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AN IDENTIFICATION PROBLEM WITH EVOLUTION ON THE BOUNDARY OF HYPERBOLIC TYPE

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Abstract. We consider an equation of the type $A(u+k*u) = f$, where A is a linear second-order elliptic operator, k is a scalar function depending on time only and $k*u$ denotes the standard time convolution of functions defined in $(-\infty, T)$ with their supports in $[0, T]$. The previous equation is endowed with second-order dynamical boundary conditions.

Assuming that the kernel k is unknown and a supplementary condition is given, k can be recovered and global existence, uniqueness and continuous dependence results can be shown.

1. Introduction

Recovering unknown functions in equations describing physical, chemical, or geological phenomena has become a routine requirement in applied sciences, although such problems are, in general, ill posed from the mathematical point of view.

The identification problem we deal with here is inspired by the direct problem in the interesting paper by Hintermann [9]. More precisely we are concerned with identifying the unknown (time) convolution *scalar* kernel k appearing in the linear second–order elliptic equation

$$A(u + k * u) = f \qquad \text{in} \quad (0, T) \times \Omega, \tag{1.1}$$

where

$$k * u(t, x) = \int_0^t k(t - s)u(s, x)\, ds$$

and Ω is a bounded (smooth) domain in $\mathbb{R}^n$.

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Equation (1.1) is endowed with a dynamical boundary condition of the flux form, as in [9],

$$D_t^2 u + b D_{\nu_A} u = g, \qquad \text{on} \quad (0,T) \times \partial\Omega,$$

D_{ν_A} denoting the conormal derivative related to the operator A. To recover k we prescribe an additional condition of the form

$$\Phi[u(t,\cdot)] = l(t), \quad t \in [0,T],$$

Φ and l being a given functional and a given function.

To our knowledge the problem of determining the memory kernel k is new under dynamical boundary conditions of second order in time. For a similar result involving first order in time dynamical boundary conditions cf. [11]. On the contrary, direct problems with $k = 0$ and dynamical boundary conditions are well studied, from the mathematical point of view (cf., e.g., the papers [1]-[9], [14]-[16] and the references therein).

Various applications of direct problems with dynamical boundary conditions can be found in different applied sciences, (cf. the introductions in [3] and [9]). Moreover, in [3] the authors consider the quasi-static problem of a viscoelastic body occupying a bounded region Ω (Ω is an open connected subset of $\mathbb{R}^3$) and subject to an elastic force on the boundary $\partial\Omega$ (which is supposed to be sufficiently regular).

Furthermore, we observe that our treatment to solve our identification problem leads to the (equivalent) identification problem (3.1)–(3.5) (cf. Section 3) related to the boundary $\partial\Omega$ of Ω, where the unknown kernel h convolves not only with the space operator D_{ν_A} but also with the time derivative $D_t w$, where $w = u_{|\partial\Omega} + k * u_{|\partial\Omega}$.

For such a kind of identification problem, to the authors' knowledge, the global in time existence, uniqueness and continuous dependence on the data of the solution (v,h) is not at all a standard result.

In this paper this difficulty is overcome by using a suitable estimate strategy and a fixed-point approach strictly related to the problem itself - we could say "suggested" by the problem itself (cf. Sections 4 and 5).

Finally, we observe that a simple treatment of the elliptic problem (1.1) (endowed with auxiliary nonhomogeneous Dirichlet conditions) forces us to choose for u - and consequently for v - an $L^2(\Omega)$-framework. In contrast to this, we have chosen for the time dependence of v and h the Hölder spaces C^α, $\alpha \in (0,1)$, to ensure a good regularity for the unknown kernel h - and consequently for k related to h by the linear convolution equation (2.7).

2. THE IDENTIFICATION PROBLEM AND THE STATEMENT OF THE FUNDAMENTAL RESULT

Let Ω be an open bounded domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class C^2. Let A be a linear (uniformly) elliptic operator of the following divergence form:

$$A = \sum_{i,j=1}^{n} D_{x_i}[a_{i,j}(x)D_{x_j}] + a_0(x),$$

where $a_{i,j} \in C^2(\overline{\Omega})$, $i,j = 1,\dots,n$, $a_0 \in C(\overline{\Omega})$ and

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \kappa|\xi|^2, \quad (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n,$$

for some positive constant κ.

Assume further that $b \in C^3(\overline{\Omega})$ satisfies the relations

$$b(x) > 0, \quad \sum_{j,k=1}^{n} D_{x_k}[a_{j,k}(x)D_{x_j}b(x)] \leq a_0(x)b(x),$$

$x \in \overline{\Omega}$, $D_{\nu_A}b(x) \geq 0$, $x \in \partial\Omega$, ν_A denoting the conormal unit vector related to A, $(\nu_A(x))_j = \sum_{i=1}^{n} \nu_i(x)a_{i,j}(x)$, $x \in \partial\Omega$, $j = 1,\dots,n$.

Remark 2.1. If b is *positive constant*, the previous conditions simplify to $a_0(x) \geq 0$ for all $x \in \overline{\Omega}$.

Consider the problem consisting of determining the pair (u,k) such that

$$(u,k) \in [C^2([0,T]:H^{(1/2)+\varepsilon}(\Omega)) \cap C([0,T];H^{1+\varepsilon}(\Omega))] \times C^1([0,T]), \quad (2.1)$$

$$A(u + k * u) = f, \quad \text{in } (0,T)\times\Omega, \quad (2.2)$$

$$D_t^2 u + bD_{\nu_A}u = g, \quad \text{on } (0,T)\times\partial\Omega, \quad (2.3)$$

$$u(0,\cdot) = u_0, \quad \text{on } \partial\Omega, \quad (2.4)$$

$$D_t u(0,\cdot) = u_1, \quad \text{on } \partial\Omega, \quad (2.5)$$

$$\Phi[u(t,\cdot)] = l(t), \quad t \in [0,T], \quad (2.6)$$

where $\varepsilon \in (1/2,3/2) \setminus \{1\}$, $f : (0,T)\times\Omega \to \mathbb{R}$, $u_0,u_1 : \partial\Omega \to \mathbb{R}$, $g : (0,T)\times\partial\Omega \to \mathbb{R}$ and $l : [0,T] \to \mathbb{R}$ are smooth prescribed functions. Finally, $\Phi \in H^{1/2+\varepsilon}(\Omega)'$, so that, for all $z \in H^{1/2+\varepsilon}(\partial\Omega)$, we can choose, e.g.,

$$\Phi[z] = \int_\Omega \left[\varphi_0(x)z(x) + \varphi_2(x)\cdot\nabla_x z(x)\right]dx + \int_{\Omega\times\Omega} |x-y|^{-(n+2\eta)}$$
$$\times \left\{\varphi_1(x,y)[z(x)-z(y)] + \varphi_3(x,y)\cdot[\nabla_x z(x) - \nabla_x z(y)]\right\} dxdy,$$

where φ_0, φ_2, φ_1 and φ_3 are given functions in $L^2(\Omega)$, $L^2(\Omega)^d$, $L^2(\Omega \times \Omega)$, and $L^2(\Omega \times \Omega)^d$, η being a constant in $(0, \min\{(2\varepsilon - 1)/2, 1\})$. Equation (2.6) stands for the additional information needed to recover the kernel k.

In Theorem 2.1 it will be shown that (2.1) is the appropriate functional framework to solve our identification problem.

The first step for determining the unknown pair (u, k) consists in changing the unknown u to $v := u + k * u$. For this purpose we recall a well-known result (cf., e.g., [12]): with any $k \in L^p((0,T))$, $p \in (1, +\infty]$, we can associate a unique function $h \in L^p((0,T))$ solving the convolution equation

$$h + k + h * k = 0 \quad \text{in } (0,T). \tag{2.7}$$

Concerning equation (2.7) also the following classic regularity result holds.

Lemma 2.1. *Let $k \in C^\alpha([0,T])$, $\alpha \in (0,1]$. Then equation (2.7) admits a unique solution $h \in C^\alpha([0,T])$, $\alpha \in (0,1]$, continuously depending on k with respect to the norm of $C^\alpha([0,T])$. In particular $h(0) = -k(0)$.*

Consequently, we get

$$v = u + k * u \quad \Longleftrightarrow \quad u = v + h * v. \tag{2.8}$$

It is easy to check that the pair (v, k) solves the following problem:

$$(v,h) \in [C^2([0,T]; H^{(1/2)+\varepsilon}(\Omega)) \cap C([0,T]; H^{1+\varepsilon})(\Omega)] \times C^1([0,T]), \tag{2.9}$$

$$Av = f \quad \text{in } (0,T) \times \Omega, \tag{2.10}$$

$$D_t^2 v + D_t h * D_t v + h(0) D_t v + u_0 D_t h + b D_{\nu_A} v + b h * D_{\nu_A} v = g, \quad \text{on } (0,T) \times \partial\Omega, \tag{2.11}$$

$$v(0,\cdot) = u_0 \quad \text{on } (0,T) \times \partial\Omega, \tag{2.12}$$

$$D_t v(0,\cdot) = u_1 - h(0) u_0, \quad \text{on } (0,T) \times \partial\Omega, \tag{2.13}$$

$$\Phi[v(t,\cdot)] + h * \Phi[v(t,\cdot)] = l(t), \quad t \in [0,T]. \tag{2.14}$$

Indeed, $v(0) = u(0) = u_0$ on $\partial\Omega$ and

$$u = v + h * v, \quad D_t u = D_t v + h * D_t v + h v(0),$$
$$D_t^2 u = D_t^2 v + D_t h * D_t v + h(0) D_t v + D_t h v(0).$$

We have thus proved the following result.

Theorem 2.1. *Problems* (2.1)–(2.6) *and* (2.9)–(2.14) *are equivalent via formulae* (2.7) *and* (2.8).

According to Theorem 4.2.3 in [13], from our assumptions on the coefficients $a_{i,j}$ and a_0 we can conclude that, for any pair $(f, w) \in L^2(\Omega) \times$

$H^{3/2}(\partial\Omega)$, the Dirichlet problem

$$Av = f, \quad \text{in } \Omega, \quad v = w \quad \text{on } \partial\Omega, \tag{2.15}$$

admits a unique solution $v \in H^2(\Omega)$ satisfying the estimate

$$\|v\|_{H^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|w\|_{H^{3/2}(\partial\Omega)}).$$

More generally, according to the fundamental result in [10, Theorem 4.2], the elliptic boundary-value problem (2.15) has a unique solution admitting the representation

$$v = L_0 w + L_1 f, \tag{2.16}$$

where $L_0 \in \mathcal{L}(H^{s-1/2}(\partial\Omega); H^s(\Omega))$, $1/2 < s < 5/2$, $s \ne 3/2$,
$L_1 \in \mathcal{L}(H^{s-1}(\Omega); H^s(\Omega))$, $0 < s < 5/2$. We emphasize that $L_0 w$ stands for the solution to problem (2.15) with $f = 0$, while $L_1 f$ stands for the solution to problem (2.15) with $w = 0$; i.e.,

$$\begin{cases} AL_0 w = 0 & \text{in } \Omega, \\ L_0 w = w & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} AL_1 f = f & \text{in } \Omega, \\ L_1 f = 0 & \text{on } \partial\Omega. \end{cases}$$

To be able to state our fundamental result we need to introduce the following Hölder vector spaces, where $\alpha \in (0,1)$, $\beta \in (0,+\infty)$, $s \in \mathbb{N}$ and $Y \in \{\partial\Omega, \Omega\}$:

$$C^{s+\alpha}([0,T]; H^\beta(Y)) := \Big\{ v \in C^s([0,T]; H^\beta(Y)) : \\ |D_t^s v|_{T,\alpha,\beta,Y} = \sup_{t\in(0,T]} t^{-\alpha} \|D_t^s v(t) - D_t^s v(0)\|_Y < +\infty \Big\}.$$

Moreover, we need the following notation:

$$\|g\|_{t,m,\beta,Y} = \|g\|_{C^m([0,t];H^\beta(Y))}, \quad \beta \ge 0, \quad \|l\|_{t,m} = \|l\|_{C^m([0,t])}, \tag{2.17}$$

$t \in (0,T]$, and

$$\|u_0\|_{\beta,Y} = \|u_0\|_{H^\beta(Y)}, \qquad \beta \ge 0. \tag{2.18}$$

In a similar way we can define the Banach space $C^{s+\alpha}([0,T])$.

