) \leq u(\xi, \tau)$, for $(\xi, \tau) \in \Omega \times [0, \tau_1]$. Since $\tau_1$ is arbitrary, we have $u_-(\xi, \tau) \leq u(\xi, \tau)$, for $(\xi, \tau) \in \Omega \times [0, \tau_0)$. The inequality $u(\xi, \tau) \leq u_+(\xi, \tau)$ is proved in a similar way. □

5.3. **Bound on the first derivative $|Du|$.** We prove the bound in two steps. First, we use Lemma 14 and interior estimates from the mean curvature flow to bound $|Dv|$. The interior estimate on $|Dv|$ gives us estimates for $|Du|$ for points staying farther away as time progresses. The second step is to consider an annulus $A_{R,\eta} = \{R < |\xi| < \eta\}$, with $\eta$ large enough to have bounds for $|Du|$ on $|\xi| = \eta$ from the first step. A maximum principle for the parabolic equation satisfied by $|Du|^2$ yields estimates in the interior of the annulus, and therefore on the whole domain.

**Lemma 15.** *Let $u(\xi, \tau)$ be a smooth solution to* (5.6a) *in the cylinder $\Omega \times [0, \tau_0)$ with boundary conditions* (5.6b). *There is a constant $M_1$ depending only on the boundary conditions $u_0$ and $f$ such that*

$$|Du(\xi, \tau)| \leq M_1 \tag{5.10}$$

*for all times $\tau$ for which $u$ exists and $|\xi| \geq R$.*

**Proof. Step 1. Away from the boundary.** Let $v(x,t)$ be the corresponding solution to the MCF defined by equation (5.5). We use the interior estimate below established by Ecker and Huisken.

**Theorem 16** (Theorem 2.3, page 551 [2])**.** *The gradient of the height function $v$ satisfies the estimate*

$$\sqrt{1+|Dv(x_0,t)|^2} \le C_1(n) \sup_{B_\rho(x_0)} \sqrt{1+|Dv(x,0)|^2}$$
$$\exp[C_2(n)\rho^{-2} \sup_{[0,T]}( \sup_{B_\rho(x_0)\times[0,T]} v - v(x_0,t))^2], \quad (5.11)$$

*where $0 \le t \le T, B_\rho(x_0)$ is a ball in $\Omega$ and $n$ is the dimension of the graph of $v$.*

The first factor is estimated by

$$|Dv_0| = |Du_0| < \|Du_0\|_{C^2(\bar{\Omega})} = M_0. \quad (5.12)$$

To bound the second factor, we recall that $u_-(\xi) \le u(\xi,\tau) \le u_+(\xi)$, therefore Corollary 12 gives us

$$|v(x,t)| \le K_2|x|. \quad (5.13)$$

The function $v(x,t)$ is defined on the domain $\{(x,t) : 0 \le t < t_0, |x| \ge \sqrt{2(1-t)}\ R\}$; in particular, it is defined for $|x| > R\sqrt{2}$ at any time $0 \le t < t_0$. Let $|x_0| \ge 2R\sqrt{2}$ and $\rho = |x_0|/4$. The estimates (5.12) and (5.13) yield

$$\sqrt{1+|Dv(x_0,t)|^2} \le C \text{ for } |x_0| \ge 2\sqrt{2}R \text{ and } t \in [0,t_0).$$

Note that $Du\big(\frac{x}{\sqrt{2(1-t)}}, -\frac{1}{2}\ln(1-t)\big) = Dv(x,t)$. With the change of variables $\tau = -\frac{1}{2}\ln(1-t)$ and $\xi = \frac{x}{\sqrt{2(1-t)}} = \frac{xe^\tau}{\sqrt{2}}$, we get

$$|Du(\xi,\tau)| \le K_3, \qquad \text{for } |\xi| \ge 2Re^\tau, \tau \in [0,\tau_0). \quad (5.14)$$

Without loss of generality, we can assume that $K_3 \ge M_0 = \|Du_0\|_{C^2(\bar{\Omega})}$.

**Step 2: Up to the boundary, using a maximum principle.** A computation shows that the function $w = |Du|^2$ satisfies the parabolic equation

$$\partial_\tau w \le g^{ij}(Du)D_{ij}w + g^{ij}_{p_l}(Du)D_{ij}u\ D_l w - \xi \cdot Dw,$$

where $g^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$ and $g^{ij}_{p_l} = \frac{\partial}{\partial p_l} g^{ij}$. Denote by $Z_l = g^{ij}_{p_l}(Du)D_{ij}u$; then

$$\partial_\tau w \le g^{ij}(Du)D_{ij}w + Z_l D_l w - \xi \cdot Dw. \quad (5.15)$$

Choose $0 < \tau_1 < \tau_0$. It follows from Lemma 13 that the coefficients in (5.15) are bounded on $\Omega \times [0,\tau_1]$. Moreover, from (5.14), we know that $w$

is bounded in the cylinder $\Omega \times [0, \tau_1]$. A maximum principle for parabolic equations on unbounded domains (see Theorem 8.1.4 in [8] for example) implies that the function $w$ in $\Omega \times [0, \tau_1)$ is smaller than its supremum on the parabolic boundary $[\partial\Omega \times (0, \tau_1)] \cup [\bar{\Omega} \times \{0\}]$. Since $\tau_1$ is arbitrary, the result is valid on the time interval $[0, \tau_0)$ as well:

$$\sup_{\Omega\times(0,\tau_0)} w \leq \sup_{[\partial\Omega\times(0,\tau_0)]\cup[\bar{\Omega}\times\{0\}]} w \leq \max(K_2^2, M_0^2). \qquad \square$$

