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MAXIMUM RECOVERABLE WORK IN THE IONOSPHERE

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Abstract. A general closed expression is derived in the frequency domain for the minimum free energy, related to a state of a linear electromagnetic conductor with memory effects in the constitutive equation of the current density, by evaluating the maximum recoverable work obtainable from the given state. The constitutive equation with memory is given by a linear relation between the current density and the electric field, expressed by a linear functional of the history of this field and a term proportional to its actual value. Another equivalent formulation is also given and used to derive explicit formulae for the particular case of a discrete spectrum model.

1. Introduction

The electromagnetic phenomena in the ionosphere can be studied by means of a hereditary theory, characterized by a local functional, which relates the current density to the electric field (see, for example, [14], [16], [15] and [2] too).

In the interesting article [12] this theory has been considered in its linear description, which occurs when the electromagnetic fields are so weak that the linear approach is justified; in particular, the constitutive equation for the current density is assumed to be a linear functional of the history of the

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electric field. In this work some thermodynamic potentials have been found as well as the maximal free energy and the maximal free enthalpy, giving several representations of each potential depending on the choice of the state variables.

In [1] the linear theory of such conductors has been considered, in the presence of thermal effects [6], to study existence, uniqueness and asymptotic stability of solutions, assuming constitutive equations with memory for the current density and the heat flux. A maximal free energy has also been introduced and used to prove the existence of a domain of dependence.

In a successive article [3] the propagation of the electromagnetic fields in the ionosphere has been studied by assuming a constitutive equation for the current density, expressed by means of two terms, one of which is a linear functional of the history of the electric field, analogous to the one considered in [12], while the other one is proportional to the actual value of this field. Therefore, this new constitutive equation reduces to the one related to a conductor without memory, if the integral kernel $\alpha'(s) \equiv 0$, while it corresponds to the conductor with memory, studied in [12], if the coefficient, which characterizes the proportionality of the current density to the instantaneous value of the electric field, is equal to zero, viz $\alpha_0 = 0$.

In this paper we consider the linear theory of electromagnetism related to the conductors considered in [3], that is, the ones characterized by the more general constitutive equation for the current density, in order to investigate the problem of finding an explicit form for the minimum free energy of these materials. This study generalizes the previous article [4], where the same problem has been examined, by using the linearized constitutive equation, which relates the current density only to the history of the electric field.

The importance of such studies, as is well known, is due to the coincidence of the minimum free energy with the maximum recoverable work, that is, the amount of energy available at any state of a material. Many works on such a subject have been done, in particular for viscoelastic solids, see for example [5], [7] and [8]; in particular, we recall [13], to which we shall refer for the method used there to solve the problem.

In this work, contrary to what occurs for viscoelastic solids, for which the minimum free energy is evaluated by means of the solution of a Wiener-Hopf integral equation of the first kind, we solve the problem by means of the solution of an analogous integral equation but now of second kind. Also such a Wiener-Hopf equation can be solved in the frequency domain by using the thermodynamic properties of the integral kernel and some theorems of factorization. Another equivalent expression for the minimum free energy is derived and used to study the discrete spectrum model material response.

The layout of the paper is as follows. In Section 2, fundamental relationships are written down. In Section 3, states, processes and prolongation of histories are defined; two particular histories are also considered. In Section 4, the notion of equivalent states is given and its characterization is derived. In Section 5, the thermodynamic restrictions on the constitutive equations are derived. The expression for the electromagnetic work, given in Section 6, is then used, in Section 7, to define another equivalence relation between states, which is proved to be equivalent to the first one. In Section 8, a first expression for the minimum free energy is obtained in the frequency domain. Another but equivalent form of this quantity is then derived in Section 9 and used in Section 10 to give the results related to the particular case of a discrete model.

2. Preliminaries and notation

Let $\mathcal{B}$ be a rigid electromagnetic solid occupying a smooth bounded, simply-connected domain $\Omega \subset \mathbb{R}^3$ with a regular boundary $\partial\Omega$. The constitutive equations are supposed to be

$$\mathbf{D}(\mathbf{x},t) = \varepsilon\mathbf{E}(\mathbf{x},t), \qquad \mathbf{B}(\mathbf{x},t) = \mu\mathbf{H}(\mathbf{x},t), \tag{2.1}$$

$$\mathbf{J}(\mathbf{x},t) = \alpha_0\mathbf{E}(\mathbf{x},t) + \int_0^{+\infty} \alpha'(s)\, {}_r\mathbf{E}^t(\mathbf{x},s)ds, \tag{2.2}$$

where $\mathbf{D}$ and $\mathbf{B}$ are the electric displacement and the magnetic induction, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, while $\mathbf{J}$ is the current density and ${}_r\mathbf{E}^t : \mathbb{R}^{++} \to \mathbb{R}^3$ denotes the past history of the electric field; that is, ${}_r\mathbf{E}^t(\mathbf{x},s) = \mathbf{E}(\mathbf{x},t-s)$ for all $s \in \mathbb{R}^{++} \equiv (0,+\infty)$; moreover, $\mathbf{x} \in \Omega$ is the position vector and t is time.

In the constitutive equation assumed for $\mathbf{J}$, together with the local functional of the past history of $\mathbf{E}$, there is the effect of the instantaneous value $\mathbf{E}(\mathbf{x},t)$; the other two equations (2.1) are supposed linear in $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$, respectively. We identify the history up to time t, $\mathbf{E}^t(\mathbf{x},s) = \mathbf{E}(\mathbf{x},t-s)$ for all $s \in \mathbb{R}^+ \equiv [0,+\infty)$, with the couple $(\mathbf{E}(\mathbf{x},t),\, {}_r\mathbf{E}^t(\mathbf{x},s))$.

The body $\mathcal{B}$ is assumed homogeneous and isotropic; thus, the electric constant and the magnetic permeability are positive constant for any point of Ω; i.e., $\varepsilon > 0$ and $\mu_0 > 0$, because of physical considerations. Moreover, we consider the relaxation function $\alpha' : \mathbb{R}^+ \to \mathbb{R}$ belonging to $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$;

therefore, the quantity

$$\alpha(t) = \alpha_0 + \int_0^t \alpha'(s)ds \tag{2.3}$$

is well defined and has the asymptotic value

$$\alpha_\infty = \lim_{t\to+\infty} \alpha(t) \neq 0. \tag{2.4}$$

The functions $\mathbf{E}$ and $\mathbf{H}$ are assumed to belong to $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, as well as ${}_r\mathbf{E}^t$.

Since later on we shall fix our attention on a specific point $\mathbf{x} \in \Omega$, we shall omit such a dependence in any physical quantity.

We shall be concerned with the frequency domain; therefore, we recall the Fourier transform and some of its properties. Thus, for any function $f : \mathbb{R} \to \mathbb{R}^n$ such a transform is defined and denoted as follows:

$$f_F(\omega) = \int_{-\infty}^{+\infty} f(s)e^{-i\omega s}ds = f_-(\omega) + f_+(\omega) \qquad \forall\omega \in \mathbb{R}, \tag{2.5}$$

where we have also introduced

$$f_-(\omega) = \int_{-\infty}^{0} f(s)e^{-i\omega s}ds, \qquad f_+(\omega) = \int_0^{+\infty} f(s)e^{-i\omega s}ds. \tag{2.6}$$

Moreover,

$$f_c(\omega) = \int_0^{+\infty} f(s)\cos\omega s\, ds, \qquad f_s(\omega) = \int_0^{+\infty} f(s)\sin\omega s\, ds \tag{2.7}$$

are the half-range Fourier cosine and sine transforms, which also hold when f is defined only on $\mathbb{R}^+$, as occurs for the transform $f_+(\omega)$; obviously, f_- holds even if f is defined on $\mathbb{R}^- \equiv (-\infty, 0]$.

If f is real, then, using * to denote the complex conjugate, we have

$$f_F^*(\omega) = f_F(-\omega). \tag{2.8}$$

It often happens that physical quantities are expressed by means of functions defined only on $\mathbb{R}^+$. Such functions can be extended in several ways on $\mathbb{R}$ and, in particular, by considering the even extension, $f(\xi) = f(-\xi)$ for all $\xi \in \mathbb{R}^{--} \equiv (-\infty, 0)$, or the odd one, $f(\xi) = -f(-\xi)$ for all $\xi \in \mathbb{R}^{--}$, or finally the causal extension, $f(\xi) = 0$ for all $\xi \in \mathbb{R}^{--}$; for these cases the Fourier transforms of the extended functions are expressed by

$$f_F(\omega) = 2f_c(\omega), \quad f_F(\omega) = -2if_s(\omega), \quad f_F(\omega) = f_c(\omega) - if_s(\omega), \tag{2.9}$$

respectively.

