

**LOCAL WELL POSEDNESS FOR THE NONLOCAL
NONLINEAR SCHRÖDINGER EQUATION
BELOW THE ENERGY SPACE**

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Abstract. We establish local well posedness for arbitrarily large initial data in the usual Sobolev spaces $H^s(\mathbb{R})$, $s > \frac{1}{2}$, for the Cauchy problem associated to the integro-differential equation

$$\partial_t u + i\alpha \partial_x^2 u = \beta u(1 + iT_h) \partial_x(|u|^2) + i\gamma |u|^2 u,$$

where $u = u(x, t) \in \mathbb{C}$, $x, t \in \mathbb{R}$, and T_h denotes the singular operator defined by

$$T_h f(x) = \frac{1}{2h} \text{p.v.} \int_{-\infty}^{\infty} \coth\left(\frac{\pi(x-y)}{2h}\right) f(y) dy,$$

when $0 < h \leq \infty$. Note that $T_\infty = \mathcal{H}$ is the Hilbert transform. Our method of proof relies on a gauge transformation localized in positive frequencies which allows us to weaken the high-low frequencies interaction in the nonlinearity.

1. INTRODUCTION

In this paper we consider the initial-value problem (IVP) associated to two general models proposed by D. Pelinovsky and R. Grimshaw [23, 24, 25] to study the evolution of quasi-harmonic wave packets at the interface of a two-layer system, where the upper layer is shallow and the lower one is deep

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if compared to length scale of quasi-harmonic wave packets: the intermediate nonlocal nonlinear Schrödinger equation (INL-NLS)

$$\partial_t u + i\alpha \partial_x^2 u = \beta u(1 + i\mathcal{T}_h) \partial_x(|u|^2) + i\gamma |u|^2 u, \quad x, t \in \mathbb{R}, \quad (1.1)$$

and the nonlocal nonlinear Schrödinger equation (NL-NLS)

$$\partial_t u + i\alpha \partial_x^2 u = \beta u(1 + i\mathcal{H}) \partial_x(|u|^2) + i\gamma |u|^2 u, \quad x, t \in \mathbb{R}, \quad (1.2)$$

where $u = u(x, t)$ is a complex-valued function, α , β and γ are real nonnegative parameters with $\alpha \neq 0$, $\beta \neq 0$, \mathcal{H} denotes the Hilbert transform defined via Fourier transform by

$$(\mathcal{H}f)^\wedge(\xi) = -i \operatorname{sgn}(\xi) \widehat{f}(\xi), \quad \forall \xi \in \mathbb{R}, \quad \forall f \in H^s(\mathbb{R}), \quad (1.3)$$

and \mathcal{T}_h denotes the singular integral operator

$$\mathcal{T}_h f(x) = \frac{1}{2h} \text{p.v.} \int_{-\infty}^{\infty} \coth\left(\frac{\pi(x-y)}{2h}\right) f(y) dy, \quad (1.4)$$

with p.v. denoting the principal value of the integral and $0 < h \leq \infty$ is a parameter proportional to the depth of the fluid. Observe that $\lim_{h \rightarrow \infty} \mathcal{T}_h f = \mathcal{H}f$ almost everywhere.

Equation (1.2) models the evolution of quasi-harmonic wave packets at the interface of a two-layer system, one being infinitely deep, while equation (1.1) describes the same phenomenon with both fluids of finite depth. The function u represents the envelope of the waves and the coefficients α , β , γ are expressed as functions of the fluid stratification (see Appendix A in [24]).

In [23], it was proved that equations (1.1) and (1.2) with $\gamma = 0$ are integrable. Their inverse scattering theory was developed in [25]. The Cauchy problem via inverse scattering and several other physical properties of these equations have been studied by Matsuno [11, 12, 13, 14, 15, 16, 17]. In [19], the first author showed that the IVP's associated to (1.1) and (1.2) are locally well posed for small initial data in the Sobolev space $H^s(\mathbb{R})$ for $s \geq 1$. Moreover, the solutions extend globally in time, taking advantage of the quantities

$$\begin{aligned} Q(u) &= \int_{\mathbb{R}} |u|^2 dx \quad \text{and} \\ E(u) &= \int_{\mathbb{R}} \left(|\partial_x u|^2 + \frac{1}{12} |u|^6 + \frac{\gamma}{2} |u|^4 + \frac{1}{2} |u|^2 \mathcal{T} \partial_x(|u|^2) \right) dx, \end{aligned} \quad (1.5)$$

conserved along the flow of (1.1) and (1.2), where \mathcal{T} denotes \mathcal{T}_h or \mathcal{H} . On the other hand, observe that, in the case $\gamma = 0$, equation (1.2) is invariant

under the scaling transformation

$$u_\lambda(x, t) = \lambda^{\frac{1}{2}} u(\lambda x, \lambda^2 t). \quad (1.6)$$

This suggests that the critical value to study the IVP (1.2) in $H^s(\mathbb{R})$ should be $s = 0$, at least in the case $\gamma = 0$, since $\|u_\lambda(0)\|_{\dot{H}^s} = \lambda^s \|u(0)\|_{\dot{H}^s}$.

The purpose of this work is to improve the local well-posedness theory obtained in [19] without assuming any restrictions on the initial data. Our main result is the following.

Theorem 1.1. *Let $s > \frac{1}{2}$. Then, for any $\phi \in H^s(\mathbb{R})$, there exist a positive time $T = T(\|\phi\|_{H^s})$ with $T(\|\phi\|_{H^s}) \rightarrow \infty$ when $\|\phi\|_{H^s} \rightarrow 0$, a space $Z_{s,T}$ such that $Z_{s,T} \hookrightarrow C([0, T]; H^s(\mathbb{R}))$, and a unique solution u to the IVP's associated to equations (1.1) and (1.2) in $Z_{s,T}$. Furthermore, for any $T' \in (0, T)$ there exists $\epsilon > 0$ such that the flow map data-solution is Lipschitz from $\{\psi \in H^s(\mathbb{R}) : \|\psi - \phi\|_{H^s} < \epsilon\}$ into $Z_{s,T'}$.*

We also prove that the flow map data-solution is smooth, provided that the H^s norm (or the L^2 norm if $\gamma = 0$) of the initial data is small enough.

Proposition 1.1. *Let $s > \frac{1}{2}$ and $0 < T \leq 1$. Then there exist $\delta > 0$ and a subspace $\tilde{Z}_{s,T} \hookrightarrow Z_{s,T}$ such that the flow map data-solution $S : B(0, \delta) \rightarrow \tilde{Z}_{s,T}$, $\phi \mapsto u$ is smooth, where $B(0, \delta)$ is the open ball of $H^s(\mathbb{R})$ centered at the origin and of radius δ (or $B(0, \delta) = \{\phi \in H^s(\mathbb{R}) : \|\phi\|_{L^2} \leq \delta\}$ in the case $\gamma = 0$).*

In [21], Ozawa and Tsutsumi studied the IVP associated to the following derivative Schrödinger equation:

$$i\partial_t u + \partial_x^2 u = i\lambda \partial_x (|u|^2)u, \quad (1.7)$$

where $u = u(x, t)$ is a complex-valued function and λ is a real constant. They combined the fact that equation (1.7) is gauge equivalent to the equation

$$i\partial_t v + \partial_x^2 v = \frac{\lambda^2}{4} |v|^4 v$$

via the transformation

$$v = e^{-i\frac{\lambda}{2}\rho} u, \quad \text{where } \rho = \rho(x, t) = \int_{-\infty}^x |u(y, t)|^2 dy,$$

and a bilinear estimate for the null gauge form $\partial_x(u\bar{u})$, to obtain local well posedness in $H^{\frac{1}{2}}(\mathbb{R})$ for the IVP associated to (1.7). Later in [26], Takaoka proved that the IVP associated to the more general derivative Schrödinger equation

$$i\partial_t u + \partial_x^2 u = i\lambda |u|^2 \partial_x u + i\mu u^2 \partial_x \bar{u}, \quad (1.8)$$

where $u = u(x, t)$ is a complex-valued function, and λ and μ are two complex constants, is locally well posed in $H^s(\mathbb{R})$ with $s \geq \frac{1}{2}$. He used the same gauge transformation as above (see also [20]) to cancel the term $\lambda|u|^2\partial_x u$ in the nonlinearity and the Fourier restriction norm method, developed by Bourgain and Kenig, Ponce and Vega, to handle the term $\mu u^2\partial_x \bar{u}$.

It is worth noticing that the methods employed by Ozawa and Tsutsumi [21] and by Takaoka [26] do not seem to apply in the case of the IVP's associated to equations (1.1) and (1.2), since these equations are gauge equivalent to

$$\partial_t v + i\partial_x^2 v = \frac{i}{4}|v|^4 v + iv\mathcal{T}\partial_x(|v|^2) + i\gamma|v|^2 v, \quad (1.9)$$

via the same change of variables as above (note that here we took $\alpha = \beta = 1$ in (1.1) and (1.2)), and as observed in the introduction of [26], the Fourier restriction norm method seems inapplicable to the nonlinearity $|v|^2\partial_x v$ appearing implicitly in the nonlinear part of (1.9).

Our strategy is to follow the recent approach introduced by Tao for the Benjamin-Ono equation [27] and further developed by Molinet and Ribaud for the generalized Benjamin-Ono equation [18]. The method combines a frequencies localized gauge transformation which allows us to weaken the high-low frequencies interaction in the nonlinearity, together with the classical energy method, and makes use of the smoothing effect associated to the linear dispersive part of the equation, and the fractional vector-valued Leibniz's rule derived by Kenig, Ponce and Vega [10] and Molinet and Ribaud [18]. We observe that since the function u is complex-valued, we need one more equation to recover u from the gauge transformed equation.

We also give another simpler proof of Theorem 1.1 in the case of small initial data, which relies on a vector-valued Leibniz's rule estimate derived by Molinet and Ribaud in [18]. Since the method is an application of the fixed-point theorem to the integral equation, it follows by a classical argument that the flow map data-solution is smooth in the case of small initial data in $H^s(\mathbb{R})$, $s > \frac{1}{2}$. It is worth noticing that the flow map cannot be C^3 in $H^s(\mathbb{R})$ whenever $s < 0$ (see [2]).

Finally, since the proof of Theorem 1.1 makes use of compactness methods, and there is, as far as we know, no result on the well posedness for the IVPs associated to equations (1.1) and (1.2) for arbitrarily large smooth initial data, we derive such results in the appendix using Kato's theory for quasi-linear evolution equations.

The work is organized as follows: in Section 2, we list the smoothing effects associated to the linear Schrödinger equation and some fractional vector-valued Leibniz's rule estimates that are used in Section 3 to derive

the proof of Theorem 1.1. In Section 4, we give the proof of Proposition 1.1. Finally, the appendix is devoted to proving the existence of smooth solutions for equations (1.1) and (1.2).

Notation. For any positive numbers a and b , the notation $a \lesssim b$ means that there exists a positive constant c independent of a and b such that $a \leq cb$.

We will denote by $\mathcal{F}\{\phi\}$ or $\hat{\phi}$, respectively by $\mathcal{F}^{-1}\{\phi\}$ or $\check{\phi}$, the Fourier transform, respectively the inverse Fourier transform, of a function ϕ with respect to the space variable x .

For $T > 0$, $1 \leq p, q \leq \infty$, we will use the Lebesgue space-time $L_x^p L_T^q$ or $L_T^q L_x^p$, whose norm is given by

$$\|f\|_{L_x^p L_T^q} = \|\|f(x, t)\|_{L_T^q}\|_{L_x^p}, \quad \text{and} \quad \|f\|_{L_T^q L_x^p} = \|\|f(x, t)\|_{L_x^p}\|_{L_T^q}.$$

We will also denote by D_x^s and J_x^s the Riesz and Bessel potentials of order $-s$, and defined via Fourier transform by

$$D_x^s \phi = \mathcal{F}^{-1}\{|\xi|^s \hat{\phi}(\xi)\}, \quad \text{and} \quad J_x^s \phi = \mathcal{F}^{-1}\{(1 + \xi^2)^{s/2} \hat{\phi}(\xi)\}.$$

We consider the usual Sobolev space $H^s(\mathbb{R})$ with norm $\|\phi\|_{H^s} = \|J^s \phi\|_{L^2}$, and we also define $\tilde{D}_x^s = \mathcal{H}D_x^s$, where \mathcal{H} is defined in (1.3).

