Advances in Differential Equations Volume 15, Numbers 9–10 (2010), 893–924

SOME QUASI-LINEAR ELLIPTIC EQUATIONS WITH INHOMOGENEOUS GENERALIZED ROBIN BOUNDARY CONDITIONS ON "BAD" DOMAINS

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(Submitted by: J.A. Goldstein)

Abstract. Let $p \in [2, N)$, $\Omega \subset \mathbb{R}^N$ an open set and let μ be a Borel measure on $\partial\Omega$. Under some assumptions on Ω, μ, f, g and β, we show that the quasi-linear elliptic equation with nonlinear inhomogeneous Robin-type boundary conditions

$$\begin{cases} -\Delta_p u + c(x)|u|^{p-2}u = f & \text{in } \Omega \\ d\,\mathsf{N}_p(u) + \beta(x,u)d\mu = g d\mu & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution which is globally bounded on $\overline{\Omega}$; that is, the weak solution u is in $L^\infty(\Omega)$ and its trace $u|_{\partial\Omega}$ belongs to $L^\infty(\partial\Omega, \mu)$. Here $\mathsf{N}_p(u)$ is a generalization of the normal derivative for bad domains. When Ω and u are smooth, then $d\,\mathsf{N}_p(u) = |\nabla u|^{p-2}(\partial u/\partial\nu)d\sigma$ where σ is the surface measure and ν the outer normal to $\partial\Omega$. A priori estimates for solutions are also obtained.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open set. Since the boundary of Ω may be so bad that no normal vector can be defined, we will use the following generalized version of a normal derivative in the weak sense. Let η be a Radon measure on $\partial\Omega$ and let $F : \Omega \to \mathbb{R}^N$ be a measurable function. If there exists a function $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that

$$\int_\Omega F\nabla\varphi\, dx = \int_\Omega f\varphi\, dx + \int_{\partial\Omega} \varphi\, d\eta$$

Accepted for publication: March 2010.
AMS Subject Classifications: 35J60, 35B45 , 35D10, 35J65.
The research of the second author was partially supported by the FIPI grant of the UPR-RP.

for all $\varphi \in C_c^1(\overline{\Omega})$, then we say that η is the *normal measure* of F which we denote by $\mathsf{N}^\star(F) := \eta$. Note that when the normal measure $\mathsf{N}^\star(F)$ exists, then it is unique and we have $d\,\mathsf{N}^\star(\psi F) = \psi d\,\mathsf{N}^\star(F)$ for all $\psi \in C^1(\overline{\Omega})$. If $u \in W^{1,1}_{\text{loc}}(\Omega)$ and $\mathsf{N}^\star(|\nabla u|^{p-2}\nabla u)$ exists, then we will denote by $\mathsf{N}_p(u) := \mathsf{N}^\star(|\nabla u|^{p-2}\nabla u)$ the p-generalized normal derivative of u. To justify this definition, let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1, ν the outer normal to $\partial\Omega$ and let σ be the surface measure on $\partial\Omega$. If $u \in C^1(\overline{\Omega})$ is such that there exist $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $g \in L^1(\partial\Omega, \sigma)$, with

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega f\varphi + \int_{\partial\Omega} g\varphi \, d\sigma$$

for all $\varphi \in C_c^1(\overline{\Omega})$, then $g = \partial u/\partial\nu$ and hence $d\,\mathsf{N}_2(u) = (\partial u/\partial\nu)d\sigma$.

In this article we are concerned with the inhomogeneous quasi-linear elliptic problem with inhomogeneous generalized Robin-type boundary conditions given by

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + c(x)|u|^{p-2}u = f & \text{in } \Omega, \\ d\,\mathsf{N}_p(u) + \beta(x,u)\,d\mu = g d\mu & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where μ is a Borel measure on $\partial\Omega$, $f \in L^{p_1}(\Omega)$, $g \in L^{q_1}(\partial\Omega, \mu)$ for some $p_1, q_1 \in [1, \infty]$, $p \in (1, \infty)$, $c \in L^\infty(\partial\Omega)$ and $\beta : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function. We remark that the classical p-Laplace operator Δ_p for $p \in (1, \infty)$ is defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The reason for considering generalized Robin boundary conditions in Equation (1.1) stems from the fact that, if $\partial\Omega$ has fractal geometry, then it may happen that the $(N-1)$-dimensional Hausdorff measure H^{N-1} is locally infinite on the whole boundary. This is the case for the domain in $\mathbb{R}^2$ bounded by the von Koch curve. In that case, by [2], there are no classical Robin boundary conditions on $\partial\Omega$. Functions in the domain of the operator will be equal to zero on $\partial\Omega$ in some sense. Hence, in this case the natural measure for defining Robin boundary conditions is the restriction to $\partial\Omega$ of the d-dimensional Hausdorff measure H^d where d is the Hausdorff dimension of the von Koch curve.

In [9], Daners has investigated Equation (1.1) for $p = 2$, $\beta(x, u) = b(x)u$, $g \equiv 0$ and $\mu = H^{N-1}$ (the linear problem with homogeneous linear boundary conditions) where he proved boundedness of weak solutions and obtained an a priori estimate on "almost all" bounded domains.

The inhomogeneous boundary conditions need a special treatment which we consider here.

The classical Laplace operator with linear and nonlinear inhomogeneous Robin boundary conditions has been investigated respectively in [28] and [5] on bounded Lipschitz domains. Most recently, Cianchi and Maz'ya [7] have considered Equation (1.1) for the case $\mu \equiv 0$ and $c \equiv 0$ (that is, the homogeneous Neumann boundary conditions) on arbitrary domains with finite measure. They have obtained beautiful a priori estimates of solutions and of their gradients in terms of the isocapacitary function. Some conditions involving the isocapacitary function have been also obtained to ensure that solutions are bounded.

The conclusions that will be presented in this paper are new for inhomogeneous Neumann and Robin-type boundary conditions on bad domains, as far as we know, and will also improve the norm estimates for weak solutions of this type of equations on smooth domains included in the literature. For example, our results read that if $\Omega \subset \mathbb{R}^N$ is a bounded (ϵ, δ)-domain and the measure μ is an upper d-Ahlfors measure for some $d \in (0, N)$, then weak solutions of (1.1) are globally bounded on $\overline{\Omega}$. More precisely, under some restrictions on p_1 and q_1 and a growth condition on β, we will show that, given $f_1, f_2 \in L^{p_1}(\Omega)$ and $g_1, g_2 \in L^{q_1}(\partial\Omega, \mu)$, if $u := u_{f_1,g_1}$ and $v := v_{f_2,g_2}$ are the corresponding weak solutions of (1.1), then, for every $p \in [2, N)$, one has the following Hölder type a priori estimates:

$$\|u - v\|^{p-1}_{L^\infty(\Omega)} \leq C \left(\|f_1 - f_2\|_{p_1,\Omega} + \|g_1 - g_2\|_{q_1,\partial\Omega}\right) \tag{1.2}$$

and

$$\|u - v\|^{p-1}_{L^\infty(\partial\Omega,\mu)} \leq C \left(\|f_1 - f_2\|_{p_1,\Omega} + \|g_1 - g_2\|_{q_2,\partial\Omega}\right) \tag{1.3}$$

for some constant $C > 0$ which depends only on Ω, N, p_1, q_1 and p.

The paper is organized as follows. In Section 2, we give some preliminary and intermediate results on Sobolev-type spaces such as a generalization of Maz'ya's spaces. In Section 3 we prove existence and uniqueness theorems of weak solutions to Equation (1.1). Section 4 contains the main results of this article where we prove that weak solutions are globally bounded on $\overline{\Omega}$ by showing the a priori estimates (1.2) and (1.3).

2. Preliminary and Intermediate Results

In this section we will give preliminary and intermediate results on Sobolev-type spaces as they are needed throughout the paper. We also introduce the notion of the classical and the relative $(1, p)$-capacity which turns out to be the right tool to study the properties of Sobolev-type spaces and the fine regularity of weak solutions to linear and nonlinear partial differential equations of elliptic type.

2.1. **The Classical and the Relative Capacities.** In this subsection we will shortly introduce the relative capacity, which plays an important role in the whole article. For example it is used to define admissible measures. Before we do this, we introduce the classical Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ be an open set and $p \in [1, \infty)$. Then the *first-order Sobolev space* $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)^N\}$$

and is equipped with the norm $\| \cdot \|_{W^{1,p}(\Omega)}$ given by

$$\|u\|^p_{W^{1,p}(\Omega)} := \int_\Omega |u|^p \, dx + \int_\Omega |\nabla u|^p \, dx.$$

Note that $W^{1,p}(\Omega)$ is a Banach space. Let $W_0^{1,p}(\Omega)$ and $\mathcal{W}^{1,p}(\Omega)$ be respectively the closure of $W^{1,p}(\Omega) \cap C_c(\Omega)$ and $W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ in $W^{1,p}(\Omega)$. It is well known that in general $\mathcal{W}^{1,p}(\Omega)$ is a proper closed subspace of $W^{1,p}(\Omega)$ but they coincide if for example Ω is of class C (see [22, Theorem 1, page 23]).

Definition 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in (1, \infty)$. Then the* relative $(1, p)$-capacity $\mathrm{Cap}_{p,\Omega}$ *with respect to Ω is defined for sets $A \subset \overline{\Omega}$ by*

$$\mathrm{Cap}_{p,\Omega}(A) := \inf \left\{ \|u\|^p_{W^{1,p}(\Omega)} \; : \; \begin{array}{l} u \in \mathcal{W}^{1,p}(\Omega), \; \exists\, O \subset \mathbb{R}^N \textit{ open}, \\ A \subset O \textit{ and } u \geq 1 \textit{ a.e. on } \Omega \cap O \end{array} \right\}.$$

- *A set $P \subset \overline{\Omega}$ is called* $\mathrm{Cap}_{p,\Omega}$-polar *if* $\mathrm{Cap}_{p,\Omega}(P) = 0$.
- *We say that a property holds* $\mathrm{Cap}_{p,\Omega}$*-quasi everywhere* (*briefly* $\mathrm{Cap}_{p,\Omega}$-q.e.) *on a set $A \subset \overline{\Omega}$, if there exists a* $\mathrm{Cap}_{p,\Omega}$*-polar set P such that it holds for all $x \in A \setminus P$.*
- *A function u is called* $\mathrm{Cap}_{p,\Omega}$*-quasi continuous on a set $A \subset \overline{\Omega}$ if for all $\varepsilon > 0$ there exists an open set O in the metric space $\overline{\Omega}$ such that* $\mathrm{Cap}_{p,\Omega}(O) \leq \varepsilon$ *and u restricted to $A \setminus O$ is continuous.*

The relative capacity $\mathrm{Cap}_{2,\Omega}$ has been introduced in [2] (see also [3]) to study the Laplace operator with linear Robin boundary conditions on arbitrary open subsets in $\mathbb{R}^N$. Biegert [4] has extended the definition of the relative capacity to every $p \in (1, \infty)$. Note that, if $\Omega = \mathbb{R}^N$, then $\mathrm{Cap}_{p,\mathbb{R}^N} = \mathrm{Cap}_p$ is the classical Wiener capacity, also called the $(1, p)$-capacity.

Definition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set. A Borel measure μ on $\partial\Omega$ is called* $\mathrm{Cap}_{p,\Omega}$-admissible *if μ does not charge $\mathrm{Cap}_{p,\Omega}$-polar Borel sets; that is, $\mathrm{Cap}_{p,\Omega}(A) = 0$ implies $\mu(A) = 0$ for every Borel set $A \subset \partial\Omega$.*

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in (1,\infty)$. Then for every $u \in \mathcal{W}^{1,p}(\Omega)$ there exists a unique (up to a $\mathrm{Cap}_{p,\Omega}$-polar set) $\mathrm{Cap}_{p,\Omega}$-quasi continuous function $\tilde{u} : \overline{\Omega} \to \mathbb{R}$ such that $\tilde{u} = u$ almost everywhere on Ω.*

Proof. The proof is contained in [4, Theorem 2.2.13]. □

2.2. **A Maz'ya-type Space.** In this subsection we define the classical and extended Maz'ya spaces and also show some properties related to these spaces.

