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ERROR ESTIMATES FOR A FINITE ELEMENT DISCRETIZATION OF THE CAHN-HILLIARD-GURTIN EQUATIONS

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Abstract. We prove optimal error estimates in energy norms and related norms for a space semidiscrete and for a fully discrete approximation of the Cahn-Hilliard-Gurtin equations with source terms. Numerical simulations in one and two space dimensions illustrate the theoretical results. We also prove convergence to equilibrium for the fully discrete scheme without source terms by the use of the Łojasiewicz inequality.

1. Introduction

We consider space and time discretizations of the Cahn-Hilliard-Gurtin equations with source terms

$$\partial_t u - a\cdot\nabla\partial_t u = \operatorname{div}(B\nabla w) + m \quad \text{in } \Omega\times(0,+\infty) \tag{1.1}$$

$$w - b\cdot\nabla w = \beta\partial_t u - \alpha\Delta u + f'(u) - \gamma \quad \text{in } \Omega\times(0,+\infty) \tag{1.2}$$

$$u(0) = u_0, \tag{1.3}$$

subject to periodic boundary conditions on the domain

$$\Omega = \Pi_{i=1}^d (0, L_i), \quad (L_i > 0, \quad i = 1,\dots,d), \quad 1\le d\le 3.$$

The nonlinearity f is typically a double-well potential, α is a prescribed positive constant and the functions $m,\gamma : \Omega\times(0,+\infty)\to\mathbb{R}$ are the source terms. The constant coefficients $\beta\in\mathbb{R}$, $a, b\in\mathbb{R}^d$ and B, a real symmetric matrix of size d, satisfy the coercivity assumption: there exists a constant $c_0 > 0$ such that for all $x\in\mathbb{R}$ and for all $y\in\mathbb{R}^d$,

$$\beta x^2 + y^t B y + y^t(a+b)x \ge c_0(x^2 + \|y\|^2), \tag{1.4}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

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These equations, which describe qualitative features of two-phases systems, were derived by M. Gurtin in [14]. The variable u is the order parameter and w is the chemical potential; m represents the external mass supply and γ is the external microforce. When $B = \kappa I$ ($\kappa > 0$), $\beta \geq 0$ and $a = b = 0$, $m = \gamma = 0$, we obtain the viscous Cahn-Hilliard equation, which was derived independently by Novick-Cohen in [29]. The subcase $\beta = 0$ corresponds to the classical Cahn-Hilliard equation; in this latter case, assumption (1.4) is not satisfied, but the classical Cahn-Hilliard equation can be seen as a singular limit of the viscous Cahn-Hilliard equation [1, 12].

Well posedness of the initial-value problem (1.1)-(1.3) with periodic boundary conditions and with source terms was studied in [24]. The case without source terms has been extensively studied [3, 4, 6, 8, 20, 21, 22, 23, 24, 25, 26, 27]; questions related to the well posedness and to the asymptotic behavior, such as the existence of attractors and convergence to equilibrium, were answered for polynomial, logarithmic or more general potentials. Global existence is a consequence of the fact that the free energy

$$\mathcal{E}(u) = \int_\Omega \frac{\alpha}{2}|\nabla u|^2 + f(u)\,\mathrm{d}x \tag{1.5}$$

is a Liapounov functional [33], and that the mass is preserved:

$$\int_\Omega u\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x \quad \text{for all } t \geq 0.$$

Note that Neumann boundary conditions are usually preferred in the Cahn-Hilliard theory, but it is not clear in general what the proper Neumann-like boundary conditions should be for this problem (see [21, 22, 24, 25] for details).

From the numerical point of view, the authors derived in [16] some space and time discretizations of the Cahn-Hilliard-Gurtin equations (1.1)-(1.3) without source terms, for a polynomial nonlinearity with subcritical growth and positive leading coefficient. They considered a Galerkin approximation (of conforming finite element type) for the space discretization and a backward Euler scheme for the time discretization, thus extending some well-known discretizations of the Cahn-Hilliard equation [9, 10] and of the viscous Cahn-Hilliard equation [1]. Stability of these schemes was proved in [16], and numerical simulations were given (see also [15, 30]).

The aim of this paper is to establish optimal error estimates for the discretizations considered in [16]. The error is evaluated in various norms and, first of all, in norms which are naturally related to the energy and to the coercivity assumption (1.4). These energy norms are first introduced in

Section 2 for the continuous problem; we simplify here the well-posedness results in [24] for our purposes, by adding an assumption on the external mass supply m.

In Section 3, we prove optimal error estimates for the space semidiscrete P^1 conforming finite element approximation of (1.1)-(1.3), assuming – as usual in such problems – enough regularity on the solution. Similar error estimates are derived in Section 4 for the fully discrete scheme. In both cases, the error estimates use maximum norm estimates in order to consider the nonlinearity as globally Lipschitz; this induces a slight restriction on the time step for the fully discrete scheme. We also show how the error estimates in the energy norm can be derived directly without using maximum norm estimates (Theorems 3.7 and 4.6). Numerical error estimates in one and two space dimensions, which illustrate the theoretical results, are presented in Section 5.

In the Appendix, we show convergence to equilibrium for the fully discrete scheme without source terms and with a polynomial nonlinearity, thus answering a question raised in [16].

2. The continuous problem

Let $V = H^1_{per}(\Omega)$ and denote $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$ and $|\cdot|_0$ the $L^2(\Omega)$-norm. Throughout the paper, $T \in (0,+\infty)$ is a fixed final time. The variational formulation of (1.1)-(1.3) reads: find $u, w : [0,T] \to V$ such that

$$\frac{d}{dt}[(u,\chi) - (a\cdot\nabla u,\chi)] + (B\nabla w,\nabla\chi) = (m,\chi), \qquad \text{for all } \chi\in V, \tag{2.1}$$

$$\beta\frac{d}{dt}(u,\chi) + \alpha(\nabla u,\nabla\chi) + (f'(u),\chi) - (w,\chi) + (b\cdot\nabla w,\chi) = (\gamma,\chi), \qquad \text{for all } \chi\in V, \tag{2.2}$$

$$u(0) = u_0. \tag{2.3}$$

The nonlinearity f satisfies

$$f \in C^2(\mathbb{R}), \tag{2.4}$$

$$c_1 s^{2p+2} - c_2 \le f(s) \le c_3 s^{2p+2} + c_4, \qquad \text{for all } s\in\mathbb{R}, \tag{2.5}$$

$$|f'(s)| \le c_5 s^{2p+1} + c_6, \qquad \text{for all } s\in\mathbb{R}, \tag{2.6}$$

$$|f''(s)| \le c_7 s^{2p} + c_8, \qquad \text{for all } s\in\mathbb{R}, \tag{2.7}$$

for some constants $c_1, c_3 > 0$ and $c_2, c_4, c_5, c_6, c_7, c_8 \ge 0$, where $p \in \mathbb{N}^\star$ if $d = 1, 2$ and $p = 1$ if $d = 3$. For example, every polynomial of degree $2p+2$

with strictly positive leading coefficient satisfies these assumptions, and in particular the double-well potential $f(s) = (s^2-1)^2$. The Sobolev imbedding $V \subset L^q(\Omega)$ holds for all $1 \leq q < \infty$ if $d = 1, 2$ and for $q = 6$ if $d = 3$; if $d = 1$, we even have $V \subset C^0(\overline{\Omega})$. We will make frequent use of the Poincaré inequality:

$$|v - \langle v \rangle|_0 \leq c_P |\nabla v|_0, \qquad \text{for all } v \in V, \tag{2.8}$$

where we denote

$$\langle v \rangle = \frac{1}{|\Omega|}(v, 1).$$

We have (cf. Miranville [24]) the following.

Theorem 2.1. *Assume that the coefficients satisfy* (1.4)*, that f satisfies* (2.4)-(2.7) *and that $m, \gamma \in L^2(0, T; L^2(\Omega))$, with $(m(t), 1) = 0$ for all $t \in [0, T]$. Then, for all $u_0 \in V$, the system* (2.1)-(2.3) *has a unique solution (u, w) such that $u \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; V)$, $u_t \in L^2(0, T; L^2(\Omega))$ and $w \in L^2(0, T; V)$. Moreover, for all $t \in [0, T]$,*

$$\mathcal{E}(u(t)) + \frac{c_0}{2} \int_0^t |u_t(s)|_0^2 + |\nabla w(s)|_0^2 \, \mathrm{d}s \leq \mathcal{E}(u_0) + \frac{1}{2c_0} \int_0^T c_P^2 |m(t)|_0^2 + |\gamma(t)|_0^2 \, \mathrm{d}s. \tag{2.9}$$

Remark 2.2. We assumed $(m(t), 1) = 0$ in order to simplify the estimates that follow (compare with [24] where this assumption is not made). Note that this assumption is satisfied for the Cahn-Hilliard-Gurtin equations without source terms, i.e., when $m = \gamma = 0$.

Proof. For uniqueness, the proof in [24], based on Gronwall's lemma, applies. It requires assumption (2.7) which is not needed elsewhere. Existence can be obtained by a Galerkin approximation as in the case without source terms [16]. We remind the reader of the a priori estimates, which differ slightly from the ones in [24] thanks to the simplifying assumption $(m(t), 1) = 0$. These estimates are valid for any solution which is regular enough, and they could be justified by a Galerkin method in the general case. Taking $\chi \equiv 1$ in (2.1), we find

$$(u_t(t), 1) = 0 \qquad \text{for all } t \geq 0. \tag{2.10}$$

Taking $\chi \equiv 1$ in (2.2), we find

$$(w(t), 1) = (f'(u(t)), 1) - (\gamma(t), 1), \qquad \text{for all } t \geq 0. \tag{2.11}$$

Taking $\chi = w$ in (2.1) and $\chi = u_t$ in (2.2), and summing, we have

$$-(a \cdot \nabla u_t, w) + (B\nabla w, \nabla w) + \beta |u_t|_0^2 + (b \cdot \nabla w, u_t)$$

$$+\frac{\alpha}{2}\frac{d}{dt}|\nabla u|_0^2 + \frac{d}{dt}\int_\Omega f(u) \quad = (m,w) + (\gamma, u_t).$$

Applying the coercivity inequality (1.4) with $x = u_t$ and $y = \nabla w$, and using the fact that $(a\cdot\nabla\cdot,\cdot)$ is skew-symmetric, we find

$$\frac{d}{dt}\mathcal{E}(u(t)) + c_0(|u_t(t)|_0^2 + |\nabla w(t)|_0^2) \le |(m(t), w(t))| + |\gamma(t)|_0 |u_t(t)|_0,$$

for all $t \ge 0$, where we recall that $\mathcal{E}$ is defined by (1.5). Noting that

$$(m(t), w(t)) = (m(t), w(t) - \langle w(t)\rangle), \qquad \text{for all } t \ge 0,$$

we find that

$$|(m(t), w(t))| \le c_P |m(t)|_0 |\nabla w(t)|_0.$$

We deduce that

$$\frac{d}{dt}\mathcal{E}(u(t)) + \frac{c_0}{2}(|u_t(t)|_0^2 + |\nabla w(t)|_0^2) \le \frac{1}{2c_0}\big(c_P^2|m(t)|_0^2 + |\gamma(t)|_0^2\big),$$

for all $t \in [0,T]$. Integrating from 0 to t, we find (2.9) for all $t \in [0,T]$. From this, we deduce the a priori estimates on u, with the help of (2.5) and of (2.10), and on u_t. We also obtain the estimate on w, with the help of the Poincaré inequality (2.8) and of (2.11), which implies with (2.6) and the injection $V \subset L^{2p+1}(\Omega)$ that

$$|(w(t),1)| \le c(\|u(t)\|_V^{2p+1} + 1 + |\gamma(t)|_0), \qquad \text{for all } t \ge 0. \qquad \square$$

3. Error estimates for the space semidiscrete problem

For the space discretization, we consider a quasiuniform family $\{\mathcal{T}^h\}$ of conforming decomposition of $\overline{\Omega}$ into d-simplices (i.e., intervals if $d = 1$, triangles if $d = 2$ and tetrahedrons if $d = 3$). The decomposition takes into account the periodic boundary conditions (i.e. that the faces of two d-simplices which are on two opposite sides of Ω and which correspond to each other are identified), so that every $\mathcal{T}^h$ is in fact a triangulation of $\simeq \mathbb{R}^d/(\Pi_{i=1}^d L_i\mathbb{Z})$. Recall that Ω is a d-parallelepiped, so that a family $\{\mathcal{T}^h\}$ of such triangulations clearly exists in dimension 1, 2 and 3.

