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ON QUASILINEAR BREZIS-NIRENBERG TYPE PROBLEMS WITH WEIGHTS

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Abstract. In this paper we study Brezis-Nirenberg type results for radial solutions of a quasilinear elliptic equation of the form

$$\begin{cases} -\Delta_p u = \lambda C(|x|)|u|^{p-2}u + B(|x|)|u|^{q-2}u, \ \textit{a.e.}\ x \in B_R(0) \subset \mathbb{R}^N, \ R > 0, \\ u = 0, \quad \text{on } \partial B_R(0), \end{cases}$$

where $\lambda \in \mathbb{R}$, $q \geq p > 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $B_R(0)$ denotes the ball of radius $R > 0$ centered at 0 in $\mathbb{R}^N$, and the weights B, C are appropriate positive measurable radially symmetric functions.

1. Introduction

In this paper we study Brezis-Nirenberg type results for quasilinear elliptic equations of the form

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u'(r))' \\ \qquad = \lambda r^{N-1}C(r)|u|^{p-2}u + r^{N-1}B(r)|u|^{q-2}u, \ r \in (0,R), \\ \lim_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0, \quad u(R) = 0, \end{cases} \tag{1.1}$$

where $R > 0$, $\lambda \in \mathbb{R}$, $q \geq p > 1$, $N \geq p$, the weights B, C are positive measurable functions defined on $(0,R)$ such that $r^{N-1}B,\ r^{N-1}C \in L^1(0,R)$ and $' = \frac{d}{dr}$.

By a solution of (1.1) we understand a function u in $AC(0,R]$ with

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$r^{(N-1)/p}u' \in L^p(0,R)$ (the usual Lebesgue space), and $r^{N-1}|u'|^{p-1} \in AC(0,R)$, which satisfies the equation in (1.1) almost everywhere and the boundary conditions at each endpoint.

The origin of this problem goes back to 1983, when Brezis and Nirenberg [4] observed that lower-order perturbations to elliptic equations involving critical exponents recovered the lost compactness. More precisely, they proved that the equation

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } B_1(0) \subset \mathbb{R}^N \\ \quad u = 0 & \text{on } \partial B_1(0) \end{cases} \tag{1.2}$$

has a positive solution if $0 < \lambda < \lambda_1$ and $N \geq 4$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(B_1(0))$. Surprisingly, for the case $N = 3$, called the *critical dimension*, they observed that (1.2) has a solution if and only if $\lambda \in (\lambda_1/4, \lambda_1)$.

In this article we will extend the concept of critical dimension to our problem; that is, we will look for the values of N (in terms of p and the weight C), so that (1.1) does not have a positive solution for any sufficiently small positive λ, and has a positive solution if $\lambda \in (\lambda^*, \lambda_1)$ for some $\lambda^* > 0$, where λ_1 is the first eigenvalue of

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u'(r))' = \lambda r^{N-1}C(r)|u|^{p-2}u \\ \lim\limits_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0, \quad u(R) = 0. \end{cases}$$

The existence of such an eigenvalue is established in section 3 under suitable assumptions on C. Roughly speaking, we will show the existence of a critical exponent ρ_b^* which plays the role of 2^* for problem (1.1), and give conditions under which, if the following critical dimension inequality (for $N > p$)

$$\frac{(N-p)p}{p-1} < \liminf_{r\to 0} \frac{|\log(\int_0^r s^{N-1}C(s)ds)|}{|\log(r)|}$$

holds, then the problem

$$\begin{cases} -\Delta_p u = \lambda C(|x|)|u|^{p-2}u + B(|x|)|u|^{\rho_b^*-2}u, & x \in B_R(0) \subset \mathbb{R}^N, \ R > 0, \\ u = 0, \quad \text{on } \partial B_R(0), \end{cases}$$

where $B_R(0)$ is the ball of radius $R > 0$ centered at 0 in $\mathbb{R}^N$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, has no positive radially symmetric solutions for $\lambda > 0$ sufficiently small, and does have a positive solution for λ in some left neighborhood of λ_1. Note that, if $C = 1$ and $p = 2$, then the above critical dimension inequality reads exactly $N < 4$. These results are contained in Theorem 5.2 and Theorem 5.3 in section 5.

An extension of the Brézis-Nirenberg result to the p-Laplacian operator was obtained by Egnell, see [8, 9, 10]. Egnell, see [10], studied the critical dimension problem for the more general elliptic operator

$$\begin{cases} -\operatorname{div}(|x|^{\alpha}|\nabla u|^{p-2}\nabla u) = \lambda |x|^{\beta}|u|^{p-2}u + |x|^{\nu}|u|^{p^*(\nu)-2}u \quad \text{in } \Omega \subseteq \mathbb{R}^N \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega = B_1(0)$ in $\mathbb{R}^N$, $N \geq p$, $\nu + p > \alpha$, $N + \alpha > p$, $\beta + p > \alpha$, and $p^*(\nu) = \frac{p(N+\nu)}{N+\alpha-p}$. Among other things, he proved the following.

Theorem EG 1. (Theorem 4 of [10]) *Let* $N - p^2 - \beta(p-1) + \alpha p \geq 0$. *Then problem* (1.3) *has a radial solution if and only if* $0 < \lambda < \lambda_1$. *Furthermore, if* $N - p^2 - \beta(p-1) + \alpha p < 0$, *then* (1.3) *has no radial solutions if*

$$\lambda \leq \frac{(\beta + N)^p (p^2 - N + \beta(p-1) - \alpha p)}{p^p (p + \beta - \alpha)}.$$

(*Here* $\lambda_1 = \inf_{\varphi \in H_0^{1,p}(\Omega)} \frac{||\nabla \varphi||^p_{p,|x|^\alpha}}{||\varphi||^p_{p,|x|^\beta}}$.)

This problem and related topics in the radial case were also studied in [5, 6, 7, 17], see for example Theorems 4.2 and 7.1 in [5]. In [5, 7] it is also shown that the critical dimension condition for this problem corresponds to

$$N - p^2 - \beta(p-1) + \alpha p < 0. \tag{1.4}$$

Other extensions of the Brézis-Nirenberg result have been achieved by Garcia Azorero and Peral Alonso in [11] and later by Guedda and Véron in [14], (see also [2]). Among other things, they proved the existence of a nontrivial positive solution of

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{p^*-2}u \quad \text{in } \Omega \subset \mathbb{R}^N, \ N > p > 1, \\ \qquad u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

provided that $1 < p^2 \leq N$ and $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of $-\Delta_p$ and $p^* = \frac{Np}{N-p}$.

In [15], Ghoussoub and Yuan consider a weight $|x|^{-s}$ in either the superlinear or linear term and use variational methods to study the existence and multiplicity of solutions for the problem

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda |u|^{r-2}u + \mu |x|^{-s}|u|^{q-2}u \quad \text{in } \Omega \subset \mathbb{R}^N, \ N > p > 1, \\ \qquad u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where λ and μ are two positive parameters, Ω is a smooth bounded domain containing 0 in its interior, $0 \leq s \leq p < N$, $q \leq p^*(-s) \equiv \frac{N-s}{N-p}p$, and $p \leq r \leq p^* \equiv p^*(0) = \frac{Np}{N-p}$. Since we want to relate their results with the previous ones and ours, we just state some of them. The authors proved that if $0 \leq s < p$, and $N \geq p^2$, then the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \mu |x|^{-s}|u|^{p^*(-s)-2}u & \text{in } \Omega \subset \mathbb{R}^N,\ N > p > 1, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least one positive solution, for any $\lambda \in (0, \lambda_1)$ and any $0 < \mu$, see [15, Theorem 1.3 (2)], and if $0 \leq s \leq p$, and $N \geq p^2 - (p-1)s$, then the problem

$$\begin{cases} -\Delta_p u = \mu |x|^{-s}|u|^{p-2}u + \lambda |u|^{p^*-2}u & \text{in } \Omega \subset \mathbb{R}^N,\ N > p > 1, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least one positive solution, for any $\lambda > 0$ and any sufficiently small $\mu > 0$, see [15, Theorem 1.4 (2)].

More recently, Xuan in [22] studied existence results in the nonradial case for problem (1.3) where now Ω is any open bounded domain in $\mathbb{R}^N$ which contains 0. They established existence of at least one positive solution if

$$N - p^2 - \beta(p-1) + \alpha p \geq 0,$$

and $\lambda \in (0, \lambda_1)$, see [22, Theorem 5.2].

We note that none of the above mentioned works (except for the original work by Brézis-Nirenberg and Egnell), contain a nonexistence result for small values of the corresponding parameter when N is below the critical dimension, and thus the criticality of the corresponding dimensions is not fully established.

Our paper is organized as follows. In section 2 we introduce the critical exponent ρ_b^*, which will play the role of $p^*(\nu)$. We defined such critical number in [12]. For the sake of completeness, in section 2, we state and explain some of the results contained in this reference.

In section 3, under appropriate assumptions on the weight C (see Theorem 3.1), we prove the existence of a first eigenvalue $\lambda_1 > 0$ and a first eigenfunction φ_1 for

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u'(r))' = \lambda r^{N-1}C(r)|u|^{p-2}u \\ \lim\limits_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0, \quad u(R) = 0, \end{cases} \tag{1.5}$$

generalizing in the radial case Theorem 1.2, [16].

In section 4, under suitable compactness assumptions (see Theorem 4.1), we prove that, for $1 < p < q < \rho_b^*$, where ρ_b^* is the critical exponent given in (2.12), the boundary-value problem (1.1) has at least one nontrivial nonnegative solution for any $\lambda < \lambda_1$, and that the same conclusion holds if $q = \rho_b^*$ and $\lim_{r\to 0} \mathcal{A}(b, \rho_b^*)(r) = 0$.

Section 5 contains our main results. It is devoted to the study of the supercritical and critical cases. As we mentioned before, it is here that we establish the critical dimension condition. In section 6 we give examples to illustrate our results and finally in section 7 we give the proofs of some auxiliary results used throughout our work.

2. Preliminary results

In this section we will introduce some basic definitions and results contained in [12]. We will give the definition of weighted L^p and Sobolev spaces, and we will define the corresponding critical number ρ_b^* which plays the role of $p^*(\nu)$. Let $a \in L^1_{loc}(0, R)$ be a positive measurable function and set

$$h_a(r) := \int_r^R a^{1-p'}(s)ds, \tag{2.1}$$

where $p' = p/(p-1)$. As in [13], it can be shown that a positive solution $u \in AC(0, R]$ with $a^{1/p}u' \in L^p(0, R)$ of

$$\begin{cases} -(a|u'|^{p-2}u')' \geq 0, \quad r \in (0, R), \\ \lim_{r\to 0} a(r)|u'(r)|^{p-1} = 0 \end{cases} \tag{2.2}$$

satisfies

$$u'(r) < 0 \text{ in } (0, R), \text{ and} \tag{2.3}$$

$$u/h_a \text{ is increasing in } (0, r_0) \text{ for some } r_0 > 0. \tag{2.4}$$

In [12] we were interested in the interplay between q and this suitable critical exponent ρ_b^* and its consequences for the existence of positive solutions to the problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = b(r)u^{q-1} \qquad \text{a.e. in} \quad (0, R), \\ \lim_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0, \quad u(R) = 0. \end{cases} \tag{2.5}$$

In order to define ρ_b^*, for $q \geq 1$ and w a positive measurable function in $(0, R)$, we denoted by

$$L^q(w) := \left\{u : w^{1/q}u \in L^q(0, R)\right\} \text{ with norm } \|u\|_{q,w} := \left(\int_0^R w(t)|u(t)|^q dt\right)^{1/q}$$

and $V^q(w) := \{u \in AC(0,R] : u(R) = 0, u' \in L^q(w)\}$.