We can introduce now the space of *admissible data:*

$$\begin{aligned} \mathcal{D}(\varepsilon,\alpha,R_0,T_0) := \Big\{ &\mathbf{d} = (u_0,u_1,f,g,l) : u_0 \in H^{2+\varepsilon}(\partial\Omega), u_1 \in H^{1+\varepsilon}(\partial\Omega), \\ &f \in C^{2+\alpha}([0,T_0]; H^\varepsilon(\Omega)),\ g \in C^\alpha([0,T_0]; H^{1/2+\varepsilon}(\partial\Omega)) \cap C^1([0,T_0]; H^\varepsilon(\partial\Omega)), \\ &l \in C^{2+\alpha}([0,T_0]),\ D_{\nu_A} L_0\big[g(0,\cdot) - bD_{\nu_A} L_1 f(0,\cdot)\big] \in H^\varepsilon(\partial\Omega),\ \big|\mathbf{d}\big|_{T_0,\alpha,\varepsilon} \le R_0, \\ &|\Phi[L_1 f(0,\cdot)]| \ge m_1 > 0,\ |\Phi[L_0 u_0 + L_1 f(0,\cdot)]| \ge m_2 > 0 \Big\}, \end{aligned} \tag{2.19}$$

where $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $R_0, T_0, m_1, m_2 \in \mathbb{R}_+$ are given and

$$\left|\mathbf{d}\right|_{T_0,\alpha,\varepsilon} := \|u_0\|_{2+\varepsilon,\partial\Omega} + \|u_1\|_{1+\varepsilon,\partial\Omega} + \|f\|_{T_0,2+\alpha,\varepsilon,\Omega} + \|g\|_{T_0,\alpha,1/2,\partial\Omega} + \|g\|_{T_0,1,\varepsilon,\partial\Omega} + \|bD_{\nu_A}L_0[g(0,\cdot) - bD_{\nu_A}L_1 f(0,\cdot)]\|_{\varepsilon,\partial\Omega} + \|l\|_{T_0,2+\alpha}.$$

Remark 2.1. The condition concerning $g(0,\cdot) - bD_{\nu_A}L_1 f(0,\cdot)$ in (2.19) is trivially satisfied if $f(0,\cdot) \in H^{1/2+\varepsilon}(\Omega)$ and $g(0,\cdot) \in H^{1+\varepsilon}(\partial\Omega)$.

We can now state our main result.

Theorem 2.2. *Let* $\mathbf{d} = (u_0, u_1, f, g, l) \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\alpha \in (0,1)$, $R_0, T_0 \in \mathbb{R}_+$. *Then there exists* $\overline{T} \in (0, T_0]$ *for which problem* (2.2)–(2.1) *admits a unique solution* $(u,k) \in [C^2([0,T]; H^{(1/2)+\varepsilon}(\Omega)) \cap C([0,T]; H^{1+\varepsilon}(\Omega))] \times C^1([0,T])$ *for all* $T \in (0, \overline{T})$, *such that* $D_{\nu_A}u \in C([0,T]; H^{-(1/2)+\varepsilon}(\partial\Omega))$, *continuously depending on the data* $\mathbf{d}$ *with respect to the natural metrics of the spaces pointed out.*

3. Equivalent problems

Problem (2.9)-(2.14) is equivalent to the following: *find a pair of functions* (w,h) *such that*

$$(w,h) \in C^2([0,T]; H^{\varepsilon}(\partial\Omega)) \cap C([0,T]; H^{(1/2)+\varepsilon}(\partial\Omega)) \times C^1([0,T]), \tag{3.1}$$

$$D_t^2 w - Bw = [g - bD_{\nu_A}L_1 f] + [-h(0)D_t w - bh * D_{\nu_A}L_1 f - D_t h * D_t w + h * Bw] - u_0 D_t h =: F_1 + F_2(w, D_t w, Bw, h, D_t h) - u_0 D_t h \quad \text{on } \partial\Omega, \tag{3.2}$$

$$w(0,\cdot) = u_0, \quad \text{on } \partial\Omega, \tag{3.3}$$

$$D_t w(0,\cdot) = u_1 - h(0)u_0, \quad \text{on } \partial\Omega, \tag{3.4}$$

$$\Phi[v(t,\cdot)] + h * \Phi[v(t,\cdot)] = l(t), \quad t \in [0,T], \tag{3.5}$$

where $w := v|_{\partial\Omega}$, $B := -bD_{\nu_A}L_0$. Then, differentiating (3.5) twice, for all $t \in [0,T]$, we get

$$\Phi[D_t v(t,\cdot)] + h * \Phi[D_t v(t,\cdot)] + h(t)\Phi[v(0,\cdot)] = l'(t), \tag{3.6}$$

$$\Phi[D_t^2 v(t,\cdot)] + D_t h * \Phi[D_t v(t,\cdot)] + h(0)\Phi[D_t v(t,\cdot)] + D_t h(t)\Phi[v(0,\cdot)] = l''(t). \tag{3.7}$$

Now we assume

$$\Phi[L_0 u_0 + L_1 f(0,\cdot)] \neq 0. \tag{3.8}$$

Consequently, from (2.16) and (3.6), with $t = 0$, recalling that $w(0,\cdot) = u_0$ and $D_t w(0,\cdot) = u_1$, we get the equation

$$\Phi[L_0 u_1 + L_1 D_t f(0,\cdot)] + h(0)\Phi[L_0 u_0 + L_1 f(0,\cdot)] = l'(0),$$

implying

$$h(0) = \{\Phi[L_0u_0 + L_1f(0,\cdot)]\}^{-1}\{l'(0) - \Phi[L_0u_1 + L_1D_tf(0,\cdot)]\} =: h_0. \quad (3.9)$$

From (3.1)–(3.5) we deduce that the pair (w, h) solves the following problem: *find a pair of functions* (w, h) *such that*

$$(w, h) \in C^2([0, T]; H^{\varepsilon}(\partial\Omega)) \cap C([0, T]; H^{(1/2)+\varepsilon}(\partial\Omega)) \times C^1([0, T]) \quad (3.10)$$

$$D_t^2 w = Bw + F_1 + F_2(w, D_tw, Bw, h, D_th) - u_0D_th, \quad \text{on } \partial\Omega, \quad (3.11)$$

$$w(0,\cdot) = u_0, \quad \text{on } \partial\Omega, \quad (3.12)$$

$$D_tw(0,\cdot) = u_1 - h_0u_0, \quad \text{on } \partial\Omega, \quad (3.13)$$

$$\begin{aligned}&\Phi[L_0D_t^2w(t,\cdot)] + h_0\Phi[L_0D_tw(t,\cdot)] + D_th * \Phi[L_0D_tw(t,\cdot)]\\ &+\Phi[L_1D_t^2f(t,\cdot)] + D_th * \Phi[L_1D_tf(t,\cdot)] + h_0\Phi[L_1D_tf(t,\cdot)]\\ &+D_th(t)\{\Phi[L_0u_0] + \Phi[L_1f(0,\cdot)]\} = l''(t), \quad t \in [0, T]. \quad (3.14)\end{aligned}$$

Now we assume

$$\chi := \Phi[L_1f(0,\cdot)] \neq 0. \quad (3.15)$$

Then we observe that equation (3.10) is equivalent to the following system, with $w \in C([0, T]; H^{1/2+\varepsilon}(\partial\Omega))$ and $D_tw \in C^1([0, T]; H^{\varepsilon}(\partial\Omega))$, $\varepsilon \in (1/2, 3/2) \setminus \{1\}$:

$$D_t\begin{pmatrix} w \\ D_tw \end{pmatrix} = \mathcal{B}_\beta\begin{pmatrix} w \\ D_tw \end{pmatrix} + \begin{pmatrix} 0 \\ F_1 + F_2(w, D_tw, Bw, h, D_th) - u_0D_th \end{pmatrix}, \quad (3.16)$$

where

$$\mathcal{D}(\mathcal{B}_\beta) = H^{\beta+1}(\partial\Omega) \times H^{\beta+1/2}(\partial\Omega), \qquad \mathcal{B}_\beta = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix}, \quad \beta \geq 0.$$

We note that $\mathcal{B}_\beta \in \mathcal{L}(H^{\beta+1}(\partial\Omega) \times H^{\beta+1/2}(\partial\Omega); H^{\beta+1/2}(\partial\Omega) \times H^{\beta}(\partial\Omega))$, for all $\beta \geq 0$. Indeed, $B \in \mathcal{L}(H^{\beta+1/2}(\partial\Omega); H^{\beta-1/2}(\partial\Omega))$, $\beta \geq 0$, (cf. [9]).

According to the results in [9] and in the Appendix, $\mathcal{B}_\beta$ generates a continuous semigroup

$$\mathcal{S}(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix}$$

of linear bounded operators from $H^{\beta+1/2}(\partial\Omega) \times H^{\beta}(\partial\Omega)$ into itself, for all $\beta \geq 0$.

Then, according to formulation (3.16) and well-known results in semigroup theory, problem (3.10) and (3.12) is equivalent to the following fixed-point equation:

$$w(t) = u_0 + \Big\{[S_{11}(t) - I]u_0 + S_{12}(t)(u_1 - h_0u_0) + \int_0^t S_{12}(t-s)F_1(s)\,ds\Big\}$$
$$+ \int_0^t S_{12}(t-s)F_2(w, D_tw, Bw, h, D_th)(s)\,ds - \int_0^t D_th(s)S_{12}(t-s)u_0\,ds$$
$$=: u_0 + w_1(t) + \widetilde{N}_1(w, D_tw, Bw, h, D_th)(t) - \int_0^t D_th(s)S_{12}(t-s)u_0\,ds$$
$$=: u_0 + w_1(t) + N_1(w, D_tw, Bw, h, D_th)(t). \tag{3.17}$$

We note that $w_1(0) = N_1(w, D_tw, Bw, h, D_th)(0) = 0$. From (9.5) and (9.6) in the Appendix we obtain

$$D_tw(t) = \Big\{S_{12}(t)Bu_0 + S_{11}(t)(u_1 - h_0u_0) + \int_0^t S_{11}(t-s)F_1(s)\,ds\Big\}$$
$$+ \int_0^t S_{11}(t-s)F_2(w, D_tw, Bw, h, D_th)(s)\,ds - \int_0^t D_th(s)S_{11}(t-s)u_0\,ds$$
$$= u_1 - h_0u_0 + \Big\{S_{12}(t)Bu_0 + [S_{11}(t) - I](u_1 - h_0u_0) + \int_0^t S_{11}(t-s)F_1(s)\,ds\Big\}$$
$$+ \int_0^t S_{11}(t-s)F_2(w, D_tw, Bw, h, D_th)(s)\,ds - \int_0^t D_th(s)S_{11}(t-s)u_0\,ds$$
$$=: u_1 - h_0u_0 + w_2(t) + \widetilde{N}_2(w, D_tw, Bw, h, D_th)(t) - \int_0^t D_th(s)S_{11}(t-s)u_0\,ds$$
$$=: u_1 - h_0u_0 + w_2(t) + N_2(w, D_tw, Bw, h, D_th)(t). \tag{3.18}$$

We note that $w_2(0) = N_2(w, D_tw, Bw, h, D_th)(0) = 0$. We need now to compute D_t^2w. Even though we could derive its expression directly from equation (3.10), it seems more convenient to differentiate (3.18) with respect to t. Using the definition of F_2 in (3.11), we get

$$D_t^2w(t) = Bu_0 + F_1(0) - h_0(u_1 - h_0u_0) + \Big\{[S_{11}(t) - I]Bu_0 \tag{3.19}$$
$$+ S_{12}(t)[Bu_1 - h_0Bu_0] + \int_0^t S_{11}(s)D_tF_1(t-s)\,ds + [S_{11}(t) - I]F_1(0)$$
$$- h_0\int_0^t S_{11}(s)bD_{\nu_A}L_1f(t-s)\,ds - h_0[S_{11}(t) - I](u_1 - h_0u_0)\Big\}$$
$$+ \Big\{\int_0^t S_{11}(t-s)F_3(w, D_tw, D_t^2w, Bw, h, D_th)(s)\,ds$$
$$- \int_0^t D_th(s)S_{12}(t-s)Bu_0\,ds\Big\} - D_th(t)u_0 =: Bu_0 + F_1(0)$$
$$- h_0(u_1 - h_0u_0) + \widetilde{w}_3(t) + \widetilde{N}_3(w, D_tw, D_t^2w, Bw, D_th)(t) - D_th(t)u_0,$$

where

$$\begin{aligned}
&F_3(w, D_t w, D_t^2 w, Bw, h, D_t h)(t)\\
=:&\, D_t F_2(w, D_t w, Bw, h, D_t h)(t) + h_0 b D_{\nu_A} L_1 f(t)\\
=&\, -h_0 D_t^2 w(t) - b D_t h * D_{\nu_A} L_1 f(t) - D_t h * D_t^2 w(t)\\
&- D_t h(t)(u_1 - h_0 u_0) + D_t h * Bw(t) + h_0 Bw(t). \qquad (3.20)
\end{aligned}$$