### 5.4. Bound on $|D^2u|$ and $|D^3u|$ away from the boundary.

**Lemma 17.** *Let $u$ be a solution of the problem* (5.6) *in $\Omega \times [0, \tau_0)$; then we have the following bounds for the second and third derivatives of $u$:*

$$|D^2_{\xi\xi}u(\xi,\tau)| \leq e^{-\delta_0}C + \max_{\Omega\times[0,\delta_0]} |D^2_{\xi\xi}u|, |\xi| \geq e(R+1), \tau \in [0, \tau_0) \tag{5.16}$$

$$|D^3_{\xi\xi}u(\xi,\tau)| \leq e^{-2\delta_0}C + \max_{\Omega\times[0,\delta_0]} |D^3_{\xi\xi}u|, |\xi| \geq e(R+1), \tau \in [0, \tau_0), \tag{5.17}$$

*where $\delta_0 = \min(\tau_0/2, 1)$ and $C$ denotes different constants depending only on $\delta_0$ , $\|Du_0\|_{C^2(\bar{\Omega})}$ and $\|f\|_{C^4}$.*

**Remark 18.** The inequality (5.16) does depend on $\tau_0$ through the constant $\delta_0$; however the dependence is very loose. In the proof, we will see that $\delta_0$ can be chosen to be $\min(1, s/2)$ for any time $0 < s < \tau_0$. Therefore, the estimate is uniform in time once we know that the solution exists past some small time $s$.

**Remark 19.** $\max_{\Omega\times[0,\delta_0]} |D^2_{\xi\xi}u|$ and $\max_{\Omega\times[0,\delta_0]} |D^3_{\xi\xi}u|$ depend on $u$ on the right-hand side, but we are primarily interested in finding a bound for $\tau$ approaching $\tau_0$ so there is no need for a better estimate near the initial time.

**Proof.** Let us tackle the second derivative first. The estimate for $\tau \leq \delta_0$ is immediate. For $\tau > \delta_0$, we use the change of variables $t = 1 - e^{-2\tau}$ and look at the function

$$v(x,t) = \sqrt{2(1-t)}\, u\Big(\frac{x}{\sqrt{2(1-t)}}, -\frac{1}{2}\ln(1-t)\Big), \tag{5.5}$$

which satisfies the mean curvature flow (5.3a). We are interested in a time $t$ such that $t > 1 - e^{-2\delta_0}$. We can use the following interior estimate on the curvature from Ecker-Huisken [2] as long as $v$ exists in the domains mentioned. Define $t_0$ to be $1 - e^{-2\tau_0}$.

**Lemma 20** (Corollary 3.2 [2]). *Let $0 < t_1 < t_0$ and $\eta > 0$ and $0 \le \theta < 1$. For $t \in [0, t_1]$, we have the estimate*

$$\sup_{B_{\theta\eta}(x_0)} |A|^2(t) \le c(n)(1-\theta^2)^{-2}\Big(\frac{1}{\eta^2}+\frac{1}{t}\Big) \sup_{B_\eta(x_0)\times[0,t]} (1+|Dv|^2)^2,$$

*where $B_\eta(x_0)$ denotes the ball of radius $\eta$ centered at $x_0$ in the plane.*

First, we show that $v$ exists in $B_\eta(x_0) \times [0, t_1]$ for $\eta = 1$, $\theta = 1/2$ and $|x_0| \ge \sqrt{2}(R+1)$: if $(x,t) \in B_\eta(x_0) \times [0,t_1]$, then $|x| \ge |x_0| - \eta$, and the corresponding $\xi = \frac{x}{\sqrt{2(1-t)}}$ satisfies $|\xi| \ge \frac{|x_0|-\eta}{\sqrt{2(1-t)}} \ge \frac{\sqrt{2}(R+1)-1}{\sqrt{2(1-t)}} \ge R$. Hence, for $|x_0| > \sqrt{2}(R+1)$ and all $t \in [1-e^{-2\delta_0}, t_0)$,

$$|A|^2(x_0,t) \le \sup_{B_{1/2}(x_0)} |A|^2(t) \le C\frac{16}{9}(1+\frac{1}{1-e^{-2\delta_0}})C \le C(\delta_0).$$

Bounds on the second fundamental form $A$ and on the first derivative $Dv$ yield a bound on the second derivative of $v$:

$$|D^2_{xx}v(x_0,t)| \le C(\delta_0), \quad |x_0| \ge \sqrt{2}(R+1), t \in [1-e^{-2\delta_0}, t_0),$$

where $C(\delta_0)$ denotes a different constant, also dependent on $\delta_0$. Therefore

$$\begin{aligned} |D^2_{\xi\xi}u(\xi,\tau)| &= \sqrt{2}e^{-\tau}|D^2_{xx}v(\sqrt{2}e^{-\tau}\xi, 1-e^{-2\tau})| \\ &\le e^{-\tau}C(\delta_0), \quad |\xi| \ge e^{\tau}(R+1), \tau \in [\delta_0, \tau_0). \end{aligned} \tag{5.18}$$