We recall that, for $q > 1$, $V^q(w)$ is a reflexive Banach space equipped with the norm

$$||u||_{V^q(w)} := \Big(\int_0^R w(t)|u'(t)|^q dt\Big)^{1/q},$$

see [18]. We handled the imbedding properties of these weighted Sobolev spaces using a Hardy-type inequality, see [18, Theorem 6.2]. For $a \in L^1_{loc}(0,R)$ and $b \in L^1(0,R)$, $a,\ b$ positive in $(0,R)$, $u \in V^p(a)$ and $1 \le p \le q < \infty$,

$$\Big(\int_0^R b(t)|u(t)|^q dt\Big)^{1/q} \le C\Big(\int_0^R a(t)|u'(t)|^p dt\Big)^{1/p}, \tag{2.6}$$

which holds if and only if the following function of r is bounded:

$$\mathcal{A}(a,b,p,q)(r) := \Big(\int_0^r b(t)dt\Big) h_a^{q/p'}(r). \tag{2.7}$$

Moreover, the best constant C in (2.6) satisfies

$$\sup_{0<r<R} \mathcal{A}(a,b,p,q)^{1/q}(r) \le C \le k \sup_{0<r<R} \mathcal{A}(a,b,p,q)^{1/q}(r)$$

where the positive constant k depends only on p,q. When $p = 1$, (2.7) is understood as

$$\mathcal{A}(a,b,1,q)(r) = \Big(\int_0^r b(t)dt\Big)\Big(\sup_{s\in(r,R)} \frac{1}{a(s)}\Big)^q.$$

Inequality (2.6) expresses the fact that the embedding

$$V^p(a) \hookrightarrow L^q(b) \tag{2.8}$$

is continuous. This embedding is compact if and only if $\lim_{r\to 0} \mathcal{A}(a,b,p,q)(r) = 0$, see [18]. In the special case that $a(r) = r^{N-1}$, we will set

$$h(r) := \begin{cases} \frac{p-1}{N-p} r^{(p-N)/(p-1)} & \text{if } N > p \\ \log(2R/r) & \text{if } N = p \\ 1 & \text{if } N < p, \end{cases} \tag{2.9}$$

and

$$\mathcal{A}(b,q)(r) := \Big(\int_0^r b(t)dt\Big) h^{q/p'}(r). \tag{2.10}$$

All our results hold with a general weight $a \in L^1(0,R)$ instead of r^{N-1} in (1.1) under the appropriate conditions on a. For the sake of a simpler notation, and since by means of a change of variables $r = r(s)$, $v(s) = u(r)$, we may transform the problem

$$\begin{cases} -(a(r)|u'|^{p-2}u'(r))' = \lambda c(r)|u|^{p-2}u + b(r)|u|^{q-2}u, \ r \in (0,R), \ R > 0, \\ \lim_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0, \quad u(R) = 0 \end{cases}$$

into one of the form (1.1), we will assume $a(r) = r^{N-1}$. We observe also that in this case we have that $h'_a = h'$ for $N \geq p$ and $\lim_{r\to 0} \frac{h_a}{h}$ is a positive constant.

Note that, in the case that $b(r) = r^\gamma$, $\gamma > -1$, we have

$$\mathcal{A}(b,q)(r) = \begin{cases} \frac{1}{\gamma+1}\left(\frac{p-1}{N-p}\right)^{q/p'} r^{\left(\frac{(\gamma+1)p}{N-p} - q\right)\frac{N-p}{p}} & \text{if } N > p, \\ \frac{1}{\gamma+1} r^{\gamma+1}\log(2R/r) & \text{if } N = p, \\ \frac{1}{\gamma+1} r^{\gamma+1} & \text{if } N < p. \end{cases}$$

We observe that the term $\mathcal{A}(b,q)$ depends also on p. With the notation in (2.10) we just emphasize the fact that we will consider p fixed and q varying.

For a general $b \in L^1(0,R)$ it might happen that $\sup_{0<r<R} \mathcal{A}(b,q)(r) = +\infty$ for all $q \geq p$, see [12]. When $\mathcal{A}(b,p)(r)$ is bounded, we set

$$\mathcal{U}_b = \mathcal{U} := \{s \geq p : \sup_{0<r<R} \mathcal{A}(b,s)(r) < \infty\},$$

which is nonempty, and we define the number (possibly ∞)

$$\rho_b^* = \rho^* = \sup \mathcal{U}. \tag{2.11}$$

We proved in [12] the following two results.

Theorem (Theorem 1.1 in [12]) *Let $p > 1$ and let $b \in L^1(0,R)$ such that $\mathcal{A}(b,p)$ is bounded. Let ρ^* be as in* (2.11); *then*

$$\rho^* = p' \liminf_{r\to 0} \frac{|\log \int_0^r b(t)dt|}{\log(h(r))}, \tag{2.12}$$

and the embedding $V^p \hookrightarrow L^q(b)$ is compact for $1 \leq q < \rho^*$. If $\rho^* < \infty$ and $q > \rho^*$, the embedding (2.8) *fails.*

Moreover, for $q = \rho^*$, we have

(i) *the embedding $V^p \hookrightarrow L^{\rho^*}(b)$ is continuous if and only if $\mathcal{A}(b,\rho^*)$ is bounded for r near zero, and*

(ii) *the embedding $V^p \hookrightarrow L^{\rho^*}(b)$ is compact if and only if*

$$\lim_{r\to 0} \mathcal{A}(b,\rho^*)(r) = 0.$$

In fact in [12] we proved (2.12) under a slightly stronger assumption, so we will give a proof of this version in the appendix, see Lemma 7.2.

For the case $b(r) = r^{N-1}$, $N > p$, we have that $\rho^* = p^* = Np/(N-p)$. From the above result we proved the following.

Theorem (Theorem 1.2 in [12]) *Let b be a positive measurable function on $(0,R)$ satisfying $b \in L^1(0,R)$ and let $1 < p < q < \rho^*$, where ρ^* is the critical exponent given in* (2.12)*. Then the boundary-value problem* (2.5) *has a nontrivial nonnegative solution. The same conclusion holds if $q = \rho^*$ and $\lim_{r\to 0} \mathcal{A}(b,\rho^*)(r) = 0$.*

3. The eigenvalue problem

We will study first the existence and basic properties of the first eigenfunction. Let us consider the problem of existence of a positive solution to (1.5), which we recall from the introduction,

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u'(r))' = \lambda c(r)|u|^{p-2}u \\ \lim\limits_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0, \quad u(R) = 0, \end{cases}$$

where from now on we set $c(r) = r^{N-1}C(r)$. In [20], Szulkin and Willem proved for $p = 2$ the existence of the first eigenvalue $\lambda_1 > 0$ and a corresponding positive eigenfunction $\varphi_1 \in W_0^{1,2}(\Omega)$, for the problem

$$\begin{cases} -\Delta_p u = \lambda C(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain, $C \in L^{N/p}(\Omega)$, with $1 < p < N$, and $C^+ \not\equiv 0$. Lucia and Prashanth in [16] treated the problem for any $p > 1$, and established the existence, uniqueness and simplicity of the first eigenvalue $\lambda_1 > 0$ for any connected open $\Omega \subset \mathbb{R}^N$. The regularity of φ_1 is studied in [14].

We have the following theorem, which, for the radial case, is an improvement of the above quoted results (see [16] and the references therein). The main tools that we will use are Hardy-type inequalities and the following Picone's identity (see [1]).

For u, v differentiable, $v > 0$, $u \ge 0$ it holds that

$$\begin{aligned} L(u,v) &:= |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u \cdot |\nabla v|^{p-2}\nabla v \\ &= |\nabla u|^p - \nabla\Big(\frac{u^p}{v^{p-1}}\Big) \cdot |\nabla v|^{p-2}\nabla v := R(u,v). \end{aligned}$$

Moreover, $L(u,v) \geq 0$, and $L(u,v) = 0$ almost everywhere in $B_R(0)$ if and only if $u = kv$ for some positive constant k.

Theorem 3.1. *Assume that* $c \in L^1(0,R)$ *and* $\mathcal{A}(c,p)(r) \to 0$ *as* $r \to 0$. *Then, there exists* $\lambda_1 > 0$ *a first eigenvalue and* φ_1 *a corresponding eigenfunction satisfying* (1.5). *Moreover:*

(i) *If* $\lambda < \lambda_1$ *there is no solution to* (1.5).
(ii) *If* $\lambda > \lambda_1$ *there is no positive solution to* (1.5).
(iii) *If* $\lambda = \lambda_1$ *there is only one (up to constant multiples) positive solution to* (1.5).

Proof. The existence of λ_1 and φ_1 follows as in the proof of Theorem 1.2 in [12] or Theorem 4.1 in the next section. Moreover, λ_1 is given by

$$\lambda_1 = \lambda_1(R) = \inf_{u \in V^p \setminus \{0\}} \frac{\int_0^R r^{N-1}|u'(r)|^p \, dr}{\int_0^R c(r)|u(r)|^p \, dr}. \tag{3.1}$$

If $\lambda < \lambda_1$, the nonexistence follows immediately from (3.1). If $\lambda \geq \lambda_1$ the result follows from Picone's identity as used in [1]: Assume that u is a positive solution to (1.5). For $\varphi \in C_0^\infty(B_R(0))$, $\varphi \geq 0$ we have

$$\begin{aligned} 0 &\leq \int_{B_R(0)} L(\varphi,u) = \int_{B_R(0)} R(\varphi,u) \\ &= \int_{B_R(0)} |\nabla\varphi|^p - \int_{B_R(0)} \nabla\Big(\frac{\varphi^p}{u^{p-1}}\Big) \cdot |\nabla u|^{p-2}\nabla u \\ &= \int_{B_R(0)} |\nabla\varphi|^p + \int_{B_R(0)} \Big(\frac{\varphi^p}{u^{p-1}}\Big)\nabla \cdot \Big(|\nabla u|^{p-2}\nabla u\Big) \\ &= \int_{B_R(0)} |\nabla\varphi|^p - \lambda\int_{B_R(0)} C(x)\varphi^p, \end{aligned}$$

and thus

$$0 \leq \int_{B_R(0)} |\nabla\varphi|^p - \lambda\int_{B_R(0)} C(x)\varphi^p.$$

Letting now $\varphi \to \varphi_1$ we obtain

$$0 \leq \int_{B_R(0)} |\nabla\varphi_1|^p - \lambda\int_{B_R(0)} C(x)\varphi_1^p.$$

Hence, for $\lambda > \lambda_1$ we obtain the contradiction

$$(\lambda_1 - \lambda)\int_{B_R(0)} C(x)\varphi_1^p \geq 0,$$

and if $\lambda = \lambda_1$ we obtain $L(\varphi_1, u) = 0$ implying $u = k\varphi_1$ for some positive constant k, and thus the theorem follows. □

Our result applies in particular to the well-known first eigenvalue problem for the p-Laplacian:

$$\begin{cases} -\Delta_p u = \lambda C(|x|)|u|^{p-2}u & \text{in } B_R(0) \\ u = 0 & \text{on } \partial B_R(0), \end{cases}$$

with $C(|x|) > 0$ for $x \in B_R(0)$.

We note that if $C \in L^{N/p}(B_R(0))$, $N > p$, then $c(r) := r^{N-1}C(r)$ satisfies the assumptions of our Theorem 3.1 since

$$\Big(\int_0^r s^{N-1}C(s)ds\Big)r^{p-N} \leq K\Big(\int_0^r s^{N-1}C^{N/p}(s)ds\Big)^{p/N}$$

which tends to 0 as $r \to 0$. Hence our theorem applies and generalizes Theorem 1.2 in [16] for radial solutions in the radial case.

