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DEGENERATE PARABOLIC EQUATION WITH CRITICAL EXPONENT DERIVED FROM THE KINETIC THEORY, IV, STRUCTURE OF THE BLOWUP SET

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Abstract. We continue to study the degenerate parabolic equation derived from the kinetic theory using Rényi-Tsallis' entropy. The finiteness of type II blowup points and several structures of the blowup set are shown for the critical exponent case.

1. Introduction

In this paper, we study the structure of the blowup set of the degenerate parabolic equation

$$\begin{aligned} & u_t = \frac{m-1}{m}\Delta u^m - \nabla\cdot(u\nabla\Gamma * u), \quad u \geq 0 && \text{in } \mathbf{R}^n \times (0,T) \\ & u|_{t=0} = u_0 && \text{in } \mathbf{R}^n \end{aligned} \tag{1.1}$$

with the critical exponent $m = 2 - \frac{2}{n}$ arising in the kinetic theory concerning the motion of the mean field of many self-gravitating particles, where

$$\Gamma = \Gamma(x) = \frac{1}{\omega_{n-1}(n-2)\,|x|^{n-2}} \tag{1.2}$$

with ω_{n-1} standing for the $(n-1)$-dimensional volume of the boundary of the unit ball in $\mathbf{R}^n$, $n \geq 3$.

For the moment, we recall the results shown in our previous papers [15, 16, 17]. We constructed the weak solution to (1.1) in [15], assuming

$$0 \leq u_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n), \qquad u_0^m \in H^1(\mathbf{R}^n). \tag{1.3}$$

This solution $u = u(x,t)$ actually has the regularity

$$\begin{aligned} & u \in C_*([0,T); L^p(\mathbf{R}^n)), \quad 1 < p \leq \infty \\ & u \in L^\infty(0,T; L^1(\mathbf{R}^n)) \cap L^\infty([0,T); L^\infty(\mathbf{R}^n)) \end{aligned}$$

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$$\nabla u^m \in L^\infty([0,T); L^2(\mathbf{R}^n))$$
$$\partial_t u^{\frac{m+1}{2}} \in L^2([0,T); L^2(\mathbf{R}^n))$$
$$\nabla\Gamma * u \in L^\infty([0,T); L^\infty(\mathbf{R}^n)) \tag{1.4}$$

for each $T \in (0, T_{\max})$, where $T_{\max}$ denotes the maximal existence time of u. This weak solution $u = u(x,t)$ is obtained as a limit of the approximate solution $u_\varepsilon = u_\varepsilon(x,t)$, $0 < \varepsilon \ll 1$. More precisely, $u_\varepsilon = u_\varepsilon(x,t)$ is the unique smooth solution to the equation

$$u_{\varepsilon t} = \frac{m-1}{m}\Delta(u_\varepsilon + \varepsilon)^m - \nabla\cdot(u_\varepsilon \nabla\Gamma * u_\varepsilon) \qquad \text{in } \mathbf{R}^n \times (0,T) \tag{1.5}$$

with

$$u_\varepsilon|_{t=0} = u_{0\varepsilon} \qquad \text{in } \mathbf{R}^n \tag{1.6}$$

for $0 < \varepsilon \ll 1$, where

$$\begin{aligned} &0 \le u_{0\varepsilon} \in L^1 \cap W^{2,p}(\mathbf{R}^n) && \text{for any } p \in [\tfrac{n}{n-1}, n+2] \\ &\|u_{0\varepsilon}\|_p \le \|u_0\|_p && \text{for any } p \in [1,\infty] \\ &\|\nabla u_{0\varepsilon}^m\|_2 \le \|\nabla u_0^m\|_2 && \\ &u_{0\varepsilon} \to u_0 && \text{strongly in } L^p(\mathbf{R}^n) \text{ as } \varepsilon \downarrow 0 \text{ for all } p \in [1,\infty), \end{aligned} \tag{1.7}$$

$u_{0\varepsilon} = \rho_\varepsilon * u_0$, and ρ_ε stands for the Friedrichs mollifier. Then, some *a priori* estimates guarantee that there exists a sequence $\{\varepsilon_j\}$ with $\varepsilon_j \downarrow 0$ such that u_{ε_j} actually converges to the weak solution u which satisfies the properties (1.4). Furthermore, the weak solution $u = u(x,t)$ is actually Hölder continuous, see [18]. Here, we note that the uniqueness of the weak solution is not guaranteed. Therefore we shall treat the one constructed in [15] in the following argument.

In the previous work [15, 16], we have pointed out the variational and scaling properties similar to the Smoluchowski-Poisson equation associated with the Boltzmann entropy in two-dimensions,

$$\begin{aligned} &u_t = \Delta u - \nabla\cdot(u\nabla\Gamma * u), \quad u \ge 0 && \text{in } \mathbf{R}^2 \times (0,T) \\ &u|_{t=0} = u_0 && \text{in } \mathbf{R}^2, \end{aligned} \tag{1.8}$$

with

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|} \tag{1.9}$$

regarded also as a simplified system of chemotaxis. We proved the total mass conservation

$$\|u(t)\|_1 = \|u_0\|_1 = \lambda \qquad \text{for } t \in (0,T), \tag{1.10}$$

$$\|u_\varepsilon(t)\|_1 = \|u_{0\varepsilon}\|_1 \le \lambda \qquad \text{for } t \in (0,T), \tag{1.11}$$

and the decrease of the free energy

$$\mathcal{F}(u(t)) \le \mathcal{F}(u_0) \qquad \text{for } t \in (0,T), \tag{1.12}$$

where

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} dx - \frac{1}{2} \langle \Gamma * u, u \rangle .$$

The relation

$$\frac{d}{dt} \int_{\mathbf{R}^n} |x|^2 u dx = 2(n-2)\mathcal{F}(u), \qquad \text{a.e. } t \in (0,T), \tag{1.13}$$

also arises in the case of

$$\int_{\mathbf{R}^n} |x|^2 u_0(x) dx < +\infty \tag{1.14}$$

with the locally absolutely continuity of $t \in [0,T) \mapsto \int_{\mathbf{R}^n} |x|^2 u(x,t)dx$. The threshold mass λ_* for the global in time existence of the solution is also determined in [16]. This value is associated with the normalized stationary state

$$-\Delta v_* = v_{*+}^q, \ v_* \le v_*(0) = 1 \ \text{ in } \mathbf{R}^n, \qquad \lambda_* = \int_{\mathbf{R}^n} v_{*+}^q dx. \tag{1.15}$$

An ε-regularity theorem concerning the local uniform bound of the solution is shown in [17]. This theorem is stated precisely in the next section, see Theorem 4. The difficulty of the proof essentially comes from the lack of decay at ∞ of the potential (1.2). To overcome this difficulty, a decomposition of $v = \Gamma * u$ used in [15, 16] is a key of the proof, that is, $v = v_1 + v_2$ as

$$v_1(x,t) = \int_{|x'-x|\ge 1} \Gamma(x-x')u(x',t)dx'$$

$$v_2(x,t) = \int_{|x'-x|<1} \Gamma(x-x')u(x',t)dx'.$$

This ε-regularity theorem is employed to show several facts in this paper, but is not sufficient to guarantee the finiteness of blowup points, see [9] for the two-dimensional case.

Henceforth, $T = T_{\max} \in (0, +\infty]$ shall denote the maximal existence time. We have showed in [15] that the weak solution $u = u(x,t)$ satisfies the standard blowup criterion

$$\lim_{t \uparrow T} \|u(t)\|_\infty = +\infty$$

if $T < +\infty$. In this paper, we shall show that $u(x,t)$ is uniformly bounded for $x \in \mathbf{R}^n$ satisfying $|x| \gg 1$ and for $t \in [0,T)$ under (1.14), and therefore the blowup set defined by $\mathcal{S} = \mathbf{R}^n \setminus \mathcal{B}$,

$$\mathcal{B} = \left\{ x_0 \in \mathbf{R}^n : \limsup_{t \uparrow T} \|u(t)\|_{L^\infty(B(x_0,r_0))} < +\infty \text{ for some } r_0 > 0 \right\},$$

is non-empty. Now, we recall that type I blowup rate comes from the ODE part. To detect this rate, we write (1.1) as

$$u_t = \frac{m-1}{m} \Delta u^m - \nabla u \cdot \nabla \Gamma * u + u^2,$$

and take the ODE part

$$\dot{\zeta} = \zeta^2.$$

Then we see that the type I blowup rate is $O((T-t)^{-1})$, and hence define the set of type I blowup points by

$$\mathcal{S}_I = \left\{ x_0 \in \mathcal{S} : \liminf_{t \uparrow T} (T-t) \|u(t)\|_{L^\infty(B(x_0,r_0))} < +\infty \quad \text{for some } r_0 > 0 \right\}.$$

The set of type II blowup points is then defined by $\mathcal{S}_{II} = \mathcal{S} \setminus \mathcal{S}_I$; that is,

$$\mathcal{S}_{II} = \left\{ x_0 \in \mathcal{S} : \lim_{t \uparrow T} (T-t) \|u(t)\|_{L^\infty(B(x_0,r_0))} = +\infty \quad \text{for any } r_0 > 0 \right\}.$$

Some results observed in [14] are shown in the present paper. One can see further discussions on other structures of the blowup set there.

The first theorem guarantees the boundedness of $\mathcal{S}$ and the finiteness of $\mathcal{S}_{II}$.

Theorem 1. *Let $u_0 = u_0(x)$ be the initial value satisfying* (1.3) *and* (1.14), *and assume $T < +\infty$ for the weak solution $u = u(x,t)$ to* (1.1) *with $m = 2 - \frac{2}{n}$. Then $\mathcal{S}$ is bounded in $\mathbf{R}^n$ and $\mathcal{S}_{II}$ is finite.*

We can show another aspect of the finiteness of type II blowup points.

Theorem 2. *Assume the hypothesis of Theorem* 1. *Then, for every $\xi > 0$, the set*

$$\begin{aligned} \mathcal{S}_{*,\xi} \quad = \quad & \Big\{ x_0 \in \mathbf{R}^n : \textit{there exist } y(t) \to x_0 \textit{ and } b > 0 \textit{ such that} \\ & |y(t) - x_0| = O((T-t)^{1/n}) \textit{ and} \\ & \liminf_{t \uparrow T} (T-t) \|u(t)\|_{L^\infty(B(y(t), b(T-t)^{1/n}))} \geq \xi \Big\} \end{aligned}$$

is finite.

The above theorem may be compared with the non-degeneracy of the blowup points concerning the semilinear parabolic equation with sub-critical nonlinearity [2]. In this case, the blowup set $\mathcal{S}$ agrees with $\mathcal{S}_{*,\xi_0}$ for some ξ_0; that is, all blowup points are nondegenerate. A different pattern may arise as for the weak solution to (1.1); that is to say, a stable blowup pattern corresponding to [3] may occur for $x_0 \in \mathcal{S}_{*,\xi}$, see Section 6.

We say that $x(t) \in \mathbf{R}^n$ attains a positive local maximum of $u(\cdot,t)$ if $u(x,t)$ is positive in a neighborhood of $x(t)$ in x-space and $x = x(t)$ is its local maximizer of $u = u(\cdot,t)$.

Theorem 3. *Assume the hypothesis of Theorem 1. Then, if $\sharp\mathcal{S} = +\infty$, there are an infinite number of $x_0 \in \mathcal{S}$ such that*

$$\limsup_{t\uparrow T} \frac{dist(x(t),x_0)}{(T-t)^{1/n}} = +\infty, \tag{1.16}$$

provided that $x(t)$ attains a positive local maximum of $u(\cdot,t)$ such that

$$\limsup_{t\uparrow T} u(x(t),t) = +\infty. \tag{1.17}$$

The above theorem implies that $\mathcal{S}$ may be a positive dimensional set if $\sharp\mathcal{S} = +\infty$. We consider this property in Section 6.

This paper is composed of six sections. Some preliminaries are done in Section 2, and then Theorem 1, 2 and 3 are proven in Sections 3, 4, and 5, respectively. Expectations for several blowup patterns are stated as concluding remarks in Section 6. In the following, C_i $(i = 1,2,\cdots)$ denote positive constants whose subscripts are renewed in each section.

2. Preliminaries

In this section, we provide some auxiliary results. First, we recall the following Lemmas 2.1-2.3, see [10, 17].

Lemma 2.1. *Let $x_0 \in \mathbf{R}^n$ and $\rho,\delta > 0$, and define $\eta = \eta_{x_0,\rho,\delta}(x)$ by*

$$\eta(x) = \begin{cases} 1, & 0 \le |x-x_0| < \rho \\ \exp\left(1 - \frac{\delta}{\rho+\delta-|x-x_0|}\right), & \rho \le |x-x_0| < \rho+\delta \\ 0, & \rho+\delta \le |x-x_0|. \end{cases}$$

Then,

$$|\nabla\eta(x)| \le \frac{4}{a^2\delta}\eta(x)^{1-a} = A_1(a,\delta)\eta(x)^{1-a} \tag{2.1}$$

$$|\Delta\eta(x)| \leq \frac{310}{a^4\delta}\eta(x)^{1-a} = A_2(a,\delta)\eta(x)^{1-a} \tag{2.2}$$

for all $x \in \mathbf{R}^n$ *and* $a \in (0,1)$.

Lemma 2.2. *Let* $1 \leq q_1 \leq q_2 \leq q_3 < \infty$ *and* $\theta = q_3/q_2 \cdot (q_2 - q_1)/(q_3 - q_1)$. *If* u *and* $v \geq 0$ *satisfy* $u \in L^{q_1}(\mathbf{R}^n)$ *and* $uv^{1/\theta} \in L^{q_3}(\mathbf{R}^n)$, *then* $uv \in L^{q_2}(\mathbf{R}^n)$ *and*

$$\|uv\|_{q_2} \leq \|u\|_{q_1}^{1-\theta}\|uv^{1/\theta}\|_{q_3}^{\theta}.$$

Lemma 2.3. *Let* $\eta = \eta_{x_0,\rho,\delta}$ *be the cut-off function as in Lemma* 2.1.

(i) *If* $0 \leq u \in L^1_{loc}(\mathbf{R}^n)$ *satisfies* $\nabla u^{\frac{r+m}{2}} \in L^2_{loc}(\mathbf{R}^n)$ *for* $r \in [2/n, \infty)$, *then* $u^{r+2}\eta \in L^1(\mathbf{R}^n)$ *and*

$$\int_{\mathbf{R}^n} u^{r+2}\eta dx \leq K_1^*(n)\|u\|_{L^1(B(x_0,\rho+\delta))}^{2/n}\int_{\mathbf{R}^n}|\nabla u^{\frac{r+m}{2}}|^2\eta dx + (C_1(n)A_1)^{n(r+2)}\|u\|_{L^1(B(x_0,\rho+\delta))}^{r+2} \tag{2.3}$$

for all $a \in (0, \frac{1}{n(r+2)}]$.

(ii) *For every* $K > \frac{3}{2}n - 4$ *and* $L > \frac{1}{2} + \frac{4}{n}$, *there is* $r^* = r^*(n,K,L) \gg 1$, *such that, if* $0 \leq u \in L^1_{loc} \cap L^{\frac{r+1}{4}}_{loc}(\mathbf{R}^n)$ *satisfies* $\nabla u^{\frac{r+m}{2}} \in L^2_{loc}(\mathbf{R}^n)$ *for* $r \in [r^*, \infty)$, *then* $u^{r+j}\eta^{1-2a} \in L^1(\mathbf{R}^n)$ *and*

$$\begin{aligned}\int_{\mathbf{R}^n} u^{r+j}\eta^{1-2a}dx &\leq \frac{\varepsilon r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n}|\nabla u^{\frac{r+m}{2}}|^2\eta dx \\ &+ C_2(n)\left\{\left(\frac{r}{\varepsilon}\right)^{\frac{3n+1}{2}} + A_1^{3n+1}\right\} \\ &\times \max\left\{\|u\|^{r-K}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}, \|u\|^{r+L}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\}\end{aligned} \tag{2.4}$$

for all $\varepsilon > 0$, $j \in [1,2]$, $a \in (0, \frac{1}{3(n+1)}]$ *and* $\delta \in (0, 36(n+1)^2]$.