For every mesh $\mathcal{T}^h = \bigcup_{T\in\mathcal{T}^h} T$, we use the associated P^1 conforming finite element space:

$$V^h = \{u \in C^0_{per}(\bar\Omega) : u_{|T} \text{ is affine for all } T \in \mathcal{T}^h\}. \tag{3.1}$$

It is clear that $V^h \subset V = H^1_{per}(\Omega)$. In the following, we denote by $\|\cdot\|_k$ the Hilbert space norm in $H^k_{per}(\Omega)$ $(k = 0,\ 1,\ 2\dots)$ and by $|\cdot|_k$ the associated seminorm; i.e., (with the multi-index notation)

$$\text{for all } v \in H^k_{per}(\Omega), \quad \|v\|_k^2 = \sum_{|\alpha|\le k} |\partial^\alpha v|_0^2, \text{ and } |v|_k^2 = \sum_{|\alpha|=k} |\partial^\alpha v|_0^2.$$

We recall the following results [7, 31]: with the assumptions made on the mesh, there exists $C > 0$ independent of h such that

$$\text{for all } v \in H^2_{per}(\Omega), \quad |v - I^h v|_0 + h|v - I^h v|_1 \le Ch^2 |v|_2, \tag{3.2}$$

where $I^h v$ is the P^1 interpolate of v, i.e., the unique element in V^h having the same values as v at the nodes of the triangulation $\mathcal{T}^h$. In fact, (3.2) holds if the family of triangulations in only assumed to be regular, i.e., if the minimal angle of the d-simplices belonging to the triangulations $\mathcal{T}^h$ is bounded from below by a positive constant independent of h. The quasiuniformity assumption provides in addition the following inverse estimates (see for instance [13, 34]), which hold for all $v^h \in V^h$:

$$\|v^h\|_{L^\infty(\Omega)} \le Ch^{-d/2} |v^h|_0, \tag{3.3}$$

$$\|v^h\|_{L^\infty(\Omega)} \le Ck_h \|v^h\|_1, \tag{3.4}$$

where $k_h = 1$ if $d = 1$, $k_h = \log(1/h)^{1/2}$ if $d = 2$ and $k_h = h^{-1/2}$ if $d = 3$. Throughout the paper, C denotes a generic constant which does not depend on h or δt. It will prove useful to define, for any $v \in V$, the Ritz projection $R^h v$ of v onto V^h as the unique element in V^h such that

$$(\nabla R^h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \text{for all } \chi \in V^h \quad \text{and} \quad (R^h v, 1) = (v, 1). \tag{3.5}$$

The space semidiscrete scheme for the resolution of (2.1)-(2.3) reads: let $u_0^h \in V^h$ and find $u^h, w^h : [0,T] \to V^h$ such that

$$(u_t^h, \chi) - (a \cdot \nabla u_t^h, \chi) + (B\nabla w^h, \nabla \chi) = (m, \chi), \quad \text{for all } \chi \in V^h, \tag{3.6}$$

$$\beta(u_t^h, \chi) + \alpha(\nabla u^h, \nabla \chi) + (f'(u^h), \chi) - (w^h, \chi) + (b \cdot \nabla w^h, \chi) = (\gamma, \chi), \quad \text{for all } \chi \in V^h, \tag{3.7}$$

$$u^h(0) = u_0^h. \tag{3.8}$$

By the same arguments as above and as in [16], we have the following.

Proposition 3.1. *Under the assumptions of Theorem* 2.1, *for all* $u_0^h \in V^h$, *the problem* (3.6)-(3.8) *has a unique solution* (u^h, w^h) *such that* $u^h \in$

$W^{1,2}(0,T;V^h)$ *and* $w^h \in L^2(0,T;V^h)$. *Moreover, for all* $t \in [0,T]$,

$$\mathcal{E}(u^h(t))+\frac{c_0}{2}\int_0^t |u_t^h(s)|_0^2+|\nabla w^h(s)|_0^2\,\mathrm{d}s \le \mathcal{E}(u_0^h)+\frac{1}{2c_0}\int_0^T c_P^2|m(t)|_0^2+|\gamma(t)|_0^2\,\mathrm{d}s. \tag{3.9}$$

From now on, we assume that the assumptions of Theorem 2.1 are satisfied. In order to estimate the errors $\|u^h-u\|$ and $\|w^h-w\|$ in appropriate (energy) norms, we write, following [10, 34],

$$\begin{aligned} u^h(t)-u(t) &= \theta^u(t)+\rho^u(t),\\ &\quad\text{with}\ \ \theta^u(t)=u^h(t)-\tilde{u}^h(t),\ \ \rho^u(t)=\tilde{u}^h(t)-u(t),\\ w^h(t)-w(t) &= \theta^w(t)+\rho^w(t),\\ &\quad\text{with}\ \ \theta^w(t)=w^h(t)-\tilde{w}^h(t),\ \ \rho^w(t)=\tilde{w}^h(t)-w(t), \end{aligned}$$

for all $t\in[0,T]$, where $\tilde{u}^h=\tilde{u}^h(t)\in V^h$ and $\tilde{w}^h=\tilde{w}^h(t)\in V^h$ are elliptic projections of $u=u(t)$ and $w=w(t)$ on V^h defined by

$$(B\nabla\tilde{w}^h,\nabla\chi) = B\nabla w,\nabla\chi) \quad \text{for all } \chi\in V^h, \tag{3.10}$$

$$(\tilde{w}^h,1)=(w,1), \tag{3.11}$$

and, for all $\chi\in V^h$,

$$\alpha(\nabla\tilde{u}^h,\nabla\chi)-(\tilde{w}^h,\chi)+(b\cdot\nabla\tilde{w}^h,\chi)=\alpha(\nabla u,\nabla\chi)-(w,\chi)+(b\cdot\nabla w,\chi), \tag{3.12}$$

$$(\tilde{u}^h,1)=(u,1). \tag{3.13}$$

To establish estimates on ρ^u and ρ^w, we first assume that u and w do not depend on time. Note that, for all $(u,w)\in V\times V$, the system (3.10)-(3.13) has a unique solution $(\tilde{u}^h,\tilde{w}^h)\in V^h\times V^h$. Indeed, let us set $\tilde{w}^h=\langle w\rangle+\delta w^h$. Then $\delta w^h\in V^h$, and (3.11) is satisfied if and only if $(\delta w^h,1)=0$; we see that the symmetric bilinear form $(\delta w^h,\delta z^h)\mapsto(B\nabla\delta w^h,\nabla\delta z^h)$ is continuous and positive definite on the vector space $\{\delta w^h\in V^h:(\delta w^h,1)=0\}$. The Lax-Milgram theorem [5] implies then existence and uniqueness of $\tilde{w}^h\in V^h$ satisfying (3.10)-(3.11). By the same arguments, $\tilde{w}^h\in V^h$ being known, $\tilde{u}^h\in V^h$ is defined uniquely by (3.12)-(3.13).

With the assumptions made on the mesh, we may state the following.

Lemma 3.2. *There exists a constant C independent of h such that, for all $u, w \in H^2_{per}(\Omega)$, the functions $(\tilde{u}^h, \tilde{w}^h) \in V^h \times V^h$ uniquely defined by* (3.10)-(3.13) *satisfy*

$$|\tilde{w}^h - w|_0 + h|\nabla(\tilde{w}^h - w)|_0 \leq Ch^2|w|_2,$$
$$|\tilde{u}^h - u|_0 + h|\nabla(\tilde{u}^h - u)|_0 \leq Ch^2(|u|_2 + |w|_2).$$

Proof. The error estimate for w is standard [31]. To estimate $\tilde{u}^h - u$, we define $v^h = R^h u$ where R^h is the Ritz projector onto V^h defined by (3.5). By the same arguments as for w, there exists $C_1 > 0$ depending only on the minimum angle of the triangulations $\mathcal{T}^h$ such that

$$|v^h - u|_0 + h|\nabla(v^h - u)|_0 \leq C_1 h^2 |u|_2.$$

Using (3.12)-(3.13), the difference $\tilde{u}^h - v^h$ satisfies

$$\alpha(\nabla(\tilde{u}^h - v^h), \nabla\chi) = (\tilde{w}^h - w, \chi) + (\tilde{w}^h - w, b \cdot \nabla\chi), \quad \text{for all } \chi \in V^h,$$
$$(\tilde{u}^h - v^h, 1) = 0,$$

where we have used the fact that the bilinear form $(b\cdot\nabla\cdot, \cdot)$ is skew-symmetric. Choosing $\chi = \tilde{u}^h - v^h$, we obtain

$$\alpha|\nabla(\tilde{u}^h - v^h)|_0^2 \leq |\tilde{w}^h - w|_0|\tilde{u}^h - v^h|_0 + \|b\||\tilde{w}^h - w|_0|\nabla(\tilde{u}^h - v^h)|_0.$$

Using the Poincaré inequality (2.8), we deduce that

$$|\nabla(\tilde{u}^h - v^h)|_0 \leq \alpha^{-1}(c_P + \|b\|)|\tilde{w}^h - w|_0 \leq C'h^2|w|_2, \tag{3.14}$$

where $C' = \alpha^{-1}(c_P + \|b\|)C$. Thus, we have

$$h|\tilde{u}^h - u|_1 \leq h|\tilde{u}^h - v^h|_1 + h|v^h - u|_1 \leq C'h^3|w|_2 + C_1h^2|u|_2,$$

on the one hand, and

$$|\tilde{u}^h - u|_0 \leq |\tilde{u}^h - v^h|_0 + |v^h - u|_0 \leq c_P|\tilde{u}^h - v^h|_1 + |v^h - u|_0$$
$$\leq c_P C'h^2|w|_2 + C_1h^2|u|_2^2,$$

on the other hand. The claim is proved. □

Note that the constant C here depends only on the minimal angle of the triangulations, and on the parameters B, b and α. In fact, Lemma 3.2 also holds if $\{\mathcal{T}^h\}$ is only assumed to be regular, i.e., if the minimal angle of the triangulations is bounded from below by a positive constant independent of h.

Similarly, we have maximum norm error estimates, which are valid for quasiuniform triangulations.

Lemma 3.3. *If $d = 1$ or 2, there exists $C > 0$ independent of h such that, for all $u, w \in W^{2,\infty}(\Omega)$,*

$$\|\tilde{w}^h - w\|_{L^\infty(\Omega)} \leq Ch^2 \log(1/h)^{d-1} \|w\|_{W^{2,\infty}(\Omega)},$$
$$\|\tilde{u}^h - u\|_{L^\infty(\Omega)} \leq Ch^2 \log(1/h)^{d-1} \left(\|u\|_{W^{2,\infty}(\Omega)} + \|w\|_2\right).$$

Proof. The error estimate for w is standard (see [7] for $d = 1$ and [28, 32] for $d = 2$). For u, we define as previously $v^h = R^h u$. Standard results show that

$$\|v^h - u\|_{L^\infty(\Omega)} \leq Ch^2 \log(1/h)^{d-1} \|u\|_{W^{2,\infty}(\Omega)}$$

for some $C > 0$. By (3.4) and (3.14),

$$\|\tilde{u}^h - v^h\|_{L^\infty(\Omega)} \leq Ck_h |\tilde{u}^h - v^h|_1 \leq C' h^2 k_h |w|_2.$$

The triangle inequality concludes the proof. □

Next, we establish energy estimates for θ^u and θ^w.