Let $\mathbb{C}$ denote the complex plane; then, it is useful to consider the following subsets $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}\, z \in \mathbb{R}^+\}$, $\mathbb{C}^{(+)} = \{z \in \mathbb{C} : \mathrm{Im}\, z \in \mathbb{R}^{++}\}$, that is, the upper half-planes, the first of which includes the real axis $\mathbb{R}$, while the second one excludes it. Analogously, the lower half-planes $\mathbb{C}^-$ and $\mathbb{C}^{(-)}$, including and excluding the real axis, respectively, can be defined.

We note that the two functions $f_\pm(\omega)$, defined in (2.6) and extended in $\mathbb{C}$, become two functions $f_\pm(z)$, which are analytic in $\mathbb{C}^{(\mp)}$; moreover, they become analytic in $\mathbb{C}^\mp$, in which the real axis is included, on assuming the analyticity of the Fourier transforms on $\mathbb{R}$ [13].

3. States and processes

The constitutive equations (2.1)-(2.2), which describe the electromagnetic behavior of $\mathcal{B}$, allow us to consider such a body as a simple material [18], which can be characterized by means of states and processes.

The electromagnetic state of the conductor $\mathcal{B}$, at any time t and any $\mathbf{x} \in \Omega$, is given by the triplet

$$\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t); \tag{3.1}$$

the electromagnetic process is assumed to be the map $P : [0, d) \to \mathbb{R}^3 \times \mathbb{R}^3$, supposed piecewise continuous on the time interval and defined by

$$P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau)) \qquad \forall \tau \in [0, d) \subset \mathbb{R}^+, \tag{3.2}$$

where d is the duration of the process.

Given a process P, its application to the body $\mathcal{B}$ can be done at any time $t \geq 0$ and its duration, d, usually has a finite value. The sets of the states and of the processes, which are possible for $\mathcal{B}$, are denoted by Σ and Π, respectively. With such spaces it is possible to introduce the state transition function $\rho : \Sigma \times \Pi \to \Sigma$, which associates to any initial state $\sigma^i \in \Sigma$ and any process $P \in \Pi$ the final state $\sigma^f = \rho(\sigma^i, P) \in \Sigma$. The restriction of any process P, with duration d, to a subset $[0, \tau) \subset [0, d)$ is denoted by P_τ, it has duration τ and also belongs to Π. It is useful to consider the notion of cycle, that is, any couple (σ, P) such that $\sigma(d) = \rho(\sigma(0), P) = \sigma(0)$ is said to be a cycle; for such a transformation the initial and final states coincide.

Let a process $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau))$ for all $\tau \in [0, d)$ be applied at time $t = 0$ when the state is $\sigma(0) = (\mathbf{E}_*(0), \mathbf{H}_*(0), {}_r\mathbf{E}^0_*) \in \Sigma$. In such a condition $\tau \equiv t$, thus $P = P(t)$ induces the family of states $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t)$ for any $t \in (0, d]$ defined by

$$\mathbf{E}(t) = \mathbf{E}_*(0) + \int_0^t \dot{\mathbf{E}}_P(\xi) d\xi, \quad \mathbf{H}(t) = \mathbf{H}_*(0) + \int_0^t \dot{\mathbf{H}}_P(\xi) d\xi, \tag{3.3}$$

$$ {}_r\mathbf{E}^t(s) = \begin{cases} \mathbf{E}(t-s) & \forall s \in (0,t], \\ {}_r\mathbf{E}^0_*(s-t) & \forall s > t. \end{cases} \tag{3.4} $$

We now examine the case where $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau))$ for all $\tau \in [0,d)$ is applied at time $t > 0$ when the state of the body is $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t)$. Such a process is related to the subsequent states expressed by means of

$$ \mathbf{E}_P : (0,d] \to \mathbb{R}^3, \ \mathbf{E}_P(\tau) \equiv \mathbf{E}(t+\tau) = \mathbf{E}(t) + \int_0^\tau \dot{\mathbf{E}}_P(\eta)d\eta, \tag{3.5} $$

$$ \mathbf{H}_P : (0,d] \to \mathbb{R}^3, \ \mathbf{H}_P(\tau) \equiv \mathbf{H}(t+\tau) = \mathbf{H}(t) + \int_0^\tau \dot{\mathbf{H}}_P(\eta)d\eta; \tag{3.6} $$

moreover, the following continuation of the past history, denoted by $(\mathbf{E}_P * \mathbf{E})^{t+\tau}$ and expressed by

$$ {}_r\mathbf{E}^{t+\tau}(s) = (\mathbf{E}_P * \mathbf{E})(t+\tau-s) = \begin{cases} \mathbf{E}_P(\tau-s) & \forall s \in (0,\tau], \\ \mathbf{E}(t+\tau-s) & \forall s > \tau, \end{cases} \tag{3.7} $$

allows us to evaluate the current density

$$ \mathbf{J}(t+\tau) = \alpha_0\mathbf{E}(t+\tau) + \int_0^\tau \alpha'(s)\mathbf{E}^\tau_P(s)ds + \int_\tau^{+\infty} \alpha'(s)\,{}_r\mathbf{E}^{t+\tau}(s)ds. \tag{3.8} $$

We now apply these relations to two particular histories of $\mathbf{E}$. Let

$$ \mathbf{E}^{t_{(a)}} = \begin{cases} \mathbf{E}(t) & \forall s \in [0,a], \\ {}_r\mathbf{E}^t(s-a) & \forall s > a \end{cases} \tag{3.9} $$

be the static continuation of a given history $(\mathbf{E}(t), {}_r\mathbf{E}^t)$ with a duration $a \in \mathbb{R}^{++}$. The expression of the current density after such acontinuation, using (3.5) and (3.7), is given by

$$ \mathbf{J}(t+a) = \alpha(a)\mathbf{E}(t) + \int_0^{+\infty} \alpha'(a+\xi)\,{}_r\mathbf{E}^t(\xi)d\xi, \tag{3.10} $$

which yields the definition of the following space for the past histories:

$$ \Gamma_\alpha = \left\{ {}_r\mathbf{E}^t : \mathbb{R}^{++} \to \mathbb{R}^3 : \left| \int_0^{+\infty} \alpha'(\tau+\xi)_r\mathbf{E}^t(\xi)d\xi \right| < +\infty \ \forall \tau \in \mathbb{R}^+ \right\}, \tag{3.11} $$

where t is a parameter.

We now consider the case when a constant history has been applied to $\mathcal{B}$; that is, $\mathbf{E}(t-s) = \mathbf{E}^\dagger(s) = \mathbf{E}$ for all $s \in \mathbb{R}^+$. Using (2.2), we have a constant current density

$$ \mathbf{J}(t) = \alpha_\infty\mathbf{E}, \tag{3.12} $$

where α_∞ is given by (2.4).

4. Equivalent states.

The constitutive equations (2.1)-(2.2) can be expressed by means of the notion of states by the following functionals:

$$\mathbf{D}(\sigma(t)) = \tilde{\mathbf{D}}(\mathbf{E}(t)), \quad \mathbf{B}(\sigma(t)) = \tilde{\mathbf{B}}(\mathbf{H}(t)), \quad \mathbf{J}(\sigma(t)) = \tilde{\mathbf{J}}(\mathbf{E}(t), {}_r\mathbf{E}^t), \quad (4.1)$$

the last of which is the functional $\tilde{\mathbf{J}} : \Gamma_\alpha \to \mathbb{R}^3$, where Γ_α is given in (3.11), and $\mathbf{E}$, $\mathbf{H}$ and ${}_r\mathbf{E}^t$ belong to $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$.

The response of the material is the triplet $(\mathbf{D}, \mathbf{B}, \mathbf{J})$, which depends on the states $\sigma \in \Sigma$. In such a space, Σ, an equivalent relation between its elements σcan be introduced by means of the response of the material.

Definition 4.1. *Two states $\sigma_j(t) = (\mathbf{E}_j(t), \mathbf{H}_j(t), {}_r\mathbf{E}_j^t) \in \Sigma$ $(j = 1, 2)$ are said to be equivalent if for any process $P_\tau \in \Pi$ and for any duration $\tau > 0$ we have*

$$\begin{aligned} \mathbf{D}(\rho(\sigma_1(t), P_\tau)) &= \mathbf{D}(\rho(\sigma_2(t), P_\tau)), \\ \mathbf{B}(\rho(\sigma_1(t), P_\tau) &= \mathbf{B}(\rho(\sigma_2(t), P_\tau)), \\ \mathbf{J}(\rho(\sigma_1(t), P_\tau)) &= \mathbf{J}(\rho(\sigma_2(t), P_\tau)). \end{aligned} \quad (4.2)$$

This definition implies that two equivalent states must give the same response of the material. This is possible if some conditions are satisfied, as the following theorem shows.