We shall write $A_i = A_i(a,\delta)$, $i = 1,2$ for (2.1)-(2.2). In the following, $\eta = \eta(x) = \eta_{x_0,\rho,\delta}(x)$ shall be as in Lemma 2.1.

Now, we consider a smooth solution to the problem

$$\begin{aligned} &w_t = \frac{m-1}{m}\Delta(w+d)^m - \nabla\cdot(w\nabla\Gamma * w),\ w \geq 0 \quad \text{in } \mathbf{R}^n \times (0,1) \\ &\int_{\mathbf{R}^n} w(x,t)dx \leq \Lambda \qquad \text{for } t \in (0,1), \end{aligned} \tag{2.5}$$

where $n \geq 3$, $m = 2 - \frac{2}{n}$, and $0 < d \leq 1$ and $\Lambda > 0$ are given constants.

The first objective in this section is to show the following lemma.

Lemma 2.4. *Let $w = w(x,t)$ be a smooth solution to* (2.5). *Then there exists $\varepsilon_0^* = \varepsilon_0^*(n) > 0$ such that, if*

$$\sup_{t\in(0,1)} \|w(t)\|_{L^1(B(x_0,R))} < \varepsilon_0^* \tag{2.6}$$

holds for $x_0 \in \mathbf{R}^n$ and $R \in (0, 72(n+1)^2]$, then

$$\sup_{t\in[1/4,1)} \|w(t)\|_{L^\infty(B(x_0,R/6))} \le C_3(n,R,\Lambda), \tag{2.7}$$

where C_3 is independent of $x_0 \in \mathbf{R}^n$.

We show Lemma 2.4 following [9].

For a smooth solution $w = w(x,t)$ to (2.5), we set

$$J_r(t) = J_r^{(x_0,\rho,\delta)}(t) = \int_{\mathbf{R}^n} w^r(x,t)\eta_{x_0,\rho,\delta}(x)dx \tag{2.8}$$

for $t \in (0,1)$ and $r \ge 1$, where $x_0 \in \mathbf{R}^n$ and $\rho, \delta > 0$. In Lemmas 2.5-2.8, their proofs and that of Lemma 2.4 below, $w = w(x,t)$ shall denote a smooth solution to (2.5).

Lemma 2.5. *For every $r \ge 1$, there exist $0 < \varepsilon_r' = \varepsilon_r'(n,r) \ll 1$ and $T_r' = T_r'(n,r,\rho,\delta,\Lambda) \in (0,1)$ such that, if*

$$\sup_{t\in(0,1)} \|w(t)\|_{L^1(B(x_0,\rho+\delta))} < \varepsilon_r', \tag{2.9}$$

then

$$J_{r+1}(t) \le t^{-(r+1)} \tag{2.10}$$

for all $t \in (0, T_r']$, $\rho > 0$ and $\delta \in (0, 4\{(n+2)(r+2)-n\}^2]$.

Proof. By multiplying (2.5) by $w^r\eta$ similarly to Lemma 3.1 in [17], we find

$$\frac{1}{r+1}\frac{d}{dt}\int_{\mathbf{R}^n} w^{r+1}(x,t)\eta(x)dx \le I_1(t) + I_2(t) + I_3(t) \tag{2.11}$$

for $t \in (0,1)$ and $r \ge 1$, where

$$I_1 = -\frac{m-1}{m}\int_{\mathbf{R}^n} \nabla w^m \cdot \nabla(w^r\eta)dx,$$

$$I_2 = \int_{\mathbf{R}^n} (w\nabla\Gamma * w)\cdot\nabla(w^r\eta)dx, \quad I_3 = \frac{(m-1)A_1^2}{4r}\int_{\mathbf{R}^n} w^{r+1}\eta^{1-2a}dx.$$

From the proof of Lemmas 3.2 (i), 3.4 and 3.3 (i) in [17], we find

$$I_1 \le \left\{-\frac{4r(m-1)}{(r+m)^2} + \frac{(m-1)K_1^*}{r+m}\|w\|_{L^1(B(x_0,\rho+\delta))}^{2/n}\right\}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}|^2\eta dx$$

$$+ (C_4(n)A_1)^{n(r+1)}\|w\|^{r+2}_{L^1(B(x_0,\rho+\delta))} + A_2^{\frac{n(r+1)}{2}}\|w\|_{L^1(B(x_0,\rho+\delta))}, \quad (2.12)$$

$$I_2 \le 3K_1^*\|w\|^{2/n}_{L^1(B(x_0,\rho+\delta))}\int_{\mathbf{R}^n}|\nabla w^{\frac{r+m}{2}}|^2\eta dx + C_5(n,r)A_1^{r+2}(\rho+\delta)^n\|w\|_1^{r+2}$$
$$+ C_6(n,r)A_1^{r+2}(1 + A_1^{2n(r+1)} + A_2^{n(r+1)})\|w\|_1^{r+2}, \quad (2.13)$$

and

$$\begin{aligned} I_3 &\le \frac{(m-1)A_1^2}{4r}\|w\|^{\frac{1}{r+1}}_{L^1(B(x_0,\rho+\delta))}\Big(\int_{\mathbf{R}^n} w^{r+2}\eta^{\frac{(1-2a)(r+1)}{r}}\Big)^{\frac{r}{r+1}} \\ &\le \int_{\mathbf{R}^n} w^{r+2}\eta dx + A_1^{2(r+1)}\|w\|_{L^1(B(x_0,\rho+\delta))} \\ &\le K_1^*\|w\|^{2/n}_{L^1(B(x_0,\rho+\delta))}\int_{\mathbf{R}^n}|\nabla w^{\frac{r+m}{2}}|^2\eta dx \\ &\quad + \Big\{(C_7(n)A_1)^{n(r+2)} + A_1^{2(r+1)}\Big\}\|w\|_{L^1(B(x_0,\rho+\delta))} \end{aligned} \quad (2.14)$$

for all $a \in (0, \frac{1}{(n+2)(r+2)-n}]$, $\delta \in (0, 4\{(n+2)(r+2)-n\}^2]$ and $r \ge 1$, where Lemma 2.3 (i) has been used for the derivation of (2.14). We apply Lemmas 2.2 and 2.3 (i) to obtain

$$\begin{aligned} &\Big(\int_{\mathbf{R}^n} w^{r+1}\eta dx\Big)^{\frac{r+2}{r+1}} = \|w\eta^{\frac{1}{r+1}}\|^{r+2}_{r+1} \\ &\le \Big(\|w\|^{\frac{1}{(r+1)^2}}_{L^1(B(x_0,\rho+\delta))}\|w\eta^{\frac{r+1}{r(r+2)}}\|^{1-\frac{1}{(r+1)^2}}_{r+2}\Big)^{r+2} \\ &\le \|w\|^{\frac{r+2}{(r+1)^2}}_{L^1(B(x_0,\rho+\delta))}\Big(\int_{\mathbf{R}^n} w^{r+2}\eta dx\Big)^{1-\frac{1}{(r+1)^2}} \\ &\le \int_{\mathbf{R}^n} w^{r+2}\eta dx + \|w\|^{r+2}_{L^1(B(x_0,\rho+\delta))} \\ &\le K_1^*\|w\|^{2/n}_{L^1(B(x_0,\rho+\delta))}\int_{\mathbf{R}^n}|\nabla w^{\frac{r+m}{2}}|^2\eta dx + (C_7(n)A_1)^{n(r+2)}\|w\|^{r+2}_{L^1(B(x_0,\rho+\delta))} \end{aligned} \quad (2.15)$$

for all $a \in (0, \frac{1}{n(r+2)}]$ provided that $A_1 \ge 1$.

Combining (2.11)-(2.15), we have

$$\begin{aligned} &\frac{1}{r+1}J'_{r+1}(t) + 2(J_{r+1}(t))^{\frac{r+2}{r+1}} \\ &\le \Big\{-\frac{4r(m-1)}{(r+m)^2} + 6K_1^*\|w(t)\|^{2/n}_{L^1(B(x_0,\rho+\delta))}\Big\}\int_{\mathbf{R}^n}|\nabla w^{\frac{r+m}{2}}|^2(x,t)\eta(x)dx \\ &\quad + C_5(n,r)A_1^{r+2}(\rho+\delta)^n\Lambda^{r+2} \end{aligned}$$

$$+C_8(n,r)(1+A_1+A_2)^{rC_9(n)}(\Lambda+\Lambda^{r+2}) \tag{2.16}$$

for all $t \in (0,1)$, $a \in (0, \frac{1}{(n+2)(r+2)-n}]$, and $\delta \in (0, 4\{(n+2)(r+2)-n\}^2]$; recall (2.1)-(2.2), (2.8) and the restriction of the mass in (2.5). We take a positive number $\varepsilon'_r = \varepsilon'_r(n,r)$ satisfying

$$(\varepsilon'_r)^{2/n} \le \frac{2r(m-1)}{3K_1^*(r+m)^2}, \tag{2.17}$$

and set $a = \frac{1}{(n+2)(r+2)-n}$. Then, inequality (2.16) becomes

$$\begin{aligned}&\frac{1}{r+1}\frac{d}{dt}J_{r+1}(t) + 2(J_{r+1}(t))^{\frac{r+2}{r+1}}\\ &\le C_{10}\delta^{-(r+2)}(\rho+\delta)^n\Lambda^{r+2} + C_{11}(n,r)(1+\delta^{-1})^{rC_{12}(n)}(\Lambda+\Lambda^{r+2})\\ &= C_{13}^*(n,r,\rho,\delta,\Lambda)\end{aligned} \tag{2.18}$$

for all $t \in (0,1)$ and $\delta \in (0, 4\{(n+2)(r+2)-n\}^2]$ if (2.9) holds for ε'_r satisfying (2.17).

It holds also that

$$\frac{1}{r+1}\frac{d}{dt}\{t^{-(r+1)}\} + 2\{t^{-(r+1)}\}^{\frac{r+2}{r+1}} = t^{-(r+2)} \tag{2.19}$$

for $t > 0$. We take $T'_r = T'_r(n,r,\rho,\delta,\Lambda) \in (0,1)$ as

$$(T'_r)^{-(r+2)} \ge C_{13}^*. \tag{2.20}$$

Taking ε'_r and T'_r satisfying (2.17) and (2.20), we can compare (2.18) with (2.19) on $(0, T'_r]$ under the restriction (2.9) and conclude that (2.9) implies (2.10). □

Henceforth, we shall use only Lemma 2.3 (ii) with $K = 2n-1$ and $L = 2$, and take $r^* = r^*(n) = r^*(n, 2n-1, 2) \gg 1$.

Lemma 2.6. *For every $\rho \in (0, 36(n+1)^2]$, there exist $0 < \varepsilon''_0 = \varepsilon''_0(n) \ll 1$ and $T''_0 = T''_0(n,\rho,\Lambda) \in (0,1)$ such that, if*

$$\sup_{t\in(0,1)} \|w(t)\|_{L^1(B(x_0,2\rho))} < \varepsilon''_0, \tag{2.21}$$

then

$$\begin{aligned}&\frac{1}{r+1}J'_{r+1}(t) + 3\{J_{r+1}(t)\}^{\frac{r+2}{r+1}} \le t^{-\frac{3(n+1)}{2}} \cdot C_{14}(n)(r+A_1)^{C_{15}(n)} \\ &\times \{1+(\rho+\delta)^n\}^{\frac{3(n+1)}{2}} \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} \cdot \max\left\{1, \|w(t)\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\}\end{aligned} \tag{2.22}$$

for all $t \in (0, T''_0]$, $a \in [\frac{1}{4(n+1)}, \frac{1}{3(n+1)}]$, $\delta \in (0,\rho]$ and $r \in [r^*, \infty)$.

Proof. Similarly to the proof of Lemmas 3.2 (ii) and 3.3 (ii) in [17], we have

$$I_1(t) \leq -\frac{11r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}(x,t)|^2\eta(x)dx + C_{16}(n)A_1^{3(n+1)} \cdot \max\left\{1, \|w(t)\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\}, \tag{2.23}$$

$$I_3(t) \leq \frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}(x,t)|^2\eta(x)dx + C_{17}(n)A_1^{3(n+1)} \cdot \max\left\{1, \|w(t)\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \tag{2.24}$$

for all $t \in (0,1)$, $a \in (0, \frac{1}{3(n+1)}]$, $\delta \in (0, 36(n+1)^2]$ and $r \in [r^*, \infty)$. We use Lemmas 2.2 and 2.3 (ii) to get

$$\begin{aligned}(J_{r+1})^{\frac{r+2}{r+1}} &\leq \int_{\mathbf{R}^n} w^{r+2}\eta dx + \|w\|^{r+2}_{L^1(B(x_0,\rho+\delta))} \\ &\leq \frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}|^2\eta dx + \|w\|^{r+2}_{L^1(B(x_0,\rho+\delta))} \\ &\quad + C_{18}(n)(r^{\frac{3n+1}{2}} + A_1^{3n+1}) \cdot \max\left\{1, \|w\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\}\end{aligned} \tag{2.25}$$

for all $a \in (0, \frac{1}{3(n+1)}]$, $\delta \in (0, 36(n+1)^2]$ and $r \in [r^*, \infty)$.