In order to prove an estimate independent of time and outside a fixed annulus, note that, for all $\tau_1 \ge 0$, the function $\tilde{u}(\xi,s) = u(\xi, s+\tau_1)$ is also a solution of $\partial_s\tilde{u} = \mathcal{E}\tilde{u}$. Moreover, $|D_\xi\tilde{u}(\xi,s)| \le M_1$ for $(\xi,s) \in \Omega \times [0, \tau_0 - \tau_1)$ so we have an estimate similar to (5.18) in this case also:

$$|D^2_{\xi\xi}\tilde{u}(\xi,s)| \le e^{-s}C(\delta_0), \quad |\xi| \ge e^{s}(R+1), s \in [\delta_0, \tau_0-\tau_1).$$

Let $s = \delta_0$ and $\tau = \tau_1 + s$; the inequality above yields

$$|D^2_{\xi\xi}u(\xi,\tau)| \le e^{-\delta_0}C(\delta_0), \quad |\xi| \ge e^{\delta_0}(R+1)$$

for all $\tau_1 \ge 0$; i.e., for all $\tau \ge \delta_0$.

The bound on the third derivative is proved in a similar fashion using

$$\sup_{B_{\theta\eta}(x_0)} |\nabla A|^2(t) \le c\Big(\frac{1}{\eta^2}+\frac{1}{t}\Big)^2,$$

with the constant $c$ depending on $\theta$ and $\sup_{B_\eta(x_0)\times[0,t]}(1+|Dv|^2)$ from Theorem 3.4 in [2]. □

5.5. **Hölder continuity of $D_\xi u$ near boundary.** Let the time $\tau_1 < \tau_0$ and denote by

$$A_{R,R_1} \text{ the annulus } \{\xi\mathbf{R}^2 : R < |\xi| < R_1\},$$
$$Q_{R,R_1,\tau_1} \text{ the domain } A_{R,R_1} \times (0,\tau_1),$$
$$\Gamma_{R,R_1,\tau_1} \text{ the set } (A_{R,R_1} \times \{0\}) \cup ((\partial B_{R_1} \cup \partial B_R) \times [0,\tau_1]),$$
$$M_1 = \max_{\Omega\times[0,\tau_0)} |D_\xi u|.$$

For a domain $Q \in \mathbf{R}^2 \times [0,\infty)$, denote by $C^{2,1}(\bar{Q})$ the set of all continuous functions in $\bar{Q}$ having continuous derivatives $D_\xi$, $D^2_{\xi\xi}u, \partial_\tau u$ in $\bar{Q}$ and by $C^{\alpha,\alpha/2}(Q)$ the set of functions on $Q$ that are $\alpha$-Hölder continuous in $\xi$ and $\alpha/2$- Hölder continuous in $\tau$.

We use the theory on quasilinear parabolic equations (see Theorem 2.3, page 533 from [9]) to obtain a bound on $\|D_\xi u\|_{C^{\alpha,\alpha/2}(\bar{A}_{R,R_1,\tau_1}\times[0,\tau_1])}$ that depends only on $\|f\|_{C^4}$, $R$, $M_1$ and $\|u\|_{C^{2,1}(\Gamma_{R,R_1,\tau_1})}$. From Lemma 17, we can estimate this last quantity uniformly in time, therefore,

$$\|D_\xi u\|_{C^{\alpha,\alpha/2}(\bar{A}_{R,R_1}\times[0,\tau_0))} \le K_4 < \infty, \tag{5.19}$$

where $K_4$ only depends on time through $\delta_0 = \min(1, \tau_0/2)$.

5.6. **Hölder continuity of $D_{\xi\xi}u$ near the boundary.** The coefficients $g^{ij}(Du)$ of (5.6a) are in $C^{\alpha,\alpha/2}(\bar{A}_{R,R_1} \times [0,\tau_0))$ hence we can apply standard theory for linear parabolic equations with Hölder coefficients (Theorem 5.2 page 320 [9] for example) to obtain

$$\|u\|_{C^{\alpha+2,\alpha/2+1}(\bar{A}_{R,R_1}\times[0,\tau_0))} \le K_5 < \infty. \tag{5.20}$$

Remark 18 explains the loose dependence of our bounds on $\delta_0$ and $\tau_0$. Let us emphasize it again: if the solution $u$ is known to exist past a small time $s > 0$, we can choose $\delta_0 = s/2$ and the constants in all the estimates following (5.16) do not depend on $\tau_0 > s$.