Theorem 4.2. *Two states $\sigma_j(t) = (\mathbf{E}_j(t), \mathbf{H}_j(t), {}_r\mathbf{E}_j^t) \in \Sigma$ $(j = 1, 2)$ are equivalent if and only if for any $P_\tau \in \Pi$ and any $\tau > 0$*

$$\mathbf{E}_1(t) = \mathbf{E}_2(t), \quad \mathbf{H}_1(t) = \mathbf{H}_2(t), \quad \int_0^{+\infty} \alpha'(\tau + \rho) \left[{}_r\mathbf{E}_1^t(\rho) - {}_r\mathbf{E}_2^t(\rho) \right] d\rho = \mathbf{0}. \quad (4.3)$$

We omit the proof since it is analogous to the one done for viscoelastic solids [8].

Two histories $(\mathbf{E}_j(t), {}_r\mathbf{E}_j^t)$ $(j = 1, 2)$, which characterize two equivalent states $\sigma_j(t)$ $(j = 1, 2)$, are said to be equivalent histories.

The zero state is the state $\sigma_0(t) = (\mathbf{0}, \mathbf{0}, \mathbf{0}^\dagger)$, where $\mathbf{0}^\dagger(s) = {}_r\mathbf{E}^t(s) = \mathbf{0}$ for all $s \in \mathbb{R}^{++}$ denotes the zero past history of $\mathbf{E}$. A state $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t)$ is equivalent to $\sigma_0(t)$ if

$$\mathbf{E}(t) = \mathbf{0}, \ \mathbf{H}(t) = \mathbf{0}, \ \int_\tau^{+\infty} \alpha'(s) \, {}_r\mathbf{E}^{t+\tau}(s) ds = \int_0^{+\infty} \alpha'(\tau + \rho) \, {}_r\mathbf{E}^t(\rho) d\rho = \mathbf{0}. \quad (4.4)$$

Hence, it follows that two equivalent states $\sigma_j(t)$ $(j = 1, 2)$ are such that their difference $\sigma_1 - \sigma_2 = (\mathbf{E}_1(t) - \mathbf{E}_2(t), \mathbf{H}_1(t) - \mathbf{H}_2(t), {}_r\mathbf{E}_1^t - {}_r\mathbf{E}_2^t)$ is a state equivalent to $\sigma_0(t)$.

Following the definition of state given by Noll in [18], two couples of histories $(\mathbf{E}_j(t), {}_r\mathbf{E}_j^t)$ $(j = 1, 2)$, such that $\mathbf{E}_1(t) = \mathbf{E}_2(t)$ while the difference of their past histories ${}_r\mathbf{E}_1^t - {}_r\mathbf{E}_2^t = {}_r\mathbf{E}^t$ satisfy $(4.3)_3$, characterize the same state $\sigma(t)$ with a given value of the magnetic field. In this sense this state can be considered as the "minimum" set of variables which yield a univocal relation between the process $P = (\dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P)$, defined in $[0, \tau)$, and the response $(\mathbf{D}(t + \tau), \mathbf{B}(t + \tau), \mathbf{J}(t + \tau))$, for every $\tau \in \mathbb{R}^{++}$. Furthermore, the state σ becomes an element of the state space Σ identified as $\Sigma = \mathbb{R}^3 \times \mathbb{R}^3 \times \Gamma_\alpha / \Gamma_\alpha^0$, where $\Gamma_\alpha / \Gamma_\alpha^0$ is the usual quotient space and Γ_α^0 is the subset of the past histories which satisfy $(4.4)_3$.

The state space, which we have now introduced, requires the boundedness of the response $(\mathbf{D}, \mathbf{B}, \mathbf{J})$ in view of (4.2). We shall consider a different point of view, which will yield a new state space related to the boundedness of the work, whence a suitable space of processes will be induced.

5. Thermodynamic restrictions on the constitutive equations

As is well known, the laws of thermodynamics yield constraints for the constitutive equations of any material ([6], [11]). In order to derive the restrictions on (2.1)-(2.2), we use the dissipation principle, which, under the hypothesis of isothermal conditions, states that the inequality

$$\oint [\dot{\mathbf{D}}(t) \cdot \mathbf{E}(t) + \dot{\mathbf{B}}(t) \cdot \mathbf{H}(t) + \mathbf{J}(t) \cdot \mathbf{E}(t)] dt \geq 0 \tag{5.1}$$

must hold for any cyclic process, the equality sign occurring only with reversible processes [9].

The consequences of the statement of this principle can be deduced by the strict inequality, to which (5.1) reduces when a periodic process is considered. Let the electromagnetic fields be given by the functions

$$\mathbf{E}(t) = \sin \omega t \, \mathbf{E}_1 + \cos \omega t \, \mathbf{E}_2, \quad \mathbf{H}(t) = \sin \omega t \, \mathbf{H}_1 + \cos \omega t \, \mathbf{H}_2 \quad \forall \omega \neq 0, \tag{5.2}$$

where $\mathbf{E}_i$ and $\mathbf{H}_i$ $(i = 1, 2)$ depend only on the spatial variable $\mathbf{x}$ and are such that $(\mathbf{E}_1, \mathbf{E}_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(\mathbf{0}, \mathbf{0})\}$ and $(\mathbf{H}_1, \mathbf{H}_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(\mathbf{0}, \mathbf{0})\}$; then a periodic history ${}_r\mathbf{E}^t$ is generated. Using (2.1)-(2.2), we can evaluate the integrand in (5.1) and integrate it over a period $d = 2\pi / \mid \omega \mid > 0$ with respect to t; the strict inequality reduces to

$$\frac{d}{2}\Big[\alpha_0 + \int_0^{+\infty} \alpha'(s)\cos\omega s\, ds\Big](\mathbf{E}_1^2 + \mathbf{E}_2^2) > 0 \qquad \forall\omega \neq 0. \tag{5.3}$$

Hence, taking into account $(2.7)_1$ and the arbitrariness of $\mathbf{E}_1^2+\mathbf{E}_2^2$, we obtain the following restriction

$$\alpha_0 + \alpha'_c(\omega) > 0 \qquad \forall\omega \in \mathbb{R}. \tag{5.4}$$

In this inequality the value $\omega = 0$ is also considered because the asymptotic value α_∞, defined by (2.4) by means of (2.3), has a nonzero value, expressible in terms of α_0 and $\alpha'_c(0) = \lim\limits_{\omega\to 0}\alpha'_c(\omega)$ as follows:

$$\alpha_\infty = \alpha_0 + \alpha'_c(0) > 0. \tag{5.5}$$

In order to consider the effect of the actual value of the electric field on the current density in (2.2), we assume

$$\alpha_0 > 0. \tag{5.6}$$

6. Electromagnetic work

The dissipation principle (5.1) involves the thermodynamic work done on any process P ([10], [11]). The work done on the material by the process $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau))$ for all $\tau \in [0, d)$, applied at time t when $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t)$ is the initial state, is a function of $(\sigma(t), P)$ given by

$$\begin{aligned} W(\sigma(t), P) &= \tilde{W}(\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) \\ &= \int_0^d \Big[\dot{\tilde{\mathbf{D}}}(\mathbf{E}_P(\tau)) \cdot \mathbf{E}_P(\tau) + \dot{\tilde{\mathbf{B}}}(\mathbf{H}_P(\tau)) \cdot \mathbf{H}_P(\tau) \\ &\quad + \tilde{\mathbf{J}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E})^{t+\tau}) \cdot \mathbf{E}_P(\tau)\Big]\, d\tau = \int_0^d \Big[\dot{\mathbf{D}}(t+\tau) \cdot \mathbf{E}_P(\tau) \\ &\quad + \dot{\mathbf{B}}(t+\tau) \cdot \mathbf{H}_P(\tau) + \mathbf{J}(t+\tau) \cdot \mathbf{E}_P(\tau)\Big]\, d\tau. \end{aligned} \tag{6.1}$$

Here, by virtue of (4.1) and (3.5)-(3.7), the derivatives with respect to τ are expressed by

$$\dot{\mathbf{D}}(t+\tau) = \varepsilon\dot{\mathbf{E}}_P(\tau), \qquad \dot{\mathbf{B}}(t+\tau) = \mu\dot{\mathbf{H}}_P(\tau), \tag{6.2}$$

while $\mathbf{J}(t+\tau)$ has been already evaluated in (3.8).