We note that $\widetilde{w}_3(0) = \widetilde{N}_3(w, D_t w, D_t^2 w, Bw, D_t h)(0) = 0$. Then we replace the expression of $D_t^2 w$ from (3.19) into (3.14). According to the definition of the operator F_3, we obtain the following fixed-point equation for $D_t h$, $t \in [0,T]$:

$$\begin{aligned}
&D_t h(t) = \chi^{-1}\Big\{ - \Phi[L_0 B u_0] - \Phi[L_0 F_1(0)] + \Phi[L_0 h_0 (u_1 - h_0 u_0)]\\
&- \Phi[L_1 D_t^2 f(0,\cdot)] - h_0 \Phi[L_1 D_t f(0,\cdot)] + l''(0)\Big\}\\
&+ \chi^{-1}\Big\{ - \Phi\{L_1 D_t^2 [f(t,\cdot) - f(0,\cdot)]\} - h_0 \Phi\{L_1 [D_t f(t,\cdot) - D_t f(0,\cdot)]\}\\
&+ l''(t) - l''(0) - \Phi[L_0 \widetilde{w}_3(t)]\Big\} + \chi^{-1}\Big\{ h_0 \Phi[L_0 S_{11} * D_t^2 w(t)]\\
&+ \Phi[L_0 D_t h * S_{11} * b D_{\nu_A} L_1 f(t)] + \Phi[L_0 D_t h * S_{11} * D_t^2 w(t)]\\
&+ \Phi[L_0 D_t h * S_{11}(u_1 - h_0 u_0)] - \Phi[L_0 D_t h * S_{11} * Bw(t)]\\
&- h_0 \Phi[L_0 S_{11} * Bw(t)]] - D_t h * \Phi[L_0 D_t w(t,\cdot)] - D_t h * \Phi[L_1 D_t f(t,\cdot)]\Big\}\\
&- h_0 \Phi[L_0 D_t w(t,\cdot)] =: \chi^{-1}\Big\{ - \Phi[L_0 B u_0] - \Phi[L_0 F_1(0)]\\
&+ \Phi[L_0 h_0 (u_1 - h_0 u_0)] - \Phi[L_1 D_t^2 f(0,\cdot)] - h_0 \Phi[L_1 D_t f(0,\cdot)] + l''(0)\Big\}\\
&+ \widetilde{h}(t) + \widetilde{N}_4(w, D_t w, D_t^2 w, Bw, D_t h)(t) - h_0 \Phi[L_0 D_t w(t,\cdot)]. \qquad (3.21)
\end{aligned}$$

We note that $\widetilde{h}(0) = \widetilde{N}_4(w, D_t w, D_t^2 w, Bw, D_t h)(0) = 0$. We introduce now the following Banach space:

$$\begin{aligned}
&\mathcal{W}(T,\varepsilon) := [C^2([0,T]; H^\varepsilon(\partial\Omega)) \cap C([0,T]; H^{(1/2)+\varepsilon}(\partial\Omega))] \times C^1([0,T];\\
&H^\varepsilon(\partial\Omega)) \times C([0,T]; H^\varepsilon(\partial\Omega)) \times C([0,T]; H^\varepsilon(\partial\Omega)) \times C^1([0,T]) \times C([0,T]).
\end{aligned}$$

Up to now we have proved that $(w,h) \in [C^2([0,T]; H^\varepsilon(\partial\Omega)) \cap C([0,T]; H^{(1/2)+\varepsilon}(\partial\Omega))] \times C^1([0,T])$ is a solution of problem (3.10)–(3.14) if and only if the sextuplet $(w, D_t w, Bw, h, D_t h)$ satisfies the problem

$$(w, D_t w, Bw, h, D_t h) \in \mathcal{W}(T,\varepsilon), \qquad (3.22)$$

$$w(t) = u_0 + w_1(t) + N_1(w, D_t w, Bw, h, D_t h)(t), \qquad (3.23)$$

$$D_t w(t) = u_1 - h_0 u_0 + w_2(t) + N_2(w, D_t w, Bw, h, D_t h)(t), \qquad (3.24)$$

$$\begin{aligned} D_t^2 w(t) + D_t h(t) u_0 &= B u_0 + F_1(0) - h_0(u_1 - h_0 u_0) + \widetilde{w}_3(t) \\ &\quad + \widetilde{N}_3(w, D_t w, D_t^2 w, Bw, D_t h)(t), \qquad (3.25) \end{aligned}$$

$$Bw(t) = Bu_0 + Bw_1(t) + BN_1(w, D_t w, Bw, h, D_t h)(t), \qquad (3.26)$$

$$h(t) = h_0 + 1 * D_t h(t), \qquad (3.27)$$

$$\begin{aligned} &D_t h(t) + h_0 \Phi[L_0 D_t w(t,\cdot)] = \chi^{-1} \Big\{ - \Phi[L_0 B u_0] - \Phi[L_0 F_1(0)] \\ &+ \Phi[L_0 h_0(u_1 - h_0 u_0)] - \Phi[L_1 D_t^2 f(0,\cdot)] - h_0 \Phi[L_1 D_t f(0,\cdot)] + l''(0) \Big\} \\ &+ \widetilde{h}(t) + \widetilde{N}_4(w, D_t w, D_t^2 w, Bw, D_t h)(t). \qquad (3.28) \end{aligned}$$

Observe now that for any $f \in C^1([0,T]; H^\varepsilon(\partial\Omega)) \cap C([0,T]; H^{1+\varepsilon}(\partial\Omega))$ we get

$$\begin{aligned} B(S_{12} * D_t w) &= S_{11} * D_t^2 w + S_{11}(\cdot) D_t w(0) - D_t w \\ &= S_{11} * D_t^2 w + S_{11}(\cdot)(u_1 - h_0 u_0) - D_t w, \qquad (3.29) \end{aligned}$$

since (cf. (9.14) in the Appendix)

$$\begin{aligned} B \int_0^t S_{12}(t-s) f(s)\, ds &= \int_0^t B S_{12}(t-s) f(s)\, ds \\ &= \int_0^t D_t[S_{11}(t-s) f(s)]\, ds = D_t \int_0^t S_{11}(t-s) f(s)\, ds - f(t) \\ &= \int_0^t S_{11}(t-s) f'(s)\, ds + S_{11}(t) f(0) - f(t). \end{aligned}$$

Let $\{f_n\} \subset C^1([0,T]; H^\varepsilon(\partial\Omega)) \cap C([0,T]; H^{1+\varepsilon}(\partial\Omega))$ be a sequence such that $f_n \to f$ in $C^1([0,T]; H^\varepsilon(\partial\Omega))$. Then

$$\begin{aligned} &\int_0^t S_{11}(t-s) f_n(s)\, ds \to \int_0^t S_{11}(t-s) f(s)\, ds, \quad \forall t \in [0,T], \text{ as } n \to +\infty, \\ &B \int_0^t S_{11}(t-s) f_n(s)\, ds = \int_0^t S_{11}(t-s) f_n'(s)\, ds + S_{11}(t) f_n(0) - f_n(t) \\ &\to \int_0^t S_{11}(t-s) f'(s)\, ds + S_{11}(t) f(0) - f(t), \quad \forall t \in [0,T], \text{ as}\, n \to +\infty. \end{aligned}$$

Indeed, if $g_n \to g$ in $C([0,T]; H^\varepsilon(\partial\Omega))$, we have (cf. (2.14a))

$$\begin{aligned} &\int_0^t \|S_{11}(t-s)\|_{\mathcal{L}(X)} \|g_n(s) - g(s)\|\, ds \le M \|g_n - g\|_{0,T,\varepsilon,\partial\Omega} \int_0^t e^{\omega s}\, ds \\ &\le \frac{M}{\omega} \big(e^{\omega T} - 1\big) \|g_n - g\|_{0,T,\varepsilon,\partial\Omega} \to 0, \quad \text{as } n \to +\infty. \end{aligned}$$

Then, from (3.29) we get

$$\begin{aligned} Bw_1(t) = \Big\{&[S_{11}(t)-I]Bu_0 + S_{12}(t)(Bu_1 - h_0Bu_0) + S_{11} * D_tF_1(t) \\ &+ [S_{11}(t)-I]F_1(0) - F_1(t) + F_1(0)\Big\}, \end{aligned}$$

$$\begin{aligned} &BN_1(w, D_tw, h)(t) \qquad (3.30)\\ &= F_2(w, D_tw, Bw, h, D_th)(0) + [S_{11}(t)-I]F_2(w, D_tw, Bw, h, D_th)(0) \\ &\quad - F_2(w, D_tw, Bw, h, D_th)(t) + S_{11} * D_tF_2(w, D_tw, Bw, h, D_th)(t) \\ &\quad - D_th(t) * S_{12}Bu_0 = -h_0(u_1 - h_0u_0) + \Big\{ - [S_{11}(t)-I][h_0(u_1 - h_0u_0)] \\ &\quad - h_0S_{11} * D_{\nu_A}L_1f(t)\Big\} + \Big\{bh * D_{\nu_A}L_1f(t) + D_th * D_tw(t) - h * Bw(t) \\ &\quad + S_{11} * F_3(w, D_tw, D_t^2w, Bw, h, D_th)(t) - D_th * S_{12}Bu_0(t)\Big\} + h_0D_tw(t) \\ &=: -h_0(u_1 - h_0u_0) + \widetilde{w}_5(t) + \widetilde{N}_5(D_tw, D_t^2w, Bw, h, D_th)(t) + h_0D_tw(t). \end{aligned}$$

We note that

$$\widetilde{w}_5(0) = \widetilde{N}_5(D_tw, D_t^2w, Bw, h, D_th)(0) = 0. \qquad (3.31)$$

Hence, we obtain the new equivalent problem for (w, D_tw, Bw, h, D_th):

$$\begin{cases} (w, D_tw, Bw, h, D_th) \in \mathcal{W}(T, \varepsilon), \\ w(t) = u_0 + w_1(t) + N_1(w, D_tw, Bw, h, D_th)(t), \\ D_tw(t) = u_1 - h_0u_0 + w_2(t) + N_2(w, D_tw, Bw, h, D_th)(t), \\ D_t^2w(t) + D_th(t)u_0 = Bu_0 + F_1(0) - h_0(u_1 - h_0u_0) + \widetilde{w}_3(t) \\ \qquad + \widetilde{N}_3(w, D_tw, D_t^2w, Bw, D_th)(t), \\ Bw(t) - h_0D_tw(t) = Bu_0 - h_0(u_1 - h_0u_0) + Bw_1(t) + \widetilde{w}_5(t) \\ \qquad + \widetilde{N}_5(D_tw, D_t^2w, Bw, h, D_th)(t), \\ h(t) = h_0 + 1 * D_th(t), \\ D_th(t) + h_0\Phi[L_0D_tw(t,\cdot)] = \chi^{-1}\Big\{ - \Phi[L_0Bu_0] - \Phi[L_0F_1(0)] \\ +\Phi[L_0h_0(u_1 - h_0u_0)] - \Phi[L_1D_t^2f(0,\cdot)] - h_0\Phi[L_1D_tf(0,\cdot)] + l''(0)\Big\} \\ +\widetilde{h}(t) + \widetilde{N}_4(w, D_tw, D_t^2w, Bw, D_th)(t). \end{cases} \qquad (3.32)$$

It is immediate to note that the two systems (3.22)–(3.28) and (3.32) are equivalent.

Let us now consider the auxiliary system

$$\begin{aligned} D_tw(t) &= f_1(t), \\ D_t^2w(t) + D_th(t)u_0 &= f_2(t), \end{aligned}$$

$$\begin{aligned} Bw(t) - h_0 D_t w(t) &= f_3(t), \\ D_t h(t) + h_0 \chi^{-1} \Phi[L_0 D_t w(t,\cdot)] &= f_4(t). \end{aligned}$$

The solution of such a system is given by

$$\begin{aligned} &D_t w(t) = f_1(t), \quad D_t^2 w(t) = f_2(t) + h_0 \chi^{-1} \Phi[L_0 f_1(t)] u_0 - f_4(t) u_0, \\ &Bw(t) = f_3(t) + h_0 f_1(t), \quad D_t h(t) = -h_0 \chi^{-1} \Phi[L_0 f_1(t)] + f_4(t). \end{aligned}$$

