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A VARIATIONAL APPROACH FOR OPTIMAL CONTROL OF THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we deal with optimal boundary control for the Navier-Stokes problem. We establish the existence of such a control in appropriate functional spaces. Then we study a stabilization problem around a steady state. In view of numerical approximation, we derive rigorously Euler equations satisfied by the control.

1. Introduction

1.1. **Setting of the problem.** The aim of this paper is to provide some new directions for solving optimal control problems for the Navier-Stokes equations. In optimal control theory, one is interested in minimizing a cost functional involving the kinetic energy and a suitable norm on the control. This kind of problem arises in fluid mechanics when one wants, for example, to increase the performance of an airplane by modifying the flow near the wall through nonhomogeneous boundary conditions (see [3], [5]). A huge literature is dedicated to the theoretical side of the subject (see [1], [2], [6], [7], [8] and [9]). To our knowledge, only heuristic numerical methods are available. Most of them are built on linearization techniques around a steady state (see [10], [11]).

In this paper, we deal with the case where the control lies on only a part of the boundary of the domain. Concerning bounded domains, as we consider an incompressible flow, we have to assume that the mean of the normal component of the control vanishes on the boundary.

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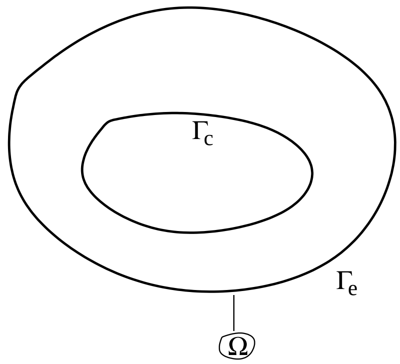

FIGURE 1. Example of domain Ω.

The situation studied here is the following one. Let Ω be a smooth bounded domain of $\mathbb{R}^2$ and $\partial\Omega$ its boundary. We assume that $\partial\Omega = \Gamma_e \cup \Gamma_c$ with $\Gamma_e \cap \Gamma_c = \emptyset$. Here Γ_e and Γ_c denote respectively the external and interior part of the boundary and, for simplicity, we define $\Gamma = \Gamma_e \cup \Gamma_c$.

The control is defined on Γ_c. On Γ_e, we impose a nonhomogeneous Dirichlet condition.

1.2. **Formulation and existence.** Suppose we are given a finite horizon time T over which the control is to be optimized, a smooth function g modeling the inflow boundary condition defined on the external boundary Γ_e and a control u_ρ acting on the interior boundary Γ_c. We introduce the quadratic functional

$$J(u_\rho) = ||u(\tau)||_Y^2 + \alpha||u_\rho(\tau)||_X^2,$$

where X and Y are two Hilbert spaces of functions $f(t,x)$ defined for $(t,x) \in [0,T] \times \Omega$ (see Section 2) and α is a positive parameter. For mathematical analysis, we may assume that $\alpha = 1$ without loss of generality.

The relationship between u and the boundary control u_ρ is given by the initial-boundary-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + (u(t,x)\cdot\nabla)u(t,x) - \dfrac{1}{\mathcal{R}_e}\Delta u(t,x) + \nabla p(t,x) = f(t,x), \\ \operatorname{div}(u)(t,x) = 0, \\ u_{|\Gamma_c}(t,x) = u_\rho(t,x), \quad u_{|\Gamma_e}(t,x) = g(x), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

for the Navier-Stokes equation posed for $(t,x) \in [0,T]\times\Omega$, where u_0 is a given initial data, f is the volume unit force and $\mathcal{R}_e$ is the Reynolds number. The function u_ρ depends on the time t and x whereas, for the sake of simplicity, g is taken to depend only on x. For the existence of such solutions see Theorem 1.2.

The first result is concerned with the existence of an optimal control u_ρ^{opt} satisfying

$$J(u_\rho^{opt}) = \inf_{u_\rho \in X} J(u_\rho). \tag{1.2}$$

The Hilbert space X is the set of admissible controls. The functional J is not a convex function. In particular, we are not able to give a uniqueness result for the minimizer. Nevertheless, we are able to derive first-order conditions for an optimal control through the derivation of Euler equations. This involves us in the study of the Fréchet derivative of the mapping $u_\rho \longmapsto u$ given by Equation (1.1).

The lack of uniqueness could present problems for the construction of good approximations of an optimal control. We provide an efficient, constructive method of approaching u_ρ^{opt} by introducing an associated linearized problem about which we derive some effective results.

The first theorem of this paper is the following one.

Theorem 1.1. *Assume that Ω is a bounded domain in $\mathbb{R}^2$ with C^1 boundary. For any function (u_0, g) in $H \times H^{\frac{1}{2}}(\Gamma_e)$, where H is defined in* (1.7) *and g satisfies the compatibility condition* (1.6)*, there exists at least one optimal control u_ρ^{opt} solution to $J(u_\rho^{opt}) = \inf_{u_\rho\in X} J(u_\rho)$, where X will be described in Section* 2 (*see Definition* 2.3).

1.3. **Existence and properties of the optimal control.** In this section, we explain briefly our method to deal with the boundary control problem and present some other results. First of all, to deal with Dirichlet boundary conditions on Γ_c, let us denote by $\mathcal{U}_\rho$ the solution to the nonstationary Stokes problem

$$\begin{cases} \dfrac{\partial \mathcal{U}_\rho}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta \mathcal{U}_\rho + \nabla p = 0, \\ \operatorname{div}(\mathcal{U}_\rho) = 0, \\ \mathcal{U}_{\rho|\Gamma_c} = u_\rho,\ \ \mathcal{U}_{\rho|\Gamma_e} = 0,\ \ \displaystyle\int_{\Gamma_c} \mathcal{U}_\rho \cdot n\, ds = 0, \\ \mathcal{U}_\rho(0) = 0. \end{cases} \tag{1.3}$$

The function $\mathcal{U}_\rho$ belongs to the Banach space

$$\begin{aligned}\mathcal{W}^L =\Big\{ & v \in L^2(0,T;H^1(\Omega)) : \frac{\partial v}{\partial t} - \frac{1}{\mathcal{R}_e}\Delta v + \nabla p = 0, \\ & \mathrm{div}(v) = 0,\ v(0,\cdot) = 0,\ \ v_{\big|\Gamma e} = 0\Big\},\end{aligned} \tag{1.4}$$

where $\frac{\partial v}{\partial t}$ and Δv are calculated in the sense of distributions in $(0,T)\times\Omega$. Define $v = u - \mathcal{U}_\rho$. Then v is a solution to the Navier-Stokes-type equations

$$\left\{\begin{array}{l} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v + (\mathcal{U}_\rho\cdot\nabla)v + (v\cdot\nabla)\mathcal{U}_\rho - \dfrac{1}{\mathcal{R}_e}\Delta v + \nabla p \\ \qquad = f - (\mathcal{U}_\rho\cdot\nabla)\mathcal{U}_\rho, \\ \mathrm{div}(v) = 0,\ \ v_{|\Gamma_c} = 0,\ \ v_{|\Gamma_e} = g,\ \ v(0) = u_0. \end{array}\right. \tag{1.5}$$

Note that, in Equation (1.5), $\mathcal{U}_\rho$ is a source term and a coefficient since it appears, for example, in the term $\mathcal{U}_\rho\cdot\nabla v$. We recall how to construct a solution of (1.5). Assume that g satisfies the compatibility conditions

$$\int_{\Gamma_e} g\cdot n\, d\sigma = 0, \qquad (u_0\cdot n)_{\big|\Gamma e} = g\cdot n, \tag{1.6}$$

and introduce the Banach spaces

$$H = \{v \in L^2(\Omega) : \mathrm{div}(v) = 0,\ (v\cdot n)_{|\Gamma_c} = 0\}, \tag{1.7}$$

$$V = \{v \in H^1_0(\Omega) : \mathrm{div}(v) = 0\}. \tag{1.8}$$

Theorem 2.4 (page 31) of [18] implies that one can construct a function $V \in H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\left\{\begin{array}{l} -\dfrac{1}{\mathcal{R}_e}\Delta V + \nabla p = 0 \text{ in } \Omega, \\ \mathrm{div}(V) = 0,\ V_{|\Gamma_e} = g,\ V_{|\Gamma_c} = 0. \end{array}\right. \tag{1.9}$$

Then $w = v - V$ is a solution to the equations

$$\left\{\begin{array}{l} \dfrac{\partial w}{\partial t} + \big((w+V)\cdot\nabla\big)(w+V) + (\mathcal{U}_\rho\cdot\nabla)(w+V) + \big((w+V)\cdot\nabla\big)\mathcal{U}_\rho \\ -\frac{1}{\mathcal{R}_e}\Delta w + \nabla p = f - (\mathcal{U}_\rho\cdot\nabla)\mathcal{U}_\rho, \\ \mathrm{div}(w) = 0,\ \ w_{|\Gamma_c} = 0,\ \ w_{|\Gamma_e} = 0,\ \ w(0) = u_0 - V. \end{array}\right. \tag{1.10}$$

Applying Theorem 3.1 and Theorem 3.2 of [18], the following result emerges.