Lemma 3.4. *Assume that the solution u of (2.1)-(2.3) takes its values in $[-\delta, \delta]$ and that $u_0^h \in V^h$ takes its values in $[-\delta-1/2, \delta+1/2]$. Let $T^h \in (0, T]$ be the maximal time such that $\|u^h(t)\|_\infty \leq \delta + 1$ for all $t \in [0, T^h]$, where u^h is the solution of (3.6)-(3.8). Then, there exists $C > 0$ independent of h such that*

$$(c_0/\alpha)(|\theta_t^u|_0^2 + |\nabla\theta^w|_0^2) + \frac{d}{dt}|\nabla\theta^u|_0^2$$
$$\leq C\left(|(\theta^u(0), 1)|^2 + |\rho^u|_0^2 + |\rho_t^u|_0^2\right) + C|\nabla\theta^u|_0^2, \qquad (3.15)$$

in the sense of distributions on $(0, T^h)$. Moreover,

$$|(\theta^w(t), 1)| \leq C(|\rho^u(t)|_0 + |\theta^u(t)|_0), \qquad \text{for all } t \in [0, T^h], \qquad (3.16)$$

and

$$(\theta^u(t), 1) = (\theta^u(0), 1), \qquad \text{for all } t \in [0, T]. \qquad (3.17)$$

Proof. Subtracting (2.1) from (3.6), we find

$$(u_t^h - u_t, \chi) - (a \cdot \nabla(u_t^h - u_t), \chi) + (B\nabla(w^h - w), \nabla\chi) = 0, \qquad \text{for all } \chi \in V^h.$$

Using the definition of θ^u, ρ^u, θ^w and $\tilde{w}^h$, this can be written

$$(\theta_t^u, \chi) - (a \cdot \nabla\theta_t^u, \chi) + (B\nabla\theta^w, \nabla\chi) = -(\rho_t^u, \chi) + (a \cdot \nabla\rho_t^u, \chi), \qquad (3.18)$$

for all $\chi \in V^h$. In the same way, subtracting (2.2) from (3.7), and using the definition of $\tilde{u}^h$, we find

$$\beta(\theta_t^u, \chi) + \alpha(\nabla\theta^u, \nabla\chi) - (\theta^w, \chi) + (b \cdot \nabla\theta^w, \chi)$$

$$= -(f'(u^h) - f'(u), \chi) - \beta(\rho_t^u, \chi), \tag{3.19}$$

for all $\chi \in V^h$.

Now we choose $\chi \equiv 1$ in (3.18):

$$(\theta_t^u, 1) = -(\rho_t^u, 1) = 0, \tag{3.20}$$

which proves equality (3.17). We choose $\chi \equiv 1$ in (3.19):

$$\beta(\theta_t^u, 1) - (\theta^w, 1) = -(f'(u^h) - f'(u), 1) - \beta(\rho_t^u, 1),$$

which, by (3.20), reduces to

$$(\theta^w, 1) = (f'(u^h) - f'(u), 1).$$

This implies that, for all $t \in [0, T^h]$,

$$|(\theta^w(t), 1)| \leq L_f |u^h(t) - u(t)|_0 |1|_0 \leq |\Omega|^{1/2} L_f (|\rho^u(t)|_0 + |\theta^u(t)|_0),$$

where L_f is the Lipschitz constant of f' on $[-\delta - 1, \delta + 1]$. This is estimate (3.16).

Finally, we choose $\chi = \theta^w$ in (3.18) and $\chi = \theta_t^u$ in (3.19), and we add. We find

$$-(a \cdot \nabla\theta_t^u, \theta^w) + (B\nabla\theta^w, \nabla\theta^w) + \beta|\theta_t^u|_0^2 + (b \cdot \nabla\theta^w, \theta_t^u) + (\alpha/2)\frac{d}{dt}|\nabla\theta^u|_0^2$$
$$= -(\rho_t^u, \theta^w) + (a \cdot \nabla\rho_t^u, \theta^w) - (f'(u^h) - f'(u), \theta_t^u) - \beta(\rho_t, \theta_t^u).$$

Using the coercivity (1.4), with $x = \theta_t^u$ and $y = \nabla\theta^w$, we obtain

$$c_0(|\theta_t^u|_0^2 + |\nabla\theta^w|_0^2) + (\alpha/2)\frac{d}{dt}|\nabla\theta^u|_0^2$$
$$\leq c_P|\rho_t^u|_0|\nabla\theta^w|_0 + \|a\||\nabla\theta^w|_0|\rho_t^u|_0 + \beta|\rho_t^u|_0|\theta_t^u|_0 + L_f|u^h - u|_0|\theta_t^u|_0,$$

in $\mathcal{D}'\left((0, T^h)\right)$, where, for the first term of the right-hand side, we have used the fact that $(\rho_t^u, 1) = 0$ and the Poincaré inequality (2.8), so that

$$|(\rho_t^u, \theta^w)| = |(\rho_t^u, \theta^w - \langle\theta^w\rangle)| \leq c_P|\rho_t^u|_0|\nabla\theta^w|_0.$$

We now apply several times Young's inequality $ab \leq \epsilon a^2 + C_\epsilon b^2$:

$$c_0(|\theta_t^u|_0^2 + |\nabla\theta^w|_0^2) + (\alpha/2)\frac{d}{dt}|\nabla\theta^u|_0^2$$
$$\leq \epsilon|\nabla\theta^w|_0^2 + C_\epsilon c_P^2|\rho_t^u|_0^2 + \epsilon|\nabla\theta^w|_0^2 + C_\epsilon\|a\|^2|\rho_t^u|_0^2 + \epsilon|\theta_t^u|_0^2 + C_\epsilon\beta^2|\rho_t^u|_0^2$$
$$+\epsilon|\theta_t^u|_0^2 + 2C_\epsilon L_f^2(|\theta^u|_0^2 + |\rho^u|_0^2).$$

Now we use the Poincaré inequality (2.8), and (3.20), which gives

$$|\theta^u|_0 \leq c_P|\nabla\theta^u|_0 + |\Omega|^{-1/2}|(\theta^u(0), 1)|.$$

Finally, we choose $\epsilon = c_0/4$, and we obtain

$$(c_0/2)(|\theta_t^u|_0^2 + |\nabla\theta^w|_0^2) + (\alpha/2)\frac{d}{dt}|\nabla\theta^u|_0^2$$
$$\leq C_1|\rho_t^u|_0^2 + C_2|\rho^u|_0^2 + C_3|\nabla\theta^u|_0^2 + C_4|(\theta^u(0),1)|^2,$$

where $C_1 = C_\epsilon c_P^2 + C_\epsilon\|a\|^2 + C_\epsilon\beta^2$, $C_2 = 2C_\epsilon L_f^2$, $C_3 = 4C_\epsilon L_f^2 c_P^2$, and, finally, $C_4 = 4C_\epsilon L_f^2|\Omega|^{-1}$. This concludes the proof. Notice that the constant C claimed in the lemma depends only on α, $|\Omega|$, c_P, c_0, $\|a\|$, β and L_f. □

We deduce from Lemmas 3.2 and 3.4 the following error estimate, where C denotes a constant independent of h, and where we use the standard notation $W^{k,2}(0,T;H)$ $k = 1, 2, \ldots$ for the L^2-Sobolev spaces with values in a Hilbert space H.

Theorem 3.5. *Assume that the solution (u, w) of (2.1)-(2.3) satisfies*

$$u, w \in W^{1,2}(0,T; H^2_{per}(\Omega))$$

and let (u^h, w^h) be the solution of (3.6)-(3.8). If $\|\theta^u(0)\|_1 \leq Ch^2$, then

$$\|u - u^h\|_{L^\infty(0,T;L^2(\Omega))} + \|w - w^h\|_{L^2(0,T;L^2(\Omega))} + \|u_t - u_t^h\|_{L^2(0,T;L^2(\Omega))} \leq Ch^2, \tag{3.21}$$

$$\|u - u^h\|_{L^\infty(0,T;H^1(\Omega))} + \|w - w^h\|_{L^2(0,T;H^1(\Omega))} \leq Ch. \tag{3.22}$$

If, furthermore, $d = 1, 2$, $u \in L^\infty((0,T); W^{2,\infty}(\Omega))$ and $w \in L^2(0,T; W^{2,\infty}(\Omega))$, then

$$\|u - u^h\|_{L^\infty(0,T;L^\infty(\Omega))} + \|w - w^h\|_{L^2(0,T;L^\infty(\Omega))} \leq Ch^2 \log(1/h)^{d-1}.$$

Proof. Since $H^2_{per}(\Omega) \subset C^0_{per}(\overline{\Omega})$, u is bounded on $[0,T]$; i.e., there exists $\delta > 0$ such that $\|u(t)\|_{L^\infty(\Omega)} \leq \delta$, for all $t \in [0,T]$. On the other hand,

$$|u_0^h - u_0|_0 \leq |\theta^u(0)|_0 + |\rho^u(0)|_0 \leq Ch^2,$$

using the assumption on $\theta^u(0)$, the Poincaré inequality and Lemma 3.2. Thus, denoting by $I^h u_0$ the P^1 interpolate of u_0, we have, with (3.3) and (3.2),

$$\begin{aligned}\|u_0^h - u_0\|_{L^\infty(\Omega)} &\leq \|u_0^h - I^h u_0\|_{L^\infty(\Omega)} + \|I^h u_0 - u_0\|_{L^\infty(\Omega)},\\ &\leq Ch^{-d/2}|u_0^h - I^h u_0|_0 + \|I^h u_0 - u_0\|_{L^\infty(\Omega)},\\ &\leq Ch^{-d/2}(|u_0^h - u_0|_0 + |I^h u_0 - u_0|_0) + \|I^h u_0 - u_0\|_{L^\infty(\Omega)},\\ &\leq Ch^{2-d/2} + Ch^\gamma,\end{aligned} \tag{3.23}$$

where $\gamma \in (0,1)$ is such that $H^2(\Omega) \subset C^{0,\gamma}(\bar{\Omega})$. For h sufficiently small, $\|u_0^h - u_0\|_{L^\infty(\Omega)} \leq 1/2$ and we can apply Lemma 3.4. In particular, for all $t \in [0, T^h]$,

$$\begin{aligned}|\nabla\theta^w|_0^2 + |\theta_t^u|_0^2 + \frac{d}{dt}|\nabla\theta^u|_0^2 \\ \leq C|\nabla\theta^u|_0^2 + C|(\theta^u(0),1)|^2 + Ch^4\left(|u|_2^2 + |w|_2^2 + |u_t|_2^2 + |w_t|_2^2\right),\end{aligned}$$

where we applied Lemma 3.2 again.

By Gronwall's inequality, by the assumption on $\theta^u(0)$ and by the regularity assumptions on u and w, for all $t \in [0, T^h]$,

$$\int_0^t |\nabla\theta^w(s)|_0^2 + |\theta_t^u(s)|_0^2 \, \mathrm{d}s + |\nabla\theta^u(t)|_0^2 \leq C|\nabla\theta^u(0)|_0^2 + Ch^4 \leq Ch^4. \quad (3.24)$$

Since $(\theta^u(t),1) = (\theta^u(0),1)$ by (3.17), we also have

$$\|\theta^u(t)\|_1 \leq Ch^2, \qquad \text{for all } t \in [0, T^h],$$

and thus, with Lemma 3.2, we obtain that, for all $t \in [0, T^h]$,

$$|u^h(t) - u(t)|_0 \leq Ch^2 + Ch^2\Big(\int_0^{T^h} |u|_2^2 + |w|_2^2 \, \mathrm{d}s\Big)^{1/2}. \quad (3.25)$$

Arguing as in (3.23), we have, for $t \in [0, T^h]$,

$$\begin{aligned}\|u^h(t) - u(t)\|_{L^\infty(\Omega)} &\leq Ch^{-d/2}|u^h(t) - I^h u(t)|_0 + \|I^h u(t) - u(t)\|_{L^\infty(\Omega)}, \\ &\leq Ch^{-d/2}(|u^h(t) - u(t)|_0 + |u(t) - I^h u(t)|_0) + Ch^\gamma, \\ &\leq Ch^{2-d/2} + Ch^\gamma.\end{aligned}$$

In the last inequality, we have used (3.2) and (3.25). Thus, for h sufficiently small, $\|u^h - u\|_{L^\infty(\Omega)} \leq 1$, and $T^h = T$. Lemma 3.4 implies then

$$|(\theta^w(t),1)| \leq Ch^2,$$

and we deduce estimates (3.21)-(3.22) with (3.24), Lemma 3.2, and the triangle inequality. The maximum norm estimates are similarly deduced from the $O(h^2)$ error estimates above for $\nabla\theta^u$ and $\nabla\theta^w$, from (3.4) and from Lemma 3.3. □

Remark 3.6. In order to have $\|\theta^u(0)\|_1 \leq Ch^2$, we can choose $u_0^h = \tilde{u}_0^h$, or simply, $u_0^h = R^h u_0$ (cf. proof of Lemma 3.2).