Definition 2.4 (Maz'ya-type spaces)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let μ be a Borel measure on $\partial\Omega$. For $p, r \in [1,\infty)$ we define the extended Maz'ya space $W^1_{p,r}(\Omega, \partial\Omega, \mu)$ and the classical Maz'ya space $W^1_{p,r}(\Omega, \partial\Omega)$ to be the completion of the spaces*

$$V^1_{p,r}(\Omega, \partial\Omega, \mu) := \left\{u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega}) : u|_{\partial\Omega} \in L^r(\partial\Omega, \mu)\right\}$$

and

$$V^1_{p,r}(\Omega, \partial\Omega) := \left\{u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega}) : u|_{\partial\Omega} \in L^r(\partial\Omega, H^{N-1})\right\},$$

respectively with respect to the norms $\|\cdot\|_{W^1_{p,r}(\Omega,\partial\Omega,\mu)}$ and $\|\cdot\|_{W^1_{p,r}(\Omega,\partial\Omega)}$ given by

$$\begin{aligned} \|u\|_{W^1_{p,r}(\Omega,\partial\Omega,\mu)} &:= \|u\|_{W^{1,p}(\Omega)} + \|u\|_{L^r(\partial\Omega,\mu)}, \\ \|u\|_{W^1_{p,r}(\Omega,\partial\Omega)} &:= \|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^r(\partial\Omega,H^{N-1})}. \end{aligned}$$

Remark 2.5. The space $W^1_{p,r}(\Omega, \partial\Omega)$ has been introduced by Maz'ya [21, Section 3.6] (see also [22, Section 2.11]). By the identification

$$\begin{aligned} &V^1_{p,r}(\Omega, \partial\Omega, \mu) \to W^{1,p}(\Omega) \oplus L^r(\partial\Omega, \mu) =: T(\Omega, \partial\Omega, \mu), \\ &u \mapsto (u|_\Omega, u|_{\partial\Omega}) \end{aligned}$$

we may consider $W^1_{p,r}(\Omega, \partial\Omega, \mu)$ as a closed subspace of $T(\Omega, \partial\Omega, \mu)$ and we will identify $(u, b) \in T(\Omega, \partial\Omega, \mu)$ with the function v defined on $\overline{\Omega}$ given by

$$v(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ b(x) & \text{if } x \in \partial\Omega. \end{cases}$$

Note that $W^{1,p}_0(\Omega)$ is a closed subspace of $W^1_{p,r}(\Omega, \partial\Omega, \mu)$.

Definition 2.6. *Let $\Omega \subset \mathbb{R}^N$ be an open set and μ be a Borel measure on $\partial\Omega$. For $p, q \in [1, \infty]$ we define the Banach space*

$$X^{p,q}(\Omega, \mu) := \{(f, g) : \ f \in L^p(\Omega) \text{ and } g \in L^q(\partial\Omega, \mu)\}$$

endowed with the norm $\|\cdot\|_{X^{p,q}(\Omega,\mu)}$ given for $1 \le p, q \le \infty$ by

$$\|(f, g)\|_{X^{p,q}(\Omega,\mu)} := \|f\|_{L^p(\Omega)} + \|g\|_{L^q(\partial\Omega,\mu)},$$

if $\min\{p, q\} < \infty$, *and*

$$\|(f, g)\|_{X^{\infty,\infty}(\Omega,\mu)} := \max\{\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega,\mu)}\}.$$

Remark 2.7. Let λ_N denote N-dimensional Lebesgue measure. Then $X^{p,p}(\Omega, \mu)$ can be identified with the space $L^p(\overline{\Omega}, \lambda_N|_\Omega \oplus \mu)$.

The following beautiful Sobolev-type embedding is due to Maz'ya.

Theorem 2.8. (Maz'ya inequality [22, Corollary 2.11.2]). *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite measure and let $p \in [1, N)$. Then there is a constant $C > 0$ such that, for all $u \in W^1_{p,r}(\Omega, \partial\Omega)$,*

$$\|u\|_{L^q(\Omega)} \le C \cdot \|u\|_{W^1_{p,r}(\Omega,\partial\Omega)}, \tag{2.1}$$

where $1 \le r \le p(N-1)/(N-p)$ and $q = rN/(N-1)$. Moreover, the above exponent q cannot be improved unless additional restrictions on Ω are supposed.

As a consequence of Maz'ya's inequality, we have the following.

Remark 2.9. If $\Omega \subset \mathbb{R}^N$ is an open set with finite volume, then we get from the previous theorem that the spaces $W^1_{p,r}(\Omega, \partial\Omega, H^{N-1})$ and $W^1_{p,r}(\Omega, \partial\Omega)$ coincide with equivalent norms whenever $p(N-1)/N \le r \le p(N-1)/(N-p)$. Moreover, $W^1_{p,r}(\Omega, \partial\Omega)$ is continuously embedded into $X^{q,r}(\Omega, H^{N-1})$ for $1 \le r \le p(N-1)/(N-p)$ and $q = rN/(N-1)$.

Remark 2.10.

(1) Let Ω be the domain bounded by the von Koch curve (also known as the snowflake) or in general a domain whose boundary is a d-set with $d \in (N-1, N]$ (see Example 2.21 below). Then

$$W^1_{p,r}(\Omega, \partial\Omega, H^{N-1}) = W^1_{p,r}(\Omega, \partial\Omega) = W^{1,p}_0(\Omega).$$

(2) Arendt and Warma [2] proved that the restriction operator

$$R : W^1_{2,2}(\Omega, \partial\Omega, H^{N-1}) \to L^{2N/N-1}(\Omega), \quad u \mapsto u|_\Omega$$

is not always injective. More precisely, they have shown that, if $H^{N-1}(\partial\Omega)$ is finite, then R is injective if and only if H^{N-1} is $\mathrm{Cap}_{2,\Omega}$-admissible. They have provided several examples of domains where H^{N-1} is not $\mathrm{Cap}_{2,\Omega}$-admissible.

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^N$ be an open set, μ be a Radon measure (μ is inner regular and finite on compact sets) on $\partial\Omega$, $p \in (1,\infty)$ and $r \in [1,\infty)$ be fixed. Then the following assertions are equivalent.*

(1) *The operator $R : W^1_{p,r}(\Omega, \partial\Omega, \mu) \to L^p(\Omega)$, $u \mapsto u|_\Omega$ is injective.*
(2) *The measure μ is $\mathrm{Cap}_{p,\Omega}$-admissible.*

Proof. (2)⇒(1). Assume that μ is $\mathrm{Cap}_{p,\Omega}$-admissible. We have to show that R is injective. Let $u \in W^1_{p,r}(\Omega, \partial\Omega, \mu)$ and suppose that $Ru = 0$. Then there exists a sequence $u_n \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ such that $u_n \to u$ in $W^1_{p,r}(\Omega, \partial\Omega, \mu)$ and therefore $Ru_n \to Ru = 0$ in $W^{1,p}(\Omega)$. By possibly passing to a subsequence, we have that u_n converges to zero $\mathrm{Cap}_{p,\Omega}$-q.e. on $\overline{\Omega}$. Since μ is $\mathrm{Cap}_{p,\Omega}$-admissible, it follows that u_n converges to zero μ-a.e. on $\partial\Omega$. As $u_n|_{\partial\Omega}$ converges to $u|_{\partial\Omega}$ in $L^r(\partial\Omega, \mu)$, the uniqueness of the limit implies that $u = 0$ μ-a.e. on $\partial\Omega$ and therefore $u = 0$.

(1)⇒(2). Assume that μ is not $\mathrm{Cap}_{p,\Omega}$-admissible. Then there is a Borel set $K \subset \partial\Omega$ such that $\mathrm{Cap}_{p,\Omega}(K) = 0$ and $\mu(K) > 0$. By the inner regularity of μ we may assume that K is compact. Since $\mathrm{Cap}_{p,\Omega}(K) = 0$, there exists a sequence $u_n \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ such that $0 \le u_n \le 1$, $u_n = 1$ on K and $\|u_n\|_{W^{1,p}(\Omega)} \to 0$. For $k \in \mathbb{N}$ we let $O_k := \{x \in \mathbb{R}^N : \mathrm{dist}(x, K) < 1/k\}$. Then

$$K \subset O_{k+1} \subset O_k, \quad \bigcap_{k\ge1} O_k = K \quad \text{and} \quad \mu(O_k \cap \partial\Omega) \to \mu(K).$$

Let $v_k \in \mathcal{D}(O_k)$ be such that $v_k = 1$ on K and $0 \le v_k \le 1$. It is clear that $v_k \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ and $\|u_n v_k\|_{W^{1,p}(\Omega)} \to 0$ as $n \to \infty$. Note that $u_n v_k \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$, $0 \le u_n v_k \le 1$ and $u_n v_k = 1$ on K for all n, k. Now, let n_k be such that $\|w_k\|_{W^{1,p}(\Omega)} \le 2^{-k}$ where $w_k := u_{n_k} v_k$. Then $w_k \to 0$ in $W^{1,p}(\Omega)$, $0 \le w_k \le 1$, $w_k = 1$ on K and $w_k \to 1_K$ everywhere on $\overline{\Omega}$. Since $w_k = 1$ on K, it follows that $\|w_k\|^r_{L^r(\partial\Omega,\mu)} \ge \mu(K) > 0$. This shows that $1_K \in W^1_{p,r}(\Omega, \partial\Omega, \mu) \setminus \{0\}$ and $R1_K = 0$, hence R is not injective. □

Proposition 2.12. *Let $\Omega \subset \mathbb{R}^N$ be an open set and μ be a $\mathrm{Cap}_{p,\Omega}$-admissible measure. Then the space*

$$\mathcal{W}^1_{p,r}(\Omega, \partial\Omega, \mu) := \{u \in \mathcal{W}^{1,p}(\Omega) : \tilde{u}|_{\partial\Omega} \in L^r(\partial\Omega, \mu)\}$$

endowed with the norm

$$\|u\|_{\mathcal{W}^1_{p,r}(\Omega,\partial\Omega,\mu)} := \|u\|_{W^{1,p}(\Omega)} + \|\tilde{u}\|_{L^r(\partial\Omega,\mu)}$$

is a Banach space. In particular, $W^1_{p,r}(\Omega,\partial\Omega,\mu)$ *is a closed subspace of* $\mathcal{W}^1_{p,r}(\Omega,\partial\Omega,\mu)$.

Proof. Let $(u_n)_n$ be a Cauchy sequence in $\mathcal{W}^1_{p,r}(\Omega,\partial\Omega,\mu)$. Then there is a function $u \in \mathcal{W}^{1,p}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. By possibly passing to a subsequence, we get that $\tilde{u}_n \to \tilde{u}$ $\mathrm{Cap}_{p,\Omega}$-q.e. on $\overline{\Omega}$. Since $u_n \to f$ in $L^r(\partial\Omega,\mu)$ for some $f \in L^r(\partial\Omega,\mu)$, using the fact that μ is $\mathrm{Cap}_{p,\Omega}$-admissible, we obtain that $\tilde{u} = f$ μ-a.e. on $\partial\Omega$ and therefore $u \in \mathcal{W}^1_{p,r}(\Omega,\partial\Omega,\mu)$. □

2.3. **Properties of Upper Ahlfors Measures.** In this subsection, we describe a class of $\mathrm{Cap}_{p,\Omega}$-admissible measures on $\partial\Omega$ for which an inequality of the type (2.1) is satisfied if the space $W^1_{p,r}(\Omega,\partial\Omega)$ is replaced by $W^1_{p,r}(\Omega,\partial\Omega,\mu)$.

Definition 2.13. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set,* $d \in [0,N]$ *and let* μ *be a Borel measure on* $\partial\Omega$ (*inner regularity is automatic*).

(1) *We say that* μ *is an* upper d-Ahlfors measure *if there exists a constant* $C_2 > 0$ *such that for every* $x \in \partial\Omega$ *and every* $r \in (0,1]$ *one has*

$$\mu(B(x,r)\cap\partial\Omega) \le C_2 r^d.$$

(2) *We say that* μ *is a* lower d-Ahlfors measure *if there exists a constant* $C_1 > 0$ *such that for every* $x \in \partial\Omega$ *and every* $r \in (0,1]$ *one has*

$$\mu(B(x,r)\cap\partial\Omega) \ge C_1 r^d.$$

Remark 2.14. Note that in the preceding definition, an upper (respectively lower) d-Ahlfors measure is called an upper (respectively lower) $(N-d)$-Ahlfors measure in [11, Definition 1.2].