Theorem 1.2. *Assume that $g \in H^{\frac{1}{2}}(\Gamma)$ satisfies (1.6), $f \in L^2(0,T;V')$ and $u_0 \in H$. Then there exists a unique solution $v \in L^\infty(0,T;H)$ to (1.5). Moreover, v is weakly continuous from $[0,T]$ into H.*

Concerning the regularity of the mapping $\mathcal{U}_\rho \longmapsto v$, we prove the following theorem.

Theorem 1.3. *Let* $\frac{1}{2} < \sigma < 1$. *The map* $\mathcal{U}_\rho \in \mathcal{W}^L \cap H^\sigma(0,T;L^2) \longmapsto v$ *where* v *is the solution to* (2.6) *is Fréchet differentiable. Furthermore, for all* $h \in \mathcal{W}^L \cap H^\sigma(0,T;L^2)$, $z = v'(\mathcal{U}_\rho)\cdot h$ *is the solution to the initial-value-boundary problem*

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta z + v\cdot\nabla z + z\cdot\nabla v + \mathcal{U}_\rho\cdot\nabla z + z\cdot\nabla\mathcal{U}_\rho \\ +h\cdot\nabla v + v\cdot\nabla h + \mathcal{U}_\rho\cdot\nabla h + h\cdot\nabla\mathcal{U}_\rho + \nabla p = 0, \\ \mathrm{div}(z) = 0, \quad z_{|\Gamma} = 0, \quad z(0) = 0. \end{array}\right.$$

In addition, we also establish the weak form of the first-order optimality condition for $\mathcal{U}_\rho^{opt}$.

Corollary 1.1. *Under the assumptions of Theorem 1.1, the extension* $\mathcal{U}_\rho^{opt}$ *defined by* (1.3) *of* u_ρ^{opt} *satisfies*

$$\forall h \in \mathcal{W}_\sigma^{NL},\ (z,\mathcal{U}_\rho^{opt})_X + ((\mathcal{U}_\rho^{opt},h))_Y = 0,$$

where z *solves the equation*

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta z + v\cdot\nabla z + z\cdot\nabla v + \mathcal{U}_\rho^{opt}\cdot\nabla z + z\cdot\nabla\mathcal{U}_\rho^{opt} \\ +h\cdot\nabla v + v\cdot\nabla h + \mathcal{U}_\rho^{opt}\cdot\nabla h + h\cdot\nabla\mathcal{U}_\rho^{opt} + \nabla\pi = 0, \\ \mathrm{div}(z) = 0, \quad z_{|\Gamma} = 0, \quad z(0) = 0, \end{array}\right.$$

and $(\cdot,\cdot)_X$ *(respectively* $((\cdot,\cdot))_Y$*) denotes the scalar product of* X *(respectively* Y*), defined in Section 2 .*

Concerning the linear version of the optimal boundary control problem, the following result is established.

Theorem 1.4. *Assume that* Ω *is a bounded domain in* $\mathbb{R}^2$ *with* C^1 *boundary. For any function* (u_0,g) *in* $H\times H^{\frac{1}{2}}(\Gamma_e)$ *satisfying the compatibility condition* (2.7)*, there exists a unique control* u_ρ^{opt} *which solves*

$$J_L(u_\rho^{opt}) = \inf_{u_\rho\in\mathcal{W}_c^L} J_L(u_\rho),$$

where $\mathcal{W}_c^L$ *is defined in Section* 2 *and* J_L *is the suitable functional fitted to the linearized version of the optimal control problem.*

Finally, we present the Euler-Lagrange equation for the optimal control solution of the linearized version of the optimal control problem.

Theorem 1.5. *Let $\mathcal{U}_\rho^{opt}$ be the optimal control of Theorem* 1.4. *Then, for all $h \in \mathcal{W}^L$,*

$$((u + \mathcal{U}_\rho^{opt}, h))_Y = 0,$$

where u is the unique solution of $a(u,v) = (\ell, v)$ for all v in $\mathcal{W}^L$ with $\ell = \mathcal{L}\,\mathcal{U}_\rho^{opt} - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}$ (see Section 5 *for the definition of a, $\bar{v}$ and $\mathcal{L}$).*

Notation. In this paper, for $1 \le p < +\infty$, L^p is the usual Lebesgue space. For simplicity, the usual norm on L^p is denoted by $\|\cdot\|_p$. For $s \ge 0$, $H^s(\Omega)$ is the usual Sobolev space defined by interpolation (see [12]). Let $\mathcal{X}$ be a Banach space. We denote by $L^p((0,T);\mathcal{X})$ $(1 \le p < +\infty)$ the space of functions $m : (0,T) \longrightarrow \mathcal{X}$ such that m is measurable and

$$\|m\|_{L^p((0,T);\mathcal{X})} = \Big(\int_0^T \|m(t)\|_{\mathcal{X}}^p dt\Big)^{\frac{1}{p}} < +\infty.$$

Let $r \ge 0$ and $s \ge 0$. Let $\mathcal{Q} = (0,T)\times\Omega$ be a time-space domain and

$$H^{r,s}(\mathcal{Q}) = L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega))$$

endowed with the norm

$$\|u\|_{H^{r,s}(\mathcal{Q})} = (\|u\|^2_{L^2(0,T;H^r)} + \|u\|^2_{H^s(0,T;L^2)})^{\frac{1}{2}}.$$

In the same way, set

$$H^{r,s}((0,T)\times\Gamma_c) = L^2(0,T;H^r(\Gamma_c)) \cap H^s(0,T;L^2(\Gamma_c)).$$

We also define $L_0^2(\Omega) = L^2(\Omega)/\mathbb{R}$, and

$$W^{-1,\infty}(0,T;L_0^2(\Omega)) = \Big\{u \in \mathcal{D}'(\mathcal{Q}) : u = \sum_{|\alpha|\le 1} \partial^\alpha f_\alpha,\ f_\alpha \in L^\infty(0,T;L_0^2(\Omega))\Big\},$$

endowed with the norm

$$\|u\|_{W^{-1,\infty}(0,T;L_0^2(\Omega))} = \inf\Big(\sup_{|\alpha|\le 1} \|f_\alpha\|_{L^\infty(0,T;L_0^2(\Omega))}\Big).$$

Denote by $(\cdot,\cdot)$ (respectively $((\cdot,\cdot))$) the usual scalar product on $L^2(\Omega)$ (respectively $H^1(\Omega)$). Throughout this paper, C will denote a generic constant which may change from one line to another.

The paper is organized as follows. In Section 2, we introduce the spaces fitted to the boundary control problem. Section 3 is devoted to the proof of Theorem 1.1 concerning the existence of a solution to the optimal boundary control problem. In Section 4, we prove the Fréchet differentiability of $\mathcal{U}_\rho \longmapsto v$, where v is a solution to (2.6) (see Theorem 1.3). Section 5 deals with the linear version of the optimal control problem (see Theorem 1.4).

2. Functional Spaces for Boundary Control

In this section the functional analytic setting for the problem introduced in Section 1.2 is presented. As usual, when we consider a Dirichlet boundary problem, it is helpful to build an extension of the boundary condition. This is achieved using appropriate trace theorems.

Lemma 2.1. *The space $\mathcal{W}^L$ defined in* (1.4) *is a closed subset of* $L^2(0,T;H^1)$ *for the induced topology.*

Proof. We first note that every $v \in \mathcal{W}^L$ satisfies $\frac{\partial v}{\partial t} \in L^2(0,T;V')$. Indeed, for any ϕ in $L^2(0,T;V)$, it transpires that

$$\Big|\Big\langle \frac{\partial v}{\partial t}, \phi \Big\rangle\Big| \leq \frac{1}{\mathcal{R}_e}\|\phi\|_{L^2(0,T;V)}\|v\|_{L^2(0,T;H^1)}, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V and $V^{'}$. Hence, the function space $\mathcal{W}^L$ is continuously embedded in $\mathcal{C}^0(0,T;V')$. Notice that, since $\mathcal{W}^L \not\subset L^2(0,T,V)$, we do not have $\mathcal{W}^L \subset C([0,T;[V,V^{'}]_{\frac{1}{2}})$ where $[\cdot,\cdot]_{\frac{1}{2}}$ denotes the interpolation space of order $\frac{1}{2}$ between V and $V^{'}$.