It is interesting to consider the work due to a process P. This work can be derived by assuming that the process $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau))$ of duration $d < +\infty$ is applied to the initial instant, supposed, for simplicity, to be

$t = 0$ when the initial state is $\sigma_0(0) = (\mathbf{0}, \mathbf{0}, \mathbf{0}^\dagger)$. In such a case $\tau \equiv t$ and (3.3)-(3.4), denoting by $\sigma(t) = (\mathbf{E}_0(t), \mathbf{H}_0(t), {}_r\mathbf{E}_0^t)$ the ensuing fields, reduce to

$$\mathbf{E}_0(t) = \int_0^t \dot{\mathbf{E}}_P(\xi)d\xi, \qquad \mathbf{H}_0(t) = \int_0^t \dot{\mathbf{H}}_P(\xi)d\xi, \tag{6.3}$$

$${}_r\mathbf{E}_0^t(s) = (\mathbf{E}_P * \mathbf{0}^\dagger)^t(s) = \begin{cases} \int_0^{t-s} \dot{\mathbf{E}}_P(\tau)d\tau & \forall s \in (0, t], \\ \mathbf{0} & \forall s > t; \end{cases} \tag{6.4}$$

consequently, (6.2) and (3.8) become

$$\dot{\mathbf{D}}(t) = \varepsilon\dot{\mathbf{E}}_0(t), \ \dot{\mathbf{B}}(t) = \mu\dot{\mathbf{H}}_0(t), \ \mathbf{J}(t) = \alpha_0\mathbf{E}_0(t) + \int_0^t \alpha'(s)\, {}_r\mathbf{E}_0^t(s)ds. \tag{6.5}$$

Substitution of (6.5) into $(6.1)_2$ and two integrations yield the required work

$$W(\sigma_0(0), P) = \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] + \alpha_0 \int_0^d \mathbf{E}_0^2(t)dt + \int_0^d \int_0^t \alpha'(s)\, {}_r\mathbf{E}_0^t(s)ds \cdot \mathbf{E}_0(t)dsdt. \tag{6.6}$$

Definition 6.1. *A process $P = (\dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P)$ with duration d, applied at $t = 0$ to the zero state $\sigma_0(0) = (\mathbf{0}, \mathbf{0}, \mathbf{0}^\dagger)$, is said to be a finite work process if*

$$W(\sigma_0(0), P) < +\infty. \tag{6.7}$$

We can give (6.6) a new useful form, by defining P on $\mathbb{R}^+$ rather than on $[0, d)$, where, generally, $d < +\infty$. Such an extension of the process is done by assuming $P(\tau) = (\mathbf{0}, \mathbf{0})$ for all $\tau \geq d$. Thus, if we also assume $\mathbf{E}_P(\tau) = \mathbf{0}$ and $\mathbf{H}_P(\tau) = \mathbf{0}$ for all $\tau > d$, (6.6) can be written as

$$W(\sigma_0(0), P) = \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] + \alpha_0 \int_0^{+\infty} \mathbf{E}_P^2(\tau)d\tau + \frac{1}{2}\int_0^{+\infty} \int_0^{+\infty} \alpha'(\mid \eta - \rho \mid)\mathbf{E}_P(\rho)d\rho \cdot \mathbf{E}_P(\eta)d\rho d\eta. \tag{6.8}$$

Theorem 6.2. *The work due to the application of any finite work process is positive.*

The proof easily follows by applying the Plancherel theorem to (6.8).

The relation (6.8) suggests the possibility of characterizing finite work processes by considering the following space in terms of the Fourier transform:

$$\tilde{H}_\alpha(\mathbb{R}^+,\mathbb{R}^3) = \Big\{\mathbf{E} : \mathbb{R}^+ \to \mathbb{R}^3 : \int_{-\infty}^{+\infty} [\alpha_0 + \alpha_c'(\omega)] \mid \mathbf{E}_{P+}(\omega) \mid^2 d\omega < +\infty\Big\}. \tag{6.9}$$

Such a space becomes a Hilbert space, $H_\alpha(\mathbb{R}^+,\mathbb{R}^3)$, by means of its completion with respect to the norm corresponding to the inner product

$$(\mathbf{E}_1,\mathbf{E}_2)_\alpha = \int_{-\infty}^{+\infty} [\alpha_0 + \alpha_c'(\omega)]\mathbf{E}_{1+}(\omega) \cdot [\mathbf{E}_{2+}(\omega)]^* d\omega.$$

We now consider the general case corresponding to the application of a process at a generic time to a given state.

Thus, let $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t)$ be the initial state at time $t > 0$, when a process $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau))$ for all $\tau \in [0, d)$ is applied to the body. We suppose that ${}_r\mathbf{E}^t \in \Gamma_\alpha$ and that a finite work is generated during any process P, related to $\mathbf{E}_P \in H_\alpha$ starting from the state $\sigma(t)$. We extend the process, of finite duration $d < +\infty$, on $\mathbb{R}^+$ by means of $P(\tau) = (\mathbf{0}, \mathbf{0})$ for all $\tau \geq d$; moreover, we put $\mathbf{E}_P(\tau) = \mathbf{0}$ for all $\tau > d$.

The work done by such a process can be evaluated by means of (6.1), where $\dot{\mathbf{D}}(t+\tau)$, $\dot{\mathbf{B}}(t+\tau)$ and $\mathbf{J}(t+\tau)$ are expressed by (6.2) and (3.8). Upon the substitution of these relations, the electromagnetic work assumes the following form:

$$\begin{aligned} W(\sigma(t),P) &= \int_0^d \Big[\varepsilon\dot{\mathbf{E}}_P(\tau)\cdot\mathbf{E}_P(\tau) + \mu\dot{\mathbf{H}}_P(\eta)\cdot\mathbf{H}_P(\tau)\Big] d\tau \\ &+\alpha_0 \int_0^{+\infty} \mathbf{E}_P^2(\tau)d\tau + \int_0^{+\infty}\int_0^{\tau} \alpha'(\tau-\rho)\mathbf{E}_P(\rho)\cdot\mathbf{E}_P(\tau)d\rho d\tau \\ &+ \int_0^{+\infty}\int_0^{+\infty} \alpha'(\tau+\eta)\,{}_r\mathbf{E}^t(\eta)\cdot\mathbf{E}_P(\tau)d\eta d\tau, \end{aligned} \tag{6.10}$$

which, by putting

$$\mathbf{I}^t(\tau,\mathbf{E}^t) = -\int_0^{+\infty} \alpha'(\tau+\eta)\,{}_r\mathbf{E}^t(\eta)d\eta \qquad \forall\tau\in\mathbb{R}^+ \tag{6.11}$$

and integrating in the first integral, can be written as

$$W(\sigma(t),P) = \frac{1}{2}[\varepsilon\mathbf{E}^2(t+d) + \mu\mathbf{H}^2(t+d)] - \frac{1}{2}[\varepsilon\mathbf{E}^2(t) + \mu\mathbf{H}^2(t)]$$

$$+\alpha_0 \int_0^{+\infty} \mathbf{E}_P^2(\tau) d\tau + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha'(| \tau - \rho |) \mathbf{E}_P(\rho) \cdot \mathbf{E}_P(\tau) d\rho d\tau$$
$$- \int_0^{+\infty} \mathbf{I}^t(\tau, \mathbf{E}^t) \cdot \mathbf{E}_P(\tau) d\tau. \tag{6.12}$$

7. Equivalence for states in terms of work

The equivalence relation, which we have already defined for two states in Section 4, requires the same response of the material, whatever may be the process applied to both of them. Another definition can be given by using the notion of work.

Definition 7.1. *Two states* $\sigma_j(t) = (\mathbf{E}_j(t), \mathbf{H}_j(t), {}_r\mathbf{E}_j^t)$ $(j = 1, 2)$ *are said to be w-equivalent if the equality*

$$W(\sigma_1(t), P) = W(\sigma_2(t), P) \tag{7.1}$$

holds for any process $P : [0, \tau) \to \mathbb{R}^3 \times \mathbb{R}^3$ *and any* $\tau > 0$*, applied to both of them.*

The two definitions so introduced yield the same equivalence relation as the following theorem states.

Theorem 7.2. *Two states are equivalent in the sense of Definition* 4.1 *if and only if they are w-equivalent.*

We omit the proof, which easily follows by considering $(6.1)_2$, written for two w-equivalent states $\sigma_1(t)$ and $\sigma_2(t)$ with the same process P, taking account of Definition 7.1.