Consequently, problem (3.32) is equivalent to the following one, where we have separated the values at $t = 0$ of the right-hand sides - which are independent of the unknowns - from the terms vanishing at $t = 0$:

$$\begin{cases} (w, D_t w, Bw, h, D_t h) \in \mathcal{W}(T, \varepsilon), \\ w(t) = z_{0,1} + w_1(t) + N_1(w, D_t w, Bw, h, D_t h)(t), \\ D_t w(t) = z_{0,2} + w_2(t) + N_2(w, D_t w, Bw, h, D_t h)(t), \\ D_t^2 w(t) = z_{0,3} + w_3(t) + N_3(w, D_t w, D_t^2 w, Bw, h, D_t h)(t), \\ Bw(t) = z_{0,4} + w_4(t) + N_4(w, D_t w, D_t^2 w, Bw, h, D_t h)(t), \\ h(t) = z_{0,5} + N_5(D_t h)(t), \\ D_t h(t) = z_{0,6} + w_6(t) + N_6(w, D_t w, D_t^2 w, Bw, h, D_t h)(t), \end{cases} \tag{3.33}$$

where we have set

$$\begin{aligned} &z_{0,1} = u_0, \quad z_{0,2} = u_1 - h_0 u_0, \quad z_{0,4} = Bu_0 + h_0(u_1 - h_0 u_0), \quad z_{0,5} = h_0, \\ &z_{0,3} = Bu_0 - h_0(u_1 - h_0 u_0) + F_1(0) + \chi^{-1} \Big\{ \Phi[L_0 B u_0(\cdot)] + \Phi[L_1 D_t^2 f(0,\cdot)] \\ &\qquad + h_0 \Phi[L_1 D_t f(0,\cdot)] + \Phi[L_0 F_1(0,\cdot)] - l''(0) \Big\} u_0, \\ &z_{0,6} = -\chi^{-1} \Phi[L_0 B u_0] + \chi^{-1} \Big\{ - \Phi[L_0 B u_0] - \Phi[L_0 F_1(0)] \\ &\qquad + \Phi[L_0 h_0 (u_1 - h_0 u_0)] - \Phi[L_1 D_t^2 f(0,\cdot)] - h_0 \Phi[L_1 D_t f(0,\cdot)] + l''(0) \Big\}, \end{aligned}$$

and

$$\begin{aligned} w_3(t) = \widetilde{w}_3(t) &+ h_0 \chi^{-1} \Phi[L_0 w_2(t)] u_0 \\ &+ \Big\{ \chi^{-1} \Phi[L_0 B w_1(t)] + \chi^{-1} \Phi[L_0 \widetilde{w}_5(t)] - \widetilde{h}(t) \Big\} u_0, \end{aligned} \tag{3.34}$$

$$w_4(t) = Bw_1(t) + \widetilde{w}_5(t) + h_0 w_2(t), \tag{3.35}$$

$$w_6(t) = -h_0 \chi^{-1} \Phi[L_0 w_1(t)] + \widetilde{h}(t), \tag{3.36}$$

$$\begin{aligned} &N_3(w, D_t w, D_t^2 w, Bw, h, D_t h)(t) = h_0 \chi^{-1} \Phi[L_0 N_1(w, D_t w, Bw, h, D_t h)(t)] u_0 \\ &- \widetilde{N}_4(w, D_t w, D_t^2 w, Bw, D_t h)(t) u_0 + \widetilde{N}_3(w, D_t w, D_t^2 w, Bw, D_t h)(t), \end{aligned} \tag{3.37}$$

$$N_4(w, D_t w, D_t^2 w, Bw, h, D_t h)(t) = h_0 N_2(w, D_t w, Bw, h, D_t h)(t)$$

$$+ \widetilde{N}_5(D_t w, D_t^2 w, Bw, h, D_t h)(t), \quad (3.38)$$

$$N_5(D_t h)(t) = 1 * D_t h(t), \quad (3.39)$$

$$N_6(w, D_t w, D_t^2 w, Bw, h, D_t h)(t) = -h_0 \chi^{-1} \Phi[L_0 N_1(w, D_t w, Bw, h, D_t h)(t)] + \widetilde{N}_4(D_t w, D_t^2 w, Bw, D_t h)(t). \quad (3.40)$$

We conclude this section with the following equivalent results.

Theorem 3.1. *Problems* (3.1)–(3.5) *and* (3.33) *are equivalent.*

Theorem 3.2. *Problems* (2.9)–(2.14) *and* (3.1)–(3.5) *are equivalent via the elliptic Dirichlet problem* (2.15).

4. Solving the fixed-point system (3.33).

To solve our fixed-point system (3.33) we will use the notation $f_\rho(t) = e^{-\rho t} f(t)$, $t \in [0, T]$, with $\rho > \max\{0, \omega\}$, ω being defined in (4.6), *for any function* f. Further we note that $(f * g)_\rho = f_\rho * g_\rho$, whenever the convolution makes sense.

We now introduce the spaces of *admissible unknowns:*

$$\mathcal{Z}(\varepsilon, \rho, R, T) := \Big\{ \mathbf{z} = (z_1, z_2, z_3, z_4, z_5, z_6) : z_j \in C([0, T]; H^\varepsilon(\partial\Omega)), \quad (4.1)$$
$$j = 1, 2, 3, 4, \ z_j \in C([0, T]), \ j = 5, 6, \ \|\mathbf{z} - \mathbf{z}_0\|_{T,\varepsilon,\rho} \le R \Big\},$$

where

$$\mathbf{z}_0 := (z_{0,1}, z_{0,2}, z_{0,3}, z_{0,4}, z_{0,5}, z_{0,6}) \quad (4.2)$$

$$\|\mathbf{z}\|_{t,\varepsilon,\rho} := \sum_{j=1}^{4} \|(z_j)_\rho\|_{t,0,\varepsilon,\partial\Omega} + \sum_{j=5}^{6} \|(z_j)_\rho\|_{t,0}. \quad (4.3)$$

Then we note that system (3.33) for $\mathbf{z}$ can be rewritten in a more symmetric form as

$$\begin{cases} \mathbf{z} \in \mathcal{Z}(\varepsilon, \rho, +\infty, T), \\ z_1(t) = z_{0,1}(\mathbf{d}) + w_1(\mathbf{d})(t) + N_1(\mathbf{z}, \mathbf{d})(t), \\ z_2(t) = z_{0,2}(\mathbf{d}) + w_2(\mathbf{d})(t) + N_2(\mathbf{z}, \mathbf{d})(t), \\ z_3(t) = z_{0,3}(\mathbf{d}) + w_3(\mathbf{d})(t) + N_3(\mathbf{z}, \mathbf{d})(t), \\ z_4(t) = z_{0,4}(\mathbf{d}) + w_4(\mathbf{d})(t) + N_4(\mathbf{z}, \mathbf{d})(t), \\ z_5(t) = z_{0,5}(\mathbf{d}) + N_5(\mathbf{z}, \mathbf{d})(t), \\ z_6(t) = z_{0,6}(\mathbf{d}) + w_6(\mathbf{d})(t) + N_6(\mathbf{z}, \mathbf{d})(t), \end{cases} \quad (4.4)$$

where we have replaced $(w, D_t w, Bw, D_t^2 w, h, D_t h)$ with $\mathbf{z}$. Moreover, we have pointed out the dependence of the functions $z_{0,j}$, w_j and the operators N_j, $j = 1, \ldots, 6$, also on the data vector $\mathbf{d}$. Further we note that some of the previous nonlinear operators N_j, $j = 1, \dots, 6$, may not depend on all the components of $(\mathbf{z}, \mathbf{d})$. In the sequel it will be useful to introduce the following vectors $\mathbf{z}_0, \mathbf{w} : \mathcal{D}(\varepsilon_0, \alpha, R_0, T_0) \to \big[C([0,T_0]; H^\varepsilon(\partial\Omega))\big]^4 \times \big[C([0,T_0])\big]^2$ and nonlinear operator $\mathbf{N} : \mathcal{Z}(\varepsilon, \rho, R, T) \times \mathcal{D}(\varepsilon, \alpha, R, T_0) \to \big[C([0,T]; H^\varepsilon(\partial\Omega))\big]^4 \times \big[C([0,T])\big]^2$, with $T \in (0, T_0]$, defined by

$$\begin{aligned}
&\mathbf{z}_0(\mathbf{d}) := \big(z_{0,1}(\mathbf{d}), z_{0,2}(\mathbf{d}), z_{0,3}(\mathbf{d}), z_{0,4}(\mathbf{d}, z_{0,5}(\mathbf{d}), z_{0,6}(\mathbf{d}))\big),\\
&\mathbf{w}(\mathbf{d}) := (w_1(\mathbf{d}), w_2(\mathbf{d}), w_3(\mathbf{d}), w_4(\mathbf{d}), 0, w_6(\mathbf{d})),\\
&\mathbf{N}(\mathbf{z}, \mathbf{d}) := \big(N_1(\mathbf{z},\mathbf{d}), N_2(\mathbf{z},\mathbf{d}), N_3(\mathbf{z},\mathbf{d}), N_4(\mathbf{z},\mathbf{d}), N_5(\mathbf{z},\mathbf{d}), N_6(\mathbf{z},\mathbf{d})\big).
\end{aligned} \tag{4.5}$$

We recall now that, for $t \in [0,T]$ and $\varepsilon \in (0, 1/2)$, we have (cf. (9.1) and (9.2) in the Appendix)

$$\begin{aligned}
&\max\Big\{\|S_{11}(t)\|_{\mathcal{L}(H^\varepsilon(\partial\Omega))}, \|S_{11}(t)\|_{\mathcal{L}(H^{1/2+\varepsilon}(\partial\Omega))}, \|S_{11}(t)\|_{\mathcal{L}(H^{1+\varepsilon}(\partial\Omega))},\\
&\|S_{12}(t)\|_{\mathcal{L}(H^{1/2+\varepsilon}(\partial\Omega); H^{1+\varepsilon}(\partial\Omega))}, \|S_{12}(t)\|_{\mathcal{L}(H^\varepsilon(\partial\Omega); H^{1/2+\varepsilon}(\partial\Omega))}\Big\} \le M e^{\omega t}.
\end{aligned} \tag{4.6}$$

Let $\mathbf{d}^i := (u_0^i, u_1^i, f^i, g^i, l^i)$, $\mathbf{z}^i := (z_1^i, z_2^i, z_3^i, z_4^i, z_5^i, z_6^i)$, $i = 1, 2$, and

$$F_1(\mathbf{d}^i)(t) = g^i(t) - b D_{\nu_A} L_1 f^i(t), \tag{4.7}$$

$$F_2(\mathbf{z}^i, \mathbf{d}^i)(t) = -h_0^i z_2^i(t) - b z_5^i * D_{\nu_A} L_1 f^i(t) - z_6^i * z_2^i(t) + z_5^i * z_3^i(t), \tag{4.8}$$

$$\begin{aligned}
F_3(\mathbf{z}^i, \mathbf{d}^i)(t) = {} & -h_0 z_4^i(t) - b z_6^i * D_{\nu_A} L_1 f^i(t) - z_6^i * z_4^i(t)\\
& - z_6^i(t)(u_1 - h_0 u_0) + z_6^i * z_3^i(t) + h_0 z_3^i(t),
\end{aligned} \tag{4.9}$$

$$h_0^i := \{\Phi[L_0 u_0^i + L_1 f^i(0,\cdot)]\}^{-1}\{(l^i)'(0) - \Phi[L_0 u_1^i + L_1 D_t f^i(0,\cdot)]\}. \tag{4.10}$$

Theorem 4.1. *Problems* (3.33) *and* (4.4) *are equivalent via* $w = z_1$ *and* $h = z_5$.

Proof. To prove the equivalence between problem (4.4) for $z \in \mathcal{Z}(\varepsilon, \rho, R, T)$ and problem (3.33) for $(w, D_t w, D_t^2 w, Bw, h, D_t h) \in \mathcal{W}(T, \varepsilon)$ first we observe that the identity

$$\begin{aligned}
&D_t\big[z_{0,1} + w_1(t) + N_1(z_1, z_2, z_3, z_4, z_5, z_6)(t)\big]\\
&= z_{0,2} + w_2(t) + N_2(z_1, z_2, z_3, z_4, z_5, z_6)(t)
\end{aligned}$$

implies $z_1 \in C^1([0,T]; H^\varepsilon(\partial\Omega))$ and $z_1' = z_2$.

Then we need to show the following relation among z_2 and z_3: $z_2 \in C^1([0,T]; H^\varepsilon(\partial\Omega))$ and $D_t z_2 = z_3$ in $[0,T]$. For this purpose we consider the difference

$$y := y_1 - y_2 = z_2 - z_{0,2} - 1 * z_3, \tag{4.11}$$

where

$$y_1 := w_2 + N_2(z_1, z_2, z_4, h_0 + 1 * z_6, z_6),$$
$$y_2 := 1 * z_{0,3} + 1 * w_3 + 1 * N_3(z_1, z_2, z_3, z_4, z_6).$$

To compute y_1 and y_2 we use the identities $S_{11} = I + 1 * S_{12}B$, $S_{12} = 1 * S_{11}$. We get

$$\begin{aligned} y_1 &= S_{12}Bu_0 + S_{11}(u_1 - h_0u_0) - (u_1 - h_0u_0) + S_{11} * F_1(t) - h_0S_{11} * z_2 \\ &\quad - h_0^1 * S_{11} * bD_{\nu_A}L_1f - 1 * S_{11} * z_6 * bD_{\nu_A}L_1f - S_{11} * z_6 * z_2 \\ &\quad + h_0^1 * S_{11} * z_4 + 1 * S_{11} * z_6 * z_4 - z_6 * S_{11}u_0 \\ &= 1 * S_{11}Bu_0 + 1 * S_{12}B(u_1 - h_0u_0) + S_{11} * \big[F_1(0) + 1 * D_tF_1\big] \\ &\quad - h_0S_{11} * z_2 - h_0^1 * S_{11} * bD_{\nu_A}L_1f - 1 * S_{11} * z_6 * bD_{\nu_A}L_1f \\ &\quad - S_{11} * z_6 * z_2 + h_0^1 * S_{11} * z_4 + 1 * S_{11} * z_6 * z_4 \\ &\quad - (1 * z_6)[u_0 + 1 * S_{12}Bu_0], \end{aligned} \tag{4.12}$$

$$\begin{aligned} y_2 &= 1 * S_{11}Bu_0 + 1 * S_{12}B(u_1 - h_0u_0) + S_{11} * \big[F_1(0) + D_tF_1\big] \\ &\quad - h_0^1 * S_{11} * bD_{\nu_A}L_1f - h_0^1 * S_{11}(u_1 - h_0u_0) \\ &\quad + 1 * S_{11} * F_3(z_1, z_3, z_4, z_6) - (1 * z_6)[u_0 + 1 * S_{12}Bu_0] \\ &= 1 * S_{11}Bu_0 + 1 * S_{12}B(u_1 - h_0u_0) + S_{11} * \big[F_1(0) + D_tF_1\big] \\ &\quad - h_0^1 * S_{11} * bD_{\nu_A}L_1f - h_0^1 * S_{11}(u_1 - h_0u_0) \\ &\quad - 1 * S_{11} * z_6 * bD_{\nu_A}L_1f - h_0^1 * S_{11} * z_3 - 1 * S_{11} * z_3 * z_6 \\ &\quad - 1 * z_6 * S_{11}(u_1 - h_0u_0) + 1 * S_{11} * z_6 * z_4 + h_0^1 * S_{11} * z_4 \\ &\quad - (1 * z_6)[u_0 + 1 * S_{12}Bu_0]. \end{aligned} \tag{4.13}$$