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{W}^L$ which converges to v in $L^2(0,T;H^1)$ for the $L^2(0,T;H^1)$-topology. By continuity of divergence and trace operators, it is clear that

$$v_{\big|\Gamma e} = 0, \quad \mathrm{div}(v) = 0.$$

We now have to check that v is a solution of the Stokes equation

$$\frac{\partial v}{\partial t} - \frac{1}{\mathcal{R}_e}\Delta v - \nabla p = 0, \tag{2.2}$$

in the distributional sense. Let ϕ be in $\mathcal{C}^1(0,T;V)$, such that $\phi(T) = 0$. Then each v_n satisfies the weak formulation

$$-\int_0^T\int_\Omega v_n(\tau)\frac{\partial\phi(\tau)}{\partial t}d\tau + \frac{1}{\mathcal{R}_e}\int_0^T\int_\Omega(\nabla v_n : \nabla\phi)\,dxd\tau = 0, \tag{2.3}$$

where " : " denotes the contract product of two second-order tensors; i.e., if $\mathcal{T} = [\mathcal{T}_{ij}]$ and $\mathcal{S} = [\mathcal{S}_{ij}]$ are two second-order tensors, $\mathcal{T} : \mathcal{S} = [\mathcal{T}_{ij}\mathcal{S}_{ij}]$. Since $(v_n)_{n\in\mathbb{N}}$ converges to v in $L^2(0,T;H^1(\Omega))$, we can take the limit as n goes to $+\infty$ in equation (2.3) to obtain

$$\mathrm{div}\phi = 0,\ -\int_0^T\int_\Omega v(\tau)\frac{\partial\phi(\tau)}{\partial t}d\tau + \frac{1}{R_e}\int_0^T\int_\Omega(\nabla v : \nabla\phi)\,dxd\tau = 0,$$

for all $\phi \in \mathcal{C}^0(0,T;V)$. Then, by de Rham's theorem (see [4], [18] for more details), there exists a distribution $p \in W^{-1,\infty}(0,T;L^2_0(\Omega))$ such that

$$\frac{\partial v}{\partial t} - \frac{1}{\mathcal{R}_e}\Delta v + \nabla p = 0.$$

Moreover, by estimate (2.1), we may suppose by passing to a subsequence that $\left(\frac{\partial v_n}{\partial t}\right)_{n\in\mathbb{N}}$ converges weakly to $\frac{\partial v}{\partial t}$ in $L^2(0,T;V')$. It follows that $(v_n)_{n\in\mathbb{N}}$ converges to v in $\mathcal{C}^0(0,T;V')$, hence $v(0) = 0$. □

Notice that the elements of $\mathcal{W}^L$ have traces on Γ_c in $L^2(0,T;H^{\frac{1}{2}}(\Gamma_c))$, as the map

$$\begin{array}{cccc} \mathcal{R}_{\Gamma_c}: & L^2(0,T;H^1(\Omega)) & \longrightarrow & L^2(0,T;H^{\frac{1}{2}}(\Gamma_c)) \\ & v & \longrightarrow & \mathcal{R}_{\Gamma_c}(v) = v_{\big|\Gamma c} \end{array}$$

is continuous.

Lemma 2.2. *The mapping $\mathcal{R}_{\Gamma_c}$ from $\mathcal{W}^L$ to $L^2(0,T;H^{\frac{1}{2}}(\Gamma_c))$ defined by $\mathcal{R}_{\Gamma_c}(v) = v_{|\Gamma_c}$ is injective.*

Proof. Let v_1 and v_2 be two functions in $\mathcal{W}^L$ vanishing on Γ_e such that $\mathcal{R}_{\Gamma_c}(v_1) = \mathcal{R}_{\Gamma_c}(v_2)$ and let $v = v_2 - v_1$. As v belongs to $L^2(0,T;V)$ and satisfies a homogeneous Stokes problem with zero initial value, by uniqueness of the weak solution, it must be the case that $v = 0$. □

In the sequel, we denote by $\mathcal{W}^L_c$ the space of the restrictions to Γ_c of the functions in $\mathcal{W}^L$

$$\mathcal{W}^L_c = \mathcal{R}_{\Gamma_c}(\mathcal{W}^L) = \left\{ w_{\big|\Gamma_c} : w \in \mathcal{W}^L \right\}.$$

Lemma 2.3. *The space $\mathcal{W}^L_c$ endowed with the norm*

$$\|\gamma\|_{\mathcal{W}^L_c} = \|v\|_{L^2(0,T;H^1)},$$

where v is the unique function of $\mathcal{W}^L$ such that $\mathcal{R}_{\Gamma_c}(v) = \gamma$, is a Hilbert space.

Proof. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{W}^L_c$. For each $n \in \mathbb{N}$, define v_n such that $\gamma_n = \mathcal{R}_{\Gamma_c}(v_n)$. Then $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0,T;H^1)$ and the result follows from the continuity of $\mathcal{R}_{\Gamma_c}$.

Remark 2.1. The set $\mathcal{W}^L_c$ is nonempty. Indeed

$$\left\{ \gamma \in H^{\frac{1}{2},1}((0,T)\times\Gamma_c) : \int_{\Gamma_c} \gamma \cdot \nu d\sigma = 0 \right\} \subset \mathcal{W}^L_c$$

(see Theorem 5.2 in [14]).

We have to notice here that we do not know that the temporal derivatives of a function in $\mathcal{W}^L$ belongs to a space with values in $H^{-1}(\Omega)$. It means that, in some way, there is no control on time evolution for the functions belonging to $\mathcal{W}_c^L$. This fact makes us think that this space is not adapted for solving optimal boundary control problem for the Navier-Stokes equations. Anyway, as we will see in the sequel, it will be a suitable one for the study of the linearized Navier-Stokes equations.

Definition 2.1. *For the study of the linearized version of the optimal control problem, the space of admissible controls is $\mathcal{W}_c^L$.*

In order to handle the optimal control problem for nonlinear Navier-Stokes equations, we introduce a new functional space. We first recall the following (see [18]).

Definition 2.2. *Let $0 < \sigma < 1$. For $v \in L^2(\mathbb{R}; L^2(\Omega))$, we denote by $D_t^\sigma(v)$ the fractional derivative of order σ with respect to time t*

$$D_t^\sigma(v)(t) = \mathcal{F}^{-1}\Big(i\tau^\sigma \mathcal{F}\big(v(\cdot)(\tau)\big)\Big)(t),$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$. We then define

$$H^\sigma(\mathbb{R}, L^2(\Omega)) = \left\{v \in L^2(\mathbb{R}; L^2(\Omega)) : D_t^\sigma(v) \in L^2(\mathbb{R}; L^2(\Omega))\right\}.$$

Moreover, $H^\sigma(0,T; L^2(\Omega))$ denotes the set of restrictions of functions in $H^\sigma(\mathbb{R}, L^2(\Omega))$ to $[0,T]$.

Definition 2.3. *Let $0 < \sigma < 1$. Define*

$$\mathcal{W}_\sigma^{NL} = \mathcal{W}^L \cap H^\sigma(0,T; L^2(\Omega))$$

equipped with the norm

$$\|v\|_{\mathcal{W}_\sigma^{NL}} = \left(\|v\|^2_{L^2(0,T;H^1(\Omega))} + \|D_t^\sigma v\|^2_{L^2(0,T;L^2(\Omega))}\right)^{\frac{1}{2}}.$$

Finally, denote by $\mathcal{W}_{\sigma,c}^{NL}$ the trace space associated through the map $\mathcal{R}_{|\Gamma_c}$.

For $\frac{1}{2} < \sigma < 1$, the space $\mathcal{W}_{\sigma,c}^{NL}$ is the set of admissible controls for the nonlinear optimal boundary control problem which is studied here.

Remark 2.2. The space $\mathcal{W}_\sigma^{NL}$ is not empty. Indeed, the stationary solutions to the Stokes equation defining the space $\mathcal{W}^L$ belong to $\mathcal{W}_\sigma^{NL}$.

The space $\mathcal{W}_\sigma^{NL}$ is a closed subspace of $\mathcal{W}^L$ and so this is a Hilbert space.

In boundary control theory, it is classical (see [14]) to consider a control in $H^{1,1}((0,T)\times\Gamma_c)$ which is more regular than $\mathcal{W}_{\sigma,c}^{NL}$. In fact,

$$\left\{\gamma \in H^{1,1}((0,T)\times\Gamma_c) : \int_{\Gamma_c} \gamma\cdot\nu d\sigma = 0\right\} \in \mathcal{W}_{\sigma,c}^{NL}.$$

The trace space above devoted to the study of the nonlinear case is more general and less regular than the one used in [7]. Furthermore, in the linear case where we consider the problem of stabilization around a steady state, the cost functional does not involve time derivatives of the control.

The bounded sets of $\mathcal{W}_\sigma^{NL}$ are weakly relatively compact in $L^2(0,T;L^2(\Omega))$ as is established by the following lemma.

Lemma 2.4. *For all $0<\sigma<1$, the embedding of $\mathcal{W}_\sigma^{NL}$ endowed with the norm $\|v\|_{\mathcal{W}_\sigma^{NL}}$ into $L^2(0,T;L^2(\Omega))$ is compact.*

Proof. We recall (see [16]) the following.

Lemma 2.5 (Aubin-Lions-Simon)**.** *Let B_1, B_0, B_{-1} be three Banach spaces such that $B_1 \hookrightarrow B_0 \hookrightarrow B_{-1}$ with compact embedding from B_1 into B_0. For $0<s\le 1$, denote*

$$W_s = \Big\{ v \in L^2(0,T;B_1) : \sup_{0<h<1} \Big(\frac{1}{h^s}\big(\tau_h(\tilde{v}) - \tilde{v}\big)\Big) \in L^2(\mathbb{R};B_{-1})\Big\},$$

where $\tau_h(v)(t) = v(t+h)$.