8. Maximum recoverable work

The maximum recoverable work, which we can obtain by starting from a given state σ of a material, is defined by ([5], [4])

$$W_R(\sigma) = \sup \{-W(\sigma, P) : \ P \in \Pi\}, \tag{8.1}$$

where Π denotes the set of finite work processes. Such a work is a non-negative function of the state, because we obtain a null work when the zero process, which also exists in Π, is considered; moreover, it is bounded above; i.e., $W_R(\sigma) < +\infty$, as a consequence of the thermodynamics. Finally, this work, as has already been shown (see for example [13], [8]), coincides with $\psi_m(\sigma)$, the minimum free energy; i.e.,

$$\psi_m(\sigma) = W_R(\sigma). \tag{8.2}$$

To derive such a quantity we consider a process $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau))$ for all $\tau \in [0, d)$ applied to the body $\mathcal{B}$ at time $t > 0$, when the state is $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t)$. Such a process will be related to $\mathbf{E}_P(\tau) \equiv \mathbf{E}(t+\tau)$ and $\mathbf{H}_P(\tau) \equiv \mathbf{H}(t+\tau)$ for all $\tau \in (0, d]$, by means of (3.5)-(3.6), and to the continuation of the past history defined in (3.7). Moreover, we extend this process for any $\tau \in [d, +\infty)$, where it is assumed such that $P(\tau) = (\mathbf{0}, \mathbf{0})$ with $\mathbf{E}_P(\tau) = \mathbf{0}$ and $\mathbf{H}_P(\tau) = \mathbf{0}$ for all $\tau \geq d$. The work done by such a process is given by (6.12), which becomes

$$W(\sigma(t), P) = -\frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \alpha_0 \int_0^{+\infty} \mathbf{E}_P^2(\tau) d\tau$$
$$+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \alpha'(\mid \tau - \rho \mid) \mathbf{E}_P(\rho) \cdot \mathbf{E}_P(\tau) d\rho d\tau$$
$$-\int_0^{+\infty} \mathbf{I}^t(\tau, \mathbf{E}^t) \cdot \mathbf{E}_P(\tau) d\tau. \tag{8.3}$$

To evaluate the maximum recoverable work we must determine the maximum of $-W(\sigma, P)$, given by (6.1), with respect to the possible processes. Such a maximum is related to an optimum process, denoted by $P^{(m)}(\tau) = (\dot{\mathbf{E}}^{(m)}(\tau), \dot{\mathbf{H}}^{(m)}(\tau))$, where $\dot{\mathbf{E}}^{(m)}$ and $\dot{\mathbf{H}}^{(m)}$ are the derivatives of $\mathbf{E}^{(m)}$ and $\mathbf{H}^{(m)}$, in terms of which we write the electromagnetic fields $(\mathbf{E}_P(\tau), \mathbf{H}_P(\tau))$ as follows:

$$\mathbf{E}_P(\tau) = \mathbf{E}^{(m)}(\tau) + \gamma \mathbf{e}(\tau), \quad \mathbf{H}_P(\tau) = \mathbf{H}^{(m)}(\tau) + \delta \mathbf{h}(\tau) \quad \forall \tau \in \mathbb{R}^+, \tag{8.4}$$

where γ and δ denote two real parameters and $\mathbf{e}$ and $\mathbf{h}$ are two arbitrary smooth functions such that $\mathbf{e}(0) = \mathbf{0}$ and $\mathbf{h}(0) = \mathbf{0}$.

Substituting (8.4) into (8.3), we obtain the following expression:

$$-W(\sigma, P) = -\tilde{W}(\mathbf{E}(t), \mathbf{H}(t), {}_r\mathbf{E}^t; \dot{\mathbf{E}}^{(m)} + \gamma \dot{\mathbf{e}}, \dot{\mathbf{H}}^{(m)} + \delta \dot{\mathbf{h}})$$
$$= \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] - \alpha_0 \int_0^{+\infty} \left\{ \left[\mathbf{E}^{(m)}(\tau)\right]^2 + 2\mathbf{E}^{(m)}(\tau)\right.$$
$$\left. \cdot \mathbf{e}(\tau)\gamma + \mathbf{e}^2(\tau)\gamma^2 \right\} d\tau - \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \alpha'(\mid \tau - \eta \mid) \left\{ \mathbf{E}^{(m)}(\eta) \cdot \mathbf{E}^{(m)}(\tau) \right.$$
$$\left. +[\mathbf{E}^{(m)}(\eta) \cdot \mathbf{e}(\tau) + \mathbf{e}(\eta) \cdot \mathbf{E}^{(m)}(\tau)]\gamma + \mathbf{e}^2(\tau)\gamma^2 \right\} d\eta d\tau$$
$$+ \int_0^{+\infty} \mathbf{I}^t(\tau, \mathbf{E}^t) \cdot \left[\mathbf{E}^{(m)}(\tau) + \mathbf{e}(\tau)\gamma\right] d\tau, \tag{8.5}$$

where we have only the effect of the process P related to $\dot{\mathbf{E}}^{(m)}$, since the effects due to $\dot{\mathbf{H}}^{(m)}$ vanish, because the history of $\mathbf{H}$ is absent in the constitutive equations (2.1)-(2.2) and $\mathbf{H}_P(d) = \mathbf{0}$, by hypothesis.

From (8.5) we obtain

$$\frac{\partial}{\partial \gamma}[-W(\sigma, P)] \mid_{\gamma=0} = \int_0^{+\infty} \Big\{ -2\alpha_0 \mathbf{E}^{(m)}(\tau) - \int_0^{+\infty} \alpha'(\mid \tau - \eta \mid) \mathbf{E}^{(m)}(\eta) d\eta + \mathbf{I}^t(\tau, \mathbf{E}^t) \Big\} \cdot \mathbf{e}(\tau) d\tau = 0, \quad (8.6)$$

whence it follows that

$$\int_0^{+\infty} \alpha'(\mid \tau - \eta \mid) \mathbf{E}^{(m)}(\eta) d\eta + 2\alpha_0 \mathbf{E}^{(m)}(\tau) = \mathbf{I}^t(\tau, \mathbf{E}^t) \quad \forall \tau \in \mathbb{R}^+, \quad (8.7)$$

because of the arbitrariness of $\mathbf{e}$.

We observe that, contrary to what occurs for viscoelastic solids [13] but also for the electromagnetic body studied in [4], where the effect of the instantaneous value of the electric field is absent in the constitutive equation for the current density (2.2), now the equation (8.7) is a Wiener-Hopf integral equation of the second kind, instead of the first one which we have for the two previous materials. Also in this new situation such an equation can be solved by virtue of the thermodynamic properties of the kernel α' and of a factorization result.

The maximum recoverable work (8.1) can be expressed in terms of the solution $\mathbf{E}^{(m)}$ of (8.7), whatever may be the quantity $\dot{\mathbf{H}}^{(m)}$ in P and the corresponding magnetic field $\mathbf{H}^{(m)}$. Thus, (8.3), by using (8.7), yields

$$\begin{aligned}\psi_m(\sigma) = W_R(\sigma) &= \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \alpha_0 \int_0^{+\infty} \Big[\mathbf{E}^{(m)}(\tau)\Big]^2 d\tau \\ &\quad + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha'(\mid \tau - \eta \mid) \mathbf{E}^{(m)}(\eta) \cdot \mathbf{E}^{(m)}(\tau) d\eta d\tau \\ &= \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} [\alpha_0 + \alpha'_c(\omega)] \mid \mathbf{E}_+^{(m)}(\omega) \mid^2 d\omega,\end{aligned} \quad (8.8)$$

by virtue of Plancherel's theorem.

In order to solve the Wiener-Hopf integral equation (8.7), we consider a function

$$\mathbf{r}(\tau) = \begin{cases} \int_{-\infty}^{+\infty} \alpha'(\mid \tau - \eta \mid) \mathbf{E}^{(m)}(\eta) d\eta & \forall \tau \in \mathbb{R}^-, \\ \mathbf{0} & \forall \tau \in \mathbb{R}^{++}, \end{cases} \quad (8.9)$$

which can be introduced into (8.7) to give the equivalent equation

$$\int_{-\infty}^{+\infty} \alpha'(\mid \tau-\eta \mid)\mathbf{E}^{(m)}(\eta)d\eta+2\alpha_0\mathbf{E}^{(m)}(\tau) = \mathbf{I}^t(\tau,\mathbf{E}^t)+\mathbf{r}(\tau) \quad \forall\tau \in \mathbb{R}. \tag{8.10}$$

We observe that $supp(\mathbf{E}^{(m)}) \subseteq \mathbb{R}^+$, $supp(\mathbf{I}^t(\cdot,\mathbf{E}^t)) \subseteq \mathbb{R}^+$, and $supp(\mathbf{r}) \subseteq \mathbb{R}^-$; consequently, the application of Fourier's transform to (8.10), by using (2.6) and $(2.9)_1$, yields

$$2H(\omega)\mathbf{E}_+^{(m)}(\omega) = \mathbf{I}_+^t(\omega,\mathbf{E}^t) + \mathbf{r}_-(\omega), \tag{8.11}$$

where we have put

$$H(\omega) = \alpha_0 + \alpha_c'(\omega). \tag{8.12}$$

This function, by virtue of the thermodynamic constraint (5.4), is such that

$$H(\omega) > 0 \qquad \forall\omega \in \mathbb{R}; \tag{8.13}$$

moreover, using (5.5)-(5.6), it satisfies

$$H(0) = \alpha_\infty > 0, \qquad H_\infty = \lim_{|\omega|\to+\infty} H(\omega) = \alpha_0 > 0. \tag{8.14}$$

Therefore, we can factor this function as follows [13]:

$$H(\omega) = H_{(+)}(\omega)H_{(-)}(\omega), \tag{8.15}$$

where the function $H_{(+)}(\omega)$ has an extension on the complex plane $\mathbb{C}$, which is analytic in $\mathbb{C}^-$, and, analogously, $H_{(-)}(\omega)$ can be extended and gives an analytic function in $\mathbb{C}^+$.