According to the proof of Lemma 3.6 in [17],

$$\|(\eta^a \nabla\Gamma * w)(t)\|_\infty \leq \{J_{4n+1}(t)\}^{\frac{1}{4n+1}} + C_{19}(n)(1 + A_1^{n(n+1)})\{1 + (\rho+\delta)^n\}\Lambda \tag{2.26}$$

for all $t \in (0,1)$, $a \in [\frac{1}{4(n+1)}, \frac{1}{3(n+1)}]$ and $\delta \in (0, 36(n+1)^2]$. Here, we apply Lemma 2.5 with $r = 4n$ and $\delta = \rho$ to obtain $0 < \varepsilon_0'' = \varepsilon_0''(n) \ll 1$ and $T_0'' = T_0''(n,\rho,\Lambda) \in (0,1)$ such that, if

$$\sup_{t\in(0,1)} \|w(t)\|_{L^1(B(x_0,2\rho))} < \varepsilon_0'', \tag{2.27}$$

then

$$\{J_{4n+1}(t)\}^{\frac{1}{4n+1}} \leq t^{-1} \tag{2.28}$$

for $t \in (0, T_0'']$ and $\delta \in (0,\rho]$. Inequalities (2.26), (2.28) and Lemma 2.3 (ii) imply

$$I_2(t) \leq \frac{A_1}{r+1}\int_{\mathbf{R}^n} w^{r+1}|\nabla\Gamma * w|\eta^{1-a}dx + \frac{r}{r+1}\int_{\mathbf{R}^n} w^{r+2}\eta dx$$

$$
\begin{aligned}
&\le \frac{A_1}{r+1}\|\eta^a\nabla\Gamma * w\|_\infty \int_{\mathbf{R}^n} w^{r+1}\eta^{1-2a}dx + \frac{r}{r+1}\int_{\mathbf{R}^n} w^{r+2}\eta dx \\
&\le \frac{A_1}{r+1}\Big[t^{-1} + C_{19}(n)(1+A_1^{n(n+1)})\{1+(\rho+\delta)^n\}\Lambda\Big] \\
&\quad \times \Big[\frac{\varepsilon_1 r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}(x,t)|^2\eta(x)dx \\
&\qquad + C_{20}(n)\Big\{(\frac{r}{\varepsilon_1})^{\frac{3n+1}{2}} + A_1^{3n+1}\Big\}\cdot\max\Big\{1, \|w\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\Big\}\Big] \\
&\quad + \frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}|^2\eta dx \\
&\quad + C_{21}(n)(r^{\frac{3n+1}{2}} + A_1^{3n+1})\cdot\max\Big\{1, \|w\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\Big\} \qquad (2.29)
\end{aligned}
$$

for all $\varepsilon_1 > 0$, $t \in (0, T_0'']$, $a \in [\frac{1}{4(n+1)}, \frac{1}{3(n+1)}]$, $\rho \in (0, 36(n+1)^2]$, $\delta \in (0,\rho]$ and $r \in [r^*, \infty)$. We take ε_1 in (2.29) as

$$
\varepsilon_1 = \frac{r+1}{A_1}\Big[t^{-1} + C_{19}(n)(1+A_1^{n(n+1)})\{1+(\rho+\delta)^n\}\Lambda\Big]^{-1}.
$$

Then the first term of the right-hand side in (2.29) is estimated by

$$
\begin{aligned}
&\frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}|^2\eta dx \\
&\quad + \varepsilon_1^{-1} C_{20}(n)\Big\{(\frac{r}{\varepsilon_1})^{\frac{3n+1}{2}} + A_1^{3n+1}\Big\}\cdot\max\Big\{1, \|w\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\Big\} \\
&\le \frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}|^2\eta dx \\
&\quad + C_{22}(n)A_1^{3n+2}\Big[t^{-1} + C_{19}(n)(1+A_1^{n(n+1)})\{1+(\rho+\delta)^n\}\Lambda\Big]^{\frac{3(n+1)}{2}} \\
&\quad\quad \times \max\Big\{1, \|w\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\Big\}
\end{aligned}
$$

for all $t \in (0, T_0'']$, $a \in [\frac{1}{4(n+1)}, \frac{1}{3(n+1)}]$, $\rho \in (0, 36(n+1)^2]$, $\delta \in (0,\rho]$ and $r \in [r^*, \infty)$. Note that $A_1 = A_1(a,\delta) \ge 1$ for all $a \in [\frac{1}{4(n+1)}, \frac{1}{3(n+1)}]$, $\rho \in (0, 36(n+1)^2]$ and $\delta \in (0,\rho]$. Hence we deduce that (2.27) (or (2.21)) implies

$$
\begin{aligned}
I_2(t) &\le \frac{r(m-1)}{2(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}(x,t)|^2\eta(x)dx \\
&\quad + C_{23}(n)(r^{\frac{3n+1}{2}} + A_1^{3n+2})\cdot[t^{-1} + C_{19}(n)(1+A_1^{n(n+1)})
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \{1+(\rho+\delta)^n\}\Lambda]^{\frac{3(n+1)}{2}} \max\left\{1, \|w\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \\
&\leq \frac{r(m-1)}{2(r+m)^2}\int_{\mathbf{R}^n} |\nabla w^{\frac{r+m}{2}}(x,t)|^2 \eta(x)dx \\
&\quad + t^{-\frac{3(n+1)}{2}} \cdot C_{24}(n)(r+A_1)^{C_{25}(n)}\{1+(\rho+\delta)^n\}^{\frac{3(n+1)}{2}} \\
&\quad \times \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} \cdot \max\left\{1, \|w\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \qquad (2.30)
\end{aligned}
$$

for all $t \in (0, T_0'']$, $a \in [\frac{1}{4(n+1)}, \frac{1}{3(n+1)}]$, $\rho \in (0, 36(n+1)^2]$, $\delta \in (0, \rho]$ and $r \in [r^*, \infty)$.

We organize (2.11), (2.23)-(2.25) and (2.30) so that (2.27) (or (2.21)) implies (2.22). □

Lemma 2.7. *For every $\rho \in (0, 36(n+1)^2]$, we have $k_0 = k_0(n) \in \mathbf{N}$, $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(n) > 0$, $\tilde{T}_0 = \tilde{T}_0(n, \rho, \Lambda) \in (0,1)$ and $C_{26}^* = C_{26}^*(n, \rho) > 1$ such that, if*

$$
\sup_{t\in(0,1)} \|w(t)\|_{L^1(B(x_0,2\rho))} < \tilde{\varepsilon}_0, \qquad (2.31)
$$

then

$$
\int_{B(x_0,\rho(1-\sum_{p=1}^{l} 1/r_p))} w^{r_l}(x,t)dx \leq M_l t^{-\theta_l} \qquad (2.32)
$$

for all $t \in (0, \tilde{T}_0]$ and $l = 1, 2, \cdots$, where

$$
r_l = 4^{k_0+l} \qquad (2.33)
$$

$$
\theta_l = \frac{3(n+1)}{2}\Big\{1 + \sum_{p=1}^{l-1} 4^p \prod_{q=1}^{p}\Big(1+\frac{1}{r_{l+1-q}}\Big)\Big\} + r_l \prod_{q=1}^{l}\Big(1+\frac{1}{r_q}\Big) \qquad (2.34)
$$

$$
\begin{aligned}
M_l = &\Big\{ r_1^{4^{l-1}\prod_{q=1}^{l-1}(1+\frac{1}{r_{l+1-q}})} \cdot r_2^{4^{l-2}\prod_{q=1}^{l-2}(1+\frac{1}{r_{l+1-q}})} \cdots r_{l-1}^{4(1+\frac{1}{r_l})} \cdot r_l \Big\}^{C_{15}(n)} \\
&\times \{C_{26}^*(n,\rho)\}^{1+\sum_{p=1}^{l-1}4^p\prod_{q=1}^{p}(1+\frac{1}{r_{l+1-q}})} \\
&\quad \times \max\Big\{1, \Lambda^{\frac{3(n+1)}{2}\left\{1+\sum_{p=1}^{l-1}4^p\prod_{q=1}^{p}(1+\frac{1}{r_{l+1-q}})\right\}}\Big\}, \qquad (2.35)
\end{aligned}
$$

$\sum_{p=1}^{0}$ is empty, $\prod_{q=1}^{0}(1 + 1/r_{2-q}) = 1$, and $C_{15}(n)$ is as in (2.22).

Proof. Let $k_1 = k_1(n)$ be the least positive integer satisfying

$$
1 < \frac{3(n+1)}{2\cdot 4^{k+l}}\Big\{1 + \sum_{p=1}^{l-1} 4^p \prod_{q=1}^{p}\Big(1+\frac{1}{4^{k+l+1-q}}\Big)\Big\} + \prod_{q=1}^{l}\Big(1+\frac{1}{4^{k+q}}\Big)
$$

$$\leq \frac{3(n+1)}{2}\Big[4^{-(k+l)} + 4^{-k}\Big(\sum_{p=1}^{\infty} 4^{-p}\Big) \cdot \Big\{\prod_{q=1}^{\infty}\Big(1 + \frac{1}{4^{k+q}}\Big)\Big\}\Big]$$

$$+ \prod_{q=1}^{\infty}\Big(1 + \frac{1}{4^{k+q}}\Big) \leq 2 \tag{2.36}$$

for all $k \geq k_1$ and $l = 1, 2, \cdots$. We take the least positive integer $k_0 = k_0(n)$ such that

$$4^{k_0} \geq \max\{4^{k_1}, r^* + 1\}. \tag{2.37}$$

Fix $\rho \in (0, 36(n+1)^2]$. We apply Lemma 2.5 with $r + 1 = 4^{k_0}$ and $\delta = \rho$ to obtain $\varepsilon_0' = \varepsilon_0'(n) > 0$ and $T_0' = T_0'(n, \rho, \Lambda) \in (0, 1)$ such that, if

$$\sup_{t \in (0,1)} \|w(t)\|_{L^1(B(x_0, 2\rho))} < \varepsilon_0',$$

then

$$\int_{B(x_0,\rho)} w^{4^{k_0}}(x,t)dx \leq t^{-4^{k_0}}$$

for $t \in (0, T_0']$. Set

$$\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(n) = \min\{\varepsilon_0', \varepsilon_0''\} > 0 \tag{2.38}$$

$$\tilde{T}_0 = \tilde{T}_0(n, \rho, \Lambda) = \min\{T_0', T_0''\} \in (0, 1), \tag{2.39}$$

where ε_0'' and T_0'' are as in Lemma 2.6. It is clear that, if

$$\sup_{t \in (0,1)} \|w(t)\|_{L^1(B(x_0, 2\rho))} < \tilde{\varepsilon}_0, \tag{2.40}$$

then

$$\int_{B(x_0,\rho)} w^{4^{k_0}}(x,t)dx \leq t^{-4^{k_0}} \tag{2.41}$$

for $t \in (0, \tilde{T}_0]$. Given $r \geq r^*$, we put

$$\tilde{A}_{1,r} = A_1(a_0, \delta_r^0), \tag{2.42}$$

where

$$a_0 = \frac{1}{3(n+1)} \quad \text{and} \quad \delta_r^0 = \frac{\rho}{r}. \tag{2.43}$$

Recalling (2.1), the definition of $A_1 = A_1(a, \delta)$, we have $C_{26}^* = C_{26}^*(n, \rho) > 1$ such that

$$C_{14}(n)(r + \tilde{A}_{1,r})^{C_{15}(n)}\{1 + (\rho' + \delta_r^0)^n\}^{\frac{3(n+1)}{2}} \max\{1, \Lambda^{\frac{3(n+1)}{2}}\}$$

$$\leq C_{14}(n)\Big(r + \frac{36(n+1)^2 r}{\rho}\Big)^{C_{15}(n)}\{1 + (2\rho)^n\}^{\frac{3(n+1)}{2}} \max\{1, \Lambda^{\frac{3(n+1)}{2}}\}$$

$$\leq r^{C_{15}(n)} C^*_{26}(n,\rho) \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} \tag{2.44}$$

for all $r \in [r^*, \infty)$ and $\rho' \in (0, \rho]$, where $C_{14}(n)$ and $C_{15}(n)$ are as in (2.22).

We shall show the lemma by an induction on $l = 1, 2, \cdots$. We take r_l, θ_l and M_l ($l = 1, 2, \cdots$) as in (2.33), (2.34) and (2.35), respectively. We put

$$\rho_l = \rho(1 - \sum_{p=1}^{l} 1/r_p) = \rho - \sum_{p=1}^{l} \delta^0_{r_p} \tag{2.45}$$

for $l = 1, 2, \cdots$, where δ^0_r is as in (2.43).

First, we show the lemma for $l = 1$. Applying Lemma 2.6 with

$$r + 1 = r_1, \quad \rho = \rho_1, \quad a = a_0 = \frac{1}{3(n+1)}, \quad \delta = \delta^0_{r_1} = \frac{\rho}{r_1},$$

and using (2.44), (2.41), (2.45) and (2.34)-(2.35), we find that (2.40) implies

$$\begin{aligned}
&\frac{1}{r_1}\frac{d}{dt} J^{(x_0,\rho_1,\delta^0_{r_1})}_{r_1}(t) + 3\Big\{J^{(x_0,\rho_1,\delta^0_{r_1})}_{r_1}(t)\Big\}^{1+1/r_1} \\
&\leq t^{-\frac{3(n+1)}{2}} C_{14}(n)(r_1 + \tilde{A}_{1,r_1})^{C_{15}(n)}\{1 + (\rho_1 + \delta^0_{r_1})^n\}^{\frac{3(n+1)}{2}} \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} \\
&\qquad \times \max\{1, \|w(t)\|^{r_1+1}_{L^{r_1/4}(B(x_0,\rho_1+\delta^0_{r_1}))}\} \\
&\leq t^{-\frac{3(n+1)}{2}} \cdot r_1^{C_{15}(n)} C^*_{26}(n,\rho) \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} \cdot t^{-(r_1+1)} = M_1 t^{-\theta_1}
\end{aligned} \tag{2.46}$$

for $t \in (0, \tilde{T}_0]$. Noting that $M_1 > 1$ and $1 \leq \theta_1/r_1 \leq 2$ by (2.33)-(2.37), it holds that

$$\begin{aligned}
&\frac{1}{r_1}\frac{d}{dt}(M_1 t^{-\theta_1}) + 3(M_1 t^{-\theta_1})^{1+1/r_1} \\
&= -\frac{\theta_1}{r_1} \cdot M_1 t^{-(\theta_1+1)} + 3M_1^{1+1/r_1} t^{-(\theta_1+\theta_1/r_1)} \\
&\geq (3 - \theta_1/r_1) \cdot M_1 t^{-(\theta_1+1)} \geq M_1 t^{-\theta_1}
\end{aligned} \tag{2.47}$$

for $t \in (0, 1)$. Comparing (2.46) with (2.47), we conclude that, if (2.40) holds, then

$$J^{(x_0,\rho_1,\delta^0_{r_1})}_{r_1}(t) \leq M_1 t^{-\theta_1}$$

for $t \in (0, \tilde{T}_0]$, which implies the lemma for $l = 1$.

Next, we assume that the lemma holds for $l - 1$ ($l \geq 2$); i.e.,

$$J^{(x_0,\rho_{l-1},\delta^0_{r_{l-1}})}_{r_{l-1}}(t) \leq M_{l-1} t^{-\theta_{l-1}} \tag{2.48}$$

for $t \in (0, \tilde{T}_0]$. A straightforward calculation shows

$$M_{l-1}^{4(1+\frac{1}{r_l})} r_l^{C_{15}(n)} C_{26}^*(n,\rho) \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} = M_l \tag{2.49}$$

$$\frac{3(n+1)}{2} + \theta_{l-1} \cdot 4\big(1 + \frac{1}{r_l}\big) = \theta_l \tag{2.50}$$

for $l = 2, 3, \cdots$. Applying Lemma 2.6 with

$$r + l = r_l, \quad \rho = \rho_l, \quad a = a_0 = \frac{1}{3(n+1)}, \quad \delta = \delta_{r_l}^0 = \frac{\rho}{r_l},$$

and using (2.44), (2.45), (2.48) and (2.49)-(2.50), we find that

$$\begin{aligned}
&\frac{1}{r_l}\frac{d}{dt} J_{r_l}^{(x_0,\rho_l,\delta_{r_l}^0)}(t) + 3\Big\{ J_{r_l}^{(x_0,\rho_l,\delta_{r_l}^0)}(t)\Big\}^{1+1/r_l} \\
&\le t^{-\frac{3(n+1)}{2}} C_{14}(n)(r_l + \tilde{A}_{1,r_l})^{C_{15}(n)}\{1 + (\rho_l + \delta_{r_l}^0)^n\}^{\frac{3(n+1)}{2}} \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} \\
&\qquad \times \max\{1, \|w(t)\|_{L^{r_l/4}(B(x_0,\rho_l+\delta_{r_l}^0))}^{r_l+1}\} \\
&\le t^{-\frac{3(n+1)}{2}} \cdot r_1^{C_{15}(n)} C_{26}^*(n,\rho) \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} \cdot \Big(J_{r_{l-1}}^{(x_0,\rho_{l-1},\delta_{r_{l-1}}^0)}(t)\Big)^{4(1+\frac{1}{r_{l-1}})} \\
&\le M_{l-1}^{4\left(1+\frac{1}{r_l}\right)} r_l^{C_{15}(n)} C_{26}^*(n,\rho) \max\{1, \Lambda^{\frac{3(n+1)}{2}}\} \cdot t^{-\left\{\frac{3(n+1)}{2} + \theta_{l-1}\cdot 4\left(1+\frac{1}{r_l}\right)\right\}} \\
&= M_l t^{-\theta_l}
\end{aligned} \tag{2.51}$$

for $t \in (0, \tilde{T}_0]$. Since $M_l > 1$ and $1 \le \theta_l / r_l \le 2$ by (2.33)-(2.37), we have

$$\begin{aligned}
&\frac{1}{r_l}\frac{d}{dt}(M_l t^{-\theta_l}) + 3(M_l t^{-\theta_l})^{1+1/r_l} \\
&= -\frac{\theta_l}{r_l} \cdot M_l t^{-(\theta_l+1)} + 3M_l^{1+1/r_l} t^{-(\theta_l+\theta_1/r_l)} \\
&\ge (3 - \theta_l/r_l) \cdot M_l t^{-(\theta_l+1)} \ge M_l t^{-\theta_l}
\end{aligned} \tag{2.52}$$

for $t \in (0, 1)$. By comparing (2.51) with (2.52), we conclude that (2.40) implies

$$J_{r_l}^{(x_0,\rho_l,\delta_{r_l}^0)}(t) \le M_l t^{-\theta_l}$$

for $t \in (0, \tilde{T}_0]$, which shows the lemma for l. The proof is complete. □

Here, we recall an ε-regularity for the weak solution obtained in [17].