Definition 2.15. *Let* $p \in (1,\infty)$. *We say that* Ω *has the* $W^{1,p}$*-extension property if for every* $u \in W^{1,p}(\Omega)$ *there exists* $U \in W^{1,p}(\mathbb{R}^N)$ *such that* $U|_\Omega = u$ *almost everywhere.*

In that case, by [19, Theorem 5], there exists a bounded linear extension operator $\mathcal{E}_p$ from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^N)$. Moreover, the spaces $W^{1,p}(\Omega)$ and $\mathcal{W}^{1,p}(\Omega)$ coincide. In particular, one also obtains that, for every $p \in (1,N)$ and $r \in [1,\infty)$, the space $W^1_{p,r}(\Omega,\partial\Omega,\mu)$ is continuously embedded into

$L^{p_s}(\Omega)$ with $p_s = pN/(N-p)$; that is, there is a constant $C > 0$ such that, for every $u \in W^1_{p,r}(\Omega, \partial\Omega, \mu)$,

$$\|u\|_{p_s,\Omega} \leq C\|u\|_{W^1_{p,r}(\Omega,\partial\Omega,\mu)}.$$

Next, we introduce a class of domains which are also known in the literature under the name Jones-domains. This class of domains has the $W^{1,p}$-extension property for every $p \in (1, \infty)$.

Definition 2.16. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We say that Ω is an* (ε, δ)-domain *if there exist $0 < \delta \leq \infty$ and $0 < \varepsilon \leq 1$ such that for all $x, y \in \Omega$ with $|x - y| \leq \delta$ there is a continuous and rectifiable curve $\gamma : [0, T] \to \Omega$ for which $\gamma(0) = x$, $\gamma(T) = y$, and*

$$l(\gamma) \leq \frac{1}{\varepsilon}|x-y|, \;\; \mathrm{dist}(z, \partial\Omega) \geq \varepsilon \cdot \min\{|x-z|, |z-y|\} \;\; \textit{for all } z \textit{ on } \gamma.$$

Remark 2.17. Note that (see [17]) bounded (ε, δ)-domains in $\mathbb{R}^2$ are the same as bounded domains in $\mathbb{R}^2$ having the $W^{1,p}$-extension property, but the situation is different when $N \geq 3$. Examples of domains in $\mathbb{R}^N$ $(N \geq 3)$ having the $W^{1,p}$-extension property and which are not (ε, δ)-domains are included in [16].

The following beautiful trace theorem for functions in $W^{1,p}(\Omega)$ by Danielli et al. is taken from [11, Theorem 10.10].

Theorem 2.18. *Let $\Omega \subset \mathbb{R}^N$ be a bounded (ε, δ)-domain, $p \in (1, N)$ and let μ be an upper d-Ahlfors measure on $\partial\Omega$ with $0 < N - d < p$. Then there is a constant $C_1 > 0$ such that*

$$\|u\|_{L^{q_s}(\partial\Omega,\mu)} \leq C_1\|u\|_{W^{1,p}(\Omega)}, \tag{2.2}$$

for every $u \in W^{1,p}(\Omega)$ where $q_s := pd/(N - p) > p$.

Corollary 2.19. *Let $\Omega \subset \mathbb{R}^N$ be a bounded (ε, δ)-domain, $p \in (1, N)$ and let μ be an upper d-Ahlfors measure on $\partial\Omega$ with $0 < N - d < p$. Then for $q_s := pd/(N - p)$ the spaces $W^1_{p,q_s}(\Omega, \partial\Omega, \mu)$ and $W^{1,p}(\Omega)$ coincide with equivalent norms. Moreover, the space $W^{1,p}(\Omega)$ is continuously embedded into $X^{p_s,q_s}(\Omega, \mu)$ for $p_s := pN/(N - p)$ and μ is* $\mathrm{Cap}_{p,\Omega}$*-admissible.*

Proof. It follows from the extension property of Ω that $W^{1,p}(\Omega) = \mathcal{W}^{1,p}(\Omega)$ and that $W^{1,p}(\Omega)$ is continuously embedded into $L^{p_s}(\Omega)$. Using (2.2) we get that the spaces coincide with equivalent norms. The fact that μ is $\mathrm{Cap}_{p,\Omega}$-admissible follows from the inner regularity of μ (see Royden [26]) and the inequality $\mu(K)^{p/q_s} \leq C_1^p \,\mathrm{Cap}_{p,\Omega}(K)$ for all compact sets $K \subset \partial\Omega$ which follows directly from (2.2). □

The following remark shows in particular that upper d-Ahlfors measure on $\partial\Omega$ are also Cap_p-admissible if $0 < N - d < p$.

Remark 2.20. Let $1 < p < \infty$, $0 < N-d < p$, and let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let μ be an upper d-Ahlfors measure on $\partial\Omega$. It follows directly from the definition of the d-dimensional Hausdorff measure H^d that there exists a nonnegative Borel function $f \in L^\infty(\partial\Omega, H^d)$ such that

$$\mu(A) = \int_A f \, dH^d$$

for every Borel set $A \subset \partial\Omega$. Using [15, Theorem 4 page 156] (see also [4, Theorem 2.5.4]) which states that $H^d(A) = 0$ whenever $\mathrm{Cap}_p(A) = 0$ we get in particular that μ is Cap_p-admissible. If, in addition, Ω has the $W^{1,p}$-extension property, then $\mathrm{Cap}_{p,\Omega}(A) \le C \cdot \mathrm{Cap}_p(A)$ for every set $A \subset \overline{\Omega}$, and hence μ is also $\mathrm{Cap}_{p,\Omega}$-admissible. An analogous result showing that the d-dimensional Hausdorff measure H^d is absolutely continuous with respect to every lower d-Ahlfors measure μ is contained in [20, Lemma 1.17].

Example 2.21. Let $\Omega \subset \mathbb{R}^2$ be the snowflake. Then, by [17], Ω is a bounded (ε, δ)-domain. Let $d := \ln(4)/\ln(3)$ be the Hausdorff dimension of $\partial\Omega$. By [27], the restriction of H^d to $\partial\Omega$ is an upper d-Ahlfors measure.

More generally, let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded (ε, δ)-domain and assume that its boundary $\partial\Omega$ is a d-set for some $d \in (0, N]$ in the sense that there are a Borel measure μ on $\partial\Omega$ and some constants $C_2 > C_1 > 0$ such that

$$C_1 r^d \le \mu(\partial\Omega \cap B(x; r)) \le C_2 r^d \ \text{ for all } x \in \partial\Omega \text{ and all } r \in (0, 1]. \tag{2.3}$$

By [18] (see also [6]), $\partial\Omega$ is a d-set if and only if (2.3) holds with μ being the restriction to $\partial\Omega$ of the d-dimensional Hausdorff measure. In that case, if μ_1 and μ_2 are two measures on $\partial\Omega$ satisfying (2.3), then there are positive constants c_1 and c_2 such that $c_1\mu_2 \le \mu_1 \le c_2\mu_2$ on $\partial\Omega$. It follows directly from (2.3) that H^d is an upper d-Ahlfors measure. A measure satisfying (2.3) is called a d-Ahlfors measure.

2.4. **Some Tools.** In this subsection, we collect some results which will be frequently used throughout the remainder of the article.

Lemma 2.22. *Let $a, b \in \mathbb{R}^N$ and $p \in (1, \infty)$. Then, there exists a constant $\hat{c}_p > 0$ such that*

$$\left(|a|^{p-2}a - |b|^{p-2}b\right)(a - b) \ge \hat{c}_p \left(|a| + |b|\right)^{p-2} |a - b|^2 \ge 0. \tag{2.4}$$

If $p \in [2,\infty)$, then there exists a constant $c_p \in (0,1]$ such that

$$\left(|a|^{p-2}a - |b|^{p-2}b\right)(a-b) \geq c_p|a-b|^p. \tag{2.5}$$

Moreover, (2.5) *implies that*

$$\left||a|^{p-2}a - |b|^{p-2}b\right| \geq c_p|a-b|^{p-1}. \tag{2.6}$$

Proof. Inequality (2.5) is included in [12, Lemma I.4.4]. In Inequality (2.4), we only need to show that the left-hand side is non-negative, which follows easily. □

The following lemma which is of an analytic nature will be useful in deriving an a priori estimate.

Lemma 2.23. *Let $\psi : [k_0,\infty) \to \mathbb{R}$ be a non-negative, non-increasing function such that there are positive constants c, α and δ ($\delta > 1$) such that*

$$\psi(h) \leq c(h-k)^{-\alpha}\psi(k)^{\delta} \qquad \forall\, h > k \geq k_0.$$

Then $\psi(k_0 + d) = 0$ with $d = c^{1/\alpha}\psi(k_0)^{(\delta-1)/\alpha}2^{\delta(\delta-1)}$.

Proof. For the proof we refer to [23, Lemma 3.11]. □

3. Existence and Uniqueness of Weak Solutions

In this section we prove some existence and uniqueness theorems for weak solutions to the quasi-linear elliptic equation (1.1). Throughout the remainder of the article, we will write " $\lesssim$ " or " $\gtrsim$ " to mean that the inequality is true up to a multiplicative constant $C > 0$ which does not depend on the functions involved.

We consider the following assumptions.

Assumption 3.1.

(A) *$\Omega \subset \mathbb{R}^N$ is open, $p \in (1,\infty)$ and μ is a $\mathrm{Cap}_{p,\Omega}$-admissible Borel measure on $\partial\Omega$.*

(B) **(i)** *$b \in L^\infty(\partial\Omega,\mu)$ is non-negative,* **(ii)** *$c \in L^\infty(\Omega)$ and $c \geq c_0$ for some constant $c_0 > 0$.*

(C) *$p_s, q_s \in (1,\infty)$ and $\|u\|_{X^{p_s,q_s}(\Omega,\mu)} \lesssim \|u\|_{W^1_{p,p}(\Omega,\partial\Omega,\mu)}$ for all $u \in \mathcal{V}_p := W^1_{p,p}(\Omega,\partial\Omega,\mu)$.*

(D) **(i)** *$b \geq b_0$ for some constant $b_0 > 0$ or* **(ii)** *$\|u\|_{L^p(\partial\Omega,\mu)} \lesssim \|u\|_{W^{1,p}(\Omega)}$ for all $u \in \mathcal{V}_p$.*

Definition 3.2. *For $p \in [1,\infty]$, we denote by p' the conjugate index of p, that is, $p' := p/(p-1)$. Given $p_s, q_s \in [1,\infty]$ we let p_h, q_h denote the conjugate index of p_s, q_s, respectively.*

3.1. **Standard nonlinear Robin-type boundary conditions.** In this subsection we investigate the following problem.

Problem 3.3. *For $p_1, q_1 \in [1,\infty]$ and $(f,g) \in X^{p_1,q_1}(\Omega,\mu)$ we consider the inhomogeneous quasi-linear elliptic boundary-value problem*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + c(x)|u|^{p-2}u = f & \text{in } \Omega, \\ d\,\mathsf{N}_p(u) + b(x)|u|^{p-2}u\, d\mu = g d\mu & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

For this problem we let $\mathcal{A}_b :\ \mathcal{V}_p \times \mathcal{V}_p \to \mathbb{R}$ be the map formally given by

$$\mathcal{A}_b(u,v) := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx + \int_\Omega c(x)|u|^{p-2}uv\, dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv\, d\mu.$$

Definition 3.4. *A function $u \in \mathcal{V}_p$ is said to be a weak solution of* (3.1) *if*

$$\mathcal{A}_b(u,\varphi) = \int_\Omega f\varphi\, dx + \int_{\partial\Omega} g\varphi\, d\mu \quad \text{for all } \varphi \in \mathcal{V}_p$$

and the integrals in the right-hand side exist.

Lemma 3.5. *Assume (A)+(C). Then for all $(f,g) \in X^{p_h,q_h}(\Omega,\mu)$ the mapping*

$$\mathcal{V}_p \to \mathbb{R}, \ \ \varphi \mapsto \int_\Omega f\varphi\, dx + \int_{\partial\Omega} g\varphi\, d\mu$$

belongs to $\mathcal{V}_p'$. For this, we simply write, $X^{p_h,q_h}(\Omega,\mu) \subset \mathcal{V}_p'$.