5.7. **Continuation of the solution.** Combining (5.16) and (5.20), we have

$$\|D_{\xi\xi}u\|_{C^{\alpha,\alpha/2}(\bar{\Omega}\times[0,\tau_0))} + \|\partial_\tau u\|_{C^{\alpha,\alpha/2}(\bar{\Omega}\times[0,\tau_0))} \le K_6 < \infty.$$

Moreover, Lemma 15 gives us $\|D_\xi u\| \le M_1 < \infty$ on $\bar{\Omega} \times [0,\tau_0)$. These estimates can be transformed into uniform bounds on the first two derivatives of the corresponding solution $v$ to the mean curvature flow (5.3a) for time $t \in [0,t_0)$. Therefore, the solution $v$ has a continuation existing for time $t \in [0,t_3)$ with $t_3 > t_0$. This contradicts the maximality of $t_0$, and proves that the solution $u$ to (5.6) exists for all time $\tau \in [0, \infty)$. Since none of the bounds on $u$, $Du$ or $D^2u$ depended on $\tau_0$, we have

$$u_-(\xi) \le u(\xi, \tau) \le u_+(\xi), \quad (\xi, \tau) \in \bar{\Omega} \times [0, \infty), \tag{5.21}$$

$$|D_\xi u(\xi, \tau)| \le M_1, \quad (\xi, \tau) \in \bar{\Omega} \times [0, \infty), \tag{5.22}$$

$$\|D_{\xi\xi} u\|_{C^{\alpha,\alpha/2}(\bar{\Omega}\times[0,\infty))} + \|\partial_\tau u\|_{C^{\alpha,\alpha/2}(\bar{\Omega}\times[0,\infty))} \le K_6. \tag{5.23}$$

## 6. A Solution to the Elliptic Equation $\mathcal{E}u=0$

Here, we show that there is a subsequence of $u(\cdot, \tau_n)$ that tends to a solution to the elliptic equation $\mathcal{E}u = 0$ as $\tau_n \to \infty$.

Let us denote by $M_\tau$ the graph of $u(\cdot, \tau)$ over $\Omega$, by $X = (\xi, u(\xi, \tau))$ the position vector of a point on $M_\tau$. From the equation $\partial_\tau u = \mathcal{E}(u)$, the variation of $X$ with respect to time is

$$\partial_\tau X = (H + X \cdot \nu)\sqrt{1 + |Du|^2}\ \mathbf{e}_3 =: Z,$$

where $\mathbf{e}_3$ is the unit vector $(0, 0, 1)$ in $\mathbf{R}^3$. It will be useful for what follows to decompose $\mathbf{e}_3$ and $Z$ into their tangent and normal components with respect to $M_\tau$:

$$\begin{aligned} \mathbf{e}_3 &= \mathbf{v} + \frac{\nu}{\sqrt{1 + |Du|^2}}, \\ Z &= Z^T + Z^\perp = (H + X \cdot \nu)\sqrt{1 + |Du|^2}\ \mathbf{v} + (H + X \cdot \nu)\nu, \end{aligned}$$

where $\mathbf{v} = \frac{1}{1+|Du|^2}(D_1 u, D_2 u, |Du|^2)$.

Consider the functional $J(u) = \int_{M_\tau} e^{-|X|^2/2} dH$, where $dH$ is the Hausdorff measure on $M_\tau$. We have

$$\begin{aligned} \frac{dJ(u)}{d\tau} &= \int_{M_\tau} \Big[(div_M Z^\perp + div_M Z^T) - X \cdot \partial_\tau X\Big] e^{-|X|^2/2}\ dH \\ &= \int_{M_\tau} e^{-|X|^2/2}(-H(H + X \cdot \nu))\ dH + \int_{M_\tau} e^{-|X|^2/2} div_M Z^T\ dH \\ &\quad - \int_{M_\tau} X \cdot \partial_\tau X e^{-|X|^2/2} dH \\ &= -\int_{M_\tau} e^{-|X|^2/2}(H + X \cdot \nu)^2\ dH + \int_{\partial M_\tau} e^{-|X|^2/2} Z^T \cdot \eta\ ds, \end{aligned}$$

where $\eta$ is the outward conormal unit vector to $\partial M_\tau$. On the circle $r = R$, $0 = \partial_\tau u = (H + X \cdot \nu)\sqrt{1 + |Du|^2}$, therefore

$$\frac{dJ(u)}{d\tau} = -\int_{M_\tau} e^{-|X|^2/2}(H + X \cdot \nu)^2\ dH \le 0.$$

Writing $\frac{dJ(u)}{d\tau}$ in terms of $\partial_\tau u$ and as an integral over $\Omega$, we obtain

$$\frac{dJ(u)}{d\tau} = -\int_\Omega \frac{(\partial_\tau u)^2}{\sqrt{1+|Du|^2}} e^{-\frac{|\xi|^2+u^2}{2}}\ d\xi \leq 0.$$

By definition, $J(u(\cdot,\tau)) \geq 0$ for all time $\tau \in [0,\infty)$, so

$$\int_0^\infty \int_\Omega \frac{(\partial_\tau u)^2}{\sqrt{1+|Du|^2}} e^{-\frac{|\xi|^2+u^2}{2}}\ d\xi < \infty$$

and

$$\lim_{n\to\infty} \int_{n-1}^{n+1} \int_\Omega \frac{(\partial_\tau u)^2}{\sqrt{1+|Du|^2}} e^{-\frac{|\xi|^2+u^2}{2}}\ d\xi = 0. \tag{6.1}$$