Theorem 4. *Suppose* $T = T_{\max} < +\infty$. *Then, we have* $\varepsilon_0 = \varepsilon_0(n) > 0$ *independent of* $x_0 \in \mathbf{R}^n$ *such that, if*

$$\limsup_{t\uparrow T} \int_{B(x_0,R)} u(x,t)dx < \varepsilon_0$$

holds for some positive $R \ll 1$, *then*

$$\limsup_{t\uparrow T} \|u(t)\|_{L^\infty(B(x_0,R/2))} \leq C_0,$$

where the positive constant C_0 *depends only on* n, T, $\lambda = \|u_0\|_1, \|u_0\|_\infty$ *and* R.

The above theorem is valid also to the approximate solution $u_\varepsilon = u_\varepsilon(x,t)$, $0 < \varepsilon \leq 1$ to (1.5) with the same constants ε_0 and C_0 provided that $\|u_{0\varepsilon}\|_1 \leq \|u_0\|_1 = \lambda$. In fact, Theorem 4 follows from this uniform result on the approximate solution, see [17]. It is clear that the equation of (2.5) is equivalent to (1.5). Consequently, the following lemma holds by a simple covering argument.

Lemma 2.8. *Assume that there exists* $t_0 \in (0, 1/4)$ *such that*

$$\|w(t_0)\|_{L^\infty(B(x_0,R))} < +\infty,$$

where $x_0 \in \mathbf{R}^n$ *and* $R > 0$. *Then, there exists* $0 < \tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(n) \ll 1$ *such that, if*

$$\sup_{t\in[t_0,1)} \|w(t)\|_{L^1(B(x_0,R))} < \tilde{\varepsilon}_1, \tag{2.53}$$

then

$$\sup_{t\in[t_0,1)} \|w(t)\|_{L^\infty(B(x_0,R/2))} \leq C_{27}(n, R, \Lambda, \|w(t_0)\|_{L^\infty(B(x_0,R))}), \tag{2.54}$$

where C_{27} *is independent of* $x_0 \in \mathbf{R}^n$.

We are in a position to prove Lemma 2.4

Proof of Lemma 2.4. Fix $R \in (0, 72(n+1)^2]$ and $x_0 \in \mathbf{R}^n$, and put $2\rho = R$. Let r_l, θ_l, M_l, $\tilde{\varepsilon}_0$ and $\tilde{T}_0$ be numbers as in Lemma 2.7. Then it follows from Lemma 2.7 that, if

$$\sup_{t\in(0,1)} \|w(t)\|_{L^1(B(x_0,2\rho))} < \tilde{\varepsilon}_0,$$

then

$$\|w(t)\|_{L^{r_l}(B(x_0,\frac{2}{3}\rho))} \leq \|w(t)\|_{L^{r_l}(B(x_0,\rho(1-\sum_{p=1}^{l} 1/r_l)))} \leq M_l^{1/r_l} \cdot t^{-\theta_l/r_l} \tag{2.55}$$

for all $t \in (0, \tilde{T}_0]$ and $l = 1, 2, \cdots$. As stated in the proof of Lemma 2.7,

$$1 \leq \theta_l/r_l \leq 2 \tag{2.56}$$

for $l = 1, 2, \cdots$. Since it holds by (2.33) that

$$\left\{ r_k^{4^{l-k}\prod_{q=1}^{l-k}(1+\frac{1}{r_{l+1-q}})} \right\}^{1/r_l} = 4^{(k_0+k)\cdot 4^{-(k_0+k)}\cdot\prod_{q=1}^{l-k}(1+\frac{1}{r_{l+1-q}})} \leq 4^{(k_0+k)\cdot 4^{-(k_0+k)}\cdot\prod_{q=1}^{\infty}(1+4^{-q})}$$

for $k = 1, 2, \cdots$ and $l = 1, 2, \cdots$, we obtain

$$\left\{ r_1^{4^{l-1}\prod_{q=1}^{l-1}(1+\frac{1}{r_{l+1-q}})} \cdot r_2^{4^{l-2}\prod_{q=1}^{l-2}(1+\frac{1}{r_{l+1-q}})} \cdots r_{l-1}^{4(1+\frac{1}{r_l})} \cdot r_l \right\}^{C_{15}(n)} \tag{2.57}$$
$$\leq \prod_{k=1}^{l} 4^{(k_0+k)\cdot 4^{-(k_0+k)}\cdot\prod_{q=1}^{\infty}(1+4^{-q})\cdot C_{15}(n)} \leq 4^{C_{28}(n)\sum_{j=k_0+1}^{k_0+l} j4^{-j}} \leq 4^{C_{29}(n)}$$

for $l = 1, 2, \cdots$. We also obtain

$$\frac{1}{r_l}\sum_{p=1}^{l-1} 4^p \prod_{q=1}^{p}\left(1+\frac{1}{r_{l+1^q}}\right) = 4^{-k_0}\sum_{p=1}^{l-1} 4^{p-l}\prod_{q=1}^{p}\left(1+\frac{1}{r_{l+1^q}}\right)$$
$$\leq 4^{-k_0}\left(\sum_{p=1}^{\infty} 4^{-p}\right)\cdot\prod_{q=1}^{\infty}(1+4^{-q}) \leq C_{30}^* \tag{2.58}$$

for $l = 1, 2, \cdots$, where C_{30}^* is an absolute constant. Recalling (2.35), the definition of M_l, inequalities (2.57) and (2.58) yield

$$M_l^{1/r_l} \leq 4^{C_{29}(n)}\{C_{26}^*(n,\rho)\}^{1+C_{30}^*}\max\{1, \Lambda^{\frac{3(n+1)}{2}(1+C_{30}^*)}\} \leq C_{31}(n,\rho,\Lambda) \tag{2.59}$$

for $l = 1, 2, \cdots$. We combine (2.55), (2.56) and (2.59), so that

$$\|w(t)\|_{L^{r_l}(B(x_0,\frac{2}{3}\rho))} \leq C_{31}(n,\rho,\Lambda)t^{-2}$$

for all $t \in (0, \tilde{T}_0]$ and $l = 1, 2, \cdots$. Letting $l \to +\infty$, we get

$$\|w(t)\|_{L^{\infty}(B(x_0,\frac{2}{3}\rho))} \leq C_{31}(n,\rho,\Lambda)t^{-2} \tag{2.60}$$

for all $t \in (0, \tilde{T}_0]$.

Now we can pick a time t_0 in $(0, \tilde{T}_0) \cap (0, 1/4)$ satisfying

$$\|w(t_0)\|_{L^\infty(B(x_0,2\rho/3))} \leq 1 + \inf_{(0,\tilde{T}_0)\cap(0,1/4)} \|w(t)\|_{L^\infty(B(x_0,2\rho/3))} < +\infty$$

by virtue of (2.60). We take $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(n)$ in Lemma 2.8 and set

$$\varepsilon_0^*(n) = \min\{\tilde{\varepsilon}_0, \tilde{\varepsilon}_1\}.$$

Then we apply Lemma 2.8 and conclude that, if

$$\sup_{t\in(0,1)} \|w(t)\|_{L^1(B(x_0,R))} < \varepsilon_0^*(n),$$

then

$$\sup_{t\in[1/4,1)} \|w(t)\|_{L^\infty(B(x_0,\rho/3))} = \sup_{t\in[1/4,1)} \|w(t)\|_{L^\infty(B(x_0,R/6))} \le C_{32}(n,R,\Lambda),$$

where C_{32} is independent of $x_0 \in \mathbf{R}^n$. The proof is complete. □

The second objective in this section is to prove the following lemma.

Lemma 2.9. *Let* $w = w(x,t)$ *be a solution to* (2.5), *and assume that there exist* $M > 0$ *and* $R > 0$ *such that*

$$\sup_{t\in[1/4,1)} \|w(t)\|_{L^\infty(x_0,R+1)} \le M. \tag{2.61}$$

Then for any $\xi > 0$ *there exists* $\hat{k} = \hat{k}(n,\Lambda,M,\xi)$*, not depending on* x_0 *and* R*, such that*

$$\sup_{t\in[1/4,1)} \|w(t)\|_{L^1(x_0,R+1)} \le \hat{k} \tag{2.62}$$

implies

$$\sup_{t\in[3/8,1)} \|w(t)\|_{L^\infty(x_0,\frac{2}{3}R)} \le \xi. \tag{2.63}$$

For our purpose, we consider a smooth solution $z = z(x,t)$ to

$$\begin{aligned} &\frac{4L}{3} z_t = \frac{m-1}{m}\Delta(z+d)^m - \nabla\cdot(z\nabla\Gamma * z),\ z\ge 0 && \text{in } \mathbf{R}^n\times(0,L) \\ &z|_{t=0} = z_0 \ge 0 && \text{in } \mathbf{R}^n, \end{aligned} \tag{2.64}$$

where $L > 1$ and $0 < d \le 1$, and z_0 is a smooth function defined on $\mathbf{R}^n$.

Lemma 2.9 will be shown by an estimation on $z = z(x,t)$ (Lemma 2.11) and a simple change of variable.

In Lemmas 2.10-2.11, their proofs and that of Lemma 2.9 below, let $L > 1$ and $0 < d < 1$ be fixed, and $z = z(x,t)$ shall denote a smooth solution to (2.64).

Lemma 2.10. *Assume that there exist* $M_0 > 0$ *and* $R > 0$ *such that*

$$\sup_{t\in[0,L)} \|z(t)\|_{L^\infty(B(x_0,R+1))} \le M_0, \tag{2.65}$$

where $x_0 \in \mathbf{R}^n$. *Then there exists* $k_L > 0$, *such that if*

$$\sup_{t\in[0,L)} \|z(t)\|_{L^1(B(x_0,R+1))} < k_L \tag{2.66}$$

holds for $x_0 \in \mathbf{R}^n$*, and if* $\rho + \frac{1}{\sqrt{r+1}} \le R$*, then*

$$\frac{4L}{3(r+1)} \frac{d}{dt} \int_{\mathbf{R}^n} z^{r+1} \eta_{x_0,\rho,1/\sqrt{r+1}} dx + 4L \left(\int_{\mathbf{R}^n} z^{r+1} \eta_{x_0,\rho,1/\sqrt{r+1}} dx \right)^{1+\frac{1}{r+1}}$$
$$\le L^{-(r+1)} + C_{33}(n, \|z_0\|_1, M_0) \{(r+1)L\}^{\frac{3(n+1)}{2}}$$
$$\times \max \left\{ \|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\frac{1}{\sqrt{r+1}}))}, \|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\frac{1}{\sqrt{r+1}}))} \right\}, \tag{2.67}$$

where C_{33} *is independent of* x_0*,* R *and* L*.*

Proof. We multiply (2.64) by $z^r \eta$ and integrate over $\mathbf{R}^n$. Then it follows that

$$\frac{4L}{3(r+1)} \frac{d}{dt} \int_{\mathbf{R}^n} z^{r+1} \eta dx \le I + II + III, \tag{2.68}$$

where

$$I = -\frac{m-1}{m} \int_{\mathbf{R}^n} \nabla z^m \cdot \nabla (z^r \eta) dx,$$
$$II = -\int_{\mathbf{R}^n} z^r \eta \nabla \cdot (z \nabla \Gamma * z) dx,$$
$$III = \frac{(m-1)A_1^2}{4r} \int_{\mathbf{R}^n} z^{r+1} \eta^{1-2a} dx.$$

The term I is estimated by

$$I = -\frac{4r(m-1)}{(r+m)^2} \int_{\mathbf{R}^n} |\nabla z^{\frac{r+m}{2}}|^2 \eta dx - \frac{2(m-1)}{r+m} \int_{\mathbf{R}^n} z^{\frac{r+m}{2}} \nabla z^{\frac{r+m}{2}} \cdot \nabla \eta dx$$
$$\le -\frac{3r(m-1)}{(r+m)^2} \int_{\mathbf{R}^n} |\nabla z^{\frac{r+m}{2}}|^2 \eta dx + \frac{(m-1)A_1^2}{r} \int_{\mathbf{R}^n} z^{r+m} \eta^{1-2a} dx. \tag{2.69}$$

We use Lemma 2.3 (ii) to obtain

$$\int_{\mathbf{R}^n} z^{r+m} \eta^{1-2a} dx \le \frac{r}{(m-1)A_1^2} \cdot \frac{r(m-1)}{4(r+m)^2} \int_{\mathbf{R}^n} |\nabla z^{\frac{r+m}{2}}|^2 \eta dx$$
$$+ C_{34}(n) A_1^{3n+1} \cdot \max \left\{ \|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}, \|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))} \right\} \tag{2.70}$$

for all $a \in (0, \frac{1}{3(n+1)}]$ and $\delta \in (0, 36(n+1)^2]$. It follows from (2.69)-(2.70) that

$$I \le -\frac{11r(m-1)}{4(r+m)^2} \int_{\mathbf{R}^n} |\nabla z^{\frac{r+m}{2}}|^2 \eta dx$$

$$+\frac{C_{35}(n)A_1^{3(n+1)}}{r}\cdot\max\left\{\|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))},\|z\|^{r+6}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \quad (2.71)$$

for all $a\in(0,\frac{1}{3(n+1)}]$ and $\delta\in(0,36(n+1)^2]$. Using Lemma 2.3 (ii) again, the term III is estimated by

$$III\le\frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n}|\nabla z^{\frac{r+m}{2}}|^2\eta dx$$
$$+\frac{C_{36}(n)A_1^{3(n+1)}}{r}\cdot\max\left\{\|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))},\|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \quad (2.72)$$

for all $a\in(0,\frac{1}{3(n+1)}]$ and $\delta\in(0,36(n+1)^2]$.