Let ψ be a real smooth even function satisfying $\psi(\xi) = 1$ if $|\xi| \leq 1$, and $\psi(\xi) = 0$ if $|\xi| \geq 2$. Set $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$. For any dyadic number $N = 2^j$, $j \in \mathbb{Z}_+$, we denote $\varphi_N(\xi) = \varphi(\xi/N)$, and $\varphi_0(\xi) = 1 - \sum_{N=2^j \geq 1} \varphi_N(\xi)$. Then, we define the Littlewood-Paley frequency localization operators $P_N f = \check{\varphi}_N * f$ and $P_0 f = \check{\varphi}_0 * f$. We will also denote by

$$P_+ f = \mathcal{F}^{-1}\{\chi_{[0, +\infty)}(\cdot) \hat{f}(\cdot)\} \quad \text{and} \quad P_- f = \mathcal{F}^{-1}\{\chi_{(-\infty, 0]}(\cdot) \hat{f}(\cdot)\}$$

the projection in positive and negative frequencies. Moreover, we define $P_{+hi} f = P_+((1 - P_0)f)$ and $P_{-hi} f = P_-((1 - P_0)f)$. Note that P_{-hi} and P_{+hi} define continuous operators on $L_x^p L_T^q$, for $1 \leq p, q \leq \infty$.

Finally, we observe that, since $1 + i\mathcal{H} = 2P_+$, the IVP associated to equations (1.1) and (1.2) can be written in the form

$$\begin{cases} \partial_t u + i\alpha \partial_x^2 u = 2\beta u P_+ \partial_x (|u|^2) - ik\beta u \mathcal{L}_h (|u|^2) + i\gamma |u|^2 u, \\ u(x, 0) = \phi(x), \end{cases} \tag{1.10}$$

with $x, t \in \mathbb{R}$, where $k \in \{0, 1\}$ and \mathcal{L}_h is the operator defined by

$$\mathcal{L}_h f = (\mathcal{H} - \mathcal{T}_h) \partial_x f. \tag{1.11}$$

Note that the equation in (1.10) becomes equation (1.1) when $k = 1$, and equation (1.2) when $k = 0$.

2. LINEAR ESTIMATES AND AUXILIARY LEMMAS

In this section, we list some estimates used in the proof of Theorem 1.1. We begin with the operator \mathcal{L}_h defined in (1.11).

Lemma 2.1. *Let \mathcal{L}_h be the operator defined in (1.11). Then there exists $C = C(h) > 0$ such that*

$$\|\mathcal{L}_h f\|_{H^s} \leq C \|f\|_{H^s}, \tag{2.1}$$

for any $f \in H^s(\mathbb{R})$, $s \in \mathbb{R}$.

Proof. The Fourier transform of $\mathcal{L}_h f$, computed for instance in [1], is given by $\widehat{\mathcal{L}_h f}(\xi) = (|\xi| - \xi \coth(h\xi) + \frac{1}{h})f(\xi)$. Then, expanding in a series of the term $\coth(h\xi)$, we see that $0 \leq |\xi| - \xi \coth(h\xi) + \frac{1}{h} \leq \frac{2}{h}$, which implies (2.1). \square

The following result for \mathcal{T}_h was proved in Lemma 3.1 of [19].

Lemma 2.2. *Let \mathcal{T}_h , $0 < h \leq \infty$, be the singular integral operator defined in (1.4) in the x -variable and let $1 < p, q < \infty$. Then $\mathcal{T}_h(\cdot) \in B(L^p(\mathbb{R}))$. Furthermore, for $f \in L_x^p L_T^q$, we have*

$$\|\mathcal{T}_h f\|_{L_x^p L_T^q} \leq C(p, q) \|f\|_{L_x^p L_T^q}. \tag{2.2}$$

Next, we turn to the linear homogeneous problem

$$\partial_t u + i\partial_x^2 u = 0, \quad u(\cdot, 0) = \phi,$$

whose solutions will be described by the unitary group $U(t) = \{e^{-it\partial_x^2}\}_{t=-\infty}^\infty$.

Definition 2.1. *We say that a triplet $(\alpha, p, q) \in \mathbb{R} \times [2, \infty]$ is*

(i) *1-admissible, if and only if,*

$$(\alpha, p, q) = (\frac{1}{2}, \infty, 2) \text{ or } p \in [4, \infty), q \in [2, \infty], \frac{2}{p} + \frac{1}{q} \leq \frac{1}{2}, \alpha = \frac{1}{p} + \frac{2}{q} - \frac{1}{2};$$

(ii) *2-admissible, if and only if,*

$$2 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad \alpha = \frac{1}{p} + \frac{3}{q} - 1.$$

If moreover, $4 \leq p < \infty$, we say that (α, p, q) is 2^* -admissible.

The following estimates were derived by Molinet and Ribaud in [18], interpolating previous results of Kenig, Ponce and Vega [7], [8].

Lemma 2.3. *Let be $(\alpha, p, q) \in \mathbb{R} \times [2, \infty]$ and $0 < T < 1$.*

(i) *If (α, p, q) is 1-admissible, then*

$$\|D_x^\alpha U(t)\phi\|_{L_x^p L_T^q} \lesssim \|\phi\|_{L^2}, \quad \forall \phi \in S(\mathbb{R}). \tag{2.3}$$

(ii) If (α, p, q) is 2-admissible, then

$$\|J_x^\alpha U(t)\phi\|_{L_x^p L_t^q} \lesssim \|\phi\|_{L^2}, \quad \forall \phi \in S(\mathbb{R}). \tag{2.4}$$

(iii) If (α, p, q) is 2^* -admissible, then

$$\|D_x^\alpha U(t)\phi\|_{L_x^p L_t^q} \lesssim T^{\frac{1}{4} - \frac{1}{2q}} \|\phi\|_{L^2}, \quad \forall \phi \in S(\mathbb{R}). \tag{2.5}$$

We will also make use of the maximal function estimate in L^p , $2 \leq p < 4$ for $U(t)$.

Lemma 2.4. For any $s > 1/2$ and $0 < T \leq 1$, we have that

$$\|U(t)\phi\|_{L_x^2 L_T^\infty} \lesssim \|\phi\|_{H^s}, \tag{2.6}$$

$$\|J_x^{-(\frac{1}{p}+)} U(t)\phi\|_{L_x^p L_T^\infty} \lesssim \|\phi\|_{L^2}, \quad \forall 2 \leq p < 4. \tag{2.7}$$

Proof. Estimate (2.6) was derived by Kenig, Ponce, and Vega in [9]. Estimate (2.7) follows by interpolation as in [18]. \square

Remark 2.1. It is interesting to observe that the restriction $s > \frac{1}{2}$ in Theorem 1.1 appears in estimate (2.6).

The following nonhomogeneous version of Lemma 2.3 was established in [18].

Lemma 2.5. Let $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, $(r_1, r_2) \in \mathbb{R}_+^2$ and $1 \leq p_1, q_1, p_2, q_2 \leq \infty$ be such that, given $\phi \in S(\mathbb{R})$,

$$\|D_x^{\alpha_1} U(t)\phi\|_{L_x^{p_1} L_T^{q_1}} \lesssim T^{r_1} \|\phi\|_{L^2}, \tag{2.8}$$

$$\|D_x^{\alpha_2} U(t)\phi\|_{L_x^{p_2} L_T^{q_2}} \lesssim T^{r_2} \|\phi\|_{L^2}. \tag{2.9}$$

Then, for all $F \in S(\mathbb{R}^2)$,

$$\left\| D_x^{\alpha_2} \int_0^t U(t-\tau) F(\cdot, \tau) d\tau \right\|_{L_T^\infty L_x^2} \lesssim T^{r_2} \|F\|_{L_x^{\tilde{p}_2} L_T^{\tilde{q}_2}}, \tag{2.10}$$

$$\left\| D_x^{\alpha_1 + \alpha_2} \int_0^t U(t-\tau) F(\cdot, \tau) d\tau \right\|_{L_x^{p_1} L_T^{q_1}} \lesssim T^{r_1 + r_2} \|F\|_{L_x^{\tilde{p}_2} L_T^{\tilde{q}_2}}, \tag{2.11}$$

provided

$$\min(p_1, q_1) > \max(\tilde{p}_2, \tilde{q}_2) \quad \text{or} \quad (q_1 = \infty \text{ and } \tilde{p}_2, \tilde{q}_2 < \infty), \tag{2.12}$$

where \tilde{p}_2, \tilde{q}_2 are defined by $\frac{1}{\tilde{p}_2} = 1 - \frac{1}{p_2}$ and $\frac{1}{\tilde{q}_2} = 1 - \frac{1}{q_2}$.

Finally, we list some vector-valued Leibniz rules for fractional derivatives. In particular estimate (2.17) will be fundamental in the proof of Theorem 1.1.

Lemma 2.6. *Let $\alpha \in (0, 1)$, $\alpha_1, \alpha_2 \in [0, \alpha]$ such that $\alpha = \alpha_1 + \alpha_2$. Then:*

(i) *If $1 < p < \infty$, we have*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p} \lesssim \|g\|_{L_x^\infty} \|D_x^\alpha f\|_{L_x^p}. \tag{2.13}$$

(ii) *If $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we have*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^q} \leq c \|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}. \tag{2.14}$$

Moreover, if $\alpha_1 = 0$, (2.14) still holds with $q_1 = \infty$.

(iii) *If $p_1, p_2, q_1, q_2 \in (1, \infty)$ with $1 = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{2} = \frac{1}{q_1} + \frac{1}{q_2}$, we have*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^1 L_T^2} \leq c \|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}. \tag{2.15}$$

Note that (i), (ii), and (iii) are still valid with \tilde{D}_x^α or J_x^α instead of D_x^α .

(iv) *If $0 \leq \beta < 1 - \alpha$, and if $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we have*

$$\|D_x^\beta([D_x^\alpha, f]g)\|_{L_x^p L_T^q} \lesssim \|g\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha+\beta} f\|_{L_x^{p_2} L_T^{q_2}}. \tag{2.16}$$

Moreover, if $\beta > 0$, then $q_1 = \infty$ is allowed.

(v) *If $\gamma_1 \geq \alpha$ and $\gamma_1 + \gamma_2 = \alpha + \beta$, and if $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we have*

$$\|D_x^\alpha P_\pm(fP_\mp D_x^\beta g)\|_{L_x^p L_T^q} \leq c \|D_x^{\gamma_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\gamma_2} g\|_{L_x^{p_2} L_T^{q_2}}. \tag{2.17}$$

Proof. Estimates (2.13), (2.14), and (2.15) were proved by Kenig, Ponce and Vega (Theorems A.8, A13 and A12 in [10]), while estimates (2.16) and (2.17) were derived by Molinet and Ribaud (Lemmas 3.4 and 3.5 in [18]). \square

The proof of the next lemma follows the lines of Lemma 3.5 in [18].

Lemma 2.7. *Let $\alpha \in (0, 1]$ and $\rho = \rho(x, t) = \int_{-\infty}^x |u(y, t)|^2 dy$. Then*

$$\|D_x^\alpha(e^{\pm i\rho} g)\|_{L_x^p L_T^q} \leq c \|u\|_{L_x^{p_1} L_T^{q_1}}^2 \|g\|_{L_x^{p_2} L_T^{q_2}} + \|J^\alpha g\|_{L_x^p L_T^q}, \tag{2.18}$$

for $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$, satisfying $\frac{1}{p} = \frac{2}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{2}{q_1} + \frac{1}{q_2}$.

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 consists in deriving an H^s energy estimate on smooth solutions to equations (1.1) and (1.2), so that the existence of solutions can be concluded by a classical compactness method as was done in [8] or in [18].