4. The subcritical case

As in the previous section, and in order to simplify our writing, we set $c(r) = r^{N-1}C(r)$ and $b(r) = r^{N-1}B(r)$. Our first result concerns existence of solutions for problem (1.1) in the subcritical case, that is, for $q < \rho^* = \rho_b^*$, or in the critical case $q = \rho^*$ if $\mathcal{A}(b,q)(r) \to 0$ as $r \to 0$, that is, when the corresponding imbedding is compact.

Theorem 4.1. *Let b, c be positive functions in $L^1(0,R)$, $\mathcal{A}(c,p) \to 0$ as $r \to 0$ and $\mathcal{A}(b,p)$ bounded. Let $1 < p < q < \rho^*$, where ρ^* is the critical exponent given in* (2.12). *Then the boundary-value problem* (1.1) *has at least one nontrivial nonnegative solution for any $\lambda < \lambda_1$. The same conclusion holds if $q = \rho^*$ and*

$$\lim_{r\to 0} \mathcal{A}(b,\rho^*)(r) = 0. \tag{4.1}$$

In our second result, we establish nonexistence of positive solutions to (1.1) for $\lambda \geq \lambda_1$.

Theorem 4.2. *Let b, c be positive measurable functions on $(0,R)$ satisfying $b, c \in L^1(0,R)$, and $\mathcal{A}(c,p) \to 0$ as $r \to 0$. Let $\lambda \geq \lambda_1$ and $p > 1$, $q > 1$. Then the problem* (1.1) *does not possess any nontrivial nonnegative solution.*

Remark 1. The above result is also true if we replace in (1.1) $b(r)|u|^{q-2}u$ for any function $W(r,u)$ such that $W \geq 0$ for $u \geq 0$, $r > 0$ and $W(r, K\varphi_1) \not\equiv 0$ for any $K > 0$.

We start by proving Theorem 4.1.

Proof of Theorem 4.1. For any $\lambda < \lambda_1$, consider the functional $S_\lambda : V^p \to \mathbb{R}$ defined by

$$S_\lambda(u) := \int_0^R r^{N-1}|u'(r)|^p dr - \lambda \int_0^R c(r)|u(r)|^p dr, \quad u \in V^p.$$

Using the homogeneity of the functions involved in the problem, we will obtain our solution by solving the constrained minimization problem for the functional S_λ restricted to the set

$$M := \{u \in V^p : \int_0^R b(r)|u(r)|^q dr = 1\}.$$

The proof consists in checking the hypotheses of Theorem 1.2 in [19]. Since V^p is a reflexive Banach space, we must show that S_λ is coercive on M with respect to V^p, that the set M is a weakly closed subset of V^p, and that S_λ is sequentially weakly lower semicontinuous on M.

The coercivity of the functional is clear since

$$S_\lambda(u) = ||u||^p_{V^p} - \lambda \int_0^R c(r)|u(r)|^p dr \geq K||u||^p_{V^p}, \tag{4.2}$$

where

$$K = \begin{cases} 1 & \text{if } \lambda \leq 0 \\ 1 - \frac{\lambda}{\lambda_1} & \text{if } \lambda \in (0, \lambda_1). \end{cases}$$

Let $\{u_n\}$ be a sequence in M which converges weakly to $u \in V^p$. By Theorem 1.1 in [12] if $q < \rho^*$ or if $q = \rho^*$ (using condition (4.1)), the imbedding $V^p \hookrightarrow L^q(b)$ is compact and hence there is a subsequence (renamed the same) such that $u_n \to u$ in $L^q(b)$ and in $L^p(c)$. This implies $||u||_{L^q(b)} = 1$, so M is weakly closed. Finally, we will prove the lower semicontinuity of S_λ. Let $\{u_n\} \in V^p$ be as above, any sequence which converges weakly in V^p to u. We have to show that

$$S_\lambda(u) \leq \liminf_{n\to\infty} S_\lambda(u_n).$$

Let $\{v_n\}$ be a subsequence of $\{u_n\}$ such that

$$\liminf_{n\to\infty} S_\lambda(u_n) = \lim_{n\to\infty} S_\lambda(v_n).$$

Since the embedding $V^p \hookrightarrow L^p(c)$ is compact we may assume that $v_n \to u$ in $L^p(c)$. Using also the fact that the norm in V^p is a lower semicontinuous function we have

$$\begin{aligned}
S_\lambda(u) &= ||u||^p_{V^p} - \lambda \int_0^R c(r)|u(r)|^p dr \\
&\leq \liminf_{n\to\infty} ||v_n||^p_{V^p} - \lambda \int_0^R c(r)|u(r)|^p dr \\
&= \liminf_{n\to\infty} ||v_n||^p_{V^p} - \lambda \lim_{n\to\infty} \int_0^R c(r)|v_n(r)|^p dr \\
&= \liminf_{n\to\infty} \Big(||v_n||^p_{V^p} - \lambda \int_0^R c(r)|v_n(r)|^p dr \Big) \\
&= \lim_{n\to\infty} S_\lambda(v_n) = \liminf_{n\to\infty} S_\lambda(u_n).
\end{aligned}$$

In this form, from Theorem 1.2 in [19] we conclude that the functional S_λ is bounded below and achieves its infimum in M. Let $\bar{u}$ denote the point where the infimum is attained. Note that, since $S_\lambda(\bar{u}) = S_\lambda(|\bar{u}|)$, we may assume that $\bar{u} \geq 0$. Let us now set

$$F(u) := \int_0^R b(r)|u(r)|^q dr.$$

Clearly, S_λ and F are Frèchet differentiable with

$$S'_\lambda(u)(h) = p\Big(\int_0^R r^{N-1}|u'|^{p-2}u'h'dr - \lambda \int_0^R c(r)|u|^{p-2}uhdr\Big),$$

and

$$F'(u)(h) = q\int_0^R b(r)|u|^{q-2}uhdr$$

for all $h \in V^p$. By the Lagrange multiplier rule ([19]) we conclude the existence of $\mu \in \mathbb{R}$ such that

$$(S'_\lambda(\bar{u}) - \mu F'(\bar{u}))(h) = 0 \quad \text{for all } h \in V^p;$$

that is,

$$\int_0^R r^{N-1}|\bar{u}'|^{p-2}\bar{u}'h'dr - \lambda\int_0^R c(r)|\bar{u}|^{p-2}\bar{u}hdr - \delta\int_0^R b(r)|\bar{u}|^{q-2}\bar{u}hdr = 0$$

for all $h \in V^p$, where $\delta := q\mu/p$. Replacing h by $\bar{u}$ we find that

$$S_\lambda(\bar{u}) = \delta\int_0^R b(r)|\bar{u}|^q dr = \delta,$$

and thus from (4.2) and since $\bar{u} \in M$, we obtain $\delta > 0$. Now, setting $\tilde{u} := \delta^{\frac{1}{q-p}} \bar{u}$ we find that $\tilde{u}$ is a weak solution to our problem. That $\tilde{u}$ is actually a solution to (1.1) follows from Lemma 7.3 in the appendix, which in turn follows by arguing as in the proof of [12, Proposition 3.1]. □

Proof of Theorem 4.2. Let $\lambda \geq \lambda_1$ and assume by contradiction that the problem (1.1) has a positive solution u. Arguing as in the proof of Theorem 3.1 and setting

$$B(x) := |x|^{1-N} b(|x|), \quad C(x) := |x|^{1-N} c(|x|),$$

we have

$$0 \leq (\lambda_1 - \lambda) \int_B C(x) \varphi_1^p dx - \int_B B(x) \varphi_1^p u^{q-p} dx,$$

a contradiction with the assumption on λ. □

Finally, we state some basic necessary conditions for existence of positive solutions to our problem.

Proposition 4.1. *Let $\lambda > 0$, $q \geq p > 1$.*

(i) *If either $c \notin L^1(0,1)$ or $b \notin L^1(0,1)$, then problem (1.1) has no positive solutions.*

(ii) *If either $\mathcal{A}(c,p)$ is unbounded near zero or $\mathcal{A}(b,p) \to \infty$ as $r \to 0$, then problem (1.1) has no positive solutions. Also, if $\lambda = 0$ and $\mathcal{A}(b,p) \to \infty$ as $r \to 0$, then problem (1.1) has no positive solutions.*

Proof. Assertion (i) follows easily from the observation that, if u is a positive solution to (1.1), then it is decreasing and, for any $0 < s < r < R$, we have

$$r^{N-1} |u'(r)|^{p-1} \geq \lambda u^{p-1}(r) \int_s^r c(t) dt + u^{q-1}(r) \int_s^r b(t) dt, \tag{4.3}$$

and the claim follows by letting $s \to 0$ in (4.3). In order to prove (ii) we assume by contradiction that problem (1.1) has a positive solution. Since, from the definition of h_a in (2.1) $r^{N-1}|u'(r)|^{p-1} = (|u'|/|h_a'|)^{p-1}$, from (4.3) and using the monotonicity of u/h_a near 0 (see (2.4)) we find that

$$1 \geq \lambda \Big(\int_0^r c(s) ds \Big) h_a^{p-1}(r) + \Big(\int_0^r b(s) ds \Big) u^{q-p}(r) h_a^{p-1}(r)$$

implying that

$$u^{q-p}(r) \leq \Big(\Big(\int_0^r b(s) ds \Big) h_a^{p-1}(r) \Big)^{-1} \tag{4.4}$$

and

$$\int_0^r c(s)ds \le \frac{1}{\lambda} h_a^{1-p}(r) \tag{4.5}$$

for all r near 0. □

5. Supercritical and critical cases

Throughout this section we will assume that $p \le \rho^* < \infty$ and that the weight functions $b(r) = r^{N-1}B(r)$ and $c(r) = r^{N-1}C(r)$ are of class $C^1(0,\infty)$. We will deal with the supercritical case, see Theorem 5.1, and as we mentioned in the introduction, we will establish the corresponding critical dimension condition for our problem, see Theorem 5.2 and Theorem 5.3. Let us set

$$\alpha(r) = p' \frac{h(r)}{|h'(r)|}, \tag{5.1}$$

where h is given in (2.9). Let

$$\rho_1^* := \limsup_{r\to 0} \frac{(c\alpha)'(r)}{c(r)}, \tag{5.2}$$

and note that by (2.12) with b changed to c and L'Hôpital's rule, $\rho_c^* \le \rho_1^*$.

We will first state and discuss our results, postponing their proofs to the remainder of the section. We have the following nonexistence result for $\lambda > 0$ sufficiently small and $q > \rho^*$.

Theorem 5.1. *Let $N \ge p$ and let the weight functions b, c be such that b, $c \in L^1(0,R)$ and $\mathcal{A}(b,p)$ bounded near 0. Assume further that ρ_1^* is finite and that $\limsup_{r\to 0} \frac{(b\alpha)'(r)}{b(r)} \le \rho^* < \infty$.*

If $q > \rho^*$, then there exists $R^* > 0$ and $\lambda^* > 0$ such that, for all $R \in (0,R^*]$ and $\lambda \in (0,\lambda^*)$, the problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda c(r)|u|^{p-2}u + b(r)|u|^{q-2}u \quad r \in (0,R) \\ \lim_{r\to 0} r^{N-1}|u'|^{p-1}(r) = u(R) = 0, \\ u(r) > 0 \quad r \in (0,R) \end{cases} \tag{5.3}$$

has no solution.

Our next result deals with the critical case. In the case that $c(r) = r^{N-1}$ it asserts that, if $p^2 > N \ge p \ge 2$, then there exists $\lambda^* > 0$ such that, for

$\lambda \in (0, \lambda^*)$, the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + |u|^{p^*-2}u & x \in B_R(0) \\ u = 0 \quad \text{in } \partial B_R(0) \end{cases}$$

has no positive radially symmetric solution for any $R > 0$.

We have the following.