Next, we estimate the term II. Using $-\Delta(\Gamma*z)=u$, we have

$$\begin{aligned}II&=-\frac{1}{r+1}\int_{\mathbf{R}^n}\eta\nabla z^{r+1}\cdot\nabla\Gamma*z dx+\int_{\mathbf{R}^n}z^{r+2}\eta dx\\&=\frac{1}{r+1}\int_{\mathbf{R}^n}z^{r+1}\nabla\Gamma*z\cdot\nabla\eta dx+\frac{r}{r+1}\int_{\mathbf{R}^n}z^{r+2}\eta dx\\&\le\frac{A_1}{r+1}\int_{\mathbf{R}^n}z^{r+1}\eta^{1-a}|\nabla\Gamma*z|dx+\int_{\mathbf{R}^n}z^{r+2}\eta dx.\end{aligned} \quad (2.73)$$

It follows from Lemma 2.3 (ii) that

$$\int_{\mathbf{R}^n}z^{r+2}\eta dx\le\frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n}|\nabla z^{\frac{r+m}{2}}|^2\eta dx+C_{37}(n)(r^{\frac{3n+1}{2}}+A_1^{3n+1})$$
$$\times\max\left\{\|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))},\|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \quad (2.74)$$

for all $a\in(0,\frac{1}{3(n+1)}]$ and $\delta\in(0,36(n+1)^2]$. To estimate the first term of the right-hand side of (2.73), we use the decomposition used in [15, 16, 17]; that is,

$$\nabla\Gamma=g_1+g_2 \quad (2.75)$$
$$g_1(x)=-\frac{x}{\omega_{n-1}|x|^n}\chi_{\mathbf{R}^n\setminus B(0,1)}(x),\quad g_2(x)=-\frac{x}{\omega_{n-1}|x|^n}\chi_{B(0,1)}(x),$$

where χ_A denotes the characteristic function of $A\subset\mathbf{R}^n$. For g_1 and g_2 in (2.75), we have

$$\int_{\mathbf{R}^n}z^{r+1}\eta^{1-a}|\nabla\Gamma*z|dx\le II'+II'', \quad (2.76)$$

where

$$II' = \int_{\mathbf{R}^n} z^{r+1}\eta^{1-a}|g_1 * z|dx, \qquad II'' = \int_{\mathbf{R}^n} z^{r+1}\eta^{1-a}|g_2 * z|dx.$$

Here, we note that the total mass conservation law holds for the solution z to (2.64). Indeed, $\tilde{z}(x,t) = z(x, \frac{4L}{3}t)$ is a solution to (1.5)-(1.6) for $\varepsilon = d$, $T = \frac{3}{4}$ and $u_{0\varepsilon} = z_0$, and therefore $\|z(x,t)\|_1 = \|z_0\|_1$ for $t \in [0, L)$ by (1.11). Then we have

$$\begin{aligned} II' &\le \int_{\mathbf{R}^n} z^{r+1}(x,t)\eta^{1-a}(x)\Big(\int_{\mathbf{R}^n} |g_1(x-x')|z(x',t)dx'\Big)dx \\ &\le \omega_{n-1}^{-1}\|z_0\|_1 \int_{\mathbf{R}^n} z^{r+1}\eta^{1-a}dx \end{aligned} \tag{2.77}$$

by the total mass conservation law and the fact that $\|g_1\|_\infty = \omega_{n-1}^{-1}$. Since

$$\mathrm{supp}\eta \subset \overline{B(x_0, \rho+\delta)}, \qquad \mathrm{supp} g_2 \subset \overline{B(0,1)}$$

the term II'' is estimated by

$$\begin{aligned} II'' &\le \int_{\mathbf{R}^n} z^{r+1}(x,t)\eta^{1-a}(x)\Big(\int_{\mathbf{R}^n} \frac{z(x',t)\chi_{B(x,1)}(x')}{\omega_{n-1}|x-x'|^{1-n}}dx'\Big)dx \\ &\le \omega_{n-1}^{-1}\Big(\int_{B(0,1)} |x'|^{-(n-1/2)}dx'\Big)^{\frac{2n-2}{2n-1}}\Big(\int_{B(x_0,\rho+\delta+1)} z^{2n-1}(x',t)dx'\Big)^{\frac{1}{2n-1}} \\ &\quad \times \int_{\mathbf{R}^n} z^{r+1}(x,t)\eta^{1-a}(x)dx \\ &\le C_{38}(n)\|z\|_{L^{2n-1}(B(x_0,\rho+\delta+1))}\int_{\mathbf{R}^n} z^{r+1}\eta^{1-a}dx. \end{aligned} \tag{2.78}$$

We apply Lemma 2.3 (ii) to obtain

$$\begin{aligned} \int_{\mathbf{R}^n} z^{r+1}\eta^{1-a}dx &\le \frac{r+1}{A_1C_{39}} \cdot \frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n} |\nabla z^{\frac{r+m}{2}}|^2\eta dx \\ &\quad + C_{40}(n)\{(A_1C_{39})^{\frac{3n+1}{2}} + A_1^{3n+1}\} \\ &\qquad \times \max\Big\{\|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}, \|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\Big\} \end{aligned} \tag{2.79}$$

for all $a \in (0, \frac{1}{3(n+1)}]$ and $\delta \in (0, 36(n+1)^2]$ satisfying $\rho + \delta \le R$, where

$$\begin{aligned} C_{39} = C_{39}(n, \|z_0\|_1, M_0) &= \omega_{n-1}^{-1}\|z_0\|_1 + C_{38}(n)\cdot(k_L M_0^{2n-2})^{\frac{1}{2n-1}} \\ &\ge \|z_0\|_1 + C_{38}\|z\|_{L^{2n-1}(B(x_0,\rho+\delta+1))} \end{aligned} \tag{2.80}$$

by the assumptions (2.65)-(2.66) provided that $\rho+\delta+1 \le R$, and k_L is a positive constant satisfying (2.66) and is determined below. Combining (2.78)-(2.80), we find

$$\frac{A_1}{r+1}\int_{\mathbf{R}^n} z^{r+1}\eta^{1-a}|\nabla\Gamma * z|dx \le \frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n}|\nabla z^{\frac{r+m}{2}}|^2\eta dx$$
$$+\frac{C_{41}}{r}A_1^{3n+2}\cdot\max\left\{\|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}, \|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \tag{2.81}$$

for all $a\in(0,\frac{1}{3(n+1)}]$ and $\delta\in(0,36(n+1)^2]$ satisfying $\rho+\delta\le R$, where

$$C_{41}=C_{41}(n,\|z_0\|_1,M_0)=C_{40}C_{39}(C_{39}^{\frac{3n+1}{2}}+1).$$

Inequalities (2.73), (2.74) and (2.81) are organized as

$$II \le \frac{r(m-1)}{2(r+m)^2}\int_{\mathbf{R}^n}|\nabla z^{\frac{r+m}{2}}|^2\eta dx \tag{2.82}$$
$$+C_{42}(r^{\frac{3n+1}{2}}+A_1^{3n+2})\cdot\max\left\{\|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}, \|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\}$$

for all $a\in(0,\frac{1}{3(n+1)}]$ and $\delta\in(0,36(n+1)^2]$ satisfying $\rho+\delta\le R$, where

$$C_{42}=C_{42}(n,\|z_0\|_1,M_0)=C_{37}+C_{41}.$$

We use Lemmas 2.2, 2.3 (ii) and (2.66) to get

$$4L\Big(\int_{\mathbf{R}^n} z^{r+1}\eta dx\Big)^{1+\frac{1}{r+1}} \le 4L\int_{\mathbf{R}^n} z^{r+2}\eta dx + 4L\|z\|^{r+2}_{L^1(B(x_0,\rho+\delta))}$$
$$\le \frac{r(m-1)}{4(r+m)^2}\int_{\mathbf{R}^n}|\nabla z^{\frac{r+m}{2}}|^2\eta dx + 4L(k_L)^{r+2}$$
$$+4LC_{43}(n)\{(4rL)^{\frac{3n+1}{2}}+A_1^{3n+1}\}$$
$$\times\max\left\{\|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}, \|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \tag{2.83}$$

for all $a\in(0,\frac{1}{3(n+1)}]$ and $\delta\in(0,36(n+1)^2]$ satisfying $\rho+\delta\le R$.

Combining (2.68), (2.71), (2.72), (2.82), and (2.83), we obtain

$$\frac{4L}{3(r+1)}\frac{d}{dt}\int_{\mathbf{R}^n} z^{r+1}\eta dx + 4L\Big(\int_{\mathbf{R}^n} z^{r+1}\eta dx\Big)^{1+\frac{1}{r+1}}$$
$$\le 4L(k_L)^{r+2}+C_{44}(n,\|z_0\|_1,M_0)L^{\frac{3(n+1)}{2}}(r^{\frac{3n+1}{2}}+A_1^{3(n+1)})$$
$$\times\max\left\{\|z\|^{r+1-2n}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}, \|z\|^{r+2}_{L^{\frac{r+1}{4}}(B(x_0,\rho+\delta))}\right\} \tag{2.84}$$

for all $a \in (0, \frac{1}{3(n+1)}]$ and $\delta \in (0, 36(n+1)^2]$. In (2.84), we take $a, \delta > 0$ and $k_L > 0$ such that

$$a = \frac{1}{3(n+1)}, \quad \delta = \frac{1}{\sqrt{r+1}}, \quad 4L(k_r)^{r+2} \le L^{-(r+1)} \text{ for all } r \ge r^*,$$

and get the estimate (2.67). □

Lemma 2.11. *Assume that there exist $M_0 > 0$ and $R > 0$ satisfying (2.65). For any $L > 1$ we have $r^{**} = r^{**}(n, L)$ satisfying the following property: given $r \ge r^{**}$, there exists $\tilde{k}_r = \tilde{k}_r(n, r, L, \|z_0\|_1, M_0) > 0$ such that*

$$\sup_{t \in [0,L)} \|z(t)\|_{L^1(B(x_0,R+1))} < \tilde{k}_r \tag{2.85}$$

implies

$$\|z(t)\|_{L^\infty(B(x_0,\frac{2}{3}R))} \le C_{45}(n, \|z_0\|_1, M_0) \cdot L^{1/\pi} t^{-2/\pi} \tag{2.86}$$

for all $t \in [1, L)$ and $x_0 \in \mathbf{R}^n$, where C_{45} is independent of x_0, R and L.

Proof. Given $R > 0$ and $x_0 \in \mathbf{R}^n$, we introduce the function

$$\begin{aligned} F_{0,r} &= F_{0,r}(t) = \|z(t)\|^{r/4}_{L^{r/4}(B(x_0,R))} \\ F_{l,r} &= F_{l,r}(t) = \int_{\mathbf{R}^n} z^{4^{l-1}r}(x,t)\eta_{x_0,\rho_l,1/\sqrt{4^{l-1}r}}(x)dx \end{aligned} \tag{2.87}$$

for $l = 1, 2, \cdots$ and $r \ge r^* + 1$, where

$$\rho_l = R - \sum_{i=1}^{l} \frac{1}{\sqrt{4^{i-1}r}}.$$

We have $k_L > 0$ such that, if

$$\sup_{t \in [0,L)} \|z(t)\|_{L^1(B(x_0,R+1))} < k_L, \tag{2.88}$$

then

$$\begin{aligned} &\frac{4^{1-l}L}{3r}\frac{d}{dt}F_{l+1,r}(t) + 4L(F_{l+1,r}(t))^{1+\frac{1}{4^l r}} \\ &\le L^{-4^l r} + C_{46}(n, \|z_0\|_1, M_0)(4^l r L)^{C_{47}(n)} \\ &\qquad \times \max\left\{F_{l,r}(t)^{4\left(1-\frac{2n}{4^l r}\right)}, F_{l,r}(t)^{4\left(1+\frac{1}{4^l r}\right)}\right\} \end{aligned} \tag{2.89}$$

for $t \in (0, L)$ and $l = 0, 1, 2, \cdots$ by virtue of Lemma 2.10, where $r \geq r^* + 1$. We take a positive constant $r^{**} = r^{**}(n, L) \geq r^*(n) + 1$ such that

$$C_{46}(4rL)^{C_{47}}\left[\left\{4 - \frac{4}{3}\prod_{j=1}^{\infty}\left(1 + \frac{1}{4^j r}\right)\right\}L - 1\right] \geq L^{-4r}, \tag{2.90}$$

$$\frac{2n}{r} \leq 1, \qquad \frac{C_{47}}{3r}\prod_{j=1}^{\infty}\left(1 + \frac{1}{4^j}\right) \leq \frac{1}{\pi} \tag{2.91}$$

for any $r \geq r^{**}$.

Fix $r \geq r^{**}$ and take $\tilde{k}_r = \tilde{k}_r(n, r, L, \|z_0\|_1, M_0) \leq k_L$ satisfying

$$L^{-r} + C_{46}(rL)^{C_{47}} \cdot (\tilde{k}_r M_0^{r/4-1})^{4\left(1-\frac{2n}{r}\right)} \leq \frac{8}{3}L^{-r}. \tag{2.92}$$

We shall show that $\tilde{k}_r$ satisfying (2.92) is a desired constant.

We put $F_l = F_{l,r}$ for $l = 0, 1, \cdots$. Note that (2.89) for $F_{l,r} = F_l$, $l = 0, 1, \cdots$, is valid because of (2.88) and the fact that $\tilde{k}_r \leq k_L$.

First,

$$F_0(t) \leq \tilde{k}_r M_0^{r/4-1} \tag{2.93}$$

by the assumptions (2.65) and (2.85). Combining (2.89) and (2.92)-(2.93), we obtain

$$\frac{4L}{3r}F_1'(t) + 4L(F_1(t))^{1+\frac{1}{r}} \leq \frac{8}{3}L^{-r} \tag{2.94}$$

for any $t \in (0, L)$. The function $g_1(t) = t^{-r}$ satisfies

$$\frac{4L}{3r}g_1'(t) + 4L(g_1(t))^{1+\frac{1}{r}} = \frac{8}{3}Lt^{-(r+1)} \geq \frac{8}{3}L^{-r} \tag{2.95}$$

for any $t \in (0, L)$. Comparing (2.94) with (2.95), we get

$$F_1(t) \leq t^{-r} \tag{2.96}$$

for any $t \in (0, L)$.

Next, we shall show

$$F_l(t) \leq \begin{cases} M_l t^{-4^{l-1}r\prod_{j=1}^{l-1}\left(1+\frac{1}{4^j r}\right)} & \text{for t} \in (0, 1] \\ M_l t^{-4^{l-1}r\prod_{j=1}^{l-1}\left(1-\frac{2n}{4^j r}\right)} & \text{for t} \in [1, L) \end{cases} \tag{2.97}$$

by an induction on $l = 2, 3, \cdots$, where

$$M_l = 4^{\left[(l-1)+\sum_{i=1}^{l-2}(l-i-1)4^i\left\{\prod_{j=1}^{i}\left(1+\frac{1}{4^{l-j}}r\right)\right\}\right]\cdot C_{47}}$$
$$\times C_{46}^{1+\sum_{i=1}^{l-2}4^i\left\{\prod_{j=1}^{i}\left(1+\frac{1}{4^{l-j}}r\right)\right\}} \times (rL)^{\left[1+\sum_{i=1}^{l-2}4^i\left\{\prod_{j=1}^{i}\left(1+\frac{1}{4^{l-j}}r\right)\right\}\right]\cdot C_{47}}, \tag{2.98}$$

and $\sum_{i=1}^{0}$ is empty.