Let $s > \frac{1}{2}$ and $\phi \in H^s(\mathbb{R})$. We will only consider the most difficult case when $\frac{1}{2} < s < \frac{9}{10}$. Then we consider a smooth solution u of the IVP (1.10) obtained in Theorem 5.1 of the appendix and defined on the time interval $[0, T]$, with $0 < T < 1$. Note that, without loss of generality, we can always assume that $\alpha = \beta = \gamma = k = 1$, so that u satisfies the integral equation

$$u(t) = U(t)\phi + \int_0^t U(t - \tau)(2uP_+\partial_x(|u|^2) - iu\mathcal{L}_h(|u|^2) + i|u|^2u)d\tau, \quad (3.1)$$

for all $t \in [0, T]$. We define our resolution space $Z_{s,T}$ by

$$Z_{s,T} = \{u \in C([0, T]; H^s(\mathbb{R})) : \|u\|_{Z_{s,T}} < \infty\}, \quad (3.2)$$

where

$$\begin{aligned} \|u\|_{Z_{s,T}} := \max \left\{ & \|u\|_{L_T^\infty H_x^s}, \|D_x^{s+\frac{1}{2}}u\|_{L_x^\infty L_T^2}, \|u\|_{L_x^2 L_T^\infty}, \|D_x^s u\|_{L_{x,T}^6}, \right. \\ & T^{-\frac{1}{10}} \|D_x^{s+\frac{1}{10}}u\|_{L_x^5 L_T^{\frac{10}{3}}}, \|\partial_x u\|_{L_x^\infty L_T^2}, \|D_x^{\frac{1}{2}}u\|_{L_{T,x}^6}, \|J_x^{\frac{9}{10}}u\|_{L_x^{20} L_T^{\frac{20}{9}}}, \\ & \|u\|_{L_x^{\frac{20}{7}} L_T^{\frac{20}{3}}}, \|u\|_{L_x^4 L_T^\infty}, \|u\|_{L_x^{\frac{40}{7}} L_T^\infty}, \|u\|_{L_x^{\frac{16}{3}} L_T^\infty}, \|u\|_{L_x^{40} L_T^\infty}, \\ & \left. \|u\|_{L_x^{80} L_T^\infty}, \|u\|_{L_x^{\frac{6}{3-2\epsilon}} L_T^\infty}, \|u\|_{L_x^{\frac{4}{2-\epsilon}} L_T^\infty}, \|J_x^{1+\frac{3\epsilon}{2}}u\|_{L_x^{\frac{6}{\epsilon}} L_T^{\frac{6}{3-2\epsilon}}} \right\}, \end{aligned}$$

and

$$0 < \epsilon < \frac{1}{3}(s - \frac{1}{2}). \quad (3.3)$$

The main estimate is the following.

Proposition 3.1. *Let $\frac{1}{2} < s < \frac{9}{10}$, $0 < T \leq 1$, $\phi \in H^\infty(\mathbb{R})$ and $u \in C([0, T]; H^\infty(\mathbb{R}))$ be a smooth solution to the IVP (1.10). Then, there exist $\theta > 0$, a polynomial q , and a polynomial p whose factors are at least of order two, such that*

$$\|u\|_{Z_{s,T}} \lesssim q(\|\phi\|_{H^s})\|\phi\|_{H^s} + T^\theta p(\|u\|_{Z_{s,T}})\|u\|_{Z_{s,T}}. \quad (3.4)$$

We will split the proof of Proposition 3.1 into various technical lemmas. First we estimate the linear part of (3.1) in $Z_{s,T}$.

Lemma 3.1. *Let $s > \frac{1}{2}$ and $0 < T \leq 1$. Then, we have that*

$$\|U(t)\phi\|_{Z_{s,T}} \lesssim \|\phi\|_{H^s}. \quad (3.5)$$

Proof. We observe that the triplets $(\frac{1}{2}, \infty, 2)$, $(0, 6, 6)$, $(-\frac{1}{4}, 4, \infty)$, $(-\frac{13}{40}, \frac{40}{7}, \infty)$, $(-\frac{5}{16}, \frac{16}{3}, \infty)$, $(-\frac{19}{40}, 40, \infty)$, and $(-\frac{39}{80}, 80, \infty)$ are 1-admissible, $(-\frac{1}{5}, \frac{20}{7}, \frac{20}{3})$, $(\frac{2}{5}, 20, \frac{20}{9})$, and $(\frac{1}{2} - \frac{5\epsilon}{6}, \frac{6}{\epsilon}, \frac{6}{3-2\epsilon})$ are 2-admissible, and $(\frac{1}{10}, 5, \frac{10}{3})$

is 2^* -admissible. Therefore estimate (3.5) follows from Lemma 2.3 and Lemma 2.4. \square

Next, we estimate the two last terms of the right-hand side of (3.1) in $Z_{s,T}$.

Lemma 3.2. *Let $s > \frac{1}{2}$. Then, we have that*

$$\left\| \int_0^t U(t-\tau)(u\mathcal{L}_h(|u|^2) + |u|^2u)d\tau \right\|_{Z_{s,T}} \lesssim T\|u\|_{Z_{s,T}}^3. \tag{3.6}$$

Proof. We have, using Minkowski’s inequality, estimate (3.5), and Hölder’s inequality in time that

$$\begin{aligned} \left\| \int_0^t U(t-\tau)(u\mathcal{L}_h(|u|^2) + |u|^2u)d\tau \right\|_{Z_{s,T}} \\ \lesssim T(\|u\mathcal{L}_h(|u|^2)\|_{L_T^\infty H_x^s} + \| |u|^2u \|_{L_T^\infty H_x^s}). \end{aligned} \tag{3.7}$$

Moreover, we deduce, using the fact that $H^s(\mathbb{R})$ is a Banach algebra for $s > \frac{1}{2}$, and estimate (2.1), that the right-hand side of (3.7) is bounded by $T\|u\|_{L_T^\infty H_x^s}^3$, which implies (3.6). \square

Finally, we decompose the remaining term in the right-hand side of (3.1) in terms of its positive and negative frequencies:

$$\int_0^t U(t-\tau)(uP_+\partial_x(|u|^2)d\tau = \sum_{\kappa=0,-hi,+hi} \int_0^t U(t-\tau)P_\kappa(uP_+\partial_x(|u|^2)d\tau. \tag{3.8}$$

Note that we use the operators $P_{\pm hi}$ instead of the operators P_\pm , since those are continuous in the space $L_x^1L_T^2$ that we will have to use in our approach. We first treat the cases of negative and low frequencies.

Lemma 3.3. *Let $\frac{1}{2} < s < \frac{9}{10}$. Then, we have that*

$$\left\| \int_0^t U(t-\tau)P_0(uP_+\partial_x(|u|^2)d\tau \right\|_{Z_{s,T}} \lesssim T^{\frac{1}{2}}\|u\|_{Z_{s,T}}^3, \tag{3.9}$$

$$\left\| \int_0^t U(t-\tau)P_{-hi}(uP_+\partial_x(|u|^2)d\tau \right\|_{Z_{s,T}} \lesssim T^{\frac{9}{40}}\|u\|_{Z_{s,T}}^3. \tag{3.10}$$

Proof. First we observe using estimate (3.5), Bernstein’s, Minkowski’s, and Hölder’s inequalities that

$$\begin{aligned} \left\| \int_0^t U(t-\tau)P_0(uP_+\partial_x(|u|^2)d\tau \right\|_{Z_{s,T}} &\lesssim T^{\frac{1}{2}}\|uP_+\partial_x(|u|^2)\|_{L_{T,x}^2} \\ &\lesssim T^{\frac{1}{2}}\|u\|_{L_x^4L_T^\infty}^2\|\partial_xu\|_{L_x^\infty L_T^2}, \end{aligned}$$

which implies estimate (3.9).

On the other hand, we get, combining Berstein’s inequality, estimate (3.5), Lemma 2.5, and the fact that the triplet $(\frac{1}{10}, 5, \frac{10}{3})$ is 2^* -admissible, that

$$\begin{aligned} \left\| \int_0^t U(t-\tau)P_{-hi}(uP_+\partial_x(|u|^2))d\tau \right\|_{Z_{s,T}} & \\ & \lesssim T^{\frac{1}{10}} \|D_x^{s-\frac{1}{10}}P_-(uP_+\partial_x(|u|^2))\|_{L_x^{\frac{5}{4}}L_T^{\frac{10}{7}}}. \end{aligned} \tag{3.11}$$

Moreover, we deduce, applying estimate (2.17), observing that $s - \frac{1}{10} < \frac{9}{10}$, and applying estimate (2.14) with $q_1 = \infty$, and Hölder’s inequality to the right-hand side of (3.11) that

$$\begin{aligned} \left\| \int_0^t U(t-\tau)P_{-hi}(uP_+\partial_x(|u|^2))d\tau \right\|_{Z_{s,T}} & \lesssim T^{\frac{1}{10}} \|D_x^{\frac{9}{10}}u\|_{L_x^{20}L_T^{\frac{20}{9}}} \|D_x^s(|u|^2)\|_{L_x^{\frac{4}{3}}L_T^4} \\ & \lesssim T^{\frac{1}{10}} \|D_x^{\frac{9}{10}}u\|_{L_x^{20}L_T^{\frac{20}{9}}} \|u\|_{L_x^2L_T^\infty} \|D_x^s u\|_{L_{T,x}^4} \\ & \lesssim T^{\frac{9}{40}} \|D_x^{\frac{9}{10}}u\|_{L_x^{20}L_T^{\frac{20}{9}}} \|u\|_{L_x^2L_T^\infty} \|D_x^s u\|_{L_{T,x}^6}^{\frac{3}{4}} \|u\|_{L_T^\infty H_x^s}, \end{aligned} \tag{3.12}$$

which concludes the proof of (3.10). □

Note that the nonlinear term $P_{+hi}(uP_+\partial_x(|u|^2))$ is more difficult to estimate in $Z_{s,T}$, since (2.17) does not apply directly. Therefore we have to rewrite it introducing the new variables

$$v = P_+(e^{i\rho}u) \text{ and } w = P_-u, \quad \text{where } \rho(x, t) = \int_{-\infty}^x |u(y, t)|^2 dy. \tag{3.13}$$

Thus we get, using Leibniz’s rule and the identities $P_+ + P_- = 1$ and $\overline{P_-f} = P_+\overline{f}$, that

$$\begin{aligned} P_{+hi}(\overline{uP_+\partial_x(|u|^2)}) &= P_{+hi}(u\partial_x(|u|^2)) - P_{+hi}(u\partial_xP_-(|u|^2)) \\ &= P_{+hi}(|u|^2e^{-i\rho}e^{i\rho}\partial_xu) + P_{+hi}(u^2\partial_xP_+\overline{u}) + P_{+hi}(u^2\partial_xP_-\overline{u}) \\ &\quad - P_{+hi}(u\partial_xP_-(|u|^2)) \\ &= P_{+hi}(|u|^2e^{-i\rho}\partial_x(e^{i\rho}u)) - iP_{+hi}(|u|^4u) + P_{+hi}(u^2\partial_xP_+\overline{u}) \\ &\quad + P_{+hi}(u^2\partial_xP_-\overline{u}) - P_{+hi}(u\partial_xP_-(|u|^2)) \\ &= P_{+hi}(|u|^2e^{-i\rho}\partial_xv) + P_{+hi}(u^2\partial_x\overline{w}) + P_{+hi}(|u|^2e^{-i\rho}\partial_xP_-(e^{i\rho}u)) \\ &\quad - iP_{+hi}(|u|^4u) + P_{+hi}(u^2P_-\partial_x\overline{u}) - P_{+hi}(u\partial_xP_-(|u|^2)). \end{aligned} \tag{3.14}$$

The interest in introducing the variables v and w resides in the following “frequencies localized” gauge transformation. Note that the same kind of idea was used by Tao [27] and Molinet, Ribaud [18] for the Benjamin-Ono and generalized Benjamin-Ono equations.