Theorem 5.2. *Let $N \geq p \geq 2$ and let the weight functions $b,\ c \in L^1(0,1)$ with $\mathcal{A}(b,p)$ bounded near 0. Assume that $p < \rho^* < \infty$, $\rho_1^* < \infty$, and*

$$pp' < \rho_c^*. \tag{5.4}$$

If $q = \rho^*$ and

$$(b\alpha)'(r) \leq \rho^* b(r) \quad \textit{on some interval } (0,R),\ R > 0, \tag{5.5}$$

then there exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$, the problem (5.3) has no solution.

Remark 2. Note that condition (5.4) is equivalent to

$$\text{there exists } \tau > p \text{ such that } \sup_{r\in[0,1]} \Big(\int_0^r c(s)ds\Big) h^\tau(r) < \infty. \tag{5.6}$$

Also, note that for the special case $c(r) = r^{N-1+\beta}$, $N + \beta > 0$, and $N > p$, condition (5.4) reads

$$p < \frac{(N+\beta)(p-1)}{N-p},$$

which is the same as (1.4) with $\alpha = 0$. This result should be compared to the second part of Theorem 4 in [10], and Theorem 4.2 in [5], see also Proposition 6.3 in the examples in section 6.

In our last theorem we give conditions under which problem (1.1) admits a positive solution for $\lambda \in (\lambda^{**}, \lambda_1)$, for some $\lambda^{**} > 0$. This, together with Theorem 5.2, establishes the criticality of the dimensions given by condition (5.4).

Theorem 5.3. *Let $N > p$ and let the weight functions $b,\ c \in L^1(0,1)$ with $\mathcal{A}(b,p)$ bounded near 0, and assume that $p < \rho^* < \infty$, $p < \rho_c^*$,*

$$\lim_{r\to 0} p' \frac{|\log(\int_0^r b(s)ds)|}{\log(h(r))} = \rho^*, \tag{5.7}$$

$$\textit{there exists a positive constant } K_1 \textit{ such that } \mathcal{A}(b,\rho^*) \leq K_1, \tag{5.8}$$

$$r^{-N}\int_0^r b(s)ds, \quad r^{-N}\int_0^r c(s)ds \in L^\tau(0,1) \quad \textit{for some } \tau > 1, \tag{5.9}$$

and there exist positive constants C_1 *and* C_2 *such that*

$$\begin{gathered} C_1\int_0^r b(s)ds \le rb(r) \le C_2\int_0^r b(s)ds, \\ rc(r) \le C_2\int_0^r c(s)ds \quad \textit{on some interval } (0,R),\ R>0. \end{gathered} \tag{5.10}$$

If $q = \rho^*$, *then there exists* $\lambda^{**} > 0$ *such that, for all* $\lambda \in (\lambda^{**}, \lambda_1)$, *the problem* (5.3) *has a nontrivial solution.*

Remark 3. This result should be compared to the first part of Theorem 4 in [10] and Theorem 7.1 in [5].

The proof of Theorem 5.1 is based on the following lemmas which can be verified by direct computation.

Lemma 5.1. *Let* $R > 0$, *and let* u *be a solution to*

$$\begin{gathered} -(r^{N-1}|u'|^{p-2}u')' = \lambda c(r)|u|^{p-2}u + b(r)|u|^{q-2}u \quad r\in(0,R) \\ u(r) > 0 \quad r \in (0,R). \end{gathered}$$

Let $\delta,\ \mu \in C^1(0,R)\cap C(0,R]$ *and set*

$$\begin{aligned} E_{\delta,\mu}(r) \;=\; & \frac{r^{N-1}\delta(r)}{p'}|u'(r)|^p + r^{N-1}\mu(r)|u'(r)|^{p-2}u'(r)u(r) \\ & + \frac{\lambda\delta(r)c(r)}{p}u^p(r) + \frac{\delta(r)b(r)}{q}u^q(r), \end{aligned}$$

then

$$\begin{aligned} E'_{\delta,\mu}(r) = & \Big[\frac{1}{p'}(a^{1-p'}\delta)'(r)a^{p'}(r) + a(r)\mu(r)\Big]|u'(r)|^p \\ & + a(r)\mu'(r)u(r)|u'|^{p-2}(r)u'(r) \\ & + \lambda\Big[\frac{(c\delta)'(r)}{p} - c(r)\mu(r)\Big]u^p(r) + \Big[\frac{(b\delta)'(r)}{q} - b(r)\mu(r)\Big]u^q(r), \end{aligned} \tag{5.11}$$

where $a(r) = r^{N-1}$.

Lemma 5.2. *Let* u, δ *and* μ *be as in the above lemma and assume that* $u(R) = 0$. *Then for* $r \in (0,R)$ *it holds that*

$$R^{N-1}\delta(R)|u'|^p(R) - E_{\delta,\mu}(r) \tag{5.12}$$

$$=\int_r^R \Big[\frac{1}{p'}(a^{1-p'}\delta)'a^{p'} + a\mu\Big]|u'|^p ds + \int_r^R a\mu' u|u'|^{p-2}u' ds$$
$$+\int_r^R \lambda\Big[\frac{(c\delta)'}{p} - c\mu\Big]u^p ds + \int_r^R \Big[\frac{(b\delta)'}{q} - b\mu\Big]u^q ds, \tag{5.13}$$

where $a(r) = r^{N-1}$.

Let in the above lemma $\delta(r) = \alpha(r)$ and $\mu = 0$, where α is given by (5.1). Then since $\delta(R) \geq 0$ we obtain the following proposition.

Proposition 5.1. *Let u be a solution to* (5.3) *and assume that*

$$\lim_{r\to 0} E_{\alpha,0}(r) = 0,$$

then

$$\int_0^R r^{N-1}|u'|^p dr \leq \limsup_{r\to 0}\Big(\frac{\lambda}{p}\int_r^R (c\alpha)'u^p ds + \frac{1}{q}\int_r^R (b\alpha)'u^q ds\Big).$$

We turn now to the proof of the theorem.

Proof of Theorem 5.1. If $\mathcal{A}(c,p)$ is unbounded near 0, then by Proposition 4.1 we can take $\lambda^* = \infty$ and $R > 0$ is arbitrary. Hence, without loss of generality we may assume that $\mathcal{A}(c,p)$ is bounded near 0. We will prove that there exists $R^* > 0$ such that if u is a nontrivial solution to (5.3) for some $R \in (0, R^*)$ and $\lambda > 0$, then there exists $D > 0$ such that

$$\int_0^R cu^p dr \leq \frac{\lambda}{D}\int_0^R cu^p dr, \tag{5.14}$$

implying that $\lambda \geq D$.

We start by fixing R^*. To this end we observe that, from $q > \rho^*$, there exists $\epsilon > 0$ such that $q > \rho^* + \epsilon$, and from the definition of ρ_1^*, ρ^*, there exists $R^* > 0$ such that for any $r \in (0, R^*]$ we have

$$(c\alpha)'(r) \leq \bar\rho_1 c(r), \quad (b\alpha)'(r) \leq \bar\rho b(r), \tag{5.15}$$

where $\bar\rho_1 = \rho_1^* + \epsilon$ and $\bar\rho = \rho^* + \epsilon$. In order to prove (5.14), we will establish that if $\lambda > 0$ is such that (5.3) has a solution u for some $R \in (0, R^*]$, then

$$\int_0^R r^{N-1}|u'|^p dr = \lambda\int_0^R cu^p dr + \int_0^R bu^q dr \tag{5.16}$$

and

$$\int_0^R r^{N-1}|u'|^p dr \leq \frac{\lambda}{p}\bar\rho_1\int_0^R cu^p dr + \frac{\bar\rho}{q}\int_0^R bu^q dr. \tag{5.17}$$

Having proved (5.16) and (5.17), and using the fact that $\mathcal{A}(c,p)$ is bounded and thus the Hardy-type inequality

$$\int_0^R cu^p dr \le K_1 \int_0^R r^{N-1}|u'|^p dr, \quad u \in V^p, \quad \Big(K_1 = \frac{1}{\lambda_1(p,R)}\Big), \tag{5.18}$$

holds, (5.14) will follow: indeed, from (5.16) and (5.17),

$$\Big(1 - \frac{\bar\rho}{q}\Big) \int_0^R bu^q dr \le \lambda\Big(\frac{\bar\rho_1}{p} - 1\Big) \int_0^R cu^p dr. \tag{5.19}$$

Therefore, since $\rho_1^* \ge p$ (see Lemma 7.2 (iii)), from (5.18), (5.19) and (5.16), we have

$$\begin{aligned} \int_0^R cu^p dr \le K_1 \int_0^R r^{N-1}|u'|^p dr &= K_1\Big(\lambda \int_0^R cu^p dr + \int_0^R bu^q dr\Big) \\ &\le K_1\lambda \int_0^R cu^p dr + \lambda K_1 K \int_0^R cu^p dr \end{aligned}$$

where $K = \big(\frac{\bar\rho_1}{p} - 1\big)/\big(1 - \frac{\bar\rho}{q}\big)$. Thus,

$$\int_0^R cu^p dr \le \lambda K_1(1+K) \int_0^R cu^p dr$$

implying that (5.14) holds with $D = \lambda_1(R)/(1+K)$.

We prove next the validity of (5.16) and (5.17). From Lemma 7.2(ii) we have

$$\lim_{r\to 0} c(r)\alpha(r) = \lim_{r\to 0} b(r)\alpha(r) = 0,$$

and from (5.15) we obtain

$$c\alpha(r) \le \bar\rho_1 \int_0^r c(s)ds, \quad b\alpha(r) \le \bar\rho \int_0^r b(s)ds. \tag{5.20}$$

Also, from the assumption $q > \bar\rho \ge \rho^*$, it can be proven that

$$\lim_{r\to 0} \mathcal{A}(b,q)(r) = \infty \tag{5.21}$$

(see Lemma 7.2(iv) in the appendix).

We will first prove (5.16). To this end we observe that from the monotonicity of u/h_a near 0 we have

$$r^{N-1}|u'|^{p-1}u(r) \le Const.\ \Big(\frac{u}{h}\Big)^p h(r) \le Const.\ (\mathcal{A}(b,q)(r))^{-p/(q-p)}, \tag{5.22}$$

and thus, by multiplying the equation in (5.3) by u, integrating over (r, R), taking the limit as $r \to 0$ and using (5.22) we conclude that

$$\int_0^R r^{N-1}|u'|^p dr = \lim_{r\to 0}\Big(\lambda \int_r^R cu^p dr + \int_r^R bu^q dr\Big),$$

implying that in fact $u \in L^p(c;(0,R)) \cap L^q(b;(0,R))$ and that (5.16) holds.

The validity of (5.17) follows from Proposition 5.1 and (5.15). We show next that each of the terms in $E_{\alpha,0}(r)$ tends to 0 as $r \to 0$, implying in particular that $\lim_{r\to 0} E_{\alpha,0}(r) = 0$. Indeed,

$$r^{N-1}|u'|^p\alpha(r) \le Const.\ \Big(\frac{u}{h}\Big)^p h(r) \le Const.\ (\mathcal{A}(b,q)(r))^{-p/(q-p)},$$

and also, from the second inequality in (5.20) and (4.4) we find that

$$b\alpha u^q(r) \le Const.\ (\mathcal{A}(b,q)(r))^{-p/(q-p)}.$$

Similarly, using the first inequality in (5.20), (4.4) and (4.5), we find that

$$\begin{aligned} c\alpha u^p(r) &\le \bar{\rho}_1 \int_0^r c(s)ds\ u^p(r) \le Const.\ \frac{u^p}{h^{p-1}}(r) \\ &\le Const.\ (\mathcal{A}(b,q)(r))^{-p/(q-p)}, \end{aligned}$$

and thus our claim follows from (5.21). □

It follows from the proof of Theorem 5.1 that the following result holds.