Lemma 3.6. *Assume* (A)+(B). *Then $\mathcal{A}_b(u,\cdot) \in \mathcal{V}_p'$ for all $u \in \mathcal{V}_p$ and $\mathcal{A}_b(u,\cdot)$ is monotone and hemicontinuous. If in addition (D) holds, then $\mathcal{A}_b$ is coercive.*

Proof. Let $u \in \mathcal{V}_p$ be fixed. We show that $\mathcal{A}_b(u,\cdot) \in \mathcal{V}_p'$. It is clear that $\mathcal{A}_b(u,\cdot)$ is linear. Moreover, for every $v \in \mathcal{V}_p$ we get that

$$\begin{aligned} |\mathcal{A}_b(u,v)| &\le \|\nabla u\|_{\Omega,p}^{p-1}\|\nabla v\|_{\Omega,p} + \|c\|_{\infty,\Omega}\|u\|_{\Omega,p}^{p-1}\|v\|_{\Omega,p} \\ &\quad + \|b\|_{\infty,\partial\Omega}\|u\|_{p,\partial\Omega}^{p-1}\|v\|_{p,\partial\Omega} \\ &\lesssim \left(\|\nabla u\|_{p,\Omega}^{p-1} + \|u\|_{\Omega,p}^{p-1} + \|u\|_{p,\partial\Omega}^{p-1}\right)\|v\|_{\mathcal{V}_p} \lesssim \|u\|_{\mathcal{V}_p}^{p-1}\|v\|_{\mathcal{V}_p}. \end{aligned} \tag{3.2}$$

Hence, $\mathcal{A}_b(u,\cdot) \in \mathcal{V}_p'$. To show that $\mathcal{A}_b$ is monotone, let $u, v \in \mathcal{V}_p$. Using (2.4) in Lemma 2.22, we obtain that

$$\begin{aligned} \mathcal{A}_b(u,u-v) \quad &- \quad \mathcal{A}_b(v,u-v) \\ &= \quad \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla(u-v)\, dx \end{aligned} \tag{3.3}$$

$$+\int_\Omega c(x)\left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\,dx$$
$$+\int_{\partial\Omega} b(x)\left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\,d\mu\geq 0.$$

By the continuity of the norm function, we get that for all $u,v,w\in\mathcal{V}$

$$\lim_{t\downarrow 0}\mathcal{A}_b(u+tv,w)=\mathcal{A}_b(u,w).$$

Hence, $\mathcal{A}_b$ is hemicontinuous. Note that if (D)(i) is satisfied then for every $u\in\mathcal{V}_p$ one has $\mathcal{A}_b(u,u)\gtrsim\|u\|^p_{\mathcal{V}_p}$ and if (D)(ii) is satisfied, then $\|u\|_{W^{1,p}(\Omega)}$ defines an equivalent norm on $\mathcal{V}_p$ and one also has $\mathcal{A}_b(u,u)\gtrsim\|u\|^p_{\mathcal{V}_p}\approx\|u\|^p_{W^{1,p}(\Omega)}$. Hence, if (D) holds, then the coercivity of $\mathcal{A}_b$ follows from the fact that

$$\lim_{\|u\|_{\mathcal{V}_p}\to+\infty}\frac{\mathcal{A}_b(u,u)}{\|u\|_{\mathcal{V}_p}}\gtrsim\lim_{\|u\|_{\mathcal{V}_p}\to+\infty}\frac{\|u\|^p_{\mathcal{V}_p}}{\|u\|_{\mathcal{V}_p}}=+\infty.\qquad\square$$

Next, we show existence and uniqueness of solutions to Equation (3.1).

Theorem 3.7. *Assume* (A)-(D). *Then for every* $(f,g)\in X^{p_h,q_h}(\Omega,\mu)$ *there exists a unique weak solution* $u\in\mathcal{V}_p$ *of Equation* (3.1).

Proof. Let $\langle\cdot,\cdot\rangle$ denote the duality between $\mathcal{V}_p'$ and $\mathcal{V}_p$. It follows from Lemma 3.6 that for each $u\in\mathcal{V}_p$ there exists $A_b(u)\in\mathcal{V}_p'$ such that

$$\mathcal{A}_b(u,v)=\langle A_b(u),v\rangle \text{ for every } v\in\mathcal{V}_p.$$

Hence, this defines an operator $A_b:\ \mathcal{V}_p\to\mathcal{V}_p'$. By Lemma 3.6 again, the operator A_b is monotone and coercive. Moreover, A_b is continuous. That A_b is bounded follows directly from (3.2). Now we can apply Browder's theorem (see [14, Theorem 5.3.22]) to deduce that $A_b(\mathcal{V}_p)=\mathcal{V}_p'$. This means that for every $h\in\mathcal{V}_p'$ there exists $u\in\mathcal{V}_p$ such that $A_b(u)=h$ and hence $\langle A_b(u),v\rangle=\mathcal{A}_b(u,v)=\langle h,v\rangle$ for all $v\in\mathcal{V}_p$. It follows from (3.3) in the proof of Lemma 3.6 (by using (2.4) in Lemma 2.22) that $\mathcal{A}_b(u,u-v)-\mathcal{A}_b(v,u-v)>0$ for all $u,v\in\mathcal{V}_p$ with $u\neq v$. Hence, for every $h\in\mathcal{V}_p'$ there exists a unique $u\in\mathcal{V}_p$ such that $A_b(u)=h$. Now, Lemma 3.5 completes the proof. $\square$

Some properties of the operator solution to Equation (3.1) are given in the following proposition.

Proposition 3.8. *Assume* (A)-(D), *and let* $p\in[2,\infty)$ *and let* $A_b:\ \mathcal{V}_p\to\mathcal{V}_p'$ *be the continuous, surjective and bounded operator defined above. Then the operator* A_b *is invertible and its inverse* A_b^{-1} *is bounded and continuous from* $\mathcal{V}_p'$ *into* $\mathcal{V}_p$ *and from* $X^{p_h,q_h}(\Omega,\mu)$ *into* $X^{p_s,q_s}(\Omega,\mu)$.

Proof. Let $u, v \in \mathcal{V}_p$. Then by (2.5) in Lemma 2.22 we obtain that

$$\begin{aligned}
&\langle A_b(u) - A_b(v), u - v\rangle = \mathcal{A}_b(u, u - v) - \mathcal{A}_b(v, u - v) \qquad (3.4)\\
&= \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla(u - v)\, dx\\
&+ \int_\Omega c(x)\left(|u|^{p-2}u - |v|^{p-2}v\right)(u - v)\, dx\\
&+ \int_{\partial\Omega} b(x)\left(|u|^{p-2}u - |v|^{p-2}v\right)(u - v)\, d\mu\\
&\gtrsim \int_\Omega |\nabla(u - v)|^p\, dx + \int_\Omega c(x)|u - v|^p\, dx + \int_{\partial\Omega} b(x)|u - v|^p\, d\mu\\
&\gtrsim \|u - v\|_{\mathcal{V}_p}^p.
\end{aligned}$$

Note that the last inequality follows from (D). This implies that $(A_b(u) - A_b(v), u - v) > 0$ for all $u, v \in \mathcal{V}_p$ with $u \neq v$. Therefore the operator A_b is injective and surjective and hence A_b^{-1} exists. Using (3.4) we get that

$$\|u - v\|_{\mathcal{V}_p}^p \lesssim \langle A_b(u) - A_b(v), u - v\rangle \leq \|A_b(u) - A_b(v)\|_{\mathcal{V}_p'}\|u - v\|_{\mathcal{V}_p}.$$

Therefore, $\|u - v\|_{\mathcal{V}_p}^{p-1} \lesssim \|A_b(u) - A_b(v)\|_{\mathcal{V}_p'}$. Letting $u = A_b^{-1}(F)$ and $v = A_b^{-1}(G)$ we obtain that

$$\|A_b^{-1}(F) - A_b^{-1}(G)\|_{\mathcal{V}_p}^{p-1} \lesssim \|F - G\|_{\mathcal{V}_p'}$$

and this shows that $A_b^{-1} : \mathcal{V}_p' \to \mathcal{V}_p$ is continuous and bounded for every $p \in [2, \infty)$. Moreover, it follows from (C) that

$$\begin{aligned}
\|A_b^{-1}(F) - A_b^{-1}(G)\|_{X^{p_s,q_s}}^{p-1} &\lesssim \|A_b^{-1}(F) - A_b^{-1}(G)\|_{\mathcal{V}_p}^{p-1}\\
&\lesssim \|F - G\|_{\mathcal{V}_p'} \lesssim \|F - G\|_{X^{p_h,q_h}}.
\end{aligned}$$

Hence, $A_b^{-1} :\ X^{p_h,q_h}(\Omega, \mu) \to X^{p_s,q_s}(\Omega, \mu)$ is continuous and bounded. □

3.2. **General nonlinear Robin-type boundary conditions.** In this subsection we consider a more general quasi-linear elliptic boundary-value problem.

Problem 3.9. *For $p_1, q_1 \in [1, \infty]$ and $(f, g) \in X^{p_1,q_1}(\Omega, \mu)$ we consider the following quasi-linear elliptic boundary-value problem:*

$$\begin{cases} -\Delta_p u + c(x)|u|^{p-2}u = f & \text{in } \Omega,\\ d\,\mathsf{N}_p(u) + \beta(x, u)d\mu = g d\mu & \text{on } \partial\Omega, \end{cases} \qquad (3.5)$$

with a measurable function $\beta :\ \partial\Omega \times \mathbb{R} \to \mathbb{R}$.

Assumption 3.10. *For a complete, non-atomic and σ-finite measure space $(\partial\Omega, \Sigma, \mu)$ and a function $\beta : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ we consider the following assumption (E):*

$$(E)\begin{cases} \beta(x,\cdot) \text{ is odd, strictly increasing,} & \text{for } \mu-a.e.\ x \in \partial\Omega, \\ \beta(x,0) = 0 & \text{for } \mu-a.e.\ x \in \partial\Omega, \\ \beta(x,\cdot) \text{ is continuous} & \text{for } \mu-a.e.\ x \in \partial\Omega, \\ \beta(\cdot,t) & \text{is measurable for all } t \in \mathbb{R} \\ \lim_{t\to\infty} \beta(x,t) = \infty & \text{for } \mu-a.e.\ x \in \partial\Omega. \end{cases}$$

Since $\beta(x,\cdot)$ is strictly increasing for μ-a.e. $x \in \partial\Omega$, it has an inverse which we denote by $\widetilde{\beta}(x,\cdot)$. Let $B, \widetilde{B} :\ \partial\Omega \times \mathbb{R} \to [0,\infty)$ be defined by

$$B(x,t) := \int_0^{|t|} \beta(x,s)\, ds \quad \text{and} \quad \widetilde{B}(x,t) := \int_0^{|t|} \widetilde{\beta}(x,s)\, ds.$$

Remark 3.11. The functions B and $\widetilde{B}$ are complementary Musielak- Orlicz functions such that $B(x,\cdot)$ and $\widetilde{B}(x,\cdot)$ are complementary $\mathcal{N}$-functions for μ-a.e. $x \in \partial\Omega$. For more details we refer to the last section.

Definition 3.12. *A $\mathcal{N}$-function $A : [0,\infty) \to \mathbb{R}$ associated with a function $a : [0,\infty) \to [0,\infty)$ is said to satisfy a* global $(\triangle_2)$-condition *(in the sense of* [1, page 32]*) if there exists a constant $c \in (0,1]$ such that*

$$cta(t) \le A(t) \le ta(t) \quad \text{for all} \ \ t \ge 0.$$

Assumption 3.13. *Let $\Omega \subset \mathbb{R}^N$ be an open set. We assume that Assumption* 3.10 *holds and that for μ-a.e. $x \in \partial\Omega$ both $B(x,\cdot)$ and $\widetilde{B}(x,\cdot)$ satisfy the global $(\triangle_2)$-condition with a constant c* independent *of x. Then the Musielak-Orlicz functions B and $\tilde{B}$ satisfy the $(\triangle_2^0)$-condition (in the sense of Definition* 5.3)*. In that case, we let*

$$L_B(\partial\Omega,\mu) := \{u : \partial\Omega \to \mathbb{R} \ \text{measurable} : B(\cdot,u(\cdot)) \in L^1(\partial\Omega,\mu)\}$$

be the Musielak-Orlicz space. $L_{\widetilde{B}}(\partial\Omega,\mu)$ is defined similarly with B replaced by $\widetilde{B}$.