Lemma 3.4. *Let $u \in C([0, T]; H^\infty(\mathbb{R}))$ be a solution of the equation in (1.10) with $\alpha = \beta = \gamma = k = 1$. Consider v and w defined as in (3.13). Then v and w satisfy the following equations:*

$$\partial_t v + i\partial_x^2 v = -2P_+(e^{i\rho}uP_-\partial_x(|u|^2)) - iP_+(e^{i\rho}u\mathcal{L}_h(|u|^2)) + iP_+(e^{i\rho}u|u|^2), \tag{3.15}$$

$$\partial_t w + i\partial_x^2 w = 2P_-(uP_+\partial_x(|u|^2)) - iP_-(u\mathcal{L}_h(|u|^2)) + iP_-(|u|^2u). \tag{3.16}$$

Proof. First, we deduce from (3.13) that

$$\partial_t v + i\partial_x^2 v = P_+(e^{i\rho}i(\partial_t\rho + i\partial_x^2\rho - (\partial_x\rho)^2)u + e^{i\rho}(\partial_t u + i\partial_x^2 u) - 2\partial_x\rho e^{i\rho}\partial_x u). \tag{3.17}$$

On the other hand we observe, using the fact that $\overline{P_+f} = P_-f$ if f is real valued, that

$$\begin{aligned} \partial_t\rho &= \int_{-\infty}^x (\partial_t u\bar{u} + u\partial_t\bar{u})dy \\ &= \int_{-\infty}^x [-i\partial_y^2 u\bar{u} + iu\partial_y^2\bar{u} + 2|u|^2P_+\partial_y(|u|^2) + 2|u|^2P_-\partial_y(|u|^2) \\ &\quad - i|u|^2\mathcal{L}_h(|u|^2) + i|u|^2\mathcal{L}_h(|u|^2) + i|u|^4 - i|u|^4] dy \\ &= \int_{-\infty}^x [-i\partial_y(\partial_y u\bar{u}) + i\partial_y(u\partial_y\bar{u}) + i\partial_y u\partial_y\bar{u} - i\partial_y u\partial_y\bar{u} + 2|u|^2\partial_y(|u|^2)] dy \\ &= -i\partial_x u\bar{u} + iu\partial_x\bar{u} + |u|^4, \end{aligned}$$

which implies that

$$\partial_t\rho + i\partial_x^2\rho - (\partial_x\rho)^2 = 2iu\partial_x\bar{u}. \tag{3.18}$$

Thus, we obtain, combining (3.17) and (3.18), that

$$\begin{aligned} \partial_t v + i\partial_x^2 v &= 2P_+(-e^{i\rho}u^2\partial_x\bar{u} + e^{i\rho}(uP_+\partial_x(|u|^2)) - e^{i\rho}|u|^2\partial_x u) \\ &\quad - iP_+(e^{i\rho}u\mathcal{L}_h(|u|^2)) + iP_+(e^{i\rho}u|u|^2), \end{aligned}$$

which yields (3.15), using the identities

$$u\partial_x(|u|^2) = u^2\partial_x\bar{u} + |u|^2\partial_x u \quad \text{and} \quad P_+ + P_- = 1.$$

Equation (3.16) follows directly applying P_- to (1.10). □

We are now ready to estimate the term $\int_0^t U(t - \tau)P_{+hi}(uP_+\partial_x(|u|^2))d\tau$ in $Z_{s,T}$ as a function of v and w .

Lemma 3.5. *Let $\frac{1}{2} < s < \frac{9}{10}$ and $0 < T \leq 1$. Then, there exist $\theta_1 > 0$ and a polynomial p_1 whose factors are at least of order two, such that*

$$\begin{aligned} & \left\| \int_0^t U(t-\tau) P_{+hi}(u P_+ \partial_x (|u|^2)) d\tau \right\|_{Z_{s,T}} \\ & \lesssim (\|D_x^{s+\frac{1}{2}} v\|_{L_x^\infty L_T^2} + \|D_x^{s+\frac{1}{2}} w\|_{L_x^\infty L_T^2}) \|u\|_{L_x^2 L_T^\infty}^2 + T^{\theta_1} p_1(\|u\|_{Z_{s,T}}) \|u\|_{Z_{s,T}}, \end{aligned} \quad (3.19)$$

where v and w are defined in (3.13).

Proof. We decompose $P_{+hi}(u P_+ \partial_x (|u|^2))$ as in (3.14) and we begin estimating the first term on the right-hand side of (3.14). Observe that

$$|u|^2 e^{-i\rho} \partial_x v = -D_x^{\frac{1}{2}} (|u|^2 e^{-i\rho} \mathcal{H} D_x^{\frac{1}{2}} v) + [D_x^{\frac{1}{2}}, |u|^2 e^{-i\rho}] \mathcal{H} D_x^{\frac{1}{2}} v. \quad (3.20)$$

Then, since $(\frac{1}{2}, \infty, 2)$ is 1-admissible and $(\frac{2}{5}, 20, \frac{20}{9})$ is 2*-admissible, it follows from Lemma 2.5 combined with estimate (3.5) and Bernstein's inequality that

$$\begin{aligned} & \left\| \int_0^t U(t-\tau) P_{+hi}(|u|^2 e^{-i\rho} \partial_x v) d\tau \right\|_{Z_{s,T}} \\ & \lesssim \|D_x^s (|u|^2 e^{-i\rho} \mathcal{H} D_x^{\frac{1}{2}} v)\|_{L_x^1 L_T^2} + T^{\frac{1}{40}} \|D_x^{s-\frac{2}{5}} [D_x^{\frac{1}{2}}, |u|^2 e^{-i\rho}] \mathcal{H} D_x^{\frac{1}{2}} v\|_{L_x^{\frac{20}{19}} L_T^{\frac{20}{11}}}. \end{aligned} \quad (3.21)$$

Moreover, using estimate (2.16), Hölder's inequality and the fact that $\mathcal{H}v = -iv$ since v is localized in positive frequencies, we get that

$$\begin{aligned} & \|D_x^s (|u|^2 e^{-i\rho} \mathcal{H} D_x^{\frac{1}{2}} v)\|_{L_x^1 L_T^2} \\ & \lesssim \|D_x^s (|u|^2 e^{-i\rho})\|_{L_x^{\frac{4}{3}} L_T^4} \|D_x^{\frac{1}{2}} v\|_{L_{x,T}^4} + \| |u|^2 e^{-i\rho} \mathcal{H} D_x^{s+\frac{1}{2}} v \|_{L_x^1 L_T^2} \\ & \lesssim \|D_x^s (|u|^2 e^{-i\rho})\|_{L_x^{\frac{4}{3}} L_T^4} \|D_x^{\frac{1}{2}} v\|_{L_{x,T}^4} + \|u\|_{L_x^2 L_T^\infty}^2 \|D_x^{s+\frac{1}{2}} v\|_{L_x^\infty L_T^2}. \end{aligned} \quad (3.22)$$

Applying estimates (2.14), (2.18), and Hölder's inequality to the first term on the right-hand side of (3.22), we deduce that

$$\begin{aligned} & \|D_x^s (|u|^2 e^{-i\rho})\|_{L_x^{\frac{4}{3}} L_T^4} \|D_x^{\frac{1}{2}} v\|_{L_{x,T}^4} \\ & \lesssim \left(\|u\|_{L_x^{\frac{16}{3}} L_T^{16}}^2 \| |u|^2 \|_{L_x^{\frac{8}{3}} L_T^8} + \|J^s (|u|^2)\|_{L_x^{\frac{4}{3}} L_T^4} \right) \|D_x^{\frac{1}{2}} (e^{-i\rho} u)\|_{L_{x,T}^4} \\ & \lesssim \left(\|u\|_{L_x^{\frac{16}{3}} L_T^{16}}^4 + \| |u|^2 \|_{L_x^{\frac{4}{3}} L_T^4} + \|D_x^s (|u|^2)\|_{L_x^{\frac{4}{3}} L_T^4} \right) (\|u\|_{L_{x,T}^{12}}^3 + \|J^{\frac{1}{2}} u\|_{L_{x,T}^4}) \\ & \lesssim \left(T^{\frac{1}{4}} \|u\|_{L_x^{\frac{16}{3}} L_T^\infty}^4 + T^{\frac{1}{4}} \|u\|_{L_x^2 L_T^\infty} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}} \right) \end{aligned}$$

$$\begin{aligned}
 &+ T^{\frac{1}{8}} \|u\|_{L_x^2 L_T^\infty} \|D_x^s u\|_{L_{x,T}^6}^{\frac{3}{4}} \|u\|_{L_T^\infty H_x^s}^{\frac{1}{4}} \\
 &\times \left(T^{\frac{1}{4}} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^3 + T^{\frac{1}{8}} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^{\frac{1}{4}} \|D_x^{\frac{1}{2}} u\|_{L_{x,T}^6}^{\frac{3}{4}} \right).
 \end{aligned} \tag{3.23}$$

The second term on the right-hand side of (3.21) is estimated using (2.14), (2.16) (note that $s < \frac{9}{10}$), (2.18), and Hölder’s inequality:

$$\begin{aligned}
 &T^{\frac{1}{40}} \|D_x^{s-\frac{2}{5}} [D_x^{\frac{1}{2}}, |u|^2 e^{-i\rho}] \mathcal{H} D_x^{\frac{1}{2}} v\|_{L_x^{\frac{20}{19}} L_T^{\frac{20}{11}}} \\
 &\lesssim T^{\frac{1}{40}} \|D_x^{\frac{1}{2}} v\|_{L_{x,T}^4} \|D_x^{s+\frac{1}{10}} (e^{-i\rho} |u|^2)\|_{L_x^{\frac{10}{7}} L_T^{\frac{10}{3}}} \\
 &\lesssim T^{\frac{1}{40}} \left(T^{\frac{1}{4}} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^3 + T^{\frac{1}{8}} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^{\frac{1}{4}} \|D_x^{\frac{1}{2}} u\|_{L_{x,T}^6}^{\frac{3}{4}} \right) \\
 &\times \left(\|u\|_{L_x^{\frac{40}{7}} L_T^{\frac{40}{3}}}^4 + \|J^{s+\frac{1}{10}}(|u|^2)\|_{L_x^{\frac{10}{7}} L_T^{\frac{10}{3}}} \right) \\
 &\lesssim T^{\frac{1}{40}} \left(T^{\frac{1}{4}} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^3 + T^{\frac{1}{8}} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^{\frac{1}{4}} \|D_x^{\frac{1}{2}} u\|_{L_{x,T}^6}^{\frac{3}{4}} \right) \\
 &\times \left(T^{\frac{3}{10}} \|u\|_{L_x^{\frac{40}{7}} L_T^\infty}^4 + \|u\|_{L_x^{\frac{20}{7}} L_T^{\frac{20}{3}}}^2 + \|u\|_{L_x^2 L_T^\infty} \|D_x^{s+\frac{1}{10}} u\|_{L_x^5 L_T^{\frac{10}{3}}} \right).
 \end{aligned} \tag{3.24}$$

Hence, we obtain a bound for $\|\int_0^t U(t-\tau) P_{+hi}(|u|^2 e^{-i\rho} \partial_x v) d\tau\|_{Z_{s,T}}$ gathering (3.20)-(3.24).

In order to treat the second term on the right-hand side of (3.14), we observe that

$$u^2 \partial_x \bar{w} = D_x^{\frac{1}{2}} (u^2 \mathcal{H} D_x^{\frac{1}{2}} \bar{w}) - [D_x^{\frac{1}{2}}, u^2] \mathcal{H} D_x^{\frac{1}{2}} \bar{w},$$

and apply the same tools as before to obtain

$$\begin{aligned}
 &\left\| \int_0^t U(t-\tau) P_{+hi} (u^2 \partial_x \bar{w}) d\tau \right\|_{Z_{s,T}} \\
 &\lesssim \|D_x^s (u^2 \mathcal{H} D_x^{\frac{1}{2}} \bar{w})\|_{L_x^1 L_T^2} + T^{\frac{1}{40}} \|D_x^{s-\frac{2}{5}} ([D_x^{\frac{1}{2}}, u^2] \mathcal{H} D_x^{\frac{1}{2}} \bar{w})\|_{L_x^{\frac{20}{19}} L_T^{\frac{20}{11}}} \\
 &\lesssim \|D_x^s (u^2)\|_{L_x^{\frac{4}{3}} L_T^4} \|D_x^{\frac{1}{2}} u\|_{L_{x,T}^4} + \|u^2 D_x^{s+\frac{1}{2}} \mathcal{H} \bar{w}\|_{L_x^1 L_T^2} \\
 &\quad + T^{\frac{1}{40}} \|D_x^{\frac{1}{2}} u\|_{L_{x,T}^4} \|D_x^{s+\frac{1}{10}} (u^2)\|_{L_x^{\frac{10}{7}} L_T^{\frac{10}{3}}} \\
 &\lesssim T^{\frac{3}{8}} \|u\|_{L_x^2 L_T^\infty} \|D_x^s u\|_{L_{x,T}^6}^{\frac{3}{4}} \|u\|_{L_T^\infty H_x^s}^{\frac{1}{4}} \|D_x^{\frac{1}{2}} u\|_{L_{x,T}^6}^{\frac{3}{4}} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^{\frac{1}{4}}
 \end{aligned}$$

$$\begin{aligned}
& + \|u\|_{L_x^2 L_T^\infty}^2 \|D_x^{s+\frac{1}{2}} w\|_{L_x^\infty L_T^2} \\
& + T^{\frac{3}{20}} \|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^{\frac{1}{4}} \|D_x^{\frac{1}{2}} u\|_{L_{x,T}^6}^{\frac{3}{4}} \|u\|_{L_x^2 L_T^\infty} \|D_x^{s+\frac{1}{10}} u\|_{L_x^{\frac{5}{2}} L_T^{\frac{10}{3}}}.
\end{aligned} \tag{3.25}$$