In the next theorem, which concludes this section, we show how to obtain the energy estimates (3.22) under weaker assumptions: instead of the L^∞ estimates, we use the polynomial growth of f, as in [34] (see also [11]). It is not clear whether the estimate (3.21) in weaker norms can be derived in a similar way.

Theorem 3.7. *Let $\{\mathcal{T}^h\}$ be a regular family of triangulations of*

$$\overline{\Omega} \simeq \Pi_{i=1}^d \mathbb{R}/(L_i\mathbb{Z}).$$

Assume that the solution (u, w) of (2.1)-(2.3) satisfies

$$u \in W^{1,2}(0,T; H^2_{per}(\Omega)) \quad \text{and} \quad w \in L^2(0,T; H^2_{per}(\Omega))$$

and let (u^h, w^h) be the solution of (3.6)-(3.8). If $\|u^h(0)\|_1 \leq R$ for some constant $R > 0$ independent of h, then

$$\|u - u^h\|_{L^\infty(0,T;H^1(\Omega))} + \|w - w^h\|_{L^2(0,T;H^1(\Omega))} \leq C\|u^h(0) - R^h u(0)\|_1 + Ch.$$

Proof. We define the elliptic projection $\tilde{w}^h \in V^h$ by (3.10)-(3.11) as previously, and $\tilde{u}^h \in V^h$ is defined now by $\tilde{u}^h = R^h u$. With these definitions, there exists (cf. [31]) $C > 0$ which depends only on $\{\mathcal{T}^h\}$ and B such that, for all $u, w \in H^2(\Omega)$,

$$|\tilde{w}^h - w|_0 + h|\tilde{w}^h - w|_1 \leq Ch^2|w|_2 \text{ and } |\tilde{u}^h - u|_0 + h|\tilde{u}^h - u|_1 \leq Ch^2|u|_2. \quad (3.26)$$

Equations (3.18) and (3.20) are still valid, and (3.19) is replaced by

$$\begin{aligned} &\beta(\theta^u_t, \chi) + \alpha(\nabla\theta^u, \nabla\chi) - (\theta^w, \chi) + (b \cdot \nabla\theta^w, \chi) \\ &\quad = -(f'(u^h) - f'(u), \chi) - \beta(\rho^u_t, \chi) + (\rho^w, \chi) - (b \cdot \nabla\rho^w, \chi), \end{aligned} \quad (3.27)$$

for all $\chi \in V^h$. Choosing here $\chi \equiv 1$, we obtain as previously

$$(\theta^w, 1) = (f'(u^h) - f(u), 1).$$

With the assumptions on f, we have

$$(f'(u^h) - f'(u), 1) = \int_\Omega q(u^h, u)(u^h - u)\,\mathrm{d}x,$$

where

$$q(u^h, u) = \int_0^1 f''(su^h + (1-s)u)\,\mathrm{d}s \quad (3.28)$$

satisfies

$$|q(u^h, u)| \leq c_7(|u^h|^{2p} + |u|^{2p}) + c_8. \quad (3.29)$$

By the energy estimates (2.9) and (3.9), we know that

$$\|u(t)\|_1 \leq K(\|u_0\|_1) \text{ and } \|u^h(t)\|_1 \leq K(\|u^h_0\|_1), \text{ for all } t \in [0,T], \quad (3.30)$$

where, here and in the following, $K : \mathbb{R}_+ \to \mathbb{R}_+$ denotes a nondecreasing function. Using the Sobolev imbedding $V \subset L^{4p}(\Omega)$, we deduce from this that $q \in L^\infty(0,T;L^2(\Omega))$, and we obtain by the Cauchy-Schwarz inequality that

$$|(\theta^w(t),1)| \leq K(R)|u^h(t)-u(t)|_0 \leq K(R)(|\rho^u(t)|_0 + |\theta^u(t)|_0), \tag{3.31}$$

for all $t \in [0,T]$.

Next we choose $\chi = \theta^w$ in (3.18) and $\chi = \theta^u_t$ in (3.27), and we add. We obtain

$$\begin{aligned} &-(a\cdot\nabla\theta^u_t,\theta^w) + (B\nabla\theta^w,\nabla\theta^w) + \beta|\theta^u_t|_0^2 + (b\cdot\nabla\theta^w,\theta^u_t) + (\alpha/2)\frac{d}{dt}|\nabla\theta^u|_0^2 \\ &\quad = -(\rho^u_t,\theta^w) + (a\cdot\nabla\rho^u_t,\theta^w) - (f'(u^h)-f'(u),\theta^u_t) - \beta(\rho^u_t,\theta^u_t) \\ &\qquad + (\rho^w,\theta^u_t) - (b\cdot\nabla\rho^w,\theta^u_t). \end{aligned} \tag{3.32}$$

In order to estimate the nonlinear term, we write

$$|(f'(u^h)-f'(u),\theta^u_t)| = \int_\Omega q(u^h,u)(u^h-u)\theta^u_t \,\mathrm{d}x,$$

where q is defined by (3.28). An application of Hölder's inequality yields

$$|(f'(u^h)-f'(u),\theta^u_t)| \leq \|q(u^h,u)\|_{L^3(\Omega))}\|u^h-u\|_{L^6(\Omega)}|\theta^u_t|_0 \quad \text{a.e. in } [0,T].$$

Using (3.29), the Sobolev imbeddings $V \subset L^{6p}(\Omega)$, $V \subset L^6(\Omega)$, and (3.30), we find

$$\|q(t)\|_{L^3(\Omega)} \leq K(R) \quad \text{and} \quad \|u^h-u\|_{L^6(\Omega)} \leq C\|u^h-u\|_1 \quad \text{a.e. in } [0,T],$$

so that

$$|(f'(u^h)-f'(u),\theta^u_t)| \leq K(R)\|u^h-u\|_1|\theta^u_t|_0 \quad \text{a.e. in } [0,T].$$

Next, in (3.32), we use the coercivity condition (1.4) with $x = \theta^u_t$ and $y = \nabla\theta^w$ for the left-hand side, and we estimate the right-hand side by the Cauchy-Schwarz inequality and Young's inequality, and we obtain

$$|\theta^u_t|_0^2 + |\theta^w|_1^2 + \frac{d}{dt}|\theta^u|_1^2 \leq C\big(|\rho^u_t|_0^2 + \|u^h-u\|_1^2 + |\rho^w|_0^2 + |\rho^w|_1^2\big).$$

By the triangle inequality and by the Poincaré inequality,

$$|\theta^u_t|_0^2 + |\theta^w|_1^2 + \frac{d}{dt}|\theta^u|_1^2 \leq C|\theta^u|_1^2 + C\big(|(\theta^u(0),1)|^2 + |\rho^u|_1^2 + |\rho^u_t|_0^2 + |\rho^w|_1^2\big).$$

Finally, we apply Gronwall's lemma and (3.26):

$$\int_0^t |\theta^u_t|_0^2 + |\theta^w|_1^2 \,\mathrm{d}s + |\theta^u(t)|_1^2 \leq C\|\theta^u(0)\|_1^2 + Ch^2\int_0^t |u|_2^2 + |u_t|_2^2 + |w|_2^2 \,\mathrm{d}s,$$

for all $t \in [0, T]$. This estimate, together with (3.20), the Poincaré inequality, and the regularity assumptions on u and w, proves that

$$\int_0^t |\theta^w|_1^2 \, \mathrm{d}s + \|\theta^u(t)\|_1^2 \leq C\|\theta^u(0)\|_1^2 + Ch^2.$$

This inequality, together with (3.31) and (3.26), concludes the proof. □

Remark 3.8. If we choose $u^h(0) = R^h u(0)$, then clearly $\|u^h(0)\|_1 \leq R$ and the energy estimate is $O(h)$ as in (3.22). Another possibility is to choose u_0^h as the $L^2(\Omega)$ projection of u_0 on V^h: if $\{\mathcal{T}^h\}$ is quasiuniform, then $\|u - u_0^h\|_1 \leq Ch\|u\|_2$ (see for instance [13, Proposition 2.9.1]), and we also obtain an $O(h)$-error estimate.

4. Error estimates for the fully discrete problem

Now we derive energy estimates for the fully discrete scheme, which is simply the implicit Euler scheme applied to the semidiscrete scheme (3.6)-(3.8). We use the assumptions and notation of the previous section; in particular, $\{\mathcal{T}^h\}$ is a quasiuniform family of triangulations of $\overline{\Omega} \simeq \mathbb{R}^d/(\Pi_{i=1}^d L_i\mathbb{Z})$. We denote $\delta t = T/N$ the time step, with $N \in \mathbb{N}^\star$. The scheme reads: let $u_h^0 \in V^h$ and, for $n = 1, 2, \ldots$, find $(u_h^n, w_h^n) \in V^h \times V^h$ such that, for all $\chi \in V^h$,

$$\begin{aligned}(\bar{\partial} u_h^n, \chi) - (a \cdot \nabla \bar{\partial} u_h^n, \chi) + (B \nabla w_h^n, \nabla \chi) &= (m(t_n), \chi), &(4.1)\\ \beta(\bar{\partial} u_h^n, \chi) + \alpha(\nabla u_h^n, \nabla \chi) + (f'(u_h^n), \chi) & \\ -(w_h^n, \chi) + (b \cdot \nabla w_h^n, \chi) &= (\gamma(t_n), \chi), &(4.2)\end{aligned}$$

where we denote by $\bar{\partial}$ the operator which, to a sequence $(v^n)_{n \geq 0}$, associates the sequence defined by

$$\bar{\partial} v^n = \frac{v^n - v^{n-1}}{\delta t}, \quad n \geq 1.$$

Throughout this section, we assume (as previously) that the assumptions of Theorem 2.1 are satisfied. In order for $m(t_n)$ and $\gamma(t_n)$ to make sense, we assume in addition that $m, \gamma \in C^0([0, T]; L^2(\Omega))$. We also assume that there exists a constant $C_{f''} \geq 0$ such that

$$f''(s) \geq -C_{f''}, \quad \text{for all } s \in \mathbb{R}. \tag{4.3}$$

Define $\lambda_1 = 1/c_P^2$ where c_P is the best constant in the Poincaré inequality (2.8). With the same arguments as in the case without source terms [16], we can state the following.

Theorem 4.1. *With the assumptions above, if* $\delta t > 0$ *satisfies* $c_0 + \delta t(\alpha\lambda_1 - C_{f''}) > 0$*, then for any choice of the initial condition* $u_h^0 \in V^h$*, the system* (4.1)-(4.2) *defines a unique sequence* $(u_h^n, w_h^n)_{1\le n\le N} \in (V^h \times V^h)^{\mathbb{N}^\star}$*. Moreover, for all* $n \ge 1$*,*

$$\mathcal{E}(u_h^n) + \frac{c_0\delta t}{4}\left(|\nabla w_h^n|_0^2 + |\bar{\partial} u_h^n|_0^2\right) \le \mathcal{E}(u_h^{n-1}) + \frac{\delta t}{c_0}\left(c_P|m(t_n)|_0^2 + |\gamma(t_n)|_0^2\right). \tag{4.4}$$

Proof. We only prove the energy estimate. Let $n \ge 1$ and let (u_h^n, w_h^n) satisfy (4.1)-(4.2). Using assumption (4.3) (see also Lemma 3.1 in [16]), we have

$$\begin{aligned}\mathcal{E}(u_h^{n-1}) - \mathcal{E}(u_h^n) \ge \alpha(\nabla u_h^n, \nabla(u_h^{n-1} - u_h^n)) + (f'(u_h^n), u_h^{n-1} - u_h^n)\\ + (1/2)(\alpha\lambda_1 - C_{f''})|u_h^{n-1} - u_h^n|_0^2.\end{aligned}$$

We choose $\chi = w_h^n$ in (4.1), $\chi = \bar{\partial} u_h^n$ in (4.2), we add the resulting equations, and we use (1.4) and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\mathcal{E}(u_h^{n-1}) - \mathcal{E}(u_h^n) \ge c_0\delta t\left(|\nabla w_h^n|_0^2 + |\bar{\partial} u_h^n|_0^2\right) - c_P\delta t|m(t_n)|_0|\nabla w_h^n|_0\\ - \delta t|\gamma(t_n)|_0|\bar{\partial} u_h^n|_0 + (1/2)(\alpha\lambda_1 - C_{f''})\delta t^2|\bar{\partial} u_h^n|_0^2.\end{aligned}$$

We use now the assumption on δt and the Cauchy-Schwarz inequality in an obvious way to obtain the energy estimate (4.4). □

To estimate the error, we decompose as previously

$$\begin{aligned}u_h^n - u(t_n) &= \theta_u^n + \rho_u^n, \text{ with } \theta_u^n = u_h^n - \tilde{u}(t_n), \quad \rho_u^n = \tilde{u}(t_n) - u(t_n),\\ w_h^n - w(t_n) &= \theta_w^n + \rho_w^n, \text{ with } \theta_w^n = w_h^n - \tilde{w}(t_n), \quad \rho_w^n = \tilde{w}(t_n) - w(t_n),\end{aligned}$$

where the projections $\tilde{u} = \tilde{u}(t_n)$ and $\tilde{w} = \tilde{w}(t_n)$ of $u = u(t_n)$ and $w = w(t_n)$ on V^h are defined as previously by (3.10)-(3.13). The error estimate for the terms ρ_u^n and ρ_w^n is already known (Lemma 3.2). It remains to estimate the terms θ_u^n and θ_w^n. We will proceed as previously, by deriving the inequalities in the energy norm. The following lemma is a time discrete version of Lemma 3.4.