Remark 3.14. If Assumptions 3.13 holds, then by [13, Theorem 1 and 2] (see also [1, Theorem 8.19] in the case of Orlicz spaces), $L_B(\partial\Omega,\mu)$ endowed with the Luxemburg norm given by

$$\|u\|_{B,\partial\Omega} := \inf\Big\{k > 0 : \int_{\partial\Omega} B\Big(x, \frac{u(x)}{k}\Big)\, d\mu \le 1\Big\}$$

is a reflexive Banach space. The same also holds for $L_{\widetilde{B}}(\partial\Omega,\mu)$. Moreover, the same proof as for Orlicz spaces (see [1, 8.11 page 234]) gives the improved Hölder inequality for Musielak-Orlicz spaces:

$$\Big|\int_{\partial\Omega} uv\, d\mu\Big| \le 2\|u\|_{B,\partial\Omega}\|v\|_{\widetilde{B},\partial\Omega}. \tag{3.6}$$

Definition 3.15. *Under the assumptions* (A) *and* 3.13 *we can define the space* $\mathcal{V}$ *by*

$$\mathcal{V} := \mathcal{V}(\Omega,\mu,B) := \{u \in \mathcal{W}^{1,p}(\Omega) : B(\cdot,u(\cdot)) \in L^1(\partial\Omega,\mu)\}$$

endowed with the norm defined by

$$\|u\|_{\mathcal{V}} := \|u\|_{W^{1,p}(\Omega)} + \|u\|_{B,\partial\Omega}.$$

Remark 3.16. In this case $\mathcal{V}$ is a reflexive Banach space which is continuously embedded into $W^{1,p}(\Omega)$. Note that under (C)+(D)(ii) we get that $\mathcal{V}$ is also continuously embedded into $X^{p_s,q_s}(\Omega,\mu)$.

Lemma 3.17. *Assume* (A) *and that Assumption* 3.13 *holds. Then* $\beta(\cdot,u(\cdot)) \in L_{\widetilde{B}}(\partial\Omega,\mu)$ *for all* $u \in L_B(\partial\Omega,\mu)$.

Proof. It follows from the assumptions that, for all $\xi \in \mathbb{R}$,

$$\widetilde{B}(x,\beta(x,\xi)) \lesssim \xi\beta(x,\xi) \lesssim B(x,\xi).$$

Hence,

$$\int_{\partial\Omega} \tilde{B}(x,\beta(x,u(x)))\, d\mu \lesssim \int_{\partial\Omega} B(x,u(x))\, d\mu < \infty. \qquad \Box$$

Lemma 3.18. *Assume that Assumption* 3.13 *holds. Then, for every* $\alpha > 0$ *there is a constant* C_α *such that for all* $u, v \in L_B(\partial\Omega,\mu)$ *with* $\|u\|_{B,\partial\Omega} \le \alpha$ *the following inequality holds:*

$$\Big|\int_{\partial\Omega} \beta(x,u)v\, d\mu\Big| \le C_\alpha \|v\|_{B,\partial\Omega}.$$

Proof. Without loss of generality we may assume that $\|u\|_{B,\partial\Omega} \ne 0$ and that $\alpha \ge 1$. Let

$$w := \begin{cases} \frac{|u|}{\|u\|_{B,\partial\Omega}} & \text{if } \|u\|_{B,\partial\Omega} \le 1 \\ |u| & \text{if } \|u\|_{B,\partial\Omega} \ge 1. \end{cases}$$

Let $K = K(\alpha) \ge 1$ be a constant (which is independent of x) such that $B(x,\xi) \le K \cdot B(x,\xi/\alpha)$ for all $\xi \in \mathbb{R}_+$ and μ-a.e. $x \in \partial\Omega$. Then (using the fact that $\|w\|_{B,\partial\Omega} \le \alpha$)

$$\Big|\int_{\partial\Omega} \beta(x,u)v\, d\mu\Big| \le \|w\|_{B,\partial\Omega}\|v\|_{B,\partial\Omega} \int_{\partial\Omega} \frac{\beta(x,w)}{\|w\|_{B,\partial\Omega}} \cdot \frac{|v|}{\|v\|_{B,\partial\Omega}}\, d\mu$$

$$\leq \|w\|_{B,\partial\Omega}\|v\|_{B,\partial\Omega}\left\{\int_{\partial\Omega}\widetilde{B}\left(x,\beta(x,w)/\|w\|_{B,\partial\Omega}\right)\,d\mu+1\right\}$$
$$\leq \|w\|_{B,\partial\Omega}\|v\|_{B,\partial\Omega}\left\{\int_{\partial\Omega}\widetilde{B}\left(x,\beta(x,w)\right)\,d\mu+1\right\}$$
$$\leq \|w\|_{B,\partial\Omega}\|v\|_{B,\partial\Omega}\left\{\int_{\partial\Omega}B\left(x,u(x)\right)\,d\mu+1\right\}$$
$$\leq \|w\|_{B,\partial\Omega}\|v\|_{B,\partial\Omega}K\cdot\left\{\int_{\partial\Omega}B\left(x,u(x)/\alpha\right)\,d\mu+1\right\}$$
$$\leq 2K\cdot\alpha\|v\|_{B,\partial\Omega}.$$

Then the claimed constant C_α is given by $2K\alpha$. □

Definition 3.19. *A function $u \in \mathcal{V}$ is said to be a weak solution of* (3.5) *if for every $\varphi \in \mathcal{V}$*

$$\mathcal{A}_\beta(u,\varphi) = \int_\Omega f\varphi\,dx + \int_{\partial\Omega} g\varphi\,d\mu,$$

provided that the integrals in the right-hand side exist and where for $u, v \in \mathcal{V}$,

$$\mathcal{A}_\beta(u,v) := \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega c(x)|u|^{p-2}uv\,dx + \int_{\partial\Omega}\beta(x,u)v\,d\mu.$$

Lemma 3.20. *Assume (A)+(B)(ii), and that Assumption 3.13 holds and let $u \in \mathcal{V}$ be fixed. Then the mapping $\mathcal{V} \to \mathbb{R}$, $v \mapsto \mathcal{A}_\beta(u,v)$ belongs to $\mathcal{V}'$ and $\mathcal{A}_\beta$ is monotone, hemicontinuous and coercive.*

Proof. Let $u \in \mathcal{V}$ be fixed. Using the classical Hölder inequality, (3.6) and Lemma 3.17 we get that, for all $v \in \mathcal{V}$,

$$\begin{aligned}|\mathcal{A}_\beta(u,v)| &\leq \|\nabla u\|_{\Omega,p}^{p-1}\|\nabla v\|_{\Omega,p} + \|c\|_{\infty,\Omega}\|u\|_{\Omega,p}^{p-1}\|v\|_{\Omega,p} + \\ &\quad 2\|\beta(\cdot,u)\|_{\widetilde{B},\partial\Omega}\|v\|_{B,\partial\Omega} \\ &\lesssim \left(\|u\|_{W^{1,p}(\Omega)}^{p-1} + \|\beta(\cdot,u)\|_{\widetilde{B},\partial\Omega}\right)\|v\|_{\mathcal{V}}.\end{aligned}$$

Since $\mathcal{A}_\beta(u,\cdot)$ is linear we have that $\mathcal{A}_\beta(u,\cdot) \in \mathcal{V}'$ for every $u \in \mathcal{V}$. Next, using (2.4) in Lemma 2.22 and the fact that $\beta(x,\cdot)$ is monotone increasing, we obtain that $\mathcal{A}_\beta$ is monotone. It follows from the continuity of the norm function and the continuity of $\beta(x,\cdot)$ that $\mathcal{A}_\beta$ is hemicontinuous. Finally, since $\widetilde{B}$ satisfies the $(\triangle_2^0)$-condition, it follows from Corollary 5.10 that

$$\lim_{\|u\|_{B,\partial\Omega}\to\infty}\frac{\int_{\partial\Omega}u\beta(x,u)\,d\mu}{\|u\|_{B,\partial\Omega}} = +\infty$$

and hence

$$\lim_{\|u\|_{\mathcal{V}}\to+\infty}\frac{\mathcal{A}_\beta(u,u)}{\|u\|_{\mathcal{V}}}=+\infty.$$

□

Next, we show existence and uniqueness of weak solutions to Equation (3.5).

Theorem 3.21. *Assume* (A)+(B)(ii) *and that Assumption* 3.13 *holds. Then for every* $f\in(\mathcal{W}^{1,p}(\Omega))'\cap L^1_{\text{loc}}(\Omega)$ *and* $g\in L_{\widetilde{B}}(\partial\Omega,\mu)$ *there exists a unique weak solution* $u\in\mathcal{V}$ *to Equation* (3.5)*.*

Proof. It follows from Lemma 3.20 that for each $u\in\mathcal{V}$ there exists $A_\beta(u)\in\mathcal{V}'$ such that $\mathcal{A}_\beta(u,v)=\langle A_\beta(u),v\rangle$ for all $v\in\mathcal{V}$. Hence, this defines an operator $A_\beta:\ \mathcal{V}\to\mathcal{V}'$. It follows from Lemmas 3.18 and 3.20 that the operator $A_\beta:\ \mathcal{V}\to\mathcal{V}'$ is hemicontinuous, monotone, coercive and bounded. Therefore $A_\beta(\mathcal{V})=\mathcal{V}'$ and hence for every $h\in\mathcal{V}'$ there exist $u\in\mathcal{V}$ such that $A_\beta(u)=h$. The fact that the mapping

$$\mathcal{V}\to\mathbb{R},\ \ v\mapsto\int_\Omega fv\,dx+\int_{\partial\Omega}gv\,d\mu$$

belongs to $\mathcal{V}'$ completes the existence part of the proof. Next, we show uniqueness. By (2.4) in Lemma 2.22, we have that for all $u,v\in\mathcal{V}$,

$$\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\nabla(u-v)\gtrsim(|\nabla u|+|\nabla v|)^{p-2}\,|\nabla u-\nabla v|^2\geq 0$$

and

$$c(x)\left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\gtrsim c(x)\,(|u|+|v|)^{p-2}\,|u-v|^2\geq 0.$$

Using the preceding two inequalities and the fact that $\beta(x,\cdot)$ is monotone; that is, for all $t,s\in\mathbb{R}$,

$$(\beta(x,t)-\beta(x,s))\,(t-s)\geq 0,$$

we obtain that, for all $u,v\in\mathcal{V}$,

$$\mathcal{A}_\beta(u,u-v)-\mathcal{A}_\beta(v,u-v)\geq 0.$$

In particular, we have that

$$\mathcal{A}_\beta(u,u-v)-\mathcal{A}_\beta(v,u-v)>0$$

for all $u,v\in\mathcal{V}$ with $u\neq v$. Hence, the weak solution is unique and this completes the proof. □

Assumption 3.22. *Here we assume that Assumption* 3.13 *holds and that the following* growth condition *(G) is satisfied with a constant* $c \in (0,1]$*:*

$$(G) \quad c|\beta(x,\xi-\eta)| \le |\beta(x,\xi)-\beta(x,\eta)|$$

for μ*-a.e.* $x \in \partial\Omega$ *and for all* $\xi,\eta \in \mathbb{R}$*.*

We have the following properties of the operator solution to Equation (3.5).