Define $u_n(\xi,\tau) = u(\xi, n+\tau)$ for $(\xi,\tau) \in \bar{\Omega}\times[-1,1]$. For any constant $\rho > R$, the $u_n$'s are uniformly bounded in $C^{\alpha+2,\alpha/2+1}(\overline{(B_\rho \cap \Omega)} \times [-1,1])$. We can therefore extract a subsequence that converges in $C^{\beta+2,\beta/2+1}(\overline{(B_\rho\cap\Omega)} \times [-1,1])$, $0 < \beta < l$. A Cantor diagonal argument gives us a further subsequence $u_{n_i}$ and a function $u_\infty$ such that $u_{n_i} \to u_\infty$ as $i \to \infty$ in the norm $C^{\beta+2,\beta/2+1}(\overline{(B_\rho\cap\Omega)}\times[-1,1])$ for every $\infty > \rho > R$. Since every $u_n$ satisfies $\partial_\tau u_n = \mathcal{E} u_n$ in $\Omega\times(-1,1)$ and $u_n(\xi,\tau) = f$ on $\partial B_R \times [-1,1]$, the limit function $u_\infty$ inherits the same properties. On the one hand, equation (6.1) and the uniform bounds (5.21) and (5.22) tell us that $\partial_\tau u_{n_i} \to 0$ pointwise. On the other hand, we know that $\partial_\tau u_{n_i} \to \partial_\tau u_\infty$ pointwise. Hence, $\partial_\tau u_\infty = 0$ and $u_\infty$ satisfies $\mathcal{E}u_\infty = 0$ in $\Omega$ and $u_\infty = f$, $\partial\Omega$. We have therefore proved the following.

**Lemma 21.** *Let $2 > R > \frac{1}{\sqrt{2}}$ and $N \geq 5$. Define $\Omega = \{\xi \in \mathbf{R}^2 : |\xi| > R\}$ to be the plane with a hole of radius $R$. There is an $\varepsilon_1 > 0$ depending on $R$ and $N$ such that for any $f : [0,2\pi] \to \mathbf{R}$ with $\|f\|_{C^4[0,2\pi]} = \varepsilon \leq \varepsilon_1$ and satisfying the symmetries $f(\theta) = -f(-\theta) = f(\pi/N - \theta)$, there exists a smooth solution $u$ to the Dirichlet problem*

$$\mathcal{E}(u) = g^{ij}(Du(\xi))D_{ij}u(\xi) - \xi\cdot Du(\xi) + u(\xi) = 0, \quad \xi\in\Omega, \tag{1.3}$$

$$u = f \text{ on } \partial B_R. \tag{1.6}$$

*In addition, we can choose the constant $\varepsilon_1 = \varepsilon_1(R_0)$ uniformly for all $R$ such that $2 > R \geq R_0 > 1/\sqrt{2}$.*

## 7. Uniqueness

For the sake of clarity let us assume $R > \sqrt{3}/2$ in this section. The same reasoning also works in the more general case $R > 1/\sqrt{2}$ although the estimates are more involved.

Let $f$ be our boundary condition, and suppose that $\|f\|_{C^4(\partial\Omega)} \leq \varepsilon_1$, where $\varepsilon_1$ is given by Lemma 21.

**Theorem 22.** *There is an $\varepsilon_2 \leq \varepsilon_1$ so that, if $\|f\|_{C^4} \leq \varepsilon_2$ and if $u$ and $\tilde{u}$ are two solutions to*

$$\mathcal{E}u = g^{ij}(Du)D_{ij}u - \xi \cdot Du + u = 0 \text{ in } \Omega \tag{7.1a}$$

$$u = f \text{ on } \partial\Omega \tag{7.1b}$$

*with $u_- \leq u, \tilde{u} \leq u_+$, then $u \equiv \tilde{u}$.*

Before we start the proof, we need some a priori estimates on the solutions $u$ and $\tilde{u}$.

7.1. **A priori estimates.** Because the solutions $u$ and $\tilde{u}$ above are not dependent on time, they satisfy the parabolic equation $\partial_\tau u = \mathcal{E}u$ as well. Estimates similar to the ones from Lemmas 15 and 17 therefore hold in this case also, with some small modifications. If $u$ is a solution to the elliptic equation (7.1a), then the function

$$v(x,t) = \sqrt{2(1-t)}\, u\Big(\frac{x}{\sqrt{2(1-t)}}\Big)$$

is a solution to the mean curvature flow on $\mathcal{Q}_1 = \{(x,t) : 0 < t < 1, |x| > \sqrt{2(1-t)R}\}$. The estimate

$$\sqrt{1+|Dv(x_0,t)|^2} \leq C_1(n) \sup_{B_\rho(x_0)} \sqrt{1+|Dv(x,0)|^2}$$
$$\exp[C_2(n)\rho^{-2} \sup_{[0,T]}( \sup_{B_\rho(x_0)\times[0,T]} v - v(x_0,t))^2] \tag{5.11}$$

where $0 \leq t \leq T, B_\rho(x_0)$ is a ball in $\Omega$ and $n$ is the dimension of the graph of $v$, is valid. The last factor of the right-hand side can be bounded as in the proof of Lemma 15; however, we do not have an initial condition independent of $v$ in this case, so at this point, our estimates depend on the value of $Dv(\cdot,0)$ over a bounded set. Taking $|x_0| = \frac{3}{2}\sqrt{2}R$ and $\rho = |x_0|/3$, we obtain

$$|Dv(x_0,t)| \leq C \sup_{2\sqrt{2}R>|x|>\sqrt{2}R} \sqrt{1+|Dv(x,0)|^2}, \quad 1 > t > 0,$$

hence, by the change of variable $x = \xi\sqrt{2(1-t)}$,

$$|Du(\xi)| \leq C \sup_{2R>|\zeta|>R} \sqrt{1+|Du(\zeta)|^2}, \quad |\xi| \geq R.$$