Then the embedding $W_s \hookrightarrow L^2(0,T;B_0)$ is compact.

Let us notice that, for $0<\sigma<1$, $\mathcal{W}_\sigma^{NL} \subset W_\sigma$ with $B_1 = H^1(\Omega)$ and $B_0 = B_{-1} = L^2(\Omega)$ from which the result follows. □

We also recall the classical result which is a direct consequence of Corollary 31 of [15].

Lemma 2.6. *For all $\frac{1}{2}<\sigma<1$ and $2\le q\le +\infty$, $\mathcal{W}_\sigma^{NL} \subset L^q(0,T;L^2)$.*

We are now able to make more precise the formulation of the optimal boundary control problem. The control u_ρ will be taken in the space $\mathcal{W}_{\sigma,c}^{NL}$ for $\frac{1}{2}<\sigma<1$. By definition of $\mathcal{W}_\sigma^{NL}$, it is obvious that, for any $u_\rho \in \mathcal{W}_{\sigma,c}^{NL}$, there exists a unique solution $\mathcal{U}_\rho \in \mathcal{W}_\sigma^{NL}$ of the Stokes equations (see Lemma 2.2)

$$\begin{cases} \dfrac{\partial \mathcal{U}_\rho}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta \mathcal{U}_\rho + \nabla p = 0, \\ \operatorname{div}(\mathcal{U}_\rho) = 0, \\ \mathcal{U}_{\rho|\Gamma_c} = u_\rho, \ \ \mathcal{U}_{\rho|\Gamma_e} = 0, \ \ \displaystyle\int_{\Gamma_c} \mathcal{U}_\rho \cdot n \, ds = 0, \\ \mathcal{U}_\rho(0) = 0. \end{cases} \tag{2.4}$$

We introduce the quadratic functional J as follows:

$$J(u_\rho) = \int_0^T ||u(\tau)||_2^2 d\tau + ||u_\rho||^2_{\mathcal{W}_{\sigma,c}^{NL}}.$$

From the above considerations, it is more convenient to work with the extension $\mathcal{U}_\rho$ given by (2.4) instead of the boundary control u_ρ. Following this idea, the problem (1.2) is reduced to finding $\mathcal{U}_\rho^{opt}$, a solution of

$$\mathcal{J}(\mathcal{U}_\rho^{opt}) = \inf_{\mathcal{U}_\rho \in \mathcal{W}_\sigma^{NL}} \mathcal{J}(\mathcal{U}_\rho), \tag{2.5}$$

where $\mathcal{J}$ is defined by

$$\mathcal{J}(\mathcal{U}_\rho^{opt}) = \int_0^T ||v(\tau)||_2^2 d\tau + ||\mathcal{U}_\rho||^2_{\mathcal{W}_\sigma^{NL}}$$

and $v = u - \mathcal{U}_\rho$ is a solution to

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v + (\mathcal{U}_\rho \cdot \nabla)v + (v \cdot \nabla)\mathcal{U}_\rho - \dfrac{1}{\mathcal{R}_e}\Delta v + \nabla p \\ \qquad = f - (\mathcal{U}_\rho \cdot \nabla)\mathcal{U}_\rho \\ \operatorname{div}(v) = 0, \quad v_{|\Gamma_c} = 0, \quad v_{|\Gamma_e} = g, \quad v(0) = u_0. \end{cases} \tag{2.6}$$

The data (u_0, g) have to be taken in $H \times H^{\frac{1}{2}}(\Gamma_e)$. Moreover, we recall that (u_0, g) has to satisfy the compatibility conditions

$$\int_{\Gamma_e} g \cdot n d\sigma = 0, \qquad (u_0 \cdot n)\Big|_{\Gamma_e} = g \cdot n.$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The proof is based on a compactness argument. Let $\frac{1}{2} < \sigma < 1$ and $(u_\rho^n)_{n\in\mathbb{N}} \in \mathcal{W}_{\sigma,c}^{NL}$ be a minimizing sequence for the nonconvex problem (1.2). According to Section 2, we introduce for each $n \in \mathbb{N}$ the extension $\mathcal{U}_\rho^n \in \mathcal{W}_\sigma^{NL}$ of u_ρ^n given by (1.3). It is clear that $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is a minimizing sequence for problem (2.5) which means that, if we denote by

$$\mu = \inf_{\mathcal{U}_\rho \in \mathcal{W}_\sigma^{NL}} \mathcal{J}(\mathcal{U}_\rho),$$

then

$$\lim_{n\to+\infty} \mathcal{J}(\mathcal{U}_\rho^n) = \mu. \tag{3.1}$$

The key point is the lower semi-continuity of $\mathcal{J}$ for the weak topology. Recall here that

$$\mathcal{J}(\mathcal{U}_\rho^n) = \int_0^T ||v^n(\tau)||_2^2 d\tau + ||\mathcal{U}_\rho^n||^2_{\mathcal{W}_\sigma^{NL}},$$

where $v^n \in L^2(0,T;V)$ is the solution to

$$\left\{\begin{array}{l} \dfrac{\partial v^n}{\partial t} + (v^n \cdot \nabla)v^n + (\mathcal{U}_\rho^n \cdot \nabla)v^n + (v^n \cdot \nabla)\mathcal{U}_\rho^n - \dfrac{1}{\mathcal{R}_e}\Delta v^n + \nabla p \\ \qquad = f - (\mathcal{U}_\rho^n \cdot \nabla)\mathcal{U}_\rho^n, \\ \mathrm{div}(v^n) = 0, \quad v^n_{|\Gamma_c} = 0, \quad v^n_{|\Gamma_e} = g, \\ v^n(0,x) = u_0(x). \end{array}\right. \tag{3.2}$$

First, since $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ satisfies (3.1), the sequence $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_\sigma^{NL}$. As $\mathcal{W}_\sigma^{NL}$ is a Hilbert space, there exists $\mathcal{U}^* \in \mathcal{W}_\sigma^{NL}$ such that, up to a subsequence, $\mathcal{U}_\rho^n$ weakly converges to $\mathcal{U}^*$ in $\mathcal{W}_\sigma^{NL}$. We deduce that

$$\|\mathcal{U}^*\|_{\mathcal{W}_\sigma^{NL}} \leq \liminf_{n\to+\infty} \|\mathcal{U}_\rho^n\|_{\mathcal{W}_\sigma^{NL}}.$$

Moreover, by Lemma 2.4, we derive

$$\mathcal{U}_\rho^n \longrightarrow \mathcal{U}^* \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.3}$$

In order to conclude that $\mathcal{U}^*$ is a solution to (2.5), we have to show that the sequence $(v^n)_{n\in\mathbb{N}}$ strongly converges to v^* in $L^2(0,T;L^2(\Omega))$ where v^* is linked to $\mathcal{U}^*$ by equation (2.6). For that purpose, we have to take the limit $n \longrightarrow +\infty$ in Equation (3.2). As usual, it is easier to work with homogeneous Dirichlet boundary conditions. We then introduce an extension of the function g as follows. Recall that g belongs to $H^{\frac{1}{2}}(\Gamma_e)$, $\int_{\Gamma_e} g \cdot \nu d\sigma = 0$ and g is taken to depend only on x. Assume G is a solution to the Stokes equation

$$\left\{\begin{array}{l} -\Delta G + \nabla p = 0, \\ \mathrm{div}(G) = 0, \quad G_{|\Gamma_c} = 0, \quad G_{|\Gamma_e} = g. \end{array}\right. \tag{3.4}$$

Consider, for each $n \in \mathbb{N}$, $w^n = v^n - G$. Then w^n belongs to $L^2(0,T;V)$ and is a solution to

$$\left\{\begin{array}{l} \dfrac{\partial w^n}{\partial t} + w^n \cdot \nabla w^n + (\mathcal{U}_\rho^n + G) \cdot \nabla w^n + w^n \cdot \nabla(\mathcal{U}_\rho^n + G) \\ -\frac{1}{\mathcal{R}_e}\Delta w^n + \nabla \pi = k_\rho^n, \\ \mathrm{div}(w^n) = 0, \quad w^n(0,x) = u_0(x) - G(x), \quad w^n_{|\Gamma_c} = 0, \quad w^n_{|\Gamma_e} = 0, \end{array}\right. \tag{3.5}$$

where $k_\rho^n = -(G \cdot \nabla G + \mathcal{U}_\rho^n \cdot \nabla G + G \cdot \nabla \mathcal{U}_\rho^n + \mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n)$. We investigate the limit $n \to +\infty$ in Equation (3.5). According to Lemma 2.6, the sequence $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2)$. Let $1 < r < 2$ be a real number which will be chosen later and define $p = \frac{2r}{2-r}$. By Hölder's inequality,

$$\|\mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n\|_r \leq C\|\nabla \mathcal{U}_\rho^n\|_2 \|\mathcal{U}_\rho^n\|_p. \tag{3.6}$$

We recall the classical Gagliardo-Nirenberg inequality.