Corollary 5.1. *Let $N \ge p$ and let the weight functions b, c be such that $b,\ c \in L^1(0,1)$ and $\mathcal{A}(b,p)$ is bounded near 0. Assume further that ρ_1^* is finite and set*

$$\rho := \sup_{r\in[0,1]} \frac{(b\alpha)'(r)}{b(r)}.$$

If $q > \rho$, then there exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$ and all $R > 0$, the problem (5.3) has no solution.

Proof of Theorem 5.2. As in the proof of Theorem 5.1 we may assume that $\mathcal{A}(c,p)$ is bounded near 0, and thus if u is a solution to (5.3), then $u \in L^p(c)$.

The proof is divided into three steps. In step 1, we choose particular functions δ and μ to use our Pohozaev-type identity (5.12). In step 2 we prove that $\lim_{r\to 0} E_{\delta,\mu}(r) = 0$. Finally in step 3 we use a Hardy type inequality for $u^{p/(p-1)}$ that will yield the existence of λ^*.

Step 1: Let R be given by (5.5). We will define at this step the functions δ and μ such that there exists a positive constant K so that for all $r \in (0, R]$

$$\int_r^R s^{N-1}\mu'(s)u(s)|u'(s)|^{p-1}ds \le E_{\delta,\mu}(r) + \lambda K \int_r^R c(s)u^p(s)ds. \tag{5.23}$$

From the definition of ρ_1^*, we have that there exists $\rho \ge \rho_1^*$ such that

$$(c\alpha)'(r) \le \rho c(r), \quad \text{for all } r \in (0, R). \tag{5.24}$$

Let us set $d := \tau - p + 1$, where τ is as in (5.6), and define the functions δ and μ by

$$\delta(r) = p'h(r)r^{\frac{N-1}{p-1}}\Big(1 - \frac{h^d(R)}{h^d(r)}\Big) \quad \mu(r) = 1 + (d-1)\frac{h^d(R)}{h^d(r)}.$$

Since $d > 1$ we have that $\mu'(r) > 0$ for all $r \in (0, R)$ and both δ, μ are positive. Moreover,

$$\lim_{r\to 0}\frac{\delta}{\alpha}(r) = 1, \quad \lim_{r\to 0}\mu(r) = 1. \tag{5.25}$$

Then from the Pohozaev-type identity (5.12) we have for $r \in (0, R)$ that

$$\begin{aligned}\int_r^R s^{N-1}\mu'u|u'|^{p-1}ds &= E_{\delta,\mu}(r) + \int_r^R n(s)|u'|^p ds\\ + \int_r^R \lambda\Big[\frac{(c\delta)'}{p} - c\mu\Big]u^p ds &+ \int_r^R \Big[\frac{(b\delta)'}{q} - b\mu\Big]u^q ds,\end{aligned} \tag{5.26}$$

where

$$n(r) = \frac{1}{p'}r^{(N-1)p'}\Big(r^{(N-1)(1-p')}\delta\Big)' + r^{N-1}\mu.$$

The choice of δ and μ implies that $n(r) \equiv 0$ and thus (5.26) becomes

$$\int_r^R s^{N-1}\mu'(s)u(s)|u'|^{p-1}(s))ds = E_{\delta,\mu}(r) \tag{5.27}$$

$$+\lambda\int_r^R\Big(\frac{(c\delta)'}{p} - c\mu\Big)u^p ds + \int_r^R\Big(\frac{(b\delta)'}{q} - b\mu\Big)u^q ds. \tag{5.28}$$

We will see next that

$$\frac{(c\delta)'(r)}{p} - c(r)\mu(r) \le Kc(r), \quad \frac{(b\delta)'(r)}{q} - b(r)\mu(r) \le 0, \tag{5.29}$$

for all $r \in (0, R]$ and for some $K > 0$.

We start by proving that the second inequality in (5.29) is satisfied.

$$(b\delta)'(r) = \Big(\frac{b}{|h'|}|h'|\delta\Big)'(r) = \frac{b(r)}{|h'(r)|}(|h'|\delta)'(r) + \Big(\frac{b}{|h'|}\Big)'(r)|h'(r)|\delta(r)$$
$$= -p'\mu(r)b(r) + \Big(\frac{b}{|h'|}\Big)'(r)|h'(r)|\delta(r).$$

From (5.5) and the definition of α in (5.1) we have that

$$\Big(\frac{b}{|h'|}\Big)'(r) \le \Big(\frac{\rho^*}{p'} + 1\Big)\frac{b(r)}{h(r)},$$

hence

$$(b\delta)'(r) \le -p'\mu(r)b(r) + \Big(\frac{\rho^*}{p'} + 1\Big)\frac{|h'(r)|}{h(r)}\delta(r)b(r).$$

We obtain thus that

$$\frac{(b\delta)'(r)}{\rho^*} - b(r)\mu(r) \le b(r)\Big(-\Big[1 + \frac{p'}{\rho^*}\Big]\mu(r) + \frac{1}{\rho^*}\Big[\frac{\rho^*}{p'} + 1\Big]\frac{|h'(r)|}{h(r)}\delta(r)\Big). \tag{5.30}$$

From the definition of δ and μ, we have

$$\frac{|h'(r)|}{h(r)}\delta(r) \le p'\mu(r), \tag{5.31}$$

and therefore the second inequality in (5.29) follows from (5.30) and (5.31).

Finally, we establish the validity of the first inequality in (5.29): As above, we find that

$$\frac{(c\delta)'(r)}{p} - c(r)\mu(r) \le c(r)\Big(-\Big[1 + \frac{p'}{p}\Big]\mu(r) + \frac{1}{p}\Big[\frac{\rho}{p'} + 1\Big]\frac{|h'(r)|}{h(r)}\delta(r)\Big),$$

hence using (5.31) we obtain

$$\frac{(c\delta)'(r)}{p} - c(r)\mu(r) \le \Big(\frac{\rho}{p} - 1\Big)c(r)\mu(r),$$

and since $\mu(r) \le d$ for all $r \in (0, R)$, we conclude that the first inequality in (5.29) holds with $K = \frac{\rho}{p}d$. Hence, (5.23) follows from (5.29) and (5.27).

Step 2: $\lim_{r\to 0} E_{\delta,\mu}(r) = 0$. From (5.5) and Lemma 7.2 (iv), the function $r \mapsto \mathcal{A}(b, \rho^*)(r)$ is decreasing, and thus either

$$(i) \quad \lim_{r\to 0} \mathcal{A}(b, \rho^*)(r) = \infty \quad \text{or} \quad (ii) \quad \lim_{r\to 0} \mathcal{A}(b, \rho^*)(r) \in (0, \infty).$$

In the first case we can argue as in the proof of Theorem 5.1 and use (5.25) to obtain $E_{\delta,\mu}(r) \to 0$ as $r \to 0$. In the second case, it must be that the Hardy inequalities

$$\int_0^r c|u|^p dt \le C \int_0^r r^{N-1}|u'|^p dt, \qquad \int_0^r b|u|^q dt \le C\Big(\int_0^r r^{N-1}|u'|^p dt\Big)^{q/p},$$

hold for some positive constant C implying that $u \in L^p(c;(0,R)) \cap L^q(b;(0,R))$ and thus, using (5.24) and the monotonicity of u, we obtain that

$$(c\alpha)(r)u^p(r) \le \rho u^p(r) \int_0^r c(s)ds \le \rho \int_0^r cu^p dt, \quad r \in (0,R)$$

and

$$(b\alpha)(r)u^q(r) \le \rho^* u^q(r) \int_0^r b(s)ds \le \rho^* \int_0^r b(s)u^q(s)ds, \quad r \in (0,R).$$

By definition of a solution $u \in V^p$, we get from the above inequalities that

$$\lim_{r\to 0}(c\alpha)(r)u^p(r) = 0 \quad \text{and} \quad \lim_{r\to 0}(b\alpha)(r)u^q(r) = 0.$$

Also, by integrating the equation in (5.3) over $(0,r)$, and then multiplying by u we obtain

$$r^{N-1}|u'|^{p-1}u(r) = \lambda u(r) \int_0^r cu^{p-1}dt + u(r) \int_0^r bu^{q-1}dt, \tag{5.32}$$

hence using again the monotonicity of u and (5.32) we find that

$$r^{N-1}|u'(r)|^{p-1}u(r) \le \lambda \int_0^r cu^p dt + \int_0^r bu^q dt,$$

implying that also $r^{N-1}|u'(r)|^{p-1}u(r) \to 0$ as $r \to 0$. Finally, using the motonicity of u/h_a near 0, we observe that

$$\alpha(r)r^{N-1}|u'(r)|^p \le Cr^{N-1}|u'(r)|^{p-1}\frac{h(r)}{|h'(r)|}|u'(r)| \le Cr^{N-1}|u'(r)|^{p-1}u(r),$$

and thus also $\alpha(r)r^{N-1}|u'(r)|^p \to 0$ as $r \to 0$. Hence using (5.25) we conclude again that $E_{\delta,\mu}(r) \to 0$ as $r \to 0$. Hence, by (5.23) we conclude

$$\int_0^R s^{N-1}\mu'(s)u(s)|u'(s)|^{p-1}ds \le \lambda K \int_0^R c(s)u^p(s)ds. \tag{5.33}$$

Step 3: Let now

$$\tilde{a}(r) := r^{N-1}\mu'(r) = Const.\, r^{\frac{(N-1)(p-2)}{p-1}} h^{-d-1}(r) \quad \text{and} \quad v(r) = u^{p/(p-1)}(r).$$

Then from (5.33) we find that

$$\int_0^R \tilde{a}(s)|v'(s)|^{p-1}ds \le \lambda\Big(\frac{p}{p-1}\Big)^{p-1} K \int_0^R c(s)v^{p-1}(s)ds. \tag{5.34}$$

We will show next that condition (5.6) (or the equivalent (5.4)) yields the continuity of the embedding $V^{p-1}(\tilde{a}) \hookrightarrow L^{p-1}(c)$; i.e., $\mathcal{A}(\tilde{a}, c, p-1, p-1)$, defined by (2.7), is bounded near 0, and thus a corresponding Hardy-type inequality holds. If $p > 2$, then $\tilde{a}^{-1/(p-2)}(r) = kr^{-\frac{N-1}{p-1}} h^{\frac{d+1}{p-2}}$, where k is a positive constant. Hence, with p replaced by $p-1$ in (2.1),

$$\begin{aligned} h_{\tilde{a}}(r) &= \int_r^1 \tilde{a}^{-1/(p-2)}(s)ds = k\int_r^1 s^{-\frac{N-1}{p-1}} h^{\frac{d+1}{p-2}}(s)ds \\ &\le kh^{\frac{d+1}{p-2}}(r)h(r) \le Ch^{\frac{d+p-1}{p-2}}(r) \end{aligned}$$

for some positive constant C. Recalling that $\tau = d+p-1$, this implies that

$$\mathcal{A}(\tilde{a}, c, p-1, p-1) = \Big(\int_0^r c(s)ds\Big) h_{\tilde{a}}^{p-2}(r) \le C\Big(\int_0^r c(s)ds\Big) h^{\tau}(r).$$

If $p = 2$, then $\tilde{a}(r) = r^{N-1}\mu'(r) = Kh^{-d-1}$ for some positive constant K, thus

$$\begin{aligned} \mathcal{A}(\tilde{a}, c, 1, 1) &= \Big(\int_0^r c(t)dt\Big) \sup_{t\in[r,R]} (\tilde{a}(t))^{-1} = K^{-1}\Big(\int_0^r c(t)dt\Big) \sup_{t\in[r,R]} h^{d+1}(t) \\ &= K^{-1}\Big(\int_0^r c(t)dt\Big) h^{d+1}(r) = K^{-1}\Big(\int_0^r c(t)dt\Big) h^{\tau}(r). \end{aligned}$$

From (5.6), in both cases we find that there exists $K_0 > 0$ such that the following Hardy inequality holds (see (2.6) and (2.10)):

$$\int_0^R c(s)v^{p-1}(s)ds \le K_0 \int_0^R \tilde{a}(s)|v'|^{p-1}(s)ds \quad \Big(K_0 = \frac{1}{\lambda_1(\tilde{a}, p-1, R)}\Big) \tag{5.35}$$

and the conclusion of the theorem follows from (5.34) and (5.35); that is,

$$\lambda \ge \lambda_1(\tilde{a}, p-1, R)\frac{p}{d\rho}\Big(\frac{p-1}{p}\Big)^{p-1}. \qquad \square$$

Finally, in this section we prove Theorem 5.3. The proof of this theorem follows the ideas in [14] and [5]. We start by stating the analogue of Proposition 7.1 in [5] and Lemma 3.3 in [14] or Proposition 7.2 in [5], which in turn follow the ideas by Aubin [3] and Trudinger [21]: For $q \in (p, \rho^*]$, let

$$S_\lambda(u,q,R) := \frac{\int_0^R s^{N-1}|u'(s)|^p ds - \lambda \int_0^R c(s)|u(s)|^p ds}{\left(\int_0^R b(s)|u(s)|^q ds\right)^{p/q}},$$

and $S_\lambda(q,R) := \inf\{S_\lambda(u,q,R) \ : \ u \in V^p \setminus \{0\}\}$.