An argument similar to the one in the proof of Lemma 17 gives us bounds on higher derivatives of $u$ away from the boundary

$$|D^2_{\xi\xi}u(\xi)| \leq C, \quad |\xi| \geq e(R+1),$$
$$|D^3_{\xi\xi}u(\xi)| \leq C, \quad |\xi| \geq e(R+1),$$

with the constant $C$ depending on $\sup_{2R\geq|\zeta|\geq R}\sqrt{1+|Du(\zeta)|^2}$. Standard elliptic theory on quasilinear equations (see Theorem 13.4 or 13.7 [4] for example) assures the existence of constants $C$ and $\alpha > 0$ depending only on $f$, $R$ and $\sup_{2R>|\zeta|>R}\sqrt{1+|Du(\zeta)|^2}$ such that $\|Du\|_{C^\alpha(A_{R,2R})} \leq C$, where $A_{R,2R}$ is the annulus $\{R < |\xi| < 2R\}$. The $C^\alpha$ Hölder norm of the coefficient $g^{ij}(Du)$, seen as a function of $\xi$, is uniformly bounded over the whole domain $\Omega$. We have a bound on $\|D^2_{\xi\xi}u\|_{C^\alpha(A_{R,2R})}$ by elliptic theory; therefore $\|D^2_{\xi\xi}u\|_{C^\alpha(\Omega)}$ is also bounded.

7.1.1. *The gradient $Du$ achieves its maximum on the boundary $\partial\Omega$.* Equation (5.15) from Section 5 implies that $w = |Du|^2$ satisfies

$$0 \leq g^{ij}(Du)D_{ij}w + Z_kD_kw - \xi\cdot Dw =: \bar{\mathcal{L}}w, \tag{7.2}$$

where $Z_k(x) = g^{ij}_k(Du)D_{ij}u$ and $g^{ij}_k(p) = \frac{\partial g^{ij}}{\partial p_k}(p)$. We will derive the following theorem as an application of the maximum principle.

**Theorem 23.** *Assume $u \in C^2(\Omega)\cap C^0(\bar{\Omega})$ is a solution of* (7.1)*; then $|Du|$ achieves its maximum on the circle $r = R$.*

From the discussion above, $|Z|$ is uniformly bounded and $\bar{\mathcal{L}}$ is a uniformly elliptic operator on $\Omega$. Moreover, the coefficients of $\bar{\mathcal{L}}$ are Hölder continuous with bounded Hölder norm on any compact subset of $\Omega$.

**Lemma 24.** *If $|\nabla h| < b$ for some constant $b$, there is a function $\psi(r) > 0$ for which $\bar{\mathcal{L}}(\psi) \leq 0$ and $\psi(r) \to \infty$ as $r \to \infty$.*

**Proof.** We consider positive increasing functions $\psi(r)$. The operator $\bar{\mathcal{L}}(\psi)$ is estimated by

$$\begin{aligned}\bar{\mathcal{L}}(\psi) &= g^{ij}(Du)\Big(\big(\frac{\delta_{ij}}{r} - \frac{\xi_i\xi_j}{r^3}\big)\psi' + \frac{\xi_i\xi_j}{r^2}\psi''\Big) + \frac{1}{r}\psi' Z\cdot\xi - r\psi' \\ &\leq g^{ij}(Du)\frac{\xi_i\xi_j}{r^2}\psi'' + \Big(\frac{1}{r} - r + |Z|\Big)\psi'.\end{aligned}$$

We have uniform bounds on $|Du|$ and $|Z|$, therefore

$$\bar{\mathcal{L}}(\psi) \leq \Big(g^{ij}(Du)\frac{\xi_i\xi_j}{r^2}\Big)\Big(\psi'' + \psi'(\frac{1}{r} - r + C)(1+b^2)\Big).$$

The solution to the differential equation $\psi'' + \psi'\left(\frac{1}{r} + C - r\right)(1+b^2) = 0$ is given by

$$\ln(\psi'(r)) - \ln(\psi'(R)) = (1+b^2)\Big(-\ln(\frac{r}{R}) - C(r-R) + \frac{1}{2}(r^2 - R^2)\Big)$$
$$\psi(r) - \psi(R) = \psi'(R)\int_R^r \left(\frac{R}{s}\right)^{1+b^2} e^{(-C(s-R)+\frac{1}{2}(s^2-R^2))(1+b^2)} ds.$$

For the choice of boundary conditions $\psi(R) = \psi'(R) = \varepsilon > 0$, the function $\psi$ and its derivative $\psi'$ are positive for $r > R$. The last thing to verify is that $\psi$ tends to infinity as $r$ approaches infinity. This is true since, for large $r$, the dominant term in the expression for $\psi'$ is of order $e^{r^2}$. □