Lemma 3.1. *For $2 \leq p \leq +\infty$, and all function $\phi \in H^1(\Omega)$,*

$$\|\phi\|_p \leq C\|\phi\|_2^{\frac{2}{p}}\|\nabla\phi\|_2^{\frac{p-2}{p}}.$$

By Lemma 3.1,

$$\|\mathcal{U}_\rho^n\|_p \leq C\|\mathcal{U}_\rho^n\|_2^{\frac{2-r}{r}}\|\nabla\mathcal{U}_\rho^n\|_2^{\frac{2(r-1)}{r}}. \tag{3.7}$$

Then, using Inequalities (3.6) – (3.7), we derive

$$\|\mathcal{U}_\rho^n \cdot \nabla\mathcal{U}_\rho^n\|_r \leq C\|\nabla\mathcal{U}_\rho^n\|_2^{\frac{3r-2}{r}}\|\mathcal{U}_\rho^n\|_2^{\frac{2-r}{r}}. \tag{3.8}$$

Choosing $r = \frac{6}{5}$, we deduce that the sequence $(\mathcal{U}_\rho^n \cdot \nabla\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $L^{\frac{3}{2}}(0,T;L^{\frac{6}{5}}(\Omega))$, so there exists a function $l \in L^{\frac{3}{2}}(0,T;L^{\frac{6}{5}}(\Omega))$ such that

$$\mathcal{U}_\rho^n \cdot \nabla\mathcal{U}_\rho^n \rightharpoonup l \text{ weakly in } L^{\frac{3}{2}}(0,T;L^{\frac{6}{5}}(\Omega)). \tag{3.9}$$

Moreover, from (3.3), we deduce that

$$\mathcal{U}_\rho^n \cdot \nabla\mathcal{U}_\rho^n \rightharpoonup \mathcal{U}^* \cdot \nabla\mathcal{U}^* \text{ weakly in } L^1(0,T;L^1(\Omega)). \tag{3.10}$$

By uniqueness of the limit in $\mathcal{D}'(]0,T[\times\Omega)$, using (3.9) – (3.10) we obtain

$$l = \mathcal{U}^* \cdot \nabla\mathcal{U}^*.$$

The other terms of k_ρ^n are estimated in the same way to obtain

$$k_\rho^n \rightharpoonup (G \cdot \nabla G + \mathcal{U}^* \cdot \nabla G + G \cdot \nabla\mathcal{U}^* + \mathcal{U}^* \cdot \nabla\mathcal{U}^*) \text{ weakly in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)).$$

We deal with the left-hand side of Equation (3.5) by performing the usual energy estimates on Navier-Stokes equations. Taking the inner product in $\mathbb{R}^2$ of Equation (3.5) with w^n and integrating over Ω gives, using the identity $\int_\Omega (w^n \cdot \nabla w^n) \cdot w^n dx = 0$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|w^n\|_2^2) + \frac{1}{\mathcal{R}_e}\|\nabla w^n\|_2^2 \leq \Big|\int_\Omega k_\rho^n \cdot w^n dx\Big| \\ + \Big|\int_\Omega ((G + \mathcal{U}_\rho^n) \cdot \nabla w^n) \cdot w^n dx\Big| + \Big|\int_\Omega (w^n \cdot \nabla(G + \mathcal{U}_\rho^n)) \cdot w^n dx\Big|. \end{aligned} \tag{3.11}$$

The sequence $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2)$ and in $L^4(0,T;L^4)$ by Lemma 3.1. We estimate each term of the right-hand-side of inequality (3.11). Since $w^n = 0$ on $\partial\Omega$ an integration by parts gives

$$\Big|\int_\Omega (\mathcal{U}_\rho^n \cdot \nabla\mathcal{U}_\rho^n) \cdot w^n dx\Big| = \Big|\int_\Omega (\mathcal{U}_\rho^n \cdot \nabla w^n) \cdot \mathcal{U}_\rho^n dx\Big|.$$

Then, by an $L^4 - L^2 - L^4$ estimate,

$$\Big|\int_\Omega (\mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n)\cdot w^n dx\Big| \le \|\mathcal{U}_\rho^n\|_4^2 \|\nabla w^n\|_2 \le C_\varepsilon \|\mathcal{U}_\rho^n\|_4^4 + \varepsilon \|\nabla w^n\|_2^2,$$

where ε will be chosen later. Using the same argument, we get

$$\Big|\int_\Omega k_\rho^n \cdot w^n dx\Big| \le f_1(t) + 4\varepsilon \|\nabla w^n\|_2^2, \tag{3.12}$$

where $f_1(t)$ is a function in $L^1(0,T)$ since $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2)$ and in $L^2(0,T;H^1)$.

For the second term, we make an $L^4 - L^2 - L^4$ estimate to obtain

$$\begin{aligned}\Big|\int_\Omega ((G+\mathcal{U}_\rho^n)\cdot \nabla w^n)\cdot w^n dx\Big| &\le C\|w^n\|_4 \|\nabla w^n\|_2 \|G+\mathcal{U}_\rho^n\|_4 \\ \le C\|w^n\|_2^{\frac{1}{2}} \|\nabla w^n\|_2^{\frac{3}{2}} \|G+\mathcal{U}_\rho^n\|_4 &\le C_\varepsilon \|G+\mathcal{U}_\rho^n\|_4^4 \|w^n\|_2^2 + \varepsilon \|\nabla w^n\|_2^2,\end{aligned} \tag{3.13}$$

by Lemma 3.1 and Hölder's inequality.

An integration by parts gives

$$\begin{aligned}\Big|\int_\Omega (w^n \cdot \nabla (G+\mathcal{U}_\rho^n))\cdot w^n dx\Big| &= \Big|\int_\Omega ((G+\mathcal{U}_\rho^n)\cdot \nabla w^n)\cdot w^n dx\Big| \\ &\le C_\varepsilon \|G+\mathcal{U}_\rho^n\|_4^4 \|w^n\|_2^2 + \varepsilon \|\nabla w^n\|_2^2.\end{aligned} \tag{3.14}$$

Collecting Inequalities (3.12), (3.13) and (3.14),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|w^n\|_2^2) &+ \frac{1}{\mathcal{R}_e}\|\nabla w^n\|_2^2 \\ &\le f_1(t) + C_\varepsilon \|G+\mathcal{U}_\rho^n\|_4^4 \|w^n\|_2^2 + 6\varepsilon \|\nabla w^n\|_2^2.\end{aligned} \tag{3.15}$$

Thus, choosing $\varepsilon \le \frac{1}{12\mathcal{R}_e}$, we obtain

$$\frac{d}{dt}(\|w^n\|_2^2) + \frac{1}{2\mathcal{R}_e}\|\nabla w^n\|_2^2 \le f_1(t) + C_\varepsilon \|G+\mathcal{U}_\rho^n\|_4^4 \|w^n\|_2^2. \tag{3.16}$$

We then conclude by Gronwall's lemmas (see [4])) that

$$(w^n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.17}$$

$$(\nabla w^n)_{n\in\mathbb{N}} \text{ is bounded in } L^2(0,T;L^2(\Omega)). \tag{3.18}$$

From (3.17) − (3.18), we deduce by classical arguments (see [4]), that

$$\Big(\frac{\partial w_n}{\partial t}\Big)_{n\in\mathbb{N}} \text{ is bounded in } L^2(0,T;V'),$$

where $V^{'}$ is the dual space of V. In the same way, there exists $w \in L^2(0,T;V)$ and a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ still denoted by $(w_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned}
&w_n \longrightarrow w \text{ strongly in } L^2(0,T;L^2(\Omega)),\\
&\nabla w_n \rightharpoonup \nabla w \text{ weakly in } L^2(0,T;L^2(\Omega)),\\
&w_n \cdot \nabla w_n \rightharpoonup w \cdot \nabla w \text{ weakly in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)).
\end{aligned}$$

Taking the limit $n \to +\infty$ into Equation (3.5), we obtain

$$\begin{cases}
\dfrac{\partial w}{\partial t} + w\cdot\nabla w + (\mathcal{U}^* + G)\cdot\nabla w + w\cdot\nabla(\mathcal{U}^* + G)\\
\qquad -\frac{1}{\mathcal{R}_e}\Delta w + \nabla\pi = k^*,\\
\operatorname{div}(w) = 0, \quad w_{|\Gamma_c} = 0, \quad w_{|\Gamma_e} = 0, \quad w(0,x) = u_0(x) - G(x),
\end{cases} \tag{3.19}$$

where $k^* = -(G\cdot\nabla G + \mathcal{U}^*\cdot\nabla G + G\cdot\nabla\mathcal{U}^* + \mathcal{U}^*\cdot\nabla\mathcal{U}^*)$. This ends the proof of Theorem 1.1. □