Proposition 5.2. *Under the assumptions of Theorem* 5.3, *the following holds:*

(i) *For fixed $R > 0$, $S_\lambda(\rho^*, R)$ is a nonincreasing concave function of $\lambda \in [0, \lambda_1]$.*
(ii) *$S_\lambda(\rho^*, R) > 0$ for all $\lambda \in [0, \lambda_1)$.*
(iii) *$S_\lambda(\rho^*, R) \le C(\lambda_1 - \lambda)$, where $C = \frac{\int_0^R c(s)|\varphi_1(s)|^p ds}{[\int_0^R b(s)|\varphi_1(s)|^{\rho^*} ds]^{p/\rho^*}}$, and φ_1 is a positive eigenfunction of* (1.5) *corresponding to λ_1.*

Proof. The proof follows step by step the one given in [5], except that we obtain (ii) by using the continuity of the imbedding $V^p \hookrightarrow L^{\rho^*}(b)$. □

Proposition 5.3. *Under the assumptions of Theorem* 5.3, *if*

$$0 < S_\lambda(\rho^*, R) < S_0(\rho^*, R),$$

then $S_\lambda(\rho^*, R)$ is achieved at some $u \in V^p$.

Proof. We will prove the existence of an increasing sequence $\{q_n\} \subset (p, \rho^*)$, $q_n \to \rho^*$ such that the corresponding solutions $u_n := u_{q_n} \in V^p$ of

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda c(r)|u|^{p-2}u + S_\lambda(q_n,R)b(r)|u|^{q_n-2}u \quad r \in (0,R) \\ u(r) > 0 \quad r \in (0,R), \quad ||u||_{L^{q_n}(b)} = 1, \quad S_\lambda(u,q,R) = S_\lambda(q_n,R), \end{cases} \tag{5.36}$$

converge in $C^{1,\eta'}[0,R]$ to $u \in C^{1,\eta}[0,R]$ for any $\eta' \in (0,\eta)$ for some $\eta \in (0,1)$. The existence of the solutions u_n to (5.36) follows from the compactness of the imbeddings $V^p \hookrightarrow L^p(c)$ and $V^p \hookrightarrow L^{q_n}(b)$ for $q_n < \rho^*$. Moreover, from the condition $\mathcal{A}(b,\rho^*)$ bounded and (5.10), we immediately obtain that (7.3) holds with $\theta = \min\{\rho^*, \rho_c^*\}$.

In order to prove our result we need some preliminaries.

Step 1. The function $q \to S_\lambda(q,R)$ is continuous from the left in some left open neighborhood $(\rho^* - \epsilon, \rho^*)$ of ρ^*. This follows by a step by step modification of the corresponding result in [14, 5] so we omit the proof.

Step 2. The solutions $u_n := u_{q_n} \in V^p$ of (5.36) are uniformly bounded in V^p and in $C^{1,\eta}[0,R]$ for some $\alpha \in (0,1)$. In order to prove this we use Lemma 7.3 in the appendix.

We first note that from Hölder's inequality we have

$$\begin{aligned}\int_0^R c(t)|u_n|^p dt &\le \Big(\int_0^R (cb^{-p/q_n})^{(q_n/p)'}dt\Big)^{1/(q_n/p)'}\Big(\int_0^R b(t)|u_n|^{q_n}dt\Big)^{p/q_n}\\ &= \Big(\int_0^R c(t)\Big(\frac{c}{b}(t)\Big)^{\frac{p}{q_n-p}}dt\Big)^{(q_n-p)/q_n}.\end{aligned}$$

Also, by (5.10), the definition of ρ_c^*, and assumption (5.7) on ρ^*, we have that for any $\epsilon > 0$ it holds that

$$\begin{aligned} rc(r) &\le C_1\int_0^r c(t)dt \le Const.\ r^{\frac{N-p}{p}(\rho-\epsilon)},\\ rb(r) &\ge C_1\int_0^r b(t)dt \ge Const.\ r^{\frac{N-p}{p}(\rho^*+\epsilon)},\end{aligned}$$

where $\rho = \rho_c^*$ if ρ_c^* is finite and ρ is any number strictly greater than p otherwise, and hence

$$\int_0^R c(t)\Big(\frac{c}{b}\Big)^{\frac{p}{q_n-p}}(t)dt \le Const.\ \int_0^R t^{\frac{N-p}{p}(\rho-\epsilon-\frac{(\rho^*+\epsilon-(\rho-\epsilon))p}{q_n-p})-1}dt.$$

Since

$$\rho-\epsilon-\frac{(\rho^*+\epsilon-(\rho-\epsilon))p}{\rho^*-p} = \frac{\rho^*(\rho-p)-\epsilon(\rho^*+p)}{\rho^*-p} > 0$$

for ϵ small enough, $q_n \to \rho^*$ and $\rho^* > p$, we have that there exists a positive constant C and $n_0 \in \mathbb{N}$ such that $||u_n||_{L^p(c)} \le C$ for all $n \ge n_0$. Since from the equation

$$\int_0^R r^{N-1}|u_n'|^p dr = \lambda\int_0^R c(r)|u_n|^p dr + S_\lambda(q_n, R),$$

by Step 1 we find that $\{u_n\}$ is uniformly bounded in V^p. By using Hardy's inequality, we obtain that

$$\int_0^R c(r)|u_n|^{p+\beta}dr \quad \text{is uniformly bounded for } 0 \le \beta < \rho_c^* - p. \tag{5.37}$$

From our assumptions (5.9), (5.10) and the definition of ρ_c^* we have that all the assumptions in Lemma 7.3 are satisfied and thus each u_n is bounded. We will show next that $||u_n||_{L^\infty(B(0,R))}$ is in fact uniformly bounded. By multiplying the equation by $|u_n|^\beta u_n$ we find that for any nonnegative β it holds that

$$\frac{\beta+1}{(1+\beta/p)^p}\int_0^R t^{N-1}|(u_n^{(1+\beta/p)})'|^p dt \tag{5.38}$$

$$= \int_0^R c(t)|u_n|^{p+\beta} dt + S_\lambda(q_n, R) \int_0^R b(t)|u_n|^{q_n+\beta} dt.$$

Let us set $v_n(r) = u_n^{1+\beta/p}(r)$. Then,

$$\begin{aligned}
\int_0^R b(r)|u_n(r)|^{q_n+\beta} dr &= \int_0^R b(r)|u_n(r)|^{p+\beta}|u_n(r)|^{q_n-p} dr \\
&\le \Big(\int_0^R b(r)|v_n(r)|^{q_n} dr\Big)^{p/q_n}\Big(\int_0^R b(r)|u_n(r)|^{q_n} dr\Big)^{(q_n-p)/q_n} \\
&= \Big(\int_0^R b(r)|v_n(r)|^{q_n} dr\Big)^{p/q_n} \\
&\le \Big(\int_0^R b(r)|v_n(r)|^{\rho^*} dr\Big)^{p/\rho^*}\Big(\int_0^R b(t)dt\Big)^{p(\frac{1}{q_n}-\frac{1}{\rho^*})}
\end{aligned}$$

and thus, by setting $B(R) := \int_0^R b(t)dt$, we find from (5.38) that

$$\begin{aligned}
&\frac{\beta+1}{(1+\beta/p)^p)} \int_0^R t^{N-1}|v_n'(t)|^p dt \\
&\le \lambda \int_0^R c(t)|v_n(t)|^p dt + S_\lambda(q_n, R)(B(R))^{p(\frac{1}{q_n}-\frac{1}{\rho^*})}\Big(\int_0^R b(r)|v_n(r)|^{\rho^*} dr\Big)^{p/\rho^*}.
\end{aligned}$$

By Hardy's inequality we obtain then that

$$\begin{aligned}
&S_0(\rho^*, R)\frac{\beta+1}{(1+\beta/p)^p}\Big(\int_0^R b(r)|v_n(r)|^{\rho^*} dr\Big)^{p/\rho^*} \\
&\le \lambda \int_0^R c(t)|v_n(t)|^p dt + S_\lambda(q_n, R)(B(R))^{p(\frac{1}{q_n}-\frac{1}{\rho^*})}\Big(\int_0^R b(r)|v_n(r)|^{\rho^*} dr\Big)^{p/\rho^*}
\end{aligned}$$

implying that

$$\begin{aligned}
&\Big(S_0(\rho^*, R)\frac{\beta+1}{(1+\beta/p)^p} - S_\lambda(q_n, R)(B(R))^{p(\frac{1}{q_n}-\frac{1}{\rho^*})}\Big)\Big(\int_0^R b(r)|v_n(r)|^{\rho^*} dr\Big)^{p/\rho^*} \\
&\le \lambda \int_0^R c(t)|v_n(t)|^p dt.
\end{aligned}$$

Hence, from our assumption $0 < S_\lambda(\rho^*, R) < S_0(\rho^*, R)$ and Step 1, we find that given $\varepsilon > 0$ there exists $\beta_1 > 0$ and $n_0 \in \mathbb{N}$ such that $\beta_1 + p < \rho_c^*$ and

$$\varepsilon\Big(\int_0^R b(r)|u_n(r)|^{(1+\beta_1/p)\rho^*} dr\Big)^{p/\rho^*} \le \lambda \int_0^R c(t)|u_n(t)|^{p+\beta_1} dt.$$

Thus, thanks to (5.37) we conclude that

$$\Big(\int_0^R b(r)|u_n(r)|^{(1+\beta_1/p)\rho^*}dr\Big)^{p/\rho^*} \quad \text{is uniformly bounded.} \tag{5.39}$$

Now that we have obtained this first $\beta_1 > 0$ such that (5.39) holds, we can iterate in (5.38) by taking

$$\beta_{i+1} := \min\{\frac{\rho^*}{p}\beta_i, \frac{\rho_c^*}{p}\beta_i\}$$

to obtain, as $q_n + \beta_{i+1} < (1+\beta_i/p)\rho^*$ and $p+\beta_{i+1} < (1+\beta_i/p)\rho_c^*$, and using Hardy's inequality, that

$$\int_0^R b(r)|u_n(r)|^{(1+\beta_{i+1}/p)\rho^*}dr \quad \text{and} \quad \int_0^R c(r)|u_n(r)|^{(1+\beta_{i+1}/p)\rho}dr$$

are uniformly bounded, where $\rho = \rho_c^*$ if ρ_c^* is finite and $\rho > p$ is arbitrary if $\rho_c^* = \infty$. Since $\beta_i \to \infty$ as $i \to \infty$, we find that

$$\{u_n\} \quad \text{is uniformly bounded in } L^\tau(b) \text{ and } L^\tau(c) \text{ for any } \tau \geq 1.$$