We estimate the third term on the right-hand side of (3.14) using Lemma 2.5 combined with (3.5) and the fact that $(\frac{1}{10}, 5, \frac{10}{3})$ is 2^* admissible:

$$\begin{aligned}
& \left\| \int_0^t U(t-\tau) P_{+hi}(|u|^2 e^{-i\rho} \partial_x P_-(e^{i\rho} u)) d\tau \right\|_{Z_{s,T}} \\
& \lesssim T^{\frac{1}{10}} \|D_x^{s-\frac{1}{10}} P_{+hi}(|u|^2 e^{-i\rho} \partial_x P_-(e^{i\rho} u))\|_{L_x^{\frac{5}{4}} L_T^{\frac{10}{7}}}.
\end{aligned} \tag{3.26}$$

Moreover, we deduce from estimates (2.14), (2.17), and (2.18) that

$$\begin{aligned}
& T^{\frac{1}{10}} \|D_x^{s-\frac{1}{10}} P_{+hi}(|u|^2 e^{-i\rho} \partial_x P_-(e^{i\rho} u))\|_{L_x^{\frac{5}{4}} L_T^{\frac{10}{7}}} \\
& \lesssim T^{\frac{1}{10}} \|D_x^s(|u|^2 e^{-i\rho})\|_{L_x^{\frac{4}{3}} L_T^4} \|D_x^{\frac{9}{10}}(e^{i\rho} u)\|_{L_x^{20} L_T^{20/9}} \\
& \lesssim T^{\frac{1}{10}} \left(T^{\frac{1}{4}} \|u\|_{L_x^{\frac{16}{3}} L_T^\infty}^4 + T^{\frac{1}{8}} \|u\|_{L_x^2 L_T^\infty} \|D_x^s u\|_{L_{x,T}^6}^{\frac{3}{4}} \|u\|_{L_T^\infty H_x^s}^{\frac{1}{4}} \right) \\
& \quad \times \left(T^{\frac{9}{20}} \|u\|_{L_x^{80} L_T^\infty}^2 \|u\|_{L_x^{40} L_T^\infty} + \|J^{\frac{9}{10}} u\|_{L_x^{20} L_T^{20/9}} \right),
\end{aligned}$$

which gives us the desired bound for the left-hand side of (3.26).

Finally, we observe that the three last terms on the right-hand side of (3.14) can be handled with similar arguments to the ones used in the proof of Lemmas 3.2 and 3.3. \square

Next, we derive a technical lemma which allows us to control $\|u\|_{L_x^2 L_T^\infty}$ with a positive power of T in front of the nonlinearity.

Lemma 3.6. *Let $\frac{1}{2} < s < \frac{9}{10}$, $0 < T \leq 1$, and $u \in C([0, T]; H^\infty(\mathbb{R}))$ be a solution of (1.10) with $\alpha = \beta = \gamma = k = 1$. Then, there exists $\theta_4 > 0$ such that*

$$\|u\|_{L_x^2 L_T^\infty} \lesssim \|\phi\|_{H^s} + T^{\theta_4} \|u\|_{Z_{s,T}}^3. \tag{3.27}$$

Proof. Fix $\epsilon > 0$ as in (3.3). Then we deduce, applying estimate (2.6) to the integral equation (3.1), that

$$\begin{aligned}
& \|u\|_{L_x^2 L_T^\infty} \lesssim \|\phi\|_{H^{\frac{1}{2}+\frac{\epsilon}{2}}} + \left\| \int_0^t U(t-\tau)(|u|^2 u + u \mathcal{L}_h(|u|^2)) d\tau \right\|_{Z_{s,T}} \\
& + \left\| \int_0^t U(t-\tau)(u P_+ \partial_x(|u|^2)) d\tau \right\|_{L_x^2 L_T^\infty}.
\end{aligned} \tag{3.28}$$

The second term on the right-hand side of (3.28) can be bounded using estimate (3.6). To handle the third term, we use the following identity:

$$\begin{aligned}
 uP_+\partial_x(|u|^2) &= P_-(uP_+\partial_x(|u|^2)) - P_+(uP_-\partial_x(|u|^2)) \\
 &\quad + P_+(|u|^2\partial_x u) + P_+(u^2\partial_x \bar{u}) \\
 &= P_-(uP_+\partial_x(|u|^2)) - P_+(uP_-\partial_x(|u|^2)) \\
 &\quad + P_+([D_x^{\frac{1}{2}}, |u|^2]D_x^{\frac{1}{2}}\mathcal{H}u) + P_+([D_x^{\frac{1}{2}}, u^2]D_x^{\frac{1}{2}}\mathcal{H}\bar{u}) \quad (3.29) \\
 &\quad - P_+D_x^{\frac{1}{2}}(|u|^2D_x^{\frac{1}{2}}\mathcal{H}u) - P_+D_x^{\frac{1}{2}}(u^2D_x^{\frac{1}{2}}\mathcal{H}\bar{u}).
 \end{aligned}$$

The nonlinearity associated to the first two terms on the right-hand side of (3.29) can be bounded as was done in Lemma 3.3, and the nonlinearity associated with the third and fourth terms can be handled as in (3.24). It remains to estimate the nonlinearity associated with the two last terms. Since $(\frac{1}{2} - \epsilon, \frac{3}{\epsilon}, \frac{6}{3-4\epsilon})$ is 1-admissible, we have from Lemmas 2.4 and 2.5 that

$$\left\| \int_0^t U(t-\tau)(P_+D_x^{\frac{1}{2}}(|u|^2D_x^{\frac{1}{2}}\mathcal{H}u))d\tau \right\|_{L_x^2L_T^\infty} \lesssim \|J_x^{\frac{1}{2}+\frac{3\epsilon}{2}}(|u|^2D_x^{\frac{1}{2}}\mathcal{H}u)\|_{L_x^{\frac{3}{3-\epsilon}}L_T^{\frac{6}{3+4\epsilon}}}.$$

Moreover, we obtain applying estimate (2.14) that

$$\begin{aligned}
 &\|J_x^{\frac{1}{2}+\frac{3\epsilon}{2}}(|u|^2D_x^{\frac{1}{2}}\mathcal{H}u)\|_{L_x^{\frac{3}{3-\epsilon}}L_T^{\frac{6}{3+4\epsilon}}} \\
 &\lesssim \|J_x^{\frac{1}{2}+\frac{3\epsilon}{2}}(|u|^2)\|_{L_x^{\frac{12}{9-4\epsilon}}L_T^{\frac{12}{3+8\epsilon}}} \|D_x^{\frac{1}{2}}u\|_{L_{x,T}^4} + \|u\|_{L_x^{\frac{4}{2-\epsilon}}L_T^{\frac{2}{\epsilon}}}^2 \|J_x^{1+\frac{3\epsilon}{2}}u\|_{L_x^{\frac{6}{\epsilon}}L_T^{\frac{6}{3-2\epsilon}}} \\
 &\lesssim T^{\frac{2\epsilon}{3}} \|u\|_{L_x^{\frac{6}{3-2\epsilon}}L_T^\infty} \|J_x^{\frac{1}{2}+\frac{3\epsilon}{2}}u\|_{L_{x,T}^4} \|D_x^{\frac{1}{2}}u\|_{L_{x,T}^4} + T^\epsilon \|u\|_{L_x^{\frac{4}{2-\epsilon}}L_T^\infty}^2 \|J_x^{1+\frac{3\epsilon}{2}}u\|_{L_x^{\frac{6}{\epsilon}}L_T^{\frac{6}{3-2\epsilon}}}.
 \end{aligned}$$

We observe that the last term on the right-hand side of (3.29) can be treated exactly as above, which concludes the proof of Lemma 3.6. \square

It remains to control the terms $\|D_x^{s+\frac{1}{2}}v\|_{L_x^\infty L_T^2}$ and $\|D_x^{s+\frac{1}{2}}w\|_{L_x^\infty L_T^2}$ which appear on the right-hand side of (3.19).

Lemma 3.7. *Let $\frac{1}{2} < s < \frac{9}{10}$, $0 < T \leq 1$, and $u \in C([0, T]; H^\infty(\mathbb{R}))$ be a solution of (1.10) with $\alpha = \beta = \gamma = k = 1$. Then, there exist $\theta_2, \theta_3 > 0$ and two polynomials p_2 and p_3 whose factors are at least of order two, such that*

$$\|D_x^{s+\frac{1}{2}}v\|_{L_x^\infty L_T^2} \lesssim (1 + \|\phi\|_{H_x^{\frac{1}{2}}}^2) \|\phi\|_{H_x^s} + T^{\theta_2} p_2(\|u\|_{Z_{s,T}}) \|u\|_{Z_{s,T}}, \quad (3.30)$$

and

$$\|D_x^{s+\frac{1}{2}}w\|_{L_x^\infty L_T^2} \lesssim \|\phi\|_{H^s} + T^{\theta_3} p_3(\|u\|_{Z_{s,T}}) \|u\|_{Z_{s,T}}, \quad (3.31)$$

where v and w are defined in (3.13).

Remark 3.1. Note that the positive power of T appearing in front of the nonlinearities on the right-hand side of (3.30) and (3.31) is fundamental to obtaining well posedness for arbitrarily large initial data in Theorem 1.1. The same does not occur if one tries to estimate directly $\|D_x^{s+\frac{1}{2}}u\|_{L_x^2L_T^\infty}$ as we will see in the next section.

Proof of Lemma 3.7. We first prove estimate (3.30). Since the triplets $(\frac{1}{2}, \infty, 2)$ and $(\frac{1}{10}, 5, \frac{10}{3})$ are respectively 1 and 2* admissible, we apply Lemma 2.5 to the integral equation for equation (3.15) to deduce that

$$\begin{aligned} \|D_x^{s+\frac{1}{2}}v\|_{L_x^\infty L_T^2} &\lesssim \|D_x^s(e^{i\rho(\cdot,0)}\phi)\|_{L_x^2} + T^{\frac{1}{2}}\|D_x^s(e^{i\rho}u|u|^2)\|_{L_{x,T}^2} \\ &+ T^{\frac{1}{2}}\|D_x^s(e^{i\rho}u\mathcal{L}_h(|u|^2))\|_{L_{x,T}^2} + T^{\frac{1}{10}}\|D_x^{s-\frac{1}{10}}P_+(e^{i\rho}uP_-\partial_x(|u|^2))\|_{L_x^{\frac{5}{2}}L_T^{\frac{10}{3}}}. \end{aligned} \tag{3.32}$$

To estimate the first term on the right-hand side of (3.32), we follow the arguments in the proof of Lemma 3.5 in [18]. We have by the triangle inequality that

$$\begin{aligned} \|D_x^s(e^{i\rho(\cdot,0)}\phi)\|_{L_x^2} &\leq \|D_x^s(P_0(e^{i\rho(\cdot,0)}\phi))\|_{L_x^2} \\ &+ \|D_x^s((1 - P_0)(e^{i\rho(\cdot,0)}\phi))\|_{L_x^2} = I + II. \end{aligned} \tag{3.33}$$

Now, using estimate (2.13), Bernstein’s inequality, and Sobolev’s embedding, we get that

$$I \lesssim \|P_0e^{i\rho(\cdot,0)}\|_{L_x^\infty}\|D_x^s\phi\|_{L_x^2} + \|\phi D_x^sP_0e^{i\rho(\cdot,0)}\|_{L_x^2} \lesssim \|D_x^s\phi\|_{L_x^2} + \|\phi\|_{L_x^2}, \tag{3.34}$$

and

$$\begin{aligned} II &\lesssim \|(1 - P_0)e^{i\rho(\cdot,0)}\|_{L_x^\infty}\|D_x^s\phi\|_{L_x^2} + \|\phi\|_{L_x^\infty}\|(1 - P_0)D_x^se^{i\rho(\cdot,0)}\|_{L_x^2} \\ &\lesssim \|D_x^s\phi\|_{L_x^2} + \|\phi\|_{H^s}\|\partial_xe^{i\rho(\cdot,0)}\|_{L_x^2} \lesssim \|D_x^s\phi\|_{L_x^2} + \|\phi\|_{H^s}\|\phi\|_{H_x^{\frac{1}{2}}}. \end{aligned} \tag{3.35}$$

Hence, we deduce by gathering (3.33)-(3.35) that

$$\|D_x^s(e^{i\rho(\cdot,0)}\phi)\|_{L_x^2} \lesssim (1 + \|\phi\|_{H_x^{\frac{1}{2}}})\|\phi\|_{H_x^s}. \tag{3.36}$$