4. Frechet Derivability of $\mathcal{U}_\rho \mapsto v$.

In this section, we prove that the solution v of equation (2.6) is Fréchet differentiable with respect to $\mathcal{U}_\rho$ (see [5] for more details). Let $\mathcal{U}_\rho$ and h be two functions in $\mathcal{W}_\sigma^{NL}$. Let v be the solution of Equation 2.6 and consider v_h, the solution to a similar equation obtained by talking $(\mathcal{U}_\rho + h)$ instead of $\mathcal{U}_\rho$. Then the difference $w = v^h - v$ is a solution to

$$\begin{cases}
\dfrac{\partial w}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta w + w\cdot\nabla w + w\cdot\nabla v + v\cdot\nabla w + w\cdot\nabla h\\
\qquad +h\cdot\nabla w + \mathcal{U}_\rho\cdot\nabla w + w\cdot\nabla\mathcal{U}_\rho + h\cdot\nabla v + v\cdot\nabla h\\
\qquad +h\cdot\nabla\mathcal{U}_\rho + \mathcal{U}_\rho\cdot\nabla h\\
\qquad +h\cdot\nabla h + \nabla\pi = 0,\\
\operatorname{div}(w) = 0, \quad w_{|\Gamma} = 0, \quad w(0,x) = 0.
\end{cases} \tag{4.1}$$

Let z be the solution to

$$\begin{cases}
\dfrac{\partial z}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta z + v\cdot\nabla z + z\cdot\nabla v + \mathcal{U}_\rho\cdot\nabla z + z\cdot\nabla\mathcal{U}_\rho\\
+h\cdot\nabla v + v\cdot\nabla h + \mathcal{U}_\rho\cdot\nabla h + h\cdot\nabla\mathcal{U}_\rho + \nabla\pi = 0,\\
\operatorname{div}(z) = 0, \quad z_{|\Gamma} = 0, \quad z(0) = 0.
\end{cases} \tag{4.2}$$

We claim that $z = v^{'}(\mathcal{U}_\rho)\cdot h$. Indeed, denoting $y = w - z$, we prove

$$\|y\|_{H^1(\Omega)} = o(\|h\|_{H^1(\Omega)}).$$

The equation satisfied by y reads

$$\left\{\begin{array}{l}\dfrac{\partial y}{\partial t}-\dfrac{1}{\mathcal{R}_e}\Delta y+(v+\mathcal{U}_\rho)\cdot\nabla y+y\cdot\nabla(v+\mathcal{U}_\rho)+\nabla\pi=-N(w,h),\\ \operatorname{div}(y)=0,\quad y_{|\Gamma}=0,\quad y(0)=0,\end{array}\right. \tag{4.3}$$

where $N(w,h)=w\cdot\nabla w+h\cdot\nabla w+w\cdot\nabla h+h\cdot\nabla h$. Multiplying Equation (4.3) by y and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|y\|_2^2)+\frac{1}{\mathcal{R}_e}\|\nabla y\|_2^2 \qquad (4.4)\\ &\le\Big|\int_\Omega\{(y\cdot\nabla v)y+(v\cdot\nabla y)y+(\mathcal{U}_\rho\cdot\nabla y)y+(y\cdot\nabla\mathcal{U}_\rho)y\}dx\Big|\\ &\qquad+\Big|\int_\Omega N(h,w)ydx\Big|,\\ &\le\|y\|_4^2\|\nabla v\|_2+\|v\|_4\|\nabla y\|_2\|y\|_4+\|\mathcal{U}_\rho\|_4\|\nabla y\|_2\|y\|_4\\ &\qquad+\|y\|_4^2\|\nabla\mathcal{U}_\rho\|_2+\Big|\int_\Omega N(h,w)ydx\Big|, \qquad (4.5)\end{aligned}$$

by the usual $L^4-L^4-L^2$-estimate. Since $w=0$ on Γ, an integration by parts gives

$$\Big|\int_\Omega(w\cdot\nabla w)ydx\Big|=\Big|\int_\Omega(w\cdot\nabla y)wdx\Big|\le\|w\|_4^2\|\nabla y\|_2, \tag{4.6}$$

using again an $L^4-L^4-L^2$-estimate. With the same arguments, one proves

$$\Big|\int_\Omega(h\cdot\nabla w)ydx\Big|=\Big|\int_\Omega(h\cdot\nabla y)wdx\Big|\le\|h\|_4\|\nabla y\|_2\|w\|_4, \tag{4.7}$$

$$\Big|\int_\Omega(w\cdot\nabla h)ydx\Big|=\Big|\int_\Omega(w\cdot\nabla y)hdx\Big|\le\|h\|_4\|\nabla y\|_2\|w\|_4, \tag{4.8}$$

$$\Big|\int_\Omega(h\cdot\nabla h)ydx\Big|=\Big|\int_\Omega(h\cdot\nabla y)hdx\Big|\le\|h\|_4^2\|\nabla y\|_2. \tag{4.9}$$

Collecting (4.4), (4.6), (4.7), (4.8) and (4.9), we derive using Lemma 3.1 and Young's inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|y\|_2^2+\frac{1}{\mathcal{R}_e}\|\nabla y\|_2^2\le\frac{1}{2\mathcal{R}_e}\|\nabla y\|_2^2+C\left\{(\|\nabla v\|_2+\|\nabla\mathcal{U}_\rho\|_2)^2\|y\|_2^2\right.\\ \left.+(\|v\|_4+\|\mathcal{U}_\rho\|_4)^4\|y\|_2^2+\|w\|_4^4+\|h\|_4^4\right\}.\end{aligned} \tag{4.10}$$

Introducing the function

$$f(t)=C(\|\nabla v\|_2+\|\nabla\mathcal{U}_\rho\|_2)^2+(\|v\|_4+\|\mathcal{U}_\rho\|_4)^4$$

belonging to $L^1_{loc}(\mathbb{R}_+)$, we get

$$\frac{d}{dt}\|y\|_2^2 + \frac{1}{\mathcal{R}_e}\|\nabla y\|_2^2 \le f(t)\|y\|_2^2 + C(\|w\|_4^4 + \|h\|_4^4). \tag{4.11}$$

Recalling that $y(0) = 0$, the Gronwall inequality implies that

$$\|y(t)\|_2^2 \le C\int_0^t e^{\int_s^t f(\tau)d\tau}\left(\|w(s)\|_4^4 + \|h(s)\|_4^4\right)ds, \tag{4.12}$$

$$\begin{aligned}\int_0^t \|\nabla y(s)\|_2^2 ds \le C\int_0^t \left(\|w(s)\|_4^4 + \|h(s)\|_4^4\right)ds \\ + C\int_0^t f(r)\left\{\int_0^r e^{\int_s^r f(\tau)d\tau}\left(\|w\|_4^4 + \|h\|_4^4\right)ds\right\}dr.\end{aligned} \tag{4.13}$$

Starting from (4.1), we derive, using the fact that $\int_\Omega (w\cdot\nabla w)\cdot w dx = 0$, the following classical energy estimates:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + \frac{1}{\mathcal{R}_e}\|\nabla w\|_2^2 \le\ & C\left(\|\nabla v\|_2 + \|\nabla \mathcal{U}_\rho\|_2 + \|h\|_2\right)\|w\|_4^2 \\ & + C\left(\|v\|_4 + \|\mathcal{U}_\rho\|_4 + \|h\|_4\right)\|\nabla w\|_2\|w\|_4 \\ & + C\left\{\|\nabla v\|_2\|h\|_4 + \|v\|_4\|\nabla h\|_2 + \|\nabla\mathcal{U}_\rho\|_2\|h\|_4\right. \\ & \left. + \|\mathcal{U}_\rho\|_4\|\nabla h\|_2 + \|h\|_4\|\nabla h\|_2\right\}\|w\|_4.\end{aligned} \tag{4.14}$$

By Lemma 3.1, we deduce from (4.14) and Young's inequality

$$\frac{d}{dt}\|w\|_2^2 + \frac{1}{2\mathcal{R}_e}\|\nabla w\|_2^2 \le m(t)\|w\|_2^2 + C(\|h\|_2^2 + \|\nabla h\|_2^2), \tag{4.15}$$

where $m(t)$ is a function depending only on the H^1-norm of v, h and $\mathcal{U}_\rho$. So we get

$$\|w\|_{L^4(0,T;L^4)} \le C\|h\|_{L^2(0,T;H^1)}.$$

In conclusion, it is clear from Estimates (4.12) and (4.13) that

$$\|y\|_{L^2(0,T;H^1)} \le C\|h\|^2_{L^2(0,T;H^1)}.$$

This ends the proof of Theorem 1.3. □

Proof of Corollary 1.1. This is a direct consequence of the above computation. Let $\mathcal{U}_\rho^{opt}$ be given by Theorem 1.1 Then one has $J'(\mathcal{U}_\rho^{opt})\cdot h = 0$ for all $h \in \mathcal{W}_\sigma^{NL}$. □

5. Stabilization result : the linear case.

In this section, we want to characterize the optimal control $\mathcal{U}_\rho$ given by Theorem 5.1. This problem is well known when one deals with ODEs (see [17]), but few has been done in the context of PDEs.