Lemma 4.2. *Let* $\delta t = T/N$ *with* $N \in \mathbb{N}^\star$ *sufficiently large. Assume that the solution* u *of* (2.1)-(2.3) *takes its values in* $[-\delta, \delta]$ *and that* $u_h^0 \in V^h$ *takes its values in* $[-\delta - 1/2, \delta + 1/2]$*. Let* $N^h \in \{1, \dots, N\}$ *be the largest integer such that* $u_h^n \in [-\delta - 1, \delta + 1]$ *for all* $n \in \{1, \dots, N^h\}$*, where* $(u_h^n, w_h^n)_{1\le n\le N}$ *denotes the solution of* (4.1)-(4.2)*. There exists* $C > 0$ *independent of* h *and*

N such that, for all $n \in \{1, \dots, N^h\}$,

$$c_1\big(|\bar\partial\theta_u^n|_0^2 + |\nabla\theta_w^n|_0^2\big) + \bar\partial|\nabla\theta_u^n|_0^2 \leq C|\nabla\theta_u^n|_0^2 + C\Big(|(\theta_u^0, 1)|^2 + |\rho_u^n|_0^2 + |\bar\partial\rho_u^n|_0^2 + |\bar\partial u(t_n) - u_t(t_n)|_0^2\Big), \quad (4.5)$$

with $c_1 = c_0/\alpha$. Moreover,

$$|(\theta_w^n, 1)| \leq C(|\rho_u^n|_0 + |\theta_u^n|_0), \quad \text{for all } n \in \{0, \dots, N^h\}, \quad (4.6)$$

and

$$(\theta_u^n, 1) = (\theta_u^0, 1), \quad \text{for all } n \in \{0, \dots, N\}. \quad (4.7)$$

Proof. We first write the equations satisfied by θ_u^n and θ_w^n. Proceeding as previously, we find (compare with (3.18)-(3.19)), for all $n \in \{1, ..., N\}$,

$$(\bar\partial\theta_u^n, \chi) - (a \cdot \nabla\bar\partial\theta_u^n, \chi) + (B\nabla\theta_w^n, \nabla\chi) = -(\bar\partial\rho_u^n, \chi) + (a \cdot \nabla\bar\partial\rho_u^n, \chi) - (\bar\partial u(t_n) - u_t(t_n), \chi) + (a \cdot \nabla[\bar\partial u(t_n) - u_t(t_n)], \chi), \quad (4.8)$$

for all $\chi \in V^h$, on the one hand, and

$$\beta(\bar\partial\theta_u^n, \chi) + \alpha(\nabla\theta_u^n, \nabla\chi) - (\theta_w^n, \chi) + (b \cdot \nabla\theta_w^n, \chi) = -(f'(u_h^n) - f'(u(t_n)), \chi) - \beta(\bar\partial\rho_u^n, \chi) - \beta(\bar\partial u(t_n) - u_t(t_n), \chi), \quad (4.9)$$

for all $\chi \in V^h$, on the other hand.

Choosing $\chi \equiv 1$ in (4.8), we find that, for $n \geq 1$,

$$(\bar\partial\theta_u^n, 1) = (\bar\partial\rho_u^n, 1) - (\bar\partial u(t_n) - u_t(t_n), 1) = 0, \quad (4.10)$$

where we have used the fact that $(u_t(t), 1) = 0$ for all t, and $(\rho_u^n, 1) = 0$. This gives equality (4.7).

Choosing $\chi \equiv 1$ in (4.9), we find

$$-(\theta_w^n, 1) = -(f'(u_h^n) - f'(u(t_n)), 1). \quad (4.11)$$

Thus, for all $n \in \{1, \dots, N^h\}$,

$$|(\theta_w^n, 1)| \leq L_f|u_h^n - u(t_n)|_0|\Omega|^{1/2} \leq L_f|\Omega|^{1/2}(|\theta_u^n|_0 + |\rho_u^n|_0),$$

where L_f denotes as previously the Lipschitz constant of f' on the interval $[-\delta - 1, \delta + 1]$. This is inequality (4.6).

Now we choose $\chi = \theta_w^n$ in (4.8) and $\chi = \bar\partial\theta_u^n$ in (4.9), and we add the resulting equations. We obtain

$$-(a \cdot \nabla\bar\partial\theta_u^n, \theta_w^n) + (B\nabla\theta_w^n, \nabla\theta_w^n) + \beta|\bar\partial\theta_u^n|_0^2 + (b \cdot \nabla\theta_w^n, \bar\partial\theta_u^n) + \alpha(\nabla\theta_u^n, \nabla\bar\partial\theta_u^n) = -(\bar\partial\rho_u^n, \theta_w^n) + (a \cdot \nabla\bar\partial\rho_u^n, \theta_w^n) - (\bar\partial u(t_n) - u_t(t_n), \theta_w^n) + (a \cdot \nabla[\bar\partial u(t_n) - u_t(t_n)], \theta_w^n) - (f'(u_h^n) - f'(u(t_n)), \bar\partial\theta_u^n) - \beta(\bar\partial\rho_u^n, \bar\partial\theta_u^n)$$

$$- \beta(\bar{\partial} u(t_n) - u_t(t_n), \bar{\partial}\theta_u^n).$$

Applying coercivity (1.4) for $\bar{\partial}\theta_u^n$ and $y = \nabla\theta_w^n$, as well as the inequality

$$\frac{1}{2}\bar{\partial}|v^n|_0^2 \leq (\bar{\partial} v^n, v^n),$$

we obtain, for $n \in \{1, \ldots, N^h\}$,

$$\begin{aligned}
&c_0\big(|\bar{\partial}\theta_u^n|_0^2 + |\nabla\theta_w^n|_0^2\big) + \frac{\alpha}{2}\bar{\partial}|\nabla\theta_u^n|_0^2 \\
&\leq c_P|\bar{\partial}\rho_u^n|_0|\nabla\theta_w^n|_0 + \|a\||\bar{\partial}\rho_u^n|_0|\nabla\theta_w^n|_0 + c_P|\bar{\partial}u(t_n) - u_t(t_n)|_0|\nabla\theta_w^n|_0 \\
&\quad + \|a\||\bar{\partial}u(t_n) - u_t(t_n)|_0|\nabla\theta_w^n|_0 + L_f|u_h^n - u(t_n)|_0|\bar{\partial}\theta_u^n|_0 + \beta|\rho_u^n|_0|\bar{\partial}\theta_u^n|_0 \\
&\quad + \beta|\bar{\partial}u(t_n) - u_t(t_n)|_0|\bar{\partial}\theta_u^n|_0.
\end{aligned}$$

Here we have used the Poincaré inequality (2.8) and the fact that $(\rho_u^n, 1) = 0$ by (3.13). Let $\epsilon > 0$ (to be determined later on). Using $ab \leq \epsilon a^2 + C_\epsilon b^2$, we find

$$\begin{aligned}
&c_0\big(|\bar{\partial}\theta_u^n|_0^2 + |\nabla\theta_w^n|_0^2\big) + \frac{\alpha}{2}\bar{\partial}|\nabla\theta_u^n|_0^2 \\
&\leq 4\epsilon|\nabla\theta_w^n|_0^2 + C_\epsilon(c_P^2 + \|a\|^2)|\bar{\partial}\rho_u^n|_0^2 + C_\epsilon(c_P^2 + \|a\|^2)|\bar{\partial}u(t_n) - u_t(t_n)|_0^2 \\
&\quad + 3\epsilon|\bar{\partial}\theta_u^n|_0^2 + 2C_\epsilon L_f^2(|\theta_u^n|_0^2 + |\rho_u^n|_0^2) + C_\epsilon\beta^2|\rho_u^n|_0^2 + C_\epsilon\beta^2|\bar{\partial}u(t_n) - u_t(t_n)|_0^2.
\end{aligned}$$

By the Poincaré inequality (2.8) and (4.10),

$$|\theta_u^n|_0 \leq c_P|\nabla\theta_u^n|_0 + |\Omega|^{-1/2}|(\theta_u^0, 1)|, \qquad \text{for all } n \geq 0.$$

Choosing $4\epsilon = c_0/2 \geq 3\epsilon$, we deduce that, for all $n \in \{1, \ldots, N^h\}$,

$$\begin{aligned}
&\frac{c_0}{2}\big(|\bar{\partial}\theta_u^n|_0^2 + |\nabla\theta_w^n|_0^2\big) + \frac{\alpha}{2}\bar{\partial}|\nabla\theta_u^n|_0^2 \\
&\leq C_1|\bar{\partial}\rho_u^n|_0^2 + C_2|\bar{\partial}u(t_n) - u_t(t_n)|_0^2 + C_3|\rho_u^n|_0^2 + C_4|(\theta_u^0, 1)|^2 + C_5|\nabla\theta_u^n|_0^2,
\end{aligned}$$

with $C_1 = C_\epsilon(c_P^2 + \|a\|^2)$, $C_2 = C_\epsilon(c_P^2 + \|a\|^2) + C_\epsilon\beta^2$, $C_3 = 2C_\epsilon L_f^2 + C_\epsilon\beta^2$, $C_4 = 4C_\epsilon L_f^2|\Omega|^{-1}$ and $C_5 = 4C_\epsilon L_f^2 c_P^2$. The claim is proved. We can notice that the constant C in the first inequality of the lemma depends only on the coefficients c_0, α, c_P, $\|a\|$, L_f, β and $|\Omega|$. □

Before establishing the final error estimate, we notice that Lemma 3.2 implies the following estimates, where we use the standard notation for the $W^{k,2}$ Sobolev spaces.

Lemma 4.3. *Assume that* $u \in W^{2,2}(0, T; H^2_{per}(\Omega))$, $w \in W^{1,2}(0, T; H^2_{per}(\Omega))$. *Then*

$$|\rho_u^n|_0 + h|\rho_u^n|_1 \leq Ch^2\left(\|u\|_{C^0([0,T]; H^2_{per})} + \|w\|_{C^0([0,T]; H^2_{per})}\right), \; 0 \leq n \leq N,$$

$$\delta t|\bar{\partial}\rho_u^n|_0^2 \le 2C^2h^4 \int_{t_{n-1}}^{t_n} |u_t(s)|_2^2 + |w_t(s)|_2^2 \,\mathrm{d}s, \quad 1 \le n \le N,$$

$$\delta t|\bar{\partial}u(t_n) - u_t(t_n)|_0^2 \le \delta t^2 \int_{t_{n-1}}^{t_n} |u_{tt}(s)|_2^2 \,\mathrm{d}s, \quad 1 \le n \le N,$$

where C is the constant of Lemma 3.2.