We estimate the second term on the right-hand side of (3.32) using (2.18), Sobolev’s embedding and Hölder’s inequality:

$$\begin{aligned} T^{\frac{1}{2}}\|D_x^s(e^{i\rho}u|u|^2)\|_{L_{x,T}^2} &\lesssim T^{\frac{1}{2}}\left(\|u\|_{L_{x,T}^{10}}^5 + T^{\frac{1}{2}}\|u\|_{L_T^\infty H_x^s}^3\right) \\ &\lesssim T\left(\|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^5 + \|u\|_{L_T^\infty H_x^s}^3\right). \end{aligned} \tag{3.37}$$

Combining the same strategy with estimates (2.1) and (2.2), we get that

$$\begin{aligned}
 & T^{\frac{1}{2}} \|D_x^s(e^{i\rho}u\mathcal{L}_h(|u|^2))\|_{L_{x,T}^2} \\
 & \lesssim T^{\frac{1}{2}} \left(\|u\|_{L_{x,T}^{10}}^2 \|u\mathcal{L}_h(|u|^2)\|_{L_{x,T}^{\frac{10}{3}}} + T^{\frac{1}{2}} \|u\mathcal{L}_h(|u|^2)\|_{L_T^\infty H_x^s} \right) \\
 & \lesssim T \left(\|u\|_{L_T^\infty H_x^{\frac{1}{2}}}^5 + \|u\|_{L_T^\infty H_x^s}^3 \right). \tag{3.38}
 \end{aligned}$$

Finally, we handle the last term on the right-hand side of (3.32) using estimates (2.14), (2.17), and (2.18), so that

$$\begin{aligned}
 & T^{\frac{1}{10}} \|D_x^{s-\frac{1}{10}} P_+(e^{i\rho}uP_-\partial_x(|u|^2))\|_{L_x^{\frac{5}{4}} L_T^{\frac{10}{9}}} \\
 & \lesssim T^{\frac{1}{10}} \|D_x^{\frac{9}{10}}(e^{i\rho}u)\|_{L_x^{20} L_T^{\frac{20}{9}}} \|D_x^s(|u|^2)\|_{L_x^{\frac{4}{3}} L_T^4} \\
 & \lesssim T^{\frac{1}{10}} \left(\|u\|_{L_x^{80} L_T^{10}}^2 \|u\|_{L_x^{40} L_T^4} + \|J_x^{\frac{9}{10}}\|_{L_x^{20} L_T^{\frac{20}{9}}} \right) \left(\|u\|_{L_x^2 L_T^\infty} \|D_x^s u\|_{L_{x,T}^4} \right) \\
 & \lesssim T^{\frac{1}{10}} \left(T^{\frac{9}{20}} \|u\|_{L_x^{80} L_T^\infty}^2 \|u\|_{L_x^{40} L_T^\infty} + \|J_x^{\frac{9}{10}}\|_{L_x^{20} L_T^{\frac{20}{9}}} \right) \\
 & \quad \times \left(T^{\frac{1}{8}} \|u\|_{L_x^2 L_T^\infty} \|D_x^s u\|_{L_{x,T}^6}^{\frac{3}{4}} \|u\|_{L_T^\infty H_x^s}^{\frac{1}{4}} \right). \tag{3.39}
 \end{aligned}$$

Therefore, we conclude the proof of (3.30) by getting together (3.32) and (3.36)–(3.39).

We apply the same argument as above to the integral equation associated with (3.16) and deduce that

$$\begin{aligned}
 & \|D_x^{s+\frac{1}{2}} w\|_{L_x^\infty L_T^2} \\
 & \lesssim \|\phi\|_{H^s} + T \|u\|_{L_T^\infty H^s}^3 + T^{\frac{9}{40}} \|D_x^{\frac{9}{10}} u\|_{L_x^{20} L_T^{20/9}} \|u\|_{L_x^2 L_T^\infty} \|D_x^s u\|_{L_{x,T}^6}^{\frac{3}{4}} \|u\|_{L_T^\infty H_x^s}^{\frac{1}{4}},
 \end{aligned}$$

which implies (3.31). □

We are now in position to give the proof of Proposition 3.1.

Proof of Proposition 3.1. We obtain estimate (3.4) applying estimates (3.5), (3.6), (3.9), (3.10), (3.19), (3.27), (3.30), (3.31), and Young’s inequality to the integral equation (3.1) satisfied by our smooth solution. □

Remark 3.2. Note that a careful examination of the proof of estimate (3.4) would lead to the following refined estimate:

$$\|u\|_{Z_{s,T}} \lesssim q(\|\phi\|_{H^{\frac{1}{2}+}}) \|\phi\|_{H^s} + T^\theta p(\|u\|_{Z_{\frac{1}{2}+,T}}) \|u\|_{Z_{s,T}},$$

where θ and p and q are as in Proposition 3.1 and $\frac{1}{2}+$ denotes a number slightly greater than $\frac{1}{2}$ but smaller than s .

Next, we derive an estimate similar to (3.4) for the difference of two solutions.

Proposition 3.2. *Let $\frac{1}{2} < s < \frac{9}{10}$, $0 < T \leq 1$, $\phi_1, \phi_2 \in H^\infty(\mathbb{R})$ and $u_1, u_2 \in C([0, T]; H^\infty(\mathbb{R}))$ be the smooth solutions to the IVP (1.10) satisfying $u_1(\cdot, 0) = \phi_1$ and $u_2(\cdot, 0) = \phi_2$. Then, there exist $\tilde{\theta} > 0$, a polynomial \tilde{q} , and a polynomial \tilde{p} whose factors are at least of order two, such that*

$$\begin{aligned} \|u_1 - u_2\|_{Z_{s,T}} &\lesssim \tilde{q}(\|\phi_1\|_{H^s} + \|\phi_2\|_{H^s})\|\phi_1 - \phi_2\|_{H^s} \\ &\quad + T^{\tilde{\theta}}\tilde{p}(\|u_1\|_{Z_{s,T}} + \|u_2\|_{Z_{s,T}})\|u_1 - u_2\|_{Z_{s,T}}. \end{aligned} \quad (3.40)$$

Proof. We define $\mu = u_1 - u_2$, $v_j = P_+(e^{i\rho_j}u_j)$, where $\rho_j = \int_{-\infty}^x |u_j|^2 dy$, $w_j = P_-u_j$, $j = 1, 2$, $\nu = v_1 - v_2$ and $\omega = w_1 - w_2$. Then, we have that

$$\begin{aligned} \partial_t \mu + i\partial_x^2 \mu &= P_0 \{ \mu (2P_+ \partial_x (|u_1|^2) - i\mathcal{L}_h (|u_1|^2) + i|u_1|^2) \\ &\quad + u_2 (P_+ \partial_x - i\mathcal{L}_h + i) (\mu \bar{u}_1) + u_2 (P_+ \partial_x - i\mathcal{L}_h + i) (u_2 \bar{\mu}) \} \\ &\quad + 2 \{ P_{-hi} (\mu P_+ \partial_x (|u_1|^2)) + P_{-hi} (u_2 P_+ \partial_x (\mu \bar{u}_1)) + P_{-hi} (u_2 P_+ \partial_x (u_2 \bar{\mu})) \} \\ &\quad + iP_{-hi} (\mu (-\mathcal{L}_h (|u_1|^2) + |u_1|^2)) + iP_{-hi} (u_2 (-\mathcal{L}_h + 1) (\mu \bar{u}_1)) \\ &\quad + iP_{-hi} (u_2 (-\mathcal{L}_h + 1) (u_2 \bar{\mu})) + 2 \{ P_{+hi} (\mu \bar{u}_1 e^{-i\rho_1} \partial_x v_1) \\ &\quad + P_{+hi} (\mu \bar{u}_1 e^{-i\rho_1} \partial_x P_-(e^{i\rho_1} u_1)) - iP_{+hi} (|u_1|^4 \mu) + P_{+hi} (\mu u_1 \partial_x \bar{w}_1) \\ &\quad + P_{+hi} (\mu u_1 P_- \partial_x \bar{u}_1) - P_{+hi} (\mu \partial_x P_-(|u_1|^2)) + P_{+hi} (u_2 \bar{u}_1 e^{-i\rho_1} \partial_x \nu) \\ &\quad + P_{+hi} (u_2 \bar{u}_1 (e^{-i\rho_1} - e^{-i\rho_2}) \partial_x v_2) + P_{+hi} (u_2 \bar{u}_1 e^{-i\rho_1} \partial_x P_-(e^{i\rho_1} \mu)) \\ &\quad + P_{+hi} (u_2 \bar{u}_1 e^{-i\rho_1} \partial_x P_-(e^{i\rho_1} - e^{i\rho_2}) u_2) + i|u_2|^2 u_2^2 \bar{\mu} \} \\ &\quad + P_{+hi} (u_2 \bar{u}_1 (e^{-i\rho_1} - e^{-i\rho_2}) \partial_x P_-(e^{-i\rho_2} u_2)) - i(|u_1|^2 u_1 + |u_2|^2 \bar{u}_1) \mu \\ &\quad + (|u_1|^2 u_2 + |u_2|^2 u_2) \bar{\mu} - P_{+hi} (u_2 P_- \partial_x (\mu \bar{u}_1)) + P_{+hi} (u_2 \mu \partial_x P_- \bar{u}_1) \\ &\quad + P_{+hi} (u_2 \mu \partial_x \bar{w}_1) + P_{+hi} (u_2 \bar{\mu} e^{-i\rho_2} \partial_x v_2) - iP_{+hi} (|u_2|^2 u_2^2 \bar{\mu}) \\ &\quad + P_{+hi} (u_2 \bar{\mu} e^{-i\rho_2} \partial_x P_-(e^{i\rho_2} u_2)) + P_{+hi} (u_2^2 P_- \partial_x \bar{\mu}) + P_{+hi} (u_2^2 \partial_x \bar{\omega}) \\ &\quad - P_{+hi} (u_2 P_- \partial_x (u_2 \bar{\mu})) + iP_{+hi} (\mu (-\mathcal{L}_h (|u_1|^2) + |u_1|^2)) \\ &\quad + iP_{+hi} (u_2 (-\mathcal{L}_h + 1) (\mu \bar{u}_1)) + iP_{+hi} (u_2 (-\mathcal{L}_h + 1) (u_2 \bar{\mu})), \end{aligned} \quad (3.41)$$

where ν and ω satisfy

$$\begin{aligned} \partial_t \nu + i\partial_x^2 \nu &= -2P_+(e^{i\rho_1} \mu P_- \partial_x (|u_1|^2)) - 2P_+((e^{i\rho_1} - e^{i\rho_2}) u_2 P_- \partial_x (|u_1|^2)) \\ &\quad - 2P_+(e^{i\rho_2} u_2 P_- \partial_x (\mu \bar{u}_1)) - 2P_+(e^{i\rho_2} u_2 P_- \partial_x (u_2 \bar{\mu})) \end{aligned}$$

$$\begin{aligned}
 & -iP_+(e^{i\rho_1}\mu\mathcal{L}_h(|u_1|^2)) - iP_+((e^{i\rho_1} - e^{i\rho_2})u_2\mathcal{L}_h(|u_1|^2)) \quad (3.42) \\
 & -iP_+(e^{i\rho_2}u_2\mathcal{L}_h(\mu\bar{u}_1)) - iP_+(e^{i\rho_2}u_2\mathcal{L}_h(u_2\bar{\mu})) \\
 & + i\gamma P_+(e^{i\rho_1}\mu|u_1|^2) + i\gamma P_+(e^{i\rho_2}u_2\mu\bar{u}_1) + i\gamma P_+(e^{i\rho_2}u_2^2\bar{\mu}),
 \end{aligned}$$

and

$$\begin{aligned}
 \partial_t\omega + i\partial_x^2\omega &= P_-(\mu(2P_+\partial_x(|u_1|^2)) - iP_-(\mu\mathcal{L}_h(|u_1|^2)) + iP_-(\mu|u_1|^2)) \\
 & + 2P_-(u_2P_+\partial_x(\mu\bar{u}_1)) - iP_-(u_2\mathcal{L}_h(\mu\bar{u}_1)) \quad (3.43) \\
 & + i(\mu\bar{u}_1) + P_-(u_2P_+\partial_x(u_2\bar{\mu})) - iP_-(u_2\mathcal{L}_h(u_2\bar{\mu})) + iP_-(u_2^2\bar{\mu}),
 \end{aligned}$$

respectively. Then, the proof of (3.40) follows applying the same arguments as the ones used in the proof of Proposition 3.1 to equation (3.41). \square

Finally, will give a sketch of the proof of Theorem 1.1 (see [18] or [8] for more references).