5.1. **Formulation and context.** Let $\bar{v}$ be a steady state of the Navier-Stokes equations; i.e., $\bar{v}$ is a solution in $H^2(\Omega)$ to

$$\begin{cases} -\dfrac{1}{\mathcal{R}_e}\Delta\bar{v} + \bar{v}\cdot\nabla\bar{v} + \nabla\pi = 0, \\ \operatorname{div}(\bar{v}) = 0, \quad \bar{v}_{|\Gamma_c} = 0, \;\; \bar{v}_{|\Gamma_e} = g, \end{cases} \tag{5.1}$$

where g belongs to $H^{\frac{3}{2}}(\Gamma_e)$. We denote by v the solution to the Navier-Stokes equation (1.1) with boundary conditions

$$v_{|\Gamma_c} = u_\rho, \;\; v_{|\Gamma_e} = g,$$

where u_ρ belongs to $\mathcal{W}_c^L$. Then $w = v - \bar{v}$ is a solution to the following equation:

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta w + w\cdot\nabla w + \bar{v}\cdot\nabla w + w\cdot\nabla\bar{v} + \nabla\pi = 0 \\ \operatorname{div}(w) = 0, \quad w_{|\Gamma_e} = 0, \;\; w_{|\Gamma_c} = u_\rho, \quad\;\; w(0) = 0. \end{cases} \tag{5.2}$$

We expect w to be small. This is why we introduce the linearized version of Equation (5.2) around 0 which reads

$$\begin{cases} \dfrac{\partial w_L}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta w_L + \bar{v}\cdot\nabla w_L + w_L\cdot\nabla\bar{v} + \nabla\pi = 0, \\ \operatorname{div}(w_L) = 0, \quad w_{L|\Gamma_e} = 0, \;\; w_{L|\Gamma_c} = u_\rho, \quad w_L(0) = 0. \end{cases} \tag{5.3}$$

As Equation (5.3) is linear in u_ρ, it is clear that the solution w_L depends linearly on the boundary value u_ρ. So there exists a linear operator $\mathcal{L}_1$ from $\mathcal{W}_c^L$ to $\mathcal{W}^L$ such that $w_L = \mathcal{L}_1(u_\rho)$. Let $\mathcal{U}_\rho$ be the extension of u_ρ given by the weak solution of

$$\begin{cases} \dfrac{\partial \mathcal{U}_\rho}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta\mathcal{U}_\rho + \nabla\pi = 0, \\ \operatorname{div}(\mathcal{U}_\rho) = 0, \;\; \mathcal{U}_{\rho|\Gamma_c} = u_\rho, \;\; \mathcal{U}_{\rho|\Gamma_e} = 0, \;\; \mathcal{U}_\rho(0) = 0. \end{cases} \tag{5.4}$$

Since Equation (5.4) is linear in $\mathcal{U}_\rho$, there exists a linear continuous operator $\mathcal{L}_2$ from $\mathcal{W}_c^L$ to $\mathcal{W}^L$ such that $\mathcal{U}_\rho = \mathcal{L}_2(u_\rho)$. By Lemma 2.2, it is obvious that $\mathcal{L}_2$ is injective and onto by construction. The operator $\mathcal{L}_2$ is one-to-one and $\mathcal{L}_2^{-1}$ is also linear continuous. It follows that w_L depends linearly on $\mathcal{U}_\rho$ in the following sense : define $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2^{-1}$, then $\mathcal{L}$ is continuous and

$$w_L = \mathcal{L}(\mathcal{U}_\rho). \tag{5.5}$$

Finally, define $\tilde{w} = w_L - \mathcal{U}_\rho$ to obtain

$$\begin{cases} \dfrac{\partial \tilde{w}}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta\tilde{w} + \bar{v}\cdot\nabla(\tilde{w}+\mathcal{U}_\rho) + (\tilde{w}+\mathcal{U}_\rho)\cdot\nabla\bar{v} + \nabla\pi = 0 \\ \mathrm{div}(\tilde{w}) = 0, \quad \tilde{w}_{|\Gamma_e} = 0, \quad \tilde{w}_{|\Gamma_c} = 0, \quad \tilde{w}(0) = 0. \end{cases} \tag{5.6}$$

Note that $\tilde{w}$ is a linear function of $\mathcal{U}_\rho$. As mentioned in the Introduction, this case is simpler than the nonlinear one. Indeed, the space $\mathcal{W}^{NL}_{\sigma,c}$ is not needed. Estimates on time derivatives in $\mathcal{U}_\rho$ are not necessary to take the limit in the Navier Stokes equation since nonlinear terms have been cancelled. Weak convergence is sufficient for that purpose. For example, one can solve the optimal control problem in $\mathcal{W}^L_c$ as it is proved in Theorem 5.1. First, introduce the cost functional

$$\mathcal{J}_L(\mathcal{U}_\rho) = \frac{1}{2}\int_0^T (||\tilde{w}+\mathcal{U}_\rho||_2^2 + ||\mathcal{U}_\rho||^2_{H^1})dt.$$

Theorem 5.1. *Assume that Ω is a bounded domain in $\mathbb{R}^2$ with C^1 boundary. For any function (u_0, g) given in $H(\Omega)\times H^{\frac{3}{2}}(\Gamma_e)$ satisfying the compatibility condition* (2.7)*, there exists a* ***unique*** *optimal control $\mathcal{U}_\rho^{opt}$ solution to*

$$\mathcal{J}_L(\mathcal{U}_\rho^{opt}) = \inf_{\mathcal{U}_\rho\in\mathcal{W}^L} \mathcal{J}_L(\mathcal{U}_\rho). \tag{5.7}$$

Moreover, $\mathcal{U}_\rho^{opt}$ satisfies the Euler-Lagrange equations

$$\mathcal{J}_L'(\mathcal{U}_\rho^{opt})\cdot h = \int_0^T \big\{(z+h, \tilde{w}+\mathcal{U}_\rho^{opt})_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1}\big\}dt = 0, \tag{5.8}$$

for all solutions h to

$$\begin{cases} \dfrac{\partial h}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta h + \nabla\pi = 0, \\ \mathrm{div}(h) = 0, \quad h_{|\Gamma_c} = k, h_{|\Gamma_e} = 0, \quad h(0) = 0, \end{cases} \tag{5.9}$$

where k is an arbitrary function given on the boundary Γ_c. The function z arising in Equation (5.8) *is equal to $\frac{\partial\tilde{w}}{\partial\mathcal{U}_\rho}(\mathcal{U}_\rho^{opt})\cdot h$ and satisfies*

$$\begin{cases} \dfrac{\partial z}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta z + \bar{v}\cdot\nabla(z+h) + (z+h)\cdot\nabla\bar{v} + \nabla\pi = 0, \\ \mathrm{div}(z) = 0, \quad z_{|\Gamma_e} = 0, \quad z_{|\Gamma_c} = 0, \quad z(0) = 0. \end{cases} \tag{5.10}$$

Remark 5.1. As $\tilde{w}$ is a linear function of $\mathcal{U}_\rho$, z is independent of the optimal control.

Note that $\mathcal{U}_\rho \in H^1(\Omega)$ which provides a bound on u_ρ in $L^2(\Gamma_c)$.

For physical applications, it is usual to replace $\|\mathcal{U}_\rho\|_{H^1}^2$ by $\|\mathrm{rot}(\mathcal{U}_\rho)\|_2^2$. Since we deal with functions which vanish on a part of the boundary $\partial\Omega$ and as $\mathcal{U}_\rho$ is a free divergence vector field, the H^1-norm and $\|\mathrm{rot}(\cdot)\|_2$ are equivalent.

In practice, in order to stabilize stationary solutions, we have to minimize the vorticity which is exactly rot(v) (see [3]).

Proof. Let $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (5.7). Then $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in the Hilbert space $L^2(0,T;H^1(\Omega))$. Thus, there exists $\mathcal{U}_\rho^* \in L^2(0,T;H^1(\Omega))$ such that $\mathcal{U}_\rho^n$ weakly converges to $\mathcal{U}^*$ in $L^2(0,T;H^1(\Omega))$. It is clear that $\mathcal{U}^*$ is a solution to problem (5.7) and a straightforward calculation gives

$$\mathcal{J}'(\mathcal{U}_\rho^{opt}) \cdot h = \int_0^T \Big\{(z+h, \tilde{w} + \mathcal{U}_\rho^{opt})_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1}\Big\} dt = 0.$$

Assume that there exist two optimal controls $\mathcal{U}_\rho^{opt,1}$ and $\mathcal{U}_\rho^{opt,2}$. Denote by $\tilde{w}^1$ and $\tilde{w}^2$ the solutions to Equation (5.6) corresponding to $\mathcal{U}_\rho^{opt,1}$ and $\mathcal{U}_\rho^{opt,2}$. By (5.8), we can write, for all $h \in \mathcal{W}^L$ and $i = 1, 2$,

$$\int_0^T \Big\{(z+h, \tilde{w}^i + \mathcal{U}_\rho^{opt,i})_{L^2} + ((\mathcal{U}_\rho^{opt,i}, h))_{H^1}\Big\} dt = 0, \tag{5.11}$$

where

$$z = \frac{\partial \tilde{w}^1}{\partial \mathcal{U}_\rho}(\mathcal{U}_\rho^{opt,1}) \cdot h = \frac{\partial \tilde{w}^2}{\partial \mathcal{U}_\rho}(\mathcal{U}_\rho^{opt,2}) \cdot h$$