Proposition 3.23. *Assume* (A)+(B)(ii), *and that Assumption* 3.22 *holds and that* $p \in [2,\infty)$. *Let* $A_\beta : \mathcal{V} \to \mathcal{V}'$ *be the continuous and bounded operator constructed in the proof of Theorem* 3.21. *Then the operator* A_β *is injective and hence invertible and its inverse* A_β^{-1} *is continuous and bounded from* $\mathcal{V}'$ *into* $\mathcal{V}$*.*

Proof. Since, for μ-a.e. $x \in \partial\Omega$ and for all $t,s \in \mathbb{R}$,

$$\beta(x,t-s)\cdot(t-s) \ge 0 \quad \text{and} \quad (\beta(x,t)-\beta(x,s))(t-s) \ge 0,$$

it follows from Assumption 3.22 that

$$(\beta(x,t)-\beta(x,s))\,(t-s) \ge c\cdot\beta(x,t-s)\cdot(t-s) \tag{3.7}$$

for μ-a.e. $x \in \partial\Omega$ and for all $t,s \in \mathbb{R}$. Let $u,v \in \mathcal{V}$. Using (2.5) in Lemma 2.22, (3.7) and the $(\triangle_2^0)$-conditions on B and $\widetilde{B}$ in Assumption 3.13, we obtain that

$$\begin{aligned}
&\langle A_\beta(u)-A_\beta(v),u-v\rangle = \mathcal{A}_\beta(u,u-v)-\mathcal{A}_\beta(v,u-v)\\
\gtrsim\;& \int_\Omega |\nabla(u-v)|^p + \int_\Omega c(x)|u-v|^p\\
&+\int_{\partial\Omega}(\beta(x,u)-\beta(x,v))(u-v)\,d\mu\\
\gtrsim\;& \|u-v\|^p_{W^{1,p}(\Omega)} + \int_{\partial\Omega}\beta(x,u-v)(u-v)\,d\mu\\
\gtrsim\;& \|u-v\|^p_{W^{1,p}(\Omega)} + \int_{\partial\Omega}B(x,u-v)\,d\mu.
\end{aligned} \tag{3.8}$$

Therefore $\langle A_\beta(u)-A_\beta(v),u-v\rangle > 0$ for all $u,v \in \mathcal{V}$ with $u \ne v$. This shows that A_β is injective and hence A_β^{-1} exists. We get from the inequality

$$\mathcal{A}_\beta(u,u) = \langle A_\beta(u),u\rangle \le \|A_\beta(u)\|_{\mathcal{V}'}\|u\|_{\mathcal{V}}$$

and the coercivity of $\mathcal{A}_\beta$ that

$$\lim_{\|u\|_{\mathcal{V}}\to\infty}\|A_\beta(u)\|_{\mathcal{V}'} = \infty.$$

Hence $A_\beta^{-1}:\ \mathcal{V}'\to\mathcal{V}$ is bounded. Next, we show that $A_\beta^{-1}:\ \mathcal{V}'\to\mathcal{V}$ is continuous. Assume that A_β^{-1} is not continuous. Then there are a sequence $F_n\in\mathcal{V}'$ with $F_n\to F$ in $\mathcal{V}'$ and a constant $K>0$ such that

$$\|A_\beta^{-1}(F_n)-A_\beta^{-1}(F)\|_{\mathcal{V}}\geq K \tag{3.9}$$

for all $n\in\mathbb{N}$. Let $u_n:=A_\beta^{-1}(F_n)$ and $u:=A_\beta^{-1}(F)$. As (F_n) is a bounded sequence and A_β^{-1} is bounded, we get that $(u_n)_n$ is bounded in $\mathcal{V}$. Since $\mathcal{V}$ is a reflexive Banach space, by possibly passing to a subsequence, we may assume that $(u_n)_n$ converges weakly to some $v\in\mathcal{V}$. Since $A_\beta(u_n)-A_\beta(v)\to F-A_\beta(v)$ in $\mathcal{V}'$ and $(u_n-v)_n$ converges weakly to zero in $\mathcal{V}$, we get that

$$\lim_{n\to\infty}\langle A_\beta(u_n)-A_\beta(v),u_n-v\rangle=0.$$

Since, by (3.8), for every $n\in\mathbb{N}$,

$$\|u_n-v\|^p_{W^{1,p}(\Omega)}+\int_{\partial\Omega}B(x,u_n-v)\,d\mu\lesssim\langle A_\beta(u_n)-A_\beta(v),u_n-v\rangle,$$

it follows that

$$\lim_{n\to\infty}\|u_n-v\|^p_{W^{1,p}(\Omega)}=0\quad\text{and}\quad\lim_{n\to\infty}\int_{\partial\Omega}B(x,u_n-v)\,d\mu=0.$$

Hence (using the Δ_2-condition) we get that $u_n\to v$ in $\mathcal{V}$. Since A_β is demicontinuous (this follows from the fact that A_β is hemicontinuous, monotone and bounded) we get that

$$F_n=A_\beta(u_n)\rightharpoonup A_\beta(v)\text{ in }\mathcal{V}'\quad\text{and}\quad F_n\to F=A_\beta(u)\text{ in }\mathcal{V}'.$$

The uniqueness of the weak limit implies that $A_\beta(u)=F=A_\beta(v)$ and therefore, by the injectivity of A_β, we obtain that $u=v$. This shows that

$$\lim_{n\to\infty}\|A_\beta^{-1}(F_n)-A_\beta^{-1}(F)\|_{\mathcal{V}}=\lim_{n\to\infty}\|u_n-u\|_{\mathcal{V}}=0,$$

and this contradicts (3.9). Hence, $A_\beta:\mathcal{V}'\to\mathcal{V}$ is continuous. □

Remark 3.24. First we remark that, given a Borel measure μ on $\partial\Omega$, the Assumptions 3.10, 3.13 and 3.22 involve only the function β. Next, let $p\in[2,\infty)$, $b:\partial\Omega\to(0,\infty)$ be a strictly positive and μ-measurable function and let

$$\beta(x,\xi):=b(x)|\xi|^{p-2}\xi,\quad\xi\in\mathbb{R}.$$

Then by [5, Example 4.17] the function β satisfies Assumption 3.22 (and hence 3.13 and 3.10).

4. Main Results

In this section, we state and prove the main results of this article. In particular, these results imply that the weak solutions of (3.1) and (3.5) are globally bounded on $\overline{\Omega}$ whenever the initial data f and g belong to certain L^p-spaces.

Definition 4.1. *For $p, q \in [1,\infty]$ such that $1/p + 1/q \leq 1$ we let $[p,q]' := s \in [1,\infty]$ be such that $1/p + 1/q + 1/s = 1$.*

Lemma 4.2. *Assume (A)+(B)+(C) and that $p \in [2,\infty)$. Then for all $u, v \in \mathcal{V}_p = W^1_{p,p}(\Omega, \partial\Omega, \mu)$ and $k \geq 0$ we have the inequality*

$$\mathcal{A}_b(w_k, w_k) \lesssim \mathcal{A}_b(u, w_k) - \mathcal{A}_b(v, w_k),$$

where w_k is defined for $u, u \in \mathcal{V}_p$ and $k \in [0,\infty)$ by $w_k := (|u - v| - k)^+ \operatorname{sgn}(u - v)$.

Proof. Let $u, v \in \mathcal{V}_p$, $w := u - v$, $w_k := (|w| - k)^+ \operatorname{sgn}(w)$ and $A_k := \{x \in \overline{\Omega} : |w(x)| \geq k\}$. Then we get directly from the definition of $\mathcal{A}_b$ that

$$\begin{aligned}\mathcal{A}_b(u, w_k) - \mathcal{A}_b(v, w_k) \quad &= \quad \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla\right) \nabla w_k \, dx \\ &\quad + \int_\Omega c(x) \left(|u|^{p-2}u - |v|^{p-2}v\right) w_k \, dx \\ &\quad + \int_{\partial\Omega} b(x) \left(|u|^{p-2}u - |v|^{p-2}v\right) w_k \, d\mu.\end{aligned}$$

Using the fact that $w_k = 0$ on $\overline{\Omega} \setminus A_k$ and $\nabla w_k = \nabla w \chi_{A_k}$ we get

$$\begin{aligned}\mathcal{A}_b(u, w_k) \quad - \quad & \mathcal{A}_b(v, w_k) = \\ & \int_{A(k)\cap\Omega} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \nabla(u - v) \, dx \\ & + \int_{A(k)\cap\Omega} c(x) \left(|u|^{p-2}u - |v|^{p-2}v\right) w_k \, dx \\ & + \int_{A(k)\cap\partial\Omega} b(x) \left(|u|^{p-2}u - |v|^{p-2}v\right) w_k \, d\mu.\end{aligned}$$

Applying (2.5) in Lemma 2.22 to the first integral we get the inequality

$$\begin{aligned}\mathcal{A}_b(u, w_k) \quad - \quad & \mathcal{A}_b(v, w_k) \geq \\ & c_p \int_{A(k)\cap\Omega} |\nabla w_k|^p + c_p \int_{A(k)\cap\Omega} c(x)|w_k|^p \, dx +\end{aligned} \tag{4.1}$$

$$c_p \int_{A(k)\cap\partial\Omega} b(x)|w_k|^p \, d\mu +$$
$$\int_{A(k)\cap\Omega} c(x) \left(|u|^{p-2}uw_k - |v|^{p-2}vw_k - c_p|w_k|^p\right) \, dx +$$
$$\int_{A(k)\cap\partial\Omega} b(x) \left(|u|^{p-2}uw_k - |v|^{p-2}vw_k - c_p|w_k|^p\right) \, d\mu.$$

Since on A_k we have that $|u-v| \geq k$, by using (2.6) in Lemma 2.22, we get (on A_k) that

$$\left||u|^{p-2}u - |v|^{p-2}v\right| \geq c_p|u-v|^{p-1} \geq c_p \left||u-v| - k\right|^{p-1} = c_p|w_k|^{p-1}$$

and hence (by multiplying both sides of the inequality by $|w_k|$)

$$|u|^{p-2}uw_k - |v|^{p-2}vw_k - c_p|w_k|^p \geq 0.$$

This, together with (4.1), shows that

$$c_p\mathcal{A}_b(w_k, w_k) \leq \mathcal{A}_b(u, w_k) - \mathcal{A}_b(v, w_k). \qquad \square$$

Lemma 4.3. *Assume* (A)+(B)(ii), *Assumption* 3.22 *and that* $p \in [2, \infty)$. *Then for all* $u, v \in \mathcal{V}$

$$\mathcal{A}_\beta(w_k, w_k) \lesssim \mathcal{A}_\beta(u, w_k) - \mathcal{A}_\beta(v, w_k),$$

where w_k *is defined for* $u, u \in \mathcal{V}$ *and* $k \in [0, \infty)$ *by* $w_k := (|u-v|-k)^+ \operatorname{sgn}(u-v)$.

Proof. Let $u, v \in \mathcal{V}_p$, $w := u - v$, $w_k := (|w| - k)^+ \operatorname{sgn}(w)$ and $A_k := \{x \in \overline{\Omega} : |w(x)| \geq k\}$. Then

$$\begin{aligned}\mathcal{A}_\beta(u, w_k) - \mathcal{A}_\beta(v, w_k) &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \nabla w_k \, dx \\ &\quad + \int_\Omega c(x) \left(|u|^{p-2}u - |v|^{p-2}v\right) w_k \, dx \\ &\quad + \int_{\partial\Omega} (\beta(x,u) - \beta(x,v))\, w_k \, d\mu.\end{aligned}$$

Using the fact that $w_k = 0$ on $\overline{\Omega} \setminus A_k$ and $\nabla w_k = \nabla w \chi_{A_k}$ we obtain that

$$\begin{aligned}\mathcal{A}_\beta(u, w_k) \quad - \quad & \mathcal{A}_\beta(v, w_k) = \\ & \int_{A(k)\cap\Omega} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \nabla(u-v) \, dx \\ & + \int_{A(k)\cap\Omega} c(x) \left(|u|^{p-2}u - |v|^{p-2}v\right) w_k \, dx\end{aligned}$$

$$+\int_{A(k)\cap\partial\Omega}(\beta(x,u)-\beta(x,v))w_k\,d\mu.$$

Using (2.5) in Lemma 2.22 and the constant c in the growth condition (G) we obtain that

$$\begin{aligned}\mathcal{A}_\beta(u,w_k) \quad - \quad & \mathcal{A}_\beta(v,w_k) \geq \\ & c_p\int_{A(k)\cap\Omega}|\nabla w_k|^p + c_p\int_{A(k)\cap\Omega}c(x)|w_k|^p\,dx + \\ & \int_{A(k)\cap\partial\Omega}c\beta(x,w_k)w_k\,d\mu + \\ & \int_{A(k)\cap\Omega}c(x)\left(|u|^{p-2}uw_k-|v|^{p-2}vw_k-c_p|w_k|^p\right)\,dx + \\ & \int_{A(k)\cap\partial\Omega}\left[\beta(x,u)-\beta(x,v)-c\beta(x,w_k)\right]w_k\,d\mu.\end{aligned}$$

As in the proof of Lemma 4.2, we have that

$$\left(|u|^{p-2}uw_k-|v|^{p-2}vw_k-c_p|w_k|^p\right)\geq 0.$$

For the last integral, using the growth condition (G) and the fact that $\beta(x,\cdot)$ is strictly increasing for μ-a.e. $x\in\partial\Omega$, we have that for μ- a.e. $x\in\partial\Omega\cap A_k^+$ with $A_k^+ := \{x\in\overline{\Omega} : u(x)-v(x)\geq k\}$,

$$\begin{aligned}c\beta(x,w_k(x)) \quad &= \quad c\beta(x,u(x)-v(x)-k)\leq c\beta(x,u(x)-v(x)) \\ &\leq \quad \beta(x,u(x))-\beta(x,v(x)).\end{aligned}$$