Proof of Theorem 1.1. Let $\frac{1}{2} < s < \frac{9}{10}$ and $\phi \in H^s(\mathbb{R})$. We denote by η a function satisfying $\eta \in C_0^\infty(\mathbb{R})$, $\eta \geq 0$, and $\int_{\mathbb{R}} \eta = 1$, and by η_ϵ , the function $\eta_\epsilon = \frac{1}{\epsilon}\eta(\cdot/\epsilon)$ for all $\epsilon > 0$. We also fix $\phi_\epsilon = \phi * \eta_\epsilon \in H^\infty(\mathbb{R})$.

It follows from Theorem 5.1 (see the appendix), that, for all $\epsilon > 0$, there exists a smooth solution u_ϵ to the equation in (1.10) satisfying $u_\epsilon(\cdot, 0) = \phi_\epsilon$ and defined on its maximal time interval $[0, T_\epsilon)$. It can be proved using Proposition 3.1, as was done in section 4.2 of [18], that there exists $T = T(\|\phi\|_{H^s}) > 0$ satisfying $T_\epsilon \geq T$ for all $\epsilon > 0$. Then Proposition 3.1 implies that (u_ϵ) is bounded, uniformly in ϵ , in $Z_{s,T}$ by $C(\|\phi\|_{H^s})$. Moreover Proposition 3.2 shows that the sequence (u_ϵ) is a Cauchy sequence in $Z_{s,T}$ as $\epsilon \rightarrow 0$, if T is chosen small enough. Hence (u_ϵ) converges to a function u in $Z_{s,T}$, a solution to the IVP (1.10).

The uniqueness and the Lipschitz continuous dependence of the flow follow directly from estimate (3.40). \square

4. PROOF OF PROPOSITION 1.1

Once again, we can assume that $\alpha = \beta = \gamma = k = 1$. The proof of Proposition 1.1 follows an argument of Bekiranov [3] based on the implicit function theorem (see also Corollary 3.1 in [19] in the case of the NL-NLS equation). Let $s > \frac{1}{2}$ and $\phi \in H^s(\mathbb{R})$. We define the following map, associated with the integral equation (3.1):

$$\Phi(u)(t) = U(t)\phi + \int_0^t U(t - \tau) (2uP_+\partial_x(|u|^2) - iu\mathcal{L}_h(|u|^2) + i|u|^2u) (\tau)d\tau, \quad (4.1)$$

and the functional space $\tilde{Z}_{s,T}$ by $\tilde{Z}_{s,T} = \{u \in C([0, T]; H^s(\mathbb{R})) : \|u\|_{\tilde{Z}_{s,T}} < \infty\}$, where

$$\|v\|_{\tilde{Z}_{s,T}} = \max\{ \|u\|_{L_T^\infty H^s}, \|D_x^{s+\frac{1}{2}}u\|_{L_x^\infty L_T^2}, \|u\|_{L_x^2 L_T^\infty}, \|D_x^{\frac{1}{2}}u\|_{L_{x,T}^6}, \|D_x^s u\|_{L_{x,T}^6}, T^{-\frac{1}{10}} \|D_x^{s+\frac{1}{10}}u\|_{L_x^5 L_T^{\frac{10}{3}}}, \|D_x^{\frac{9}{10}}u\|_{L_x^{20} L_T^{20/9}} \}.$$

For $a > 0$, we consider the closed ball $\tilde{Z}_{s,T}(a)$ of $\tilde{Z}_{s,T}$ centered at the origin, of radius a , and equipped with the metric induced by the norm $\|\cdot\|_{\tilde{Z}_{s,T}}$.

First we prove that, if $\|\phi\|_{H^s}$ and $a = a(\|\phi\|_{H^s})$ are small enough, then Φ is a map from $\tilde{Z}_{s,T}(a)$ into $\tilde{Z}_{s,T}(a)$.

Lemma 4.1. *Let $s > \frac{1}{2}$ and $0 < T \leq 1$. Then there exists $\delta > 0$ such that if $\|\phi\|_{H^s} < \delta$ then there exists $a = a(\|\phi\|_{H^s})$ such that*

$$\|\Phi(u)\|_{\tilde{Z}_{s,T}} \leq a, \quad \forall u \in \tilde{Z}_{s,T}(a). \tag{4.2}$$

Proof. We get applying estimates (3.5), (3.6), and (3.9) to (4.1) that

$$\|\Phi(u)\|_{\tilde{Z}_{s,T}} \lesssim \|\phi\|_{H^s} + T\|u\|_{\tilde{Z}_{s,T}}^3 + \left\| \int_0^t U(t-\tau)(1-P_0)(uP_+\partial_x(|u|^2))d\tau \right\|_{\tilde{Z}_{s,T}}.$$

In order to estimate the last term of the right-hand side of (4.3), we use the decomposition in (3.29). The nonlinearity associated to the four first terms of the right-hand side of (3.29) can be bounded as in the proof of Lemmas 3.3 and Lemma 3.5. We use the identities $P_+ = iP_+\mathcal{H}$ and $\mathcal{H}^2 = -1$, the fact that P_{+hi} is bounded in L_x^1 , Lemma 2.5 and estimate (2.15) with $\tilde{D}_x^s = D_x^s \mathcal{H}$ to handle the nonlinearity associated to the fifth term. Then

$$\begin{aligned} \left\| \int_0^t U(t-\tau)P_{+hi}D_x^{\frac{1}{2}}(|u|^2 D_x^{\frac{1}{2}}\mathcal{H}u)d\tau \right\|_{\tilde{Z}_{s,T}} &\lesssim \|\tilde{D}_x^s(|u|^2 \tilde{D}_x^{\frac{1}{2}}u)\|_{L_x^1 L_T^2} \\ &\lesssim \|D_x^s(|u|^2)\|_{L_x^{\frac{4}{3}} L_T^4} \|D_x^{\frac{1}{2}}u\|_{L_{x,T}^4} + \| |u|^2 D_x^{s+\frac{1}{2}}u \|_{L_x^1 L_T^2} \\ &\lesssim \|u\|_{L_x^2 L_T^\infty} \|D_x^s u\|_{L_{x,T}^4} \|D_x^{\frac{1}{2}}u\|_{L_{x,T}^4} + \|u\|_{L_x^2 L_T^\infty}^2 \|D_x^{s+\frac{1}{2}}u\|_{L_x^\infty L_T^2}. \end{aligned}$$

Note that the nonlinearity associated to the term $P_{+hi}D_x^{\frac{1}{2}}(u^2 D_x^{\frac{1}{2}}\mathcal{H}\bar{u})$ can be estimated exactly as above. Therefore, we deduce the existence of two positive constants θ and C such that

$$\|\Phi(u)\|_{\tilde{Z}_{s,T}} \leq C\|\phi\|_{H^s} + C(1 + T^\theta)\|u\|_{\tilde{Z}_{s,T}}^3,$$

and it is enough to choose

$$a = \frac{C}{2} \|\phi\|_{H^s} \quad \text{and} \quad 0 < a \leq \frac{1}{\sqrt{4C}}, \tag{4.3}$$

to conclude the proof of Lemma 4.1. □

Remark 4.1. We can deduce by a similar argument that Φ is also a contraction on $\tilde{Z}_{s,T}(a)$. Thus, we obtain an easier proof of Theorem 1.1 in the case where the initial data satisfies the smallness condition $\|\phi\|_{H^s} < \delta$. Therefore, for every $0 < T \leq 1$, the flow map data-solution $S : B(0, \delta) \rightarrow \tilde{Z}_{s,T}(a)$, $\phi \mapsto u(\cdot) = S(\cdot)\phi$, is well defined, where $B(0, \delta)$ denotes the ball of $H^s(\mathbb{R})$ centered at the origin and of radius δ .

Now, we give the proof of Proposition 1.1.

Proof of Proposition 1.1. For $0 < T \leq 1$, let us define $G : B(0, \delta) \times \tilde{Z}_{s,T} \rightarrow \tilde{Z}_{s,T}$ by

$$G(\phi, u)(t) = u(t) - U(t)\phi - \int_0^t U(t-\tau)(2\nu P_+ \partial_x(|u|^2) - iu\mathcal{L}_h(|u|^2) + i|u|^2 u)(\tau) d\tau.$$

Then, G is well defined, G is smooth, and $G(\phi, S(t)\phi) \equiv 0$. Moreover, its partial derivative with respect to u is given by

$$\begin{aligned} \partial_u G(\phi, u)\nu(t) = \nu(t) - \int_0^t U(t-\tau) & (2\nu P_+ \partial_x(|u|^2) + 4uP_+ \Re \partial_x(u\bar{\nu}) \tag{4.4} \\ & - i\nu\mathcal{L}_h(|u|^2) - 2i\nu\mathcal{L}_h \Re(\bar{u}\nu) + 2i|u|^2\nu + iu^2\bar{\nu})(\tau) d\tau. \end{aligned}$$

Hence, we deduce by a similar argument to the one used in the proof of Lemma 4.1 that

$$\|(I - \partial_u G(\phi, S\phi))\nu\|_{\tilde{Z}_{s,T}} \leq C(1 + T^\theta)a^2\|\nu\|_{\tilde{Z}_{s,T}},$$

so that by the choice of a in (4.3), $\partial_u G(\phi, S\phi)$ is an isomorphism. Thus, we conclude by the implicit function theorem that there exist a neighborhood V of ϕ in $B(0, \delta)$ and a smooth application $g : V \rightarrow \tilde{Z}_{s,T}$ such that $G(\psi, g(\psi)) = 0$, for all $\psi \in V$. This means that $S|_V = g$ is smooth, and since smoothness is a local property, we conclude that the flow map is smooth in $B(0, \delta)$. □

Remark 4.2. We observe, using the scaling property (1.6), that Proposition 1.1 still holds with the weaker assumption $\|\phi\|_{L^2} \ll 1$ in the case $\gamma = 0$ (for details see [19], Corollary 2.1).

5. APPENDIX: EXISTENCE OF SMOOTH SOLUTIONS
FOR EQUATIONS (1.1) AND (1.2)

In this appendix we prove the existence of smooth solutions to the IVPs associated with equations (1.1) and (1.2) for arbitrarily large initial data. Our main result is the following.

Theorem 5.1. *Let $s \geq 2$. Then, given $\phi \in H^s(\mathbb{R})$, there exists $T > 0$ and a unique function*

$$u \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R})) \quad (5.1)$$

that is a solution to (1.1) or (1.2), satisfying $u(\cdot, 0) = \phi$. Moreover, the flow map data-solution : $\phi \mapsto u$ is continuous from $H^s(\mathbb{R})$ into $C([0, T]; H^s(\mathbb{R}))$.

The strategy adopted to prove Theorem 5.1 is to employ the following classical Kato's theorem for quasi-linear equations of evolution (see Theorems 6 and 7 of section 7 in [5]).

Theorem 5.2. *Let X and Y be two real reflexive Banach spaces such that*

- (X) *Y is continuously and densely embedded in X , and there exists an isometric isomorphism S from Y onto X .*

Let us denote by $G(X)$ the set of all generators of strongly continuous semigroups on X , and for every $R > 0$, by $B_Y(0, R)$ the ball of Y centered at the origin and of radius R . We consider the abstract evolution Cauchy problem

$$\begin{cases} \partial_t u = A(u)u, & t > 0 \\ u(0) = \phi \in Y, \end{cases} \quad (5.2)$$

where $A : Y \rightarrow G(X)$ is an operator satisfying the following assumptions:

- (A1) *For all $R > 0$, there exists $\delta \geq 0$ such that, if $u \in B_Y(0, R)$, then the strongly continuous semigroup $e^{tA(u)}$ generated by $A(u)$ satisfies*

$$\|e^{tA(u)}\|_{\mathcal{B}(X)} \leq e^{3\delta t}, \quad t \geq 0. \quad (5.3)$$

- (A2) *For all $R > 0$, there exists $\lambda > 0$ such that, if $u, v \in B_Y(0, R)$, then the operator $B(u)$ defined by $B(u) = [S, A(u)]S^{-1}$ satisfies*

$$B(0) = 0 \quad \text{and} \quad \|B(u)w - B(v)w\|_X \leq \lambda \|u - v\|_Y \|w\|_X, \quad \forall w \in X.$$

- (A3) *For all $R > 0$, $Y \subseteq \mathcal{D}(A(u))$, $A(u) \in \mathcal{B}(Y, X)$ for all $u \in B_Y(0, R)$, and there exists $\eta > 0$ such that, if $u, v \in B_Y(0, R)$, then*

$$\|A(u) - A(v)\|_{\mathcal{B}(Y, X)} \leq \eta \|u - v\|_X.$$

Then, for each $\phi \in Y$, there exists $T > 0$ and a unique solution u of (5.2) satisfying $u \in C([0, T]; Y) \cap C^1([0, T]; X)$. Moreover, the flow map : $\phi \mapsto u$ is continuous from $H^s(\mathbb{R})$ to $C([0, T]; Y)$.