Proof. The first estimate is straightforward. For the second, we write, with $\rho^u(t) = \tilde{u}^h(t) - u(t)$,

$$\delta t|\bar{\partial}\rho_u^n|_0^2 = \frac{1}{\delta t}\Big|\int_{t_{n-1}}^{t_n} \rho_t^u(t)\Big|_0^2 \mathrm{d}t \le \int_{t_{n-1}}^{t_n} |\rho_t^u(t)|_0^2 \,\mathrm{d}t,$$

and the result follows. For the last inequality, the Taylor formula gives

$$\delta t[\bar{\partial}u(t_n) - u_t(t_n)] = u(t_n) - u(t_{n-1}) - \delta t u_t(t_n) = -\int_{t_{n-1}}^{t_n} (s - t_{n-1})u_{tt}(s)\,\mathrm{d}s,$$

and thus

$$\delta t^2|\bar{\partial}u(t_n) - u_t(t_n)|_0^2 \le \Big(\int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 \,\mathrm{d}s\Big)\Big(\int_{t_{n-1}}^{t_n} |u_{tt}(s)|_0^2 \,\mathrm{d}s\Big),$$

and the result follows. □

We finally obtain the following estimates, where C denotes a constant independent of h and δt.

Theorem 4.4. *Assume that the solution (u, w) of* (2.1) -(2.3) *satisfies*

$$u \in W^{2,2}(0, T; H^2_{per}(\Omega)) \text{ and } w \in W^{1,2}((0, T); H^2_{per}(\Omega)),$$

and let $(u_h^n, w_h^n)_{n\ge1}$ be the solution of (4.1)-(4.2). *If $\|\theta_u^0\|_1 \le Ch^2$ and if $\delta t = o(k_h)$, then*

$$\max_{0\le n\le N} |u_h^n - u(t_n)|_0 + \Big(\sum_{k=1}^N \delta t|w_h^n - w(t_n)|_0^2\Big)^{1/2} \le C(h^2 + \delta t),$$

$$\max_{0\le n\le N} \|u_h^n - u(t_n)\|_1 + \Big(\sum_{k=1}^N \delta t\|w_h^n - w(t_n)\|_1^2\Big)^{1/2} \le C(h + \delta t).$$

If, furthermore, $d = 1$ or 2, $u \in C^0([0, T]; H^2_{per}(\Omega))$ and $w \in C^0([0, T]; H^2_{per}(\Omega)$ then

$$\max_{0\le n\le N} \|u_h^n - u(t_n)\|_{L^\infty(\Omega)} + \Big(\sum_{k=1}^N \delta t\|w_h^n - w(t_n)\|_{L^\infty(\Omega)}^2\Big)^{1/2}$$

$$\leq Ch^2 \log(1/h)^{d-1} + C\delta t k_h.$$

Proof. Since $H^2_{per}(\Omega) \subset C^0_{per}(\bar{\Omega})$, u is bounded on $[0,T]$; i.e., there exist $\delta > 0$ such that $\|u(t)\|_{L^\infty(\Omega)} \leq \delta$, for all $t \in [0,T]$. On the other hand,

$$|u_h^0 - u_0|_0 \leq |\theta_u^0|_0 + |\rho_u^0|_0 \leq Ch^2,$$

using the assumption on θ_u^0 and Lemma 4.3. Proceeding as in (3.23), for h sufficiently small, $\|u_h^0 - u_0\|_{L^\infty(\Omega)} \leq 1/2$ and we can apply Lemmas 4.2 and 4.3. In particular, for all $n \in \{1, \ldots, N^h\}$,

$$c_1\big(|\bar{\partial}\theta_u^n|_0^2 + |\nabla\theta_w^n|_0^2\big) + \bar{\partial}|\nabla\theta_u^n|_0^2 \quad \leq \quad C|\nabla\theta_u^n|_0^2 + Ch^4 + Ch^4 v_1^n + C\delta t v_2^n,$$

where

$$v_1^n = \frac{1}{\delta t}\int_{t_{n-1}}^{t_n} |u_t(s)|_2^2 + |w_t(s)|_2^2 \,\mathrm{d}s, \text{ and } v_2^n = \int_{t_{n-1}}^{t_n} |u_{tt}(s)|_2^2 \,\mathrm{d}s.$$

We deduce, with the discrete Gronwall lemma (see Lemma 4.5 below), that, for all $n \in \{1, \ldots, N^h\}$,

$$|\nabla\theta_u^n|_0^2 + c_1\delta t\sum_{k=1}^{n}\big(|\bar{\partial}\theta_u^n|_0^2 + |\nabla\theta_w^n|_0^2\big) \leq C\Big(|\nabla\theta_u^0|_0^2 + \delta t\sum_{k=1}^{n}\big(h^4 + h^4 v_1^n + \delta t v_2^n\big)\Big),$$

where we have used the fact that $e^{2Cn\delta t} \leq e^{2CT}$ remains bounded independently of δt. For $n \in \{1, \ldots, N\}$, we have

$$\sum_{k=1}^{n} \delta t v_1^n = \int_0^{t_n} |u_t(s)|_2^2 + |w_t(s)|_2^2 \,\mathrm{d}s \leq C,$$

and

$$\sum_{k=1}^{n} v_2^n = \int_0^{t_n} |u_{tt}(s)|_2^2 \,\mathrm{d}s \leq C,$$

so, for $n \in \{1, \ldots, N^h\}$,

$$|\nabla\theta_u^n|_0^2 + c_1\delta t\sum_{k=1}^{n}\big(|\bar{\partial}\theta_u^n|_0^2 + |\nabla\theta_w^n|_0^2\big) \leq C\left(|\nabla\theta_u^0|_0^2 + h^4 + \delta t^2\right). \tag{4.12}$$

Since $(\theta_u^n, 1) = (\theta_u^0, 1)$ (cf. Lemma 4.2), we deduce from (4.12) that

$$\|\theta_u^n\|_1 \leq C(h^2 + \delta t), \qquad \text{for all } n \in \{0, \ldots, N^h\}.$$

Thus, using (3.4), Lemma 4.3 and (3.2), for all $n \in \{0, \ldots, N^h\}$,

$$\|u_h^n - u(t_n)\|_{L^\infty(\Omega)} \quad \leq \quad Ck_h\|u_h^n - I^h u(t_n)\|_1 + \|I^h u(t_n) - u(t_n)\|_{L^\infty(\Omega)},$$

$$\leq \quad Ck_h\Big(\|\theta_u^n\|_1 + \|\rho_n^u\|_1 + \|u(t_n) - I^h u(t_n)\|_1\Big) + Ch^\gamma,$$
$$\leq \quad Ck_h\big(h^2 + \delta t + h\big) + Ch^\gamma,$$

where $\gamma \in (0,1]$ is such that $H^2(\Omega) \subset C^{0,\gamma}(\bar{\Omega})$. Since $hk_h \to 0$ and $\delta t k_h \to 0$, for h sufficiently small,

$$\|u_h^n - u(t_n)\|_{L^\infty(\Omega)} \leq 1, \qquad \text{for all } n \in \{0,\dots,N^h\},$$

and thus $N^h = N$. Lemma 4.2 implies that $|(\theta_w^n, 1)| \leq C(h^2 + \delta t)$. We deduce the estimates in energy norms and weaker norms with (4.12) and Lemma 4.3. Note that we also obtain the estimate

$$\Big(\sum_{k=1}^{N} \delta t \left|\bar{\partial}(u_h^n - u(t_n))\right|_0^2\Big)^{1/2} \leq C(h^2 + \delta t).$$

Maximum norm estimates are deduced from (3.4) and Lemma 3.3. □

We used the following discrete version of Gronwall's lemma.

Lemma 4.5. *Let $(\delta^n)_{0\leq n\leq N}$, $(\epsilon^n)_{0\leq n\leq N}$ and $(\gamma^n)_{0\leq n\leq N}$ be three sequences of nonnegative real numbers satisfying*

$$\gamma^n + \bar{\partial}\delta^n \leq C\delta^n + \epsilon^n, \qquad \textit{for all } n \in \{1,\dots,N\},$$

where C is a constant independent of δt. Then, if $\delta t \leq 1/(2C)$, we have, for all $n \in \{1,\dots,N\}$,

$$\delta^n + \delta t \sum_{k=1}^{n} \gamma^k \leq e^{2Cn\delta t}\Big(\delta^0 + \delta t \sum_{k=1}^{n} \epsilon^k\Big).$$

Proof. The induction assumption is equivalent to

$$(1 - C\delta t)\delta^n \leq \delta^{n-1} + \delta t(\epsilon^n - \gamma^n), \qquad \text{for all } n \in \{1,\dots,N\}.$$

If $\delta t < 1/C$, this is equivalent to

$$\delta^n \leq \frac{1}{1 - C\delta t}\delta^{n-1} + \frac{\delta t(\epsilon^n - \gamma^n)}{1 - C\delta t}, \qquad \text{for all } n \in \{1,\dots,N\}.$$

By induction, we obtain, for all $n \in \{1,\dots,N\}$,

$$\delta^n \leq \frac{1}{(1 - C\delta t)^n}\delta^0 + \delta t \sum_{k=1}^{n} \frac{\epsilon^k - \gamma^k}{(1 - C\delta t)^{n+1-k}}.$$

Now we use the fact that, if $\delta t \in (0, 1/(2C)]$, then $1 \leq (1 - C\delta t)^{-1} \leq 1 + 2C\delta t \leq e^{2C\delta t}$, and the lemma is proved. □

As in the semidiscrete case, the following result shows that we can obtain the energy estimates under weaker assumptions. In particular, there is no restriction on the time step any more.

Theorem 4.6. *Let $\{\mathcal{T}^h\}$ be a regular family of triangulations of*

$$\overline{\Omega} \simeq \Pi_{i=1}^{d} \mathbb{R}/(L_i\mathbb{Z}).$$

Assume that the solution of (2.1)-(2.3) *satisfies*

$$u \in W^{2,2}(0,T;H^2_{per}(\Omega)) \quad \textit{and} \quad w \in L^2(0,T;H^2_{per}(\Omega)),$$

and let $(u_h^n, w_h^n)_{n\geq 1}$ be the solution of (4.1)-(4.2) *with initial condition u_h^0. If $\|u_h^0\|_1 \leq R$ for some constant $R > 0$ independent of h, then*

$$\max_{0\leq n\leq N} \|u_h^n - u(t_n)\|_1 + \Big(\sum_{k=1}^{N} \delta t \|w_h^n - w(t_n)\|_1^2\Big)^{\frac{1}{2}} \leq C\|u_h^0 - R^h u(0)\|_1 + C(h+\delta t).$$

Proof. We adapt the proof of Theorem 4.4 as in Theorem 3.7: the elliptic projection of $u(t_n)$ onto V^h is defined by $\tilde{u}_h^n = R^h u(t_n)$, and the nonlinear terms are estimated thanks to the polynomial growth of f. Details are left to the reader. □

5. SOME NUMERICAL ILLUSTRATIONS

In this section we illustrate the error estimates of Theorem 4.4 with some numerical simulations in one and two space dimensions, for the double-well potential $f(u) = (u^2-1)^2/4$. In one space dimension, we used the SCILAB[1] software and in two space dimensions, FREEFEM++[2].