The relations (2.89) and (2.96) give

$$\frac{L}{3r}F_2'(t)+4L(F_2(t))^{1+\frac{1}{4r}}\le L^{-4r}+C_{46}(4rL)^{C_{47}}\cdot\max\{F_1^{4\left(1-\frac{2n}{4r}\right)},F_1^{4\left(1+\frac{1}{4r}\right)}\}$$
$$\le\begin{cases}L^{-4r}+C_{46}(4rL)^{C_{47}}t^{-4r\left(1+\frac{1}{4r}\right)} & \text{for t}\in(0,1]\\ L^{-4r}+C_{46}(4rL)^{C_{47}}t^{-4r\left(1-\frac{2n}{4r}\right)} & \text{for t}\in[1,L).\end{cases} \tag{2.99}$$

Setting

$$g_2(t)=\begin{cases}C_{46}(4rL)^{C_{47}}t^{-4r\left(1+\frac{1}{4r}\right)} & \text{for t}\in(0,1]\\ C_{46}(4rL)^{C_{47}}t^{-4r\left(1-\frac{2n}{4r}\right)} & \text{for t}\in[1,L),\end{cases}$$

we find that

$$\begin{aligned}&\frac{L}{3r}g_2'(t)+4L(g_2(t))^{1+\frac{1}{4r}}\\ &\ge C_{46}(4rL)^{C_{47}}\cdot\left\{4-\frac{4}{3}\left(1+\frac{1}{4r}\right)\right\}L\cdot t^{-4r\left(1+\frac{1}{4r}\right)-1}\\ &\ge C_{46}(4rL)^{C_{47}}\cdot t^{-4r\left(1+\frac{1}{4r}\right)-1}+C_{46}(4rL)^{C_{47}}\cdot\left[\left\{4-\frac{4}{3}\left(1+\frac{1}{4r}\right)\right\}L-1\right]\\ &\ge L^{-4r}+C_{46}(4rL)^{C_{47}}\cdot t^{-4r\left(1+\frac{1}{4r}\right)}\end{aligned} \tag{2.100}$$

for $t\in(0,1]$ by (2.90) and the fact that $L>1$, and that

$$\begin{aligned}&\frac{L}{3r}g_2'(t)+4L(g_2(t))^{1+\frac{1}{4r}}\\ &\ge C_{46}(4rL)^{C_{47}}\cdot\left\{4-\frac{4}{3}\left(1-\frac{2n}{4r}\right)\right\}L\cdot t^{-4r\left(1-\frac{2n}{4r}\right)-1}\\ &\ge C_{46}(4rL)^{C_{47}}L\cdot L^{-1}t^{-4r\left(1-\frac{2n}{4r}\right)}+C_{46}(4rL)^{C_{47}}\cdot\frac{5}{3}L\cdot L^{-4r\left(1-\frac{2n}{4r}\right)-1}\\ &\ge C_{46}(4rL)^{C_{47}}t^{-4r\left(1-\frac{2n}{4r}\right)}+\frac{5}{3}C_{46}(4r)^{C_{47}}L^{-4r+2n+C_{47}}\\ &\ge L^{-4r}+C_{46}(4r)^{C_{47}}t^{-4r\left(1-\frac{2n}{4r}\right)}\end{aligned} \tag{2.101}$$

for $t\in[1,L)$ by $L>1$. Comparing (2.99) with (2.100)-(2.101), we obtain (2.97) for $l=2$.

Suppose that the lemma holds for $l\ge 2$. It is not difficult to see that

$$M_{l+1}=C_{46}(4^l rL)^{C_{47}}M_l^{4(1+\frac{1}{4^l r})} \tag{2.102}$$

by (2.98). Using (2.89), (2.102) and the hypothesis of the induction (2.97), we deduce

$$\begin{aligned}&\frac{4^{1-l}L}{3r}F'_{l+1}(t)+4L(F'_{l+1}(t))^{1+\frac{1}{4^l r}}\\ &\leq\begin{cases}L^{-4^l r}+M_{l+1}t^{-4^l r\prod_{j=1}^{l}(1+\frac{1}{4^j r})} & \text{for } t\in(0,1]\\ L^{-4^l r}+M_{l+1}t^{-4^l r\prod_{j=1}^{l}(1-\frac{2n}{4^j r})} & \text{for } t\in[1,L).\end{cases}\end{aligned} \tag{2.103}$$

Setting

$$g_{l+1}(t)=\begin{cases}M_{l+1}t^{-4^l r\prod_{j=1}^{l}(1+\frac{1}{4^j r})} & \text{for } t\in(0,1]\\ M_{l+1}t^{-4^l r\prod_{j=1}^{l}(1-\frac{2n}{4^j r})} & \text{for } t\in[1,L),\end{cases}$$

we find that

$$\begin{aligned}&\frac{4^{1-l}L}{3r}g'_{l+1}(t)+4L(g_{l+1}(t))^{1+\frac{1}{4^l r}}\\ &\geq M_{l+1}\Big\{4-\frac{4}{3}\prod_{j=1}^{l}\Big(1+\frac{1}{4^j r}\Big)\Big\}L\cdot t^{-1-4^l r\prod_{j=1}^{l}\left(1+\frac{1}{4^j r}\right)}\\ &\geq M_{l+1}t^{-1-4^l r\prod_{j=1}^{l}(1+\frac{1}{4^j r})}+M_{l+1}\Big[\Big\{4-\frac{4}{3}\prod_{j=1}^{l}\Big(1+\frac{1}{4r}\Big)\Big\}L-1\Big]\\ &\geq M_{l+1}t^{-4^l r\prod_{j=1}^{l}(1+\frac{1}{4^j r})}+C_{46}(4rL)^{C_{47}}\Big[\Big\{4-\frac{4}{3}\prod_{j=1}^{l}\Big(1+\frac{1}{4r}\Big)\Big\}L-1\Big]\\ &\geq L^{-4^l r}+M_{l+1}t^{-4^l r\prod_{j=1}^{l}(1+\frac{5}{4^j r})}\end{aligned} \tag{2.104}$$

for $t\in(0,1]$ by (2.90) and the fact that $L>1$, and that

$$\begin{aligned}&\frac{4^{1-l}L}{3r}g'_{l+1}(t)+4L(g_{l+1}(t))^{1+\frac{1}{4^l r}}\\ &\geq M_{l+1}\Big\{4-\frac{4}{3}\prod_{j=1}^{l}\Big(1-\frac{2n}{4^j r}\Big)\Big\}L\cdot t^{-1-4^l r\prod_{j=1}^{l}(1-\frac{2n}{4^j r})}\\ &\geq M_{l+1}L\cdot L^{-1}t^{-4^l r\prod_{j=1}^{l}(1-\frac{2n}{4^j r})}+\frac{5}{3}M_{l+1}L\cdot L^{-1-4^l r\prod_{j=1}^{l}(1-\frac{2n}{4^j r})}\\ &\geq L^{-4^l r}+M_{l+1}t^{-4^l r\prod_{j=1}^{l}\left(1-\frac{2n}{4^j r}\right)}\end{aligned} \tag{2.105}$$

for $t\in[1,L)$ by the fact that $L>1$. Comparing (2.103) with (2.104)-(2.105), we obtain (2.97) for $l+1$. Thus, the estimate (2.97) is shown completely.

It follows from (2.87) and (2.97) that

$$\|z(t)\|_{L^{4^l r}(B(x_0,\rho_{l+1}))} \le \begin{cases} M_{l+1}^{\frac{1}{4^l r}} t^{-\prod_{j=1}^{l}\left(1+\frac{1}{4^j r}\right)} & \text{for } t\in(0,1] \\ M_{l+1}^{\frac{1}{4^l r}} t^{-\prod_{j=1}^{l}\left(1-\frac{2n}{4^j r}\right)} & \text{for } t\in[1,L) \end{cases} \tag{2.106}$$

for $l = 1, 2, \cdots$. Wallis' product says that

$$\prod_{j=1}^{\infty}\left(1-\frac{1}{4j^2}\right) = \frac{2}{\pi},$$

and so

$$\prod_{j=1}^{\infty}\left(1-\frac{1}{4^j}\right) \ge \frac{2}{\pi}. \tag{2.107}$$

We compute

$$\begin{aligned} M_{l+1}^{\frac{1}{4^l r}} &\le 4^{\frac{C_{47}l}{4^l r}+\frac{C_{47}}{r}\prod_{j=1}^{\infty}\left(1+\frac{1}{4^j}\right)\times\sum_{i=1}^{\infty} i4^{-i}} \\ &\quad \times C_{46}^{\frac{1}{4^l r}+\frac{1}{r}\prod_{j=1}^{\infty}\left(1+\frac{1}{4^j}\right)\times\sum_{i=1}^{\infty}4^{-i}} \cdot (rL)^{\frac{C_{47}}{4^l r}+\frac{C_{47}}{r}\prod_{j=1}^{\infty}\left(1+\frac{1}{4^j}\right)\times\sum_{i=1}^{\infty}4^{-i}} \\ &= 4^{\frac{C_{47}l}{4^l r}+\frac{4C_{47}}{9r}\prod_{j=1}^{\infty}\left(1+\frac{1}{4^j}\right)} \times C_{46}^{\frac{1}{4^l r}+\frac{1}{3r}\prod_{j=1}^{\infty}\left(1+\frac{1}{4^j}\right)} \\ &\quad \times L^{\frac{C_{47}}{4^l r}+\frac{C_{47}}{3r}\prod_{j=1}^{\infty}\left(1+\frac{1}{4^j}\right)} \times r^{\frac{1}{r}\left(\frac{C_{47}}{4^l}+\frac{C_{47}}{3}\prod_{j=1}^{\infty}\left(1+\frac{1}{4^j}\right)\right)}. \end{aligned} \tag{2.108}$$

Organizing (2.106)-(2.108) and (2.91), and letting $l \to +\infty$, we conclude that (2.85) implies (2.86) for $\tilde{k}_r$ satisfying (2.92). $\square$

Proof of Lemma 2.9. We put $z(x,t) = w(x, 1/4 + \frac{3}{4L}t)$ and choose $L = L(n,\Lambda,M,\xi) > 1$ satisfying

$$C_{45}(n,\Lambda,M)\cdot L^{1/\pi}\left(\frac{L}{6}\right)^{-2/\pi} \le \xi, \tag{2.109}$$

where C_{45} is as in Lemma 2.11. Since $z = z(x,t)$ is a solution to (2.64) for $z_0(x) = w(x,1/4)$, we apply Lemma 2.11 to find that there is $\hat{r} = \hat{r}(n,L) = \hat{r}(n,\Lambda,M,\xi)$ satisfying the following property: given $r \ge \hat{r}$, there exists $\hat{k}_r = \hat{k}_r(n,\Lambda,M,\xi,r)$ such that, if

$$\sup_{t\in[0,L)} \|z(t)\|_{L^1(B(x_0,R))} < \hat{k}_r, \tag{2.110}$$

then

$$\|z(t)\|_{L^\infty(B(x_0,R))} \le C_{45}(n,\Lambda,M)\cdot L^{1/\pi} t^{-2/\pi} \tag{2.111}$$

for $t \in [1, L)$. Setting $\hat{k} = \hat{k}_{\hat{r}}$, we summarize the relations (2.109)-(2.111) and conclude that (2.62) implies (2.63). $\square$

3. Proof of Theorem 1

In this section, we assume that $u_0 = u_0(x)$ is the initial value satisfying (1.3) and (1.14), and that $T = T_{\max} < +\infty$ for the weak solution $u = u(x,t)$ to (1.1) with $m = 2 - \frac{2}{n}$.

We begin with the boundedness of the blowup set.

Lemma 3.12. *We have*

$$\limsup_{t\uparrow T} \|u(t)\|_{L^\infty(|x|>R)} \leq C_1 \tag{3.1}$$

for $R \gg 1$; that is, the blowup set $\mathcal{S}$ is bounded in $\mathbf{R}^n$.

Proof. By (1.12) and (1.13), we have

$$\int_{\mathbf{R}^n} |x|^2 \, u(x,t)dx \leq C_2(T,u_0) \tag{3.2}$$

for

$$C_2(T,u_0) = 2(n-2)T\mathcal{F}(u_0) + \int_{\mathbf{R}^n} |x|^2 \, u_0 dx,$$

and hence it follows that

$$\sup_{t\in[0,T)} \int_{|x|>R} u(x,t)dx \leq \frac{1}{R^2} C_2(T,u_0).$$

Taking $R \gg 1$ as $C_2(T,u_0)R^{-2} < \varepsilon_0$, we obtain (3.1) by Theorem 4. □

Given $x_0 \in \mathcal{S}$ and $0 < R \ll 1$, we take $0 \leq \varphi = \varphi_{x_0,R}(x) \in C_0^\infty(\mathbf{R}^n)$ such that

$$\begin{aligned} &0 \leq \varphi \leq 1 \quad \text{in } \mathbf{R}^n, \qquad \varphi = 1 \quad \text{on } B(x_0,R) \\ &\text{supp } \varphi \subset \overline{B(x_0,2R)}, \qquad \|\nabla\varphi\|_\infty \leq C_3(n)R^{-1}, \end{aligned} \tag{3.3}$$

and put

$$A(t) = \int_{\mathbf{R}^n} \varphi(x)u(x,t)dx,$$

where $C_3(n)$ is a positive constant determined only by n.

For the moment, we shall present the formal calculation. First, we have

$$\begin{aligned} \left|\frac{d}{dt}\int_{\mathbf{R}^n} \varphi u dx\right|^2 &= \left|\int_{\mathbf{R}^n} u\nabla(u^{m-1} - \Gamma * u)\cdot\nabla\varphi dx\right|^2 \\ &\leq \int_{\mathbf{R}^n} u \left|\nabla(u^{m-1} - \Gamma * u)\right|^2 dx \cdot \int_{\mathbf{R}^n} u \left|\nabla\varphi\right|^2 dx \leq -\left\|\nabla\varphi\right\|_\infty^2 \lambda \frac{d}{dt}\mathcal{F}(u), \end{aligned} \tag{3.4}$$

and therefore,

$$(A')^2 \le -\frac{\|\nabla\varphi\|_\infty^2 \lambda}{2(n-2)} H'' \tag{3.5}$$

for almost every $t \in (0,T)$ by (1.13), where

$$H(t) = \int_{\mathbf{R}^n} |x|^2 u(x,t)dx. \tag{3.6}$$

If

$$\lim_{t\uparrow T} \mathcal{F}(u(t)) > -\infty \tag{3.7}$$

is the case, it follows from (1.12) and (3.4) that

$$\int_0^T \left| \frac{d}{dt} \int_{\mathbf{R}^n} \varphi u dx \right| dt \le T^{1/2} \left\{ \int_0^T \left| \frac{d}{dt} \int_{\mathbf{R}^n} \varphi u dx \right|^2 dt \right\}^{1/2} < +\infty,$$

and hence

$$\lim_{t\uparrow T} A(t) = \lim_{t\uparrow T} \int_{\mathbf{R}^n} \varphi(x) u(x,t) dx \tag{3.8}$$

exists. Since Theorem 4 guarantees

$$\liminf_{t\uparrow T} A(t) = \limsup_{t\uparrow T} A(t) \ge \limsup_{t\uparrow T} \int_{B(x_0,R)} u(x,t)dx \ge \varepsilon_0,$$

we obtain

$$\liminf_{t\uparrow T} \int_{B(x_0,R)} u(x,t)dx \ge \varepsilon_0$$

for any $x_0 \in \mathcal{S}$ and $0 < R \ll 1$, and therefore the finiteness of $\mathcal{S}$ by the total mass conservation (1.10). This formal argument is actually justified by the following lemma.

Lemma 3.13. *We have*

$$\int_t^{t'} A'(s)^2 ds \le \lambda \|\nabla\varphi\|_\infty^2 (\mathcal{F}(u(t)) - \mathcal{F}(u(t'))) \tag{3.9}$$

$$\int_t^{t'} (t'-s) A'(s)^2 ds \le \frac{\lambda\|\nabla\varphi\|_\infty^2}{2(n-2)} (H(t) - H(t')) + \lambda\|\nabla\varphi\|_\infty^2 (t'-t)\mathcal{F}(u(t)), \tag{3.10}$$

for $t \in [0,T)$ *and* $0 \le t \le t' < T$, *and furthermore,* (3.8) *holds under the assumption of* (3.7).