Hence, by arguing as in the proof of Lemma 7.3 we obtain from (7.8) that

$$||u_n||^{p-1}_{L^\infty(B(0,R))} \leq C(N,p,R)||f_n||_{L^s(B(0,R))} \leq Const. \text{ independent of } n$$

where $f_n = b(r)r^{1-N}u_n^{q_n-1} + \lambda c(r)r^{1-N}u_n^{p-1}$, and furthermore, the constant appearing in (7.11) is independent of n. Therefore we obtain that $\{u_n\}$ is uniformly bounded in $C^{1,\eta}(0,R)$ for some $0 < \eta < 1$, hence there exists $u \in C^{1,\eta}(0,R)$ such that $u_n \to u$ in $C^{1,\eta'}(0,R)$ for all $0 < \eta' < \eta$. We obtain thus that $||u||_{L^{\rho^*}} = 1$, implying that $u \neq 0$. Also, by letting $n \to \infty$ in (5.36) we see that $u \in V^p$ satisfies $S_\lambda(u,\rho^*,R) = S_\lambda(\rho^*,R)$, $u > 0$, and thus also

$$-(r^{N-1}|u'|^{p-2}u')' = \lambda c(r)|u|^{p-2}u + S_\lambda(\rho^*,R)b(r)|u|^{\rho^*-2}u \quad r \in (0,R). \qquad \square$$

Proof of Theorem 5.3. From Proposition 5.2, it follows that either $0 < S_\lambda(\rho^*,R) < S_0(\rho^*,R)$ for all $\lambda \in (0,\lambda_1)$, or there exists $\lambda^{**} \in (0,\lambda_1)$ such that $S_\lambda \equiv S_0(\rho^*,R)$ for $\lambda \in (0,\lambda^{**})$ and $0 < S_\lambda(\rho^*,R) < S_0(\rho^*,R)$ for $\lambda \in (\lambda^{**},\lambda_1)$. Hence the result follows from Proposition 5.3. $\square$

6. Examples

In this section we give some examples that illustrate our results. To this end we will assume now that $N \geq p$ and choose the weight functions to be such that there exists a positive constant K such that for $r \in (0,1)$ it holds that

$$\begin{cases} 0 \le c(r) \le K r^{N-p-1}(\log(2/r))^{\omega} \\ \text{with } \omega < 0 \text{ if } N > p \text{ and } \omega < -p \text{ if } N = p, \end{cases} \tag{6.1}$$

and

$$\begin{cases} 0 \le b(r) \le K r^{N-1+\nu}(\log(2/r))^{\sigma} \\ \text{with } N+\nu > 0 \text{ and any } \sigma \in \mathbb{R}, \text{ or } N+\nu = 0 \text{ and } \sigma < -1. \end{cases} \tag{6.2}$$

Then we have that $c, b \in L^1(0,1)$ and $\lim_{r\to 0} \mathcal{A}(c,p)(r) = 0$. Moreover, in the case $N > p$ the imbedding $V^p \hookrightarrow L^q(b)$, with $q \ge p$, is compact in the following cases:

(i) $N+\nu > 0$ and $q < \frac{p(N+\nu)}{N-p}$, $\sigma \in \mathbb{R}$,
(ii) $N+\nu > 0$ and $q = \frac{p(N+\nu)}{N-p}$, and $\sigma < 0$,

and in the case that $N = p$, the imbedding $V^p \hookrightarrow L^q(b)$ is compact in the following cases:

(i) $N+\nu > 0$, $q \ge p$,
(ii) $N+\nu = 0$ and $p \le q < -p'(\sigma+1)$.

Moreover, if b satisfies (6.2) and

$$\lim_{r\to 0} \frac{b(r)}{r^{N-1+\nu}|\log(r)|^{\sigma}} = b_0 > 0, \tag{6.3}$$

then

(i) for $N > p$, we have $\rho^* = \frac{p(N+\nu)}{N-p}$, if either $\nu + p > 0$, $\sigma \in \mathbb{R}$, or $\nu + p = 0$ and $\sigma < 0$;
(ii) for $N = p$, we have $\rho^* = +\infty$ if $N+\nu > 0$, and $\rho^* = -p'(\sigma+1)$ if $N+\nu = 0$ and $\sigma + p < 0$.

As a direct application of Theorem 4.1 and the calculations above we have the following result for the problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda c(r)|u|^{p-2}u + b(r)|u|^{q-2}u, \quad r \in (0,1), \\ \lim\limits_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0, \quad u(1) = 0. \end{cases} \tag{6.4}$$

Proposition 6.1. *Let c, b satisfy* (6.1), (6.2), *and* (6.3) *and let $\lambda < \lambda_1$.*

(i) *If $N > p$, assume $\omega < 0$. If $\nu + p > 0$, then $\rho^* = \frac{p(N+\nu)}{N-p}$ and problem* (6.4) *has a positive solution for any $\sigma \in \mathbb{R}$ and $p < q < \rho^*$. If in addition $\sigma < 0$ then problem* (6.4) *has a positive solution for $q = \rho^*$.*

(ii) *If $N = p$, assume $\omega < -p$. If $N + \nu > 0$, then $\rho^* = \infty$ and problem (6.4) has a positive solution for any $q > p$. If $N + \nu = 0$ and $\sigma < -p$, then $\rho^* = -p'(\sigma + 1)$ and problem (6.4) has a positive solution for any $p < q \leq \rho^*$.*

(iii) *If $N < p$, and either $N + \nu > 0$ or $N + \nu = 0$ and $\sigma < -1$, then $\rho^* = \infty$ and problem (6.4) has a positive solution for any $q > p$.*

As another application of our results we treat a more general problem than the one considered in [5]. We consider, for $r \in (0, 1)$, the problem

$$
\begin{aligned}
&-(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1+\beta}(\log(\frac{2}{r}))^{\omega}|u|^{p-2}u + r^{N-1+\nu}(\log(\frac{2}{r}))^{\sigma}|u|^{q-2}u,\\
&\lim_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0, \quad u(1) = 0.
\end{aligned}
\tag{6.5}
$$

Our next result is a direct application of Theorem 4.2 and Proposition 4.1, and generalizes Theorem 3.1 in [5].

Proposition 6.2. *Let $\lambda \geq \lambda_1$. If $N > p$ assume that*

$$
\beta + p > 0, \quad \omega \in \mathbb{R}, \tag{6.6}
$$

and if $N = p$ assume $(\beta, \omega) \neq (-N, -p)$.

Then problem (6.5) has no positive solutions for any $q > p$.

Proof. We observe first that, if $N < p$, $\mathcal{A}(c, p) \to 0$ as $r \to 0$ independently of the parameters involved. Let now $N > p$. Then

$$
\mathcal{A}(c, p) \sim r^{\beta+p}\big(\log(2/r)\big)^{\omega}
$$

which tends to 0 by (6.6). Finally, assume $N = p$. If either $N + \beta < 0$ or $N + \beta = 0$ and $\omega + 1 \geq 0$, then $c \notin L^1(0, 1)$ and thus there is nonexistence for any $\lambda > 0$, so we are left with the cases $N + \beta > 0$ or $N + \beta = 0$ and $\omega + 1 < 0$. In the first case it is clear that $\mathcal{A}(c, p) \to 0$ as $r \to 0$. In the second case

$$
\mathcal{A}(c, p) \sim \big(\log(2/r)\big)^{\omega+p},
$$

which tends to 0 as $r \to 0$ if $\omega + p < 0$ and thus in both cases Theorem 4.2 yields nonexistence of positive solutions for $\lambda \geq \lambda_1$. If $\omega + p > 0$, then $\mathcal{A}(c, p) \to \infty$ as $r \to 0$ and from Proposition 4.1 we obtain nonexistence of positive solutions for any $\lambda > 0$. □

From Theorem 5.2 we have the following result which generalizes Theorem 4.2 in [5], see also the second part of Theorem 4 in [10].

Proposition 6.3. *Assume that $N > p \geq 2$. Assume*

$$\beta + p > 0, \quad \nu + p > 0, \quad \sigma \geq 0, \quad \omega \in \mathbb{R}, \tag{6.7}$$

and

$$N - p^2 - \beta(p-1) < 0. \tag{6.8}$$

Then problem (6.5) *has no positive solutions if $q = \frac{(N+\nu)p}{N-p}$ and λ is small enough.*

Proof. Here $c(r) = r^{N-1+\beta}\big(\log(2/r)\big)^{\omega}$, $b(r) = r^{N-1+\nu}\big(\log(2/r)\big)^{\sigma}$ and hence the first two inequalities in assumption (6.7) imply that $b, c \in L^1(0,1)$ and $\mathcal{A}(b,p)$ is bounded. Moreover, $\rho_c^* = \rho_1^* = \frac{p(N+\beta)}{N-p} < \infty$ (see (5.2)) and $\rho^* = \frac{p(N+\nu)}{N-p} < \infty$. Also, (6.8) implies that the critical dimension condition (5.4) is satisfied. Finally, we observe that

$$(b\alpha)'(r) = b(r)\frac{p}{N-p}\Big(N + \nu - \frac{\sigma}{\log(2/r)}\Big) \leq \rho^*$$

by the third in (6.7). $\square$

The following example is an application of Theorem 5.3 and generalizes Theorem 7.1 in [5] (see also the first part of Theorem 4 in [10]).

Proposition 6.4. *Assume that $N > p > 1$ and*

$$\beta + 1 > 0, \quad \nu + 1 > 0, \quad \sigma,\ \omega \in \mathbb{R}. \tag{6.9}$$

Then there exixts $\lambda^{**} > 0$ such that problem (6.5) *has a positive solution if $q = \frac{p(N+\nu)}{N-p}$ and $\lambda \in (\lambda^{**}, \lambda_1)$.*

7. Appendix

We give here the proofs of some of the auxiliary results that we use throughout this work. We start with those related to the critical exponents ρ^* and ρ_c^*.

Lemma 7.1. *Let g be any positive function in $L^1(0,1)$ with $\mathcal{A}(g,p)$ bounded. Then the following statements are equivalent:*

(i) $\sup\Big\{q \geq p : \sup_{r\in(0,1)}\mathcal{A}(g,q)(r) < \infty\Big\} = p' \liminf_{r\to 0} \dfrac{|\log(\int_0^r g(s)ds)|}{\log(h(r))}$.

(ii) $p' \liminf_{r\to 0} \dfrac{|\log(\int_0^r g(s)ds)|}{\log(h(r))} \geq p$.