From (4.11)–(4.13) we easily deduce the following homogeneous linear convolution equation for y:

$$\begin{aligned} y &= -h_0S_{11} * z_2 - S_{11} * z_6 * z_2 + h_0^1 * S_{11}(u_1 - h_0u_0) + h_0^1 * S_{11} * z_3 \\ &\quad + 1 * z_6 * S_{11}(u_1 - h_0u_0) + 1 * S_{11} * z_3 * z_6 = -h_0S_{11} * y - S_{11} * z_6 * y \\ &= -\big[h_0S_{11} + S_{11} * z_6\big] * y. \end{aligned}$$

Since $h_0S_{11} + S_{11} * z_6 \in L^\infty((0,T); \mathcal{L}(H^\varepsilon(\partial\Omega)))$, we deduce (cf. (2.18)) that

$$\|y(t)\|_{\varepsilon,\partial\Omega} \le C \int_0^t \|y(s)\|_{\varepsilon,\partial\Omega}\, ds, \qquad \text{for a.e. } t \in (0,T).$$

From Gronwall's inequality it follows that $y = 0$ in $[0,T]$. This implies $z_2 = u_1 - h_0u_0 + 1 * z_3$; i.e., $z_2 \in C^1([0,T]; H^\varepsilon(\partial\Omega))$, $D_tz_2 = z_3$ in $[0,T]$ and $z_2(0) = u_1 - h_0u_0$.

Summing up, we have proved that $z_1 \in C^2([0,T]; H^\varepsilon(\partial\Omega))$ and $z_1' = z_2$, $z_1'' = z_3$.

Observe now that formula (3.30) holds when $(w, D_t w, h)$ is replaced with (z_1, z_2, z_5). Therefore, from the equations for z_1 and z_4 in (4.4), via the equivalence between problems (3.32) and (3.33) with $(w, D_t w, D_t^2 w, Bw, h, D_t h)$ being replaced with $(z_1, z_2, z_3, z_4, z_5, z_6)$, we deduce that $z_1 \in C([0,T]; H^{1+\varepsilon}(\partial\Omega))$ and $Bz_1 = z_4$. □

We can now state the main result for problem (4.4).

Theorem 4.2. *Let* $\mathbf{d} = (u_0, u_1, f, g, l) \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\alpha \in (0,1)$, $R_0, T_0 \in \mathbb{R}_+$. *Then there exists* $\overline{T} \in (0, T_0]$ *for which the fixed-point system (4.4) admits a unique solution* $\mathbf{z} \in \big[C([0,T]; H^\varepsilon(\partial\Omega)\big]^4 \times \big[C([0,T])\big]^2$ *such that* $z_1 \in C([0,T]; H^{1/2+\varepsilon}(\partial\Omega)\big]$. *Moreover, for all* $T \in (0, \overline{T})$, $\mathbf{z}$ *continuously depends on the data* $\mathbf{d}$ *with respect to the metrics of the spaces pointed out.*

Proof. First we estimate the functions $\mathbf{z}_0(\mathbf{d})$ and $\mathbf{w}(\mathbf{d})$ and their increments in $\mathcal{Z}(\varepsilon, 0, +\infty, T)$ and $\mathcal{Z}(\varepsilon, \rho, +\infty, T)$. From the first two formulae in (4.5) and from Lemmata 5.1, 5.2, 6.5–6.9 in Sections 5 and 6 we get

$$\|\mathbf{z}_0(\mathbf{d})\|_{\varepsilon,0} \le C(1+R_0)^3(1+m_1)(1+m_2)^2,$$

$$\|\mathbf{z}_0(\mathbf{d}^1) - \mathbf{z}_0(\mathbf{d}^2)\|_{\varepsilon,0} \le C(1+R_0)^4(1+m_1)^2(1+m_2)^4 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon},$$

$$\|\mathbf{w}(\mathbf{d})\|_{T,\varepsilon,\rho} \le C\Big[(\rho-\omega)^{-1} + \rho^{-\alpha}\Big](1+M)(1+R_0)^4(1+m_1)(1+m_2)^2,$$

$$\begin{aligned}&\|\mathbf{w}(\mathbf{d}^1) - \mathbf{w}(\mathbf{d}^2)\|_{T,\varepsilon,\rho}\\ &\le C\Big[(\rho-\omega)^{-1} + \rho^{-\alpha}\Big]\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}(1+M)(1+R_0)^4(1+m_1)^2(1+m_2)^4.\end{aligned}$$

Now we estimate the operator $\mathbf{N}(\mathbf{z}, \mathbf{d})$ and its increment in $\mathcal{Z}(\varepsilon, \rho, +\infty, T)$. From the third formula in (4.5) and from the lemmata in Sections 5, 6, 7, 8 we get

$$\begin{aligned}&\|\mathbf{N}(\mathbf{z}, \mathbf{d})\|_{T,\varepsilon,\rho}\\ &\le C\Big[(\rho-\omega)^{-1} + \rho^{-1}\Big](1+M)(R + \|\mathbf{z}_0\|_{T,\varepsilon,0})^2(1+R_0)^2(1+m_1)(1+m_2)\\ &+ C(\rho-\omega)^{-1}\big(1+\rho^{-1}\big)M(1 + R + \|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)^3(1+m_1)(1+m_2)^2\\ &+ C\rho^{-1}(1 + \|\mathbf{z}_0\|_{T,\varepsilon,0})(R + \|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)(1+m_1)\\ &+ CR^2T(1+R_0)(1+m_1)\\ &\le C\Big[(\rho-\omega)^{-1} + \rho^{-1}\Big](1+M)(1+R)^2(1+R_0)^8(1+m_1)^3(1+m_2)^5\end{aligned}$$

$$
\begin{aligned}
&+ C(\rho-\omega)^{-1}(1+\rho^{-1})(1+M)(1+R)(1+R_0)^6(1+m_1)^2(1+m_2)^4\\
&+ CTR^2(1+R_0)(1+m_1) =: \rho^{-1}K_{0,0}(\rho,R,R_0) + TR^2K_{0,1}(R_0),\\
&\|\mathbf{N}(\mathbf{z}^1,\mathbf{d}^1)-\mathbf{N}(\mathbf{z}^2,\mathbf{d}^2)\|_{T,\varepsilon,0} \le C(\rho-\omega)^{-1}\big(1+\rho^{-1}\big)(1+M)\\
&\times\Big[(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2(1+R_0)^3(1+m_1)^2(1+m_2)^3\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\\
&+(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)^2(1+m_1)(1+m_2)^3\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big]\\
&+C\big(RT+\rho^{-1}\|\mathbf{z}_0\|_{T,\varepsilon,0}\big)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\\
&+C\Big[R^2T+\rho^{-1}\|\mathbf{z}_0\|_{T,\varepsilon,0}(R+\|\mathbf{z}_0\|_{T,\varepsilon,\rho})\Big](1+R_0)(1+m_2)^2\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\\
&\le C(\rho-\omega)^{-1}\big(1+\rho^{-1}\big)(1+M)\\
&\times\Big[(1+R)^2(1+R_0)^9(1+m_1)^4(1+m_2)^7\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\\
&+(1+R)(1+R_0)^5(1+m_1)^2(1+m_2)^5\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big]\\
&+C\rho^{-1}\Big[(1+R_0)^3(1+m_1)(1+m_2)^2\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\\
&+(1+R_0)^7(1+m_1)^2(1+m_2)^4\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\Big]\\
&+CRT\Big[\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}+(1+R)(1+R_0)(1+m_2)^2\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\Big]\\
&\le\big[\rho^{-1}K_1(\rho,R,R_0)+CTR\big]\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}+K_2(\rho,R,R_0,T_0)\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}.
\end{aligned}
$$

From the previous formula we deduce that, for any fixed $\mathbf{d}\in\mathcal{D}(\varepsilon_0,\alpha,R_0,T_0)$, the map $\mathbf{z}\to\mathbf{z}_0+\mathbf{w}(\mathbf{d})+\mathbf{N}(\mathbf{z},\mathbf{d})$ is a contraction mapping in $\mathcal{Z}(\varepsilon,\rho,R,T)$ provided that the following inequalities are satisfied:

$$\rho^{-1}K_{0,0}(\rho,R,R_0)+TR^2K_{0,1}(R_0)\le R, \tag{4.14}$$

$$\rho^{-1}K_1(\rho,R,R_0)+CRT<1. \tag{4.15}$$

Taking the limit in (4.14) and (4.15) as $\rho\to+\infty$, (R_0,R,T) being kept fixed, we get $CTR^2K_0(R_0)\le R$, $CRT\le 1$. Choose now T such that

$$0<T<\min\big\{R^{-2}K_0(R_0)^{-1},C^{-1}R^{-1}\big\}=:\overline{T}. \tag{4.16}$$

Observe that, for any fixed triple (R,R_0,T), with $T\in(,\overline{T})$, $K_{0,1}(\rho,R,R_0)$ and $K_1(R_0)$ have finite limits as $\rho\to+\infty$. Therefore, if $T\in(0,\overline{T})$, we can choose ρ so large as to satisfy (4.14) and (4.15). Consequently, for any fixed $T\in(0,\overline{T})$, the system (4.4) admits a unique solution $\mathbf{z}=\mathbf{Z}(\mathbf{d})\in\mathcal{Z}(\varepsilon,\rho,R,T)$, for any large enough ρ satisfying (4.14) and (4.15).

Assume now that $\mathbf{z}^j = (z_1^j, z_2^j, z_3^j, z_4^j, z_5^j, z_6^j)$, $j = 1,2$, are two solutions of

$$\mathbf{z}(\mathbf{d}) = \mathbf{z}_0(\mathbf{d}) + \mathbf{w}(\mathbf{d}) + \mathbf{N}\big(\mathbf{z}(\mathbf{d}), \mathbf{d}\big),$$

where

$$z_k^j \in C([0,T]; H^\varepsilon(\partial\Omega)),\ k = 1,2,3,4, \quad z_k^j \in C([0,T]), \quad k = 5,6, \quad j = 1,2.$$

According to Lemma 5.5 in Section 5, for any fixed $T \in (0, \overline{T})$ we can choose a large enough $\rho > \max\{0, \omega\}$ such that $\max_{j=1,2} \|\mathbf{z}^j - \mathbf{z}_0\|_{T,\varepsilon,\rho} \le R$. We can now apply the previous part of the proof related to $\mathcal{Z}(\varepsilon, \rho, R, T)$ and conclude that $\mathbf{z}^1 = \mathbf{z}^2$ in $[0,T]$. We have thus proved the existence of a unique fixed point $\mathbf{z} \in \big[C([0,T]; H^\varepsilon(\partial\Omega)\big]^4 \times \big[C([0,T])\big]^2$ for system (4.4).

Since $u_0 \in C([0,T]; H^\varepsilon(\partial\Omega))$, from the first equation in (4.4) and Lemmata 6.5 and 7.1 in the following Sections 6 and 7 we easily deduce $z_1 \in C([0,T]; H^{(1/2)+\varepsilon}(\partial\Omega))$.

To prove the continuous dependence of $\mathbf{z}$ on the data, we note that, from the identities

$$\mathbf{Z}(\mathbf{d}^j) = \mathbf{z}_0(\mathbf{d}^j) + \mathbf{w}(\mathbf{d}^j) + \mathbf{N}\big(\mathbf{Z}(\mathbf{d}^j), \mathbf{d}^j\big), \quad j = 1,2,$$

we easily deduce the estimates

$$\begin{aligned} &\|\mathbf{Z}(\mathbf{d}^1) - \mathbf{Z}(\mathbf{d}^2)\|_{T,\varepsilon,\rho} \le \|\mathbf{z}_0(\mathbf{d}^1) - \mathbf{z}_0(\mathbf{d}^2)\|_{T,\varepsilon,\rho} + \|\mathbf{w}(\mathbf{d}^1) - \mathbf{w}(\mathbf{d}^2)\|_{T,\varepsilon,\rho} \\ &+ \Big\|\mathbf{N}\big(\mathbf{Z}(\mathbf{d}^1), \mathbf{d}^1\big) - \mathbf{N}\big(\mathbf{Z}(\mathbf{d}^2), \mathbf{d}^2\big)\Big\|_{T,\varepsilon,\rho} \\ &\le \big[K_2(\rho, R, R_0, T_0) + K_3(\rho, R, R_0)\big]\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} \\ &+ \big[K_1(\rho, R, R_0) + CRT\big]\|\mathbf{Z}(\mathbf{d}^1) - \mathbf{Z}(\mathbf{d}^2)\|_{T,\varepsilon,\rho}, \end{aligned} \tag{4.17}$$

where (ρ, T) is fixed so as to satisfy inequalities (4.14)–(4.16) and

$$\begin{aligned} K_3(\rho, R, R_0) = C\Big\{&(1+R_0)^4(1+m_1)^2(1+m_2)^4 + \big[(\rho-\omega)^{-1} + \rho^{-\alpha}\big] \\ &\times (1+M)(1+R_0)^4(1+m_1)^2(1+m_2)^4\Big\}. \end{aligned}$$

From (4.17) we deduce

$$\begin{aligned} \|\mathbf{Z}(\mathbf{d}^1) - \mathbf{Z}(\mathbf{d}^2)\|_{T,\varepsilon,\rho} &\le [1 - K_1(\rho, R, R_0, T_0) - CRT]^{-1} \\ &\times \big[K_2(\rho, R, R_0, T_0) + K_3(\rho, R, R_0)\big]\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}. \end{aligned}$$

This conclude the proof. □

Proof of Theorem 2.2. This is an immediate consequence of Theorems 2.1, 3.1, 3.2, 4.1 and 4.2.