Such a factorization allows us to derive from (8.11) the following relation:

$$H_{(+)}(\omega)\mathbf{E}_+^{(m)}(\omega) = \frac{1}{2H_{(-)}(\omega)}[\mathbf{I}_+^t(\omega,\mathbf{E}^t) + \mathbf{r}_-(\omega)], \tag{8.16}$$

where, by applying the Plemelj formulae [17],

$$\frac{\mathbf{1}}{2H_{(-)}^{(\beta)}(\omega)}\mathbf{I}_+^t(\omega,\mathbf{E}^t) = \mathbf{p}_{(-)}^t(\omega)-\mathbf{p}_{(+)}^t(\omega), \tag{8.17}$$

with, for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\mathbf{p}^t(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\mathbf{I}_+^t(\omega,\mathbf{E}^t)/[2H_{(-)}(\omega)]}{\omega - z}d\omega \tag{8.18}$$

and

$$\mathbf{p}_{(\pm)}^t(\omega) = \lim_{\beta\to0^\mp} \mathbf{p}^t(\omega + i\beta). \tag{8.19}$$

Using (8.17), (8.16) yields the equality

$$H_{(+)}(\omega)\mathbf{E}_+^{(m)}(\omega) + \mathbf{p}_{(+)}^t(\omega) = \mathbf{p}_{(-)}^t(\omega) + \frac{1}{2H_{(-)}(\omega)}\mathbf{r}_-(\omega) \equiv \mathbf{M}(\omega), \tag{8.20}$$

thus defining the function $\mathbf{M}(\omega)$, which can be considered as a function of $z \in \mathbb{C}$ and is analytic in $\mathbb{C}^-$, by virtue of its definition by means of the left-hand side, and in $\mathbb{C}^+$, because of its second definition. The Liouville theorem implies that $\mathbf{M}(\omega)$ must vanish, since both the two expressions in (8.20) vanish at infinity. Therefore, we obtain, in particular, the solution

$$\mathbf{E}_+^{(m)}(\omega) = -\frac{\mathbf{p}_{(+)}^t(\omega)}{H_{(+)}(\omega)}, \tag{8.21}$$

which, taking into account (8.12) and (8.15), can be substituted into $(8.8)_2$ together with its complex conjugate and gives the required expression of the minimum free energy

$$\psi_m(t) = \frac{1}{2}[\varepsilon\mathbf{E}^2(t) + \mu\mathbf{H}^2(t)] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mid \mathbf{p}_{(+)}^t(\omega) \mid^2 d\omega. \tag{8.22}$$

It is important to observe that (8.22) gives an expression, which coincides with the one we have derived in [4], where the same letter is used for $\mathbf{p}^t$, but this coincidence is only formal. In fact, now the quantity $\mathbf{p}^t$ is quite different, because it is influenced by the presence of the instantaneous value of the electric field in the constitutive equation for the current density, contrary to what occurs in [4]. Such an effect is evident, in particular, in the expression (8.12) for $H(\omega)$, where the coefficient α_0 appears; thus, also the factorization with $H_{(\pm)}(\omega)$, by means of which $\mathbf{p}_{(+)}^t$ has to be derived, is modified by the presence of α_0.

9. A different form for ψ_m

The expression now derived for the minimum free energy can assume a new form by considering the relation which exists between $\mathbf{p}_{(+)}^t(\omega)$ and $\mathbf{E}_+^t(\omega)$.

In order to give this equivalent formulation for the minimum free energy, we identify the past history ${}_r\mathbf{E}^t : \mathbb{R}^{++} \to \mathbb{R}^3$ with its causal extension, denoted by $\mathbf{E}^t$, by putting $\mathbf{E}^t(s) = \mathbf{0}$ for all $s \in \mathbb{R}^-$; moreover, we consider the even extensions of $\alpha'(s)$ by introducing the function $\alpha'^{(e)}(s)$ defined by

$$\alpha'^{(e)}(s) = \begin{cases} \alpha'(s) & \forall s \in \mathbb{R}^+, \\ \alpha'(-s) & \forall s \in \mathbb{R}^{--}. \end{cases} \tag{9.1}$$

Consequently, the expression (6.11) of $\mathbf{I}^t(\tau,\mathbf{E}^t)$ can be rewritten as

$$\mathbf{I}^t(\tau,\mathbf{E}^t) = -\int_0^{+\infty} \alpha'(\tau+\xi)\, {}_r\mathbf{E}^t d\xi = -\int_{-\infty}^{+\infty} \alpha'^{(e)}(\tau+\xi)\mathbf{E}^t(\xi)d\xi \quad \forall \in \mathbb{R}^+, \tag{9.2}$$

because of the causal extension, $\mathbf{E}^t : \mathbb{R} \to \mathbb{R}^3$, of ${}_r\mathbf{E}^t : \mathbb{R}^+ \to \mathbb{R}^3$. Such a function, defined for any $\tau \in \mathbb{R}^+$, can be extended on $\mathbb{R}$ by means of

$$\mathbf{I}^{t(\mathbb{R})}(\tau,\mathbf{E}^t) = -\int_{-\infty}^{+\infty} \alpha'^{(e)}(\tau+\xi)\mathbf{E}^t(\xi)d\xi = \begin{cases} \mathbf{I}^t(\tau,\mathbf{E}^t) & \forall\tau \in \mathbb{R}^+, \\ \mathbf{I}^{t(n)}(\tau,\mathbf{E}^t) & \forall\tau \in \mathbb{R}^{--}, \end{cases} \tag{9.3}$$

where

$$\mathbf{I}^{t(n)}(\tau,\mathbf{E}^t) = -\int_{-\infty}^{+\infty} \alpha'^{(e)}(\tau+\xi)\mathbf{E}^t(\xi)d\xi \qquad \forall\tau \in \mathbb{R}^{--}. \tag{9.4}$$

Moreover, if we introduce the function

$$\mathbf{E}^t_n(s) = \begin{cases} \mathbf{E}^t(-s) & \forall s \in \mathbb{R}^-, \\ \mathbf{0} & \forall s \in \mathbb{R}^{++}, \end{cases} \tag{9.5}$$

we can write (9.3) as follows:

$$\mathbf{I}^{t(\mathbb{R})}(\tau,\mathbf{E}^t) = -\int_{-\infty}^{+\infty} \alpha'^{(e)}(\tau-s)\mathbf{E}^t_n(s)ds. \tag{9.6}$$

The Fourier transform of (9.5), which is given by

$$\mathbf{E}^t_{n_F}(\omega) = \mathbf{E}^t_{n-}(\omega) = \left[\mathbf{E}^t_+(\omega)\right]^*, \tag{9.7}$$

allows us to write the Fourier transform of (9.6) in the form

$$\mathbf{I}^{t(\mathbb{R})}_F(\omega,\mathbf{E}^t) = -2\alpha'_c(\omega)\left[\mathbf{E}^t_+(\omega)\right]^*, \tag{9.8}$$

where we have used $(2.9)_1$ and, by virtue of (8.12) and (8.15),

$$\alpha'_c(\omega) = H_{(+)}(\omega)H_{(-)}(\omega) - \alpha_0. \tag{9.9}$$

Thus, (9.8) and (9.9) yield the following quantity:

$$\frac{1}{2H_{(-)}(\omega)}\mathbf{I}^{t(\mathbb{R})}_F(\omega,\mathbf{E}^t) = -H_{(+)}(\omega)\left[\mathbf{E}^t_+(\omega)\right]^* + \frac{\alpha_0}{H_{(-)}(\omega)}\left[\mathbf{E}^t_+(\omega)\right]^*. \tag{9.10}$$