In one space dimension, the space discretization by conforming P^1 finite elements is based on a regular subdivision of $\Omega = (0,1)$ with mesh stepsize $h = 1/N$ for an integer N. For the resolution of the fully discrete system (4.1)-(4.2), we write its matrix formulation in the nodal basis, and we use a Newton algorithm (see [16] for more details). We choose as a test case

$$u(x,t) = (1+t)\cos(2\pi x) \text{ and } w(x,t) = \cos(2\pi x),$$

with $B = 1$, so that

$$m(x,t) = \cos(2\pi x)(1+4\pi^2) + 2a\pi \sin(2\pi x)$$

and

$$\gamma(x,t) = (\alpha(1+t)4\pi^2 - (2+t) + \beta)\cos(2\pi x) + \cos(2\pi x)^3(1+t)^3 - 2b\pi\sin(2\pi x).$$

[1] `SCILAB` is freely available at `http://www.scilab.org/`

[2] `FreeFem++` is freely available at `http://www.freefem.org/ff++/`

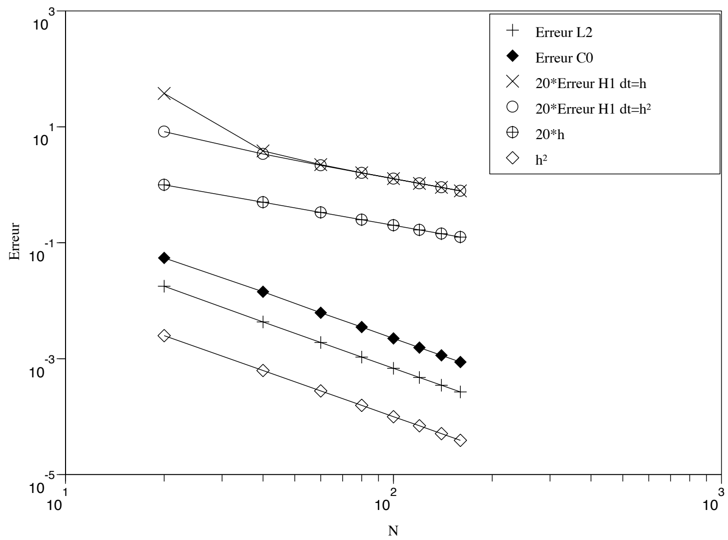

Figure 1. $1d$-error estimates for u

In particular, m satisfies the required assumption $(m(t), 1) = 0$ for all t. The integrals for the matrix formulation of (4.1)-(4.2) are computed by a Gauss quadrature formula of order 5, so that the nonlinear term is computed exactly. There is, however, a small error in the computation of the source terms, which is expected to be small in comparison with the error due to the discretization. The initial condition is chosen as the P^1 interpolate of $u(0)$ onto the mesh.

In Figure 1, we have represented error estimates for u and for various N in the $C^0([0,T]; L^2(\Omega))$-norm (simply denoted $L2$ in the figure), in the $C^0([0,T]; H^1(\Omega))$-norm (denoted $H1$ in the figure), and also in the $C^0([0,T]; L^\infty(\Omega))$-norm (denoted $C0$). The final time is $T = t_{n_{\max}} = 0.1$. The error estimate for the $C^0([0,T]; L^\infty(\Omega))$-norm (or more simply, the L^∞-norm) is more precisely computed by its following approximation:

$$\max_{0 \le n \le n_{\max}} \max_{1 \le i \le N} |u(x_i, t_n) - u_h^n(x_i)|,$$

where $x_i = ih$, $i = 0, 1, \ldots, N$ and $t_n = n\delta t$, $n = 0, 1, \ldots$ Similarly, the error estimate for the $C^0([0,T]; L^2(\Omega))$-norm is computed by

$$\max_{0 \le n \le n_{max}} \|u(t_n) - u_n^h\|_{L^2(\Omega)},$$

where the $L^2(\Omega)$-norm is computed by a Gauss quadrature formula of order 5: the error due to the quadrature formula is therefore small in comparison with the error which was obtained. In both cases, the time step is $\delta t = h^2$, since we expect a $O(h^2 + \delta t)$-error. This is confirmed in Figure 1: we see a $O(h^2)$-error (we have also represented the values h^2 as a comparison): this shows that there is no need to choose a smaller time step for this choice of norm.

The $C^0([0,T]; H^1(\Omega))$-error is computed similarly, and the result is plotted in Figure 1, for both $\delta t = h^2$ and $\delta t = h$ (since we expect a $O(h + \delta t)$-error). We see a $O(h)$-error, as expected; we have also represented the values $20h$ as a comparison, and we have represented 20 times the $C^0([0,T]; H^1(\Omega))$ error estimate for a nicer figure. For $N = 20$ and for $\delta t = h$; i.e., for h and δt relatively large, we notice that the error is a little worse than expected. This is partly related to the choice of the time step: if we choose $\delta t = h^2$, we find a better result. Otherwise the results for $\delta t = h$ and $\delta t = h^2$ coincide.

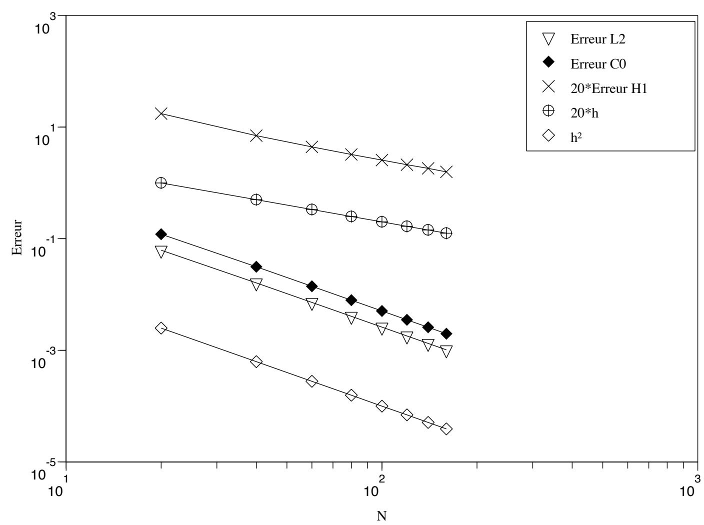

FIGURE 2. $2d$-error estimate for u

In two space dimensions, we use the test case

$$u(x, y, t) = (1 + t)\cos(2\pi x)\cos(2\pi y) \text{ and } w(x, y, t) = \cos(2\pi x)\cos(2\pi y)$$

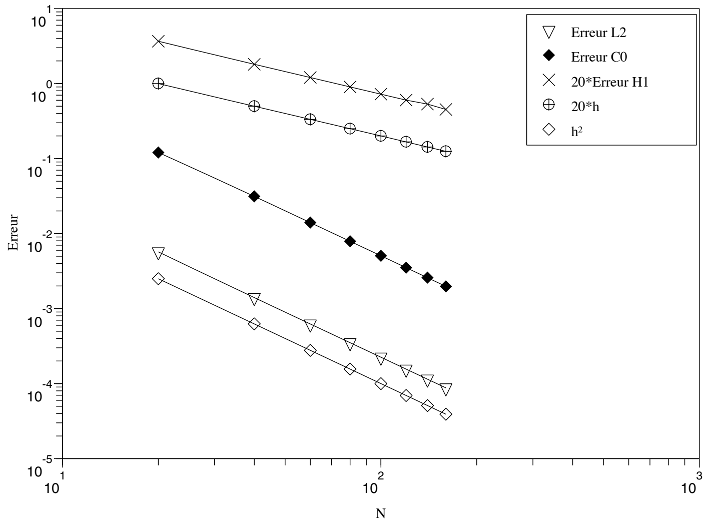

FIGURE 3. $2d$-error estimate for w

on the unit square $\Omega = (0,1) \times (0,1)$. We set

$$B = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} \text{ and } b = \begin{pmatrix} -0.1 \\ -0.1 \end{pmatrix},$$

and the source terms are defined accordingly. The other coefficients are $\alpha = 0.01$ and $\beta = 0.01$. The square Ω is divided into $N \times N$ squares of the same size $h \times h$ ($h = 1/N$), each of which is divided into two triangles. The nonlinear system (4.1)-(4.2) is linearized by a Newton algorithm, and we only need to write the corresponding variational formulation. The quadrature formula which was used in FREEFEM++ has order 5, so that the same considerations as above apply (exact computation of the nonlinear term, quadrature error for the source terms).

Figure 2 is the $2d$ version of Figure 1. The final time is again $T = 0.1$. Figure 3 is the analogue for w, except that the $C^0([0,T]; L^2(\Omega))$-norm is replaced by the $L^2(0,T; L^2(\Omega))$-norm, and the $C^0([0,T]; H^1(\Omega))$-norm is replaced with the $L^2(0,T; H^1(\Omega))$-norm. In agreement with the theoretical results of Theorem 4.4, we observe a $O(h^2)$-error for u in the $C^0([0,T]; L^2(\Omega))$ and the L^∞ norms, and for w in the $L^2(0,T; L^2(\Omega))$ and the L^∞ norms. We also observe a $O(h)$-error for the $C^0([0,T]; H^1(\Omega))$-norm and for the $L^2(0,T; H^1(\Omega))$-norm.

As a conclusion about these simulations, we note that it would be interesting to derive error estimates which take into account the quadrature error on the source terms.

APPENDIX

The purpose of this section is to prove convergence to equilibrium, as time goes to infinity, of the solution of the fully discrete scheme (4.1)-(4.2) without source terms, under some natural assumptions on the nonlinearity. Convergence to equilibrium for the solution of the continuous problem (2.1)-(2.3) without source terms was proved in [27] by the use of a Łojasiewicz inequality. In [16], we similarly proved convergence to equilibrium for the solution of the space semidiscrete scheme (3.6)-(3.8) (without source terms). For the solution of the fully discrete problem, we only obtained convergence to equilibrium under the assumption that the steady states are isolated; such an assumption is generally impossible to check. In [19], convergence to equilibrium for the solution of the backward Euler scheme applied to a gradient flow was proved by the use of a Łojasiewicz inequality (see also [2, Theorem 24]. Here, we prove convergence to equilibrium by adapting the proof of [19, Theorem 2.4].

The fully discrete problem without source terms reads: let $u_h^0 \in V^h$ and for $n = 1, 2, \ldots$, find $(u_h^n, w_h^n) \in V^h \times V^h$ such that

$$(\frac{u_h^n - u_h^{n-1}}{\delta t}, \chi) - (a \cdot \nabla \frac{u_h^n - u_h^{n-1}}{\delta t}, \chi) + (B\nabla w_h^n, \nabla\chi) = 0, \quad \text{for all } \chi \in V^h, \tag{5.1}$$

$$\begin{aligned} &\beta(\frac{u_h^n - u_h^{n-1}}{\delta t}, \chi) + \alpha(\nabla u_h^n, \nabla\chi) \\ &+(f'(u_h^n), \chi) - (w_h^n, \chi) + (b \cdot \nabla w_h^n, \chi) = 0, \qquad \text{for all } \chi \in V^h. \end{aligned} \tag{5.2}$$

Here V^h is defined by (3.1); V^h can more generally be defined as a finite-dimensional subspace of $H^1_{per}(\Omega) \cap L^\infty(\Omega)$ which contains the constants.

By the same arguments as in [16], we have the following existence, uniqueness and stability result.

Theorem 5.1. *Assume that f satisfies* (2.4)-(2.6) *and* (4.3) *and let $\delta t > 0$ be such that $c_0 + \delta t(\alpha\lambda_1 - C_{f''}) > 0$. Then, for every $u_h^0 \in V^h$,* (5.1)-(5.2) *defines a unique sequence $(u_h^n, w_h^n)_{n\geq 1} \in (V^h \times V^h)^{\mathbb{N}^\star}$. Moreover, for all*

$n \geq 1$,

$$\mathcal{E}(u_h^n) + c_0\delta t|\nabla w_h^n|_0^2 + c_0|u_h^n - u_h^{n-1}|_0^2/(2\delta t) \leq \mathcal{E}(u_h^{n-1}). \tag{5.3}$$

Recall that a steady state for (5.1)-(5.2) with initial data u_h^0 is $(u_h^\infty, w_h^\infty) \in V^h \times \mathbb{R}$ such that

$$\begin{cases} (u_h^\infty, 1) = (u_h^0, 1), \\ \alpha(\nabla u_h^\infty, \nabla\chi) + (f'(u_h^\infty), \chi) = (w_h^\infty, \chi), \qquad \text{for all } \chi \in V^h. \end{cases} \tag{5.4}$$

We have the following.