Multiplying this inequality by $w_k(x)\geq 0$ implies that for μ- a.e. $x\in\partial\Omega\cap A_k^+$:

$$[\beta(x,u(x))-\beta(x,v(x))-c\beta(x,w_k(x))]\cdot w_k(x)\geq 0. \tag{4.2}$$

Similarly, for μ-a.e. $x\in\partial\Omega\cap A_k^-$, where $A_k^- := \{x\in\overline{\Omega} : u(x)-v(x)\leq -k\}$, we have that

$$\begin{aligned}c\beta(x,w_k(x)) \quad &= \quad c\beta(x,u(x)-v(x)+k)\geq c\beta(x,u(x)-v(x)) \\ &\geq \quad \beta(x,u(x))-\beta(x,v(x)).\end{aligned}$$

Hence, multiplying this inequality by $w_k(x)\leq 0$, we get that for μ-a.e. $x\in\partial\Omega\cap A_k^-$

$$[\beta(x,u(x))-\beta(x,v(x))-c\beta(x,w_k(x))]\cdot w_k(x)\geq 0. \tag{4.3}$$

Hence, using (4.2) and (4.3), we obtain that

$$\min\{c,c_p\}\mathcal{A}_\beta(w_k,w_k)\leq\mathcal{A}_\beta(u,w_k)-\mathcal{A}_\beta(v,w_k). \qquad \square$$

Lemma 4.4. *Assume* (A)+(B)+(C)+(D), $p \in [2,\infty)$, *and* $\chi_{\overline{\Omega}} \in X^{1,1}(\Omega,\mu)$ (*that is,* Ω *has finite Lebesgue measure and* $\mu(\partial\Omega) < \infty$) *and let*

$$1 \le p_3 < p_s/(p-1) \quad \textit{and} \quad 1 \le q_3 < q_s/(p-1).$$

If there exists a constant C_1 *such that*

$$\mathcal{A}_b(u,w_k) - \mathcal{A}_b(v,w_k) \le C_1 \|w_k\|_{\mathcal{V}_p} \|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)} \tag{4.4}$$

for all $u, v \in \mathcal{V}_p$, *then there exists a constant* $C_2 \ge 0$ (*independent of* C_1) *such that*

$$\|u-v\|^{p-1}_{X^{\infty,\infty}(\Omega,\mu)} \le C_2 \cdot C_1^{1/(1-p)},$$

where w_k *is defined for* $u, v \in \mathcal{V}_p$ *and* $k \in [0,\infty)$ *by* $w_k := (|u-v|-k)^+ \operatorname{sgn}(u-v)$ *and* A_k *is defined by* $A_k := \{x \in \overline{\Omega} : |u(x)-v(x)| \ge k\}$.

Proof. Let $u, v \in \mathcal{V}_p$, $w := u - v$, $w_k := (|w|-k)^+ \operatorname{sgn}(w)$ and $A_k := \{x \in \overline{\Omega} : |w| \ge k\}$. From (D), and Lemma 4.2 and (4.4), we get that, for every $k > 0$,

$$\begin{aligned} \|w_k\|^p_{\mathcal{V}_p} &\lesssim \mathcal{A}_b(w_k,w_k) \lesssim \mathcal{A}_b(u,w_k) - \mathcal{A}_b(v,w_k) \\ &\le C_1 \|w_k\|_{\mathcal{V}_p} \|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)}. \end{aligned}$$

Hence, for every $k > 0$, $\|w_k\|^{p-1}_{\mathcal{V}_p} \lesssim C_1 \|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)}$. Using assumption (C) we obtain that

$$\|w_k\|^{p-1}_{X^{p_s,q_s}(\Omega,\mu)} \lesssim C_1 \|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)},$$

for every $k > 0$. Let $h > k$. Then $A_h \subset A_k$ and on A_h we have that $|w_k| \ge h - k$. Therefore,

$$\|(h-k)\chi_{A_h}\|^{p-1}_{X^{p_s,q_s}(\Omega,\mu)} \lesssim C_1 \|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)}.$$

This shows that

$$\|\chi_{A_h}\|^{p-1}_{X^{p_s,q_s}(\Omega,\mu)} \lesssim C_1 (h-k)^{-(p-1)} \|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)}.$$

Let $C_3 := \|1_{\overline{\Omega}}\|_{X^{p_s,q_s}(\Omega,\mu)}$, and

$$\delta := \min\left\{\frac{p_s}{p_3}, \frac{q_s}{p_3}\right\} > p-1 \text{ and } \delta_0 := \frac{\delta}{p-1} > 1.$$

Then

$$\begin{aligned} \|C_3^{-p_s/p_3}\chi_{A_k}\|_{L^{p_3}(\Omega)} &= \|C_3^{-1}\chi_{A_k}\|^{p_s/p_3}_{L^{p_s}(\Omega)} \le \|C_3^{-1}\chi_{A_k}\|^{\delta}_{L^{p_s}(\Omega)} \\ &\le \|\chi_{A_k}\|^{\delta}_{X^{p_s,q_s}(\Omega,\mu)} C_3^{-\delta} \end{aligned}$$

and

$$\begin{aligned} \|C_3^{-q_s/q_3}\chi_{A_k}\|_{L^{q_3}(\partial\Omega,\mu)} &= \|C_3^{-1}\chi_{A_k}\|^{q_s/q_3}_{L^{q_s}(\partial\Omega,\mu)} \le \|C_3^{-1}\chi_{A_k}\|^{\delta}_{L^{q_s}(\partial\Omega,\mu)} \\ &\le \|\chi_{A_k}\|^{\delta}_{X^{p_s,q_s}(\Omega,\mu)} C_3^{-\delta}. \end{aligned}$$

This shows that for $C_\Omega := C_3^{p_s/p_3-\delta} + C_3^{q_s/q_3-\delta}$ we have

$$\|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)} \le C_\Omega \|\chi_{A_k}\|^{\delta}_{X^{p_s,q_s}(\Omega,\mu)}.$$

Hence,

$$\begin{aligned} \|\chi_{A_h}\|^{p-1}_{X^{p_s,q_s}(\Omega,\mu)} &\lesssim C_1(h-k)^{-(p-1)} \|\chi_{A_k}\|^{\delta}_{X^{p_s,q_s}(\Omega,\mu)} \\ &= C_1(h-k)^{-(p-1)} \left[\|\chi_{A_k}\|^{p-1}_{X^{p_s,q_s}(\Omega,\mu)}\right]^{\delta_0}. \end{aligned}$$

It follows from Lemma 2.23 with $\psi(h) := \|\chi_{A_h}\|^{p-1}_{X^{p_s,q_s}(\Omega,\mu)}$ that there exists a constant C_2 (independent of C_1) such that

$$\|\chi_{A_K}\|^{p-1}_{X^{p_s,q_s}(\Omega,\mu)} = 0 \ \text{ with } \ K := C_2 \cdot C_1^{1/(p-1)},$$

and this completes the proof. □

Lemma 4.5. *Assume* (A)+(B)(ii), *Assumption* 3.22 *holds,* $\chi_{\overline{\Omega}} \in X^{1,1}(\Omega,\mu)$ *and* $p \in [2,\infty)$*. Let* $1 \le p_3 < p_s/(p-1)$ *and* $1 \le q_3 < q_s/(p-1)$ *and assume that*

$$(C') \ \|u\|_{X^{p_s,q_s}(\Omega,\mu)} \lesssim \|u\|_{W^{1,p}(\Omega)} \ \textit{for all} \ \ u \in \mathcal{V}.$$

If there exists a constant $C_1 \ge 0$ *such that*

$$\mathcal{A}_\beta(u,w_k) - \mathcal{A}_\beta(v,w_k) \le C_1 \|w_k\|_{W^{1,p}(\Omega)} \|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)}, \tag{4.5}$$

for all $k > 0$ *and all* $u, v \in \mathcal{V}$*, then there exists a constant* $C_2 \ge 0$ *(independent of* C_1*) such that*

$$\|u-v\|^{p-1}_{X^{\infty,\infty}(\Omega,\mu)} \le C_2 \cdot C_1^{1/(p-1)},$$

where w_k *is defined for* $u, v \in \mathcal{V}$ *and* $k \in [0,\infty)$ *by* $w_k := (|u-v|-k)^+ \operatorname{sgn}(u-v)$ *and* A_k *is defined by* $A_k := \{x \in \overline{\Omega} : |u(x)-v(x)| \ge k\}$*.*

Proof. Let $u, v \in \mathcal{V}$, $w := u - v$, $w_k := (|w|-k)^+ \operatorname{sgn}(w)$ and $A_k := \{x \in \overline{\Omega} : |w| \ge k\}$. From Lemma 4.3 and (4.5) we get that

$$\begin{aligned} \|w_k\|^p_{W^{1,p}(\Omega)} &\lesssim \mathcal{A}_\beta(w_k,w_k) \lesssim \mathcal{A}_\beta(u,w_k) - \mathcal{A}_\beta(v,w_k) \\ &\le C_1 \|w_k\|_{W^{1,p}(\Omega)} \|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)}. \end{aligned}$$

Hence, for $k > 0$ we have $\|w_k\|^{p-1}_{W^{1,p}(\Omega)} \lesssim C_1\|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)}$. Using the assumption (C') we get that

$$\|w_k\|^{p-1}_{X^{p_s,q_s}(\Omega,\mu)} \lesssim C_1\|\chi_{A_k}\|_{X^{p_3,q_3}(\Omega,\mu)},$$

for every $k > 0$. The remainder of the proof follows the lines in the proof of Lemma 4.4. □

Now we are ready to state and prove the main results of this article which are the following two theorems.

Theorem 4.6. *Assume* (A)+(B)+(C)+(D), $p \in [2,\infty)$, $\chi_{\overline{\Omega}} \in X^{1,1}(\Omega,\mu)$, $p_s > p$, $q_s > p$ *and*

$$p_1 > \frac{p_s}{p_s - p}, \quad q_1 > \frac{q_s}{q_s - p}.$$

Then there exists a constant $C_2 \geq 0$ *such that*

$$\|A_b^{-1}(f_1,g_1) - A_b^{-1}(f_2,g_2)\|_{X^{\infty,\infty}(\Omega,\mu)} \leq C_2\|(f_1-f_2,g_1-g_2)\|^{1/(p-1)}_{X^{p_1,q_1}(\Omega,\mu)}$$

for all $(f_1,g_1),(f_2,g_2) \in X^{p_h,q_h}(\Omega,\mu)$ *with* $(f_1-f_2,g_1-g_2) \in X^{p_1,q_1}(\Omega,\mu)$.

Theorem 4.7. *Assume* (A)+(B)(ii), *Assumption* 3.22 *holds,* $p \in [2,\infty)$, $\chi_{\overline{\Omega}} \in X^{1,1}(\Omega,\mu)$, $p_s > p$, $q_s > p$ *and assume that Assumption (C') in Lemma* 4.5 *holds and let*

$$p_1 > \frac{p_s}{p_s - p}, \quad q_1 > \frac{q_s}{q_s - p}.$$

Then there exists a constant $C_2 \geq 0$ *such that*

$$\|A_\beta^{-1}(f_1,g_1) - A_\beta^{-1}(f_2,g_2)\|_{X^{\infty,\infty}(\Omega,\mu)} \leq C_2\|(f_1-f_2,g_1-g_2)\|^{1/(p-1)}_{X^{p_1,q_1}(\Omega,\mu)}$$

for all $(f_1,g_1),(f_2,g_2) \in X^{p_h,q_h}(\Omega,\mu)$ *with* $(f_1-f_2,g_1-g_2) \in X^{p_1,q_1}(\Omega,\mu)$.

Proof of Theorems 4.6 and 4.7. Let $p_3, q_3 \in [1,\infty)$ be such that $1/p_3 + 1/p_s + 1/p_1 = 1$ and $1/q_3 + 1/q_s + 1/q_1 = 1$. Then

$$\frac{1}{p_3} = 1 - \frac{1}{p_s} - \frac{1}{p_1} > \frac{p_s}{p_s} - \frac{1}{p_s} - \frac{p_s - p}{p_s} = \frac{p-1}{p_s} \Rightarrow p_3 < \frac{p_s}{p-1}.$$

Similarly, we get that $q_3 < q_s/(p-1)$. In view of Lemma 4.4 and Lemma 4.5 we have to show that for $C_1 := \|(f,g)\|_{X^{p_1,q_1}(\Omega,\mu)}$ we have (with $f := f_1 - f_2$ and $g := g_1 - g_2$)

$$\int_\Omega f\varphi\,dx + \int_{\partial\Omega} g\varphi\,d\mu \leq C_1\|\varphi\| \cdot \|\chi_{\mathrm{supp}(\varphi)}\|_{X^{p_3,q_3}(\Omega,\mu)}$$

for all $\varphi \in \mathcal{V}_p$ with $\|\varphi\| := \|\varphi\|_{\mathcal{V}_p}$ (for Theorem 4.6) or for all $\varphi \in \mathcal{V}$ with $\|\varphi\| := \|\varphi\|_{W^{1,p}(\Omega)}$ (for Theorem 4.7). But this follows from the classical Hölder inequality. □

The following corollary shows that all the assumptions involving the regularity of the open set Ω in our main theorems are satisfied if $\Omega \subset \mathbb{R}^N$ is a bounded (ε, δ)-domain.