First, we rewrite (1.1) in the form (5.2), where $X = L^2(\mathbb{R})$, $Y = H^s(\mathbb{R})$, $s \geq 2$,

$$A(u) = -i\partial_x^2 + \sum_{j=1}^3 A_j(u), \tag{5.4}$$

with

$$A_1(u)w = 4uP_+\Re(\bar{u}\partial_x w), \quad A_2(u) = |u|^2w, \quad A_3(u)w = -2iu\mathcal{L}_h\Re(\bar{u}w). \tag{5.5}$$

Here \Re denotes the real part of a complex number and \mathcal{L}_h is defined in (1.11). Note that we have fixed $\alpha = \beta = \gamma = 1$ in (1.1) without loss of generality.

Finally we observe, as commented in [5], that since the function u is complex valued, we have to consider $L^2(\mathbb{R})$ equipped with the scalar product

$$\langle u, v \rangle = \Re(u, v)_{L^2} = \Re \int_{-\infty}^{\infty} u\bar{v}dx, \tag{5.6}$$

so that $L^2(\mathbb{R})$ and $H^s(\mathbb{R})$ become real Hilbert spaces.

To verify the hypothesis (A1)-(A3), we will make use of some commutator estimates. The first one is the Kato-Ponce commutator estimate (see Lemma XI in the appendix of [6]).

Lemma 5.1. *Let $s, r \in \mathbb{R}$ be such that $s \geq 1$ and $r > \frac{1}{2}$. Then for any $1 < p < \infty$, there exists a positive constant $c = c(s)$ such that*

$$\|[J^s, f]g\|_{L^p} \leq c (\|\partial_x f\|_{H^r} \|J^{s-1}g\|_{L^p} + \|J^s f\|_{L^p} \|g\|_{H^r}). \tag{5.7}$$

The second one is Calderón’s first commutator estimate obtained in [4].

Lemma 5.2. *There exists a positive constant c such that*

$$\|[\mathcal{H}, a]\partial_x f\|_{L^2} \leq c\|\partial_x a\|_{L^\infty}\|f\|_{L^2}. \tag{5.8}$$

Now we derive some technical lemmas useful in the proof of Theorem 5.1.

Lemma 5.3. *Given $s > \frac{1}{2}$ and $R > 0$, there exists $C = C(s, R) > 0$ such that*

$$\|A_j(u)w\|_{L^2} \leq C\{(1 - \epsilon^{-1})\|w\|_{L^2} + \epsilon\|\partial_x^2 w\|_{L^2}\}, \quad j = 1, 2, 3, \tag{5.9}$$

for all $u \in B_{H^s}(0, R)$ and $w \in H^2(\mathbb{R})$.

Proof. Estimate (5.9) is a direct application of Sobolev’s embedding and Hölder’s and Young’s inequalities to (5.4). □

The next lemma is the key one. It relies strongly on Calderon’s first commutator estimate to handle the Hilbert transform in $A_1(u)$.

Lemma 5.4. *Let $s > \frac{3}{2}$ and $R > 0$. Then there exists $\delta = \delta(s, R) > 0$ such that*

$$|\langle A_j(u)v, v \rangle| \leq \delta \|v\|_{L^2}^2, \quad j = 1, 2, 3, \quad (5.10)$$

for all $u \in B_{H^s}(0, R)$ and $v \in H^s$.

Proof. We have from the definition of A_1 that

$$\begin{aligned} \langle A_1(u)v, v \rangle &= \langle |u|^2 \partial_x v, v \rangle + \langle u^2 \partial_x \bar{v}, v \rangle + \langle iu\mathcal{H}(\bar{u}\partial_x v), v \rangle + \langle iu\mathcal{H}(u\partial_x \bar{v}), v \rangle \\ &= \langle |u|^2 \partial_x v, v \rangle + \langle u^2 \partial_x \bar{v}, v \rangle + \langle iu[\mathcal{H}, \bar{u}]\partial_x v, v \rangle \\ &\quad + \langle i|u|^2 \mathcal{H}\partial_x v, v \rangle + \langle iu\mathcal{H}(u\partial_x \bar{v}), v \rangle. \end{aligned} \quad (5.11)$$

Now, integrating by parts and using Hölder's inequality, we get

$$\langle |u|^2 \partial_x v, v \rangle = -\frac{1}{2} \int_{-\infty}^{\infty} \partial_x(|u|^2)|v|^2 dx \lesssim \|\partial_x(|u|^2)\|_{L^\infty} \|v\|_{L^2}^2, \quad (5.12)$$

and

$$\langle u^2 \partial_x \bar{v}, v \rangle = -\frac{1}{2} \Re \int_{-\infty}^{\infty} \partial_x(u^2)\bar{v}^2 dx \lesssim \|\partial_x(u^2)\|_{L^\infty} \|v\|_{L^2}^2. \quad (5.13)$$

On the other hand we obtain, using the fact that $(\mathcal{H}f, g)_{L^2} = -(f, \mathcal{H}g)_{L^2}$ and integration by parts, that

$$\begin{aligned} (i|u|^2 \mathcal{H}\partial_x v, v)_{L^2} &= -(i[\mathcal{H}, |u|^2]\partial_x v, v)_{L^2} + (i\mathcal{H}(|u|^2 \partial_x v), v)_{L^2} \\ &= -(i[\mathcal{H}, |u|^2]\partial_x v, v)_{L^2} + (i\partial_x(|u|^2)v, \mathcal{H}v)_{L^2} - \overline{(i|u|^2 \mathcal{H}\partial_x v, v)_{L^2}}. \end{aligned} \quad (5.14)$$

Thus, from (5.14) we deduce that

$$\langle i|u|^2 \mathcal{H}\partial_x v, v \rangle = -\frac{1}{2} (i[\mathcal{H}, |u|^2]\partial_x v, v)_{L^2} + \frac{1}{2} (i\partial_x(|u|^2)v, \mathcal{H}v)_{L^2}. \quad (5.15)$$

Finally, by the Plancherel identity

$$\langle iu\mathcal{H}\partial_x(u\bar{v}), v \rangle = \Re \int_{-\infty}^{\infty} i\mathcal{H}\partial_x(u\bar{v})u\bar{v} dx = \Re \int_{-\infty}^{\infty} i|\xi| \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(u\bar{v})} d\xi = 0,$$

so that

$$\langle iu\mathcal{H}(u\partial_x \bar{v}), v \rangle = -\langle iu\mathcal{H}(\partial_x u\bar{v}), v \rangle \leq \|u\|_{L^\infty} \|\partial_x u\|_{L^\infty} \|v\|_{L^2}^2. \quad (5.16)$$

Therefore, we conclude the proof of estimate (5.10) for $j = 1$ by gathering (5.11)-(5.16), using estimate (5.8), Sobolev's embedding and the fact that $u \in B_{H^s}(0, R)$ with $s > \frac{3}{2}$.

The cases $j = 2$ and $j = 3$ follow directly from Sobolev's embedding and estimate (2.1). \square

The next lemmas show that the maps $u \mapsto A_j(u)$ and $u \mapsto [J^s, A_j(u)]$ are locally Lipschitz from $H^s(\mathbb{R})$ to $\mathcal{B}(H^s(\mathbb{R}), L^2(\mathbb{R}))$.

Lemma 5.5. *Let $s > \frac{3}{2}$ and $R > 0$. Then there exists $C = C(s, R) > 0$ such that*

$$\|A_j(u)w - A_j(v)w\|_{L^2} \leq C\|u - v\|_{L^2}\|w\|_{H^s}, \quad j = 1, 2, 3, \quad (5.17)$$

for all $u, v \in B_{H^s}(0, R)$.

Proof. We get using Sobolev’s embedding, Lemma 2.1 and (5.4) that

$$\begin{aligned} & \|A_1(u)w - A_1(v)w\|_{L^2} \\ & \leq 4\|(u - v)P_+\Re(\bar{u}\partial_x w) + v(P_+\Re(\bar{u}\partial_x w) - P_+\Re(\bar{v}\partial_x w))\|_{L^2} \\ & \lesssim (\|P_+(\bar{u}\partial_x w)\|_{L^\infty} + \|v\|_{L^\infty}\|\partial_x w\|_{L^\infty})\|u - v\|_{L^2} \\ & \lesssim (\|u\|_{H^s} + \|v\|_{H^s})\|w\|_{H^s}\|u - v\|_{L^2}, \\ & \|A_2(u)w - A_2(v)w\|_{L^2} \leq \|w\|_{L^\infty}\| |u|^2 - |v|^2 \|_{L^2} \\ & \lesssim (\|u\|_{H^s} + \|v\|_{H^s})\|w\|_{H^s}\|u - v\|_{L^2}, \end{aligned}$$

and

$$\|A_3(u)w - A_3(v)w\|_{L^2} \lesssim (\|u\|_{H^s} + \|v\|_{H^s})\|w\|_{H^s}\|u - v\|_{L^2},$$

which yield (5.17), since $u, v \in B_{H^s}(0, R)$. □

Lemma 5.6. *Let $s > \frac{3}{2}$ and $R > 0$. Then there exists $C = C(s, R) > 0$ such that*

$$\|[J^s, A_j(u) - A_j(v)]w\|_{L^2} \leq C\|u - v\|_{H^s}\|w\|_s, \quad j = 1, 2, 3, \quad (5.18)$$

for all $u, v \in B_{H^s}(0, R)$.

Proof. Observe that

$$\begin{aligned} [J^s, A_1(u) - A_1(v)]w &= 4\{[J^s, u - v]P_+\Re(\bar{u}\partial_x w) + [J^s, v]P_+\Re((\bar{u} - \bar{v})\partial_x w) \\ & \quad + (u - v)P_+\Re([J^s, \bar{u}]\partial_x w) + vP_+\Re([J^s, \bar{u} - \bar{v}]\partial_x w)\}, \end{aligned} \quad (5.19)$$

which implies (5.18) for $j = 1$, applying estimate (5.7) to each term of the right side of (5.19).

The cases $j = 2$ and $j = 3$ are a direct application of the fact that $H^s(\mathbb{R})$ is a Banach algebra, since $s > \frac{1}{2}$. □

We are now in position to give the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $s \geq 2$ and $\phi \in H^s(\mathbb{R})$. We recall that we have already chosen $X = L^2(\mathbb{R})$, $Y = H^s(\mathbb{R})$ with the scalar product $\langle \cdot, \cdot \rangle$ defined in (5.6), and $S = J^s$. Thus assumption (X) is valid and it remains to show that the operator A defined in (5.4) satisfies assumptions (A1)-(A3) of Theorem 5.2.

In order to see that (A1) holds we proceed as follows. We write $A(u)$ as

$$A(u) - 3\delta = -i\partial_x^2 + \sum_{j=1}^3 (A_j(u) - \delta). \quad (5.20)$$

By (5.9) we deduce that $A_j(u) - \delta$ is $A(0)$ -bounded with zero as $A(0)$ -bound. Moreover, it follows from (5.10) that $A_j(u) - \delta$ is dissipative. Therefore, by Corollary 3.3 from Chapter 3 of [22], it follows that $A(u) - 3\delta$ generates a contractions semi-group, thus $A(u) \in G(L^2(\mathbb{R}))$ and $e^{tA(u)}$ satisfies (5.3).

Assumption (A2) follows from Lemma 5.6 and the fact that $[J^s, \partial_x^2] = 0$. Lemma 5.5 implies (A3).

Then we deduce from Theorem 5.2 that there exists a unique solution u of (1.1) satisfying $u \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; L^2(\mathbb{R}))$. Moreover, since the map $u \mapsto A(u)u$ is continuous from $H^s(\mathbb{R})$ to $H^{s-2}(\mathbb{R})$, we obtain that $\partial_t u \in C([0, T]; H^{s-2}(\mathbb{R}))$ so that (5.1) holds. \square

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