Theorem 5.2. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is real analytic and satisfies* (2.5), (2.6) *and* (4.3). *Let* $\delta t > 0$ *be such that* $c_0 + \delta t(\alpha\lambda_1 - C_{f''}) > 0$, *and let* $u_h^0 \in V^h$. *The sequence* $(u_h^n, w_h^n)_{n\geq 1}$ *uniquely defined by* (5.1)-(5.2) *converges to a steady state, i.e., a solution* $(u_h^\infty, w_h^\infty) \in V^h \times \mathbb{R}$ *of* (5.4), *as* n *tends to infinity.*

Proof. Let $(\varphi_i)_{1\leq i\leq m}$ be an orthonormal basis of V^h for the $L^2(\Omega)$-scalar product, with φ_1 a constant. We write

$$u_h^n = \sum_{i=1}^m u_i^n \varphi_i \quad \text{and} \quad w_h^n = \sum_{i=1}^m w_i^n \varphi_i.$$

Define the square matrices A, M_B, M_a, M_b and I as follows:

$$\begin{gathered} (A)_{ij} = (\nabla\varphi_i, \nabla\varphi_j), \quad (M_B)_{ij} = (B\nabla\varphi_i, \nabla\varphi_j), \\ (M_a)_{ij} = (a \cdot \nabla\varphi_i, \varphi_j), \quad (M_b)_{ij} = (b \cdot \nabla\varphi_i, \varphi_j), \\ (I)_{ij} = (\varphi_i, \varphi_j) = \delta_{ij}, \end{gathered}$$

for $1 \leq i, j \leq m$; define also

$$U^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_m^n \end{pmatrix}, \quad W^n = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}, \quad G^h(U^n) = \begin{pmatrix} (f'(u_h^n), \varphi_1) \\ \vdots \\ (f'(u_h^n), \varphi_m) \end{pmatrix}.$$

With these notations, equations (5.1)-(5.2) may be rewritten, by choosing $\chi = \varphi_i$, $i = 1, 2, \ldots$,

$$\begin{pmatrix} M_B^t & I + M_a \\ M_b^t - I & \beta I \end{pmatrix} \cdot \begin{pmatrix} W^n \\ (U^n - U^{n-1})/\delta t \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha A U^n - G^h(U^n) \end{pmatrix}. \tag{5.5}$$

It is difficult to work with this system because the matrix M_B is not invertible. However, denoting by M the square matrix of size $2m$ in the left-hand side of (5.5), we get rid of the constants by noticing that

$$M = \begin{pmatrix} 0 & 0 \cdots 0 & 1 & 0 \cdots 0 \\ 0 & & 0 & \\ \vdots & \dot{M}_B^t & \vdots & \dot{I} + \dot{M}_a \\ 0 & & 0 & \\ -1 & 0 \cdots 0 & \beta & 0 \cdots 0 \\ 0 & & 0 & \\ \vdots & \dot{M}_b^t - \dot{I} & \vdots & \beta \dot{I} \\ 0 & & 0 & \end{pmatrix},$$

where, for any square matrix $C = (C_{ij})_{1 \leq i,j \leq m}$ of size m, we denote by $\dot{C}$ the submatrix of size $m-1$ defined by $\dot{C} = (C_{ij})_{2 \leq i,j \leq m}$. The nonlinear system (5.5) is therefore equivalent to

$$u_1^n = u_1^{n-1}, \tag{5.6}$$

$$-w_1^n = -(G^h(u_1^{n-1}, \dot{U}^n))_1, \tag{5.7}$$

$$\dot{M} \cdot \begin{pmatrix} \dot{W}^n \\ (\dot{U}^n - \dot{U}^{n-1})/\delta t \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha \dot{A} \dot{U}^n - \dot{G}^h(u_1^{n-1}, \dot{U}^n) \end{pmatrix}, \tag{5.8}$$

where $\dot{X} = (x_2, \ldots, x_m)$ for any vector $X = (x_1, \ldots, x_m) \in \mathbb{R}^m$, and

$$\dot{M} = \begin{pmatrix} \dot{M}_B^t & (\dot{I} + \dot{M}_a) \\ \dot{M}_b^t - \dot{I} & \beta \dot{I} \end{pmatrix}.$$

With the notations above, the affine space $\{u_h \in V^h \,:\, (u_h, \varphi_1) = (u_h^0, \varphi_1)\}$ is identified to $\mathbb{R}^{m-1}$. The discrete energy on this space is

$$E(\dot{V}) = \mathcal{E}\Big(u_1^0 \varphi_1 + \sum_{i=2}^m v_i \varphi_i\Big), \qquad \text{for all } \dot{V} = (v_2, \ldots, v_m) \in \mathbb{R}^{m-1}.$$

Differentiating, we see that

$$(\nabla E_{\dot{V}})_i = \alpha(\dot{A}\dot{V})_i + (\dot{G}^h(u_1^0, \dot{V}))_i, \qquad i = 2, \ldots, m. \tag{5.9}$$

Now we take the Euclidean norm of equality (5.8) and we use (5.9) to obtain

$$\mu_1\big(\|\dot{W}^n\|^2 + \frac{\|\dot{U}^n - \dot{U}^{n-1}\|^2}{\delta t^2}\big) \leq \|\nabla E_{\dot{U}^n}\|^2 \leq \mu_2\big(\|\dot{W}^n\|^2 + \frac{\|\dot{U}^n - \dot{U}^{n-1}\|^2}{\delta t^2}\big), \tag{5.10}$$

where $0 < \mu_1 < \mu_2 < \infty$ are respectively the smallest and the largest eigenvalues of the symmetric positive definite matrix $\dot{M}^t \cdot \dot{M}$. By (5.3), $(E(\dot{U}^n))_n$ is nonincreasing, and since E is bounded from below, $E(\dot{U}^n) \to E^*$ when $n \to \infty$. Without loss of generality, we assume that $E^* = 0$. Since E is coercive on $\mathbb{R}^{m-1}$, $(\dot{U}^n)_n$ is bounded, so there exist $\dot{U}^\infty \in \mathbb{R}^{m-1}$ and a subsequence $(\dot{U}^{n_k})_{k\geq 0}$ such that $\dot{U}^{n_k} \to \dot{U}^\infty$ as $k \to \infty$. By (5.3) again, we have $\|\dot{U}^n - \dot{U}^{n-1}\| \to 0$ and $\dot{W}^n \to 0$. Furthermore, by (5.7), $w_1^{n_k}$ has a limit $w_1^\infty \in \mathbb{R}$. Choosing $n = n_k$ in (5.1)-(5.2), we find that the couple (u_h^∞, w_h^∞) associated to $(\dot{U}^\infty, W^\infty)$ is a steady state, i.e., a solution of (5.4) with $w_h^\infty \in \mathbb{R}$.

Now we use the analyticity assumption on f: by Lemma 5.3 in [17], the map $u \mapsto f'(u)$ is real analytic from $L^\infty(\Omega)$ into $L^\infty(\Omega)$. Since $V^h \subset L^\infty(\Omega)$, this implies that G^h is real analytic from $\mathbb{R}^m$ into $\mathbb{R}^m$, and, thus, E is real analytic on $\mathbb{R}^{m-1}$, as a function of the variables $(v_2, ..., v_m) = \dot{V}$. By the Łojasiewicz inequality [18], there exist $\theta \in (0,1)$, $\sigma > 0$ and $c_L > 0$ such that

$$\text{for all } \dot{V} \in \mathbb{R}^{m-1}, \quad \|\dot{V} - \dot{U}^\infty\| < \sigma \Rightarrow |E(\dot{V})|^{1-\theta} \leq c_L \|\nabla E_{\dot{V}}\|, \tag{5.11}$$

where we have used the fact that $E(\dot{U}^\infty) = E^\star = 0$. Let n be such that $\|\dot{U}^n - \dot{U}^\infty\| < \sigma$. We consider two cases.

Case 1: $E(\dot{U}^n) > E(\dot{U}^{n-1})/2$. Then

$$\begin{aligned}
&[E(\dot{U}^{n-1})]^\theta - [E(\dot{U}^n)]^\theta \\
&\geq \int_{E(\dot{U}^n)}^{E(\dot{U}^{n-1})} \theta . x^{\theta-1}\,\mathrm{d}x \geq \int_{E(\dot{U}^n)}^{E(\dot{U}^{n-1})} \theta . [E(\dot{U}^{n-1})]^{\theta-1}\,\mathrm{d}x, \\
&\overset{\text{case 1}}{\geq} 2^{\theta-1}\theta[E(\dot{U}^n)]^{\theta-1} . [E(\dot{U}^{n-1}) - E(\dot{U}^n)], \\
&\overset{(5.3)}{\geq} \frac{2^{\theta-2}\theta c_0 \delta t}{[E(\dot{U}^n)]^{1-\theta}} \Big(\|\dot{W}^n\|^2 + \frac{\|\dot{U}^n - \dot{U}^{n-1}\|^2}{\delta t^2} \Big), \\
&\overset{(5.10)}{\geq} \frac{2^{\theta-2}\theta c_0}{\mu_2^{1/2} c_L} \|\dot{U}^n - \dot{U}^{n-1}\|.
\end{aligned}$$

Case 2: $E(\dot{U}^n) \leq E(\dot{U}^{n-1})/2$. Then

$$\begin{aligned}
\|\dot{U}^n - \dot{U}^{n-1}\| &\overset{(5.3)}{\leq} \sqrt{\frac{2\delta t}{c_0}} [E(\dot{U}^{n-1}) - E(\dot{U}^n)]^{1/2} \leq \sqrt{\frac{2\delta t}{c_0}} [E(\dot{U}^{n-1})]^{1/2}, \\
&\overset{\text{case 2}}{\leq} \Big(1 - \frac{1}{\sqrt{2}}\Big)^{-1} \sqrt{\frac{2\delta t}{c_0}} \Big([E(\dot{U}^{n-1})]^{1/2} - [E(\dot{U}^n)]^{1/2} \Big).
\end{aligned}$$

In both cases, for all n such that $\|\dot{U}^n - \dot{U}^\infty\| < \sigma$, we have

$$\|\dot{U}^n - \dot{U}^{n-1}\| \leq \frac{2^{2-\theta}\mu_2^{1/2} c_L}{\theta c_0}\left([E(\dot{U}^{n-1})]^\theta - [E(\dot{U}^n)]^\theta\right) + 5\sqrt{\frac{\delta t}{c_0}}\left([E(\dot{U}^{n-1})]^{1/2} - [E(\dot{U}^n)]^{1/2}\right). \quad (5.12)$$

Let now $\tilde{E} > 0$ be small enough so that

$$\frac{2^{2-\theta}\mu_2^{1/2} c_L}{\theta c_0}\tilde{E}^\theta + 5\sqrt{\frac{\delta t}{c_0}}\tilde{E}^{1/2} \leq \sigma/3. \quad (5.13)$$

We choose $\bar{n} = n_k$ large enough so that $\|\dot{U}^{\bar{n}-1} - \dot{U}^\infty\| < \sigma/3$ and $E(\dot{U}^{\bar{n}-1}) \leq \tilde{E}$. Let $N \geq \bar{n} - 1$ be the largest integer (including $+\infty$) such that $\|\dot{U}^n - \dot{U}^\infty\| < 2\sigma/3$ for all $\bar{n} - 1 \leq n \leq N$. Assume by contradiction that N is finite. Then

$$\begin{aligned}\|\dot{U}^{N+1} - \dot{U}^\infty\| &\leq \|\dot{U}^{N+1} - \dot{U}^N\| + \|\dot{U}^N - \dot{U}^\infty\| \\ &\overset{(5.3)}{\leq} \sqrt{\frac{2\delta t}{c_0}}[E(\dot{U}^N)]^{1/2} + \|\dot{U}^N - \dot{U}^\infty\| \overset{(5.13)}{<} \sigma.\end{aligned}$$

We may therefore apply (5.12) to every $\bar{n} \leq n \leq N+1$, and we obtain, since $(E(\dot{U}^n))_n$ is nonincreasing,

$$\sum_{n=\bar{n}}^{N+1}\|\dot{U}^n - \dot{U}^{n-1}\| \leq \frac{2^{2-\theta}\mu_2^{1/2} c_L}{\theta c_0}[E(\dot{U}^{\bar{n}-1})]^\theta + 5\sqrt{\frac{\delta t}{c_0}}[E(\dot{U}^{\bar{n}-1})]^{1/2} \overset{(5.13)}{\leq} \sigma/3. \quad (5.14)$$

Thus

$$\|\dot{U}^{N+1} - \dot{U}^\infty\| \leq \|\dot{U}^{N+1} - \dot{U}^{\bar{n}-1}\| + \|\dot{U}^{\bar{n}-1} - \dot{U}^\infty\| < 2\sigma/3,$$

and this contradicts the definition of N. So $N = +\infty$, the estimate (5.14) still holds, and the whole sequence $(\dot{U}^n)_n$ converges to $\dot{U}^\infty$. We also know that $u_1^n = u_1^0$ for all n, that $\dot{W}^n \to 0$, and from (5.7) we deduce that $w_1^n \to w_1^\infty$ as $n \to \infty$. This concludes the proof. □

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