Corollary 4.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded (ε, δ)- domain, $p \in [2, N)$ and let μ be an upper d-Ahlfors measure on $\partial\Omega$ with $0 < N - d < p$ such that $\mu(\partial\Omega) < \infty$. Let $b \in L^\infty(\partial\Omega, \mu)$ be non-negative and $c \in L^\infty(\Omega)$ be such that $c \geq c_0$ for some constant $c_0 > 0$. Let $p_1 > N/p$ and $q_1 > pd/(p^2 - p(N-d))$. Then there exists a constant $C_2 \geq 0$ such that*

$$\|A_b^{-1}(f_1, g_1) - A_b^{-1}(f_2, g_2)\|_{X^{\infty,\infty}(\Omega,\mu)} \leq C_2 \|(f_1 - f_2, g_1 - g_2)\|^{1/(p-1)}_{X^{p_1,q_1}(\Omega,\mu)}$$

for all $(f_1, g_1), (f_2, g_2) \in X^{p_h,q_h}(\Omega, \mu)$ with $(f_1 - f_2, g_1 - g_2) \in X^{p_1,q_1}(\Omega, \mu)$.

Proof. The above hypotheses imply that Assumption (B) is satisfied. Assumption (A) is satisfied by Remark 2.20. For $p_s := pN/(N - p)$ and $q_s := pd/(N - p)$ Assumption (C) follows from Corollary 2.19. Assumption (D)(ii) and hence (D) follows from Theorem 2.18 and the assumption that $\mu(\partial\Omega) < \infty$. Finally, since $p_s := pN/(N-p) > p$, $q_s := pd/(N-p) > p$, and $N/p = p_s/(p_s - p)$ and $pd/(p^2 - p(N - d)) = q_s/(q_s - p)$, the corollary follows from Theorem 4.6. □

Remark 4.9. Recently, we have been notified that Daners and Drábek [10] have considered Problem (3.1) for the case where $\mu := H^{N-1}$ and $g \equiv 0$, that is, the homogeneous boundary conditions. Using Moser's iterations, they have shown in particular that if Ω has finite Lebesgue measure and if $f \in L^{p_1}(\Omega)$ with $p_1 > N/p$, then a weak solution u of (3.1) (recall that in this case $\mu := H^{N-1}$ and $g \equiv 0$) satisfies the estimate

$$\|u\|^{p-1}_{L^\infty(\Omega)} \leq C\|f\|_{L^{p_1}(\Omega)}.$$

Moreover, for the inhomogeneous boundary conditions (as our case) one has to be more careful than for the homogeneous boundary conditions considered in [10]. Our results also include *inhomogeneous Neumann boundary conditions* by taking $b \equiv 0$ in (3.1).

5. Appendix: Some properties of Musielak-Orlicz spaces

For the convenience of the reader we introduce in this section the Musielak-Orlicz spaces, some well-known definitions, and prove some properties of these spaces which have been used in the previous sections.

Definition 5.1. *Let (X, Σ, μ) be a complete measure space. We call a function $B : X \times \mathbb{R} \to [0, \infty]$ a* Musielak-Orlicz function *on X if*

(1) *$B(x, \cdot)$ is non-trivial, even, convex for μ-a.e. $x \in X$;*
(2) *$B(x, \cdot)$ is vanishing and continuous at 0 for μ-a.e. $x \in X$;*
(3) *$B(x, \cdot)$ is left continuous on $[0, \infty)$;*
(4) *$B(\cdot, t)$ is Σ-measurable for all $t \in [0, \infty)$;*
(5) $\lim_{t\to\infty} B(x, t)/t = \infty$.

The complementary Musielak-Orlicz function *$\widetilde{B}$ in the sense of Young is defined by $\widetilde{B}(x, t) := \sup\{s|t| - B(x, s) : s > 0\}$.*

Remark 5.2. It follows directly from the definition that for $t, s \geq 0$ (and hence for all $t, s \in \mathbb{R}$)

$$st \leq B(x, t) + \widetilde{B}(x, s).$$

Definition 5.3. *We will say that a Musielak-Orlicz function B satisfies the $(\triangle_\alpha^0)$-condition $(\alpha > 1)$ if there exists a set X_0 of μ-measure zero and a constant $C_\alpha > 1$ such that*

$$B(x, \alpha t) \leq C_\alpha B(x, t)$$

for all $t \in \mathbb{R}$ and every $x \in X \setminus X_0$. We will say that B satisfies the (∇_2^0)-condition if there is a set X_0 of μ-measure zero and a constant $c > 1$ such that

$$B(x, t) \leq \frac{1}{2c} B(x, ct)$$

for all $t \in \mathbb{R}$ and all $x \in X \setminus X_0$.

Definition 5.4. *A function $\Phi : \mathbb{R} \to [0, \infty)$ is called an $\mathcal{N}$-function if*

- *Φ is even and convex;*
- *$\Phi(t) = 0$ if and only if $t = 0$;*
- $\lim_{t\to 0} t^{-1}\Phi(t) = 0$ *and* $\lim_{t\to\infty} t^{-1}\Phi(t) = \infty$.

We say that an $\mathcal{N}$-function Φ satisfies the $(\triangle_2)$- condition if there exists a constant $C_2 > 1$ such that

$$\Phi(2t) \leq C_\alpha \Phi(t) \quad \text{for all } \ t \in \mathbb{R},$$

and it satisfies the (∇_2)*-condition if there is a constant* $c > 1$ *such that*

$$\Phi(t) \le \Phi(ct)/(2c) \quad \textit{for all} \ \ t \in \mathbb{R}.$$

For more details on $\mathcal{N}$*-functions, we refer to the monograph by Adams* [1, Chapter VIII] (*see also* [24, Chapter I] *and* [25, Chapter I]).

Remark 5.5. In this case we let φ be the left derivative of Φ. Then φ is left continuous and non-decreasing on $(0,\infty)$ and $\varphi(0,\infty) = (0,\infty)$. Let ψ be given by $\psi(s) := \inf\{t > 0 : \varphi(t) > s\}$. Then

$$\Phi(t) = \int_0^{|t|} \varphi(s)\, ds; \qquad \Psi(t) := \int_0^{|t|} \psi(s)\, ds = \sup\{|t|s - \Phi(s) : s > 0\}.$$

As before, $st \le \Phi(t) + \Psi(s)$. Moreover, if $s = \varphi(t)$ or $t = \psi(s)$ then we have equality; that is,

$$\Psi(\varphi(t)) = t\varphi(t) - \Phi(t).$$

Lemma 5.6. *Let* Φ *be an* $\mathcal{N}$*-function which satisfies the* $(\triangle_2)$*- condition with the constant* $C_2 > 1$ *and let* Ψ *be the complementary* $\mathcal{N}$*-function. Then* Ψ *satisfies the* (∇_2)*-condition with the constant* $c := 2^{C_2-1}$.

Proof. We have

$$t\varphi(t) \le \int_t^{2t} \varphi(s)\, ds \le \int_0^{2t} \varphi(s)\, ds = \Phi(2t) \le C_2\Phi(t).$$

Since $\varphi(\psi(s)) \ge s$ for all $s \ge 0$ and $s/\Psi(s)$ and $s/(s-1)$ are decreasing, we get for $t := \psi(s)$

$$\frac{s\psi(s)}{\Psi(s)} \ge \frac{\varphi(\psi(s))\psi(s)}{\Psi(\varphi(\psi(s)))} = \frac{t\varphi(t)}{\Psi(\varphi(t))} = \frac{t\varphi(t)}{t\varphi(t) - \Phi(t)} \ge \frac{C_2}{C_2 - 1}.$$

Now let $c := 2^{C_2-1}$. Then for $t \ge 0$

$$\begin{aligned} \ln\left(\frac{\Psi(ct)}{\Psi(t)}\right) &= \int_t^{ct} \frac{\psi(s)}{\Psi(s)}\, ds \ge \int_t^{ct} \frac{C_2}{s(C_2-1)}\, ds \\ &= \frac{C_2}{C_2-1}\ln(c) = C_2\log(2) = \ln(2\cdot 2^{C_2-1}). \end{aligned}$$

Hence, $\Psi(t)2c \le \Psi(ct)$. □

Corollary 5.7. *Let* B *be a Musielak-Orlicz function such that* $B(x,\cdot)$ *is an* $\mathcal{N}$*-function for* μ*-a.e.* x*. If* B *satisfies the* $(\triangle_2^0)$*-condition, then* $\widetilde{B}$ *satisfies the* (∇_2^0)*-condition.*

Definition 5.8. *Let B be a Musielak-Orlicz function. Then the Musielak-Orlicz space $L^B(X)$ associated with B is given by*

$$L^B(X) := \{u : X \to \mathbb{R} \text{ measurable } : \rho_B(u/\alpha) < \infty \text{ for some } \alpha > 0\}$$

where $\rho_B(v) := \int_X B(x, v(x))\, d\mu(x)$. On this space we consider the Luxemburg norm $\|\cdot\|_{X,B}$ defined by

$$\|u\|_{X,B} := \inf\{\alpha > 0 : \rho_B(u/\alpha) \le 1\}.$$

Proposition 5.9. *Let B be a Musielak-Orlicz function satisfying the (∇_2^0)-condition. Then*

$$\lim_{\|u\|_{X,B}\to\infty} \frac{\rho_B(u)}{\|u\|_{X,B}} = \infty.$$

Proof. If B satisfies the (∇_2^0)-condition, then there exists a set $X_0 \subset X$ of measure zero such that for every $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon) > 0$ such that

$$B(x, \alpha t) \le \alpha\varepsilon B(x, t)$$

for all $t \in \mathbb{R}$ and all $x \in X \setminus X_0$. Let $\lambda \in (0, \infty)$ be fixed. For $\varepsilon := 1/\lambda$ there exists $\alpha > 0$ satisfying the above inequality. We will show that $\rho_B(u) \ge \lambda\|u\|_{X,B}$ whenever $\|u\|_{X,B} > 1/\alpha$. Assume that $\|u\|_{X,B} > 1/\alpha$ and let $\delta > 0$ be such that $\alpha = (1+\delta)/\|u\|_{X,B}$. Then

$$\begin{aligned}\rho_B(\alpha u) &= \int_X B(x, u(1+\delta)/\|u\|_{X,B})\, d\mu \\ &\ge (1+\delta)^{1-1/n} \int_X B(x, u(1+\delta)^{1/n}/\|u\|_{X,B})\, d\mu \ge (1+\delta)^{1-1/n}\end{aligned}$$

for all $n \in \mathbb{N}$. If we assume that the last inequality is untrue, then

$$\|u\|_{X,B}/(1+\delta) \in \{\alpha > 0 : \rho(u/\alpha) \le 1\},$$

a contradiction to the definition of $\|u\|_{X,B}$. Therefore

$$\rho_B(\alpha u) \ge 1 + \delta = \alpha\|u\|_{X,B}.$$

Now we obtain that

$$\rho_B(u) = \int_X B(x, u(x))\, d\mu \ge \frac{\lambda}{\alpha}\int_X B(x, \alpha u(x))\, d\mu = \frac{\lambda}{\alpha}\rho_B(\alpha u) \ge \lambda\|u\|_{X,B}.$$

□

Corollary 5.10. *Let B be a Musielak-Orlicz function such that $B(x,\cdot)$ is an $\mathcal{N}$-function for μ-a.e. x. If $\widetilde{B}$ satisfies the $(\triangle_2^0)$-condition, then B satisfies the (∇_2^0)- condition and $\lim_{\|u\|_{X,B}\to\infty} \frac{\rho_B(u)}{\|u\|_{X,B}} = \infty$.*

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