5. Estimates for the function h_0 and operators F_j, $j = 1, 2, 3$.

From now on we will point out explicitly the dependence of the operators F_j, $j = 1, 2, 3$, on the vector datum $\mathbf{d}$ and on the vector unknown $\mathbf{z}$.

Lemma 5.1. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then*

$$|h_0^1 - h_0^2| \leq C(1 + R_0)(1 + m_2)^2 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \quad |h_0| \leq C(1 + R_0)(1 + m_2).$$

Proof. From (3.9) and the second inequality in (2.19) we get

$$\begin{aligned} |h_0^1 - h_0^2| &\leq C m_2^2 \Big[\big(|(l^1)'(0) - (l^2)'(0)| + \|u_1^1 - u_1^2\|_{\varepsilon,\partial\Omega} + \|f^1 - f^2\|_{T_0,0,\varepsilon,\Omega}\big) \\ &\quad \times \big(\|u_0^2\|_{\varepsilon,\partial\Omega} + \|f^2\|_{T_0,0,\varepsilon,\Omega}\big) + \big(|(l^2)'(0)| + \|u_1^2\|_{\varepsilon,\partial\Omega} + \|f^2\|_{T_0,0,\varepsilon,\Omega}\big) \\ &\quad \times \big(\|u_0^1 - u_0^2\|_{\varepsilon,\partial\Omega} + \|f^1 - f^2\|_{T_0,0,\varepsilon,\Omega}\big)\Big] \leq 3C m_2^2 R_0 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}. \end{aligned}$$

Lemma 5.2. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $t \in (0, T_0]$,

$$\|[F_1(\mathbf{d}^1) - F_1(\mathbf{d}^2)]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \leq C\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \tag{5.1}$$

$$\|[D_t F_1(\mathbf{d}^1) - D_t F_1(\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq C\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \tag{5.2}$$

$$\|[D_t F_1(\mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq C R_0, \quad \|[F_1(\mathbf{d})]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \leq C R_0. \tag{5.3}$$

Proof. From (3.1) we get

$$\begin{aligned} &\|[F_1(\mathbf{d}^1) - F_1(\mathbf{d}^2)]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \\ &\leq \|(g^1 - g^2)_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} + C\|(f^1 - f^2)_\rho\|_{t,0,\varepsilon,\Omega} \\ &\leq \|g^1 - g^2\|_{t,0,1/2+\varepsilon,\partial\Omega} + C\|f^1 - f^2\|_{t,0,\varepsilon,\Omega} \leq C\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \\ &\|[D_t F_1(\mathbf{d}^1) - D_t F_1(\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \\ &\leq \|(g^1 - g^2)_\rho\|_{t,1,\varepsilon,\partial\Omega} + C\|(f^1 - f^2)_\rho\|_{t,1,\varepsilon,\Omega} \leq C\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}. \end{aligned}$$

Lemma 5.3. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T_0]$,

$$\begin{aligned} &\|[F_2(\mathbf{z}^1, \mathbf{d}^1) - F_2(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \\ &\leq C(1 + R_0)(1 + m_2)^2 (R + \|\mathbf{z}_0\|_{T,\varepsilon,0}) \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} \\ &\quad + (1 + R + \|\mathbf{z}_0\|_{T,\varepsilon,0}) C(1 + R_0)(1 + m_2) \|\mathbf{z}^1 - \mathbf{z}^2\|_{T,\varepsilon,\rho} \\ &\quad + C\rho^{-1}\Big[(R + \|\mathbf{z}_0\|_{T,\varepsilon,0}) \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} + R_0 \|\mathbf{z}^1 - \mathbf{z}^2\|_{T,\varepsilon,\rho}\Big], \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\|[F_2(\mathbf{z}, \mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega} \\ &\leq C(1 + \rho^{-1})(1 + R + \|\mathbf{z}_0\|_{T,\varepsilon,0})(1 + R_0)^2 (1 + m_2) + \big(R + \|\mathbf{z}_0\|_{T,\varepsilon,0}\big)^2. \end{aligned} \tag{5.5}$$

Proof. From (3.1) and (3.9) we get

$$\begin{aligned}
&\|[F_2(\mathbf{z}^1,\mathbf{d}^1)-F_2(\mathbf{z}^2,\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega}\\
&\le |h_0^1|\|(z_2^1-z_2^2)_\rho\|_{t,0,\varepsilon,\partial\Omega}+|h_0^1-h_0^2|\|(z_2^2)_\rho\|_{t,0,\varepsilon,\partial\Omega}\\
&\quad + C\|f^1\|_{t,\varepsilon,\Omega}\int_0^t e^{-\rho(t-s)}\|(z_5^1-z_5^2)_\rho\|_{s,0}\,ds\\
&\quad + C\|f^1-f^2\|_{t,\varepsilon,\Omega}\int_0^t e^{-\rho(t-s)}\|(z_5^2)_\rho\|_{s,0}\,ds\\
&\quad + \int_0^t \|(z_2^1)_\rho\|_{s,0,\varepsilon,\partial\Omega}\|(z_6^1-z_6^2)_\rho\|_{t-s,0}\,ds\\
&\quad + \int_0^t \|(z_2^1-z_2^2)_\rho\|_{s,0,\varepsilon,\partial\Omega}\|(z_6^2)_\rho\|_{t-s,0}\,ds\\
&\quad + \int_0^t \|(z_4^1)_\rho\|_{s,0,\varepsilon,\partial\Omega}\|(z_5^1-z_5^2)_\rho\|_{t-s,0}\,ds\\
&\quad + \int_0^t \|(z_4^1-z_4^2)_\rho\|_{s,0,\varepsilon,\partial\Omega}\|(z_5^2)_\rho\|_{t-s,0}\,ds.
\end{aligned}$$

Inequalities (5.4) and (5.5) follow immediately from Lemma 5.1. □

Lemma 5.4. *Let* $\mathbf{d},\mathbf{d}^1,\mathbf{d}^2\in\mathcal{D}(\varepsilon,\alpha,R_0,T_0)$ *and* $\mathbf{z},\mathbf{z}^1,\mathbf{z}^2\in\mathcal{Z}(\varepsilon,\rho,R,T)$. *Then, for all* $t\in(0,T_0]$,

$$\begin{aligned}
&\|[F_3(\mathbf{z}^1,\mathbf{d}^1)-F_3(\mathbf{z}^2,\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega}\\
&\le C(1+R+|\!|\!|\mathbf{z}_0|\!|\!|_{T,\varepsilon})(1+R_0)^2(1+m_2)|\!|\!|\mathbf{z}^1-\mathbf{z}^2|\!|\!|_{T,\varepsilon,\rho}\\
&+C(1+R+|\!|\!|\mathbf{z}_0|\!|\!|_{T,\varepsilon,0})(1+R_0)^2(1+m_2^2)\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\\
&+C\rho^{-1}\Big[(R+|\!|\!|\mathbf{z}_0|\!|\!|_{T,\varepsilon,0})\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}+R_0|\!|\!|\mathbf{z}^1-\mathbf{z}^2|\!|\!|_{T,\varepsilon,\rho}\Big],
\end{aligned}\tag{5.6}$$

$$\|[F_3(\mathbf{z},\mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega}\le C(1+\rho^{-1})(1+R+|\!|\!|\mathbf{z}_0|\!|\!|_{T,\varepsilon,0})(1+R_0)^2(1+m_2).\tag{5.7}$$

Proof. From (3.20) we get

$$\begin{aligned}
&\|[F_3(\mathbf{z}^1,\mathbf{d}^1)-F_3(\mathbf{z}^2,\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega}\\
&\le |h_0^1|\|(z_4^1-z_4^2)_\rho\|_{t,0,\varepsilon,\partial\Omega}+|h_0^1-h_0^2|\|(z_4^2)_\rho\|_{t,0,\varepsilon,\partial\Omega}.\\
&\quad + C\|f^1\|_{t,\varepsilon,\Omega}\int_0^t e^{-\rho(t-s)}\|(z_6^1-z_6^2)_\rho\|_{t-s,0}\,ds\\
&\quad + C\|f^1-f^2\|_{t,\varepsilon,\Omega}\int_0^t e^{-\rho(t-s)}\|(z_6^2)_\rho\|_{s,0}\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \|(z_3^1)_\rho\|_{s,0,\varepsilon,\partial\Omega} \|(z_6^1 - z_6^2)_\rho\|_{t-s,0}\, ds \\
&+ \int_0^t \|(z_3^1 - z_3^2)_\rho\|_{s,0,\varepsilon,\partial\Omega} \|(z_6^2)_\rho\|_{t-s,0}\, ds \\
&+ (\|u_1^1\|_{\varepsilon,\partial\Omega} + |h_0^1| \|u_0^1\|_{\varepsilon,\partial\Omega}) \|(z_6^1 - z_6^2)_\rho\|_{t,0} \\
&+ (\|u_1^1 - u_1^2\|_{\varepsilon,\partial\Omega} + |h_0^1| \|u_0^1 - u_0^2\|_{\varepsilon,\partial\Omega} + |h_0^1 - h_0^2| \|u_0^2\|_{\varepsilon,\partial\Omega}) \|(z_6^2)_\rho\|_{t,0} \\
&+ \int_0^t \|(z_4^1)_\rho\|_{s,0,\varepsilon,\partial\Omega} \|(z_6^1 - z_6^2)_\rho\|_{t-s,0}\, ds \\
&+ \int_0^t \|(z_4^1 - z_4^2)_\rho\|_{s,0,\varepsilon,\partial\Omega} \|(z_6^2)_\rho\|_{t-s,0}\, ds \\
&+ |h_0^1 - h_0^2| \|(z_3^1)_\rho\|_{t,0,\varepsilon,\partial\Omega} + |h_0^2| \|(z_3^1 - z_3^2)_\rho\|_{t,0,\varepsilon,\partial\Omega}. \qquad (5.8)
\end{aligned}
$$

Estimate (5.6) follows immediately from (5.8) and Lemma 5.1, while (5.7) can be obtained likewise. □

Lemma 5.5. *Let Y be a Banach space and f in $C([0,T];Y)$ such that $f(0) = 0$. Then* $\sup_{t\in[0,T]} e^{-\rho t}\|f\|_Y \to 0$, as $\rho \to +\infty$.

Proof. For any $\delta > 0$, there exists $0 < \eta < T$ such that $\|f(t)\|_Y \le \delta$, $0 \le t \le \eta$. Then

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\rho t}\|f\|_Y = \sup_{t\in[0,\eta]} e^{-\rho t}\|f\|_Y + \sup_{t\in[\eta,T]} e^{-\rho t}\|f\|_Y \\
&\le \|f\|_{C[0,T];Y)}\big(\delta \sup_{t\in[0,\eta]} e^{-\rho t} + \sup_{t\in[\eta,T]} e^{-\rho t}\big) \le \|f\|_{C[0,T];Y)}\big(\delta + e^{-\rho\eta}\big).
\end{aligned}
$$

Hence, for any $\delta > 0$, we get

$$
\limsup_{\rho\to+\infty} \sup_{t\in[0,T]} e^{-\rho t}\|f\|_Y \le \delta \|f\|_{C[0,T];Y)}.
$$

This implies the assertion. □

6. Estimating the functions $\widetilde{w}_3, \widetilde{w}_5, \chi, \widetilde{h}, w_1, \dots, w_6$.

This section is devoted to estimating $\widetilde{w}_3, \widetilde{w}_5, w_1, \dots, w_6$ and their increments in terms of $\mathbf{d}, h_0, \chi^{-1}$ and of their increments in suitable spaces. Since the proofs are similar to the ones in Section 5, the lemmata will be stated with no proofs. The interested reader can find them on the site http://www.mat.unimi.it/∼lorenzi.