**Proof of Theorem 23.** Let us consider the function $w - \eta\psi$, with $\psi$ as in Lemma 24 and $w = |Dh|^2$. This function is negative for large values of $r$. From equation (7.2), we have

$$\bar{\mathcal{L}}(w - \eta\psi) \geq 0,$$

for any positive constant $\eta$. By the maximum principle, $w - \eta\psi$ does not achieve an interior maximum. Since the function $w - \eta\psi$ goes to $-\infty$ as $r$ goes to infinity, it has to achieve its maximum on the boundary $\partial\Omega$:

$$w(\xi) - \eta\psi(|\xi|) \leq \max_{\partial\Omega} w - \eta\varepsilon, \quad \xi \in \Omega, \eta > 0$$

Letting $\eta$ tend to 0, we obtain $w(\xi) \leq \max_{\partial\Omega} w$ for $\xi \in \Omega$. □

Our subsolution and supersolution provide barriers for the function $u$, therefore,

$$|Du| \leq C\|f\|_{C^4},$$

with the constant $C$ independent of $f$ and $u$. With this bound, we can run through the arguments in the beginning of this section again and obtain estimates for the second derivative $D^2u$ that are independent of the solution $u$.

### 7.2. **Proof of uniqueness.**

**Proof of Theorem 22.** Let $u$ and $\tilde{u}$ be two solutions as described in the hypotheses. We subtract equation (7.1a) for $\tilde{u}$ from the equation for $u$ and use the notation $w = u - \tilde{u}$ to obtain

$$g^{ij}(Du)D_{ij}w + [g^{ij}(Du) - g^{ij}(D\tilde{u})]D_{ij}\tilde{u} - \xi \cdot Dw + w = 0.$$

The mean value theorem implies that

$$g^{ij}(Du)D_{ij}w + [g^{ij}_k(D\tilde{u} + \theta(Du - D\tilde{u}))]D_{ij}\tilde{u}D_k w - \xi \cdot Dw + w = 0,$$

for some $\theta \in (0,1)$ and where $g_k^{ij}(p) = \frac{\partial g^{ij}}{\partial p_k}$ for $p \in \mathbf{R}^2$. Denoting by $\bar{a}$ the vector with components $a_k = [g_k^{ij}(D\tilde{u} + \theta(Du - D\tilde{u}))]D_{ij}\tilde{u}$, $k = 1, 2$, we have

$$\bar{\mathcal{L}}(w) = g^{ij}(Du)D_{ij}w + \bar{a} \cdot Dw - \xi \cdot Dw + w = 0. \tag{7.3}$$

A computation shows that $g_k^{ij}(p) \leq 6|p|$ if $|p| < 1$. From Section 7.1, we know that there exists a constant $C$ so that $|D_{ij}\tilde{u}| \leq C$ in $\Omega$. Moreover, the maximum of the first derivative of any solution to (7.1a) is achieved on the boundary $\partial\Omega$ hence $\max_\Omega |Du|$ and $\max_\Omega |D\tilde{u}|$ are controlled by $\varepsilon = \|f\|_{C^4}$ and go to zero as $\varepsilon$ tends to 0. The same fact is true for $\eta = \sup |\bar{a}|$ by the definition of $\bar{a}$.

The maximum principle does not apply immediately to (7.3) because of the positive coefficient in front of $w$. To circumvent this, we use a positive supersolution $\psi$ to (7.3) which grows faster than linearly at infinity. The existence of such a supersolution $\psi$ is given below and is the main difficulty in this proof.

Assuming such a positive supersolution $\psi$ exists, we define $\tilde{w} = w/\psi$ and look at

$$\tilde{\mathcal{L}}(\tilde{w}) := \bar{\mathcal{L}}(\psi\tilde{w}) = \bar{\mathcal{L}}w = 0.$$

The coefficient in front of $\tilde{w}$ in the explicit expression of $\tilde{\mathcal{L}}(\tilde{w})$ is now $\mathcal{L}(\psi)$, which is nonpositive, so the maximum principle applies to $\tilde{\mathcal{L}}$. It implies that, for any subdomain $\Omega'$ of $\Omega$, $\sup_{\Omega'} \tilde{w} \leq \sup_{\partial\Omega'} \tilde{w}$. From the hypotheses, $w = 0$ on $\partial\Omega$ and from the growth of $\psi$ at infinity, $|\tilde{w}(\xi)|$ tends to 0 as $|\xi|$ tends to infinity. Choosing the domains $\Omega'$ to be increasingly large annuli, we obtain that $\sup_\Omega \tilde{w} = 0$. Since $\tilde{w} = w/\psi$ and $\psi > 0$, $\sup_\Omega w = 0$, therefore $u \leq \tilde{u}$ in the entire domain $\Omega$. Switching the roles of $u$ and $\tilde{u}$ we show that $\tilde{u} \leq u$ and conclude that $u \equiv \tilde{u}$ in $\Omega$.

We are left to prove the existence of the function $\psi$.