(see Remark 5.1). Since $\mathcal{U}_\rho^{opt,1}$ and $\mathcal{U}_\rho^{opt,2}$ satisfy Equation (5.4), $h = \mathcal{U}_\rho^{opt,1} - \mathcal{U}_\rho^{opt,2}$ satisfies Equation (5.9) with $k = u_\rho^1 - u_\rho^2$. Subtracting Equations (5.11) for $i = 1, 2$ and taking $h = \mathcal{U}_\rho^{opt,1} - \mathcal{U}_\rho^{opt,2}$, we obtain

$$\int_0^T \Big\{\|h\|_2^2 + ((z+h, \tilde{w}^1 - \tilde{w}^2 + h))_{H^1}\Big\} dt = 0. \tag{5.12}$$

Using (5.9) and (5.10), we derive that the following equation is satisfied by $z + h$:

$$\begin{cases} \dfrac{\partial(z+h)}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta(z+h) + \bar{v}\cdot\nabla(z+h) + (z+h)\cdot\nabla\bar{v} + \nabla\pi = 0 \\ \mathrm{div}(z) = 0, \quad z_{|\Gamma_e} = 0, \quad z_{|\Gamma_c} = 0, \quad z(0) = 0. \end{cases} \tag{5.13}$$

By a direct calculation on Equation (5.6) and (5.9), one shows that $\tilde{w}^1 - \tilde{w}^2 + h$ satisfies (5.13). Since this equation is linear, it is clear that

$$\tilde{w}^1 - \tilde{w}^2 + h = z + h.$$

Then (5.12) gives

$$\int_0^T \left(\|h\|_2^2 + \|z + h\|_{H^1}^2\right) = 0,$$

which proves $\mathcal{U}_\rho^{opt,1} = \mathcal{U}_\rho^{opt,2}$. □

5.2. **Euler-Lagrange equations for optimal control.** In this section, we make precise the Euler-Lagrange equation satisfied by the optimal control of Theorem 5.1 by proving the following.

Theorem 5.2. *Let $\mathcal{U}_\rho^{opt}$ be the optimal control of Theorem* 5.1*. Then, for all $h \in \mathcal{W}^L$,*

$$\int_0^T ((u + \mathcal{U}_\rho^{opt}, h))_{H^1} dt = 0,$$

where u is the unique solution of $a(u, v) = (\ell, v)$ for all v in $\mathcal{W}^L$ with $\ell = \mathcal{L}\,\mathcal{U}_\rho^{opt} - \bar{v} \cdot \nabla p - {}^t(\nabla p)\bar{v}$. The bilinear form a is defined by (5.19)*.*

We first give the following definition.

Definition 5.1. *Let A be the unbounded operator from V to V' defined by*

$$\forall p \in V,\ < A(r), p >= \int_\Omega \Big(\frac{1}{\mathcal{R}_e} \nabla r \cdot \nabla p + (\bar{v} \cdot \nabla r) \cdot p + (r \cdot \nabla \bar{v}) \cdot p\Big).$$

We compute the adjoint of the operator A as follows.

Lemma 5.1. *The adjoint of the operator A is given by*

$$\forall r \in V,\ < {}^tA(p), r >= \int_\Omega \Big(\frac{1}{\mathcal{R}_e} \nabla p \cdot \nabla r - (\bar{v} \cdot \nabla p) \cdot r + ({}^t(\nabla \bar{v}) \cdot p) \cdot r\Big).$$

Proof. Let r and p belong to V. Using the fact that $\mathrm{div}(\bar{v}) = 0$ and the $H^2(\Omega)$-regularity for the strong stationary solution $\bar{v}$, we have

$$\begin{aligned}\int_\Omega (\bar{v} \cdot \nabla r) \cdot p &= \sum_{i,j=1}^2 \int_\Omega (\bar{v}_j \frac{\partial r_i}{\partial x_j}) p_i = \sum_{i,j=1}^2 \int_\Omega \frac{\partial(\bar{v}_j r_i)}{\partial x_j} p_i \\ &= -\sum_{i,j=1}^2 \int_\Omega \bar{v}_j r_i \frac{\partial p_i}{\partial x_j} + \sum_{i,j=1}^2 \int_\Gamma \bar{v}_j r_i p_i n_j = -\int_\Omega (\bar{v} \cdot \nabla p) \cdot r,\end{aligned}$$

since $p = 0$ on Γ. In the same way, we get

$$\int_\Omega (r \cdot \nabla \bar{v}) \cdot p = \sum_{i,j=1}^{2} \int_\Omega r_j p_i \frac{\partial \bar{v}_i}{\partial x_j} = \int_\Omega \left({}^t(\nabla \bar{v}) p\right) \cdot r.$$

Next, let us introduce the adjoint problem

$$\begin{cases} -\dfrac{dp}{dt}(t) + {}^tAp(t) = \tilde{w} + \mathcal{U}_\rho, \\ \operatorname{div}(p) = 0, \quad p(T) = 0, \quad p_{|\Gamma} = 0. \end{cases} \tag{5.14}$$

In order to derive an equation for $\mathcal{U}_\rho^{opt}$, we recall that $\mathcal{U}_\rho^{opt}$ satisfies

$$\mathcal{J}_L'(\mathcal{U}_\rho^{opt}) \cdot h = \int_0^T \left\{ (z + h, \tilde{w} + \mathcal{U}_\rho^{opt})_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1} \right\} dt = 0, \tag{5.15}$$

where h and z satisfy respectively (5.9) and (5.10). As $w_L = \tilde{w} + \mathcal{U}_\rho^{opt}$, one gets

$$\int_0^T \left\{ (z + h, \tilde{w} + \mathcal{U}_\rho^{opt})_{L^2} \right\} dt = \int_0^T (z, w_L)_{L^2} dt + \int_0^T (h, w_L)_{L^2} dt.$$

Using Equation (5.14) we have

$$\begin{aligned} \int_0^T (z, w_L) dt &= \int_0^T (z, -\frac{dp}{dt} + {}^tAp)_{L^2} dt \\ &= \int_0^T (\frac{dz}{dt} + Az, p)_{L^2} dt - (z(T), p(T))_{L^2} + (z(0), p(0))_{L^2}. \end{aligned}$$

Since $z(0) = 0$, $P(T) = 0$, and the function z solves Equation (5.10), we obtain

$$\begin{aligned} \int_0^T (z, w_L)_{L^2} dt &= -\int_0^T (\bar{v} \cdot \nabla h + h \cdot \nabla \bar{v}, p)_{L^2} dt, \\ &= -\int_0^T (\bar{v} \cdot \nabla p + {}^t(\nabla \bar{v}) p, h)_{L^2} dt. \end{aligned} \tag{5.16}$$

Hence,

$$\int_0^T \left\{ (z + h, \tilde{w} + \mathcal{U}_\rho^{opt})_{L^2} \right\} dt = -\int_0^T (\bar{v} \cdot \nabla p + {}^t(\nabla \bar{v}) p - w_L, h)_{L^2} dt.$$

Collecting (5.15) – (5.16) we derive

$$\int_0^T \left\{ (w_L - \bar{v} \cdot \nabla p - {}^t(\nabla p) \bar{v}, h)_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1} \right\} dt = 0. \tag{5.17}$$

Using the linearity of the map $\mathcal{U}_\rho^{opt} \mapsto w_L$ through the operator $\mathcal{L}$, we finally obtain

$$\int_0^T \Big\{(\mathcal{L}\,\mathcal{U}_\rho^{opt} - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}, h)_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1}\Big\} dt = 0. \qquad (5.18)$$

In order to give a better interpretation of the above formula, we introduce the bilinear continuous and coercive form

$$\begin{aligned} a : \mathcal{W}^L \times \mathcal{W}^L &\longrightarrow \mathbb{R} \\ u, v \quad &\longmapsto \int_\Omega (uv + \nabla u \cdot \nabla v)_{L^2} = ((u,v))_{H^1}, \end{aligned} \qquad (5.19)$$

and define $\ell = \mathcal{L}\mathcal{U}_\rho^{opt} - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}$. It is obvious that $\ell \in L^2(\Omega)$ and that the function $h \longmapsto (\ell, h)_{L^2}$ is linear continuous from $\mathcal{W}^L$ to $\mathbb{R}$. Then, by the Lax-Milgram theorem, there exists a unique $u \in \mathcal{W}^L$ such that, for all $h \in \mathcal{W}^L$, $a(u,h) = (\ell, h)_{L^2}$. So the optimal control $\mathcal{U}_\rho^{opt}$ is characterized by

$$\int_0^T ((u + \mathcal{U}_\rho^{opt}, h))_{H^1} dt = 0,$$

for all h in $\mathcal{W}^L$, where u is the unique solution of $a(u,v) = (\ell, v)_{L^2}$ for all v in $\mathcal{W}^L$ with $\ell = \mathcal{L}\mathcal{U}_\rho^{opt} - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}$. □

Conclusion. In this paper, we have introduced a set of functional spaces to handle optimal boundary control for Navier-Stokes equations. In this context, we establish the existence of an optimal control. In the linear case, we consider the problem of stabilization around a steady state. In this situation, the optimal control is unique. Furthermore, we bring to the fore the Euler equations associated with this problem.

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