Proof. The last assertion is an immediate consequence of (3.9) as can be seen from the above argument. To show (3.9), we recall the proof of Lemma 2.2 in [16], and follow the notation. We find the estimate

$$\frac{d}{dt}\mathcal{F}_{\varepsilon,l} \leq -(1-\delta)J_1 + \int_{\mathbf{R}^n} u_\varepsilon(u_\varepsilon+\varepsilon)^m(1-\psi_l)dx + K^\delta_{\varepsilon,l}$$

presented there, with $0<\delta<1$, where $\psi_l=\psi_l(x)$ is the cut-off function satisfying

$$\begin{aligned} &0\leq\psi_l\leq 1, \quad \psi_l\equiv 1 \text{ in } B(0,l), \quad \mathrm{supp}\psi_l\subset B(0,2l) \\ &x\cdot\nabla\psi_l\leq 0, \quad |\nabla\psi_l|\leq C_4\ell^{-1}\psi_l^{1/2}, \quad |\nabla^2\psi_l|\leq C_5\ell^{-2}, \end{aligned} \tag{3.11}$$

with C_i, $i=4,5$ depending only on the space-dimension n, and

$$\begin{aligned} \mathcal{F}_{\varepsilon,l} &= \int_{\mathbf{R}^n}\frac{(u_\varepsilon+\varepsilon)^m}{m}\psi_l dx - \frac{1}{2}\langle\Gamma * u_\varepsilon, u_\varepsilon\rangle \\ J_1 &= \int_{\mathbf{R}^n}(u_\varepsilon+\varepsilon)|\nabla W_\varepsilon|^2\psi_l dx, \quad W_\varepsilon=(u_\varepsilon+\varepsilon)^m - v_\varepsilon \\ v_\varepsilon &= \Gamma * u_\varepsilon, \quad D_l=\mathrm{supp}\psi_l \\ K^\delta_{\varepsilon,l} &= l^{-2}\delta^{-1}\cdot C_6(\|u_\varepsilon\|^{2m-1}_{2m-1}+\varepsilon^{2m-1}|D_l|) \\ &\quad + l^{-1}\cdot C_7\|\nabla v_\varepsilon\|_\infty(\|u_\varepsilon\|^m_m+\varepsilon^m|D_l|)+\varepsilon\|u_\varepsilon\|_1\|v_\varepsilon\|_\infty \end{aligned}$$

with C_i, $i=6,7$, independent of ε, l, δ, and t. Thus, we have

$$J_1 \leq -\frac{1}{1-\delta}\frac{d}{dt}\mathcal{F}_{\varepsilon,l} + \frac{1}{1-\delta}\int_{\mathbf{R}^n}u_\varepsilon(u_\varepsilon+\varepsilon)^m(1-\psi_l)dx + \frac{K^\delta_{\varepsilon,l}}{1-\delta}. \tag{3.12}$$

From the proof of Lemma 2.5 in [16], it follows that

$$\lim_{\varepsilon\downarrow 0}\frac{d}{dt}\int_{\mathbf{R}^n}\varphi(x)u_\varepsilon(x,t)dx = \frac{d}{dt}\int_{\mathbf{R}^n}\varphi(x)u(x,t)dx \tag{3.13}$$

for almost every t, and therefore,

$$\int_t^{t'}\left|\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u dx\right|^2 dt \leq \liminf_{\varepsilon\downarrow 0}\int_t^{t'}\left|\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u_\varepsilon dx\right|^2 dt \tag{3.14}$$

for $t\in[0,T)$ and $t'\in[t,T)$. In a manner similar to (3.4), we get

$$\begin{aligned} \left|\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u_\varepsilon dx\right|^2 &\leq \left|\int_{\mathbf{R}^n}\psi_l\Big\{(u_\varepsilon+\varepsilon)\nabla W_\varepsilon+\varepsilon\nabla v_\varepsilon\Big\}\cdot\nabla\varphi dx\right|^2 \\ &\leq (1+\varepsilon)\left|\int_{\mathbf{R}^n}\psi_l(u_\varepsilon+\varepsilon)\nabla W_\varepsilon\cdot\nabla\varphi dx\right|^2 + \varepsilon(1+\varepsilon)\left|\int_{\mathbf{R}^n}\nabla v_\varepsilon\cdot\nabla\varphi dx\right|^2 \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&\leq (1+\varepsilon)\int_{\mathbf{R}^n}(u_\varepsilon+\varepsilon)|\nabla W_\varepsilon|^2\psi_l dx \\
&\qquad \times \int_{\mathbf{R}^n}(u_\varepsilon+\varepsilon)|\nabla\varphi|^2\psi_l dx+\varepsilon(1+\varepsilon)\left|\int_{\mathbf{R}^n}\nabla v_\varepsilon\cdot\nabla\varphi dx\right|^2 \\
&\leq (1+\varepsilon)J_1\cdot\|\nabla\varphi\|_\infty^2(\|u_\varepsilon\|_1+\varepsilon\|\psi_l\|_1)+\varepsilon(1+\varepsilon)\left|\int_{\mathbf{R}^n}\nabla v_\varepsilon\cdot\nabla\varphi dx\right|^2
\end{aligned}
$$

for almost every t provided that $l \gg 1$. Inequalities (3.12) and (3.15) imply

$$
\begin{aligned}
\left|\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u_\varepsilon dx\right|^2 &\leq -\frac{1+\varepsilon}{1-\delta}\|\nabla\varphi\|_\infty^2(\|u_{0\varepsilon}\|_1+\varepsilon\|\psi_l\|_1)\frac{d}{dt}\mathcal{F}_{\varepsilon,l} \\
&\quad+\frac{1+\varepsilon}{1-\delta}\|\nabla\varphi\|_\infty^2(\|u_{0\varepsilon}\|_1+\varepsilon\|\psi_l\|_1)\cdot\int_{\mathbf{R}^n}u_\varepsilon(u_\varepsilon+\varepsilon)^m(1-\psi_l)dx \\
&\quad+\frac{1+\varepsilon}{1-\delta}\|\nabla\varphi\|_\infty^2(\|u_{0\varepsilon}\|_1+\varepsilon\|\psi_l\|_1)\cdot K_{\varepsilon,l}^\delta+\varepsilon(1+\varepsilon)\left|\int_{\mathbf{R}^n}\nabla v_\varepsilon\cdot\nabla\varphi dx\right|^2,
\end{aligned}
$$

where we used the property (1.11). We integrate this over (t,t') and operate with $\lim_{l\uparrow\infty}\liminf_{\varepsilon\downarrow 0}$, so that

$$
\liminf_{\varepsilon\downarrow 0}\int_t^{t'}\left|\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u_\varepsilon dx\right|^2 dt \leq \frac{\lambda}{1-\delta}\|\nabla\varphi\|_\infty^2(\mathcal{F}(u(t))-\mathcal{F}(u(t')))
$$

for $t\in[0,T)$ and $0\leq t\leq t'<T$. Letting $\delta\downarrow 0$, (3.14) reduces to (3.9).

Next, we integrate $(A')^2$ twice to obtain

$$
\int_t^{t'}d\xi\int_t^\xi A'(s)^2 ds=\int_t^{t'}(t'-s)A'(s)^2 dx \tag{3.16}
$$

for $t\in[0,T)$ and $0\leq t\leq t'<T$. Moreover,

$$
\int_t^\xi A'(s)^2 ds \leq \lambda\|\nabla\varphi\|_\infty^2\mathcal{F}(u(t))-\frac{\lambda\|\nabla\varphi\|_\infty^2}{2(n-2)}H'(\xi) \tag{3.17}
$$

for almost every $\xi\in[t,t']$ by (3.9) and (1.13). Combining (3.16) with (3.17), we get the inequality (3.10). □

In the other case

$$
\lim_{t\uparrow T}\mathcal{F}(u(t))=-\infty, \tag{3.18}
$$

we have $\mathcal{F}(u(t_0))<0$ for some $t_0\in[0,T)$. We may assume $t_0=0$ without loss of generality. Inequality (1.13) then implies

$$
\frac{dH}{dt}<0, \tag{3.19}
$$

and hence there is $H(T)=\lim_{t\uparrow T}H(t)\geq 0$, and, in particular,

$$\int_0^T \mathcal{F}(u(t))dt=\frac{1}{2(n-2)}(H(T)-H(0))>-\infty.$$

If $H(T)=0$ is the case, then

$$\lim_{t\uparrow T}\int_{|x|>\varepsilon}u(x,t)dx=0$$

for any $\varepsilon>0$, which implies $\mathcal{S}\subset\{0\}$ by Theorem 4. Thus we see that the problem is the case (3.18) and $H(T)>0$.

Lemma 3.14. *Assume (3.18) and*

$$\mathcal{F}(u_0)<0. \tag{3.20}$$

Then we have

$$\sup_{t'\in[t,\frac{T+t}{2}]}A(t')\leq A(t)+R^{-1}C_8(n,\lambda)(H(t)-H(T))^{1/2} \tag{3.21}$$

with

$$C_8(n,\lambda)=C_3(n)\cdot\sqrt{\frac{\lambda\log 2}{4(n-2)}}. \tag{3.22}$$

Proof. It holds that

$$\begin{aligned}
\left|A(\frac{t+t'}{2})-A(t)\right|^2 &= \left|\int_t^{\frac{t+t'}{2}}A'(s)ds\right|^2\\
&\leq \frac{1}{2}\int_t^{\frac{t+t'}{2}}(t'-s)^{-1}ds\int_t^{t'}(t'-s)A'(s)^2ds\\
&\leq \frac{\log 2}{2}\cdot\frac{\lambda\|\nabla\varphi\|_\infty^2}{2(n-2)}\cdot(H(t)-H(T))\\
&\leq \left(\frac{C_3}{R}\right)^2\frac{\lambda\log 2}{4(n-2)}\cdot(H(t)-H(T))
\end{aligned}$$

by (3.10), (3.3), and the fact that $\mathcal{F}(u(t))\leq 0$ derived from the assumption and (1.12). Varying $t'\in[t,T)$, we obtain (3.21). □

In the following proof, the scaling property of (1.1) arising from $m=2-\frac{2}{n}$ takes an important role. In fact, if $u=u(x,t)$ is a solution to (1.1), then $u_\mu(x,t)=\mu^n u(\mu x,\mu^n t)$ satisfies

$$u_{\mu t}=\frac{m-1}{m}\Delta u_\mu^m-\nabla\cdot(u_\mu\nabla\Gamma * u_\mu),\ u_\mu\geq 0 \qquad \text{in } \mathbf{R}^n\times(0,T_\mu)$$

$$\int_{\mathbf{R}^n} u_\mu dx = \int_{\mathbf{R}^n} u dx = \lambda \qquad \text{for } t \in [0, T_\mu) \tag{3.23}$$

where $\mu > 0$ is a constant and $T_\mu = \mu^{-n}T$. Moreover, a similar fact holds for the approximate solution $u_\varepsilon = u_\varepsilon(x,t)$ to (1.5); that is, $u_{\varepsilon,\mu}(x,t) = \mu^n u_\varepsilon(\mu x, \mu^n t)$ satisfies

$$(u_{\varepsilon,\mu})_t = \frac{m-1}{m}\Delta(u_{\varepsilon,\mu} + \mu^n\varepsilon)^m - \nabla\cdot(u_{\varepsilon,\mu}\nabla\Gamma * u_{\varepsilon,\mu}),\ u_{\varepsilon,\mu} \geq 0 \qquad \text{in } \mathbf{R}^n \times (0, T_\mu)$$

$$\int_{\mathbf{R}^n} u_{\varepsilon,\mu} dx = \int_{\mathbf{R}^n} u_\varepsilon dx \leq \lambda \qquad \text{for } t \in [0, T_\mu), \tag{3.24}$$

recall (1.11).

Lemma 3.15. *Suppose* (3.18) *and* (3.20). *Then, each* $r_0 > 0$ *admits* $t_0 = t_0(n, T, \lambda, r_0) \in [0, T)$ *and* $C_9 = C_9(n, \lambda) > 0$ *such that, if*

$$\int_{B(x_0, 2r_0)} u(x, t_1) < \varepsilon_0^*/2 \tag{3.25}$$

holds for $x_0 \in \mathbf{R}^n$ *and* $t_1 \in [t_0, T)$, *then*

$$\sup_{B(x_0,(T-t_1)^{1/n})\times[t_1+\frac{1}{8}(T-t_1), t_1+\frac{1}{2}(T-t_1)]} (T-t)u(x,t) \leq C_9, \tag{3.26}$$

where ε_0^* *is as in Lemma* 2.4.

Proof. Fix $r_0 > 0$ and $x_0 \in \mathbf{R}^n$, and set

$$A(t) = \int_{\mathbf{R}^n} \varphi_{x_0, r_0}(x) u(x,t) dx$$
$$A_\varepsilon(t) = \int_{\mathbf{R}^n} \varphi_{x_0, r_0}(x) u_\varepsilon(x,t) dx$$

for $0 < \varepsilon \ll 1$. We take $t_0' = t_0'(n, T, \lambda, r_0) \in [0, T)$ satisfying

$$r_0^{-1} C_8(n, \lambda)(H(t_0') - H(T)) < \varepsilon_0^*/4, \tag{3.27}$$

and find that, if

$$A(t_1) < \varepsilon_0^*/2, \tag{3.28}$$

then

$$\sup_{t' \in [t_1, \frac{T+t_1}{2}]} A(t') < 3\varepsilon_0^*/4 \tag{3.29}$$

for all $t_1 \in [t_0', T)$ by Lemma 3.14 and (3.19) derived from the assumption (3.20). In addition, we take $t_0'' = t_0''(n, T, r_0) \in [0, T)$ satisfying

$$t_0'' \geq T - \left(\frac{r_0}{6}\right)^n, \tag{3.30}$$

and set

$$t_0 = \max\{t_0', t_0''\}. \tag{3.31}$$

Fix $t_1 \in [t_0, T)$ and suppose (3.28). We take the approximate solutions $\{u_{\varepsilon_j}\}$ such that u_{ε_j} converges to the weak solution u as $\varepsilon_j \downarrow 0$. We have

$$u_{\varepsilon_j} \to u \quad \text{in } C([0, T']; L^1_{loc}(\mathbf{R}^n))$$

for any $T' \in (0, T)$, see [15]. Therefore, we can get a subsequence, still denoted by ε_j, such that

$$\sup_{t' \in [t_1, \frac{T+t_1}{2}]} A_{\varepsilon_j}(t') < \varepsilon_0^* \tag{3.32}$$

for all j by (3.29).