**Claim 25.** *For $R > \sqrt{3}/2$ and $\varepsilon_2$ small enough, there exists a function $\psi$ on $\Omega$ satisfying*

$$\psi > 0, \quad \bar{\mathcal{L}}\psi \leq 0, \quad \text{and } \lim_{|\xi|\to\infty} \frac{|\xi|}{\psi(\xi)} = 0.$$

The supersolution is found among functions depending on the radius $r$ only, $\psi = \psi(r)$, that are increasing $\psi'(r) > 0$. The operator $\mathcal{L}$ applied to a radial function is estimated by

$$\bar{\mathcal{L}}(\psi) = g^{ij}(Du)\Big[\big(\frac{\delta_{ij}}{r} - \frac{\xi_i\xi_j}{r^3}\big)\psi' + \frac{\xi_i\xi_j}{r^2}\psi''\Big] + r^{-1}\psi'\bar{a}\cdot\xi - r\psi' + \psi$$

$$\leq g^{ij}(Du)\frac{\xi_i\xi_j}{r^2}\psi'' + \left[\frac{1}{r} - r + \eta\right]\psi' + \psi.$$

Let us consider functions of the form $\psi(r) = r^\alpha - br^{\alpha-2}$ with constants $2 > \alpha > 1$ to be chosen later and $b = 3/4$. This function is positive on the set $r > R$ provided $b < R^2$. To simplify the notation, we work with the case $b = 3/4$ and imposed $R^2 > 3/4$; however, the following argument is also valid in general for $R^2 > b > 1/2$ with slightly more subtle estimates. The first and second derivatives are given by $\psi' = \alpha r^{\alpha-1} - b(\alpha-2)r^{\alpha-3} > 0$ and $\psi'' = \alpha(\alpha-1)r^{\alpha-2} - b(\alpha-2)(\alpha-3)r^{\alpha-4}$. We then have

$$g^{ij}(Du)\frac{\xi_i\xi_j}{r^2}\psi'' \leq \alpha(\alpha-1)r^{\alpha-2} - \frac{1}{1+|Du|^2}b(\alpha-2)(\alpha-3)r^{\alpha-4}.$$

By choosing $\varepsilon$ small enough, we can assume that $|Du|^2 \leq 1/8$. A computation gives

$$\begin{aligned}\bar{\mathcal{L}}(\psi) \leq (1-\alpha)r^\alpha + \eta\alpha r^{\alpha-1} + (\alpha^2 + b\alpha - 3b)r^{\alpha-2} \\ + \eta b(2-\alpha)r^{\alpha-3} + b(2-\alpha)(1 - \tfrac{8}{9}(3-\alpha))r^{\alpha-4}.\end{aligned}$$

The coefficient of $r^{\alpha-2}$ is negative if $1 < \alpha \leq 9/8 < \frac{-3+\sqrt{153}}{8}$ and $b = 3/4$. Such a choice of $\alpha$ also guarantees that the coefficient of $r^{\alpha-4}$ is negative. Therefore,

$$\bar{\mathcal{L}}(\psi) \leq r^{\alpha-4}\big((1-\alpha)r^4 + \eta\alpha r^3 + \eta\tfrac{3}{4}(2-\alpha)r\big) \leq r^{\alpha-3}\big((1-\alpha)r + 2\eta\big) \quad (7.4)$$

since $\frac{3}{4}r \leq r^3$. The right-hand side of (7.4) is negative if we choose $\varepsilon$ small enough and $\alpha$ so that $1 + \frac{4\eta}{\sqrt{3}} < \alpha < \frac{9}{8}$. □

Let $\|f\|_{C^4} \leq \varepsilon$ and let $u$ be the unique solution to (7.1) such that $u_- \leq u \leq u_+$.

**Remark 26.** Because of uniqueness, the function $u$ has to satisfy the symmetries

$$u(r,\theta) = -u(r,-\theta) = u(r,\pi/N - \theta) \text{ for } r > R, \theta \in [0,2\pi). \quad (1.7)$$

Taking $\varepsilon_0 = \varepsilon_2$, Theorem 1 from the introduction is then a corollary of Lemma 21 and Theorem 22.

**Remark 27.** From the explicit formulas in Lemma 11, we have

$$u_- \sim u_1 - \varepsilon^3 v_C \leq u \leq u_+ \sim u_1 + \varepsilon^3 v_C,$$

where the constant $C$ and the function $v_C$ do not depend on $\varepsilon$. Recall that $u_1$ is the constructed solution to the linear problem $\mathcal{L}u_1 = 0$, $u_1|_{\partial\Omega} = f$. This implies that $D_r u = D_r u_1 + O(\varepsilon^3)$ on the boundary circle $r = R$; in other words, the radial derivative of $u$ on the circle $r = R$ is the radial derivative of the solution to the linear problem with an error of order $\varepsilon^3$.

**Remark 28.** Considering equation (5.18) with $|\xi| = e^\tau(R+1)$ gives us

$$|D^2_{\xi\xi}u(\xi)| \leq \tfrac{C}{|\xi|}, \quad |\xi| \geq e(R+1).$$

Since $|Du|$ is uniformly bounded in $\Omega$, the above estimate implies a bound on the mean curvature of the graph, and therefore on $X \cdot \nu$:

$$|X \cdot \nu| = |H| \leq \tfrac{C}{|\xi|}, \quad |\xi| \geq e(R+1).$$

The graph of the solution $u$ is therefore asymptotic to a cone at infinity.

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