Lemma 6.1. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon_0, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$\|[\widetilde{w}_3(\mathbf{d}^1) - \widetilde{w}_3(\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+M)(1+R_0)^2(1+m_2)^2 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \tag{6.1}$$

$$\|[\widetilde{w}_3(\mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+M)(1+R_0)^2(1+m_2). \tag{6.2}$$

Lemma 6.2. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$\|[\widetilde{w}_5(\mathbf{d}^1) - \widetilde{w}_5(\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+M)(1+R_0)^3(1+m_2)^3 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \tag{6.3}$$

$$\|[\widetilde{w}_5(\mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+M)(1+R_0)^3(1+m_2)^2. \tag{6.4}$$

Lemma 6.3. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$|[\chi(\mathbf{d}^1)]^{-1} - [\chi(\mathbf{d}^2)]^{-1}| \leq C m_1^2 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \quad |[\chi(\mathbf{d})]^{-1}| \leq m_1.$$

Lemma 6.4. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$\|[\widetilde{h}(\mathbf{d}^1) - \widetilde{h}(\mathbf{d}^2)]_\rho\|_{t,0} \leq C\rho^{-\alpha}(1+R_0)^2(1+m_2)^2 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} + C(\rho-\omega)^{-1}(1+M)(1+R_0)^2(1+m_2)^2 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \tag{6.5}$$

$$\|[\widetilde{h}(\mathbf{d})]_\rho\|_{t,0} \leq C\rho^{-\alpha}(1+R_0)^2(1+m_2) + C(\rho-\omega)^{-1}(1+M)(1+R_0)^2(1+m_2). \tag{6.6}$$

Lemma 6.5. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$\|[w_1(\mathbf{d}^1) - w_1(\mathbf{d}^2)]_\rho\|_{t,0,1+\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+R_0)^2(1+m_2)^2 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \tag{6.7}$$

$$\|[w_1(\mathbf{d})]_\rho\|_{t,0,1+\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+R_0)^2(1+m_2). \tag{6.8}$$

Lemma 6.6. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$\|[w_2(\mathbf{d}^1) - w_2(\mathbf{d}^2)]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+R_0)^2(1+m_2)^2 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \tag{6.9}$$

$$\|[w_2(\mathbf{d})]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+R_0)^2(1+m_2). \tag{6.10}$$

Lemma 6.7. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$\begin{aligned} &\|[w_3(\mathbf{d}^1) - w_3(\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \le C\rho^{-\alpha}(1+R_0)^3(1+m_2)^2 \big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} \\ &\quad + C(\rho-\omega)^{-1}(1+M)(1+R_0)^4(1+m_1)^2(1+m_2)^3\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \quad (6.11) \\ &\|[w_3(\mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega} \le C\rho^{-\alpha}(1+R_0)^3(1+m_2) \\ &\quad + C(\rho-\omega)^{-1}(1+M)(1+R_0)^4(1+m_1)(1+m_2)^2. \quad (6.12) \end{aligned}$$

Lemma 6.8. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$\begin{aligned} &\|[w_4(\mathbf{d}^1) - w_4(\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \\ &\le C(\rho-\omega)^{-1}(1+M)(1+R_0)^3(1+m_2)^3\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \quad (6.13) \\ &\|[w_4(\mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega} \le C(\rho-\omega)^{-1}(1+M)(1+R_0)^3(1+m_2)^2. \quad (6.14) \end{aligned}$$

Lemma 6.9. *Let* $\mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$. *Then, for all* $\varepsilon \in (1/2, 3/2) \setminus \{1\}$, $\beta \in \mathbb{R}_+$ *and* $t \in (0, T_0]$,

$$\begin{aligned} &\|[w_6(\mathbf{d}^1) - w_6(\mathbf{d}^2)]_\rho\|_{T,0} \le C\rho^{-\alpha}(1+R_0)^2(1+m_2)^2\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} \\ &\quad + C(\rho-\omega)^{-1}(1+M)(1+R_0)^3(1+m_1)^2(1+m_2)^3\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \quad (6.15) \\ &\|[w_6(\mathbf{d})]_\rho\|_{T,0} \le C\rho^{-\alpha}(1+R_0)^2(1+m_2) \\ &\quad + C(\rho-\omega)^{-1}(1+M)(1+R_0)^3(1+m_1)(1+m_2)^2. \quad (6.16) \end{aligned}$$

7. Estimates for operators $\widetilde{N}_j$, $j = 1, \ldots, 5$.

Since the proofs are similar to the ones in Section 5, the lemmata will be stated with no proofs. The interested reader can find them on the site http://www.mat.unimi.it/∼lorenzi.

Lemma 7.1. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T]$,

$$\begin{aligned} &\|[\widetilde{N}_1(\mathbf{z}^1, \mathbf{d}^1) - \widetilde{N}_1(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \\ &\le CM(\rho-\omega)^{-1}\Big[(1+\rho^{-1})(1+R_0)(1+m_2)^2(R + \|\mathbf{z}_0\|_{T,\varepsilon,0})\big|\mathbf{d}^1 - \mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} \\ &\quad + (1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)(1+m_2)\|\mathbf{z}^1 - \mathbf{z}^2\|_{T,\varepsilon,\rho}\Big], \quad (7.1) \\ &\|[\widetilde{N}_1(\mathbf{z}, \mathbf{d})]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \\ &\le CM(\rho-\omega)^{-1}(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2(1+R_0)^2(1+m_2). \quad (7.2) \end{aligned}$$

Lemma 7.2. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T]$,

$$\begin{aligned}
&\|[\widetilde{N}_2(\mathbf{z}^1, \mathbf{d}^1) - \widetilde{N}_2(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \\
&\le CM(\rho-\omega)^{-1}\Big[(1+\rho^{-1})(1+R_0)(1+m_2)^2(R+\|\mathbf{z}_0\|_{T,\varepsilon,0})\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} \\
&\quad + (1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)(1+m_2)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big], \qquad (7.3)
\end{aligned}$$

$$\begin{aligned}
&\|[\widetilde{N}_2(\mathbf{z}, \mathbf{d})]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega} \\
&\le CM(\rho-\omega)^{-1}(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2(1+R_0)^2(1+m_2). \qquad (7.4)
\end{aligned}$$

Lemma 7.3. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T]$,

$$\begin{aligned}
&\|[\widetilde{N}_3(\mathbf{z}^1, \mathbf{d}^1) - \widetilde{N}_3(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \le CM(\rho-\omega)^{-1} \\
&\quad \times \Big[(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)^2(1+m_2)^2\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} \\
&\quad + (1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon})(1+R_0)^2(1+m_2)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big], \qquad (7.5)
\end{aligned}$$

$$\begin{aligned}
&\|[\widetilde{N}_3(\mathbf{z}, \mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega} \\
&\le CM(\rho-\omega)^{-1}(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)^2(1+m_2). \qquad (7.6)
\end{aligned}$$

Lemma 7.4. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T]$,

$$\begin{aligned}
&\|[\widetilde{N}_4(\mathbf{z}^1, \mathbf{d}^1) - \widetilde{N}_4(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0} \\
&\le Cm_1^2\Big[(\rho-\omega)^{-1}(R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)(1+m_2)^2 \\
&+R^2T+\rho^{-1}(1+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)\Big]\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} \\
&+Cm_1\Big[\rho^{-1}+(\rho-\omega)^{-1}\Big](1+R_0)(1+m_2)(R+\|\mathbf{z}_0\|_{T,\varepsilon,0})\|\mathbf{z}^1-\mathbf{z}^2\|_{\rho,\varepsilon}, \quad (7.7)
\end{aligned}$$

$$\begin{aligned}
\|[\widetilde{N}_4(\mathbf{z}, \mathbf{d})]_\rho\|_{t,0} \le Cm_1&\Big[(\rho-\omega)^{-1}(1+R_0)(1+m_2)(R+\|\mathbf{z}_0\|_{T,\varepsilon,0}) \\
&+R^2T+\rho^{-1}(1+\|\mathbf{z}_0\|_{T,\varepsilon,0})(R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)\Big]. \quad (7.8)
\end{aligned}$$

Lemma 7.5. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T]$,

$$\|[\widetilde{N}_5(\mathbf{z}^1, \mathbf{d}^1) - \widetilde{N}_5(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \le CM(\rho-\omega)^{-1}$$

$$\times\Big[(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)^2(1+m_2)^2|\mathbf{d}^1-\mathbf{d}^2|_{T_0,\alpha,\varepsilon}$$
$$+(1+R+\|\mathbf{z}_0\|_{T,\varepsilon})(1+R_0)^2(1+m_2)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big]$$
$$+C\rho^{-1}(R+\|\mathbf{z}_0\|_{T,\varepsilon,0})|\mathbf{d}^1-\mathbf{d}^2|_{T_0,\alpha,\varepsilon}$$
$$+C(1+R_0)\big(RT+\rho^{-1}\|\mathbf{z}_0\|_{T,\varepsilon,0}\big)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}, \tag{7.9}$$

$$\|[\widetilde{N}_5(\mathbf{z},\mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega}$$
$$\le M(\rho-\omega)^{-1}(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)^2(1+m_2)$$
$$+C\rho^{-1}(1+R_0)(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2+CR^2T. \tag{7.10}$$

8. Estimates for the operators $N_j,\ j=1,\dots,6$.

This section is devoted to estimating the increments of $N_1,\dots,N_6$ in terms of $\mathbf{d},\widetilde{N}_1,\widetilde{N}_2,\widetilde{N}_3,\widetilde{N}_4,\widetilde{N}_5,h_0,\chi^{-1},F_2,F_3$ and of their increments in suitable spaces.

Since the proofs are similar to the ones in Section 5, the lemmata will be stated with no proofs. The interested reader can find them on the site http://www.mat.unimi.it/∼lorenzi.

Lemma 8.1. *Let* $\mathbf{d},\mathbf{d}^1,\mathbf{d}^2\in\mathcal{D}(\varepsilon,\alpha,R_0,T_0)$ *and* $\mathbf{z},\mathbf{z}^1,\mathbf{z}^2\in\mathcal{Z}(\varepsilon,\rho,R,T)$. *Then, for all* $t\in(0,T]$,

$$\|[N_1(\mathbf{z}^1,\mathbf{d}^1)-N_1(\mathbf{z}^2,\mathbf{d}^2)]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega}$$
$$\le C(\rho-\omega)^{-1}M\Big[(1+\rho^{-1})(1+R_0)(1+m_2)^2(R+\|\mathbf{z}_0\|_{T,\varepsilon,0})|\mathbf{d}^1-\mathbf{d}^2|_{T_0,\alpha,\varepsilon}$$
$$+(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)(1+m_2)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big], \tag{8.1}$$

$$\|[N_1(\mathbf{z},\mathbf{d})]_\rho\|_{t,0,1/2+\varepsilon,\partial\Omega}$$
$$\le C(\rho-\omega)^{-1}M(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2(1+R_0)^2(1+m_2). \tag{8.2}$$

Lemma 8.2. *Let* $\mathbf{d},\mathbf{d}^1,\mathbf{d}^2\in\mathcal{D}(\varepsilon,\alpha,R_0,T_0)$ *and* $\mathbf{z},\mathbf{z}^1,\mathbf{z}^2\in\mathcal{Z}(\varepsilon,\rho,R,T)$. *Then, for all* $t\in(0,T]$,

$$\|[N_2(\mathbf{z}^1,\mathbf{d}^1)-N_2(\mathbf{z}^2,\mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega}$$
$$\le C(\rho-\omega)^{-1}M\Big[(1+\rho-1)(1+R_0)(1+m_2)^2(R+\|\mathbf{z}_0\|_{T,\varepsilon,0})|\mathbf{d}^1-\mathbf{d}^2|_{T_0,\alpha,\varepsilon}$$
$$+(1+\rho-1)(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)(1+m_2)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big], \tag{8.3}$$

$$\|[N_2(\mathbf{z},\mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega}$$
$$\le C(\rho-\omega)^{-1}M(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2(1+R_0)^2(1+m_2). \tag{8.4}$$

Lemma 8.3. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T]$,

$$\begin{aligned}
&\|[N_3(\mathbf{z}^1, \mathbf{d}^1) - N_3(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+M)\Big\{(1+\rho^{-1})\\
&\quad \times (1+R_0)^3(1+m_1)^2(1+m_2)^3(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\\
&\quad + (1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)^2(1+m_1)(1+m_2)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big\}\\
&\quad + C\rho^{-1}(1+R_0)^2(1+m_1)(1+m_2)\Big[(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\\
&\quad + (R+\|\mathbf{z}_0\|_{T,\varepsilon,0})\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big]\\
&\quad + C(1+R_0)(1+m_1)^2R^2T\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}, \qquad (8.5)\\
&\|[N_3(\mathbf{z}, \mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq C(\rho-\omega)^{-1}(1+M)(1+R_0)^3(1+m_1)(1+m_2)^2\\
&\quad \times \Big[(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2\Big]\\
&\quad + C\rho^{-1}(1+R_0)^2m_1(1+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0}). \qquad (8.6)
\end{aligned}$$

Lemma 8.4. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T]$,

$$\begin{aligned}
&\|[N_4(\mathbf{z}^1, \mathbf{d}^1) - N_4(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0,\varepsilon,\partial\Omega} \leq CM(\rho-\omega)^{-1}\\
&\quad \times \Big[(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2(1+R_0)^3(1+m_2)^3\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\\
&\quad + (1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon})(1+R_0)^2(1+m_2)^2\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho}\Big]\\
&\quad + C\rho^{-1}(R+\|\mathbf{z}_0\|_{T,\varepsilon,0})\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}\\
&\quad + C(1+R_0)\big(RT+\rho^{-1}\|\mathbf{z}_0\|_{T,\varepsilon,0}\big)\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho},\\
&\quad \|[N_4(\mathbf{z}, \mathbf{d})]_\rho\|_{t,0,\varepsilon,\partial\Omega}\\
&\leq M(\rho-\omega)^{-1}(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2(1+R_0)^3(1+m_1)(1+m_2)\\
&\quad + C\rho^{-1}(1+R_0)(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2 + CR^2T.
\end{aligned}$$