Moreover, taking the Fourier transform of (9.3), we have

$$\mathbf{I}^{t(\mathbb{R})}_F(\omega,\mathbf{E}^t) = \mathbf{I}^{t(n)}_-(\omega,\mathbf{E}^t) + \mathbf{I}^t_+(\omega,\mathbf{E}^t), \tag{9.11}$$

whence, multiplying by $\frac{1}{2H_{(-)}(\omega)}$ and using (8.17), it follows that

$$\frac{1}{2H_{(-)}(\omega)}\mathbf{I}_F^{t(\mathbb{R})}(\omega,\mathbf{E}^t)=\frac{1}{2H_{(-)}(\omega)}\mathbf{I}_-^{t(n)}(\omega,\mathbf{E}^t)+\mathbf{p}^t_{(-)}(\omega)-\mathbf{p}^t_{(+)}(\omega). \quad (9.12)$$

Let us apply the Plemelj formulae to the left-hand side of this equation; we obtain

$$\frac{1}{2H_{(-)}(\omega)}\mathbf{I}_F^{t(\mathbb{R})}(\omega)=\mathbf{p}^t_{1(-)}(\omega)-\mathbf{p}^t_{1(+)}(\omega), \quad (9.13)$$

where $\mathbf{p}^t_{1(\pm)}(\omega)$, using the function

$$\mathbf{p}^t_1(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\mathbf{I}_F^{t(\mathbb{R})}(\omega,\mathbf{E}^t)/[2H_{(-)}(\omega)]}{\omega-z}d\omega, \quad (9.14)$$

are given by

$$\mathbf{p}^t_{1(\pm)}(\omega)=\lim_{\beta\to 0^{\mp}}\mathbf{p}^t_1(\omega+i\beta). \quad (9.15)$$

Substituting (9.13) into (9.12) we get the following function:

$$-\mathbf{p}^t_{1(+)}(\omega)+\mathbf{p}^t_{(+)}(\omega)=-\mathbf{p}^t_{1(-)}(\omega)+\mathbf{p}^t_{(-)}(\omega)+\frac{1}{2H_{(-)}(\omega)}\mathbf{I}_-^{t(n)}(\omega,\mathbf{E}^t)\equiv\mathbf{N}(\omega), \quad (9.16)$$

which must be equal to zero, $\mathbf{N}(\omega)=\mathbf{0}$, since it is defined in two different ways, one of which, given in the left-hand side, is analytic in $\mathbb{C}^-$, while the other one is analytic in $\mathbb{C}^+$ and, moreover, each of them vanishes at infinity.

Consequently, in particular we have

$$\mathbf{p}^t_{(+)}(\omega)=\mathbf{p}^t_{1(+)}(\omega), \quad (9.17)$$

whence we can evaluate $\mathbf{p}^t_{(+)}(\omega)$, by substituting (9.10) into (9.14)-(9.15); thus, we obtain

$$\begin{aligned}\mathbf{p}^t_{(+)}(\omega) &= -\lim_{z\to\omega^-}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{H_{(+)}(\omega')\left[\mathbf{E}^t_+(\omega')\right]^*}{\omega'-z}d\omega' \\ &\quad +\lim_{z\to\omega^-}\frac{\alpha_0}{2\pi i}\int_{-\infty}^{+\infty}\frac{\left[\mathbf{E}^t_+(\omega')\right]^*/H_{(-)}(\omega')}{\omega'-z}d\omega', \end{aligned} \quad (9.18)$$

and hence

$$\begin{aligned}\left[\mathbf{p}^t_{(+)}(\omega)\right]^* &= \lim_{w\to\omega^+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{H_{(-)}(\omega')\mathbf{E}^t_+(\omega')}{\omega'-w}d\omega' \\ &\quad -\lim_{w\to\omega^+}\frac{\alpha_0}{2\pi i}\int_{-\infty}^{+\infty}\frac{\mathbf{E}^t_+(\omega')/H_{(+)}(\omega')}{\omega'-w}d\omega'. \end{aligned} \quad (9.19)$$

We now observe that in this expression the integral with the factor α_0 vanishes because we can evaluate this integral by closing the contour in $\mathbb{C}^{(-)}$, where both the extension of $\mathbf{E}^t_+(\omega)$ and the one of $H_{(+)}(\omega)$ have no singularities and zeros and, therefore, are analytic functions in this subset $\mathbb{C}^{(-)}$, but also on $\mathbb{R}$ by virtue of our assumption for the Fourier transforms.

Thus, (9.19) reduces to

$$\left[\mathbf{p}^t_{(+)}(\omega)\right]^* = \lim_{z\to\omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{H_{(-)}(\omega')\mathbf{E}^t_+(\omega')}{\omega' - z} d\omega'. \tag{9.20}$$

In this integral we have a function, which, by applying the Plemelj formulae, is expressed by

$$H_{(-)}(\omega)\mathbf{E}^t_+(\omega) = \mathbf{q}^t_{(-)}(\omega) - \mathbf{q}^t_{(+)}(\omega), \tag{9.21}$$

where

$$\mathbf{q}^t_{(\pm)}(\omega) = \lim_{z\to\omega^\mp} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{H_{(-)}(\omega')\mathbf{E}^t_+(\omega')}{\omega' - z} d\omega'. \tag{9.22}$$

Finally, comparison of (9.22) with (9.20) yields

$$\left[\mathbf{p}^t_{(+)}(\omega)\right]^* = \mathbf{q}^t_{(-)}(\omega), \tag{9.23}$$

which allows us to obtain the new required expression for the minimum free energy,

$$\psi_m(t) = \frac{1}{2}[\varepsilon\mathbf{E}^2(t) + \mu\mathbf{H}^2(t)] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mid \mathbf{q}^t_{(-)}(\omega) \mid^2 d\omega. \tag{9.24}$$

Also for this expression, equivalent to the first one in (8.22), we see that only a formal coincidence occurs with the analogous expression derived in [4], because the same observations related to $\mathbf{p}^t_{(+)}(\omega)$, done at the end of the previous section, still hold. Therefore, the expression of ψ_m now changes since the instantaneous value of $\mathbf{E}$ is present in the constitutive equation for $\mathbf{J}$.

10. A particular form for the integral kernel

We consider the particular class of relaxation functions characterizing the discrete spectrum model, expressed by

$$\alpha(t) = \begin{cases} \alpha_\infty - \sum_{i=1}^{n} h_i e^{-\alpha_i t} & \forall t \in \mathbb{R}^+, \\ 0 & \forall t \in \mathbb{R}^{--}, \end{cases} \tag{10.1}$$

where n is a positive integer, the coefficients h_i $(i = 1, 2, ..., n)$ and the inverse decay times α_i $(i = 1, 2, ..., n)$ are assumed positive. We consider α_i ordered as follows:

$$\alpha_j < \alpha_{j+1} \quad (j = 1, 2, ..., n-1). \tag{10.2}$$

From (10.1) we obtain

$$\alpha_\infty - \alpha_0 = \sum_{i=1}^{n} h_i > 0, \tag{10.3}$$

where $\alpha_0 = \alpha(0) > 0$; that is, (5.5)-(5.6) are satisfied. Moreover, we have

$$\alpha'(t) = \sum_{i=1}^{n} \alpha_i h_i e^{-\alpha_i t}, \qquad \alpha'_F(\omega) = \sum_{i=1}^{n} \frac{\alpha_i h_i}{\alpha_i + i\omega}, \tag{10.4}$$

whence, using $(2.9)_3$, it follows that

$$\alpha'_c(\omega) = \sum_{i=1}^{n} \frac{\alpha_i^2 h_i}{\alpha_i^2 + \omega^2}, \tag{10.5}$$

which, evaluated for $\omega = 0$, yields (10.3) again, by virtue of (5.5).

From (8.12) and (10.5) we obtain

$$H(\omega) = \alpha_0 + \sum_{i=1}^{n} \frac{\alpha_i^2 h_i}{\alpha_i^2 + \omega^2} \quad \forall \omega \in \mathbb{R}, \qquad H_\infty = \alpha_0 > 0. \tag{10.6}$$

We now observe that the function $f(y) = H(\omega)$ with $y = -\omega^2$ is such that

$$f(0) = \alpha_0 + \sum_{i=1}^{n} h_i = \alpha_\infty, \quad \lim_{y \to \pm\infty} f(y) = \alpha_0^{\mp}, \quad \lim_{y \to (\alpha_i^2)^{\mp}} f(y) = \pm\infty; \tag{10.7}$$

therefore, it has n simple poles at α_i^2 $(i = 1, 2, ..., n)$ and n simple zeros at ν_i^2 $(i = 1, 2, ..., n)$, which are ordered as

$$\alpha_1^2 < \nu_1^2 < \alpha_2^2 < \nu_2^2 < ... < \alpha_n^2 < \nu_n^2. \tag{10.8}$$