Proof. To prove this lemma it is enough to prove that (ii) implies (i). If

$$p' \liminf_{r\to 0} \frac{|\log(\int_0^r g(s)ds)|}{\log(h(r))} > p,$$

the proof is the same as in [12], hence we may assume that

$$p' \liminf_{r\to 0} \frac{|\log(\int_0^r g(s)ds)|}{\log(h(r))} = p.$$

We will prove that $\sup \mathcal{G} = p$ where

$$\mathcal{G} := \Big\{ q \geq p : \sup_{r\in(0,1)} \mathcal{A}(g,q)(r) < \infty \Big\}.$$

Let $q > p$ and let $p < \bar{q} < q$. Since there exists a sequence $\{r_n\} \to 0$ such that

$$p' \liminf_{n\to\infty} \frac{|\log(\int_0^{r_n} g(s)ds)|}{\log(h(r_n))} = p,$$

we may assume that

$$p' \frac{|\log(\int_0^{r_n} g(s)ds)|}{\log(h(r_n))} < \bar{q} < q$$

for all $n \in \mathbb{N}$, and hence

$$\log\Big[((h(r_n))^{\bar{q}/p'} \int_0^{r_n} g(s)ds\Big] \geq 1.$$

We deduce then that

$$(h(r_n))^{q/p'} \int_0^{r_n} g(s)ds = \Big((h(r_n))^{\bar{q}/p'} \int_0^{r_n} g(s)ds\Big)(h(r_n))^{(q-\bar{q})/p'},$$

and hence

$$(h(r_n))^{q/p'} \int_0^{r_n} g(s)ds \geq e(h(r_n))^{(q-\bar{q})/p'} \to \infty$$

as $n \to \infty$, which yields $q \notin \mathcal{G}$, and therefore $\mathcal{G} = \{p\}$. □

Lemma 7.2. *Let $N \geq p$ and let g be any positive function in $L^1(0,1)$.*

(i) *If $\mathcal{A}(g,p)(r)$ is bounded, then*

$$p' \liminf_{r\to 0} \frac{|\log(\int_0^r g(s)ds)|}{\log(h(r))} = \sup\Big\{ q \geq p : \sup_{r\in(0,1)} \mathcal{A}(g,q)(r) < \infty \Big\} := \rho.$$

(ii) *Let $g \in L^1(0,1)$ and assume that there exists a positive constant q such that $(g\alpha)'(r) \leq qg(r)$ for r sufficiently small; then $g\alpha(r) \to 0$ as $r \to 0$.*

(iii) *Let $g \in C^1(0,1)$ such that $\mathcal{A}(g,p)(r)$ is bounded; then*

$$\liminf_{r\to 0}\frac{(g\alpha)'(r)}{g(r)} \le \rho \le \limsup_{r\to 0}\frac{(g\alpha)'(r)}{g(r)}.$$

(iv) *If for some $q_0>0$ it holds that $(g\alpha)'(r)\le q_0 g(r)$ for r sufficiently small, then $\mathcal{A}(g,q)$ is decreasing for all $q\ge q_0$ and $\lim_{r\to 0}\mathcal{A}(g,q)(r)=\infty$ for all $q>q_0$.*

Proof. (i) From the definition of $\mathcal{A}(g,p)$, we have that there exists a positive constant K such that

$$h^{p/p'}(r)\int_0^r g(s)ds \le K,$$

hence,

$$p \le p'\frac{\log(K)}{\log(h(r))} + p'\frac{|\log(\int_0^r g(s)ds)|}{\log(h(r))},$$

and from here the conclusion follows from the above lemma since $\log(h(r))\to\infty$ as $r\to 0$ ($N\ge p$).

(ii) Since $g\in L^1(0,1)$ and $1/\alpha\notin L^1(0,1)$, there must exist a sequence $\{r_n\}\to 0$ such that $g\alpha(r_n)\to 0$. By integrating $(g\alpha)'(r)\le qb$ over (r_n,r), for r sufficiently small, we obtain that

$$g\alpha(r)\le g\alpha(r_n) + q\int_{r_n}^r g(s)ds,$$

and thus

$$g\alpha(r)\le q\int_0^r g(s)ds, \quad r\in(0,R). \tag{7.1}$$

(iii) By the generalized L'Hôpital's rule we have

$$\begin{aligned}\liminf_{r\to 0}\frac{(g\alpha)'(r)}{g(r)} &\le \liminf_{r\to 0}\frac{g\alpha(r)}{\int_0^r g(s)ds} \le p'\liminf_{r\to 0}\frac{|\log(\int_0^r g(s)ds)|}{\log(h(r))}\\ &\le p'\limsup_{r\to 0}\frac{|\log(\int_0^r g(s)ds)|}{\log(h(r))} \le \limsup_{r\to 0}\frac{g\alpha(r)}{\int_0^r g(s)ds} \le \limsup_{r\to 0}\frac{(g\alpha)'(r)}{g(r)},\end{aligned}$$

and thus the result follows.

(iv) Let $q\ge q_0$; we have that

$$\frac{d}{dr}\mathcal{A}(g,q)(r) = \frac{h^{q/p'}}{g\alpha}(r)\Big[\alpha(r) - q\frac{\int_0^r g(s)ds}{g(r)}\Big] \le 0,$$

where the last inequality follows from (7.1). □

Finally we prove some facts concerning the regularity of solutions to (1.1) that are used in the proof of some of our results. We have the following.

Lemma 7.3. *Let b, c be weight functions, $b,\ c \in L^1(0,1)$ and let $1 < p \leq q$ be such that the imbeddings $V^p \hookrightarrow L^p(c)$, $V^p \hookrightarrow L^q(b)$are continuous. Then we have the following.*

(i) $u \in V^p(a)$ is a weak solution of the problem (1.1) if and only if

$$-r^{N-1}|u'(r)|^{p-2}u'(r) = \lambda \int_0^r c(s)|u(s)|^{p-2}u(s)ds + \int_0^r b(s)|u(s)|^{q-2}u(s)ds, \quad r \in (0, R). \tag{7.2}$$

Furthermore, in this case,

$$\lim_{r\to 0} r^{N-1}|u'(r)|^{p-1} = 0.$$

(ii) If $q < \rho^*$, and there exists $\theta > p$ such that

$$b(r) \leq Const.r^{\frac{(N-p)\theta}{p}-1},\ c(r) \leq Const.r^{\frac{(N-p)\theta}{p}-1} \quad \text{for all } r > 0 \text{ sufficiently small} \tag{7.3}$$

and $u \in V^p(a)$ is a weak solution of the problem (1.1), then u is bounded.

(iii) If u is a bounded weak solution to (1.1) and

$$\lim_{r\to 0} r^{1-N} \int_0^r c(s)ds = \lim_{r\to 0} r^{1-N} \int_0^r b(s)ds = 0, \tag{7.4}$$

then u is a classical solution, and if in addition there exists a constant $C > 0$ such that

$$rc(r) \leq C \int_0^r c(s)ds, \quad rb(r) \leq C \int_0^r b(s)ds \tag{7.5}$$

for $r > 0$ sufficiently small, and

$$r^{-N} \int_0^r c(s)ds,\ r^{-N} \int_0^r b(s)ds \in L^\tau(0,1) \quad \text{for some } \tau > 1, \tag{7.6}$$

then $u \in C^{1,\eta}[0, R]$, for some $\eta \in (0,1)$. (Note that (7.5) and (7.6) imply (7.4).)

Proof. Part (i) follows exactly as in [12], Proposition 3.1, hence we only prove (ii) and (iii). Let $\bar{p}$ and $\bar{q}$ be such that $p \leq q < \bar{q} < \rho^*$ and $p < \bar{p} < \rho_c^*$.

We have, by the continuity of the imbeddings $V^p \hookrightarrow L^s(c)$ for $p \le s < \rho_c^*$ and $V^p \hookrightarrow L^s(b)$ for $p \le s < \rho^*$, that

$$\int_0^R c(t)|u|^{p+\beta}dt, \qquad \int_0^R c(t)|u|^{q+\beta}dt$$

are finite for $0 < \beta < \min\{\rho_c^* - p, \rho^* - q\}$, hence, by multiplying the equation in (1.1) by $|u|^\beta u$ and then integrating over $(0, R)$, we find that

$$\frac{\beta+1}{(1+\beta/p)^p)}\int_0^R t^{N-1}|(u^{(1+\beta/p)})'|^p dt = \lambda \int_0^R c(t)|u|^{p+\beta}dt + \int_0^R c(t)|u|^{q+\beta}dt, \tag{7.7}$$

implying that, for $\beta = \beta_1 = \min\{\bar{p} - p, \bar{q} - q\}$, $u^{(1+\beta_1/p)} \in V^p$. Hence, again using the continuity of the above mentioned imbeddings, we find that $u^{1+\beta_1/p} \in L^{\bar{q}}(b) \cap L^{\bar{p}}(c)$. Let now $\beta_2 := \min\{\beta_1 \frac{\bar{q}}{p}, \beta_1 \frac{\bar{p}}{p}\}$. We then obtain that

$$q + \beta_2 \le \bar{q} + \beta_1 \frac{\bar{q}}{p}, \qquad p + \beta_2 \le \bar{p} + \beta_1 \frac{\bar{p}}{p},$$

and thus, by setting $\beta = \beta_2$ in (7.7), we find that $u^{1+\beta_2/p} \in L^{\bar{q}}(b) \cap L^{\bar{p}}(c)$. In general, setting $\beta_{n+1} = \min\{\beta_n \frac{\bar{q}}{p}, \beta_n \frac{\bar{p}}{p}\}$, we find that $u^{1+\beta_{n+1}/p} \in L^{\bar{q}}(b) \cap L^{\bar{p}}(c)$, and since $\beta_n \to \infty$ as $n \to \infty$, we conclude that $u \in L^\tau(b) \cap L^\tau(c)$ for all $\tau \ge 1$.

Let now

$$f = b(r)r^{1-N}u^{q-1} + \lambda c(r)r^{1-N}u^{p-1}.$$

We will show that $f \in L^s(B(0,R))$ for some $s > N/p$, and then, using Proposition 1.3 in [14], we will obtain that

$$u \in L^\infty(B_R(0)) \quad \text{and} \ ||u||^{p-1}_{L^\infty(B_R(0))} \le C(N,p,R)||f||_{L^s(B_R(0))}. \tag{7.8}$$

We do so for the term $b(r)r^{1-N}u^{q-1}$, the term $c(r)r^{1-N}u^{p-1}$ is handled similarly. Since for any $t > 1$ it holds that

$$\begin{aligned}\int_0^R r^{(1-s)(N-1)}b^s u^{(q-1)s}dr &= \int_0^R b^{s-\frac{1}{t}}r^{(1-s)(N-1)}u^{(q-1)s}b^{\frac{1}{t}}dr \\ &\le \Big(\int_0^R b^{(s-\frac{1}{t})t'}r^{(1-s)(N-1)t'}dr\Big)^{\frac{1}{t'}}\Big(\int_0^R bu^{(q-1)ts}dr\Big)^{\frac{1}{t}},\end{aligned}$$

by using the first inequality in (7.3) we see that in order that the integral

$$\int_0^R b^{(s-\frac{1}{t})t'}r^{(1-s)(N-1)t'}dr \tag{7.9}$$

converge for some $s > N/p$ and some large t, it is sufficient that

$$\Big((N-p)\frac{\theta}{p} - N\Big)s + N > 0 \quad \text{for some } s > N/p, \tag{7.10}$$

which is clearly true since for $s = N/p$ (7.10) reads $\frac{\theta}{p} > 1$. Indeed, by fixing $s > N/p$ so that (7.10) holds, we can choose t large enough so that

$$\Big((N-p)\frac{\theta}{p} - N\Big)s + N > \frac{1}{t}(N-p)\frac{\theta}{p},$$

implying that

$$\Big((N-p)\frac{\theta}{p} - 1\Big)(s - \frac{1}{t}) + (1-s)(N-1) > -\frac{1}{t'}$$

implying the convergence of the integral in (7.9).

As for the $C^{1,\eta}$ regularity we proceed as in [5]: Clearly, assumption (7.4) and the regularity of $b,\ c$ in $(0,R)$ imply that $u \in C^1[0,R] \cap C^2(0,R]$ and $u'(0) = 0$. Also, by observing that, from (7.2),

$$g(r) := |u'|^{p-1} = r^{1-N}\Big(\lambda\int_0^r c(s)|u(s)|^{p-2}u(s)ds + \int_0^r b(s)|u(s)|^{q-2}u(s)ds\Big),$$

and thus, using the boundedness of u and (7.5),

$$|g'(r)| \le Const\ r^{-N}\Big(\int_0^r c(s)ds + \int_0^r b(s)ds\Big), \tag{7.11}$$

assumption (7.6) yields our claim by using Hölder's inequality. □

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