Lemma 8.5. *Let* $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\varepsilon, \alpha, R_0, T_0)$ *and* $\mathbf{z}, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{Z}(\varepsilon, \rho, R, T)$. *Then, for all* $t \in (0, T]$,

$$\begin{aligned}
&\|[N_6(\mathbf{z}^1, \mathbf{d}^1) - N_6(\mathbf{z}^2, \mathbf{d}^2)]_\rho\|_{t,0} \leq C(\rho-\omega)^{-1}M(1+\rho^{-1})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})^2\\
&\quad \times (1+R_0)^3(1+m_1)^2(1+m_2)^3\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon} + C(\rho-\omega)^{-1}M(1+\rho^{-1})\\
&\quad \times (1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)^2(1+m_1)(1+m_2)^2\|\mathbf{z}^1-\mathbf{z}^2\|_{T,\varepsilon,\rho} + Cm_1^2\\
&\quad \times \Big[R^2T+\rho^{-1}(1+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R+\|\mathbf{z}_0\|_{T,\varepsilon,0})(1+R_0)\Big]\big|\mathbf{d}^1-\mathbf{d}^2\big|_{T_0,\alpha,\varepsilon}
\end{aligned}$$

$$+ Cm_1\rho^{-1}(1+R_0)(1+m_2)(R+|||\mathbf{z}_0|||_{T,\varepsilon,0})|||\mathbf{z}^1-\mathbf{z}^2|||_{\rho,\varepsilon}, \tag{8.7}$$

$$|||[N_6(\mathbf{z},\mathbf{d})]_\rho|||_{t,0}$$
$$\le C(\rho-\omega)^{-1}M(1+\rho^{-1})(1+R+|||\mathbf{z}_0|||_{T,\varepsilon,0})^2(1+R_0)^3(1+m_1)(1+m_2)^2$$
$$+ Cm_1\Big[R^2T+\rho^{-1}(1+|||\mathbf{z}_0|||_{T,\varepsilon,0})(R+|||\mathbf{z}_0|||_{T,\varepsilon,0})(1+R_0)\Big]. \tag{8.8}$$

9. Appendix

First we show that $\mathcal{B}_\beta$ generates a C_0-semigroup in $H^\beta(\partial\Omega)\times H^{\beta-1/2}(\partial\Omega)$. To show this property we need to generalize the result on page 57 in [9] where the author deals with the simpler case related to the pair $(-\Delta, D_\nu)$. In the author's notation, in our case we have

$$\mathcal{A}=A,\quad \mathcal{B}=bD_{\nu_A},\quad \mathcal{B}_0u=u \text{ on } \partial\Omega,\quad \mathcal{R}_0=(\mathcal{A},\mathcal{B}_0)^{-1}\equiv L_0.$$

We can follow the same procedure in the quoted paper, provided we make the following change, where $g,h\in H^{1/2}(\partial\Omega)$:

$$|\langle bD_{\nu_A}g,h\rangle_{\partial\Omega}| = \Big|\int_\Omega AL_0g(bL_0h)\,dx\Big|$$
$$= \Big|\int_\Omega \Big[\sum_{j,k=1}^n a_{j,k}(D_{x_k}L_0g)D_{x_j}(bL_0h)+a_0b(L_0g)(L_0h)\Big]\,dx\Big|$$
$$\le \kappa\|\nabla L_0g\|_{0,\Omega}\|\nabla(bL_0h)\|_{0,\Omega}+\|a_0\|_{C(\overline{\Omega})}\|b\|_{C(\overline{\Omega})}\|L_0g\|_{0,\Omega}\|L_0h\|_{0,\Omega}$$
$$\le \|L_0g\|_{1,\Omega}\|L_0h\|_{1,\Omega}\|b\|_{C^1(\overline{\Omega})}\big(2\kappa^2+\|a_0\|^2_{C(\overline{\Omega})}\big)^{1/2}$$
$$\le C(\kappa,\|a_0\|_{C(\overline{\Omega})},\|b\|_{C^1(\overline{\Omega})})\|g\|_{1/2,\partial\Omega}\|h\|_{1/2,\partial\Omega}.$$

This implies that $bD_{\nu_A}\in\mathcal{L}(H^{1/2}(\partial\Omega);H^{-1/2}(\partial\Omega))$.

Finally, for any positive μ and $g\in H^{1/2}(\partial\Omega)$, taking assumption (2.1) into account, we have

$$(\mu\mathcal{B}\mathcal{R}_0g,g)_{\partial\Omega} = \int_\Omega \Big[\sum_{j,k=1}^n a_{j,k}(D_{x_k}L_0g)D_{x_j}(bL_0g)+a_0b(L_0g)^2\Big]\,dx$$
$$+\mu\|g\|^2_{0,\partial\Omega} = \int_\Omega b\sum_{j,k=1}^n a_{j,k}(D_{x_k}L_0g)D_{x_j}(L_0g)\,dx$$
$$+\int_\Omega L_0g\sum_{j,k=1}^n [a_{j,k}(D_{x_k}L_0g)D_{x_j}b+a_0b(L_0g)^2]\,dx+\mu\|g\|^2_{0,\partial\Omega}$$

$$= \int_\Omega b \sum_{j,k=1}^n a_{j,k}(D_{x_k} L_0 g) D_{x_j}(L_0 g)\, dx$$
$$+ \int_\Omega \Big[a_0 b - \sum_{j,k=1}^n D_{x_k}[a_{j,k} D_{x_j} b]\Big](L_0 g)^2\, dx$$
$$+ \int_{\partial\Omega} g^2 D_{\nu_A} b\, d\sigma + \mu\|g\|^2_{0,\partial\Omega} \geq \mu\|g\|^2_{0,\partial\Omega}.$$

From now on we can follow the proof in [9] showing that actually $\mathcal{B}_\beta$ generates a C_0-semigroup in $H^\beta(\partial\Omega) \times H^{\beta-1/2}(\partial\Omega))$ that we have denoted by

$$\mathcal{S}(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix}.$$

According to the results in [9] $\{\mathcal{S}(t)\}_{t\geq 0}$ turns out to be also a semigroup of linear bounded operators from $H^{\beta+1/2}(\partial\Omega) \times H^\beta(\partial\Omega)$ into itself, for all $\beta \geq 0$. Now we recall that, for $(x, y) \in H^{\beta+1/2}(\partial\Omega) \times H^\beta(\partial\Omega)$, $\beta \geq 0$, $w \geq 0$ (cf. [9]).

$$\|S_{11}(t)x + S_{12}(t)y\|_{H^{\beta+1/2}(\partial\Omega)} \leq Me^{wt}(\|x\|_{H^{\beta+1/2}(\partial\Omega)} + \|y\|_{H^\beta(\partial\Omega)}),$$
$$\|S_{21}(t)x + S_{22}(t)y\|_{H^\beta(\partial\Omega)} \leq Me^{wt}(\|x\|_{H^{\beta+1/2}(\partial\Omega)} + \|y\|_{H^\beta(\partial\Omega)});$$

as a consequence, setting $y = 0$ or $x = 0$, we get the formulae

$$\|S_{11}(t)x\|_{H^{\beta+1/2}(\partial\Omega)} \leq Me^{wt}\|x\|_{H^{\beta+1/2}(\partial\Omega)}, \quad x \in H^{\beta+1/2}(\partial\Omega), \tag{9.1}$$
$$\|S_{12}(t)y\|_{H^{\beta+1/2}(\partial\Omega)} \leq Me^{wt}\|y\|_{H^\beta(\partial\Omega)}, \quad y \in H^\beta(\partial\Omega), \tag{9.2}$$
$$\|S_{21}(t)x\|_{H^\beta(\partial\Omega)} \leq Me^{wt}\|x\|_{H^{\beta+1/2}(\partial\Omega)}, \quad x \in H^{\beta+1/2}(\partial\Omega), \tag{9.3}$$
$$\|S_{22}(t)y\|_{H^\beta(\partial\Omega)} \leq Me^{wt}\|y\|_{H^\beta(\partial\Omega)}, \quad y \in H^\beta(\partial\Omega). \tag{9.4}$$

Now we note that, for $(x, y) \in H^{\beta+1}(\partial\Omega) \times H^{\beta+1/2}(\partial\Omega)$, $\beta \geq 0$,

$$\begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_{12}(t)B & S_{11}(t) \\ S_{22}(t)B & S_{21}(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix}\begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_{21}(t) & S_{22}(t) \\ BS_{11}(t) & BS_{12}(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Consequently, for $\beta \geq 0$

$$S_{21}(t)x = S_{12}(t)Bx, \quad x \in H^{\beta+1}(\partial\Omega), \tag{9.5}$$
$$S_{22}(t)y = S_{11}(t)y, \quad y \in H^{\beta+1/2}(\partial\Omega), \tag{9.6}$$
$$S_{22}(t)Bx = BS_{11}(t)x, \quad x \in H^{\beta+1}(\partial\Omega), \tag{9.7}$$

$$S_{21}(t)y = BS_{12}(t)y, \quad y \in H^{\beta+1/2}(\partial\Omega). \tag{9.8}$$

In particular, for all $y \in H^{\beta}(\partial\Omega)$ we have

$$\begin{aligned}\|BS_{12}(t)y\|_{H^{\beta}(\partial\Omega)} &\le \|B\|_{\mathcal{L}(H^{\beta}(\partial\Omega);H^{\beta+1}(\partial\Omega))}\|S_{12}(t)y\|_{\beta+1,\partial\Omega} \\ &\le M\|B\|_{\mathcal{L}(H^{\beta}(\partial\Omega);H^{\beta+1}(\partial\Omega))}e^{wt}\|y\|_{\beta+1/2,\partial\Omega}, \quad y \in H^{\beta+1/2}(\partial\Omega). \end{aligned} \tag{9.9}$$

When the domain of $\mathcal{S}(t)$ is restricted to $H^{\beta+1}(\partial\Omega) \times H^{\beta+1/2}(\partial\Omega)$, $\beta \ge 0$, the following formulae hold:

$$\mathcal{S}(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{12}(t)B & S_{11}(t) \end{pmatrix}, \qquad \mathcal{B}\mathcal{S}(t) = \begin{pmatrix} S_{12}(t)B & S_{11}(t) \\ BS_{11}(t) & BS_{12}(t) \end{pmatrix}.$$

Finally, from $\mathcal{S}(0) = I$, we deduce $S_{11}(0) = I$, $S_{12}(0) = O$. Moreover we note that, for $(x, y) \in H^{\beta+1}(\partial\Omega) \times H^{\beta+1/2}(\partial\Omega)$, $\beta \ge 0$,

$$\begin{aligned} D_t\Big[\mathcal{S}(t)\begin{pmatrix} x \\ y \end{pmatrix}\Big] &= D_t\begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \mathcal{B}\mathcal{S}(t)\begin{pmatrix} x \\ y \end{pmatrix} \in H^{\beta+1/2}(\partial\Omega) \times H^{\beta}(\partial\Omega). \end{aligned}$$

As a consequence, for $(x, y) \in H^{\beta+1}(\partial\Omega) \times H^{\beta+1/2}(\partial\Omega)$, $\beta \ge 0$, we get

$$D_t[S_{11}(t)x] + D_t[S_{12}(t)y] = S_{12}(t)Bx + S_{11}(t)y, \tag{9.10}$$

$$D_t[S_{21}(t)x] + D_t[S_{22}(t)y] = BS_{11}(t)x + BS_{12}(t)y. \tag{9.11}$$

Setting $y = 0$ or $x = 0$, we get the formulae

$$D_t[S_{11}(t)x] = S_{12}(t)Bx, \quad D_t[S_{21}(t)x] = BS_{11}(t)x, \quad x \in H^{\beta+1}(\partial\Omega), \tag{9.12}$$

$$D_t[S_{12}(t)y] = S_{11}(t)y, \quad y \in H^{\beta+1/2}(\partial\Omega), \tag{9.13}$$

$$D_t[S_{22}(t)y] = BS_{12}(t)y, \quad y \in H^{\beta+1/2}(\partial\Omega). \tag{9.14}$$

Observe now that, from formulae (9.5), (9.12) and (9.13), for $x \in H^{\beta+1}(\partial\Omega)$,

$$BS_{11}(t)x = D_t[S_{21}(t)x] = D_t[S_{12}(t)Bx] = S_{11}(t)Bx.$$

Moreover, from formula (9.6), we observe that

$$D_t[S_{22}(t)y] = D_t[S_{11}(t)y], \quad y \in H^{\beta+1/2}(\partial\Omega),$$

then, from formulae (9.12)–(9.14), we deduce

$$BS_{12}(t)y = S_{12}(t)By, \quad y \in H^{\beta+1/2}(\partial\Omega).$$

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