Here we use the scaling property (3.23) and (3.24), and put

$$\tilde{u}(x, t) = \mu^n u(\mu x + x_0, \mu^n t + t_1)$$
$$\tilde{u}_{\varepsilon_j}(x, t) = \mu^n u_{\varepsilon_j}(\mu x + x_0, \mu^n t + t_1),$$

where

$$\mu^n + t_1 = \frac{T + t_1}{2}, \qquad \text{i.e., } \mu^n = \frac{T - t_1}{2}. \tag{3.33}$$

It holds that

$$\tilde{u}_t = \frac{m-1}{m} \Delta \tilde{u}^m - \nabla \cdot (\tilde{u} \nabla \Gamma * \tilde{u}), \ \tilde{u} \geq 0 \qquad \text{in } \mathbf{R}^n \times (0, 1), \tag{3.34}$$

$$\tilde{u}_{\varepsilon_j t} = \frac{m-1}{m} \Delta (\tilde{u}_{\varepsilon_j} + \mu^n \varepsilon_j)^m - \nabla \cdot (\tilde{u}_{\varepsilon_j} \nabla \Gamma * \tilde{u}_{\varepsilon_j}), \ \tilde{u}_{\varepsilon_j} \geq 0 \quad \text{in } \mathbf{R}^n \times (0, 1). \tag{3.35}$$

Noting (3.30)-(3.31) and $t_1 \in [t_0, T) \subset [t_0'', T)$, we have

$$\sup_{t \in (0,1)} \|\tilde{u}_{\varepsilon_j}(t)\|_{L^1(B(0, 6 \cdot 2^{1/n}))} < \varepsilon_0^*$$

for all j by (3.32), and therefore

$$\sup_{t \in [1/4, 1)} \|\tilde{u}_{\varepsilon_j}(t)\|_{L^\infty(B(0, 2^{1/n}))} \leq C_{10}(n, \lambda) \tag{3.36}$$

for all j by virtue of Lemma 2.4. We have

$$u_{\varepsilon_j} \to u \qquad \text{locally uniformly in } \mathbf{R}^n \times (0, T)$$

as $j \to +\infty$, see [18], and so (3.36) establishes that

$$\sup_{t\in[1/4,1)} \|\tilde{u}(t)\|_{L^\infty(B(0,2^{1/n}))} \le \liminf_j \sup_{t\in[1/4,1)} \|\tilde{u}_{\varepsilon_j}(t)\|_{L^\infty(B(0,2^{1/n}))} \le C_{10},$$

which implies

$$\sup_{B(x_0,(T-t_1)^{1/n})\times[t_1+\frac{1}{8}(T-t_1),t_1+\frac{1}{2}(T-t_1))} (T-t_1)u(x,t) \le C_{10}.$$

Since $x_0 \in \mathbf{R}^n$ and $t_1 \in [t_0, T)$ are arbitrary, we obtain a desired constant $C_9 = \frac{7}{8}C_{10}$ such that (3.25) (or (3.28)) implies (3.26). □

Proof of Theorem 1. The finiteness of $\mathcal{S}_{II}$ follows from

$$\inf_{x_0\in\mathcal{S}_{II}} \lim_{r\downarrow 0} \liminf_{t\uparrow T} \int_{B(x_0,r)} u(x,t)dx \ge \varepsilon_0^*/2$$

because of the total mass conservation (1.10). Assuming the contrary, we have $x_0 \in \mathcal{S}_{II}$, $r_0 > 0$, and $t_j \uparrow T$ such that

$$\int_{B(x_0,3r_0)} u(x,t_j)dx < \varepsilon_0^*/2$$

for $j = 1, 2, \cdots$. Then we obtain

$$\sup_{y\in B(x_0,r_0)} \int_{B(y,2r_0)} u(x,t_j)dx < \varepsilon_0^*/2$$

for j sufficiently large, and, therefore,

$$\sup_{B(y,(T-t_j)^{1/n})\times[t_j+\frac{1}{8}(T-t_j),\frac{1}{2}(T-t_j))} (T-t)u(x,t) \le C_9$$

by Lemma 3.15, where $y \in B(x_0, r_0)$ is arbitrary. Then, it follows that

$$\sup_{B(x_0,r_0)\times[t_j+\frac{1}{8}(T-t_j),\frac{1}{2}(T-t_j))} (T-t)u(x,t) \le C_9$$

and hence $\liminf_{t\uparrow T}(T-t)\|u(t)\|_{L^\infty(B(x_0,r_0))} < +\infty$, a contradiction. □

4. Proof of Theorem 2

Proof of Theorem 2: For every $\xi > 0$, we have only to show

$$\inf_{x_0\in\mathcal{S}_{*,\xi}} \lim_{r\downarrow 0} \liminf_{t\uparrow T} \int_{B(x_0,r)} u(x,t)dx > 0 \tag{4.1}$$

by virtue of the total mass conservation (1.10). Suppose that there is $\xi_0 > 0$ such that (4.1) is false. Then there are $x_k \in \mathcal{S}_{*,\xi_0}$, $r_k > 0$ and $t_{jk} \uparrow T$ for $j, k = 1, 2, \cdots$ such that $0 < T - t_{jk} < \frac{1}{jk}$ and

$$\int_{B(x_k,4r_k)} u(x, t_{jk})dx < \min\left\{\frac{\varepsilon_0^*}{2}, \frac{1}{2k}\right\}, \tag{4.2}$$

where ε_0^* is as in Lemma 2.4. Since $x_k \in \mathcal{S}_{*,\xi_0}$, we have $y_k(t) \to x_k$, $b_k > 0$ and $L_k > 0$ satisfying

$$|y_k(t) - y_k(s)| \le L_k|t - s|^{1/n} \qquad \text{for } 0 < T - t, T - s \ll 1 \tag{4.3}$$

$$\liminf_{t\uparrow T}(T - t)\|u(t)\|_{L^\infty(B(y_k(t),b_k(T-t)^{1/n}))} \ge \xi_0. \tag{4.4}$$

We may assume $b_k \gg 1$, if necessary, by the definition of $\mathcal{S}_{*,\xi}$. Since $y_k(t_{jk}) \to x_k$ as $j \to \infty$, for each k there is $j(k) \gg 1$ satisfying

$$B(y_k(t_{jk}), 3r_k) \subset B(x_k, 4r_k) \tag{4.5}$$

for $j \ge j(k)$. It follows from (4.2) and (4.5) that

$$\sup_{y\in B(y_k(t_{jk}),r_k)} \int_{B(y,2r_k)} u(x, t_{jk})dx < \min\left\{\frac{\varepsilon_0^*}{2}, \frac{1}{2k}\right\} \tag{4.6}$$

for $j \ge j(k)$. We introduce the functions

$$A_{y,r_k}(t) = \int_{\mathbf{R}^n} \varphi_{y,r_k}(x)u(x, t)dx$$
$$A_{y,r_k,\varepsilon}(t) = \int_{\mathbf{R}^n} \varphi_{y,r_k}(x)u_\varepsilon(x, t)dx$$

for each $y \in \mathbf{R}^n$, $k = 1, 2, \cdots$, and $0 < \varepsilon \ll 1$, where $\varphi = \varphi_{x_0,R}(x)$ is a smooth function provided with (3.3). Let $u_{\varepsilon_l} = u_{\varepsilon_l}(x, t)$ with $\varepsilon_l \downarrow 0$ (as $l \to \infty$) be the approximate solution converging to the weak solution $u = u(x, t)$. Since $t_{jk} \uparrow T$ as $j \to \infty$, we can show, similarly to the proof of Lemma 3.15 by using (4.6) and Lemma 3.14, that for each k and $j \ge j(k)$ there exists a subsequence $\{\varepsilon_l^{(j,k)}\} \subset \{\varepsilon_l\}$ such that

$$\sup_{t'\in[t_{jk},\frac{T+t_{jk}}{2}]} A_{y,r_k,\varepsilon_l^{(j,k)}}(t') < \min\left\{\varepsilon_0^*, \frac{1}{k}\right\} \tag{4.7}$$

for all $l = 1, 2, \cdots$ and $y \in B(y_k(t_{jk}), r_k)$. Similarly to the proof of Lemma 3.15, for each k and $j \ge j(k)$, with $j(k)$ larger if necessary,

$$\sup_{t\in[1/4,1)} \|\tilde{u}_{\varepsilon_l^{(j,k)},y}(t)\|_{L^\infty(0,2^{1/n})} \le C_1(n, \lambda) \tag{4.8}$$

for all $l = 1, 2, \cdots$ and $y \in B(y_k(t_{jk}), r_k)$, where

$$\tilde{u}_{\varepsilon_l^{(j,k)},y}(x,t) = \mu_{jk}^n u_{\varepsilon_l^{(j,k)}}(\mu_{jk}x + y, \mu_{jk}^n t + t_{jk}), \quad \mu_{jk}^n = \frac{T - t_{jk}}{2}. \tag{4.9}$$

Relations (4.8)-(4.9) imply

$$\sup_{t\in[1/4,1)} \|\tilde{u}_{\varepsilon_l^{(j,k)}}(t)\|_{L^\infty(B(0,\mu_{jk}^{-1}r_k))} \le C_2(n,\lambda) \tag{4.10}$$

for all k, $j \ge j(k)$, and $l = 1, 2, \cdots$, where

$$\tilde{u}_{\varepsilon_l^{(j,k)}}(x,t) = \mu_{jk}^n u_{\varepsilon_l^{(j,k)}}(\mu_{jk}x + y_k(t_{jk}), \mu_{jk}^n t + t_{jk}), \quad \mu_{jk}^n = \frac{T - t_{jk}}{2}.$$

It follows from (4.7) that

$$\sup_{t\in[0,1]} \|\tilde{u}_{\varepsilon_l^{(j,k)}}(t)\|_{L^1(B(0,2\mu_{jk}^{-1}r_k))} \le C_3(n) \min\{\varepsilon_0^*, \frac{1}{4}\}. \tag{4.11}$$

for all k, $j \ge j(k)$, and $l = 1, 2, \cdots$.

Noting that $\mu_{jk}^{-1} \to +\infty$ as $j \to \infty$, inequalities (4.10)-(4.11) and Lemma 2.9 ensure that, for any $R \gg 1$, there are $k_0 = k_0(n,\lambda)$ and $j(k,R)$ such that

$$\sup_{t\in[3/8,1)} \|\tilde{u}_{\varepsilon_l^{(j,k)}}(t)\|_{L^\infty(B(0,2^{1/n}R))} \le \xi_0/2$$

for all $k \ge k_0$, $j \ge j(k,R)$ and $l = 1, 2, \cdots$, which implies

$$\sup_{(x,t)\in B(y_k(t_{jk}),R(T-t_{jk})^{1/n})\times[t_{jk}+\frac{3}{16}(T-t_{jk}),t_{jk}+\frac{1}{2}(T-t_{jk}))} (T-t)u_{\varepsilon_l^{(j,k)}}(x,t) \le \xi_0/2. \tag{4.12}$$

Inequality (4.12) for $R = L_k + b_k$ yields

$$\begin{aligned}&\liminf_{t\uparrow T}(T-t)\|u(t)\|_{L^\infty(B(y_k(t),b_k(T-t)^{1/n}))}\\ &\le \liminf_{l\to\infty}\liminf_{t\uparrow T}(T-t)\|u_{\varepsilon_l^{(j,k)}}(t)\|_{L^\infty(B(y_k(t),b_k(T-t)^{1/n}))} \le \xi_0/2\end{aligned}$$

for $k \gg 1$ by letting $j \to \infty$. This relation contradicts (4.4). □

5. Proof of Theorem 3

Proof of Theorem 3. If $\sharp\mathcal{S} = +\infty$ is the case, then $\sharp(\mathcal{S} \setminus \mathcal{S}_{*,\xi}) = +\infty$ for each $\xi > 0$ by Theorem 2. Suppose that $x(t) \in \mathbf{R}^n$ attains a positive local maximum of $u(\cdot,t)$ and satisfies (1.17). We rewrite the equation of (1.1) as

$$u_t = \frac{m-1}{m}\Delta u^m - \nabla u \cdot \nabla\Gamma * u + u^2.$$

Since $x(t)$ attains a positive local maximum of $u(\cdot,t)$, $u=u(x,t)$ is smooth near $(x(t),t)$ in x-t space by the continuity of the weak solution and the parabolic-elliptic regularity. Thus

$$\dot{U} \leq U^2 \tag{5.1}$$

for almost every t, see [1], where $U(t)=u(x(t),t)$. Inequality (5.1) reads

$$-\frac{d}{dt}(U^{-1}) \leq 1.$$

We have $\liminf_{t\uparrow T} U^{-1}(t)=0$ by (1.17), and hence

$$U(t)=u(x(t),t) \geq (T-t)^{-1}.$$

In particular, we have

$$\liminf_{t\uparrow T}(T-t)\|u(t)\|_{L^\infty(B(x(t),b(T-t)^{1/n}))} \geq 1$$

for any $b>0$. Therefore $x_0 \in \mathcal{S}_{*,1}$ if $x(t) \to x_0$ does not satisfy (1.16). This implies the conclusion since $\sharp(\mathcal{S} \setminus \mathcal{S}_{*,1}) = +\infty$. □

6. Concluding Remarks

Theorem 3 evokes the study of several blowup patterns. A natural question in this direction is the existence of a radially symmetric shock wave concentrating toward a blowup point. A formal dimension analysis of [3] applies to formulate such a solution. Thus, in (1.1) with general $1<m<2$, we assume a radially symmetric bulk moving to the origin of which distance from the origin, the height, and the thickness are $R(t)$, $h(t)$, and $\mu(t)$, respectively, provided with the property $0<\mu(t) \ll R(t)=o(1)$. At this bulk we have $\frac{\partial}{\partial r} \sim \frac{1}{\mu}$, $r \sim R$, and $u \sim h$ so that $\left|\frac{u_r}{r}\right| \sim \frac{h}{\mu R} \ll |u_{rr}| \sim \frac{h}{\mu^2}$. We have $\frac{1}{r}|v_r| \ll |v_{rr}|$ similarly. Then (1.1) is reduced to

$$u_t = u^m_{rr} - (uv_r)_r, \qquad -v_{rr}=u, \tag{6.1}$$

which implies

$$-v_{rrt} = u^m_{rr} + \tfrac{1}{2}(v_r^2)_{rr},$$

and hence

$$(v_r)_t + [u^m + \tfrac{1}{2}v_r^2]_r = 0. \tag{6.2}$$

Since $r=R(t)$ is regarded as a wavefront of u, the propagation speed of this wave is formulated by $c=\dot{R}(t)$. Then the Rankine-Hugoniot condition to equation (6.2) describing a conservation law reads

$$c[v_r]_{R(t)} = [u^m + \frac{1}{2}v_r^2]_{R(t)} = [\frac{1}{2}v_r^2]_{R(t)},$$

where

$$[\zeta]_{R(t)} = \lim_{r\downarrow R(t)} \zeta(r) - \lim_{r\uparrow R(t)} \zeta(r) = \zeta(R(t)^+) - \zeta(R(t)^-),$$

see [7]. Using the second equation of (6.1) assumed for $u = \frac{M}{\omega_{n-1}R^{n-1}}\chi_{r=R}$ with $M = \|u(t)\|_1$, we can derive

$$v_r(R(t)^+, t) = -\frac{M}{\omega_{n-1}R(t)^{n-1}}, \qquad v_r(R(t)^-, t) = 0,$$

from classical potential theory. Therefore, it follows that

$$\dot{R}(t) = -\frac{M}{2\omega_{n-1}R(t)^{n-1}}$$

and hence $R(t) \sim (T-t)^{1/n}$, $t \uparrow T$.

Next, we plug in $u \sim h$ and $r \sim \mu$ to (6.1), which should be valid at the bulk. Then we obtain $v \sim \mu^2 h$ from the second equation, and hence $(uv_r)_r \sim h^2$. Now the first equation ensures

$$\frac{h^m}{\mu^2} \sim h^2,$$

and hence $h \sim \mu^{-n}$ if $m = 2 - \frac{2}{n}$. We have, on the other hand,

$$\omega_{n-1}R^{n-1}\mu h \sim M,$$

and therefore, it follows that

$$\mu \sim (T-t)^{\frac{n-1}{n}\cdot\frac{2-m}{m}}, \quad h \sim (T-t)^{-\frac{n-1}{n}\cdot\frac{2}{m}}, \qquad t \uparrow T.$$

This condition is compatible to the ansatz $0 < \mu \ll R$ if and only if $1 < m < 2 - \frac{2}{n}$, that is, the super-critical case. In the critical case $m = 2 - \frac{2}{n}$, the above formal argument is valid when we replace $\frac{u_r}{r}$ by u_{rr} since $\mu \sim R$.

In the critical case, this blowup rate is type I and $x_0 \in \mathcal{S}_{*,\xi} \setminus \mathcal{S}_{II}$ for some $\xi > 0$. In this connection, we recall that the non-existence of the non-trivial (backward) self-similar solution with a finite mass to (1.1) is proven similarly to [5]. A type II blowup point, on the other hand, will be realized using the stationary state provided with the quantized mass. Such a blowup pattern is also examined in the higher-dimensional Smoluchowski-Poisson equation, see [4]. Similarly, we expect that these type I and type II blowup patterns are stable and unstable, respectively. However, it seems to be difficult to realize an infinite number of blowup points to (1.1) by using a combination of essentially radially symmetric blowup profiles.

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