Thus, (10.6) can be written as

$$H(\omega) = H_\infty \prod_{i=1}^{n} \left\{ \frac{\nu_i^2 + \omega^2}{\alpha_i^2 + \omega^2} \right\} = \alpha_0 \prod_{i=1}^{n} \left\{ \frac{(\omega - i\nu_i)(\omega + i\nu_i)}{(\omega - i\alpha_i)(\omega + i\alpha_i)} \right\}, \tag{10.9}$$

whence we obtain

$$H_{(-)}(\omega) = \alpha_0^{1/2} \prod_{i=1}^{n} \left\{ \frac{\omega + i\nu_i}{\omega + i\alpha_i} \right\} \equiv \alpha_0^{1/2} \Big(1 + i \sum_{i=1}^{n} \frac{S_i}{\omega + i\alpha_i} \Big), \tag{10.10}$$

where

$$S_i = (\nu_i - \alpha_i) \prod_{j=1, j\neq i}^{n} \left\{ \frac{\nu_j - \alpha_i}{\alpha_j - \alpha_i} \right\} \quad (i = 1, 2, ..., n). \tag{10.11}$$

Substituting $(10.10)_2$ into (9.22), we have

$$\begin{aligned} \mathbf{q}^t_{(-)}(\omega) &= \frac{\alpha_0^{1/2}}{2\pi i} \Big[\int_{-\infty}^{+\infty} \frac{\mathbf{E}^t_+(\omega')}{\omega' - \omega^+} d\omega' + i \sum_{i=1}^{n} S_i \int_{-\infty}^{+\infty} \frac{\mathbf{E}^t_+(\omega')/(\omega' - \omega^+)}{\omega' - (-i\alpha_i)} d\omega' \Big] \\ &= i\alpha_0^{1/2} \sum_{r=1}^{n} \frac{S_r}{\omega + i\alpha_r} \mathbf{E}^t_+(-i\alpha_r), \end{aligned} \tag{10.12}$$

because the first integral vanishes by closing in $\mathbb{C}^{(-)}$, where the integrand function $\mathbf{E}^t_+$ is analytic, and the other integrals are evaluated always by closing in $\mathbb{C}^{(-)}$ and taking account of the sense of integration.

From $(10.12)_2$ we have

$$[\mathbf{q}^t_{(-)}(\omega)]^* = -i\alpha_0^{1/2} \sum_{l=1}^{n} \frac{S_l}{\omega - i\alpha_l} \mathbf{E}^t_+(-i\alpha_l), \tag{10.13}$$

because, by virtue of the definition $(2.5)_1$, we obtain a real quantity for

$$\mathbf{E}^t_+(-i\alpha_l) = \int_0^{+\infty} e^{-\alpha_i s}\, {}_r\mathbf{E}^t(s) ds \equiv \left[\mathbf{E}^t_+(-i\alpha_l) \right]^*. \tag{10.14}$$

Using (10.12)-(10.13) and closing in $\mathbb{C}^{(+)}$, we derive

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mid \mathbf{q}^t_{(-)}(\omega) \mid^2 d\omega \\ &\quad = \alpha_0 \sum_{r,l=1}^{n} S_r S_l \mathbf{E}^t_+(-i\alpha_r) \cdot \mathbf{E}^t_+(-i\alpha_l) \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i/(\omega + i\alpha_r)}{\omega - i\alpha_l} d\omega \\ &\quad = \alpha_0 \sum_{r,l=1}^{n} \frac{S_r S_l}{\alpha_r + \alpha_l} \mathbf{E}^t_+(-i\alpha_r) \cdot \mathbf{E}^t_+(-i\alpha_l). \end{aligned} \tag{10.15}$$

Substituting such a result into (9.24) and using $(10.14)_1$, we deduce the required expression

$$\psi_m(t) = \frac{1}{2} [\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] \tag{10.16}$$

$$+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} 2\alpha_0 \sum_{r,l=1}^{n} \frac{S_r S_l}{\alpha_r+\alpha_l} e^{-(\alpha_r s_1+\alpha_l s_2)}\, {}_r\mathbf{E}^t(s_1)\cdot\, {}_r\mathbf{E}^t(s_2) ds_1 ds_2.$$

It is interesting to derive this result for the particular case when $n = 1$ in (10.1). In such a case (10.6) becomes

$$H(\omega) = \alpha_0 \frac{\omega^2+\alpha_1^2(1+\frac{h_1}{\alpha_0})}{\alpha_1^2+\omega^2}, \tag{10.17}$$

which has the zeros

$$\omega_{1,2} = \pm i\alpha_1\Big(1+\frac{h_1}{\alpha_0}\Big)^{1/2} = \pm i\nu_1. \tag{10.18}$$

Consequently, we have

$$H_{(-)}(\omega) = \alpha_0^{1/2}\frac{\omega+i\alpha_1(1+\frac{h_1}{\alpha_0})^{1/2}}{\omega+i\alpha_1} \equiv \alpha_0^{1/2}\Big(1+i\frac{S_1}{\omega+i\alpha_1}\Big), \tag{10.19}$$

where, by virtue of (10.3) with $n = 1$,

$$S_1 = \alpha_1\Big(\sqrt{1+\frac{h_1}{\alpha_0}}-1\Big) = \alpha_1\Big(\sqrt{\frac{\alpha_\infty}{\alpha_0}}-1\Big) = \nu_1-\alpha_1. \tag{10.20}$$

Therefore, we obtain for the minimum free energy (10.16) the simpler form

$$\psi_m(t) = \frac{1}{2}[\varepsilon\mathbf{E}^2(t)+\mu\mathbf{H}^2(t)] + \frac{1}{2}\alpha_0\alpha_1\Big(\sqrt{1+\frac{h_1}{\alpha_0}}-1\Big)^2\Big[\int_0^{+\infty} e^{-\alpha_1 s}\, {}_r\mathbf{E}^t(s) ds\Big]^2. \tag{10.21}$$

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REFERENCES

[1] G. Amendola, *Linear stability for a thermoelectromagnetic material with memory*, Quarterly of Applied Mathematics, **LIX**(1) (2001), 67–84.

[2] G. Amendola, *On a conducting electromagnetic medium*, Applicable Analysis, **81** (2002), 663–679.

[3] G. Amendola, *Asymptotic behavior for electromagnetic fields in the ionosphere*, Annali dell'Università di Ferrara, Sez. VII - Sci. Mat., **XLVIII** (2002), 165–187.

[4] G. Amendola, *The minimum free energy for an electromagnetic conductor*, Applicable Analysis, **84**(1) (2005), 67–87.

[5] S. Breuer and E.T. Onat, *On recoverable work in linear viscoelasticity*, Zeitschrift fur Angewandte Mathematik and Physik, **15** (1964), 12–21.

[6] B.D. Coleman and E.H. Dill, *Thermodynamic restrictions on the constitutive equations of electromagnetic theory*, Zeitschrift fur Angewandte Mathematik and Physik, **22** (1971), 691–702.
[7] W.A. Day, *Reversibility, recoverable work and free energy in linear viscoelasticity*, The Quarterly Journal of Mechanics and Applied Mathematics, **23** (1970), 1–15.
[8] L. Deseri, M. Fabrizio and J.M. Golden, *The concept of minimum state in viscoelasticity: new free energies and applications to PDE_S*, Archive for Rational Mechanics and Analysis, **181**(1) (2006), 43–96.
[9] M. Fabrizio, *Teoremi di approssimazione e restrizioni termodinamiche per le equazioni costitutive del campo elettromagnetico*, Atti Accademia delle Scienze dell'Istituto di Bologna, **XII**(10) (1973), 97.
[10] M. Fabrizio and A. Morro, *Thermodynamics of electromagnetic isothermal systems with memory*, J. Non-Equilibrium Thermodynanics, **22** (1997), 110–128.
[11] M. Fabrizio and A. Morro, "Electromagnetism of Continuous Media," University Press, Oxford, 2003.
[12] G. Gentili, *Thermodynamic potentials for electromagnetic fields in the ionosphere*, International Journal of Engineering Science, **11**(33) (1995), 1561–1575.
[13] J.M. Golden, *Free energy in the frequency domain: the scalar case*, Quarterly of Applied Mathematics, **LVIII**(1) (2000), 127–150.
[14] D. Graffi, *Una teoria ereditaria dell'effetto Lussemburgo*, Rendiconti del Seminario Matematico dell'Università di Padova, **7** (1956), 36–55.
[15] A.V. Gurevich, "Non-linear Phenomena in the Ionosphere," Berlin, Springer, 1994.
[16] M. Marziani, *The hereditary theoretical approach to nonlinear phenomena in the ionosphere*, Accademia Scienze Ferrara, **66** (1989), 211.
[17] N.I. Muskhelishvili, "Singular Integral Equations," Groningen, Noordhoff, 1953.
[18] W. Noll, *A new mathematical theory of simple materials*, Archive for Rational Mechanics and Analysis, **48** (1972), 1–50.